INTEGRATION OF A CATEGORICAL OPERAD

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ABSTRACT. We describe a Grothendieck construction for non-symmetric operads with values in categories, and hence in groupoids and posets. The construction produces a 2-category which is operadically fibered over the category Δ_s of finite non-empty ordinals and surjections. We describe an inverse for the construction, yielding an equivalence of constant-free non-symmetric categorical operads and operadic 2-categories (split-)fibered over Δ_s, which resembles the correspondence of categorical presheaves and fibered categories. The result provides a new characterization of non-symmetric categorical operads and tools to study them.

1. Introduction

To integrate means to make something *whole.* An operad $\mathcal{P}$ is a compositional structure consisting of separate objects $\mathcal{P}_n$ of abstract n-ary operations, and rules how to compose them. By integrating an operad we mean producing a new structure which contains all the operations of the operad and faithfully encodes the composition of operations. Formally, we wish to describe operadic integration as a fully faithful functor from the category of operads. The aim of this text is to find a suitable codomain of the integration functor and characterize its essential image for operads valued in the category of categories.

We admit that a more suitable name for integration is perhaps *operadic Grothendieck construction*, following the terminology of [BM15, p. 17]. However the term integration is shorter and still meaningful. Loc. cit., the operadic Grothendieck construction is introduced for any Set-valued operad, resulting in an *operadic category.* The article [BM15] develops the theory of operadic categories, which is a unifying framework for general operadic structures and their comparison. The integration is thus one of the features of their theory and it is available for a general Set-valued $\mathbb{O}$-operad together with an equivalence

$$\mathbb{O}-\mathrm{oper}(\mathrm{Set}) \simeq \mathrm{DoFib}(\mathbb{O})$$

between Set-valued $\mathbb{O}$-operads and discrete operadic fibrations over the operadic category $\mathbb{O}$. The $\mathbb{O}$-operads include classical symmetric and non-symmetric operads, colored operads, cf. [BM15, ex. 1.15], or graph-based operads (hyperoperads) governing cyclic or modular operads, wheeled properads, dioperads, $\frac{1}{2}$PROPs, permutads, and more, cf. [BM23, s. 5-7].

To draw a connection, we recall the classical Grothendieck construction (i.e. integration) of a categorical presheaf [JY20, s. 10.1]. For a functor $F:\mathscr{C}^{\mathrm{op}} \to \mathrm{Cat}$, the category $\int_{\mathscr{C}} F$ has objects $[C,a]$, where $C \in \mathscr{C}$ and $a \in FC$. The morphisms $[C,a] \to [D,b]$ are pairs $[f;\alpha]$ with $f:C \to D$ in $\mathscr{C}$ and $\alpha:Ff(b) \to a$ in FC. The term integration is already suggested by the commonly used symbol $\int$ for the construction, however it is not a common terminology. There is a fully faithful functor

$$\int_{\mathscr{C}}:\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, Cat) \to \mathrm{Cat}/\mathscr{C},$$

which establishes a 2-equivalence

$$[\mathscr{C}^{\mathrm{op}}, \mathrm{Cat}] \simeq \mathrm{Fib}(\mathscr{C})$$

of categorical presheaves and categorical fibrations over $\mathscr{C}$. For a detailed treatment cf. [JY20, ch. 9 & 10].

The current text is a follow-up to [T24], where the integration is described for categorical operads of *unary* operadic categories, i.e. operadic categories whose cardinality functor has a constant value 1. Our long term goal is to develop the integration for categorical operads of operadic categories of general cardinalities. The current text makes a step in this direction by deriving the results for a relatively simple non-unary operadic category Δ_s of finite non-empty ordinals and surjections. Its operads are constant-free non-symmetric operads, i.e. non-symmetric operads that have no operations of arity zero ($\mathcal{P}_0 = \varnothing$). We chose this particular setup to reduce any unnecessary technicalities.

As an application, the operadic integration can be used to combine two compatible structures on one set. In Example 2.7 we deal with planar rooted trees together with the operation of grafting (operadic structure) and edge contraction (poset structure). As a result, the integration of the categorical operad of trees $\int \mathcal{T}$ is a 2-category whose morphisms combine the two structures in a natural way. Further, there is a strict factorization system on $\int \mathcal{T}$ which factors a general morphisms as *contractions* followed by *cuts*, cf. Proposition 2.9 and the diagrams below it.

This work builds on the operadic Grothendieck construction of [BM15, p. 17], and the classical Grothendieck construction for categorical presheaves [JY20, s. 10.1]. It was however necessary to develop a new framework of (non-symmetric) operadic 2-categories, which is novel. We believe that the operadic integration could be alternatively approached by other operadic frameworks, that is, using the language of T-multicategories [Lein04], polynomial monads [BB17], or Feynman categories [KW17]. However, the author admits his lack of knowledge of such results in these contexts. The results of this text are summarized below.

RESULTS: For a constant-free non-symmetric operad $\mathcal{P}$, a 2-category $\int \mathcal{P}$ is constructed, together with a projection $\pi:\mathcal{P} \to \Delta_s$. We describe its properties and introduce a 2-categorical generalization of (non-symmetric) operadic categories in Definition 2.12. We also introduce a non-discrete version of operadic fibrations and show that the projection π is a splitting operadic fibration, cf. Definitions 2.17, 2.18 and Theorem 3.1. We arrive

at Theorem 3.8 which gives the equivalence of constant-free non-symmetric categorical operads and split-fibered (non-symmetric) operadic 2-categories. Lastly, Proposition 4.3 relates the results to the standard case of categorical presheaves and categorical fibrations.

To close the introductory section, we comment briefly on categorical operads and where to find them. In the literature, categorical operads often appear either as operads with values in groupoids, or operads valued in partially ordered sets. Some of the most classical operads also carry a natural structure of a poset or groupoid. Namely, there is the Tamari order on the set of planar binary rooted trees, e.g. [Lod02, s. 2.8]. A remarkable application of operads valued in groupoids is [Fre17] on Grothendieck–Teichmüller groups. Operads valued in posets appear e.g. in [Ber97, def. 1.4 & ex. 1.15(b)] and many poset-valued operads are described in [Bash24]. Categorical operads are further considered, for instance, in [Bat08, CG14, Elme17]. A concrete example of a (non-strict) categorical operad is the operad of leveled trees [T23, def. 5.1]: the operations of arity n are the leveled planar rooted trees with n leaves, and there is a unique isomorphism of trees if they differ by an admissible change of leveling, cf. Figure 1. This operad plays a key rôle in the construction of free operads, cf. [BM23, s. 3.2] or [T23, s. 5].
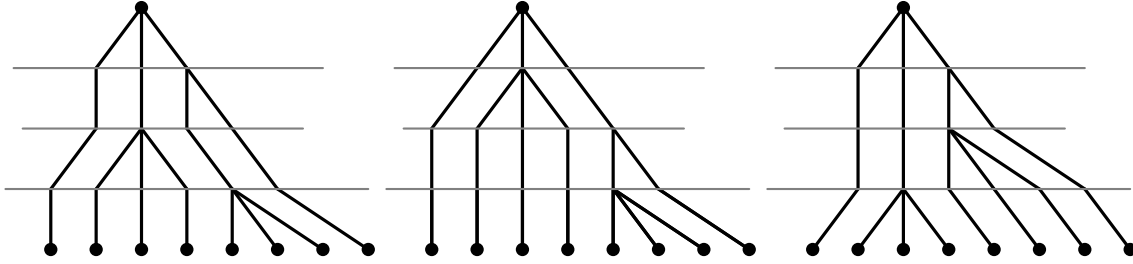

Figure 1: Three isomorphic leveled trees.

2. Definitions and Examples

Briefly, a constant-free non-symmetric categorical operad $\mathcal{P}$ is a collection of categories $\{\mathcal{P}_n\}_{n \geq 1}$ equipped with composition functors (1) which are strictly associative and unital.

$$\mathcal{P}_n \times \mathcal{P}_{k_1} \times \cdots \times \mathcal{P}_{k_n} \xrightarrow{\mu} \mathcal{P}_{k_1 + \cdots + k_n} \tag{1}$$

To state the full definition we follow [BM15] and we use the language of finite ordinals and order preserving maps. By constant-free we mean that the operad $\mathcal{P}$ does not contain nullary operations, i.e. we do not consider $\mathcal{P}_0$. Hence, we will work only with non-empty finite ordinals and order preserving surjections.

