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A NOTE ON THE ATOMICITY OF ARITHMETICITY

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Abstract. The main aim of this note is to show that, in the regular context, every matrix property in the sense of [13] either implies the Mal'tsev property, or is implied by the majority property. When the regular category $\mathbb{C}$ is arithmetical, i.e., both Mal'tsev and a majority category, then we show that $\mathbb{C}$ satisfies every non-trivial matrix property.

1. Introduction

Consider an extended matrix of variables

$$\mathsf{M} = \left[\begin{array}{ccc|c} x_{11} & \cdots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nm} & y_n \end{array}\right]$$

where the x_{ij}'s and the y_i's are (not necessarily distinct) variables from $\{x_1, \ldots, x_k\}$. We refer to the last column (the column of y_i's in M) as the *right column* of M, and every other column as a *left column*. A *row-wise interpretation* of M of type $(X_1, \ldots, X_n)$ is a matrix of the form

$$\left[\begin{array}{ccc|c} f_1(x_{11}) & \cdots & f_1(x_{1m}) & f_1(y_1) \\ \vdots & & \vdots & \vdots \\ f_n(x_{n1}) & \cdots & f_n(x_{nm}) & f_n(y_n) \end{array}\right]$$

where the $f_i \colon \{x_1, \ldots, x_k\} \to X_i$ are functions. Given any relation $R \subseteq X_1 \times \cdots \times X_n$, we say that it is strictly M-closed if for every row-wise interpretation N of M of type $(X_1, \ldots, X_n)$ if the left columns of N are elements of R then the right column of N is also an element of R. This set-theoretic property of relations can be internalised (via the Yoneda embedding) in any finitely complete category $\mathbb{C}$, so that $\mathbb{C}$ is then said to have M-closed relations if every internal relation in $\mathbb{C}$ is strictly M-closed. We will also refer to the property of M-closedness of internal relations in $\mathbb{C}$ as simply the *matrix property* M.

Let us formulate this property in a way which we will use for the remainder of this paper. Given two morphisms f and g in a category $\mathbb{C}$ with the same codomain, we will write $f \sqsubset g$ if f factors through g. This relation on morphisms in $\mathbb{C}$ defines a *cover relation* in the sense of [15, 16]. Then we can reformulate the matrix property

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M of a finitely complete category $\mathbb{C}$ in the following way: given any internal relation $r\colon R \rightarrowtail X_1 \times \cdots \times X_n$ in $\mathbb{C}$ and any row-wise interpretation

$$N = \left[\begin{array}{ccc|c} c_1 & \cdots & c_m & y \end{array}\right]$$

of the matrix M of type $(\hom(S, X_1), \ldots, \hom(S, X_n))$ where the c_i's are the left columns of N and the y is the right column of N (viewed as morphisms $S \to X_1 \times \cdots \times X_n$), then, if for every $i \in \{1, \ldots, m\}$ we have $c_i \sqsubset r$, then we have $y \sqsubset r$.

The first and most well-known example of a matrix property is the property of a finitely complete category to be a Mal'tsev category [5, 4], since a category $\mathbb{C}$ with finite limits is a Mal'tsev category if and only if every internal relation is *difunctional* [19], i.e., every internal relation is strictly Mal-closed where

$$\mathsf{Mal} = \left[\begin{array}{ccc|c} x_1 & x_2 & x_2 & x_1 \\ x_2 & x_2 & x_1 & x_1 \end{array}\right].$$

Similarly to what was done in [9], given integers $n, k > 0$ and $m \geqslant 0$, we write $\mathsf{matr}(n, m, k)$ for the set of all matrices with n rows, $m+1$ columns and whose entries are in the set $\{x_1, \ldots, x_k\}$. Then matr is the union of all such $\mathsf{matr}(n, m, k)$ for $n, k > 0$ and $m \geqslant 0$. Corresponding to a matrix $\mathsf{M} \in \mathsf{matr}$, we will write $\mathsf{mclex}\{\mathsf{M}\}$ for the collection of finitely complete categories which satisfy the matrix property M, and refer to these collections as *matrix classes*. The collection of all such matrix classes is then denoted by Mclex, i.e.,

