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EQUIVALENCES OF CATEGORIES AND A MODEL STRUCTURE ON RELATIVE CATEGORIES

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ABSTRACT. We show that there is a model structure on the category RelCat of small relative categories such that for a morphism f in RelCat, f is a weak equivalence iff the associated functor on homotopy 1-categories is an equivalence of categories. In this model category (i) every object is cofibrant and (ii) the homotopy category functor becomes a fibrant replacement. The model structure is left-induced from the model category on small categories with equivalences of categories as weak equivalences by the homotopy category functor in a Quillen equivalent way.

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1. Introduction

The purpose of this note is to show that there is a model structure on the category RelCat ([BK12b]) of small relative categories such that for a morphism $f : (C, V) \to (D, W)$ in RelCat, f is a weak equivalence iff the associated functor

$$C[V^{-1}] \xrightarrow{\simeq} D[W^{-1}] \tag{1}$$

on homotopy categories[1] ([GZ67]) is an equivalence of categories. Furthermore, if the category Cat of small categories is equipped with the model structure whose weak equivalences are precisely the equivalences of categories, then RelCat equipped with this model

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[1] $C[V^{-1}]$ is the 1-categorical localization, not the simplicial localization of Dwyer and Kan ([DK80b],[DK80a]).

structure is Quillen equivalent to Cat. In fact, the model structure on RelCat is lifted from Cat as a left-induced model structure ([BHK+15]) along the functor associated with localizations of relative categories. Before we state the main result, we fix some notations.

For an object (C, W) of RelCat (Definition 2.27), we denote by $L(C, W)$ the associated homotopy category $C[W^{-1}]$ ([GZ67]). Given a morphism $f : (C, V) \to (D, W)$ in RelCat, we denote by $Lf : L(C, V) \to L(D, W)$ the functor associated with f. Then we have a functor $L : \mathrm{RelCat} \to \mathrm{Cat}$ which is a left adjoint functor of an adjunction

$$L : \mathrm{RelCat} \rightleftarrows \mathrm{Cat} : R \tag{2}$$

where R is a functor mapping C to $(C, \mathrm{iso}(C))$ where $\mathrm{iso}(C)$ is the subcategory of isomorphisms in C, i.e., the maximal groupoid of C.

For a model category M we denote its underlying category by

$$u(M) \tag{3}$$

and its model structure by

$$\big(w(M), c(M), f(M)\big) \tag{4}$$

where $w(M), c(M), f(M)$ is the class of weak equivalences, cofibrations and fibrations in M respectively. We use the following abbreviations.

$$wc(M) = w(M) \cap c(M) \qquad wf(M) = w(M) \cap f(M) \tag{5}$$

We denote by $\mathrm{Cat_c}$ the model category on $\mathrm{Cat}(= u(\mathrm{Cat_c}))$ whose model structure

$$\big(w(\mathrm{Cat_c}), c(\mathrm{Cat_c}), f(\mathrm{Cat_c})\big) \tag{6}$$

satisfies the following properties.

- $w(\mathrm{Cat_c})$ is the class of equivalences of small categories.
- $c(\mathrm{Cat_c})$ is the class of functors injective on objects.
- $f(\mathrm{Cat_c})$ is the class of isofibrations.

Now we state our main result.

1.1. Theorem. *There is a model category* $\mathrm{RelCat_h}$ *on* RelCat *whose model structure*

$$\big(w(\mathrm{RelCat_h}), c(\mathrm{RelCat_h}), f(\mathrm{RelCat_h})\big) \tag{7}$$

satisfies the following properties.

1. *For a morphism f in* RelCat*,*
   (a) *f is in $w(\mathrm{RelCat_h})$ iff Lf is in $w(\mathrm{Cat_c})$.*
   (b) *f is in $c(\mathrm{RelCat_h})$ iff Lf is in $c(\mathrm{Cat_c})$.*

2. With the model structures (6) *and* (7)*, the adjunction* (2) *is a Quillen equivalence.*

3. Every object of $\mathrm{RelCat}_{\mathrm{h}}$ *is cofibrant.*

4. For an object (C, W) *of* RelCat*, the following are equivalent.*

(a) (C, W) *is fibrant in* $\mathrm{RelCat}_{\mathrm{h}}$*.*

(b) $(C, W) = RC$*, i.e.,* $W = \mathrm{iso}(C)$ *holds.*

The proofs for (1) and (2) are in Section 3.

Section 4 is about the two weak factorization systems associated with (7). The cofibrations and trivial fibrations are characterized explicitly with a generating set for cofibrations. For the pair of trivial cofibrations and fibrations we only prove (4). In this note, we are concerned only with 1-categorical localization. As such, (2) and (4) say that the model structure (7) is exactly what we need. When a category H is viewed as a homotopy category, a relative category (C, W) satisfying

$$H = C[W^{-1}] \tag{8}$$

provides a presentation of H. (4) says that the fibrant objects are precisely the homotopy categories. Another consequence of (4) is that for every object (C, W) of RelCat, the morphism

$$(C, W) \to RL(C, W) \tag{9}$$

associated with the localization of (C, W) is a fibrant replacement of (C, W) in $\mathrm{RelCat}_{\mathrm{h}}$.

The enriched model structures on $\mathrm{RelCat}_{\mathrm{h}}$ parallel to those on $\mathrm{Cat}_{\mathrm{c}}$ are explained in Section 5. $\mathrm{RelCat}_{\mathrm{h}}$ has one monoidal model structure and two simplicial model structures.

In Section 6 we compare our model structure with the Barwick-Kan model structure ([BK12b]). We observe that our model structure is not a Bousfield localization of Barwick-Kan model structure.

The model structure (6) is the unique one with $w(\mathrm{Cat}_{\mathrm{c}})$ as the class of weak equivalences. Unlike (6), the model structure (7) is not the unique one with $w(\mathrm{RelCat}_{\mathrm{h}})$ as the class of weak equivalences. There is a model structure one can obtain by mixing ([Col06]) the model structure (7) and the Barwick-Kan model structure. It is a left Bousfield localization of the Barwick-Kan model structure. We discuss about it in Section 7.

I would like to thank Professor Clark Barwick for answering my question and raising interesting questions. I would also like to thank the referee for pointing my attention to a Mathoverflow question by Tim Campion ([Cam16]). There the existence of the model structure (7) is claimed without proof. The fibrant objects described there is the fibrant objects of a different but related model category of which a left Bousfield localization is $\mathrm{RelCat}_{\mathrm{h}}$.