Let $\underline{n} = [1 < 2 < \cdots < n]$ denote the finite ordinal for $n \geq 1$, and let $g : \underline{k} \to \underline{n}$ be an order preserving surjective map. These form a category Δ_s. For $1 \leq i \leq n$ we identify the preimage $g^{-1}(i)$ with a finite ordinal $\underline{k_i}$. For two composable maps

$$\underline{m} \xrightarrow{f} \underline{k} \xrightarrow{g} \underline{n}$$

and $i \in \underline{n}$, there is an induced map between the preimages $(gf)^{-1}(i) \to g^{-1}(i)$, which is denoted by f^i. For $i \in \underline{n}$ and $j \in \underline{k}$ with $g(j) = i$ it holds $(f^i)^{-1}(j) = f^{-1}(j)$. This setup allows us to label the composition map of (1) by the unique order preserving surjection

$$g: \underline{k_1 + \cdots + k_n} \to \underline{n}$$

with the preimages $\underline{k_1}, \ldots, \underline{k_n}$. The associativity and unitality conditions can then be stated using composites of surjections and induced maps between preimages, which is the content of the following definition.

2.1. DEFINITION. *A* constant-free non-symmetric categorical operad *is a collection of categories* $\{\mathcal{P}_n\}_{n \geq 1}$, *equipped with composition functors*

$$\mathcal{P}_n \times \mathcal{P}_{k_1} \times \cdots \times \mathcal{P}_{k_n} \xrightarrow{\mu_g} P_k,$$

indexed by the maps $g: \underline{k} \to \underline{n}$ *of* Δ_s, *which are associative and unital in the following sense.*
(Associativity.) *Let* $f: \underline{m} \to \underline{k}$ *and* $g: \underline{k} \to \underline{n}$ *be two composable order preserving surjections. For* $i \in \underline{n}$ *and* $j \in \underline{k}$ *with* $g(j) = i$ *we identify*

$$g^{-1}(i) = \underline{k_i}, \ (gf)^{-1}(i) = \underline{m_i}, \ and \ (f^i)^{-1}(j) = f^{-1}(j) = \underline{m_j^i}.$$

The following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{P}_n \times \prod_i (\mathcal{P}_{k_i} \times \prod_j \mathcal{P}_{m_j^i}) & \xrightarrow{\mathcal{P}_n \times \prod_i \mu_{f^i}} & \mathcal{P}_n \times \prod_i \mathcal{P}_{m_i} \\
{\scriptstyle \text{shuffle}} \downarrow & & \downarrow {\scriptstyle \mu_{gf}} \\
(\mathcal{P}_n \times \prod_i \mathcal{P}_{k_i}) \times \prod_{i,j} \mathcal{P}_{m_j^i} & & \\
{\scriptstyle \mu_g \times \prod_{i,j} \mathcal{P}_{m_j^i}} \downarrow & & \\
\mathcal{P}_k \times \prod_{i,j} \mathcal{P}_{m_j^i} & \xrightarrow{\mu_f} & \mathcal{P}_m
\end{array}
$$

(Unitality.) *The operad is equipped with a unit*

$$1 \xrightarrow{\eta} \mathcal{P}_1,$$

where **1** *is the terminal category. For each* $n \geq 1$, $\mathbb{1}_n: \underline{n} \to \underline{n}$ *is the identity map and* $!_n: \underline{n} \to \underline{1}$ *is the unique map to* $\underline{1}$. *The following diagrams commute.*

$$
\begin{array}{ccc}
\mathcal{P}_n \times \mathbf{1}^{\times n} & \xrightarrow{\cong} & \mathcal{P}_n \\
{\scriptstyle \mathcal{P}_n \times \eta^{\times n}} \downarrow & \nearrow {\scriptstyle \mu_{\mathbb{1}_n}} & \\
\mathcal{P}_n \times \mathcal{P}_1^{\times n} & &
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{1} \times \mathcal{P}_n & \xrightarrow{\cong} & \mathcal{P}_n \\
{\scriptstyle \eta \times \mathcal{P}_n} \downarrow & \nearrow {\scriptstyle \mu_{!_n}} & \\
\mathcal{P}_1 \times \mathcal{P}_n & &
\end{array}
$$

Every operad in this text is assumed to be constant-free, non-symmetric, and valued in categories, and so we will skip these adjectives.

2.2. DEFINITION. *Let $\mathcal{P}$ be an operad with composition μ and unit η, and let $\mathcal{Q}$ be an operad with composition ν and unit ζ. A morphism of operads $F:\mathcal{P} \to \mathcal{Q}$ is a collection of functors $F_n:\mathcal{P}_n \to \mathcal{Q}_n$, for each $n \geq 1$, which respect the composition and unit. That is, for any $g:\underline{k} \to \underline{n}$ of Δ_s, the following two diagrams commute.*

$$\begin{array}{ccc}
\mathcal{P}_n \times \mathcal{P}_{k_1} \times \cdots \times \mathcal{P}_{k_n} & \xrightarrow{\mu_g} & \mathcal{P}_k \\
\scriptstyle{F_n \times F_{k_1} \times \cdots \times F_{k_n}} \downarrow & & \downarrow \scriptstyle{F_k} \\
\mathcal{Q}_n \times \mathcal{Q}_{k_1} \times \cdots \times \mathcal{Q}_{k_n} & \xrightarrow{\nu_g} & \mathcal{Q}_k
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{\eta} & \mathcal{P}_1 \\
 & \scriptstyle{\zeta} \searrow & \downarrow \scriptstyle{F_1} \\
 & & \mathcal{Q}_1
\end{array}$$

We denote the category of operads and their morphisms by Δ_s–oper(Cat).

Let $g:\underline{k} \to \underline{n}$ be a map of Δ_s. A sequence of objects $a_1, \ldots, a_k$ can be cut into n blocks by the map f, which we write as follows.

$$\{a_i\}_{1 \leq i \leq k} = \left\{\{a_j^i\}_{1 \leq j \leq k_i}\right\}_{1 \leq i \leq n} \tag{2}$$

In terms of elements, the associativity is written as an equation

$$\mu_f\big(\mu_g(c, b_1, \ldots, b_n), a_1, \ldots, a_k\big) = \mu_{gf}\big(c, \mu_{f^1}(b_1, a_1^1, \ldots, a_{k_1}^1), \ldots, \mu_{f^n}(b_n, a_1^n, \ldots, a_{k_n}^n)\big), \tag{3}$$

and the unitality gives two equations
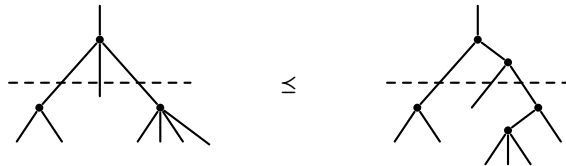
$$\mu_{\mathbb{1}_n}(a, e, \ldots, e) = a, \tag{4}$$

$$\mu_{!_n}(e, a) = a. \tag{5}$$

2.3. EXAMPLE. Any strict monoidal category $(\mathscr{V}, \otimes, 1)$ gives an operad with $\mathcal{P}_1 = \mathscr{V}$ and $\mathcal{P}_{n \geq 2} = \varnothing$. The only nontrivial composition functor of the operad is $\mu_{\mathbb{1}_1}$, indexed by the identity on $\underline{1}$, and it is given by the monoidal product $\otimes:\mathcal{P}_1 \times \mathcal{P}_1 \to \mathcal{P}_1$.

2.4. EXAMPLE. We describe a categorical operad which combines two classical operations on planar rooted trees: grafting and edge contraction. Let $\mathcal{T}_n$ be a poset of planar rooted trees with n leaves, with the partial order

$$t \leq s \text{ if } t \text{ is obtained from } s \text{ by edge contractions.}$$

The grafting operation preserves the partial order $\leq$, which can be seen on the diagram below. Hence, interpreting posets as categories, the collection $\mathcal{T}_n$ with grafting is a non-symmetric categorical operad.

