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FROBENIUS STRUCTURE AND THE BECK–CHEVALLEY CONDITION FOR ALGEBRAIC WEAK FACTORIZATION SYSTEMS

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Abstract. If a locally cartesian closed category carries a weak factorization system, then the left maps are stable under pullback along right maps if and only if the right maps are closed under pushforward along right maps. In this paper we state and prove an analogous result for algebraic weak factorization systems. These algebraic weak factorization systems are an explicit variant of the more traditional weak factorization systems in that the factorization and the lifts are part of the structure of an algebraic weak factorization system and are not merely required to exist. Our work has been motivated by the categorical semantics of type theory, where the closure of right maps under pushforward provides a useful tool for modelling dependent function types. We illustrate our ideas using split fibrations of groupoids, which are the backbone of the groupoid model of Hofmann and Streicher.

1. Introduction

In this paper we contribute to the theory of *algebraic weak factorization systems* (or AWFS, for short). The notion of an AWFS is due to Garner [Gar09], building on earlier ideas by Grandis and Tholen [GT06]. The notion of an AWFS is a refinement of the well-known notion of a weak factorization system (WFS), which has become quite important in homotopy theory and category theory. In particular, Quillen's influential notion of a model category involves two weak factorization systems which interact in a suitable manner [Hir03].

The notion of an algebraic weak factorization system can be seen as a more explicit version of the more familiar notion of a WFS. To explain this, recall that for morphisms f, g in a category $\mathcal{C}$, we say that f has the right lifting property against g and g the left lifting property against f, denoted $f \pitchfork g$, if for any solid commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "g"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet \end{tikzcd}$$

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in $\mathcal{C}$ there exists a dotted lift as shown. For a class of maps A in a category $\mathcal{C}$ we write:

$$\begin{aligned} A^{\pitchfork} &= \{f : (\forall g \in A)\, g \pitchfork f\}, \\ {}^{\pitchfork}A &= \{g : (\forall f \in A)\, g \pitchfork f\}. \end{aligned}$$

A WFS on a category $\mathcal{C}$ consists of a pair (L, R) of classes of maps in $\mathcal{C}$, such that the following two axioms are satisfied:

1. Any map in $\mathcal{C}$ factors as a map in L followed by a map in R.
2. ${}^{\pitchfork}R = L$ and $L^{\pitchfork} = R$.

The theory of AWFS is more "explicit" in that when we are given an AWFS, we will have explicit witnesses for the truth of axioms (1) and (2): that is, for an AWFS both the factorization in (1) as well as the lift in (2) can be obtained in an explicit manner and are not just assumed to exist. As a result, the theory of algebraic weak factorization systems has a number of advantages over the usual theory:

1. Due to its explicit nature, it is more constructive than the usual theory.
2. In an AWFS, the coalgebras are closed under colimits and the algebras under limits, in an appropriate sense.
3. They can be used to describe situations that are not WFS, because algebras and coalgebras need not be closed under retracts. For example, split Grothendieck fibrations are the algebras for a monad in an AWFS, while not being closed under retracts.

For the purposes of this paper, points (1) and (3) are the important ones. Indeed, we feel that point (1) has recently become especially important due to the influence of *homotopy type theory* [HoTT23]. Here, type theory refers to formal systems, like the Calculus of Constructions or Martin-Löf's intuitionistic type theory, which are both constructive foundations for mathematics and functional programming languages. Recent developments have amply demonstrated how ideas from homotopy theory and higher category theory are highly relevant to type theory. In particular, Voevodsky has shown how one can exploit the Kan–Quillen model structure on simplicial sets to construct a model of type theory which validates principles like his Univalence Axiom [KL21].

This paper focuses on an important aspect of model constructions like those of Voevodsky which make use of Quillen's model categories to obtain models of type theory: how they model Π-types. Π-types, or dependent function types, are arguably the most important type former in type theory and the argument for why these model categories model them is basically always as follows. The underlying category is a presheaf category and hence locally cartesian closed: this means that pullback functors have right adjoints, which we will call *pushforward functors*. To show that we have a model of Π-types, we need that pushforward functors along R-maps (fibrations) preserve R-maps (fibrations); we refer to this as the *pushforward property*. Using general facts about adjoints, this property is equivalent to the pullback functors along R-maps preserving L-maps. We will refer to the latter property as the *Frobenius property*. The Frobenius property is often easier to check than the pushforward property; for instance, the Frobenius property will hold in a model category if it is right proper and the cofibrations are stable under pullback.

The main contribution of this paper is to find suitable analogues of these statements for algebraic weak factorization systems. That is, we will formulate appropriate forms of the statement that algebras are closed under pushforward along algebras and the Frobenius property for AWFS, and we will prove that these are equivalent. In order to model type theory, one needs suitably "stable" versions of both the pushforward and the Frobenius property. This amounts to requiring an additional property, a version of the well-known Beck–Chevalley condition, and we will also formulate an appropriate version of the Beck–Chevalley conditions for AWFS. In order to do so, we use the definition of an AWFS that was given in recent work of John Bourke [Bou23], who makes extensive use of the language of double categories.

We start this paper by giving this definition of an AWFS in terms of double categories in Section 2. In Section 3 we will formulate the pushforward and Frobenius property for an AWFS and prove their equivalence; in order to do so, we will recast the usual equivalence as a statement about an adjunction between two slice WFS. It turns out that this formulation can be rather straightforwardly generalised to AWFS and in this form, it follows fairly directly from the work on Bourke and Garner [BG16]. In Section 4 we illustrate these ideas on our running example, an AWFS on groupoids in which the right class is formed by split fibrations of groupoids. These maps are precisely the maps which interpret the dependent types in the groupoid model of Hofmann and Streicher [HS98], which was a fundamental contribution to the development of Homotopy Type Theory. In Section 5 we state the Beck–Chevalley condition for AWFS, and also verify it for our running example. In Section 6 we consider various natural strengthenings of the Frobenius and Beck–Chevalley conditions, and we will find that most, but not all of them are satisfied in our running example. Finally, in Section 7, we show how one could use our framework to obtain models of type theory with dependent function types, as well as other type formers, such as identity and dependent sum types. This is again applied to groupoids, thus recovering the groupoid model of Hofmann and Streicher.

This paper is based on the MSc thesis of the first author, which was supervised by the second author [vW21].

2. Preliminaries

We begin by recalling some definitions and results that underlie our work. We will often work in the context of some (small) ambient category $\mathcal{C}$ which is (sufficiently) locally cartesian closed, and write $\mathcal{C}_1$ for its set (or class) of morphisms.

2.1. Weak factorization systems. Of central importance will be the notion of *lifting*. Given two subsets $J, K \subseteq \mathcal{C}_1$ we write $J \pitchfork K$ when for all $f \in J, g \in K$ every commutative square $(u,v) : f \longrightarrow g$ has a *lift* $\phi : \operatorname{cod} f \longrightarrow \operatorname{dom} g$ which makes both of the induced triangles commute:

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "v"'] \arrow[ur, dotted, "\phi"] & \bullet \end{tikzcd} . \tag{1}$$

Given $J \subseteq \mathcal{C}_1$ we define ${}^{\pitchfork}J = \{f \in \mathcal{C}_1 \mid \{f\} \pitchfork J\}$ and $J^{\pitchfork} = \{f \in \mathcal{C}_1 \mid J \pitchfork \{f\}\}$. For a pair $L, R \subseteq \mathcal{C}_1$ we can now ask whether the following axioms hold:

1. *Axiom of lifting*: $L = {}^{\pitchfork}R$ and $R = L^{\pitchfork}$.
2. *Axiom of factorization*: Every morphism $f \in \mathcal{C}_1$ admits a factorization $f = r.l$ with $l \in L$ and $r \in R$.

2.2. Definition. A pair (L, R) of subsets of $\mathcal{C}_1$ is a *weak factorization system* (wfs) if it satisfies the axioms of lifting and factorization, and a *pre*-wfs if it only satisfies the axiom of lifting.

In the context of a (pre-)wfs (L, R) the maps in L and R are referred to as left and right maps, respectively.

2.3. Categorical lifting operations. As an intermediate step towards the definition of awfs we discuss a categorical version of lifting, which is phrased with respect to functors with codomain $\mathbf{Ar}(\mathcal{C})$, where $\mathbf{Ar}(\mathcal{C})$ denotes the familiar category of arrows of $\mathcal{C}$. First some remarks on notation. We use the name of the domain of a functor $\mathcal{J} \longrightarrow \mathbf{Ar}(\mathcal{C})$ also to refer to the functor itself. Furthermore, following [BG16, Bou23], given a functor $\mathcal{J} \longrightarrow \mathbf{Ar}(\mathcal{C})$ we use bold type to denote an element $\mathbf{f} \in \mathcal{J}$, and italic type for the corresponding image f of $\mathbf{f}$ under $\mathcal{J}$. Similarly, we use double letters for morphisms in $\mathcal{J}$: given object $\mathbf{f}$ and $\mathbf{g}$ in $\mathcal{J}$, a map $\mathbf{f} \longrightarrow \mathbf{g}$ will usually be written $\mathbf{uv}$ and the image of this map under $\mathcal{J}$ will be $(u,v) : f \longrightarrow g$.

The idea of a category of maps is formalised as a functor $\mathcal{J} \longrightarrow \mathbf{Ar}(\mathcal{C})$; indeed, we think of $\mathcal{J}$ as consisting of a category of *structured* morphisms in $\mathcal{C}$ and structure-preserving maps between those. There is a notion of lifting of categories of maps, which is a more structured version of the one we saw in Section 2.1, as it asks for a specific and coherent choice of lifts.

2.4. Definition. Let $\mathcal{J}, \mathcal{K}$ be functors over $\mathbf{Ar}(\mathcal{C})$, then a $(\mathcal{J}, \mathcal{K})$-*lifting operation* ϕ assigns to each $\mathbf{f} \in \mathcal{J}$, $\mathbf{g} \in \mathcal{K}$, and lifting problem $(u,v) : f \longrightarrow g$, a solution $\phi_{\mathbf{f},\mathbf{g}}(u,v)$ as in (1). This choice is required to respect morphisms of $\mathcal{J}$ and $\mathcal{K}$, in the sense that if $\mathbf{wx} : \mathbf{f}' \longrightarrow \mathbf{f}$ in $\mathcal{J}$, $(u,v) : f \longrightarrow g$, and $\mathbf{yz} : \mathbf{g} \longrightarrow \mathbf{g}'$ in $\mathcal{K}$, then $y.\phi_{\mathbf{f},\mathbf{g}}(u,v).x = \phi_{\mathbf{f}',\mathbf{g}'}(w.u.y, x.v.z)$:

$$\begin{tikzcd} \bullet \arrow[r, "w"] \arrow[d, "f'"'] & \bullet \arrow[r, "u"] \arrow[d, "f"'] & \bullet \arrow[r, "y"] \arrow[d, "g"] & \bullet \arrow[d, "g'"] \\ \bullet \arrow[r, "x"'] & \bullet \arrow[r, "v"'] \arrow[ur, dotted, "\phi_{\mathbf{f},\mathbf{g}}"] & \bullet \arrow[r, "z"'] & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, "y.u.w"] \arrow[d, "f'"'] & \bullet \arrow[d, "g'"] \\ \bullet \arrow[r, "z.v.x"'] \arrow[ur, dotted, "\phi_{\mathbf{f}',\mathbf{g}'}"] & \bullet \end{tikzcd} .$$

Given $\mathcal{J} \longrightarrow \mathbf{Ar}(\mathcal{C})$, we can define a category $\mathcal{J}^{\pitchfork} \longrightarrow \mathbf{Ar}(\mathcal{C})$. The objects of $\mathcal{J}^{\pitchfork}$ are pairs (f, ϕ) with f a morphism in $\mathcal{C}$ and ϕ a $(\mathcal{J}, f)$-lifting operation (where f is considered a functor $f : 1 \longrightarrow \mathbf{Ar}(\mathcal{C})$), and a morphism $(u, v) : (f, \phi) \longrightarrow (g, \psi)$ is a commutative square $(u, v) : f \longrightarrow g$ that respects the lifting operations. The category ${}^{\pitchfork}\mathcal{J}$ over $\mathbf{Ar}(\mathcal{C})$ is defined analogously, and these assignments give rise to an adjunction:

$$(\mathbf{Cat}/\mathbf{Ar}(\mathcal{C}))^{\mathrm{op}} \underset{(-)^{\pitchfork}}{\overset{{}^{\pitchfork}(-)}{\underset{\perp}{\rightleftarrows}}} \mathbf{Cat}/\mathbf{Ar}(\mathcal{C}) \ . \tag{2}$$

For categories $\mathcal{J}, \mathcal{K}$ over $\mathbf{Ar}(\mathcal{C})$, any $(\mathcal{J}, \mathcal{K})$-lifting operation ϕ gives rise to two functors $\phi_l : \mathcal{J} \longrightarrow {}^{\pitchfork}\mathcal{K}$ and $\phi_r : \mathcal{K} \longrightarrow \mathcal{J}^{\pitchfork}$, which are each other's transpose according to (2).

This notion of lifting can be further expanded to a *double* categorical notion, and it is in those terms that AWFS can be conveniently defined, so we now specify what we mean by a double category.

2.5. Double categories. A double category $\mathbb{C}$ can be succinctly defined as an internal category in $\mathbf{Cat}$. This amounts to the following data, satisfying the usual axioms of a category:

$$\begin{tikzcd} \mathbb{C}_0 \arrow[r, "\mathrm{id}" description] & \mathbb{C}_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \arrow[l, "c"'] \end{tikzcd} .$$

More specifically, a double category $\mathbb{C}$ consists of a category $\mathbb{C}_0$ of objects and a category $\mathbb{C}_1$ of arrows, with operations id, s, t, c for identity arrow assignment, source, target, and composition operations, respectively. A double category has two types of morphisms: There are the morphisms of $\mathbb{C}_0$, called the *horizontal* morphisms of $\mathbb{C}$; and the objects of $\mathbb{C}_1$, called the *vertical* morphisms of $\mathbb{C}$. The morphisms of $\mathbb{C}_1$ are called *squares* of $\mathbb{C}$. There is an associated notion of double functor, and so we have a category $\mathbf{Dbl}$ of double categories, with evident projections $(-)_0, (-)_1 : \mathbf{Dbl} \longrightarrow \mathbf{Cat}$. There is a functor $\mathbb{S}\mathbf{q} : \mathbf{Cat} \longrightarrow \mathbf{Dbl}$ sending a category $\mathcal{C}$ to its *double category of squares* $\mathbb{S}\mathbf{q}(\mathcal{C})$. Its object category $(\mathbb{S}\mathbf{q}(\mathcal{C}))_0$ is $\mathcal{C}$ itself, and its arrow category $(\mathbb{S}\mathbf{q}(\mathcal{C}))_1$ is the familiar category $\mathbf{Ar}(\mathcal{C})$, whose objects are the arrows of $\mathcal{C}$ and whose morphisms are commutative squares in $\mathcal{C}$. Note that the $\mathbf{Ar}$ operation can be considered as a (2-)functor $\mathbf{Cat} \longrightarrow \mathbf{Cat}$.