2. Review on basic notions

The only purpose of Section 2 is to make this note self-contained. We recall some well-known notions and facts and fix notations convenient for us.

2.1. Model category. Here we review model categories ([Qui67], [Hov99], [Hir03], [MP12]).

2.2. Definition. *Let K be a category. Let f, g be two morphisms in K. We say that f (resp g)* ***has a right (resp. left) lifting property with respect to*** *g (resp. f), denoted by*

$$g \boxslash f, \tag{10}$$

if every commutative square of solid arrows (11) has a dotted morphism h that makes the whole diagram commute.

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle g}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle f} \\ \cdot & \longrightarrow & \cdot \end{array} \tag{11}$$

2.3. Definition. *Let K be a category. Let A be a class of morphisms in K.*

1. *We denote by $A^{\boxslash}$ the class of morphisms in K satisfying right lifting property with respect to every morphism in A.*
2. *We denote by ${}^{\boxslash}A$ the class of morphisms in K satisfying left lifting property with respect to every morphism in A.*

2.4. Definition. *Let K be a category. A pair (A, B) of classes of morphisms in K is called a* ***weak factorization system*** *on K if the following two properties hold.*

1. *$A^{\boxslash} = B$ and $A = {}^{\boxslash}B$.*
2. *Every morphism f in K can be factored as $f = hg$ where $h \in B$ and $g \in A$.*

2.5. Definition. *A* ***model category*** *M consists of*

1. *a category, called the* ***underlying category*** *of M and denoted by*

$$u(M), \tag{12}$$

with all small limits and small colimits and

2. *three classes of morphisms in $u(M)$, called the* ***model structure*** *of M and denoted by*

$$\big(w(M), c(M), f(M)\big), \tag{13}$$

such that

   (a) *$w(M)$ satisfies the two-out-of-three property: for every composable morphisms f, g in $u(M)$ if two of f, g, gf are in $w(M)$ then so is the third.*
   (b) *$(w(M) \cap c(M), f(M))$ and $(c(M), w(M) \cap f(M))$ are weak factorization systems.*

We call the elements of $w(M), c(M)$ and $f(M)$ weak equivalences, cofibrations and fibrations in M respectively.

2.6. REMARK. For a model category M, we use the following abbreviations.

$$wc(M) = w(M) \cap c(M) \qquad wf(M) = w(M) \cap f(M) \tag{14}$$

2.7. MODEL CATEGORY $\mathrm{Cat_c}$. It is known that Cat has an unique model structure such that the weak equivalences are precisely the equivalences of small categories ([Rez96], [Joy], [mmm10]). On Cat, we only work with this model structure. Here we review its property.

2.8. THEOREM. *There is a combinatorial model category* $\mathrm{Cat_c}$ *on* Cat *whose model structure*

$$\big(w(\mathrm{Cat_c}), c(\mathrm{Cat_c}), f(\mathrm{Cat_c})\big) \tag{15}$$

satisfies the following properties.

1. $w(\mathrm{Cat_c})$ *is the class of equivalences of categories.*
2. $c(\mathrm{Cat_c})$ *is the class of functors injective on objects.*
3. $f(\mathrm{Cat_c})$ *is the class of isofibrations.*

Below we collect some properties of $\mathrm{Cat_c}$ without proof.

2.9. LEMMA. *Every object of* $\mathrm{Cat_c}$ *is cofibrant and fibrant.*

2.10. LEMMA. *Let f be a morphism in* Cat*. Then the following hold.*

1. $f \in wc(\mathrm{Cat_c})$ *iff* $f \in w(\mathrm{Cat_c})$ *and f is injective on objects.*
2. $f \in wf(\mathrm{Cat_c})$ *iff* $f \in w(\mathrm{Cat_c})$ *and f is surjective on objects.*

2.11. DEFINITION. *We define two objects P and E of* Cat *as follows.*

1. *We denote by P the category with two objects* $0, 1$ *and two distinct morphisms* $0 \rightrightarrows 1$.

$$P = \big\{0 \rightrightarrows 1\big\} \tag{16}$$

2. *We denote by E the groupoid with two objects* $0, 1$ *and two non-identity morphisms* $0 \rightleftarrows 1$.

$$E = \big\{0 \rightleftarrows 1\big\} \tag{17}$$

2.12. Definition. *We define morphisms t, b, p, j in* Cat *as follows.*

1. *We denote by*
$$t : \emptyset \to [0] \tag{18}$$
the unique functor.

2. *We denote by*
$$b : \mathrm{dis}([1]) \to [1] \tag{19}$$
the canonical embedding where $\mathrm{dis}([1]) = \{0, 1\}$ *is the discrete subcategory of* $[1]$.

3. *We denote by*
$$p : P \to [1] \tag{20}$$
the functor mapping 0 *and* 1 *in* P *to* 0 *and* 1 *in* $[1]$ *respectively.*

4. *We denote by*
$$e : [0] \to E \tag{21}$$
the functor mapping 0 *in* $[0]$ *to* 0 *in* E.

2.13. Lemma. *With the notations in Definition 2.11 and Definition 2.12 the following hold.*

1. $\{t, b, p\}$ *is a generating set for* $c(\mathrm{Cat_c})$.

2. $\{e\}$ *is a generating set for* $wc(\mathrm{Cat_c})$.

2.14. Left-induced model structure. Here we recall the left-induced model structure in [BHK+15].

2.15. Definition. *Let K be a category with small limits and small colimits. Let M be a model category. Let $L : K \rightleftarrows u(M) : R$ be an adjunction. If the triple of classes of morphisms in K*
$$\left(L^{-1}w(M), L^{-1}c(M), \left(L^{-1}(wc(M))\right)^{\boxslash}\right) \tag{22}$$
satisfies the axioms of model category with

- $L^{-1}w(M)$ *the class of weak equivalences.*
- $L^{-1}c(M)$ *the class of cofibrations.*
- $(L^{-1}(wc(M)))^{\boxslash}$ *the class of fibrations*

then it is called a ***left-induced model structure*** *on K.*

The cofibrant generation in [BHK+15] is weaker than the one in [Hir03]. It allows the generation by a class of morphisms instead of a set, and the smallness condition is not required.