2.5. DEFINITION. *For a categorical operad* $\mathcal{P}$, *its* integration *(or* operadic Grothendieck construction) $\int \mathcal{P}$ *is a 2-category defined by the following data.*

- *0-cells of* $\int \mathcal{P}$ *are pairs* $[m, a]$, *where* $m \geq 1$ *and* $a \in \mathcal{P}_m$,

- *1-cells of* $\int \mathcal{P}$, *that is objects of* $\int \mathcal{P}\big([m, a], [k, b]\big)$, *are tuples* $[f; a_1, \ldots, a_k; \alpha]$, *where*

  - $f \colon \underline{m} \to \underline{k} \in \Delta_s$
  - $a_i \in \mathcal{P}_{f^{-1}(i)}$, *and*
  - $\alpha \colon \mu_f(b, a_1, \ldots, a_k) \to a \in \mathcal{P}_m$.

- *A 2-cell* δ *of* $\int \mathcal{P}$, *that is morphisms in* $\int \mathcal{P}\big([m, a], [k, b]\big)$,

$$[f; a_1', \ldots, a_k'; \alpha'] \overset{\delta}{\Longrightarrow} [f; a_1'', \ldots, a_k''; \alpha''],$$

is a sequence of morphisms $\{\delta_i \colon a_i' \to a_i'' \in \mathcal{P}_{f^{-1}(i)}\}_{1 \leq i \leq k}$, *such that*

$$\alpha'' \circ \mu_f(\mathbb{1}; \delta_1, \ldots, \delta_k) = \alpha'.$$

There are no 2-cells between morphisms which differ in the first component.

The horizontal composition of 1-cells is given as follows. Let

$$\underline{m} \overset{f}{\to} \underline{k} \overset{g}{\to} \underline{n},$$

$$\alpha \colon \mu_f(b, a_1, \ldots, a_k) \to a,$$
$$\beta \colon \mu_g(c, b_1, \ldots, b_n) \to b.$$

The composite $[g; b_1, \ldots b_n; \beta] \circ [f; a_1, \ldots, a_k; \alpha]$ is defined as

$$\Big[gf; \mu_{f^1}(b_1, a_1^1, \ldots, a_{k_1}^1), \ldots, \mu_{f^n}(b_n, a_1^n, \ldots, a_{k_n}^n); \alpha \circ \mu_f(\beta, a_1, \ldots, a_k)\Big].$$

The source of $\alpha \circ \mu_f(\beta, a_1, \ldots, a_k)$ has a correct form thanks to associativity of $\mathcal{P}$. The identity maps for horizontal composition come from the operad unit $e \in \mathcal{P}_1$ (i.e. the image of η),
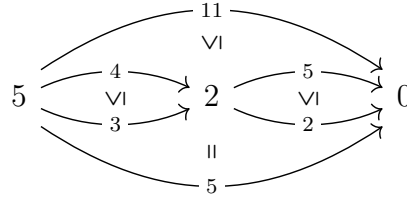
$$\mathbb{1}_{[m,a]} = [\mathbb{1}_m, e \ldots, e; \mathbb{1}_a] \colon [m, a] \to [m, a].$$

It is straightforward to check that $\int \mathcal{P}$ is indeed a 2-category. We will use the term *integration* of the operad $\mathcal{P}$ for this 2-category.

2.6. EXAMPLE. Consider the poset $(\mathbb{N}, \geq)$ of natural numbers as a category with $a \to b$ if $a \geq b$. The addition respects the order, so $(\mathbb{N}, \geq, +)$ is a strict monoidal category, and hence a non-symmetric operad concentrated in arity 1, cf. Example 2.3. Its integration $\int \mathbb{N}$ is the 2-category given as follows. Objects of $\int \mathbb{N}$ are natural numbers, the maps are
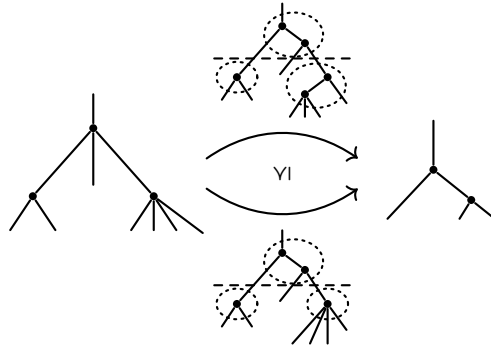
$$\int \mathbb{N}(a, b) = \{p \in \mathbb{N}, b + p \geq a\}$$

and 2-cells are $p' \geq p''$. The horizontal composite of $p : a \to b$ and $q : b \to c$ is $(p + q) : a \to c$, since $p + q + c \geq p + b \geq a$. The following is an example of a 2-dimensional diagram in $\int \mathbb{N}$.

Notice, that for any $a \in \mathbb{N}$, the poset $\int \mathbb{N}(a, 0)$ has a terminal object, namely the map $a : a \to 0$. In fact, for $a \geq b$, $\int \mathbb{N}(a, b)$ has a terminal object $c = a - b$, and $\int \mathbb{N}(b, a) = \varnothing$ if $b < a$.

2.7. EXAMPLE. Let $\mathcal{T}$ be the operad of trees of Example 2.4. Its integration $\int \mathcal{T}$ has as object planar rooted trees. A morphism $s \to t$ is a new tree p with (i) a *cut*, such that the upper part (containing the root) is the tree t, and (ii) there exists a sequence of edge contractions of p producing the tree s. There is a 2-cell $p' \Rightarrow p''$ if and only if $p'' \leq p'$. An example of two morphisms and a 2-cell is given by the following diagram.

   Analogously to the classical Grothendieck construction for categorical presheaves, there is a canonical factorization of morphisms of $\int \mathcal{P}$. We recall the definition of a strict factorization system on a category.

2.8. DEFINITION. *A* strict factorization system *on a category $\mathscr{C}$ is a pair of wide subcategories* E *and* M *of $\mathscr{C}$ such that every morphism f of $\mathscr{C}$ factors uniquely as $f = m \circ e$, where $m \in$ M *and $e \in$* E.
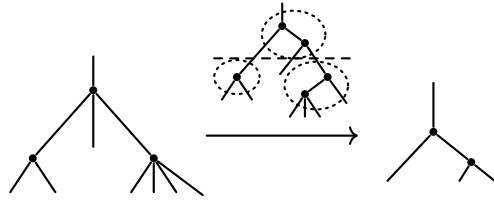
2.9. PROPOSITION. *Let $\mathcal{P}$ be a non-symmetric categorical operad. There is a strict factorization system on $\int \mathcal{P}$ given by*

$$[f; a_1, \ldots, a_k; \alpha] = [f; a_1, \ldots, a_k; \mathbb{1}] \circ [\mathbb{1}; e, \ldots, e; \alpha].$$
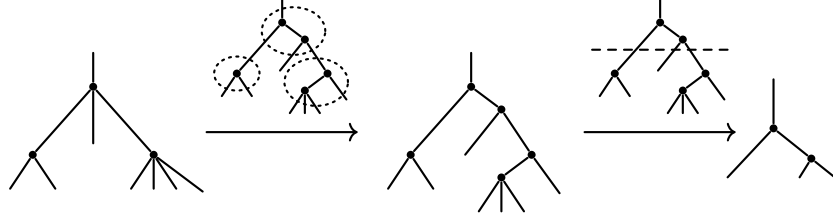
The subcategory $\mathtt{E}$ consists of morphisms $[f; a_1, \ldots, a_k; \alpha]$ with $f = \mathbb{1}$ and all a_i's are the operad unit e, and $\mathtt{M}$ consists of morphisms with $\alpha = \mathbb{1}$.

PROOF. Straightforward. ∎

In Example 2.6 a morphism $a \xrightarrow{p} b$ factors as $a \xrightarrow{p} b = a \xrightarrow{0} p+b \xrightarrow{p} b$, and in Example 2.7, the morphism

factors as

where the first map cuts only leaves and the second map does not contract anything.

Our next goal is to describe additional properties of the 2-category $\int \mathcal{P}$ and the projection

$$\pi \colon \int \mathcal{P} \to \Delta_s.$$

2.10. DEFINITION. *Let $\mathscr{C}$ be a 2-category and x an object of $\mathscr{C}$. The* lax slice 2-category *$\mathscr{C}/x$ has the following structure.*

- *Objects are maps $y \xrightarrow{\varphi} x$ of $\mathscr{C}$ with codomain x.*

- *For objects $z \xrightarrow{\theta} x$ and $y \xrightarrow{\varphi} x$, a map $\theta \to \varphi$ of $\mathscr{C}/x$ is a pair (ψ, α), where $z \xrightarrow{\psi} y$ is a map of $\mathscr{C}$ and $\varphi \circ \psi \xRightarrow{\alpha} \theta$ is a 2-cell of $\mathscr{C}$. The pair (φ, α) is drawn as:*

$$z \xrightarrow{\ \ \psi\ \ } y.$$

(6)

We omit the 2-cell arrows "⇒" in triangles as above.