Of particular interest in the context of AWFS are double functors of the form $\mathbb{J} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$ (which we will, as before, refer to simply as $\mathbb{J}$). Often, this double functor will be such that $\mathbb{J}_0$ is the identity and $\mathbb{J}_1$ is faithful. Following [BG16, Bou23], we refer to these as *concrete*. Note that a concrete double functor $\mathbb{J} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$ equips morphisms of $\mathcal{C}$ with a structure, via its arrow component $\mathbb{J}_1 \longrightarrow \mathbf{Ar}(\mathcal{C})$. The double functoriality of this assignment provides a composition operation for this structure, and ensures all identity arrows are assigned a structure.

By internalising the definition of natural transformations the category $\mathbf{Dbl}$ can be made a 2-category. Spelling this out, a *double* natural transformation $\alpha : F \Rightarrow G$ between double functors $F, G : \mathbb{C} \longrightarrow \mathbb{D}$ is a functor $\alpha : \mathbb{C}_0 \longrightarrow \mathbb{D}_1$ subject to various conditions. This means that, by virtue of how we defined horizontal and vertical morphisms, a double natural transformation would assign to each object of $\mathbb{C}$ a *vertical* morphism of $\mathbb{D}$. However, a second candidate for the 2-cells of **Dbl** is given by first *transposing* the double categories and functors involved. This is a type of duality for double categories in which the sets of vertical and horizontal arrows are switched. We thus obtain a notion of double natural transformation which assigns to each object of $\mathbb{C}$ a *horizontal* morphism of $\mathbb{D}$. Following [BG16], we will consider **Dbl** as a 2-category by taking the latter of these two options for the 2-cells.

2.6. Algebraic weak factorization systems. An AWFS is a significantly more structured object than a WFS, and at first sight their definitions appear quite distinct, because AWFS are usually defined in terms of an interacting monad-comonad pair, making no explicit mention of liftings as in diagram (1). However, it was recently shown in [Bou23] that AWFS admit an equivalent definition—there called *lifting* AWFS for distinction—very much akin to Definition 2.2 of WFS. In this definition of AWFS, the role of the subclasses $L, R \subseteq \mathcal{C}_1$ in the definition of a WFS are played by double functors $\mathbb{L}, \mathbb{R} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$ over the double category $\mathbb{S}\mathbf{q}(\mathcal{C})$ of squares in $\mathcal{C}$.

A lifting operation for a pair of double functors $\mathbb{J}, \mathbb{K} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$ is one for their underlying functors over $\mathbf{Ar}(\mathcal{C})$, satisfying a further coherence condition with respect to composition in $\mathbb{J}$ and $\mathbb{K}$.

2.7. Definition. Let $\mathbb{J}, \mathbb{K} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$; a $(\mathbb{J}, \mathbb{K})$-*lifting operation* ϕ is a $(\mathbb{J}_1, \mathbb{K}_1)$-lifting operation which respects composition in $\mathbb{J}$ and $\mathbb{K}$, in the sense that given vertical composable morphisms $\mathbf{f}, \mathbf{f}'$ in $\mathbb{J}$, and $\mathbf{g}, \mathbf{g}'$ in $\mathbb{K}$, as well as a lifting problem $(u, v) : f.f' \longrightarrow g.g'$, we have $\phi_{\mathbf{f},\mathbf{g}'}(\phi_{\mathbf{f}',\mathbf{g}.\mathbf{g}'}(u, v.f), \phi_{\mathbf{f}.\mathbf{f}',\mathbf{g}}(g'.u, v)) = \phi_{\mathbf{f}.\mathbf{f}',\mathbf{g}.\mathbf{g}'}(u, v)$:

$$
\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "f'"'] & \bullet \arrow[d, "g'"] \\
\bullet \arrow[d, "f"'] \arrow[ur, dotted] \arrow[r, phantom, "\phi_{\mathbf{f},\mathbf{g}'}"] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "v"'] \arrow[ur, dotted] \arrow[uur, dotted] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[dd, "f.f'"'] & \bullet \arrow[dd, "g.g'"] \\
\arrow[r, phantom, "\phi_{\mathbf{f}.\mathbf{f}',\mathbf{g}.\mathbf{g}'}"] & \\
\bullet \arrow[r, "v"'] \arrow[uur, dotted] & \bullet
\end{tikzcd} \,. \tag{3}
$$

Given $\mathbb{J} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$ we can define a double category $\mathbb{J}^{\pitchfork} \longrightarrow \mathbf{Ar}(\mathcal{C})$. Its vertical morphisms are pairs (f, ϕ) where f is a morphism in $\mathcal{C}$ and ϕ is a $(\mathbb{J}, f)$-lifting operation (where f is considered a double functor $f : \mathbf{2}_v \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$, with $\mathbf{2}_v$ denoting the free double category on a vertical arrow), and where vertical composition is performed according to (3). Squares of $\mathbb{J}^{\pitchfork}$ are commutative squares of $\mathcal{C}$ that respect the lifting operations. Similarly there is a double category ${}^{\pitchfork}\mathbb{J}$ over $\mathbf{Ar}(\mathcal{C})$, and these assignments give rise to an adjunction:

$$
(\mathbf{Dbl}/\mathbb{S}\mathbf{q}(\mathcal{C}))^{\mathrm{op}} \underset{(-)^{\pitchfork}}{\overset{{}^{\pitchfork}(-)}{\underset{\perp}{\leftrightarrows}}} \mathbf{Dbl}/\mathbb{S}\mathbf{q}(\mathcal{C}) \,. \tag{4}
$$

Given double categories $\mathbb{J}, \mathbb{K}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$, any $(\mathbb{J}, \mathbb{K})$-lifting operation ϕ gives rise to two double functors $\phi_l : \mathbb{J} \longrightarrow {}^{\pitchfork}\mathbb{K}$ and $\phi_r : \mathbb{K} \longrightarrow \mathbb{J}^{\pitchfork}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$, which are each other's transpose according to (4).

We now have the definitions in place to recall [Bou23, Definition 3] of a lifting AWFS. Given two double functors $\mathbb{L}$, $\mathbb{R}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$ with a $(\mathbb{J}, \mathbb{K})$-lifting operation ϕ we can phrase two axioms:

1. *Axiom of lifting*: The induced double functors ϕ_l and ϕ_r are invertible.
2. *Axiom of factorization*: Every morphism $f \in \mathcal{C}_1$ admits a bi-universal factorization $f = r.l$ for vertical maps $\mathbf{l} \in \mathbb{L}$ and $\mathbf{r} \in \mathbb{R}$.

The axiom of factorization will not play a big role in our work so we will not discuss it further; the reader is referred to [Bou23] for more details.

2.8. Definition. A triple $(\mathbb{L}, \phi, \mathbb{R})$ of double functors $\mathbb{L}, \mathbb{R}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$ with an $(\mathbb{L}, \mathbb{R})$-lifting operation ϕ is an *algebraic weak factorization system* (AWFS) if it satisfies the axioms of lifting and of factorization, and a *pre*-AWFS if it only satisfies the axiom of lifting.

We will often leave the reference to the lifting operation associated with an AWFS implicit, and simply refer to the pair $(\mathbb{L}, \mathbb{R})$ of double categories as the AWFS.

3. The Frobenius property

When $\mathcal{C}$ is locally cartesian closed, every morphism $f : A \longrightarrow B \in \mathcal{C}_1$ is *exponentiable*, meaning there is an adjunction $f^* : \mathcal{C}/B \longrightarrow \mathcal{C}/A \dashv f_*$ of pullback and pushforward along f, respectively. A *Frobenius property* of a WFS can be phrased with respect to these adjunctions, and one of the primary objectives of this work is to develop an algebraic version of this property, which we do in this section. We start by discussing the Frobenius property of a WFS in Section 3.1, and then define an analogous version of this property for an AWFS in Section 3.8.

3.1. The Frobenius property for a WFS. The Frobenius property of a WFS (L, R) is commonly phrased as the statement that "L is stable under pullback along R" (see e.g. [GS17, Definition 2.3]). Closely related to this is the statement that "R is closed under pushforward along itself," to which we will refer as the pushforward property of (L, R). To find suitable analogs for AWFS we will work with equivalent rephrasings of these conditions, and we begin by introducing these.

Consider a WFS (L, R); to say that L is stable under pullback along R means that for any right map $f : A \longrightarrow B \in R$ and any left map g with codomain B, the pullback of g along f is again a left map, as illustrated in the diagram on the left below:

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "g"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
,\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "f^\circ g"] \arrow[dd, bend right, "f^*(b.g)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r] \arrow[d, "f^*b"] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "b"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
.
$$

In a sense, this is to require that the action of the pullback functor f^* *on objects* (of the slice category $\mathcal{C}/B$) preserves left maps. We will instead require that the action of f^* *on arrows* (of $\mathcal{C}/B$) preserves left maps. Recall that an arrow of $\mathcal{C}/B$ is a pair (g, b) of composable morphisms in $\mathcal{C}$ such that b has codomain B. In this context, the action of f^* on arrows produces an arrow $f^*(g, b)$ of $\mathcal{C}/A$ which we (abusively) denote by $(f^\circ g, f^*b)$, as illustrated on the right above. When confusion is unlikely to arise we will simply write f^*g instead of $f^\circ g$. When we say that the arrow component of f^* preserves left maps we mean that $g \in L$ implies $f^\circ g \in L$, for any such g.

To state this succinctly we use the notion of *slicing* for classes of maps. Given a subclass $J \subseteq \mathcal{C}_1$ and an object $A \in \mathcal{C}$, we have a slice category $\mathcal{C}/A$, and define a subclass $J/A \subseteq (\mathcal{C}/A)_1$ of morphisms in $\mathcal{C}/A$ by $J/A = \{(f, a) \in (\mathcal{C}/A)_1 \mid f \in J\}$. Now we can say that the action on arrows of a pullback functor $f^* : \mathcal{C}/B \longrightarrow \mathcal{C}/A$ preserves left maps when $(g, b) \in L/B$ implies $f^*(g, b) \in L/A$, i.e. when $f^*(L/B) \subseteq L/A$.

3.2. Definition. Let (L, R) be a WFS on $\mathcal{C}$ and $f : A \longrightarrow B$ an exponentiable morphism in $\mathcal{C}$, then f has the *Frobenius property* if $f^*(L/B) \subseteq L/A$, and it has the *pushforward property* if $f_*(R/A) \subseteq R/B$. We say that (L, R) has the *Frobenius (pushforward) property* if every map in R has the Frobenius (pushforward) property.

3.2.1. Equivalence with conventional phrasing. We now show that this definition of the Frobenius property is indeed equivalent to its conventional phrasing. To do this, we rely on the fact that both classes of a WFS are closed under isomorphisms, and that the right class is closed under pullbacks, both of which are well-known consequences of the axiom of lifting of WFS.

3.3. Proposition. *Let (L, R) be a* WFS *on a category $\mathcal{C}$; then (L, R) satisfies the Frobenius property if and only if L is stable under pullback along R.*

Proof. For left-to-right, consider a right map $f : A \longrightarrow B$ and a left map g with codomain B. We then have $(g, 1)$ in L/B and so we get a left map $f^\circ g \in L$ as on the left below:

$$\begin{tikzcd}
\bullet \arrow[dd, "f^*g"'] & \bullet \arrow[l, "i"'] \arrow[r] \arrow[d, "f^\circ g"] & \bullet \arrow[d, "g"] \\
\arrow[r, phantom, "\cong"] & \bullet \arrow[r] \arrow[d, "f^*1"] & \bullet \arrow[d, "1"] \\
A & A \arrow[l, "1"'] \arrow[r, "f"] & B ,
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "j"] \arrow[d, "\varepsilon_b^* g"'] \arrow[dr, phantom, "\cong"] & \bullet \arrow[r] \arrow[d, "f^\circ g"] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "1"'] & \bullet \arrow[r, "\varepsilon_b"'] \arrow[d, "f^*b"] & \bullet \arrow[d, "b"] \\
& A \arrow[r, "f"] & B .
\end{tikzcd}
\tag{5}$$

There is an induced isomorphism $(i, f^*1) : f^\circ g \cong f^*g$ and so we indeed have that $f^*g \in L$ as desired. For the other direction, assume that L is stable under pullback along R. We consider a right map $f : A \longrightarrow B$ and a morphism $(g, b) \in L/B$, as on the right of (5). Letting ε denote the counit of $f_! \dashv f^*$, we have that $\varepsilon_b \in R$ because R is closed under pullbacks and $f \in R$. Therefore, it follows by assumption that $\varepsilon_b^* g \in L$, and thus by the induced isomorphism $(j, 1) : \varepsilon_b^* g \cong f^\circ g$ that $f^\circ g \in L$ as desired. ∎

Similarly, we have the following result for the pushforward property.

3.4. Proposition. *Let (L, R) be a* WFS *on a category $\mathcal{C}$; then (L, R) satisfies the pushforward property if and only if R is stable under pushforward along R.*

This statement is a bit cumbersome to prove directly. Instead, we will first establish a useful equivalence between the Frobenius and pushforward properties, as this will then allow us to derive Proposition 3.4 from Proposition 3.3.

3.4.1. Correspondence between pullback and pushforward. The pushforward property is relevant in the context of modelling type theory with WFS [AW09]. The idea is to interpret types as the right maps in a WFS, and the dependent product using the pushforward functors, so then it is necessary that the right class is closed under pushforward. It can be difficult to show that a particular WFS has the pushforward property. Instead, one can show that it has the Frobenius property, which tends to be a bit easier and is equivalent to the pushforward property. We now show that this equivalence follows readily from two well-known results on WFS.

Firstly, we recall that WFS can be constructed by slicing.

3.5. Proposition. *Let (L, R) be a* WFS *on $\mathcal{C}$ and $A \in \mathcal{C}$ an object, then $(L/A, R/A)$ is a* WFS *on $\mathcal{C}/A$.*

Secondly, we recall that WFS can be related along adjunctions, in the sense that a left adjoint preserves left maps if and only if its right adjoint preserves right maps.