2.16. Definition. *We say that a weak factorization system (A, B) on a category K is* ***cofibrantly generated*** *by a class I of morphisms in K iff $I^{\boxslash} = B$.*

2.17. Definition. *A model category M is* ***cofibrantly generated*** *by a pair of classes of morphisms I and J in $u(M)$ iff the weak factorization systems $(c(M), wf(M))$ and $(wc(M), f(M))$ are cofibrantly generated by I and J respectively.*

The following theorem relies on the work of Makkai and Rosický ([MR14]).

2.18. Theorem. [Theorem 2.23 in [BHK+15]] *Let K be a locally presentable category with small limits and small colimits. Let M be a locally presentable model category that is cofibrantly generated by a pair of sets of morphisms in M. Let $L : K \rightleftarrows u(M) : R$ be an adjunction. If*

$$\left(L^{-1}c(M)\right)^{\boxslash} \subseteq L^{-1}w(M) \tag{23}$$

then the left-induced model structure on K exists and is cofibrantly generated by a pair of sets.

2.19. Localization of category. There is a general procedure to obtain a localization $C[W^{-1}]$ of a category C with respect to a class Σ of morphisms in C ([GZ67], [DHKS04]). Here we review it.

2.20. Definition. *Let C be a category and let Σ be a class of morphisms in C.*

1. *A* ***zigzag*** *of (C, Σ) is a finite sequence*

$$a_0 \overset{\varphi_1}{\text{———}} a_1 \overset{\varphi_2}{\text{———}} \cdots \overset{\varphi_m}{\text{———}} a_m \tag{24}$$

of morphisms in C, $m \geq 1$, such that whenever $\varphi_i : a_i \to a_{i-1}$ is a backward morphism φ_i belongs to Σ. We call m the ***length*** *of the zigzag.*

2. *Given a zigzag (24), its* ***type*** *is the partition $P \cup N$ of $\{1, \ldots, m\}$ such that for every i in $\{1, \ldots, m\}$, i belongs to N iff φ_i is a backward morphism.*

3. *By a* ***hat*** *we mean a zigzag*

$$a \overset{\varphi}{\to} b \overset{\varphi}{\leftarrow} a \quad \text{or} \quad a \overset{\varphi}{\leftarrow} b \overset{\varphi}{\to} a \tag{25}$$

for some φ in Σ.

The category $C[W^{-1}]$ has the same object as C. For morphisms, we have the following

2.21. Remark. Let C be a category and let Σ be a class of morphisms in C. Given two objects x, y of C the hom-set $C[W^{-1}](x, y)$ consists of equivalence classes of zigzags where two zigzags are equivalent iff one is obtained from the other by a finite sequence of the following six operations.

1. Inserting an identity morphism.

2. Deleting an identity morphism.

3. Composing two adjacent morphisms in the same direction.

4. Factorizing a morphism into two morphism in the same direction.

5. Inserting a hat (25).

6. Deleting a hat.

2.22. Definition. *Let C be a category and let Σ be a class of morphisms in C. Given a zigzag* (24)*, we denote by*

$$\left[\ a_0 \xrightarrow{\ \varphi_1\ } a_1 \xrightarrow{\ \varphi_2\ } \cdots \xrightarrow{\ \varphi_m\ } a_m\ \right] \tag{26}$$

the element of the hom-set $C[W^{-1}](a_0, a_m)$ that (24) *represents.*

2.23. Lemma. *Let C be a category and let Σ be a class of morphisms in C. Let $\varphi : x \to y$ be an isomorphism in C. If φ and φ^{-1} are in Σ then they represents the same morphism in $L(C, W)(x, y)$.*

$$\left[x \xrightarrow{\varphi} y\right] = \left[x \xleftarrow{\varphi^{-1}} y\right] \tag{27}$$

2.24. Relative categories. Here we review relative categories ([BK12b]).

2.25. Definition. *Let C be a category. We denote by* $\operatorname{dis}(C)$ *the discrete subcategory of C.*

2.26. Definition. *Let W be a subcategory of a category C. We call W* ***wide*** *if* $\operatorname{dis}(W) = \operatorname{dis}(C)$.

2.27. Definition. *A* ***relative category*** *is a pair (C, W) of a category C and a wide subcategory W of C. We call C the* ***underlying category*** *of (C, W) and W the* ***category of weak equivalences****. The morphisms of W are called* ***weak equivalences****. A relative category (C, W) is* ***small*** *if C is small.*

2.28. Remark. Let (C, W) be a relative category. By an abuse of notation, we will often denote W by $w(C)$.

$$(C, W) = (C, w(C)) \tag{28}$$

Also, when it is convenient, we will abbreviate (C, W) with the bold $\mathbf{C}$.

$$(C, W) = \mathbf{C} \tag{29}$$

2.29. Definition. *A relative functor from a relative category $\mathbf{C}$ to $\mathbf{D}$ is a functor $f : C \to D$ that preserves weak equivalences.*

2.30. Definition. *We denote by* RelCat *the category*

1. *whose objects are small relative categories and*
2. *whose morphisms are relative functors between small relative categories.*

We need to use the Dwyer maps ((3.5) in [BK12b]) in Section 6. But we don't recall the definition here. We just need the following property of Dwyer maps.

2.31. Remark. If a morphism $\mathbf{A}' \to \mathbf{B}$ in RelCat is a Dwyer map then it admits a factorization $\mathbf{A}' \xrightarrow{\cong} \mathbf{A} \to \mathbf{B}$ such that

1. the first morphism is an isomorphism in RelCat.
2. the second morphisms is a relative inclusion, i.e.,
$$A \subset B \quad \text{and} \quad w(A) = A \cap w(B) \tag{30}$$
where $\mathbf{A} = (A, w(A))$ and $\mathbf{B} = (B, w(B))$.
3. if $Z(\mathbf{A}, \mathbf{B})$ is the full relative subcategory of $\mathbf{B}$ spanned by the objects $b \in B$ for which there exists a morphism $a \to b$ in B with $a \in A$ then there is a morphism $r : Z(\mathbf{A}, \mathbf{B}) \to \mathbf{A}$ in RelCat such that for every $a \in A$, $ra = a$.

We will use the following two adjunctions.