- *A 2-cell γ of $\mathscr{C}/x$,*

$$\theta \underset{(\psi'',\alpha'')}{\overset{(\psi',\alpha')}{\Downarrow\gamma}} \varphi$$

is a 2-cell $\psi' \overset{\gamma}{\Rightarrow} \psi''$ of $\mathscr{C}$, such that $\alpha'' \circ (\mathbb{1}_\varphi \square \gamma) = \alpha'$. The symbol $\square$ denotes horizontal composition of 2-cells in $\mathscr{C}$.

More details can be found for instance in [JY20, Definition 7.1], but note the opposite orientation of the triangle interior. For any $x \in \mathscr{C}$, a lax functor $F: \mathscr{C} \to \mathscr{D}$ induces a lax functor $\mathscr{C}/x \to \mathscr{D}/Fx$. For a lax triangle (ψ, α) as in (6) we shall use $d_2\alpha = \psi$, $d_1\alpha = \theta$, and $d_0\alpha = \varphi$.

2.11. DEFINITION. *An object v in a 2-category $\mathscr{C}$ is* lali-terminal, *if for every object x of $\mathscr{C}$, the category $\mathscr{C}(x,v)$ has a terminal object. We require that the terminal object of $\mathscr{C}(v,v)$ is the identity on v. An object v is* local lali-terminal *if it is lali-terminal in its connected component.*

The prefix *lali* stands for *left adjoint left inverse*. This terminology appears in [Štěp24, ex. 4.27], however the condition on the terminal object of $\mathscr{C}(v,v)$ being identity is not required there.

2.12. DEFINITION. *A* non-symmetric operadic 2-category *is a 2-category $\mathbb{O}$ equipped with*

- *a 2-functor $|-|: \mathbb{O} \to \Delta_s$, called* cardinality,

- *2-functors $\mathrm{fib}_x: \mathbb{O}/x \to \mathbb{O}^{|x|}$ from the lax slice, for every object x of $\mathbb{O}$, and*

- *for each connected component c of $\mathbb{O}$ that contains at least one object of non-zero cardinality we require a choice of a local lali-terminal object u_c in this component.*

This data satisfies axioms analogous to those of classical operadic categories [BM15], and we state them below.

To present the axioms we first introduce necessary notation and terminology. The functors fib_x are called fiber functors. For $(\varphi: y \to x) \in \mathbb{O}/x$, $\mathrm{fib}_x(\varphi)$ is a tuple of objects

$$\mathrm{fib}_x^i(\varphi), 1 \le i \le |x|,$$

which we call the *fibers* of φ. We denote them simply by φ_i. For a lax triangle

$$\alpha: (\varphi \circ \psi) \Rightarrow \theta$$

in $\mathbb{O}$, the *induced maps* between fibers $\mathrm{fib}_x^i(\alpha): \mathrm{fib}_x^i(\theta) \to \mathrm{fib}_x^i(\varphi)$ will be denoted by $\alpha^i: \theta_i \to \varphi_i$. For any object x of $\mathbb{O}$ with $|x| \ne 0$, lying in a connected component c, we denote the terminal object of $\mathbb{O}(x, u_c)$ by ε_x. The object $u_c = u_{\pi(x)}$ will be sometimes denoted simply by u_x. The promised axioms of Definition 2.12 are the following.

(i) The fiber functors preserve cardinality, that is, the following diagram commutes for any object x of $\mathbb{O}$.

$$\begin{array}{ccc} \mathbb{O}/x & \xrightarrow{\mathrm{fib}_x} & \mathbb{O}^{|x|} \\ \downarrow{\scriptstyle |-|/x} & & \downarrow{\scriptstyle |-|^{|x|}} \\ \Delta_s/|x| & \xrightarrow[\mathrm{fib}_{|x|}]{} & \Delta_s^{|x|} \end{array}$$

The bottom functor $\mathrm{fib}_{|x|}$ is given by preimages and induced maps as on page 132.

(ii) For every connected component c of $\mathbb{O}$, $|u_c| = \underline{1}$, and fib_{u_c} is the domain functor.

(iii) For any object x of $\mathbb{O}$, fibers of the identity $\mathbb{1}_x$ are the chosen local lali-terminal objects and we denote them by $u_x^1, \ldots, u_x^k$, where $|x| = \underline{k}$.

(iv) For a map $\varphi \colon y \to x$ of $\mathscr{C}$, denote the lax triangle $\mathbb{1}_x \circ \varphi \overset{\mathbb{1}_\varphi}{\Longrightarrow} \varphi$ by ε_φ, then $(\varepsilon_\varphi)^i = \varepsilon_{(\varphi_i)}$.

(v) THE FIBER AXIOM. For any $\varphi \colon y \to x$, the following diagram commutes.

$$\begin{array}{ccccc} (\mathbb{O}/x)/\varphi & \xrightarrow{\mathrm{fib}_x/\varphi} & \mathbb{O}^{|x|}/\mathrm{fib}_x(\varphi) & \xrightarrow{\cong} & \displaystyle\prod_{i=1}^{|x|} \mathbb{O}/\mathrm{fib}_x^i(\varphi) \\ \downarrow{\scriptstyle \mathrm{dom}/\varphi} & & & & \downarrow{\scriptstyle \prod_{i=1}^{|x|} \mathrm{fib}_{\mathrm{fib}_x^i(\varphi)}} \\ \mathbb{O}/y & \xrightarrow[\mathrm{fib}_y]{} & \mathbb{O}^{|y|} & \xrightarrow[\cong]{} & \displaystyle\prod_{i=1}^{|x|} \mathbb{O}^{|\mathrm{fib}_x^i(\varphi)|} \end{array}$$

The bottom isomorphism comes from the equation

$$|\mathrm{fib}_x^1(\varphi)| + \cdots + |\mathrm{fib}_x^k(\varphi)| = |y|.$$

2.13. REMARK. The fiber axiom of [BM15, s. 1] for operadic 1-categories was reformulated to the above compact form (v) in [Lack18, def. 2.2]. Note that we do not require the existence of local lali-terminal objects in every component, but only in those where the cardinality functor is not constantly zero. This is a mild generalization of the standard definition of [BM15, s. 1], which allows us to view ordinary (2-)categories as *nullary* operadic (2-)categories, i.e. those with constantly zero cardinality, similarly to [Lack18, ex. 2.11]. In some form the weakening of unitality of operadic categories was studied in [Lack18, prop. 2.4] and [BM24, def. 17]. The generalization will be important for comparison of operadic fibrations and classical categorical fibrations in Proposition 4.3.

Since Δ_s is considered as a 2-category with only identity 2-cells, the cardinality functor sends any 2-cell of $\mathbb{O}$ to an identity 2-cell.

2.14. EXAMPLE. Every non-symmetric operadic 1-category $\mathbb{O}$, by which we mean a classical operadic category of [BM15] where the cardinality factors through Δ_s, is a non-symmetric operadic-2-category, viewing $\mathbb{O}$ as a 2-category with only identity 2-cells. In particular Δ_s is an example.

2.15. DEFINITION. *An* operadic 2-functor *between non-symmetric operadic 2-categories is a 2-functor which respects the fiber 2-functors, preserves cardinality and chosen local lali-terminal objects.*

2.16. DEFINITION. *Let $p: \mathbb{O} \to \mathbb{P}$ be an operadic 2-functor and let $\varphi: s \to t$ in $\mathbb{O}$ with fibers $\varphi_1, \ldots, \varphi_k$. We say that φ is* operadic p-cartesian *if for any morphism $\theta: r \to t$ with fibers $\theta_1 \ldots \theta_k$, morphisms $\psi^i: \theta_i \to \varphi_i$ in $\mathbb{P}$ and a lax triangle α in $\mathbb{P}$, with $\alpha^i = p\psi^i$, $d_1\alpha = p\theta$ and $d_0\alpha = p\varphi$, there exists a unique lax triangle $\tilde{\alpha}$ in $\mathbb{O}$ with $\tilde{\alpha}^i = \psi^i$ and $p\tilde{\alpha} = \alpha$.*

The operadic p-cartesian property is depicted on the following diagram.