3.6. Proposition. *Let $F \dashv G : \mathcal{D} \longrightarrow \mathcal{C}$ be an adjunction, (L, R) a* WFS *on $\mathcal{C}$, and (L', R') a* WFS *on $\mathcal{D}$, then $F(L) \subseteq L'$ if and only if $G(R') \subseteq R$.*

Putting these together, we obtain the aforementioned equivalence.

3.7. Theorem. *Let (L, R) be a* WFS *on $\mathcal{C}$ and $f : A \longrightarrow B$ an exponentiable morphism in $\mathcal{C}$, then f has the Frobenius property if and only if it has the pushforward property.*

Proof. By Proposition 3.5 there are slice WFS $(L/A, R/A)$ and $(L/B, R/B)$ on $\mathcal{C}/A$ and $\mathcal{C}/B$, respectively. Furthermore, by assumption that f is exponentiable, there is an adjunction $f^* \dashv f_* : \mathcal{C}/A \longrightarrow \mathcal{C}/B$ of pullback and pushforward along f. Proposition 3.6 thus tells us that $f^*(L/B) \subseteq L/A$ if and only if $f_*(R/A) \subseteq R/B$, as desired. ∎

A similar equivalence holds for the conventional phrasing of the Frobenius and pushforward properties: given a WFS (L, R), L is stable under pullback along R iff R is stable under pushforward along itself; see e.g. [GS17, Proposition 2.4].[1] It should be noted that this version, in contrast with Theorem 3.7, does not seem to hold "point-wise." In other words, fixing a map $f : A \longrightarrow B$, it does not seem to hold in general that the action of f^* on objects preserves L maps if and only if the action of f_* on objects preserves R maps (even if we assumed in addition that $f \in R$).

We can now give a simple proof of Proposition 3.4.

[1] The equivalence of (i) and (ii) in [GS17, Proposition 2.4] does not depend on freeness of the WFS.

Proof of Proposition 3.4. For a WFS (L,R) we have the following string of equivalences:

$$\begin{aligned}(L,R) \text{ has the pushforward property} &\text{ iff } (L,R) \text{ has the Frobenius property}\\ &\text{ iff } L \text{ is stable under pullback along } R\\ &\text{ iff } R \text{ is stable under pushforward along } R,\end{aligned}$$

by Theorem 3.7, Proposition 3.3, and [GS17, Proposition 2.4], respectively. ∎

Propositions 3.3 and 3.4 show that our "arrow versions" of the Frobenius and pushforward properties indeed coincides with the usual "object versions" that are used in the literature. The reason we opt for the arrow versions is that they are readily adapted to AWFS, as the next section shows.

3.8. The Frobenius property for AWFS. The Frobenius property of a morphism $f : A \longrightarrow B$ with respect to a WFS (L,R) states that (the action on arrows of) its pullback functor f^* preserves left maps, i.e., the restriction of $(f^*)_1 : (\mathcal{C}/B)_1 \longrightarrow (\mathcal{C}/A)_1$ to L/B should land in L/A. This can be expressed diagrammatically as the requirement that there exists a (necessarily unique) filler as on the left below:

$$\begin{tikzcd} L/B \arrow[r, dotted] \arrow[d, hook] & L/A \arrow[d, hook] \\ (\mathcal{C}/B)_1 \arrow[r, "(f^*)_1"] & (\mathcal{C}/A)_1 \end{tikzcd} , \qquad \begin{tikzcd} R/A \arrow[r, dotted] \arrow[d, hook] & R/B \arrow[d, hook] \\ (\mathcal{C}/A)_1 \arrow[r, "(f_*)_1"] & (\mathcal{C}/B)_1 \end{tikzcd} . \tag{6}$$

The diagram on the right above corresponds to the pushforward property of f. There are evident analogs of these diagrams for an AWFS $(\mathbb{L},\mathbb{R})$. To phrase them we first need an analog of slicing for double categories of maps:

3.9. Definition. Given a double functor $\mathbb{J} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$ and an object $A \in \mathcal{C}$ we define the *slice* double functor $\mathbb{J}/A \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C}/A)$ by the following pullback of double functors:

$$\begin{tikzcd} \mathbb{J}/A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{J} \arrow[d] \\ \mathbb{S}\mathbf{q}(\mathcal{C}/A) \arrow[r, "\mathbb{S}\mathbf{q}(\mathrm{dom})"] & \mathbb{S}\mathbf{q}(\mathcal{C}) \end{tikzcd} .$$

Here $\mathrm{dom} : \mathcal{C}/A \longrightarrow \mathcal{C}$ denotes the forgetful functor that sends a morphism to its domain.

Note that the definition of the slice $J/A \subseteq (\mathcal{C}/A)_1$ that we used for classes of maps can similarly be seen as the pullback of $J \subseteq \mathcal{C}_1$ along $\mathrm{dom}_1 : (\mathcal{C}/A)_1 \longrightarrow \mathcal{C}_1$, which shows that Definition 3.9 is just the double categorical version of the slicing operation that we used with WFS. Similarly, we now have a double categorical version of Definition 3.2:

3.10. Definition. Let $(\mathbb{L}, \mathbb{R})$ be an AWFS on a category $\mathcal{C}$, and $f : A \longrightarrow B$ an exponentiable morphism in $\mathcal{C}$. A *Frobenius structure* on f for $(\mathbb{L}, \mathbb{R})$ is a double functor $\mathbf{f}^*$ that fits into a commutative diagram as on the left below:

$$\begin{tikzcd} \mathbb{L}/B \arrow[r, dotted, "\mathbf{f}^*"] \arrow[d] & \mathbb{L}/A \arrow[d] \\ \mathbb{S}\mathbf{q}(\mathcal{C}/B) \arrow[r, "\mathbb{S}\mathbf{q}(f^*)"] & \mathbb{S}\mathbf{q}(\mathcal{C}/A) \end{tikzcd}, \qquad \begin{tikzcd} \mathbb{R}/A \arrow[r, dotted, "\mathbf{f}_*"] \arrow[d] & \mathbb{R}/B \arrow[d] \\ \mathbb{S}\mathbf{q}(\mathcal{C}/A) \arrow[r, "\mathbb{S}\mathbf{q}(f_*)"] & \mathbb{S}\mathbf{q}(\mathcal{C}/B) \end{tikzcd}. \tag{7}$$

Similarly, a *pushforward structure* on f is a double functor $\mathbf{f}_*$ as on the right above. A *Frobenius (pushforward) structure* for $(\mathbb{L}, \mathbb{R})$ is an assignment of a Frobenius (pushforward) structure to every vertical morphism $\mathbf{f} \in \mathbb{R}$.

Note the similarity between the diagrams in (6) and (7). Both phrase the requirement that the pullback functor f^* preserves left maps; however, in line with the view of an AWFS as a more structured analog of a WFS, we have that (6) is a property of f whereas (7) is additional structure, because the Frobenius structure $\mathbf{f}^*$ needs to produce a specific witness for the $\mathbb{L}$-membership of the pullback. Moreover, this assignment should respect the squares and vertical composition of left maps, which is encoded as the double functoriality of $\mathbf{f}^*$.

3.11. Remark. In the context of viewing an AWFS as an interacting monad-comonad pair (L, R), it follows from [BG16, Proposition 2] that a Frobenius structure on f makes f^* an oplax morphism of slice AWFS $(\mathsf{L}/B, \mathsf{R}/B) \longrightarrow (\mathsf{L}/A, \mathsf{R}/A)$, while a pushforward structure on f makes f_* a lax morphism of slice AWFS $(\mathsf{L}/A, \mathsf{R}/A) \longrightarrow (\mathsf{L}/B, \mathsf{R}/B)$.

3.11.1. Correspondence between pullback and pushforward. An argument that is very similar to the proof of Theorem 3.7 shows that Frobenius and pushforward structures are in bijection with each other. It relies on the same two ingredients: slice AWFS, and change of base along an adjunction. Both of these ingredients were established in [BG16], and are phrased as follows.

3.12. Proposition. *Let $(\mathbb{L}, \mathbb{R})$ be an* AWFS *on $\mathcal{C}$ and $A \in \mathcal{C}$ an object, then there is a slice* AWFS $(\mathbb{L}/A, \mathbb{R}/A)$ *on $\mathcal{C}/A$.*

Proof. The slice AWFS is obtained as the injective (or projective) lift of $(\mathbb{L}, \mathbb{R})$ along the forgetful functor $\mathrm{dom} : \mathcal{C}/A \longrightarrow \mathcal{C}$; see [BG16, Section 4.5] for more details. ∎

3.13. Proposition. *Let $F \dashv G : \mathcal{D} \longrightarrow \mathcal{C}$ be an adjunction, $(\mathbb{L}, \mathbb{R})$ an* AWFS *on $\mathcal{C}$, and $(\mathbb{L}', \mathbb{R}')$ an* AWFS *on $\mathcal{D}$, then there is a bijection between lifts $\mathbf{F}$ and $\mathbf{G}$ as in:*

$$\begin{tikzcd} \mathbb{L} \arrow[r, dotted, "\mathbf{F}"] \arrow[d] & \mathbb{L}' \arrow[d] \\ \mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd}, \qquad \begin{tikzcd} \mathbb{R}' \arrow[r, dotted, "\mathbf{G}"] \arrow[d] & \mathbb{R} \arrow[d] \\ \mathbb{S}\mathbf{q}(\mathcal{D}) \arrow[r, "\mathbb{S}\mathbf{q}(G)"] & \mathbb{S}\mathbf{q}(\mathcal{C}) \end{tikzcd}.$$

Proof. This is a straightforward consequence of [BG16, Proposition 21]. ∎

Putting these together, we obtain an algebraic analog of Theorem 3.7:

3.14. THEOREM. *Let* $(\mathbb{L}, \mathbb{R})$ *be an* AWFS *on* $\mathcal{C}$ *and* $f : A \longrightarrow B$ *an exponentiable morphism in* $\mathcal{C}$*, then there is a bijection between Frobenius and pushforward structures on* f*.*

PROOF. By Proposition 3.12 there are slice AWFS $(\mathbb{L}/A, \mathbb{R}/A)$ and $(\mathbb{L}/B, \mathbb{R}/B)$ on $\mathcal{C}/A$ and $\mathcal{C}/B$, respectively. Furthermore, by assumption that f is exponentiable, there is an adjunction $f^* \dashv f_*$ of pullback and pushforward along f. Proposition 3.13 thus tells us that there is a bijection between lifts $\mathbf{f}^* : \mathbb{L}/B \longrightarrow \mathbb{L}/A$ of $\mathbb{S}\mathbf{q}(f^*)$, and lifts $\mathbf{f}_* : \mathbb{R}/A \longrightarrow \mathbb{R}/B$ of $\mathbb{S}\mathbf{q}(f_*)$, as desired. ∎

We mention that Theorem 3.14 could be phrased slightly more generally. An AWFS is *cofibrantly generated* by a double category $\mathbb{J} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$ if $\mathbb{R} \cong \mathbb{J}^{\pitchfork}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$. The reasoning in the proof of Theorem 3.14 shows that for an AWFS which is cofibrantly generated by $\mathbb{J}$ the pushforward structures on f are in bijection with lifts $\mathbf{f}^* : \mathbb{J}/B \longrightarrow \mathbb{L}/A$ of $\mathbb{S}\mathbf{q}(f^*)$. Since any AWFS is cofibrantly generated by its left class, this can be seen as a more general statement. Furthermore, neither the definition of a Frobenius/pushforward structure nor the proof of Theorem 3.14 rely on the axiom of factorization of an AWFS, which means they apply more generally to pre-AWFS.

4. An example of reflections and fibrations

To exemplify our definitions we will use the AWFS of *split reflections* and *split fibrations*. This AWFS is also discussed by [GL23, Bou23]; some of the explanations below are based on those works, and the reader is referred there for additional details.

A split reflection is a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ paired with a left inverse $F : \mathcal{C} \longrightarrow \mathcal{D}$ such that $F \dashv G$ with identity counit. The unit of such an adjunction is necessarily unique, so that the split reflection is actually already fully specified by (G, F) (see [BG16, Section 4.2]). Split reflections can be given categorical structure by considering commutative squares of functors that also commute with their left adjoints. More specifically, a morphism between split reflections $(G, F, \eta) \longrightarrow (G', F', \eta')$ is a commutative square of functors $(U, V) : G \longrightarrow G'$ such that also $U.F = F'.V$:

$$\begin{tikzcd} \bullet \arrow[r, "U"] \arrow[d, shift left, "G"] & \bullet \arrow[d, shift left, "G'"] \\ \bullet \arrow[r, "V"'] \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"] & \bullet \arrow[u, shift left, "F'"] \arrow[u, phantom, "\dashv"] \end{tikzcd} .$$

This is sufficient to ensure that (U, V) also commutes with the units, in the sense that $V \circ \eta = \eta' \circ V$, where $\circ$ denotes horizontal composition. This gives a functor $\mathbf{SplRef}(\mathbf{Cat}) \longrightarrow \mathbf{Ar}(\mathbf{Cat}) : (G, F) \mapsto G$ that forgets the left adjoint. Together with the usual composition operation on adjunctions, it can be made the arrow component of a concrete double functor $\mathbb{S}\mathbf{plRef}(\mathbf{Cat}) \longrightarrow \mathbb{S}\mathbf{q}(\mathbf{Cat})$.

A fibration is called split if it comes equipped with a specified cleavage (i.e. a choice of cartesian lifts) that preserves identities and compositions on the nose. A morphism of split fibrations is a commutative square that also commutes with the splitting. This yields a category $\mathbf{SplFib}(\mathbf{Cat})$ together with a (faithful) forgetful functor $\mathbf{SplFib}(\mathbf{Cat}) \longrightarrow \mathbf{Ar}(\mathbf{Cat})$. Split fibrations can be composed, by composing the splittings, and so this forgetful functor fits into a concrete double category $\mathbb{S}\mathbf{plFib}(\mathbf{Cat}) \longrightarrow \mathbb{S}\mathbf{q}(\mathbf{Cat})$.