2.32. Definition. *We define functors*
$$\pi_0, \pi_1 : \mathrm{RelCat} \to \mathrm{Cat} \tag{31}$$
by $\pi_0(C, W) = C$ *and* $\pi_1(C, W) = W$*, and functors*
$$\iota_0, \iota_1 : \mathrm{Cat} \to \mathrm{RelCat} \tag{32}$$
by $\iota_0(C) = (C, \mathrm{dis}(C))$ *and* $\iota_1(C) = (C, C)$*.*

2.33. Lemma. *There are following two adjunctions.*

1. *An adjunction*
$$\iota_0 : \mathrm{Cat} \rightleftarrows \mathrm{RelCat} : \pi_0. \tag{33}$$
2. *An adjunction*
$$\iota_1 : \mathrm{Cat} \rightleftarrows \mathrm{RelCat} : \pi_1. \tag{34}$$

2.34. Remark. Even though for a morphism f in RelCat, f and $\pi_0(f)$ are essentially the same, it will be convenient for us to distinguish them systematically.

3. Existence and Quillen equivalence

Here we prove (1) and (2) in Theorem 1.1. We first prove (1).

The model category $\mathrm{Cat_c}$ (Theorem 2.8) is a combinatorial model category. The category RelCat has small limits and small colimits, and locally finitely presentable. So by applying Theorem 2.23 in [BHK+15] (Theorem 2.18) to the adjunction (2) we can lift a model structure from $\mathrm{Cat_c}$ once we verify the acyclicity condition

$$(L^{-1}c(\mathrm{Cat_c}))^{\boxslash} \subseteq L^{-1}w(\mathrm{Cat_c}). \tag{35}$$

We first characterize the morphisms in $L^{-1}c(\mathrm{Cat_c})$ and $(L^{-1}c(\mathrm{Cat_c}))^{\boxslash}$.

3.1. Lemma. *Let g be a morphism in* RelCat*. Then the following are equivalent.*

1. *Lg is a cofibration in* $\mathrm{Cat_c}$*.*
2. *Lg is injective on objects.*
3. *$\pi_0(g)$ is injective on objects.*
4. *$\pi_0(g)$ is a cofibration in* $\mathrm{Cat_c}$*.*

Proof. It follows from the definition of cofibrations in $\mathrm{Cat_c}$. ∎

3.2. Lemma. *Let g be a morphism in* Cat*. If $g \in c(\mathrm{Cat_c})$ then*

$$\iota_0(g),\ \iota_1(g) \in L^{-1}c(\mathrm{Cat_c}). \tag{36}$$

Proof. It follows from Lemma 3.1 because $\pi_0(\iota_i(g)) = g$ for $i = 1, 2$. ∎

3.3. Lemma. *Let f be a morphism in* RelCat*. If*

$$f \in (L^{-1}(c(\mathrm{Cat_c})))^{\boxslash} \tag{37}$$

then

$$\pi_0(f),\ \pi_1(f) \in wf(\mathrm{Cat_c}). \tag{38}$$

Proof. Let $i = 1, 2$. Let g be a cofibration in $\mathrm{Cat_c}$. Then $g \boxslash \pi_i(f)$ iff $\iota_i(g) \boxslash f$ by the adjunction (33) and (34). Thus (37) and Lemma 3.2 imply that $\pi_i(f)$ is a trivial fibration in $\mathrm{Cat_c}$. ∎

3.4. Remark. The converse of Lemma 3.3 also holds (Proposition 4.5).

We also need the following lemma.

3.5. Lemma. *Let* $f : (C, w(C)) \to (D, w(D))$ *be a morphism in* RelCat*. We assume that*

1. $\pi_0(f)$ *is full and faithful.*
2. $\pi_1(f)$ *is full.*

Let φ *be a morphism in* C*. If* $\pi_0(f)(\varphi)$ *is an identity morphism in* D *then*

1. φ *is an isomorphism in* C *and*
2. φ, φ^{-1} *are in* $w(C)$*.*

Proof. Since $\pi_0(f)$ is full and faithful, φ is an isomorphism in C. Since $\pi_1(f)$ is full and $w(D)$ is a wide subcategory of D, there is a morphism φ' in $w(C)$ mapped to the identity morphism. Since $\pi_0(f)$ is faithful, $\varphi = \varphi'$. Thus φ is in $w(C)$. The inverse φ^{-1} of φ also mapped to the identity in D. Thus φ^{-1} is also in $w(C)$. ■

Now we show that (35) holds.

3.6. Lemma. *Let* L *be the left adjoint functor in* (2)*. Then the inclusion* (35)

$$(L^{-1}c(\mathrm{Cat_c}))^{\boxslash} \subseteq L^{-1}w(\mathrm{Cat_c}) \tag{39}$$

holds.

Proof. Let $f : (C, w(C)) \to (D, w(D))$ be in RelCat. We assume that

$$f \in (L^{-1}c(\mathrm{Cat_c}))^{\boxslash}. \tag{40}$$

Below we show that Lf is a weak equivalence in $w(\mathrm{Cat_c})$, i.e. an equivalence of categories.

Lemma 2.10 and Lemma 3.3 imply that $\pi_0(f)$ is surjective on objects and $\pi_0(f), \pi_1(f)$ are full. Thus Lf is surjective on objects and is full. So it remains to show that Lf is faithful.

Let m, n be positive integers. Let x, y be objects of C. Let α and β be two morphisms in $L(C, w(C))(x, y)$. Let the zigzags (Definition 2.20)

$$(x =)a_0 \overset{\varphi_1}{\text{———}} a_1 \overset{\varphi_2}{\text{———}} \cdots \overset{\varphi_m}{\text{———}} a_m(= y) \tag{41}$$

and

$$(x =)b_0 \overset{\rho_1}{\text{———}} b_1 \overset{\rho_2}{\text{———}} \cdots \overset{\rho_n}{\text{———}} b_n(= y) \tag{42}$$

represent α and β in $L(C, w(C))(x, y)$ respectively. We assume that

$$Lf(\alpha) = Lf(\beta). \tag{43}$$

Below we will show that $\alpha = \beta$.

First, because of (43), we may assume that two zigzags (41) and (42) have the same length and same type and their images in D by $\pi_0(f)$ coincide. It is because we can lift the six operations (Remark 2.21) on the images of (41) and (42) in D to C. It follows from the following observations (i) - (vi).