2.17. DEFINITION. *An operadic 2-functor $p: \mathbb{O} \to \mathbb{P}$ is an* operadic fibration *if it induces a surjection $\pi_0\mathbb{O} \twoheadrightarrow \pi_0\mathbb{P}$ on the sets of connected components, and for any $b, a_1, \ldots, a_k \in \mathbb{O}$, with $|b| = k$, and $f: x \to pb$ in $\mathbb{P}$ with fibers pa_j, there exists an operadic p-cartesian map $\tilde{f}$ in $\mathbb{O}$ with $p\tilde{f} = f$ and fibers a_j. We say that $\tilde{f}$ is an* (operadic cartesian) lift *of f.*

From now on, we will be interested only in operadic fibrations over $\mathbb{P} = \Delta_s$. Since it is the terminal non-symmetric operadic 2-category, the only functors with target Δ_s are the cardinality functors.

2.18. DEFINITION. *A non-symmetric operadic 2-category $\mathbb{O}$ is called* fibered *if its cardinality functor is an operadic fibration. It is called* split-fibered *if it is equipped with a choice of operadic cartesian lifts $\tilde{f} = \ell(f, b, a_1 \cdots a_k)$, for any map $f: \underline{m} \to \underline{k}$ in Δ_s and objects $b, a_1, \ldots, a_k$ in $\mathbb{O}$, with $|b| = k$, such that the following conditions hold for any $f: \underline{m} \to \underline{k}$, $g: \underline{k} \to \underline{n}$ and objects $c, b_i, a_j \in \mathbb{P}$, $0 \le i \le n, 0 \le j \le k$.*

- $\ell(\mathbb{1}_n, c, e \cdots e) = \mathbb{1}_c$,

- $\ell(!_n, e, c) = \varepsilon_c$, *and*

- *let $\overline{a}^i_j$ denote the sequence $a^i_1 \cdots a^i_{k_i}$, then*

$$\ell\big(g, \ell_1(f, c, b_1 \cdots b_n), a_1 \cdots a_k\big) = \ell\big(f \circ g, c, \ell_1(f^1, b_1, \overline{a}^1_j) \cdots \ell_1(f^n, b_n, \overline{a}^n_j)\big),$$

where ℓ_1 denotes the domain of the lift.

3. The Equivalence

3.1. THEOREM. *For any constant-free non-symmetric categorical operad $\mathcal{P}$, the 2-category $\int \mathcal{P}$ is a non-symmetric operadic 2-category which is split-fibered over Δ_s.*

Before proving the theorem we describe the lax slice 2-category $(\int \mathcal{P})/[n,c]$, for some object $[n,c] \in \int \mathcal{P}$. Its objects are maps $[g; b_1, \ldots, b_n; \beta]: [k,b] \to [n,c]$. A 1-cell

$$[h; c_1, \ldots, c_n; \gamma] \to [g; b_1, \ldots, b_n; \beta]$$

is a lax triangle in $\int \mathcal{P}$, i.e. a 2-cell

$$[g; b_1, \ldots b_n; \beta] \circ [f; a_1, \ldots, a_k; \alpha] \overset{\delta}{\Longrightarrow} [h; c_1, \ldots, c_n; \gamma],$$

with $h = g \circ f$, which amounts to a tuple of morphisms

$$\mu_{f^i}(b_i, a_1^i, \ldots, a_{k_i}^i) \overset{\delta_i}{\to} c_i, 1 \le i \le n, \tag{7}$$

in $\mathcal{P}_n$ (recalling the notation a_j^i from (2)), such that

$$\gamma \circ \mu_h(c, \delta_1, \ldots, \delta_n) = \alpha \circ \mu_f(\beta, a_1, \ldots, a_k). \tag{8}$$

A 2-cell $\xi: \delta' \to \delta''$ in $\int \mathcal{P}/[n,c]$ is a tuple of maps

$$\xi_j: a_j' \to a_j'', 1 \le j \le k, \tag{9}$$

such that

$$\alpha'' \circ \mu_f(b, \xi_1, \ldots, \xi_k) = \alpha'$$

and

$$\delta_i'' \circ \mu_{f^i}(b_i, \xi_1^i, \ldots, \xi_{k_i}^i) = \delta_i', 1 \le i \le n. \tag{10}$$

We will also need the following

3.2. LEMMA. *The data of maps $g: \underline{k} \to \underline{n}$, $h: \underline{m} \to \underline{n}$, and maps $f^i: h^{-1}(i) \to g^{-1}(i)$, for $0 \le i \le n$, determine a unique (strict) triangle α in Δ_s with $d_0\alpha = g$, $d_1\alpha = h$ and $\alpha^i = f^i$.*

PROOF. The triangle is $\alpha: g \circ f = h$ with $f = f^1 + \cdots + f^n$, the ordinal sum of finite maps. ∎

PROOF PROOF OF THEOREM 3.1. We begin by describing the operadic structure of $\int \mathcal{P}$. The cardinality is the projection π on the first component. Let $[n,c]$ be an object of $\int \mathcal{P}$. The fiber 2-functor

$$\mathrm{fib}_{[n,c]}: \left(\int \mathcal{P} \right)/[n,c] \longrightarrow \left(\int \mathcal{P} \right)^{\times n}$$

is defined as follows.

The fibers of a map $[g; b_1, \ldots, b_m; \beta]$ are the objects $[g^{-1}(i), b_i]$, we write

$$\mathrm{fib}_{[n,c]}^i \big([g; b_1, \ldots, b_m; \beta] \big) = [g; b_1, \ldots, b_m; \beta]^i = [g^{-1}(i), b_i].$$

For a lax triangle

$$[g; b_1, \ldots b_n; \beta] \circ [f; a_1, \ldots, a_k; \alpha] \xrightarrow{\delta} [h; c_1, \ldots, c_n; \gamma],$$

that is $\delta = (\delta_1, \ldots, \delta_n)$ as in (7), its fibers are maps $[f^i; a_1^i, \ldots, a_{k_i}^i; \delta_i]$ and we write

$$\mathrm{fib}_{[n,c]}^i(\delta) = \delta^i = [f^i; a_1^i, \ldots, a_{k_i}^i; \delta_i], \tag{11}$$

which is indeed a map $[h^{-1}(i), c_i] \to [g^{-1}(i), b_i]$. The fibers of a 2-cell $\xi: \delta' \to \delta''$, that is $\xi = (\xi_1, \ldots, \xi_k)$ as in (9), are 2-cells

$$\mathrm{fib}_{[n,c]}^i(\xi) = \xi^i = (\xi_1^i, \ldots, \xi_{k_i}^i).$$

By equations (10), each ξ^i is a 2-cell $: \delta_i' \Rightarrow \delta_i''$. It is straightforward to check that the fiber assignment is functorial. The 2-category $\int \mathcal{P}$ has only one connected component and the chosen lali-terminal object is the object $[1, e \in \mathcal{P}_1]$ given by the unit e of $\mathcal{P}$. Indeed, for any $[n, c]$, there is a map

$$[!_n, c, \mathbb{1}_c]: [n, c] \to [1, e], \tag{12}$$

with the unique map $!_n: n \to 1$ and $\mu_{!_n}(e, c) = c$. For any other map $[!_n, b, \alpha]: [n, c] \to [1, e]$, that is, $\alpha: \mu_{!_n}(e, b) = b \to c$, there is a unique 2-cell $\alpha: [!_n, b, \alpha] \Rightarrow [!_n, c, \mathbb{1}_c]$. It is clear that the fiber functor preserves cardinality and the cardinality of $[1, e]$ is $\underline{1}$. By definition, the fibers of horizontal identities $\mathbb{1}_{[n,a]}$ are the objects $[1, e]$. This gives axioms (i)-(iii) of a non-symmetric operadic 2-category. For the axiom (iv) we compute that for any

$$[f; a_1, \ldots, a_k; \alpha]: [m, a] \to [k, b]$$

the fibers of the lax triangle

$$\mathbb{1}_{[f;a_1,\ldots,a_k;\alpha]}: [\mathbb{1}_m; e, \ldots, e; \mathbb{1}_b] \circ [f; a_1, \ldots, a_k; \alpha] \Longrightarrow [f; a_1, \ldots, a_k; \alpha]$$

are the maps $[!_{f^{-1}(i)}; a_i, \mathbb{1}_{a_i}]$, which are the terminal maps (12). The fiber axiom (v) is monstrous, but straightforward to check.