$$\mathsf{Mclex} = \{\mathsf{mclex}\{\mathsf{M}\} \mid \mathsf{M} \in \mathsf{matr}\}$$

and it has a poset structure given by inclusion of matrix classes. We will also write $\mathsf{Mclex}[n, m, k]$ for the sub-poset of Mclex of matrix classes $\mathsf{mclex}\{\mathsf{M}\}$ determined by a matrix M in $\mathsf{matr}(n, m, k)$. Among the elements of the poset Mclex are two *trivial* ones, i.e., the matrix class of preorders with a single isomorphism class and the matrix class of finitely complete preorders. These are respectively the bottom element and the unique atom of Mclex. They are determined by the so-called *trivial matrices* (see [11]). The top element of Mclex, i.e., the matrix class of all finitely complete categories is called the *anti-trivial* element and is determined by the so-called *anti-trivial matrices*. The *degenerate matrix classes* (respectively the *degenerate matrices*) are the ones which are either trivial or anti-trivial.

As another example of a collection of categories determined by a matrix property, consider the matrix property corresponding to the matrix Maj where

$$\mathsf{Maj} = \left[\begin{array}{ccc|c} x_1 & x_1 & x_2 & x_1 \\ x_1 & x_2 & x_1 & x_1 \\ x_2 & x_1 & x_1 & x_1 \end{array}\right].$$

Then, $\mathsf{mclex}\{\mathsf{Maj}\}$ is the collection of all finitely complete majority categories [7]. For another example, consider the matrix Ari where

$$\mathsf{Ari} = \left[\begin{array}{ccc|c} x_1 & x_2 & x_2 & x_1 \\ x_2 & x_2 & x_1 & x_1 \\ x_1 & x_2 & x_1 & x_1 \end{array}\right].$$

Then mclex{Ari} is the collection of all finitely complete arithmetical categories as defined in [11]. Note that in the Barr-exact context [1] with coequalisers this matrix property determines arithmetical categories as introduced first in [18] and later generalised to wider contexts in [2, 6].

Some matrix properties imply others, which is to say that some matrix classes are contained (as sub-collections) in other matrix classes. For example, we have that mclex{Ari} $\subseteq$ mclex{Maj} and also mclex{Ari} $\subseteq$ mclex{Mal}. Thus, the general question: is there a procedure for determining when a given matrix class contains another? This question has been recently answered in the paper [11], where an algorithm was given which determines inclusions of the form mclex{N} $\subseteq$ mclex{M}, i.e., which determines implications of matrix properties. Computer implementation of this algorithm allows us to determine, for relatively small n, m, k, the posets Mclex$[n, m, k]$. For instance, Figure 1 (which describes the same poset as Figure 2 in [11] and Figure 1 in [12]) gives a visual depiction of the poset of non-degenerate elements of Mclex$[3, 7, 2]$ as obtained by the computer, where each integer entry i corresponds to the variable x_i and the shaded column is the right column in the representing matrix. Note that most of these matrix classes are not represented here by $3 \times (7 + 1)$ matrices but up to duplication of rows and left columns, they can be turned to such matrices. One of the main results of [12] shows that among the non-trivial matrix classes in Mclex represented by a matrix with at most two variables (i.e., binary matrices) the matrix class mclex{Ari} is the least. In fact, Figure 1 already illustrates this fact, as the matrix class farthest to the left is mclex{Ari}. However, in Mclex we have non-trivial matrix classes which are strictly contained in mclex{Ari}.