The pair $(\mathbb{S}\mathbf{plRef}(\mathbf{Cat}), \mathbb{S}\mathbf{plFib}(\mathbf{Cat}))$ form an AWFS on $\mathbf{Cat}$ (see e.g. [Bou23, Example 4] for a proof of this). This AWFS does not seem to admit a Frobenius structure, however, and so we will instead consider reflections and fibrations of groupoids. By taking the injective (or, equivalently, the projective) lift of $(\mathbb{S}\mathbf{plRef}(\mathbf{Cat}), \mathbb{S}\mathbf{plFib}(\mathbf{Cat}))$ along the inclusion $\mathbf{Gpd} \longrightarrow \mathbf{Cat}$ we obtain the AWFS $(\mathbb{S}\mathbf{plRef}(\mathbf{Gpd}), \mathbb{S}\mathbf{plFib}(\mathbf{Gpd}))$ of split reflections and split fibrations of groupoids. This AWFS does admit a Frobenius structure, as the proposition below shows, which extends [GL23, Proposition 5.3].

Before we proceed we fix some notation. Note that a split fibration of groupoids $P : \mathcal{A} \longrightarrow \mathcal{B}$ is also an opfibration. More specifically, for an object $a \in \mathcal{A}$, any morphism $f : Pa \longrightarrow b$ in $\mathcal{B}$ has a cocartesian lift along P, which we denote by $\underline{f} : a \longrightarrow f_!a$, that is obtained as the inverse of the chosen cartesian lift of f^{-1} at a:

$$\begin{tikzcd} a \arrow[r, "\underline{f}"] \arrow[d, dash, dashed] & f_!a \arrow[d, dash, dashed] \\ Pa \arrow[r, "f"] & b \end{tikzcd} .$$

Note that this notation is imprecise, since the lift $\underline{f}$ also depends on a. Each morphism $f : b \longrightarrow b' \in \mathcal{B}$ thus induces a functor $f_! : \mathcal{A}(b) \longrightarrow \mathcal{A}(b')$, where $\mathcal{A}(b)$ denotes the preimage category under P, sending an element $a \in \mathcal{A}(b)$ to the codomain of $\underline{f}$.

4.1. PROPOSITION. $(\mathbb{S}\mathbf{plRef}(\mathbf{Gpd}), \mathbb{S}\mathbf{plFib}(\mathbf{Gpd}))$ *admits a Frobenius structure.*

PROOF. We show that a split fibration of groupoids $P : \mathcal{A} \longrightarrow \mathcal{B}$ has a lift $\mathbf{P}^*$ as in:

$$\begin{tikzcd} \mathbb{S}\mathbf{plRef}(\mathbf{Gpd})/\mathcal{B} \arrow[r, "\mathbf{P}^*"] \arrow[d] & \mathbb{S}\mathbf{plRef}(\mathbf{Gpd})/\mathcal{A} \arrow[d] \\ \mathbb{S}\mathbf{q}(\mathbf{Gpd}/\mathcal{B}) \arrow[r, "\mathbb{S}\mathbf{q}(P^*)"] & \mathbb{S}\mathbf{q}(\mathbf{Gpd}/\mathcal{A}) \end{tikzcd} .$$

To this end, consider a split reflection $L \dashv R$ with unit η as on the left below; the goal is to construct a corresponding split reflection $G \dashv P^*R$:

$$\begin{tikzcd} \mathcal{A} \times_{\mathcal{B}} \mathcal{D} \arrow[r] \arrow[d, shift left, "P^*R"] & \mathcal{D} \arrow[d, shift left, "R"] \\ \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \arrow[r, "\pi_{\mathcal{C}}"'] \arrow[u, shift left, dotted, "G"] \arrow[d, shift left, "P^*U"] & \mathcal{C} \arrow[u, shift left, "L"] \arrow[d, "U"] \\ \mathcal{A} \arrow[r, "P"'] \arrow[u, shift left, dotted, "F"] & \mathcal{B} \end{tikzcd} , \qquad \begin{tikzcd} a \arrow[r, "\underline{U\eta_c}"'] \arrow[d, dash, dashed] & (U\eta_c)_!a \arrow[d, dash, dashed] \\ Uc \arrow[r, "U\eta_c"'] & URLc \end{tikzcd} , \qquad \begin{tikzcd} a_1 \arrow[r, "\underline{U\eta_{c_1}}"'] \arrow[d, "x"'] & F(a_1, c_1) \arrow[d, dotted, "F(x,y)"] \\ a_2 \arrow[r, "\underline{U\eta_{c_2}}"'] & F(a_2, c_2) \end{tikzcd} . \tag{8}$$

To achieve this we first define an auxiliary functor $F : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \longrightarrow \mathcal{A}$, which is defined on objects by $F(a, c) = (U\eta_c)_! a$, and on an arrow $(x, y) : (a_1, c_1) \longrightarrow (a_2, c_2)$ by the universal property of the cocartesian lift of $U\eta_{c_1}$; see the illustration on the right above. We can now define G by $(a, c) \mapsto (F(a, c), Lc)$, i.e. $G = F \times_{\mathcal{B}} L.\pi_{\mathcal{C}}$. Lastly, we need a unit $\theta : 1 \Rightarrow P^*R.G$, for which we take $\theta_{(a,c)} = (\underline{U\eta_c}, \eta_c) : (a, c) \longrightarrow (F(a, c), RLc)$. We omit the straightforward verification that (G, P^*R) is indeed a split reflection.

Next we should check that $\mathbf{P}^*$ preserves morphisms of split reflections; to this end, we consider such a morphism $(X, Y) : (L', R') \longrightarrow (L, R)$ as drawn below:

$$\begin{tikzcd}
\mathcal{A} \times_{\mathcal{B}} \mathcal{F} \arrow[r, "P^*X"] \arrow[d, shift left, "P^*R'"] & \mathcal{A} \times_{\mathcal{B}} \mathcal{D} \arrow[d, shift left, "P^*R"] & \mathcal{F} \arrow[r, "X"] \arrow[d, shift left, "R'"] & \mathcal{D} \arrow[d, shift left, "R"] \\
\mathcal{A} \times_{\mathcal{B}} \mathcal{E} \arrow[r, "P^*Y"] \arrow[u, shift left, "G'"] \arrow[d, shift right, "F'"'] \arrow[d, shift left, "P^*U'"] & \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \arrow[u, shift left, "G"] \arrow[d, shift right, "F"'] \arrow[d, shift left, "P^*U"] & \mathcal{E} \arrow[r, "Y"] \arrow[u, shift left, "L'"] \arrow[d, "U'"'] & \mathcal{C} \arrow[u, shift left, "L"] \arrow[d, "U"] \\
\mathcal{A} \arrow[r, "1"'] & \mathcal{A} \arrow[r, "P"'] & \mathcal{B} \arrow[r, "1"'] & \mathcal{B} .
\end{tikzcd}$$

We should show $G.P^*Y = P^*X.G'$, for which it suffices to prove $F.P^*Y = F'$ because:

$$\begin{aligned}
G.P^*Y &= (F \times_{\mathcal{B}} L.\pi_{\mathcal{C}}).P^*Y & P^*X.G' &= (\pi_{\mathcal{A}} \times_{\mathcal{B}} X.\pi_{\mathcal{F}}).G' \\
&= F.P^*Y \times_{\mathcal{B}} L.\pi_{\mathcal{C}}.P^*Y & &= \pi_{\mathcal{A}}.G' \times_{\mathcal{B}} X.\pi_{\mathcal{F}}.G' \\
&= F.P^*Y \times_{\mathcal{B}} X.L'.\pi_{\mathcal{E}}, & &= F' \times_{\mathcal{B}} X.L'.\pi_{\mathcal{E}}.
\end{aligned}$$

Note that we have $Y \circ \eta' = \eta \circ Y$ and $U' = U.Y$ by assumption, from which it follows that for any $e \in \mathcal{E}$ we have $U\eta_{Ye} = U'\eta'_e$. From this it is straightforward to derive $F.P^*Y = F'$, and we leave the details to the reader.

Lastly, we should check that $\mathbf{P}^*$ respects vertical composition. To this end, we consider the situation sketched below:

$$\begin{tikzcd}
\mathcal{A} \times_{\mathcal{B}} \mathcal{E} \arrow[r, "\pi_{\mathcal{E}}"] \arrow[d, shift left, "P^*R'"] & \mathcal{E} \arrow[d, shift left, "R'"] & \mathcal{A} \times_{\mathcal{B}} \mathcal{E} \arrow[l, "\pi_{\mathcal{E}}"'] \arrow[dd, shift left, "P^*(R.R')"] \\
\mathcal{A} \times_{\mathcal{B}} \mathcal{D} \arrow[r, "\pi_{\mathcal{D}}"] \arrow[u, shift left, "G'"] \arrow[d, shift left, "P^*R"] \arrow[dd, bend right=60, "F'"'] & \mathcal{D} \arrow[u, shift left, "L'"] \arrow[d, shift left, "R"] & \\
\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \arrow[r, "\pi_{\mathcal{C}}"] \arrow[u, shift left, "G"] \arrow[d, shift right, "F"'] \arrow[d, shift left, "P^*U"] & \mathcal{C} \arrow[u, shift left, "L"] \arrow[d, "U"] & \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \arrow[l, "\pi_{\mathcal{C}}"'] \arrow[uu, shift left, "G''"] \arrow[d, shift right, "F''"'] \arrow[d, shift left, "P^*U"] \\
\mathcal{A} \arrow[r, "P"'] & \mathcal{B} & \mathcal{A} . \arrow[l, "P"]
\end{tikzcd}$$

On the left we have the composition of the pullbacks $G'.G \dashv P^*R.P^*R'$, and on the right the pullback of the composition $G'' \dashv P^*(R.R')$. It suffices to show that $G'' = G'.G$ because the unit of a split reflection is unique. In turn, this follows once we establish $F'.G = F''$, because then we have

$$G'.G = (F' \times_{\mathcal{B}} L'.\pi_{\mathcal{D}}).G = F'.G \times_{\mathcal{B}} L'.\pi_{\mathcal{D}}.G = F'' \times_{\mathcal{B}} L'.L.\pi_{\mathcal{C}} = G''.$$

To show that $F'.G = F''$ on objects we consider $(a,c) \in \mathcal{A} \times_{\mathcal{B}} \mathcal{C}$. We denote the unit of the composite adjunction $L'L \dashv RR'$ by $\theta = (R \circ \eta' \circ L).\eta$. We have:

$$\begin{aligned}
F''(a,c) &= (U\theta_c)_! a && \text{(def. of } F''\text{)} \\
&= (UR\eta'_{Lc}.U\eta_c)_! a && \text{(def. of } \theta\text{)} \\
&= (UR\eta'_{Lc})_!(U\eta_c)_! a && \text{(as } P \text{ is split)} \\
&= F'(F(a,c), Lc) && \text{(def. of } F'\text{)} \\
&= F'G(a,c). && \text{(def. of } G\text{)}
\end{aligned}$$

For morphisms we consider $(x,y) : (a_1,c_1) \longrightarrow (a_2,c_2)$, and show that $F'(G(x,y))$ is equal to $F''(x,y)$ by using the uniqueness property of $F''(x,y)$ and a diagram chase:

$$\begin{tikzcd}
a_1 \arrow[rr, "x"] \arrow[d, "\underline{U\eta_{c_1}}"] \arrow[dd, bend right=60, "\underline{U\theta_{c_1}}"'] && a_2 \arrow[d, "\underline{U\eta_{c_2}}"'] \arrow[dd, bend left=60, "\underline{U\theta_{c_2}}"] \\
F(a_1,c_1) \arrow[rr, "F(x,y)"] \arrow[d, "\underline{UR\eta'_{Lc_1}}"] && F(a_2,c_2) \arrow[d, "\underline{UR\eta'_{Lc_2}}"'] \\
F''(a_1,c_1) \arrow[rr, "F'(G(x,y))"'] && F''(a_2,c_2)
\end{tikzcd} .$$

More specifically, we have:

$$\begin{aligned}
F'(G(x,y)).\underline{U\theta_{c_1}} &= F'(G(x,y)).\underline{UR\eta'_{Lc_1}}.\underline{U\eta_{c_1}} && \text{(as } P \text{ is split)} \\
&= \underline{UR\eta'_{Lc_2}}.F(x,y).\underline{U\eta_{c_1}} && \text{(by def. of } F'\text{)} \\
&= \underline{UR\eta'_{Lc_2}}.\underline{U\eta_{c_2}}.x && \text{(by def. of } F\text{)} \\
&= \underline{U\theta_{c_2}}.x. && \text{(as } P \text{ is split)}
\end{aligned}$$

■

5. The Beck–Chevalley condition

Ultimately we are interested in constructing models of type theory using the right adjoint method for splitting comprehension categories, as used in [GL23]. In order for this method to work the right maps should satisfy a number of stability conditions, as outlined in [Lar18, Chapter 2]. The stability condition for Π-types is ensured by a Beck–Chevalley condition (cf. Proposition 7.5) which we phrase in this section.

We will work up to the Beck–Chevalley condition of a Frobenius structure of an AWFS by first considering a simpler case in Section 5.1 where the double categories of the AWFS are freely obtained from ordinary categories. Then, in Section 5.8, we phrase the double categorical version of the Beck–Chevalley condition, and show that the Frobenius structure of our running-example AWFS satisfies this condition.

5.1. The categorical case. The functor $(-)_1 : \mathbf{Dbl} \longrightarrow \mathbf{Cat}$ that sends a double category to its category of vertical arrows has a left adjoint [vdBF22, Proposition 2.3]. This means that we can consider the following special case of the definition of pre-AWFS with a Frobenius structure, phrased in terms of ordinary categories and the adjunction (2).

5.2. Definition. A *lifting structure* $(\mathcal{J}, \phi, \mathcal{K})$ on a category $\mathcal{C}$ consists of two functors $\mathcal{J}, \mathcal{K} \longrightarrow \mathbf{Ar}(\mathcal{C})$ with a $(\mathcal{J}, \mathcal{K})$-lifting operation ϕ. This structure is *closed* if both of the induced functors $\phi_l : \mathcal{J} \longrightarrow {}^{\pitchfork}\mathcal{K}$ and $\phi_r : \mathcal{K} \longrightarrow \mathcal{J}^{\pitchfork}$ are invertible.

We get similar definitions of slicing and Frobenius structures for this case.

5.3. Definition. Given a functor $\mathcal{J} \longrightarrow \mathbf{Ar}(\mathcal{C})$ and an object $A \in \mathcal{C}$ we define the *slice* functor $\mathcal{J}/A \longrightarrow \mathbf{Ar}(\mathcal{C}/A)$ as the pullback of $\mathcal{J}$ along $\mathbf{Ar}(\mathrm{dom}) : \mathbf{Ar}(\mathcal{C}/A) \longrightarrow \mathbf{Ar}(\mathcal{C})$.