Next we show that the projection is an operadic fibration. For a map $g: \underline{k} \to \underline{n}$ of Δ_s and objects $[n, c]$ and $[g^{-1}(i), b_i] \in \int \mathcal{P}$, there is a lift

$$\ell\big(g, [n, c], [g^{-1}(1), b_1] \cdots [g^{-1}(n), b_n]\big) := [g; b_1, \ldots, b_n; \mathbb{1}_{\mu_g(c; b_1 \ldots, b_n)}], \tag{13}$$

with source $[k, \mu_g(c, b_1 \ldots, b_n)]$. We prove it is operadic cartesian. Assume any map

$$[h; c_1, \ldots, c_n; \gamma]: [m, a] \to [n, c],$$

and maps

$$[f^i; a_1^i \ldots, a_{k_i}^i; \psi_i]: [(h)^{-1}(i); c_i] \to [g^{-1}(i); b_i], 0 \le i \le n.$$

By Lemma 3.2 the data of h, g and f^i determine a commutative triangle $\alpha: g \circ f = h$ in Δ_s with $\alpha^i = f^i$. This has a lift $\tilde{\alpha} = (\psi_1, \ldots, \psi_n)$ with $d_2\tilde{\alpha} = [f; a_1, \ldots, a_k; \gamma \circ \mu_{gf}(c, \psi_1, \ldots, \psi_n)]$

which is forced to be unique. Indeed, assume any filling lax triangle $\delta = (\delta_1, \ldots, \delta_n)$ with $d_2\delta = [\varphi; \chi_1, \ldots, \chi_k; \omega]$, i.e.

$$
\begin{array}{ccc}
[g^{-1}(i), b_i] & \rhd_i & [k, \mu_g(c, b_1, \ldots, b_n)] \xrightarrow{\;[g;b_1,\ldots,b_n;\mathbb{1}\mu_g(c,b_1,\ldots,b_n)]\;} [n, c]. \\[2pt]
{\scriptstyle[f^i;a_1^i,\ldots,a_{k_i}^i;\psi_i]}\Big\uparrow & & {\scriptstyle[\varphi;\chi_1,\ldots,\chi_k;\omega]}\Big\uparrow \qquad \Downarrow{\scriptstyle \delta} \\[2pt]
[h^{-1}(i), c_i] & \rhd_i & [m, a] \xrightarrow{\qquad\qquad [h;c_1,\ldots,c_n;\gamma]\qquad\qquad}
\end{array}
$$

Since δ lifts the strict triangle $g \circ f = h$, it must hold $\varphi = f$. The fibers of δ have to match the prescribed fibers, hence $\chi_j = a_j$ and $\delta_i = \psi_i$ by (11). Finally, the equation (8) with $\beta \equiv \mathbb{1}_{\mu_g(c;b_1\ldots,b_n)}$ and $\alpha \equiv \omega$ gives

$$
\omega = \gamma \circ \mu_{gf}(c, \psi_1, \ldots, \psi_n).
$$

The splitting conditions follow from the associativity and unitality of the operad $\mathcal{P}$, cf. the defining equation (13) together with the correspondence of the equations (3)-(5) and the splitting conditions of Definition 2.18. ∎

3.3. DEFINITION. *A morphism of split-fibered non-symmetric operadic 2-categories is an operadic 2-functor which preserves the chosen operadic cartesian lifts. The category of split-fibered non-symmetric operadic 2-categories will be denoted by* $\mathrm{sFib}(\Delta_s)$.

3.4. PROPOSITION. *The construction $\int \mathcal{P}$ of Definition 2.5 extends to a fully faithful functor*

$$
\int : \Delta_s\text{--oper}(\mathrm{Cat}) \longrightarrow \mathrm{sFib}(\Delta_s).
$$

PROOF. Let $F: \mathcal{P} \to \mathcal{Q}$ be a map of operads. We write briefly Fa for the value of F_n on an operation $a \in \mathcal{P}_n$. We define $\int F$ on objects, morphisms, and 2-cell as follows. Let $[m, a] \in \int \mathcal{P}$, we put

$$
\int F[m, a] := [m, Fa] \in \int \mathcal{Q}.
$$

Let $[f; a_1, \ldots, a_k; \alpha]: [m, a] \to [k, b]$ be a map in $\int \mathcal{P}$, we put

$$
\int F[f; a_1, \ldots, a_k; \alpha] := [f; Fa_1, \ldots, Fa_k; F\alpha]: [m, Fa] \to [k, Fb]. \tag{14}
$$

Since F preserves the operad compositions μ, the morphism is well defined:

$$
F\alpha: \mu_f(Fb; Fa_1, \ldots, Fa_k) = F\mu_f(b; a_1, \ldots, a_k) \to Fa.
$$

For a 2-cell

$$
[f; a_1', \ldots, a_k'; \alpha'] \xRightarrow{\delta} [f; a_1'', \ldots, a_k''; \alpha''],
$$

i.e. a sequence of morphisms $\{\delta_i: a_i' \to a_i'' \in \mathcal{P}_{f^{-1}(i)}\}_{1 \le i \le k}$, we define $\int F\delta$ to be the sequence $\{F\delta_i: Fa_i' \to Fa_i'' \in \mathcal{Q}_{f^{-1}(i)}\}_{1 \le i \le k}$. The assignment is functorial and it gives an operadic

2-functor, i.e. $\int F$ commutes with the projections to Δ_s, preserves the only lali-terminal object

$$\int F[1, e_{\mathcal{P}}] = [1, F(e_{\mathcal{P}})] = [1, e_{\mathcal{Q}}],$$

and preserves fibers, which can be seen from (14). The 2-functor $\int F$ further preserves the chosen lifts (13), which is shown by the following computation.

$$\int F\ell\big(g, [n, c], [g^{-1}(1), b_1]\cdots[g^{-1}(n), b_n]\big) = \int F[g; b_1, \ldots, b_n; \mathbb{1}_{\mu_g(c; b_1, \ldots, b_n)}] =$$
$$= [g; Fb_1, \ldots, Fb_n; F\mathbb{1}_{\mu_g(c; b_1, \ldots, b_n)}] =$$
$$= [g; Fb_1, \ldots, Fb_n; \mathbb{1}_{\mu_g(Fc; Fb_1, \ldots, Fb_n)}] =$$
$$= \ell\big(g, [n, Fc], [g^{-1}(1), Fb_1]\cdots[g^{-1}(n), Fb_n]\big)$$

We now show that the functor $\int$ is fully faithful. Let $h: \int \mathcal{P} \to \int \mathcal{Q}$ be a morphism of split-fibered non-symmetric operadic 2-categories. It uniquely determines a map of operads $H: \mathcal{P} \to \mathcal{Q}$ as follows. For $a \in \mathcal{P}_n$, the value Ha is given by

$$h[n, a] = [n, Ha].$$

Since h preserves the lali-terminal object, it holds $He_{\mathcal{P}} = e_{\mathcal{Q}}$, and since h preserves fibers, for a general morphism $[f; a_1, \ldots, a_k; \alpha]: [m, a] \to [k, b]$ we write

$$h[f; a_1, \ldots, a_k; \alpha] = [f; Ha_1, \ldots, Ha_k; H\alpha]$$

with

$$H\alpha: \mu_f(Hb; Ha_1, \ldots, Ha_k) \to Ha.$$

For a morphism $\alpha: b \to a$ in $\mathcal{P}_n$, the value $H\alpha$ is given by

$$h[\mathbb{1}_n; e_{\mathcal{P}}, \ldots, e_{\mathcal{P}}, \alpha] = [\mathbb{1}_n; e_{\mathcal{Q}}, \ldots, e_{\mathcal{Q}}, H\alpha],$$

with

$$H\alpha: \mu_{\mathbb{1}_n}(Hb; e_{\mathcal{Q}}, \ldots, e_{\mathcal{Q}}) = Hb \to Ha \text{ in } \mathcal{Q}_n.$$

This is clearly functorial. Since h further preserves the operadic cartesian lifts (13), there is a chain of equalities

$$[g; Hb_1, \ldots, Hb_n; \mathbb{1}_{H\mu_g(c; b_1 \ldots, b_n)}] = [g; Hb_1, \ldots, Hb_n; H\mathbb{1}_{\mu_g(c; b_1 \ldots, b_n)}]$$
$$= h[g; b_1, \ldots, b_n; \mathbb{1}_{\mu_g(c; b_1 \ldots, b_n)}] =$$
$$= h\ell\big(g, [n, c], [g^{-1}(1), b_1]\cdots[g^{-1}(n), b_n]\big) =$$
$$= \ell\big(g, [n, Hc], [g^{-1}(1), Hb_1]\cdots[g^{-1}(n), Hb_n]\big) =$$
$$= [g; Hb_1, \ldots, Hb_n; \mathbb{1}_{\mu_g(Hc; Hb_1 \ldots, Hb_n)}],$$

and hence

$$H\mu_g(c; b_1 \ldots, b_n) = \mu_g(Hc; Hb_1 \ldots, Hb_n).$$

Together with $He_{\mathcal{P}} = e_{\mathcal{Q}}$ this shows that H is indeed a map of operads. Since H is uniquely determined by h, the functor $\int$ is fully faithful.   ∎

Next we investigate the inverse to the integration $\int$. Let us introduce the following concept.