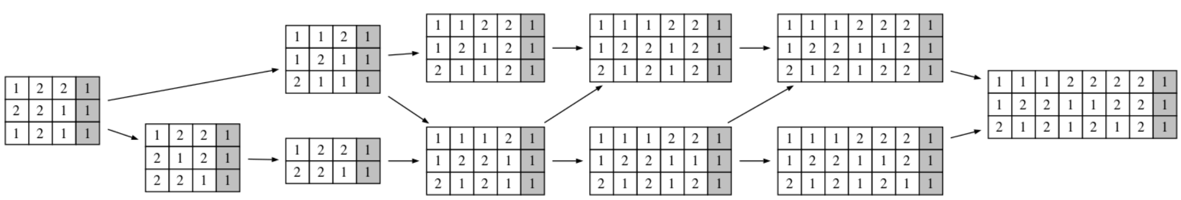

Figure 1: The poset of non-degenerate elements of Mclex$[3, 7, 2]$. The matrix furthest to the left represents the matrix class mclex{Ari} and the only matrix with two rows represents mclex{Mal}. The matrix which is just above this latter matrix represents the matrix class mclex{Maj}.

Let us now write mcreg{M} for the collection of all regular categories [1] satisfying the matrix property M, i.e., mcreg{M} is the intersection of mclex{M} and the collection of all regular categories. Such classes of regular categories we refer to as *regular* matrix classes. We may then consider the poset Mcreg of all regular matrix classes ordered by inclusion, and ask the analagous question: is there an algorithm for determining whether or not mcreg{M} $\subseteq$ mcreg{N}? As it stands, this question is still open, and it is known that mcreg{M} $\subseteq$ mcreg{N} need not imply that mclex{M} $\subseteq$ mclex{N} — see Section 5 of [11].

In this paper we will show that the regular matrix classes corresponding to Mal, Maj and Ari play a special role in Mcreg. For one thing, we will show that every regular matrix class in Mcreg is either contained in $\mathsf{mcreg}\{\mathsf{Mal}\}$ or contains $\mathsf{mcreg}\{\mathsf{Maj}\}$. We will also show that among the non-trivial members of Mcreg, the least is the regular matrix class corresponding to Ari. While, as shown in [12], both results extend to the finitely complete context for binary matrices, we know they do not extend to that context in full generality.

Notation. In order to simplify notation, we will borrow the notation of [10], and write $\mathsf{M} \Rightarrow_{\mathsf{lex}} \mathsf{N}$ if $\mathsf{mclex}\{\mathsf{M}\} \subseteq \mathsf{mclex}\{\mathsf{N}\}$ and likewise write $\mathsf{M} \Rightarrow_{\mathsf{reg}} \mathsf{N}$ if $\mathsf{mcreg}\{\mathsf{M}\} \subseteq \mathsf{mcreg}\{\mathsf{N}\}$.

2. A strong majority property

Given a category $\mathbb{C}$ with binary products, in what follows we will write $\pi_{i,j}$ for the two-fold projection $(\pi_i, \pi_j)\colon X_1 \times \cdots \times X_n \to X_i \times X_j$ determined by π_i and π_j, where $i, j \in \{1, 2, \ldots, n\}$. For a natural number $n \geqslant 3$, we define the following property on a finitely complete category $\mathbb{C}$.

(M_n) For any morphism $y\colon S \to X_1 \times \cdots \times X_n$ and any monomorphism $r\colon R \rightarrowtail X_1 \times \cdots \times X_n$, if $\pi_{i,j} y \sqsubset \pi_{i,j} r$ for any $i, j \in \{1, 2, \ldots, n\}$, then $y \sqsubset r$.

Note that for such $y\colon S \to X_1 \times \cdots \times X_n$, $r\colon R \rightarrowtail X_1 \times \cdots \times X_n$ and $i, j \in \{1, \ldots, n\}$, if $\pi_{i,j} y \sqsubset \pi_{i,j} r$, then $\pi_{j,i} y \sqsubset \pi_{j,i} r$ and $\pi_{i,i} y \sqsubset \pi_{i,i} r$; so that in the above description of (M_n), it is equivalent to ask $\pi_{i,j} y \sqsubset \pi_{i,j} r$ only for all $i, j \in \{1, \ldots, n\}$ with $i < j$. As we will see shortly, the property (M_n) for any integer $n \geqslant 3$ is equivalent to a matrix property. Define a matrix M_n with n rows, $m = \binom{n}{2}$ left columns $c_{i,j}$ indexed by all pairs of integers (i, j) where $1 \leqslant i < j \leqslant n$ and whose right column is the column vector containing only the variable x_1. Order the left columns $c_{i,j} < c_{i',j'}$ from left to right according to the lexicographic order $(i, j) < (i', j')$ on $\mathbb{N}^2$. In each column $c_{i,j}$ place a x_1 at the i^{th} and j^{th} entry. Then in each row, insert the variables $x_2, \ldots, x_k$ (where $k = \binom{n-1}{2} + 1$) in increasing order (of index) at each position which does not contain a x_1. For example, in the case $n = 3$, the matrix M_3 is nothing but