5.4. Definition. Let $(\mathcal{J}, \phi, \mathcal{K})$ be a lifting structure on $\mathcal{C}$, and $f : A \longrightarrow B$ an exponentiable morphism in $\mathcal{C}$, then a *Frobenius structure* on f for $(J, \phi, \mathcal{K})$ is a functor $\mathbf{f}^*$ that fits into a commutative diagram as on the left below:

$$\begin{tikzcd} \mathcal{J}/B \arrow[r, dotted, "\mathbf{f}^*"] \arrow[d] & \mathcal{J}/A \arrow[d] \\ \mathbf{Ar}(\mathcal{C}/B) \arrow[r, "\mathbf{Ar}(f^*)"] & \mathbf{Ar}(\mathcal{C}/A) \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{K}/A \arrow[r, dotted, "\mathbf{f}_*"] \arrow[d] & \mathcal{K}/B \arrow[d] \\ \mathbf{Ar}(\mathcal{C}/A) \arrow[r, "\mathbf{Ar}(f_*)"] & \mathbf{Ar}(\mathcal{C}/B) \end{tikzcd}. \tag{9}$$

Similarly, a *pushforward structure* on f is a functor $\mathbf{f}_*$ as on the right above. A *Frobenius (pushforward) structure* for $(\mathcal{J}, \phi, \mathcal{K})$ is an assignment of a Frobenius (pushforward) structure to every $\mathbf{f} \in \mathcal{K}$.

As in the double categorical case, Frobenius and pushforward structures are in bijection. Furthermore, given an AWFS $(\mathbb{L}, \phi, \mathbb{R})$ with a Frobenius structure, an application of $(-)_1$ yields a lifting structure $(\mathbb{L}_1, \phi, \mathbb{R}_1)$ in the sense of Definition 5.2 (that need not be closed), with a Frobenius structure. As before, we may omit reference to the lifting operation ϕ.

Now, consider two functors $\mathcal{J}, \mathcal{K} \longrightarrow \mathbf{Ar}(\mathcal{C})$ that are part of a closed lifting structure $(\mathcal{J}, \phi, \mathcal{K})$ with a Frobenius structure. We will consider the stability conditions outlined in [Lar18, Chapter 2] for this case. These relate to the coherence axioms of type theory with respect to substitution. Under the interpretation of types as the right maps of an AWFS, and substitution as pullbacks, this translates to a condition on the pushforward structure of the AWFS with respect to morphisms of right maps whose underlying square in the ambient category $\mathcal{C}$ is a pullback square.

More specifically, we consider $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathcal{K}$ such that g and h are composable, and a morphism $\mathbf{uv} : \mathbf{f} \longrightarrow \mathbf{g}$ whose underlying square in $\mathcal{C}$ is a pullback square:

$$\begin{tikzcd} & \bullet \arrow[d, "h"] \\ A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}, \qquad \begin{tikzcd} \bullet \arrow[rr, "\beta_h^{-1}"] \arrow[dr, "f_* u^* h"'] & & \bullet \arrow[dl, "v^* g_* h"] \\ & B & \end{tikzcd}. \tag{10}$$

This gives composite parallel adjunctions $u_! f^* \dashv f_* u^*$ and $g^* v_! \dashv v^* g_*$:

$$\begin{tikzcd} \mathcal{C}/B \arrow[r, shift left=2, "f^*"] \arrow[r, phantom, "\perp"] & \mathcal{C}/A \arrow[l, shift left=2, "f_*"] \arrow[r, shift left=2, "u_!"] \arrow[r, phantom, "\perp"] & \mathcal{C}/C \arrow[l, shift left=2, "u^*"] \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{C}/B \arrow[r, shift left=2, "v_!"] \arrow[r, phantom, "\perp"] & \mathcal{C}/D \arrow[l, shift left=2, "v^*"] \arrow[r, shift left=2, "g^*"] \arrow[r, phantom, "\perp"] & \mathcal{C}/C \arrow[l, shift left=2, "g_*"] \end{tikzcd}. \tag{11}$$

These induce a mateship correspondence between natural transformations with signatures $u_! f^* \Rightarrow g^* v_!$ and $v^* g_* \Rightarrow f_* u^*$. There is a canonical map $\alpha : u_! f^* \Rightarrow g^* v_!$ (itself obtained as a mate of the identity natural transformation $g_! u_! = v_! f_!$) of which the component α_w at an object w in $\mathcal{C}/B$ is given as follows:

$$\begin{tikzcd}
\bullet \arrow[drrr, bend left, "g^* v_! w"] \arrow[ddr, bend right] & & & \\
& \bullet \arrow[r, "f^* w"] \arrow[d] \arrow[ul, dotted, "\alpha_w"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
& \bullet \arrow[r, "w"'] & B \arrow[r, "v"'] & D .
\end{tikzcd}$$

We denote the mate of α with signature $v^* g_* \Rightarrow f_* u^*$ by β. These transformations α and β exist regardless of whether (u, v) is a pullback, but when it is they are both isomorphisms; see e.g. [HR24, Lemma 3.5]. We call β the Beck–Chevalley isomorphism.

In the setup of the left diagram of (10), we thus get a triangle pictured as the right diagram of (10). When $(\mathcal{J}, \phi, \mathcal{K})$ is a closed lifting structure, the right class is always closed under pullbacks, and so if it comes with a Frobenius structure then the morphisms $f_* u^* h$ and $v^* g_* h$ have $\mathcal{K}$-structure. What we would need for the interpretation of type-theoretic dependent products is that $(\beta_h^{-1}, 1)$ underlies a morphism of $\mathcal{K}$ maps. The goal of this section is to precisely state this condition, and to show it is equivalent to a condition that is easier to check in practice. To do this we follow the approach of the statement and proof of [GS17, Proposition 6.7].

The first step will be to generalize to the case where h is not an object but an arrow of $\mathcal{C}/C$, i.e. we consider $(\mathbf{h}, w) \in \mathcal{K}/C$:

$$\begin{tikzcd}
& \bullet \arrow[d, "h"] \\
& \bullet \arrow[d, "w"] \\
A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
B \arrow[r, "v"'] & D ,
\end{tikzcd} \qquad \begin{tikzcd}
\bullet \arrow[rr, "\beta_{w.h}^{-1}"] \arrow[d, "f_* u^* h"'] & & \bullet \arrow[d, "v^* g_* h"] \\
\bullet \arrow[rr, "\beta_w^{-1}"] \arrow[dr, "f_* u^* w"'] & & \bullet \arrow[dl, "v^* g_* w"] \\
& B . &
\end{tikzcd} \tag{12}$$

The square $(\beta_{w.h}^{-1}, \beta_w^{-1})$ on the right above is the component $\mathbf{Ar}(\beta^{-1})_{(h,w)}$ of the natural transformation $\mathbf{Ar}(\beta^{-1}) : \mathbf{Ar}(f_* u^*) \Rightarrow \mathbf{Ar}(v^* g_*)$. For a closed lifting structure $(\mathcal{J}, \phi, \mathcal{K})$

an isomorphism in $\mathbf{Ar}(\mathcal{C})$ underlies a morphism of $\mathcal{J}$ or $\mathcal{K}$ maps if and only if its inverse does, so we will focus on β instead of β^{-1}. To state what it means for $\mathbf{Ar}(\beta)_{(h,w)}$ to underlie a morphism of $\mathcal{K}$ maps we use the following definition.

5.5. Definition. Let $\mathbf{F}, \mathbf{G} : \mathcal{J} \longrightarrow \mathcal{K}$ be lifts of $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ for some $\mathcal{J} \longrightarrow \mathbf{Ar}(\mathcal{C})$ and $\mathcal{K} \longrightarrow \mathbf{Ar}(\mathcal{D})$, and $\mu : F \Rightarrow G$ a natural transformation. We say that $\boldsymbol{\mu} : \mathbf{F} \Rightarrow \mathbf{G}$ is a *lift* of μ if $\mathcal{K} \circ \boldsymbol{\mu} = \mathbf{Ar}(\mu) \circ \mathcal{J}$, meaning its components are over those of μ:

$$\begin{tikzcd} \mathcal{J} \arrow[r, shift left=2, "\mathbf{F}"] \arrow[r, shift right=2, "\mathbf{G}"'] \arrow[r, phantom, "\Downarrow\boldsymbol{\mu}"] \arrow[d] & \mathcal{K} \arrow[d] \\ \mathbf{Ar}(\mathcal{C}) \arrow[r, shift left=2, "\mathbf{Ar}(F)"] \arrow[r, shift right=2, "\mathbf{Ar}(G)"'] \arrow[r, phantom, "\Downarrow\mathbf{Ar}(\mu)"] & \mathbf{Ar}(\mathcal{D}) \end{tikzcd} .$$

Note that a composition functor like $u_! : \mathcal{C}/A \longrightarrow C/C$ always lifts to a functor $\mathbf{u}_! : \mathcal{J}/A \longrightarrow \mathcal{J}/C : (\mathbf{f}, a) \mapsto (\mathbf{f}, u.a)$. Since $(\mathcal{J}, \phi, \mathcal{K})$ has a Frobenius structure, we thus obtain lifts $\mathbf{u}_!\mathbf{f}^*$ and $\mathbf{g}^*\mathbf{v}_!$ as on the left of:

$$\begin{tikzcd} \mathcal{J}/B \arrow[r, shift left=2, "\mathbf{u}_!\mathbf{f}^*"] \arrow[r, shift right=2, "\mathbf{g}^*\mathbf{v}_!"'] \arrow[r, phantom, "\Downarrow\boldsymbol{\alpha}"] \arrow[d] & \mathcal{J}/C \arrow[d] \\ \mathbf{Ar}(\mathcal{C}/B) \arrow[r, shift left=2, "\mathbf{Ar}(u_!f^*)"] \arrow[r, shift right=2, "\mathbf{Ar}(g^*v_!)"'] \arrow[r, phantom, "\Downarrow\mathbf{Ar}(\alpha)"] & \mathbf{Ar}(\mathcal{C}/C) \end{tikzcd} , \qquad \begin{tikzcd} \mathcal{K}/C \arrow[r, shift left=2, "\mathbf{v}^*\mathbf{g}_*"] \arrow[r, shift right=2, "\mathbf{f}_*\mathbf{u}^*"'] \arrow[r, phantom, "\Downarrow\boldsymbol{\beta}"] \arrow[d] & \mathcal{K}/B \arrow[d] \\ \mathbf{Ar}(\mathcal{C}/C) \arrow[r, shift left=2, "\mathbf{Ar}(v^*g_*)"] \arrow[r, shift right=2, "\mathbf{Ar}(f_*u^*)"'] \arrow[r, phantom, "\Downarrow\mathbf{Ar}(\beta)"] & \mathbf{Ar}(\mathcal{C}/B) \end{tikzcd} . \tag{13}$$

Applying [BG16, Proposition 21] (restricted to categories) with respect to the adjunctions in (11) we obtain corresponding lifts $\mathbf{v}^*\mathbf{g}_*$ and $\mathbf{f}_*\mathbf{u}^*$ as on the right above. The condition that the square $(\beta_{w.h}, \beta_w)$ is a morphism of $\mathcal{K}$ maps is now expressed by requiring the existence of a lift $\boldsymbol{\beta}$ as depicted on the right above. Note that, since the lifting structure $(\mathcal{J}, \phi, \mathcal{K})$ is closed, the existence of this lift is not additional structure but is instead just a property of β. In sum, we have the following definition.

5.6. Definition. A Frobenius structure for a closed structure $(\mathcal{J}, \phi, \mathcal{K})$ satisfies the *Beck–Chevalley condition* when for every $\mathbf{uv} : \mathbf{f} \longrightarrow \mathbf{g}$ in $\mathcal{K}$ overlying a pullback square (u, v), the Beck–Chevalley isomorphism β lifts to a transformation $\boldsymbol{\beta} : \mathbf{v}^*\mathbf{g}_* \Rightarrow \mathbf{f}_*\mathbf{u}^*$.

Rather than lifting β we may lift its mate α, as the following proposition shows. This can be useful in practice because the definition of α is simpler.

5.7. Proposition. *For a closed lifting structure $(\mathcal{J}, \phi, \mathcal{K})$ with a Frobenius structure, the Beck–Chevalley isomorphism β lifts in the sense of (13) iff its mate α does.*

Proof. By [GS17, Proposition 5.8] and the fact that the lifting structure is closed. ∎

For future reference we expand on what it means for α to lift in the sense of Proposition 5.7. Considering the pullback square (u, v) in (12) and $(w, b) \in \mathbf{Ar}(\mathcal{C}/B)$, the component $\mathbf{Ar}(\alpha)_{(w,b)}$ is given by the square $(\alpha_{b.w}, \alpha_b) : u_! f^*(w, b) \longrightarrow g^* v_!(w, b)$:

$$\begin{tikzcd}
\bullet \arrow[d, "w"'] & \bullet \arrow[l] \arrow[r, dotted, "\alpha_{b.w}"] \arrow[d, "f^*w"'] & \bullet \arrow[d, "g^*w"] \\
\bullet \arrow[d, "b"'] & \bullet \arrow[l] \arrow[r, dotted, "\alpha_b"] \arrow[d, "f^*b"'] & \bullet \arrow[d, "g^*(v.b)"] \\
B \arrow[dr, "v"'] & A \arrow[l, "f"'] \arrow[r, "u"] & C \arrow[dl, "g"] \\
& D . &
\end{tikzcd} \tag{14}$$

If, in addition, $(\mathbf{w}, b) \in \mathcal{J}/B$ then f^*w and g^*w underlie $\mathcal{J}$ maps $\mathbf{f}^*\mathbf{w}$ and $\mathbf{g}^*\mathbf{w}$, and $\boldsymbol{\alpha}$ should produce a $\mathcal{J}$ morphism $\boldsymbol{\alpha}_{(\mathbf{w},b)} : \mathbf{f}^*\mathbf{w} \longrightarrow \mathbf{g}^*\mathbf{w}$ lying over $(\alpha_{b.w}, \alpha_b)$.

5.8. The double categorical case. Next, we phrase an analogous version of Definition 5.6 for a pair of double categories. More specifically, we consider the scenario in which (12) underlies a square $\mathbf{uv} : \mathbf{f} \longrightarrow \mathbf{g}$ in $\mathbb{R}$ for an AWFS $(\mathbb{L}, \mathbb{R})$ with a Frobenius structure. To start, we define lifts of double natural transformations.