(i) Inserting an identity morphism at $\pi_0(f)(a_i)$ for $0 \leq i \leq m$ can be lifted by inserting the identity morphism at a_i.

(ii) Deleting an identity morphism in D can be lifted by the following argument: Assume that φ_i is a backward morphism $a_{i-1} \xleftarrow{\varphi_i} a_i$ and it is mapped to an identity morphism in D for some $1 \leq i \leq m$. By Lemma 3.3 and Lemma 3.5, φ_i is an isomorphism in C and $\varphi_i, \varphi_i^{-1}$ are in $w(C)$. By Lemma 2.23,

$$[\varphi_i] = [\varphi_i^{-1}]. \tag{44}$$

Then we can remove φ_i by composing φ_i or φ_i^{-1} with an adjacent morphism. When φ_i is a forward morphism we argue similarly.

(iii) Factoring a morphism $\pi_0(f)(\varphi_i)$ for $1 \leq i \leq m$ can be lifted because $\pi_0(f)$ and $\pi_1(f)$ are full and $\pi_0(f)$ is surjective on objects by Lemma 3.3.

(iv) Composing two morphisms $\pi_0(f)(\varphi_{i-1})$ and $\pi_0(f)(\varphi_i)$ in the same direction for $2 \leq i \leq m$ can be lifted by composing φ_{i-1} and φ_i.

(v) Inserting a hat (25)

$$\pi_0(f)(a_i) \xrightarrow{\psi} d \xleftarrow{\psi} \pi_0(f)(a_i) \tag{45}$$

or

$$\pi_0(f)(a_i) \xleftarrow{\psi} d \xrightarrow{\psi} \pi_0(f)(a_i) \tag{46}$$

for $0 \leq i \leq m$ can be lifted because $\pi_1(f)$ is full and $\pi_0(f)$ is surjective on objects by Lemma 3.3.

(vi) Deleting a hat can be lifted. Suppose that $\pi_0(f)(a_i) = \pi_0(f)(a_{i+2})$ and $\pi_0(f)(\varphi_{i+1}) = \pi_0(f)(\varphi_{i+2})$ for some $0 \leq i \leq m-2$ so that we have a hat (25)

$$\pi_0(f)(a_i) \xleftarrow{\pi_0(f)(\varphi_{i+1})} \pi_0(f)(a_{i+1}) \xrightarrow{\pi_0(f)(\varphi_{i+2})} \pi_0(f)(a_{i+2}) \tag{47}$$

or

$$\pi_0 f(a_i) \xrightarrow{\pi_0(f)(\varphi_{i+1})} \pi_0(f)(a_{i+1}) \xleftarrow{\pi_0(f)(\varphi_{i+2})} \pi_0(f)(a_{i+2}). \tag{48}$$

We assume for a moment that (47) holds. Since $\pi_0(f)$ is full and faithful by Lemma 3.3, there is a morphism $\varphi : a_i \to a_{i+2}$ in C such that

(a) φ is mapped to the identity morphism at $\pi_0(f)(a_i)$.

(b) $\varphi_{i+2} = \varphi_{i+1} \cdot \varphi$.

By (b) we can replace

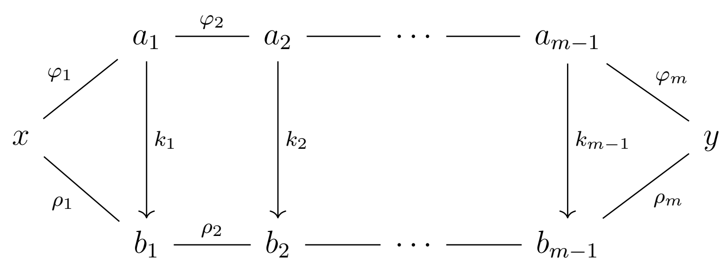
$$a_i \xleftarrow{\varphi_{i+1}} a_{i+1} \xrightarrow{\varphi_{i+2}} a_{i+2} \tag{49}$$

with

$$a_i \xrightarrow{\varphi} a_{i+2}. \tag{50}$$

Lemma 3.5 and (a) imply that φ is an isomorphism and φ, φ^{-1} are in $w(C)$. Then by Lemma 2.23, we can compose φ or φ^{-1} with an adjacent morphism and can omit (49). A similar argument holds for (48).

Finally, we show that (43) implies $\alpha = \beta$. Since a_i and b_i are mapped to the same object in D and $\pi_0(f)$ is full, there is a morphism $k_i : a_i \to b_i$ in C mapped to the identity morphism at $\pi_0(f)(a_i)$. Consider the hammock diagram in C made of two zigzags (41), (42) and k_i.



(51)

Note that since $\pi_0(f)$ is faithful by Lemma 3.3, the collection of k_i makes the diagram (51) commute. By Lemma 3.5, k_i is an isomorphism and k, k_i^{-1} are in $w(C)$. So, by Lemma 2.23,

$$[k_i] = [k_i^{-1}] \tag{52}$$

for $1 \le i \le m-1$. Then the two zigzags represent the same morphism in $L(C, w(C))$ because α is represented by the zigzag (42) by inserting

$$b_i \xleftarrow{k_1} a_i \xrightarrow{k_i} b_i \tag{53}$$

at every b_i for $1 \le i \le m-1$. ■

Now we prove (2). Let η and ε be the unit and counit of the adjunction (2). Every object of $\mathrm{Cat_c}$ is fibrant by Lemma 2.9. Every object of $\mathrm{RelCat_h}$ is cofibrant by the definition and Lemma 2.9 (Corollary 4.3). So because of Proposition 1.3.13 in [Hov99], it is enough to show that

(a) for every object (C, W) of RelCat,

$$\eta_{(C,W)} : (C, W) \to RL(C, W) \tag{54}$$

is a weak equivalence in $\mathrm{RelCat_h}$ and

(b) for every object C of Cat,

$$\varepsilon_C : LR(C) \to C \tag{55}$$

is a weak equivalence in $\mathrm{Cat_c}$.

(a) holds because

$$RL(C,W) = \Big(C[W^{-1}], \mathrm{iso}\big(C[W^{-1}]\big)\Big) \tag{56}$$

and $\eta_{(C,W)}$ is the functor associated with the localizations $C \to C[W^{-1}]$ of C with respect to W. (b) holds because ε_C is an isomorphism in Cat.