$$\mathsf{M}_3 = \mathsf{Maj} = \left[\begin{array}{ccc|c} x_1 & x_1 & x_2 & x_1 \\ x_1 & x_2 & x_1 & x_1 \\ x_2 & x_1 & x_1 & x_1 \end{array}\right]$$

as defined in the Introduction. In the case $n = 4$, we have

$$\mathsf{M}_4 = \left[\begin{array}{cccccc|c} x_1 & x_1 & x_1 & x_2 & x_3 & x_4 & x_1 \\ x_1 & x_2 & x_3 & x_1 & x_1 & x_4 & x_1 \\ x_2 & x_1 & x_3 & x_1 & x_4 & x_1 & x_1 \\ x_2 & x_3 & x_1 & x_4 & x_1 & x_1 & x_1 \end{array}\right].$$

2.1. Proposition. *Let $n \geqslant 3$ be an integer. A finitely complete category $\mathbb{C}$ satisfies (M_n) if and only if $\mathbb{C}$ has M_n-closed relations, i.e., $\mathbb{C}$ satisfies the matrix property corresponding to M_n.*

Proof. Suppose that $\mathbb{C}$ satisfies (M_n) and let $r\colon R \rightarrowtail X_1\times\cdots\times X_n$ be any monomorphism and

$$M' = \left[\begin{array}{ccc|c} x_{11} & \cdots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nm} & y_n \end{array}\right]$$

be a row-wise interpretation of the matrix M_n of type $(\hom(S,X_1),\dots,\hom(S,X_n))$ where $m=\binom{n}{2}$. Each column of the matrix M' above determines a (unique) morphism $S\to X_1\times\cdots\times X_n$, and we will write $c'_{i,j}\colon S\to X_1\times\cdots\times X_n$ for the morphism corresponding to the left column $c_{i,j}$ of M_n. We write $y=(y_1,\dots,y_n)\colon S\to X_1\times\cdots\times X_n$ for the morphism determined by the right column of M'. We suppose that each $c'_{i,j}$ factorises through r and we must show that y also does. As earlier, we write $\pi_{i,j}\colon X_1\times\cdots\times X_n\to X_i\times X_j$ for the two-fold projection determined by π_i and π_j. It may then be seen that $\pi_{i,j}c'_{i,j}=(y_i,y_j)$, since M' is a row-wise interpretation of M_n. Therefore $\pi_{i,j}y=\pi_{i,j}c'_{i,j}\sqsubset \pi_{i,j}r$ for all $i,j\in\{1,2,\dots,n\}$ with $i<j$ so that $y\sqsubset r$ by (M_n).

Conversely, suppose that $\mathbb{C}$ has M_n-closed relations and that we are given a monomorphism $r\colon R\rightarrowtail X_1\times\cdots\times X_n$ and a morphism $y=(y_1,\dots,y_n)\colon S\to X_1\times\cdots\times X_n$ such that $\pi_{i,j}y\sqsubset\pi_{i,j}r$ for each $i,j\in\{1,2,\dots,n\}$. Thus, there are factorisations $f'_{i,j}\colon S\to R$ such that $\pi_{i,j}rf'_{i,j}=(y_i,y_j)$. Form the matrix M' whose left columns are determined by the morphisms $c'_{i,j}=rf'_{i,j}$ (again ordering them via the lexicographic order on $\mathbb{N}^2$) and whose right column is determined by the morphism y. This matrix M' is then a row-wise interpretation of M_n of type $(\hom(S,X_1),\dots,\hom(S,X_n))$. Since $\mathbb{C}$ has M_n-closed relations, we deduce that $y\sqsubset r$. ∎

The matrix M_4 is identical (up to replacement of variables) to the matrix in Remark 2.4 of [12]. For this reason, we know we do *not* have $\mathsf{M}_3\Rightarrow_{\mathsf{lex}}\mathsf{M}_4$ although we do have $\mathsf{M}_4\Rightarrow_{\mathsf{lex}}\mathsf{M}_3$. We can actually generalise this.