5.9. Definition. Let $\mathbf{F}, \mathbf{G} : \mathbb{J} \longrightarrow \mathbb{K}$ be lifts of $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ for some $\mathbb{J} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$ and $\mathbb{K} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{D})$, and $\mu : F \Rightarrow G$ a natural transformation. We say that $\boldsymbol{\mu} : \mathbf{F} \Rightarrow \mathbf{G}$ is a *lift* of μ if $\mathbb{K} \circ \boldsymbol{\mu} = \mathbb{S}\mathbf{q}(\mu) \circ \mathbb{J}$:

$$\begin{tikzcd}
\mathbb{J} \arrow[r, bend left, "\mathbf{F}"] \arrow[r, bend right, "\mathbf{G}"'] \arrow[r, phantom, "\Downarrow\boldsymbol{\mu}"] \arrow[d] & \mathbb{K} \arrow[d] \\
\mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, bend left, "\mathbb{S}\mathbf{q}(F)"] \arrow[r, bend right, "\mathbb{S}\mathbf{q}(G)"'] \arrow[r, phantom, "\Downarrow\mathbb{S}\mathbf{q}(\mu)"] & \mathbb{S}\mathbf{q}(\mathcal{D}) .
\end{tikzcd}$$

Note that a composition functor such as $u_! : \mathcal{C}/A \longrightarrow \mathcal{C}/C$ in fact lifts to a double functor $\mathbf{u}_! : \mathbb{L}/A \longrightarrow \mathbb{L}/C$, as the assignment $(\mathbf{f}, a) \mapsto (\mathbf{f}, u.a)$ (trivially) preserves vertical composition. Using the Frobenius structure of $(\mathbb{L}, \mathbb{R})$ and [BG16, Proposition 21] we thus obtains lifts $\mathbf{u}_!\mathbf{f}^*$ and $\mathbf{g}^*\mathbf{v}_!$ as on the left below:

$$\begin{tikzcd}
\mathbb{J}/B \arrow[r, bend left, "\mathbf{u}_!\mathbf{f}^*"] \arrow[r, bend right, "\mathbf{g}^*\mathbf{v}_!"'] \arrow[r, phantom, "\Downarrow\boldsymbol{\alpha}"] \arrow[d] & \mathbb{J}/C \arrow[d] \\
\mathbb{S}\mathbf{q}(\mathcal{C}/C) \arrow[r, bend left, "\mathbb{S}\mathbf{q}(u_! f^*)"] \arrow[r, bend right, "\mathbb{S}\mathbf{q}(g^* v_!)"'] \arrow[r, phantom, "\Downarrow\mathbb{S}\mathbf{q}(\alpha)"] & \mathbb{S}\mathbf{q}(\mathcal{C}/B) ,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K}/C \arrow[r, bend left, "\mathbf{v}^*\mathbf{g}_*"] \arrow[r, bend right, "\mathbf{f}_*\mathbf{u}^*"'] \arrow[r, phantom, "\Downarrow\boldsymbol{\beta}"] \arrow[d] & \mathbb{K}/B \arrow[d] \\
\mathbb{S}\mathbf{q}(\mathcal{C}/C) \arrow[r, bend left, "\mathbb{S}\mathbf{q}(v^* g_*)"] \arrow[r, bend right, "\mathbb{S}\mathbf{q}(f_* u^*)"'] \arrow[r, phantom, "\Downarrow\mathbb{S}\mathbf{q}(\beta)"] & \mathbb{S}\mathbf{q}(\mathcal{C}/B) .
\end{tikzcd} \tag{15}$$

Another application of [BG16, Proposition 21] lets us transpose the lifts $\mathbf{u}_!\mathbf{f}^*$ and $\mathbf{g}^*\mathbf{v}_!$ to lifts $\mathbf{v}^*\mathbf{g}_*$ and $\mathbf{f}_*\mathbf{u}^*$, as on the right above. Now we can phrase the Beck–Chevalley condition for double categories completely analogously to that for categories.

5.10. Definition. A Frobenius structure for an AWFS $(\mathbb{L}, \mathbb{R})$ satisfies the *Beck–Chevalley condition* when for every square $\mathbf{uv} : \mathbf{f} \longrightarrow \mathbf{g}$ of $\mathbb{R}$ lying over a pullback square (u, v) the Beck–Chevalley isomorphism β lifts to a transformation $\boldsymbol{\beta} : \mathbf{v}^*\mathbf{g}_* \Rightarrow \mathbf{f}_*\mathbf{u}^*$.

As before, lifting β in this way can be done by lifting α, which is easier in practice.

5.11. Proposition. *For an* AWFS $(\mathbb{L}, \mathbb{R})$ *with a Frobenius structure, the Beck–Chevalley isomorphism* β *lifts in the sense of* (15) *iff its mate* α *does.*

Proof. Immediate from [BG16, Proposition 21]. ■

In fact, as we will now show, for concrete double functors Definition 5.9 reduces to Definition 5.5. In particular, this means that to check whether a Frobenius structure for an AWFS $(\mathbb{L}, \phi, \mathbb{R})$ satisfies the Beck–Chevalley condition, it suffices to check that its corresponding Frobenius structure for the lifting structure $(\mathbb{J}_1, \phi, \mathbb{K}_1)$, obtained by an application of $(-)_1 : \mathbf{Dbl} \longrightarrow \mathbf{Cat}$, satisfies the Beck–Chevalley condition.

5.12. Lemma. *Let* $\mathbb{J} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$ *and* $\mathbb{K} \longrightarrow \mathbb{S}\mathbf{q}(\mathcal{D})$ *be concrete double functors, and* $\mathbf{F}, \mathbf{G} : \mathbb{J} \longrightarrow \mathbb{K}$ *lifts of* $F, G : \mathcal{C} \longrightarrow \mathcal{D}$*; then lifts of* $\mu : F \longrightarrow G$ *to* $\mathbf{F} \Rightarrow \mathbf{G}$ *are in bijection with its lifts to* $\mathbf{F}_1 \Rightarrow \mathbf{G}_1$*:*

$$\begin{tikzcd}
\mathbb{J} \arrow[r, shift left=2, "\mathbf{F}"] \arrow[r, shift right=2, "\mathbf{G}"'] \arrow[r, phantom, "\Downarrow\boldsymbol{\mu}"] \arrow[d] & \mathbb{K} \arrow[d] \\
\mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, shift left=2, "\mathbb{S}\mathbf{q}(F)"] \arrow[r, shift right=2, "\mathbb{S}\mathbf{q}(G)"'] \arrow[r, phantom, "\Downarrow\mathbb{S}\mathbf{q}(\mu)"] & \mathbb{S}\mathbf{q}(\mathcal{D})
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathbb{J}_1 \arrow[r, shift left=2, "\mathbf{F}_1"] \arrow[r, shift right=2, "\mathbf{G}_1"'] \arrow[r, phantom, "\Downarrow\boldsymbol{\mu}"] \arrow[d] & \mathbb{K}_1 \arrow[d] \\
\mathbf{Ar}(\mathcal{C}) \arrow[r, shift left=2, "\mathbf{Ar}(F)"] \arrow[r, shift right=2, "\mathbf{Ar}(G)"'] \arrow[r, phantom, "\Downarrow\mathbf{Ar}(\mu)"] & \mathbf{Ar}(\mathcal{D})
\end{tikzcd}.$$

Proof. By spelling out the definitions. ■

5.13. Lemma. *For an* AWFS $(\mathbb{L}, \mathbb{R})$ *and an object* $A \in \mathcal{C}$ *the double functors* $\mathbb{L}/A$ *and* $\mathbb{R}/A$ *are concrete.*

Proof. The double functors $\mathbb{J}^{\pitchfork}$ and ${}^{\pitchfork}\mathbb{J}$ are always concrete for any $\mathbb{J}$, and so, by the axiom of lifting of AWFS, so are $\mathbb{L}$ and $\mathbb{R}$. Furthermore, slicing preserves concreteness. ■

5.14. Proposition. *A Frobenius structure for an* AWFS $(\mathbb{L}, \phi, \mathbb{R})$ *satisfies the Beck–Chevalley condition when its associated Frobenius structure for* $(\mathbb{L}_1, \phi, \mathbb{R}_1)$ *does.*

Proof. By Lemmas 5.12 and 5.13. ■

Lastly, we show that our running example from Section 4 satisfies this condition.

5.15. PROPOSITION. *The Frobenius structure for the* AWFS $(\mathbb{S}\mathbf{plRef}(\mathbf{Gpd}), \mathbb{S}\mathbf{plFib}(\mathbf{Gpd}))$ *of Proposition 4.1 satisfies the Beck–Chevalley condition.*

PROOF. Consider split fibrations $P : \mathcal{A} \longrightarrow \mathcal{B}$ and $Q : \mathcal{C} \longrightarrow \mathcal{D}$ of groupoids, and a pullback square $(X, Y) : P \longrightarrow Q$ that commutes with their splittings, which means that for any morphism f in $\mathcal{B}$ we have $X\underline{f} = \underline{Yf}$. By Proposition 5.14 it suffices to show that the induced transformation $\alpha : X_!P^* \Rightarrow Q^*Y_!$ has a lift $\boldsymbol{\alpha}$ as on the left of:

$$
\begin{tikzcd}
\mathbf{SplRef}(\mathbf{Gpd})/\mathcal{B} \arrow[r, bend left, "\mathbf{X_!P^*}"] \arrow[r, bend right, "\mathbf{Q^*Y_!}"'] \arrow[r, phantom, "\Downarrow\boldsymbol{\alpha}"] \arrow[d] & \mathbf{SplRef}(\mathbf{Gpd})/\mathcal{C} \arrow[d] \\
\mathbf{Ar}(\mathbf{Cat}/\mathcal{B}) \arrow[r, bend left, "\mathbf{Ar}(X_!P^*)"] \arrow[r, bend right, "\mathbf{Ar}(Q^*Y_!)"'] \arrow[r, phantom, "\Downarrow\mathbf{Ar}(\alpha)"] & \mathbf{Ar}(\mathbf{Cat}/\mathcal{C})\ ,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F} \arrow[d, shift left, "R"] & \mathcal{A} \times_{\mathcal{B}} \mathcal{F} \arrow[l, "\pi_{\mathcal{F}}"'] \arrow[r, dotted, "\alpha_{U.R}"] \arrow[d, shift left, "P^*R"] & \mathcal{C} \times_{\mathcal{D}} \mathcal{F} \arrow[d, shift left, "Q^*R"] \\
\mathcal{E} \arrow[u, shift left, "L"] \arrow[d, "U"'] & \mathcal{A} \times_{\mathcal{B}} \mathcal{E} \arrow[u, shift left, "G"] \arrow[l, "\pi_{\mathcal{E}}"'] \arrow[r, dotted, "\alpha_U"] \arrow[d, shift right, "F"'] \arrow[d, shift left, "P^*U"] & \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \arrow[u, shift left, "G'"] \arrow[d, shift right, "F'"'] \arrow[d, shift left, "Q^*(Y.U)"] \\
\mathcal{B} \arrow[dr, "Y"'] & \mathcal{A} \arrow[l, "P"'] \arrow[r, "X"] & \mathcal{C} \arrow[dl, "Q"] \\
& \mathcal{D}\ . &
\end{tikzcd}
$$

In other words, the square $(\alpha_{U.R}, \alpha_U)$, as on the right above, should be a morphism of split reflections: $\alpha_{U.R}.G = G'.\alpha_U$. Note that

$$
\begin{aligned}
\alpha_{U.R}.G &= (X.\pi_{\mathcal{A}} \times_{\mathcal{D}} \pi_{\mathcal{F}}).G \\
&= X.\pi_{\mathcal{A}}.G \times_{\mathcal{D}} \pi_{\mathcal{F}}.G \\
&= X.F \times_{\mathcal{D}} L.\pi_{\mathcal{E}},
\end{aligned}
\qquad
\begin{aligned}
G'.\alpha_U &= (F' \times_{\mathcal{D}} L.\pi_{\mathcal{E}}).\alpha_U \\
&= F'.\alpha_U \times_{\mathcal{D}} L.\pi_{\mathcal{E}}.\alpha_U \\
&= F'.\alpha_U \times_{\mathcal{D}} L.\pi_{\mathcal{E}},
\end{aligned}
$$

so that we are done if $X.F = F'.\alpha_U$ (where $\alpha_U = X.\pi_{\mathcal{E}} \times_{\mathcal{D}} \pi_{\mathcal{E}}$).

To show equality on objects we consider $(a, e) \in \mathcal{A} \times_{\mathcal{B}} \mathcal{E}$. Note that $X\underline{U\eta_e} = \underline{YU\eta_e}$ by assumption and so $X((U\eta_e)_!a) = (YU\eta_e)_!(Xa)$. We therefore have

$$XF(a, e) = X((U\eta_e)_!a) = (YU\eta_e)_!(Xa) = F'(Xa, e) = F'\alpha_U(a, e).$$

Similarly, given a morphism $(x, y) : (a_1, e_1) \longrightarrow (a_2, e_2)$ in $\mathcal{A} \times_{\mathcal{B}} \mathcal{E}$, we have

$$(XF(x, y)).\underline{YU\eta_{e_1}} = (XF(x, y)).X\underline{U\eta_{e_1}} = X\underline{U\eta_{e_2}}.Xx = \underline{YU\eta_{e_2}}.Xx,$$

so that $XF(x, y) = F'\alpha_U(x, y)$ follows from the uniqueness property of $F'(Xx, y)$. ∎

6. Other properties of Frobenius structures

In Section 3 we defined a notion of Frobenius structure for AWFS, and in Section 5 we looked at a property of such structures which we will need in Section 7 where we discuss modelling type theory. Before we continue, we will briefly consider some other properties of Frobenius structures (which we will not need for modelling type theory).

6.1. Adjoint Frobenius structures. A property of a WFS (L, R) that is sometimes considered in the literature is called the *strong* Frobenius property; see e.g. [Lar18, Definition B.6.2] and [vdBF22, Proposition 5.2]. Recall that the Frobenius property for a WFS (L, R) is usually defined as the statement that L is closed under pullback along R. If, in addition, L comes with a notion of structure preserving squares of L maps, then the Frobenius property can be strengthened to demand that the pullback squares of L maps along R maps are structure preserving. We argue that this strengthened Frobenius property can be understood in terms of the notion of natural transformation lifting of Definition 5.5.