3.5. DEFINITION. *Let x, y be objects of a non-symmetric operadic 2-category with $|x| = |y|$. A morphism $\varphi \colon y \to x$ is* trivial *if for any $\psi \colon z \to y$ and $1 \leq i \leq |y|$,*

$$\mathrm{fib}_x^i(\mathbb{1}_{\varphi \circ \psi}) = \varepsilon_{\mathrm{fib}_y^i(\psi)}, \tag{15}$$

where $\mathbb{1}_{\varphi \circ \psi}$ is the triangle

$$z \xrightarrow{\;\;\psi\;\;} y.$$

with $\varphi \circ \psi$ and $\mathbb{1}_{\varphi \circ \psi}$ and φ forming a triangle to x

The condition $|x| = |y|$ implies $|\varphi| = \mathbb{1}_{|x|}$ and it follows from (15) that the fibers of a trivial morphism $\varphi \colon y \to x$ are the chosen local lali-terminal objects. In the case of classical (non-symmetric) operadic categories of [BM15], viewed as non-symmetric operadic 2-categories, the trivial morphisms of the above definition recover quasibijections of [BM23, s. 1.1].

3.6. LEMMA. *For a trivial morphism $\varphi \colon y \to x$ and any $\psi \colon z \to y$, the fibers of $\varphi \circ \psi$ are the same as fibers of ψ, i.e. $\mathrm{fib}_x^i(\varphi \circ \psi) = \mathrm{fib}_y^i(\psi)$, for $0 \leq i \leq |y|$.*

PROOF. Since $\mathrm{fib}_x^i(\varphi)$ is the codomain of $\mathrm{fib}_x^i(\mathbb{1}_{\varphi \circ \psi}) = \varepsilon_{\mathrm{fib}_y^i(\psi)}$, it is a chosen local lali-terminal object u. By axiom (ii), $\mathrm{fib}_u^1(\varepsilon_x) = x$ for any x, and by the fiber axiom (v),

$$\mathrm{fib}_u^1(\mathrm{fib}_x^i(\mathbb{1}_{\varphi \circ \psi})) = \mathrm{fib}_y^i(\psi).$$

Thus

$$\mathrm{fib}_x^i(\varphi \circ \psi) = \mathrm{fib}_u^1(\varepsilon_{\mathrm{fib}_x^i(\varphi \circ \psi)}) = \mathrm{fib}_u^1(\mathrm{fib}_x^i(\mathbb{1}_{\varphi \circ \psi})) = \mathrm{fib}_y^i(\psi).$$

∎

3.7. LEMMA. *Trivial morphisms form a subcategory.*

PROOF. By axiom (iv) of an operadic 2-category, all identities are trivial. Let $\varphi' \colon y \to x$ and $\varphi'' \colon x \to w$ be two composable trivial morphisms, and $\psi \colon z \to y$ any morphism. Since

$$\mathrm{fib}_x^i(\varphi' \circ \psi) = \mathrm{fib}_y^i(\psi)$$

by Lemma 3.6, we have

$$\mathrm{fib}_w^i(\mathbb{1}_{(\varphi'' \circ \varphi') \circ \psi}) = \mathrm{fib}_w^i(\mathbb{1}_{\varphi'' \circ (\varphi' \circ \psi)}) = \varepsilon_{\mathrm{fib}_x^i(\varphi' \circ \psi)} = \varepsilon_{\mathrm{fib}_y^i(\psi)},$$

hence $\varphi'' \circ \varphi'$ is trivial.

∎

3.8. Theorem. *The operadic integration functor $\int$ gives an equivalence between the categories of split-fibered non-symmetric operadic 2-categories and constant-free non-symmetric categorical operads*

$$\Delta_s\text{-oper}(\mathrm{Cat}) \cong \mathrm{sFib}(\Delta_s).$$

Proof. We will prove that the functor $\int$ is essentially surjective. Let $\mathbb{O}$ be split-fibered operadic 2-category. We define a non-symmetric categorical operad $\mathcal{P}$, where each $\mathcal{P}_n$ is a subcategory of trivial morphisms (cf. Lemma 3.7) of $\mathbb{O}$ above n. The operad multiplication is defined using the operadic lifts as follows. Let $g\colon \underline{k} \to \underline{n}$, $c \in \mathcal{P}_n$, and $b_i \in \mathcal{P}_{g^{-1}(i)}$, for $1 \le i \le n$. We then define

$$\mu_g(c, b_1, \ldots, b_n) := \ell_1(g, c, b_1\cdots b_n),$$

which stands for the domain of the lift $\ell(g, c, b_1\cdots b_n)$. Further, let $\varphi\colon c' \to c'' \in \mathcal{P}_n$ and $\psi_i\colon b_i' \to b_i'' \in \mathcal{P}_{g^{-1}(i)}$. Since φ is trivial, the composite

$$\varphi \circ \ell(g, c', b_1'\cdots b_n'')\colon \mu_g(c', b_1', \ldots, b_n') \to c''$$

has also fibers b_i', by Lemma 3.6. With the prescribed maps ψ_i, the operadic cartesian property of the lift $\ell(g, c'', b_1''\cdots b_n'')$ produces a unique lax triangle

$$\ell_1(g, c', b_1'\cdots b_n') \xrightarrow{\quad d_2\tilde{\alpha} \quad} \ell_1(g, c'', b_1''\cdots b_n'')$$

with vertices involving $\tilde{\alpha}$, $\varphi\circ\ell(g,c',b_1'\cdots b_n')$, b'', $\ell(g,c'',b_1''\cdots b_n'')$

with fibers $\tilde{\alpha}^i = \psi_i$, which lifts the (strict) triangle $\mathbb{1}_{\underline{n}} \circ g = g$ in Δ_s. We define

$$\mu_g(\varphi, \psi_1, \ldots, \psi_n) := d_2\tilde{\alpha}.$$

Since the triangle

$$\mathbb{1}_{\ell(g,c,b_1\cdots b_n)}\colon \ell(g, c, b_1\cdots b_n) \circ \mathbb{1}_{\ell(g,c,b_1\cdots b_n)} \Rightarrow \ell(g, c, b_1\cdots b_n)$$

lifts the triangle $\mathbb{1}_g\colon g \circ \mathbb{1}_k = g$ with given endpoints $\mathbb{1}_c, \mathbb{1}_{b_1}, \ldots, \mathbb{1}_{b_n}$, and there is unique such, it holds

$$\mu_g(\mathbb{1}_c, \mathbb{1}_{b_1}, \ldots, \mathbb{1}_{b_n}) = \mathbb{1}_{\ell(g,c,b_1\cdots b_n)}.$$

By a similar argument, it holds

$$\mu_g(\varphi'', \psi_1'', \ldots, \psi_n'') \circ \mu_g(\varphi', \psi_1', \ldots, \psi_n') = \mu_g(\varphi'' \circ \varphi', \psi_1'' \circ \psi_1', \ldots, \psi_n'' \circ \psi_n'),$$

hence μ_g is indeed a functor. The splitting conditions of Definition 2.18 ensure associativity and unitality of the operad $\mathcal{P}$. There a canonical isomorphism $\int \mathcal{P} \cong \mathbb{O}$, sending $[m, a] \in \int \mathcal{P}$ to $a \in \mathbb{O}$ with $|a| = n$. The functor $\int$ is thus essentially surjective which together with Proposition 3.4 yields an equivalence. ∎

3.9. REMARK. The equivalence of Theorem 3.8 extends the equivalence of [BM15, prop. 2.5] for $\mathbb{P} = \Delta_s$. Indeed, for a non-symmetric categorical operad $\mathcal{P}$ with each $\mathcal{P}_n$ discrete, the integration $\int$ reduces to the construction of [BM15, above prop. 2.5] and further, every discrete operadic fibration of [BM15, def. 2.1] over Δ_s is a split-fibered non-symmetric operadic 2-category.