2.2. Proposition. *We have a sequence of strict implications*

$$\cdots\Rightarrow_{\mathsf{lex}}\mathsf{M}_{n+1}\Rightarrow_{\mathsf{lex}}\mathsf{M}_n\Rightarrow_{\mathsf{lex}}\cdots\Rightarrow_{\mathsf{lex}}\mathsf{M}_5\Rightarrow_{\mathsf{lex}}\mathsf{M}_4\Rightarrow_{\mathsf{lex}}\mathsf{M}_3$$

and all these matrices are non-degenerate.

Proof. According to Theorem 2.5 in [11], a matrix is anti-trivial if and only if its right column appears among its left columns. Clearly, this is not the case for these matrices M_n. Moreover, we can immediately deduce from Theorem 2.3 in [11] that these matrices are not trivial neither and so not degenerate.

Let us now prove that for $n\geqslant 3$, we have $\mathsf{M}_{n+1}\Rightarrow_{\mathsf{lex}}\mathsf{M}_n$. To fix notation, let $m=\binom{n}{2}$, $k=\binom{n-1}{2}+1$, $m'=\binom{n+1}{2}$ and $k'=\binom{n}{2}+1$ so that $\mathsf{M}_n\in\mathsf{Mclex}[n,m,k]$ and $\mathsf{M}_{n+1}\in\mathsf{Mclex}[n+1,m',k']$. According to the algorithm from [11], to prove $\mathsf{M}_{n+1}\Rightarrow_{\mathsf{lex}}\mathsf{M}_n$ it is enough to prove that the matrix M' formed by the first n rows of M_{n+1} admits a row-wise interpretation of type $(\{x_1,\dots,x_k\},\dots,\{x_1,\dots,x_k\})$ whose left columns can be found among the left columns of M_n and whose right column is the right column of M_n. This can be easily seen by interpreting in M' each x_1 by x_1 and each other variable (which appears only once in each row) by the only variable such that the left column $c'_{i,j}$ of M' for $1 \leqslant i < j \leqslant n$ is interpreted as the left column $c_{i,j}$ of M_n, and the left column $c'_{i,n+1}$ of M' for $1 \leqslant i \leqslant n$ is interpreted as the left column $c_{i,n}$ if $i < n$ or $c_{1,n}$ if $i = n$ of M_n.

It remains to prove that we do not have $\mathsf{M}_n \Rightarrow_{\mathsf{lex}} \mathsf{M}_{n+1}$. Again according to the algorithm from [11], it is enough to prove that for any matrix M' with $n+1$ rows and whose each row is a row of M_n, and for any row-wise interpretation M'' of M' of type $(\{x_1, \dots, x_{k'}\}, \dots, \{x_1, \dots, x_{k'}\})$, if the left columns of M'' can be found among the left columns of M_{n+1}, then so can its right column. By contradiction, suppose M' and M'' are such matrices such that the left columns of M'', but not its right column, are among the left columns of M_{n+1}. In each row of M_n (and so of M'), there are exactly $n-1$ many x_1's in its left part (i.e., not counting its rightmost entry). If, for some $j \in \{1, \dots, n+1\}$, these x_1's in the j^{th} row of M' are interpreted in M'' as x_l with $1 < l \leqslant k'$, then the left columns of M' containing x_1 in the j^{th} row are all interpreted the same in M'' since x_l appears exactly once in the j^{th} row of M_{n+1}. In that case, since for each $j' \in \{1, \dots, n+1\}$, there is a left column of M' with x_1 as j^{th} and $j'^{\text{ th}}$ entries, the right column of M' (constituted only of x_1's) is interpreted in M'' in the same way as its left columns containing x_1 in the j^{th} row. This would imply that the right column of M'' must be found among the left columns of M_{n+1}, which is a contradiction. Therefore, each x_1 in the left part of M' is interpreted as x_1 in M''. Thus, there are at least $(n+1)(n-1)$ many x_1's in the left part of M''. However, each left column of M'' being a left column of M_{n+1}, they contain exactly two x_1's each. There are thus exactly $2m = 2\binom{n}{2} = n(n-1)$ many x_1's in the left part of M'', which is a contradiction. ∎