To see this, consider an AWFS $(\mathbb{L}, \mathbb{R})$ and an exponentiable morphism $f : A \longrightarrow B$ with a Frobenius structure $\mathbf{f}$. A vertical morphism $(\mathbf{g}, b)$ of $\mathbb{L}/B$ then gives rise to a pullback square as depicted below, where ε is the counit of $f_! \dashv f^*$:

$$\begin{tikzcd}
\bullet \arrow[r, "\varepsilon_{b.g}"] \arrow[d, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "\varepsilon_b"] \arrow[d, "f^*b"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "b"] \\
A \arrow[r, "f"'] & B .
\end{tikzcd}$$

As mentioned previously, the functor $f_!$ has a lift $\mathbf{f}_! : \mathbb{L}/A \longrightarrow \mathbb{L}/B : (\mathbf{h}, a) \mapsto (\mathbf{h}, f.a)$, and so the demand that $(\varepsilon_{b.g}, \varepsilon_b)$ underlies a square $\mathbf{f}^*\mathbf{g} \longrightarrow \mathbf{g}$ in $\mathbb{L}$ can be phrased as the requirement that ε has a lift $\boldsymbol{\varepsilon} : \mathbf{f}_!\mathbf{f}^* \longrightarrow 1$ in the sense of Definition 5.9. What is called the strong Frobenius property in the literature can then be seen as the special case of this where $b = 1_B$ and $f^*b = 1_A$.

6.2. Definition. A *strong Frobenius structure* $(\mathbf{f}^*, \boldsymbol{\varepsilon})$ on a morphism $f : A \longrightarrow B$ with respect to an AWFS $(\mathbb{L}, \mathbb{R})$ is a Frobenius structure for f together with a lift $\boldsymbol{\varepsilon} : \mathbf{f}_!\mathbf{f}^* \longrightarrow 1$ of the counit of $f_! \dashv f^*$. A *strong Frobenius structure* for an AWFS $(\mathbb{L}, \mathbb{R})$ is an assignment of a strong Frobenius structures to every vertical map $\mathbf{f}$ of $\mathbb{R}$.

Thinking of the strong Frobenius property in this way shows that it has an evident counterpart, which is the requirement that the unit η of $f_! \dashv f^*$ lifts to $\boldsymbol{\eta} : 1 \longrightarrow \mathbf{f}^*\mathbf{f}_!$. Together, these lifts could then be further required to satisfy the triangle identities, so that $\mathbf{f}_! \dashv \mathbf{f}^*$. Of course, using the reasoning behind Theorem 3.14 we can also transpose $\mathbf{f}^*$ and $\mathbf{f}_!$ to obtain a pushforward structure $\mathbf{f}_* : \mathbb{R}/A \longrightarrow \mathbb{R}/B$ and (by abuse of notation) a pullback functor $\mathbf{f}^* : \mathbb{R}/B \longrightarrow \mathbb{R}/A$. These can then be subjected to similar conditions, asking for a lifted adjunction $\mathbf{f}^* \dashv \mathbf{f}_*$. In sum, we have the following definition.

6.3. Definition. An *adjoint Frobenius structure* $(\mathbf{f}^*, \boldsymbol{\eta}, \boldsymbol{\varepsilon})$ on a morphism $f : A \longrightarrow B$ with respect to an AWFS $(\mathbb{L}, \mathbb{R})$ is a Frobenius structure on f together with lifts $\boldsymbol{\eta}$ and $\boldsymbol{\varepsilon}$

of the unit and counit of $f_! \dashv f^*$ giving an adjunction $\mathbf{f}_! \dashv \mathbf{f}^*$ as on the left of:

$$
\begin{tikzcd}
\mathbb{L}/B \arrow[r, shift left, "\mathbf{f}^*"] \arrow[r, phantom, "\top"] \arrow[d] & \mathbb{L}/A \arrow[l, shift left, "\mathbf{f}_!"] \arrow[d] \\
\mathbb{S}\mathbf{q}(\mathcal{C}/B) \arrow[r, shift left, "\mathbb{S}\mathbf{q}(f^*)"] \arrow[r, phantom, "\top"] & \mathbb{S}\mathbf{q}(\mathcal{C}/A) \arrow[l, shift left, "\mathbb{S}\mathbf{q}(f_!)"]
\end{tikzcd}
\quad , \qquad
\begin{tikzcd}
\mathbb{R}/A \arrow[r, shift left, "\mathbf{f}_*"] \arrow[r, phantom, "\top"] \arrow[d] & \mathbb{R}/B \arrow[l, shift left, "\mathbf{f}^*"] \arrow[d] \\
\mathbb{S}\mathbf{q}(\mathcal{C}/A) \arrow[r, shift left, "\mathbb{S}\mathbf{q}(f_*)"] \arrow[r, phantom, "\top"] & \mathbb{S}\mathbf{q}(\mathcal{C}/B) \arrow[l, shift left, "\mathbb{S}\mathbf{q}(f^*)"]
\end{tikzcd}
\quad .
$$

Similarly, an *adjoint pushforward structure* $(\mathbf{f}_*, \bar{\boldsymbol{\eta}}, \bar{\boldsymbol{\varepsilon}})$ on f is a pushforward structure on f with lifts $\bar{\boldsymbol{\eta}}$ and $\bar{\boldsymbol{\varepsilon}}$ of the unit and counit of $f^* \dashv f_*$ such that $\mathbf{f}^* \dashv \mathbf{f}_*$.

6.4. Proposition. *For an exponentiable map $f : A \longrightarrow B$ in $\mathcal{C}$ and an* AWFS $(\mathbb{L}, \mathbb{R})$ *there is a bijection between adjoint Frobenius and pushforward structures on f.*

Proof. The units and counits of $f_! \dashv f^* \dashv f_*$ are mates [HR24, Lemma 3.1], and so—due to how the various lifts of $f_!$, f^*, and f_* are constructed—their lifts are in bijective correspondence by [BG16, Proposition 21]. The triangle equalities do not add any extra requirements because the projections $\mathbb{L}_1, \mathbb{R}_1 \longrightarrow \mathbf{Ar}(\mathcal{C})$ are faithful. ∎

The Frobenius structures we constructed for our running example appears to be strong, but not adjoint.

6.5. Proposition. *The Frobenius structure of* $(\mathbb{S}\mathbf{plRef}(\mathbf{Gpd}), \mathbb{S}\mathbf{plFib}(\mathbf{Gpd}))$ *is strong.*

Proof. We expand on Proposition 4.1 by showing that the Frobenius structure $\mathbf{P}^*$ of a split fibration $P : \mathcal{A} \longrightarrow \mathcal{B}$ of groupoids comes with the required lift. We consider:

$$
\begin{tikzcd}
\mathcal{A} \times_{\mathcal{B}} \mathcal{D} \arrow[r, "\varepsilon_{U.R}"] \arrow[d, shift left, "P^*R"] & \mathcal{D} \arrow[d, shift left, "R"] \\
\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \arrow[r, "\varepsilon_U"] \arrow[u, shift left, "G"] \arrow[d, shift right, "F"'] \arrow[d, shift left, "P^*U"] & \mathcal{C} \arrow[u, shift left, "L"] \arrow[d, "U"] \\
\mathcal{A} \arrow[r, "P"'] & \mathcal{B}
\end{tikzcd}
\quad ,
$$

and note that it is immediate from $G = F \times_{\mathcal{B}} L.\pi_{\mathcal{C}}$ that $\varepsilon_{U.R}.G = L.\varepsilon_U$, as desired. ∎

The requirement that the unit $\theta : 1 \longrightarrow P^*P_!$ lifts seems to be quite a bit stronger for this case, and indeed it does not appear to hold. To see why, consider the situation:

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, "\theta_{U.R}"] \arrow[d, shift left, "R"] & \mathcal{A} \times_{\mathcal{B}} \mathcal{D} \arrow[r] \arrow[d, shift left, "P^*R"] & \mathcal{D} \arrow[d, shift left, "R"] \\
\mathcal{C} \arrow[r, "\theta_U"] \arrow[u, shift left, "L"] \arrow[d, "U"'] & \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \arrow[r] \arrow[u, shift left, "G"] \arrow[d, shift right, "F"'] \arrow[d, shift left, "P^*P_!U"] & \mathcal{C} \arrow[u, shift left, "L"] \arrow[d, "P_!U"] \\
\mathcal{A} \arrow[r, "1"'] & \mathcal{A} \arrow[r, "P"'] & \mathcal{B}
\end{tikzcd}
\quad .
$$

To have a lift of $\theta : 1 \longrightarrow P^*P_!$ we would need that $G.\theta_U = \theta_{U.R}.L$, which would follow from $F.\theta_U = U.R.L$. However, this is asking that for any $c \in \mathcal{C}$ we have $URLc = (PU\eta_c)_!Uc$, and this is not guaranteed. To make this true we would need further requirements of our fibration. For example, it seems that the statement would hold for a *discrete* fibration P, as then we would have $\underline{PU\eta_c} = U\eta_c$.

6.6. Pseudofunctorial Frobenius structures. As is well-known, the assignment $f \mapsto f^*$ that sends a morphism $f : A \longrightarrow B$ to its pullback functor $f^* : A \longrightarrow B$ is *pseudofunctorial*, which is to say that it preserves units and compositions, but only up to coherent isomorphisms. More specifically, for any identity morphism 1_A in a category $\mathcal{C}$ with pullbacks we have a functor $1_A^* : \mathcal{C}/A \longrightarrow \mathcal{C}/A$ and an isomorphism $\delta : 1_A^* \cong 1_{\mathcal{C}/A}$. Similarly, for composable morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ we have an isomorphism $\gamma : g^*.f^* \cong (f.g)^*$.

Now, suppose that we have a Frobenius structure for an awfs $(\mathbb{L}, \mathbb{R})$, so that any right map $\mathbf{f}$ of the awfs is assigned a Frobenius structure $\mathbf{f}^*$. In particular, any identity map 1_A is assigned the structure of a right map $\mathbf{1}_A$ by the identity functor of $\mathbb{R}$, and thus there is a Frobenius structure $\mathbf{1}_A^* : \mathbb{L}/A \longrightarrow \mathbb{L}/A$. It is therefore natural to ask whether there is a lift of the isomorphism $\delta : 1_A^* \cong 1_{\mathcal{C}/A}$ to $\boldsymbol{\delta} : \mathbf{1}_A^* \cong 1_{\mathbb{L}/A}$, as depicted on the left below:

$$
\begin{tikzcd}
\mathbb{L}/A \arrow[r, shift left=2, "\mathbf{1}_A^*"] \arrow[r, shift right=2, "1"'] \arrow[r, phantom, "\Downarrow\boldsymbol{\delta}"] \arrow[d] & \mathbb{L}/A \arrow[d] \\
\mathbb{S}\mathbf{q}(\mathcal{C}/A) \arrow[r, shift left=2, "\mathbb{S}\mathbf{q}(1_A^*)"] \arrow[r, shift right=2, "1"'] \arrow[r, phantom, "\Downarrow\mathbb{S}\mathbf{q}(\delta)"] & \mathbb{S}\mathbf{q}(\mathcal{C}/A)
\end{tikzcd}
,
\qquad
\begin{tikzcd}
\mathbb{L}/C \arrow[rr, "(\mathbf{f}.\mathbf{g})^*"] \arrow[dr, "\mathbf{g}^*"'] \arrow[ddd] & & \mathbb{L}/A \arrow[ddd] \\
& \mathbb{L}/B \arrow[ur, "\mathbf{f}^*"'] \arrow[d] \arrow[u, phantom, "\Uparrow\boldsymbol{\gamma}"] & \\
& \mathbb{S}\mathbf{q}(\mathcal{C}/B) \arrow[dr, "\mathbb{S}\mathbf{q}(f^*)"] \arrow[d, phantom, "\Downarrow\mathbb{S}\mathbf{q}(\gamma)"] & \\
\mathbb{S}\mathbf{q}(\mathcal{C}/C) \arrow[ur, "\mathbb{S}\mathbf{q}(g^*)"] \arrow[rr, "\mathbb{S}\mathbf{q}((f.g)^*)"'] & & \mathbb{S}\mathbf{q}(\mathcal{C}/A)
\end{tikzcd}
.
$$

Similarly, given composable right maps $\mathbf{f} : A \longrightarrow B$ and $\mathbf{g} : B \longrightarrow C$, the Frobenius structure gives parallel double functors $(\mathbf{f}.\mathbf{g})^*$ and $\mathbf{f}^*.\mathbf{g}^* : \mathbb{L}/C \longrightarrow \mathbb{L}/A$, and we can ask whether the coherence isomorphism $\gamma : (f.g)^* \cong g^*.f^*$ lifts to $\boldsymbol{\gamma} : \mathbf{f}^*.\mathbf{g}^* \cong (\mathbf{f}.\mathbf{g})^*$, as depicted on the right above. In other words, we can ask whether the pseudofunctoriality of $f \mapsto f^*$ lifts to that of the assignment $\mathbf{f} \mapsto \mathbf{f}^*$, and this gives rise to the following definition.

6.7. Definition. A Frobenius structure of an awfs $(\mathbb{L}, \mathbb{R})$ is *pseudofunctorial* if the assignment $\mathbf{f} \mapsto \mathbf{f}^*$ of a right map $\mathbf{f}$ to its Frobenius structure $\mathbf{f}^*$ is pseudofunctorial, and such that its coherence isomorphisms are lifts of those of $f \mapsto f^*$.

Again, we can check that our running example satisfies this condition.