4. Closing Remarks

In this last part we relate operadic fibrations to classical categorical fibrations and make suggestions for the generalization of our results. We first introduce operads for an arbitrary operadic 2-category.

4.1. DEFINITION. *Let $\mathbb{O}$ be a non-symmetric operadic 2-category. A categorical $\mathbb{O}$-operad $\mathcal{P}$ is a collection of categories $\mathcal{P}_x$, indexed by objects $x \in \mathbb{O}$, equipped with functors*

$$\mathcal{P}_x \times \mathcal{P}_{\mathrm{fib}_x^1(\varphi)} \times \cdots \times \mathcal{P}_{\mathrm{fib}_x^{|x|}(\varphi)} \xrightarrow{\mu_\varphi} \mathcal{P}_y,$$

for every map $\varphi \colon y \to x$ in $\mathbb{O}$ which satisfy the following associativity and unit laws.

- *For any lax triangle $\alpha \colon \psi \circ \varphi \Rightarrow \theta$ in $\mathbb{O}$*

with $|y| = \underline{k}$ and $|x| = \underline{n}$, and objects

$$a \in \mathcal{P}_x, b_i \in \mathcal{P}_{\varphi_i}, 1 \le i \le n, c_j^i \in \mathcal{P}_{(\alpha^i)_j}, 1 \le j \le k_i,$$

$$\mu_f\big(\mu_g(a, b_1, \ldots, b_k), c_1^1, \ldots, c_{k_n}^n\big) = \mu_h\big(a, \mu_{\alpha^1}(b_1, c_1^1, \ldots, c_{k_1}^1), \ldots, \mu_{\alpha^n}(b_n, c_1^n, \ldots, c_{k_n}^n)\big),$$

where $\underline{k_i} = |\varphi|^{-1}(i)$ are the fibers of the map $|\varphi| \colon \underline{k} \to \underline{n}$. It follows from the axioms of operadic 2-categories that for $j \in |y|$, $|\psi|^{-1}(j) = |\alpha^i|^{-1}(\varepsilon j)$, where $i = |\varphi|(j)$, and $\varepsilon j \in |\varphi|^{-1}(i)$ is the element corresponding to $j \in |y|$. Hence the associativity equation above is well defined.

- *For any object $x \in \mathbb{O}$ of non-zero cardinality, there are units $e_x \in \mathcal{P}_{u_x}$ and $e_i^x \in \mathcal{P}_{u_i^x}$, such that for any $a \in \mathcal{P}_x$,*

$$\mu_{\mathbb{1}_x}(a, e_1^x, \ldots, e_n^x) = a,$$

$$\mu_{\varepsilon_x}(e_x, a) = a.$$

4.2. EXAMPLE. Recall the non-symmetric operadic 2-category $\int \mathbb{N}$ of Example 2.6. For simplicity, we describe $\int \mathbb{N}$-operads with values in Set. A set-valued $\int \mathbb{N}$-operad is a collection of sets $\mathcal{P}_n$, $n \geq 0$, together with maps

$$\mathcal{P}_m \times \mathcal{P}_p \xrightarrow{\mu} \mathcal{P}_n, \tag{16}$$

whenever $m + p \geq n$, subject to associativity and unitality conditions. By Proposition 2.9 any morphism $n \xrightarrow{p} m$ of $\int \mathbb{N}$ factors as

$$
\begin{array}{ccc}
n & \xrightarrow{\quad p \quad} & m \\
 & \searrow_{0} \qquad \nearrow_{p} & \\
 & m + p &
\end{array}
$$

and the associativity gives the following commutative diagram.

$$
\begin{array}{ccc}
\mathcal{P}_m \times \mathcal{P}_p \times \mathcal{P}_0 & \xrightarrow{\mathbb{1} \times \mu} & \mathcal{P}_m \times \mathcal{P}_p \\
{\scriptstyle \mu_p \times \mathbb{1}} \downarrow & & \downarrow {\scriptstyle \mu} \\
\mathcal{P}_{m+p} \times \mathcal{P}_0 & \xrightarrow{\mu_0} & \mathcal{P}_n
\end{array}
$$

Let us further assume that an $\int \mathbb{N}$-operad $\mathcal{P}$ is reduced, i.e. that the component of the lali-terminal object 0 contains only the operad unit ($P_0 \cong \{e\}$). Note that any map $n \xrightarrow{0} m$ factors as $n \xrightarrow{0} n+1 \xrightarrow{0} \cdots \xrightarrow{0} m-1 \xrightarrow{0} m$. The composition maps (16) are then generated only by the maps

$$\mathcal{P}_m \times \mathcal{P}_p \xrightarrow{\mu_p} \mathcal{P}_{m+p}$$

for any $m, p \in \mathbb{N}$ and maps

$$\partial \colon \mathcal{P}_m \to \mathcal{P}_{m-1},$$

given as the composites

$$\mathcal{P}_m \xrightarrow{\cong} \mathcal{P}_m \times \mathcal{P}_0 \xrightarrow{\mu_0} \mathcal{P}_{m-1}.$$

The comparison to classical categorical fibrations and Grothendieck construction is provided by the following

4.3. PROPOSITION. *Any 1-category $\mathscr{C}$, viewed as a 2-category with only identity 2-cells, is a trivially non-symmetric operadic 2-category with constantly zero cardinality functor. A categorical $\mathscr{C}$-operad $\mathcal{P}$ in this case is an ordinary functor $\mathcal{P} \colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Cat}$. The integration recovers the classical Grothendieck construction on a categorical presheaf and operadic fibrations over $\mathscr{C}$ recover categorical fibration over $\mathscr{C}$.*

PROOF. Straightforward.   ∎

For completeness, we close this article by stating the splitting conditions for a general operadic fibration and a general conjectural equivalence.

4.4. DEFINITION. *An operadic fibration $p\colon \mathbb{O} \to \mathbb{P}$ is called* split *if it is equipped with a choice of operadic p-cartesian lifts $\tilde{g} = \ell(b, a_1 \cdots a_k, g)$ for any map g of $\mathbb{P}$ and endpoints $b, a_1, \ldots, a_k \in \mathbb{O}$, satisfying the following conditions. Denote the domain of $\ell(b, a_1 \cdots a_k, g)$ by $\ell_1(b, a_1 \cdots a_k, g)$.*

- *For every object $x \in \mathbb{O}$, $\ell(u_x, x, u_{px}) = u_x$ and $\ell(x, u_1^x \cdots u_k^x, \mathbb{1}_{px}) = \mathbb{1}_x$.*

- *For any lax triangle $\delta\colon fg \Rightarrow h$ in $\mathbb{P}$,*

$$
\begin{array}{ccc}
m & \xrightarrow{\ \ g\ \ } & py, \\
& \searrow{\scriptstyle h} \quad \delta \quad \swarrow{\scriptstyle f} & \\
& px &
\end{array}
$$

with $|y| = kl$ and $|x| = n$, and objects $a, b_i, c_j^i \in \mathbb{O}, 1 \le i \le n, 1 \le j \le k_i$,

$$
\ell\big(\ell_1(z, b_1 \cdots b_n, g), c_1^1 \cdots c_{k_n}^n, g\big) = \ell\big(z, \ell_1(b_1, c_1^1 \cdots c_{k_1}^1, \delta^1) \cdots \ell_1(b_n, c_n^1 \cdots c_{k_n}^n, \delta^n), h\big),
$$

where $\underline{k_i}$ are the fibers the map $|f|$.

4.5. CONJECTURE. *There is an equivalence of split operadic fibrations over $\mathbb{O}$ and categorical $\mathbb{O}$-operads.*

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