4. Two weak factorization systems

Here we prove (3) and (4) in Theorem 1.1. (3) is Corollary 4.3. (4) is Proposition 4.11.

4.1. Cofibration and trivial fibration.

4.2. Proposition. *Let f be a morphism in* RelCat. *The following are equivalent.*

1. *f is a cofibration in* $\mathrm{RelCat_h}$.
2. *$\pi_0(f)$ is a cofibration in* $\mathrm{Cat_c}$.

Proof. Because $L^{-1}(c(\mathrm{Cat_c}))$ is the class of cofibrations in RelCat, it follows from Lemma 3.1. ∎

4.3. Corollary. *In* $\mathrm{RelCat_h}$, *every object is cofibrant.*

Proof. Every object in $\mathrm{Cat_c}$ is cofibrant by Lemma 2.9. Thus it follows from Proposition 4.2. ∎

4.4. Definition. *We define a set I_h of morphisms in* RelCat *by*

$$I_h = \iota_0(I) \cup \iota_1(I) \tag{57}$$

where $I = \{t, b, p\}$ is a generating set for $c(\mathrm{Cat_c})$ in Lemma 2.13(1).

4.5. Proposition. *Let f be a morphism in* RelCat. *The following are equivalent.*

1. *f is a trivial fibration in* $\mathrm{RelCat_h}$.
2. *$\pi_0(f)$ and $\pi_1(f)$ are trivial fibration in* $\mathrm{Cat_c}$.
3. *f has a right lifting property with respect to I_h.*

PROOF. (1)⇒ (2) Because $\left(L^{-1}(c(\mathrm{Cat_c}))\right)^{\boxslash}$ is the class of trivial fibrations in $\mathrm{RelCat_h}$, it follows from Lemma 3.3.

(1)⇐ (2) Let $f : (C, w(C)) \to (D, w(D))$. Let $g : (A, w(A)) \to (B, w(B))$ be in $c(\mathrm{RelCat_h})$. Let

$$\begin{array}{ccc} (A, w(A)) & \longrightarrow & (C, w(C)) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ (B, w(B)) & \longrightarrow & (D, w(D)) \end{array} \tag{58}$$

be a commuting square in RelCat. By Proposition 4.2, $\pi_0(g)$ is a cofibration in $\mathrm{Cat_c}$. Then there is a lifting $h : B \to C$ in the square

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle \pi_0(g)}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle \pi_0(f)} \\ B & \longrightarrow & D \end{array} \tag{59}$$

associated with (58) in Cat. It remains to show that $h(w(B)) \subseteq w(C)$ holds.

Let $\varphi : b \to b'$ be a morphism in $w(B)$. Since $\pi_1(f)$ is full, there is a morphism $\rho : h(b) \to h(b')$ in $w(C)$ such that $\pi_1(f)(\rho)$ is the image of φ in $w(D)$. Then, since $\pi_0(f)$ is faithful, ρ must be $h(\varphi)$ in C. Thus $h(\varphi)$ is in $w(C)$.

(2)⇔(3) The set $\{t, b, p\}$ is a generating set for $c(\mathrm{Cat_c})$ by Lemma 2.13. Then (2) and (3) are equivalent by Lemma 2.33. ∎

4.6. TRIVIAL COFIBRATION AND FIBRATION.

4.7. PROPOSITION. *Let f be a morphism in* RelCat*. Then the following are equivalent.*

1. *f is a trivial cofibration in* $\mathrm{RelCat_h}$.
2. *Lf is a trivial cofibration* $\mathrm{Cat_c}$.
3. *Lf is injective on objects and is an equivalence of categories.*

PROOF. It follows from Lemma 2.10 and the definitions. ∎

4.8. LEMMA. *Let f be a morphism in* RelCat*. If f is a fibration in* $\mathrm{RelCat_h}$ *then* $\pi_0(f)$ *and* $\pi_1(f)$ *are fibrations in* $\mathrm{Cat_c}$.

PROOF. It follows from Lemma 2.13 and Lemma 2.33 because $\iota_0([0] \to E)$ and $\iota_1([0] \to E)$ are trivial cofibrations in $\mathrm{RelCat_h}$. ∎

4.9. Remark. In $\mathrm{Cat_c}$ every object is fibrant by Lemma 2.9. So, $\pi_0(f)$ and $\pi_1(f)$ being fibrations in $\mathrm{Cat_c}$ does not imply that f is a fibration in $\mathrm{RelCat_h}$ because of Proposition 4.11.

4.10. Lemma. *Let f be a morphism in* Cat. *Then the following are equivalent.*

1. *f is a fibration in* $\mathrm{Cat_c}$.
2. *Rf is a fibration in* $\mathrm{RelCat_h}$.

Proof. (1) $\Rightarrow$ (2) holds because R is a right Quillen functor.

(1) $\Leftarrow$ (2) holds by Lemma 4.8. ■

Every object in $\mathrm{RelCat_h}$ is cofibrant. But not every object is fibrant. Fibrant objects are precisely $(C, \mathrm{iso}(C))$.

4.11. Proposition. *Let $(C, w(C))$ be an object of* RelCat. *Then the following are equivalent.*

1. *$(C, w(C))$ is fibrant in* $\mathrm{RelCat_h}$.
2. *$w(C) = \mathrm{iso}(C)$.*

Proof. (1)$\Leftarrow$(2) It follows from Lemma 4.10 because every object of $\mathrm{Cat_c}$ is fibrant by Lemma 2.9.

(1)$\Rightarrow$(2) First, we show that $w(C) \subseteq \mathrm{iso}(C)$.

Let $\varphi : x \to y$ be a morphism in $w(C)$. Let $g : ([1], [1]) \to (C, w(C))$ be the morphism mapping the morphism $0 \to 1$ in $[1]$ to φ. Consider

$$k : ([1], [1]) \to (E, [1]) \tag{60}$$

where $\pi_0(k) : [1] \to E$ is the functor mapping $0, 1$ in $[1]$ to $0, 1$ in E (17) respectively. Lk is an isomorphism. k is in $wc(\mathrm{RelCat_h})$. Then there is an extension $(E, [1]) \to (C, w(C))$ of g because $(C, w(C))$ is fibrant. Hence $\varphi \in \mathrm{iso}(C)$.