6.8. Proposition. *The Frobenius structure of* $(\mathbb{S}\mathbf{plRef}(\mathbf{Gpd}), \mathbb{S}\mathbf{plFib}(\mathbf{Gpd}))$ *is pseudofunctorial.*

Proof. The requirement that the coherence isomorphism δ lifts in this way is a particular instance of Proposition 6.5. To check whether γ lifts, we consider composable split fibrations of groupoids $P : \mathcal{A} \longrightarrow \mathcal{B}$ and $Q : \mathcal{E} \longrightarrow \mathcal{A}$, and a split reflection $L \dashv R$ as in:

$$\begin{tikzcd}
\mathcal{E} \times_{\mathcal{B}} \mathcal{D} \arrow[d, shift right, "(P.Q)^*R"'] & \mathcal{E} \times_{\mathcal{A}} (\mathcal{A} \times_{\mathcal{B}} \mathcal{D}) \arrow[l, dotted, "\gamma_{U.R}"'] \arrow[r] \arrow[d, shift right, "Q^*P^*R"'] & \mathcal{A} \times_{\mathcal{B}} \mathcal{D} \arrow[r] \arrow[d, shift right, "P^*R"'] & \mathcal{D} \arrow[d, shift right, "R"'] \\
\mathcal{E} \times_{\mathcal{B}} \mathcal{C} \arrow[u, shift right, "G''"'] \arrow[d, shift right, "F''"'] \arrow[d, shift left, "(P.Q)^*U"] & \mathcal{E} \times_{\mathcal{A}} (\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \arrow[l, dotted, "\gamma_U"'] \arrow[r] \arrow[u, shift right, "G'"'] \arrow[d, shift right, "F'"'] \arrow[d, shift left, "Q^*P^*U"] & \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \arrow[r] \arrow[u, shift right, "G"'] \arrow[d, shift right, "F"'] \arrow[d, shift left, "P^*U"] & \mathcal{C} \arrow[u, shift right, "L"'] \arrow[d, "U"'] \\
\mathcal{E} \arrow[r, "1"] & \mathcal{E} \arrow[r, "Q"] & \mathcal{A} \arrow[r, "P"] & \mathcal{B}\,.
\end{tikzcd}$$

The goal is now to show that $G''.\gamma_U = \gamma_{U.R}.G'$. Similarly as in the other proofs, we can do this by showing $F''.\gamma_U = F'$ because:

$$\begin{aligned}
G''.\gamma_U &= (F'' \times_{\mathcal{B}} L.\pi_{\mathcal{C}}).\gamma_U \\
&= F''.\gamma_U \times_{\mathcal{B}} L.\pi_{\mathcal{C}}.\gamma_U \\
&= F''.\gamma_U \times_{\mathcal{B}} L.\pi_{\mathcal{C}}.\pi_{\mathcal{A}\times_{\mathcal{B}}\mathcal{C}},
\end{aligned}
\qquad
\begin{aligned}
\gamma_{U.R}.G' &= (Q^*P^*(U.R) \times_{\mathcal{B}} \pi_{\mathcal{D}}.\pi_{\mathcal{A}\times_{\mathcal{B}}\mathcal{D}}).G' \\
&= Q^*P^*(U.R).G' \times_{\mathcal{B}} \pi_{\mathcal{D}}.\pi_{\mathcal{A}\times_{\mathcal{B}}\mathcal{D}}.G' \\
&= F' \times_{\mathcal{B}} L.\pi_{\mathcal{C}}.\pi_{\mathcal{A}\times_{\mathcal{B}}\mathcal{C}}.
\end{aligned}$$

The equality $F''.\gamma_U = F'$ is straightforward to derive from the fact that the cleavage of the composed fibration $P.Q$ is just the composition of the cleavages of P and Q. For example, for an object $(e, a, c) \in \mathcal{E} \times_{\mathcal{A}} (\mathcal{A} \times_{\mathcal{B}} \mathcal{C})$, writing $\theta_{(a,c)} = (\underline{U\eta_c}, \eta_c)$ for the unit of $G \dashv P^*R$, we have that

$$\begin{aligned}
F'(e, a, c) &= ((P^*U)\theta_{(a,c)})_!e \\
&= (\pi_A(\underline{U\eta_c}, \eta_c))_!e \\
&= (\underline{U\eta_c})_!e,
\end{aligned}
\qquad
\begin{aligned}
F''(\gamma_U(e, a, c)) &= F''(e, c) \\
&= (U\eta_c)_!e.
\end{aligned}$$

First taking the lift of $U\eta_c$ along P and then along Q is the same as lifting it along the composed fibration $P.Q$, and so we indeed have that $F'(e, a, c) = F''(\gamma_U(e, a, c))$. Equality on arrows is equally straightforward, and so we omit the details. ■

7. An algebraic model of type theory from groupoids

Comprehension categories are one of the categorical structures used to model various forms of type theory. Substitution is modeled using pullbacks, and one difficulty with this is that pullbacks are usually only associative up to isomorphism, wheras substitution in type theory is strictly associative. So, in order to faithfully model type theory, the comprehension category in question should be split. One way to do this is to use the right adjoint of the forgetful functor **SplFib** $\longrightarrow$ **Fib**. In [GL23, Theorem 2.6], Gambino and Larrea give a coherence theorem for this method, by identifying pseudo-stability conditions on a comprehension category that ensure that its splitting has strict interpretations of the Σ-, Π, and Id-types of Martin-Löf type theory. Any AWFS $(\mathbb{L}, \mathbb{R})$ induces a comprehension category $\mathbb{L}_1^{\pitchfork} \longrightarrow \mathbf{Ar}(\mathcal{C})$, and conditions are identified in [GL23, Theorem 4.11] on $(\mathbb{L}, \mathbb{R})$ that ensure that the coherence theorem can be applied to it. They then apply this to the category of groupoids by showing that the AWFS of split reflections and split fibrations of groupoids satisfies these conditions [GL23, Theorem 5.5].

An AWFS $(\mathbb{L}, \mathbb{R})$ induces a second (closely related) comprehension category, namely $\mathbb{R}_1 \longrightarrow \mathbf{Ar}(\mathcal{C})$, and our goal in this section is to reproduce the aforementioned results in [GL23] for this second option—that is, to identify conditions on the AWFS $(\mathbb{L}, \mathbb{R})$ that ensure that the coherence theorem can be applied to $\mathbb{R}_1 \longrightarrow \mathbf{Ar}(\mathcal{C})$. (Note that there is a functor $\mathbb{R}_1 \cong (\mathbb{L}^{\pitchfork\!\!\pitchfork})_1 \hookrightarrow \mathbb{L}_1^{\pitchfork}$ over $\mathbf{Ar}(\mathcal{C})$, which is generally not invertible; $\mathbb{L}_1^{\pitchfork}$ corresponds to the retract closure of $\mathbb{R}_1$.) We phrase these conditions and then show that the same AWFS used by Gambino and Larrea on **Gpd** satisfies our conditions. The majority of the work needed for this has been done in Sections 3 through 5, and what remains is a straightforward adaptation of the approach of [GL23]. Therefore, we only outline the proofs, and the reader is referred to [GL23] for the remaining details.

7.1. Coherence for the comprehension category of right maps. As mentioned, the forgetful functor of the right class of an AWFS is always a comprehension category.

7.2. Proposition. *If* $(\mathbb{L}, \mathbb{R})$ *is an* AWFS *then* $\mathbb{R}_1 \longrightarrow \mathbf{Ar}(\mathcal{C})$ *is a comprehension category.*

Proof. See [BG16, Proposition 8]. ∎

In fact, no further requirements on the AWFS are needed to choose the Σ-types.

7.3. Proposition. *The comprehension category* $\mathbb{R}_1 \longrightarrow \mathbf{Ar}(\mathcal{C})$ *associated with an* AWFS *admits a pseudo-stable choice of* Σ*-types.*

Proof. The choice of Σ-types is given simply by the composition operation of $\mathbb{R}$, which is functorial and preserves pullback squares. The rest of the choices of structure are made as in [GL23, Proposition 4.3] (and see [Lar18, Lemma 2.7.7]). ∎

The Π-types are interpreted using a pushforward structure on the AWFS, and the Beck–Chevalley condition ensures this choice is pseudo-stable. Following terminology of [Joy17] we introduce a shorthand for an AWFS with this required structure.

7.4. Definition. A π-AWFS is an AWFS with a Frobenius structure satisfying the Beck–Chevalley condition as in Definition 5.10.

7.5. Proposition. *The comprehension category* $\mathbb{R}_1 \longrightarrow \mathbf{Ar}(\mathcal{C})$ *associated with a* π-AWFS *admits a pseudo-stable choice of* Π*-types.*

Proof. Consider composable vertical morphisms $\mathbf{g}, \mathbf{f}$ of $\mathbb{R}$. As the AWFS has a Frobenius structure, $\mathbf{f}$ has a pushforward structure $\mathbf{f}_*$ by Theorem 3.14. We can consider $\mathbf{g}$ as an object $(\mathbf{g}, 1) \in \mathbb{R}/(\operatorname{dom} f)$ and apply $\mathbf{f}_*$ to obtain $\mathbf{f}_*(\mathbf{g}, 1) \in \mathbb{R}/(\operatorname{cod} f)$, whence f_*g gets an $\mathbb{R}$-structure which we denote by $\mathbf{f}_*\mathbf{g}$; this is the choice of Π-type for $\mathbf{g}$ and $\mathbf{f}$. The

Beck–Chevalley condition then ensures that the assignment $\Pi : (\mathbf{g}, \mathbf{f}) \mapsto \mathbf{f}_*\mathbf{g}$ is functorial:

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "\mathbf{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "\mathbf{i}"] \\ \bullet \arrow[r, "v"] \arrow[d, "\mathbf{f}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "\mathbf{h}"] \\ \bullet \arrow[r, "w"'] & \bullet \end{tikzcd} \quad \overset{\Pi}{\longmapsto} \quad \begin{tikzcd} \bullet \arrow[r, "f_*\alpha"] \arrow[d, "\mathbf{f}_*\mathbf{g}"'] & \bullet \arrow[r, "\beta_i"] \arrow[d, "\mathbf{f}_*\mathbf{v}^*\mathbf{i}"'] & \bullet \arrow[r, "w^+"] \arrow[d, "\mathbf{w}^*\mathbf{h}_*\mathbf{i}"] & \bullet \arrow[d, "\mathbf{h}_*\mathbf{i}"] \\ \bullet \arrow[r, "1"'] & \bullet \arrow[r, "1"'] & \bullet \arrow[r, "w"'] & \bullet \end{tikzcd} .$$

The outer rectangle on the right above is a pullback, and is a square of $\mathbb{R}$ because all three of the squares comprising it are. The rest of the choices of structure are made as in [GL23, Proposition 4.6] (and see [Lar18, Lemma 2.7.8]). ∎

The idea for interpreting identity types using WFS originates from [AW09] and was further developed in [vdBG12] by the introduction of the notion of stable functorial choice of path objects. We adapt this definition to the right maps of an AWFS as follows.

7.6. Definition. A *functorial factorization of the diagonal* on a category $\mathcal{C}$ is a functor $P = (r, \rho) : \mathbf{Ar}(\mathcal{C}) \longrightarrow \mathbf{Ar}(\mathcal{C}) \times_{\mathcal{C}} \mathbf{Ar}(\mathcal{C})$ such that $\rho f . r f = \delta_f$ for any map $f : A \longrightarrow B$ in $\mathcal{C}$, where $\delta_f : A \longrightarrow A \times_B A$ is the diagonal morphism. Such a factorization is called *stable* if its right leg ρ preserves pullback squares.

7.7. Definition. A *stable functorial choice of path objects* (SFPO) on an AWFS $(\mathbb{L}, \mathbb{R})$ is a lift $\mathbf{P}$ of a stable functorial factorization of the diagonal P:

$$\begin{tikzcd} \mathbb{R}_1 \arrow[r, "\mathbf{P}"] \arrow[d] & \mathbb{L}_1 \times_{\mathcal{C}} \mathbb{R}_1 \arrow[d] \\ \mathbf{Ar}(\mathcal{C}) \arrow[r, "P"] & \mathbf{Ar}(\mathcal{C}) \times_{\mathcal{C}} \mathbf{Ar}(\mathcal{C}) \end{tikzcd} .$$

7.8. Proposition. *The comprehension category* $\mathbb{R}_1 \longrightarrow \mathbf{Ar}(\mathcal{C})$ *associated with an* AWFS *with an* SFPO *admits a pseudo-stable choice of* Id*-types.*

Proof. The choice of Id-type for an $\mathbb{R}$ map $\mathbf{f}$ is given by the right leg $\boldsymbol{\rho}\mathbf{f}$ of the SFPO $\mathbf{P} = (\boldsymbol{r}, \boldsymbol{\rho})$; this assignment is functorial and preserves cartesian morphisms by definition. The rest of the choices of structure are made as in [GL23, Proposition 4.9]. ∎

We thus obtain the desired analog of [GL23, Theorem 4.11].

7.9. Theorem. *A* π-AWFS *with an* SFPO *induces a comprehension category with strictly stable choices of* Σ-, Π-, *and* Id*-types.*

Proof. By Propositions 7.3, 7.5, and 7.8 the comprehension category associated with the AWFS has pseudo stable choices of these types, and so its right adjoint splitting has strictly stable choices by [GL23, Theorem 2.6]. ∎

7.10. The groupoid model. Finally, we mention that our running example, the AWFS of split reflections and fibrations of groupoids, satisfies the conditions of Theorem 7.9.

7.11. Proposition. *$(\mathbb{S}\mathbf{plRef}(\mathbf{Gpd}), \mathbb{S}\mathbf{plFib}(\mathbf{Gpd}))$ is a π-AWFS with an SFPO.*

Proof. It is a π-AWFS by Propositions 4.1 and 5.15, and that it admits an SFPO follows from a straightforward adaptation of [GL23, Proposition 3.5]. ∎

This gives us the following analog of [GL23, Theorem 5.5].

7.12. Theorem. *The right adjoint splitting of $\mathbf{SplRef}(\mathbf{Gpd}) \longrightarrow \mathbf{Ar}(\mathbf{Gpd})$ yields a model of dependent type theory with Σ-, Π-, and Id-types.*

Proof. By Theorem 7.9 and Proposition 7.11. ∎

8. Conclusion

In this paper we have formulated suitable versions of the pushforward and Frobenius properties of algebraic weak factorization systems (AWFS) and showed that these are equivalent. In addition, we have formulated a Beck–Chevalley condition for AWFS. We have shown how these notions can be used to obtain models of type theory with dependent function types, as illustrated by split fibrations of groupoids, the basis for the groupoid model of Hofmann and Streicher [HS98].

In this way our work is similar to [Lar18, GL23], where the authors also show an equivalence between a pushforward and a Frobenius property of an AWFS, which is stated and proven in [GS17]. However, their version differs from the one we propose here in several ways. First of all, their version holds for AWFS which are cofibrantly generated by a category; we make no such assumption. Also, the works of [GS17, Lar18, GL23] use an interpretation of type theory in which the dependent types are interpreted using the algebras of the pointed endofunctor associated with an AWFS, rather than the algebras of the monad associated with an AWFS. In that way their framework targets a different class of examples and does not cover examples like split fibrations of groupoids.

In future work we hope to find more examples. In particular, we would like to show how our ideas apply to the *effective Kan fibrations* from [vdBF22]. These are intended to be a good constructive analogue of the Kan fibrations from simplicial homotopy theory and to lead to a constructive account of Voevodsky's model of homotopy type theory in simplicial sets [KL21]. In [vdBG25] the second author has shown together with Freek Geerligs how the effective Kan fibrations appear as the right class in an AWFS, so our current framework is the appropriate one for these maps.

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