Next we show that $w(C) \supseteq \mathrm{iso}(C)$. Let $\varphi : x \to y$ be in $\mathrm{iso}(C)$. Let $g : (E, \mathrm{dis}(E)) \to (C, w(C))$ be the morphism such that $\pi_0(g)$ maps $0 \to 1$ in E to φ in C. Let

$$k : (E, \mathrm{dis}(E)) \to (E, [1]) \tag{61}$$

be a morphism satisfying $\pi_0(k) = \mathrm{id}_E$. Lk is an isomorphism. k is a trivial cofibration in $\mathrm{RelCat_h}$. Then there is an extension $(E, [1]) \to (C, w(C))$ of g because $(C, w(C))$ is fibrant. Hence $\varphi \in w(C)$. ■

4.12. Remark. Let (C, W) be a relative category. Then the localization

$$\gamma : C \to L(C, W) \tag{62}$$

has an associated morphism

$$\eta_{(C,W)} : (C, W) \to RL(C, W) \tag{63}$$

It is a trivial cofibration in RelCat. Thus Proposition 4.11 implies that $\eta_{(C,W)}$ is a fibrant replacement of (C, W).

5. Enriched model structures

Here we show that RelCat has three enriched model structures, one monoidal model structure and two simplicial model structures. The proof is a relative version of what was proved in [Rez96].

5.1. Monoidal model structure. Here we show that $\mathrm{RelCat}_{\mathrm{h}}$ is a monoidal model category. First, we show that RelCat is a closed symmetric monoidal category.

5.2. Definition. *Let* $\mathbf{C} = (C, w(C))$ *and* $\mathbf{D} = (D, w(D))$ *be objects in* RelCat.

1. *The product* $\mathbf{C} \otimes \mathbf{D}$ *of* $\mathbf{C}$ *and* $\mathbf{D}$ *is defined by*

$$\mathbf{C} \otimes \mathbf{D} = (C \times D, w(C) \times w(D)). \tag{64}$$

2. *The hom-object* $\mathrm{Map}(\mathbf{C}, \mathbf{D})$ *in* Cat *has*
   (a) *as objects the morphisms in* $\mathrm{RelCat}(\mathbf{C}, \mathbf{D})$.
   (b) *as morphisms from* f *to* g *the natural transformations*

$$\alpha : \pi_0(f) \to \pi_0(g). \tag{65}$$

3. *The hom-object* $\mathrm{RelMap}(\mathbf{C}, \mathbf{D})$ *in* RelCat *is the pair*

$$\Big(\mathrm{Map}(\mathbf{C}, \mathbf{D}), w\Big(\mathrm{Map}(\mathbf{C}, \mathbf{D})\Big)\Big) \tag{66}$$

   where $w\Big(\mathrm{Map}(\mathbf{C}, \mathbf{D})\Big)$ *is the wide subcategory of* $\mathrm{Map}(\mathbf{C}, \mathbf{D})$ *such that for a morphism* $\alpha : f \to g$ *in* $\mathrm{Map}(\mathbf{C}, \mathbf{D})$, α *belongs to* $w\Big(\mathrm{Map}(\mathbf{C}, \mathbf{D})\Big)$ *iff* α_c *is in* $w(D)$ *for all* $c \in C$.

5.3. Proposition. *The bifunctors* $\otimes$ *and* RelMap *make* RelCat *a closed symmetric monoidal category.*

Proof. $([0],[0])$ is the unit object of RelCat. The associativity isomorphism and the unit isomorphisms are the pairs of the corresponding isomorphisms in the monoidal category Cat. The product $\otimes$ is clearly symmetric. Let $\mathbf{C} = (C, w(C))$, $\mathbf{D} = (D, w(D))$ and $\mathbf{E} = (E, w(E))$ be objects of RelCat. By definition, we have

$$\mathrm{RelCat}(\mathbf{C} \otimes \mathbf{D}, \mathbf{E}) \cong \mathrm{RelCat}(\mathbf{C}, \mathrm{RelMap}(\mathbf{D}, \mathbf{E})) \tag{67}$$

and

$$\mathrm{RelCat}(\mathbf{C} \otimes \mathbf{D}, \mathbf{E}) \cong \mathrm{RelCat}(\mathbf{D}, \mathrm{RelMap}(\mathbf{C}, \mathbf{E})) \tag{68}$$

In particular, the monoidal structure is closed. ■

5.4. Proposition. *With the monoidal structure in Proposition 5.3,* $\mathrm{RelCat_h}$ *is a monoidal model category.*

Proof. SM0: By (67) and (68), RelCat has tensors $\mathbf{C} \otimes \mathbf{D}$, cotensors $\mathrm{RelMap}(\mathbf{C}, \mathbf{D})$ and hom-objects $\mathrm{RelMap}(\mathbf{C}, \mathbf{D})$ in RelCat where $\mathbf{C}, \mathbf{D} \in \mathrm{RelCat}$.

SM7: The functor L in (2) preserves pushouts. The functor L preserves and reflects (trivial) cofibrations by definition. So it follows from Theorem 5.1 in [Rez96]. ■

5.5. Simplicial model structure. Here we show that there are two simplicial model structure on RelCat associated with the two adjunctions in (33) and (34). We use the Quillen adjunction

$$\tau : \mathrm{sSet} \rightleftarrows \mathrm{Cat} : N \tag{69}$$

where N is the nerve functor.

5.6. Proposition. *Let* $i = 1, 2$. *Then* $\mathrm{RelCat_h}$ *is a simplicial model category with tensors*

$$\mathbf{C} \otimes K = \mathbf{C} \otimes \iota_i(\tau K) \tag{70}$$

cotensors

$$\mathbf{C}^K = \mathrm{RelMap}(\iota_i(\tau K), \mathbf{C}) \tag{71}$$

and hom-objects in sSet

$$\underline{\mathrm{RelCat}}(\mathbf{C}, \mathbf{D}) = N\pi_i(\mathrm{RelMap}(\mathbf{C}, \mathbf{D})) \tag{72}$$

where $\mathbf{C}, \mathbf{D} \in \mathrm{RelCat}$ *and* $K \in \mathrm{Cat}$.

Proof. SM0: Using (67) and (68), we have

$$\begin{aligned}
&\mathrm{RelCat}(\mathbf{C} \otimes K, \mathbf{D}) && (73)\\
=&\mathrm{RelCat}(\mathbf{C} \otimes \iota_i(\tau K), \mathbf{D}) && (74)\\
\cong&\mathrm{RelCat}(\iota_i(\tau K), \mathrm{RelMap}(\mathbf{C}, \mathbf{D})) && (75)\\
\cong&\mathrm{sSet}(K, N\pi_i \mathrm{RelMap}(\mathbf{C}, \mathbf{D})) && (76)\\
=&\mathrm{sSet}(K, \underline{\mathrm{RelCat}}(\mathbf{C}, \mathbf{D})) && (77)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{RelCat}(\mathbf{C}, \mathbf{D}^K) && (78)\\
=&\mathrm{RelCat}(\mathbf{C}, \mathrm{RelMap}(\iota_i(\tau K), \mathbf{D})) && (79)\\
\cong&\mathrm{RelCat}(\iota_i(\tau K), \mathrm{RelMap}(\mathbf{C}, \mathbf{D})) && (80)\\
\cong&\mathrm{sSet}(K, N\pi_i\mathrm{RelMap}(\mathbf{C}, \mathbf{D})) && (81)\\
=&\mathrm{sSet}(K, \underline{\mathrm{RelCat}}(\mathbf{C}, \mathbf{D})) && (82)
\end{aligned}$$

SM7: It follows from SM7 of the monoidal model category $\mathrm{RelCat_h}$ because τ and ι_i preserve (trivial) cofibrations and colimits. ■

6. Comparison with the Barwick and Kan model structure

We denote by $\mathrm{RelCat_{BK}}$ the Barwick-Kan model category ([BK12b]). Here we compare $\mathrm{RelCat_h}$ with $\mathrm{RelCat_{BK}}$.

It is known that a weak equivalence in $\mathrm{RelCat_{BK}}$ is a weak equivalence in $\mathrm{RelCat_h}$ ([BK10],[BK12a]).

$$w(\mathrm{RelCat_{BK}}) \subset w(\mathrm{RelCat_h}) \tag{83}$$

So, we may ask if $\mathrm{RelCat_h}$ is obtained from $\mathrm{RelCat_{BK}}$ as a Bousfield localization. The answer is negative for both of the left and the right.

A Dwyer map (Remark 2.31) in RelCat is a cofibration in $\mathrm{RelCat_h}$. Every cofibration in $\mathrm{RelCat_{BK}}$ is a Dwyer map by Theorem 6.1(iii) in [BK12b]. Thus $c(\mathrm{RelCat_{BK}})$ is a subcategory of $c(\mathrm{RelCat_h})$. Every cofibrant object in $\mathrm{RelCat_{BK}}$ is a relative poset by Theorem 6.1(iv) in [BK12b]. But, in $\mathrm{RelCat_h}$, every object is cofibrant (Corollary 4.3). Thus

$$c(\mathrm{RelCat_{BK}}) \subsetneqq c(\mathrm{RelCat_h}) \tag{84}$$

So, $\mathrm{RelCat_h}$ is not a left Bousfield localization of $\mathrm{RelCat_{BK}}$.

Let $k : [1] \to E$ be the canonical inclusion in Cat where E is the category (17). The morphism $\iota_1(k) : ([1], [1]) \to (E, E)$ is a trivial cofibration in RelCat because $L\iota_1(k)$ is an isomorphism. But $\iota_1(k)$ is not a trivial cofibration in $\mathrm{RelCat_{BK}}$ because it is not a Dwyer map (Remark 2.31(3)). We have $Z(([1], [1]), (E, E)) = (E, E)$. But there is no morphism $r : (E, E) \to ([1], [1])$ such that $\pi_0(r)0 = 0$ and $\pi_0(r)1 = 1$. Thus $f(\mathrm{RelCat_h})$ is a proper subcategory of $f(\mathrm{RelCat_{BK}})$.

$$f(\mathrm{RelCat_{BK}}) \supsetneqq f(\mathrm{RelCat_h}) \tag{85}$$

So, $\mathrm{RelCat_h}$ is not a right Bousfield localization of $\mathrm{RelCat_{BK}}$.

7. Non-uniqueness and the mixed model structure on RelCat

The model structure (6) on $\mathrm{Cat_c}$ is the unique model structure with $w(\mathrm{Cat_c})$ as the class of weak equivalences. In contrast, the model structure in Theorem 1.1 is not an unique one whose class of weak equivalences is $w(\mathrm{RelCat_h})$. It is an immediate consequence of the following theorem of Cole on mixed model structures.

7.1. THEOREM. [Theorem 2.1 in [Col06]] *Let* (W_1, C_1, F_1) *and* (W_2, C_2, F_2) *be two model structures on a category* M*. If* $W_1 \subseteq W_2$ *and* $C_1 \subseteq C_2$ *then there exists a model structure* (W_m, C_m, F_m) *such that* $W_m = W_2$ *and* $C_m = C_1$*.*

Thus, we know from (83), (84) and Theorem 7.1 that there is a different model structure on RelCat such that

- $w(\mathrm{RelCat})$ is the class of weak equivalences and
- $c(\mathrm{RelCat_{BK}})$ is the class of cofibrations.

Now we denote by $\mathrm{RelCat_{mix}}$ the model category with the mixed model structure above so that

- $w(\mathrm{RelCat_{mix}}) = w(\mathrm{RelCat_h})$.
- $c(\mathrm{RelCat_{mix}}) = c(\mathrm{RelCat_{BK}})$.
- $f(\mathrm{RelCat_{mix}}) = \big(w(\mathrm{RelCat_h}) \cap c(\mathrm{RelCat_{BK}})\big)^{\boxslash}$.

The left adjoint $L : \mathrm{RelCat} \to \mathrm{Cat}$ factors through the category SimCat of simplicially enriched categories

$$L : \mathrm{RelCat} \xrightarrow{L^H} \mathrm{SimCat} \xrightarrow{\pi_0} \mathrm{Cat} \tag{86}$$

where L^H is the Hammock localization ([DK80a]). $\mathrm{RelCat_{mix}}$ is a left Bousfield localization of $\mathrm{RelCat_{BK}}$. So, it would be interesting to know if there are intermediate model structures between $\mathrm{RelCat_{BK}}$ and $\mathrm{RelCat_{mix}}$ in which for a morphism f in RelCat, f is a weak equivalence iff the functor on homotopy $(n, 1)$-categories is an equivalence.

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