Let us recall now that from Corollary 2.4 in [11] and Corollary 2.5 in [10] we have the following two propositions.

2.3. Proposition. *[11] If* $\mathsf{M} \in \mathsf{matr}(2, m, k)$ *is a two-row matrix (for integers* $m \geqslant 0$ *and* $k > 0$*), then* $\mathsf{mclex}\{\mathsf{M}\}$ *is trivial, anti-trivial or the matrix class of Mal'tsev categories.*

2.4. Proposition. *[10] For any matrix* $\mathsf{M} \in \mathsf{matr}$*, the implication* $\mathsf{M} \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$ *does* not *hold if and only if every selection of two rows from* M *forms an anti-trivial matrix.*

We are now able to prove the following.

2.5. Proposition. *Given integers* $n \geqslant 3$*,* $m \geqslant 0$ *and* $k > 0$ *and a matrix* $\mathsf{M} \in \mathsf{matr}(n, m, k)$*, we have that either* $\mathsf{M} \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$ *or* $\mathsf{M}_n \Rightarrow_{\mathsf{lex}} \mathsf{M}$*, and these two implications cannot occur simultaneously.*

Proof. By construction, every selection of two rows from M_n forms an anti-trivial matrix and so $\mathsf{M}_n \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$ does not hold by Proposition 2.4. This already proves that the two implications of the statement cannot occur simultaneously. Suppose now that $\mathsf{M} \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$ does not hold. Then every selection of two rows of M forms an anti-trivial matrix by Proposition 2.4. Let $\mathbb{C}$ be any finitely complete category in $\mathsf{mclex}\{\mathsf{M}_n\}$ and let us prove it is in $\mathsf{mclex}\{\mathsf{M}\}$. Let $r\colon R \rightarrowtail X_1 \times \cdots \times X_n$ be any relation in $\mathbb{C}$ and let

$$M' = \left[\begin{array}{ccc|c} c'_1 & \cdots & c'_m & y \end{array}\right]$$

be any row-wise interpretation of M of type $(\hom(S,X_1),\ldots,\hom(S,X_n))$ where the left columns $c'_1,\ldots,c'_m$ of M' (viewed as morphisms $S\to X_1\times\cdots\times X_n$) satisfy $c'_l\sqsubset r$ for each $l\in\{1,\ldots,m\}$. We must show that the right column y of M' (also viewed as a morphism $S\to X_1\times\cdots\times X_n$) satisfy $y\sqsubset r$. Since, by Proposition 2.1, $\mathbb{C}$ satisfies the property (M_n), we only have to show that $\pi_{i,j}y\sqsubset\pi_{i,j}r$ for any $i,j\in\{1,\ldots,n\}$. For such i and j, since the matrix obtained by selecting the i^{th} and the j^{th} row of M is anti-trivial, we know that $\pi_{i,j}y=\pi_{i,j}c'_l$ for some $l\in\{1,\ldots,m\}$. Therefore, $\pi_{i,j}y=\pi_{i,j}c'_l\sqsubset\pi_{i,j}r$ as desired. ■

3. The regular context

Recall that a category is *regular* [1] if it is finitely complete, has coequalisers of kernel pairs and regular epimorphisms are stable under pullbacks. In that case, each morphism factorises as a regular epimorphism followed by a monomorphism. Although, in the finitely complete context, the implications of Proposition 2.2 are strict, this is not the case any more in the regular context as attested by the following theorem.

3.1. Theorem. *We have a sequence of equivalences*

$$\cdots\Leftrightarrow_{\mathsf{reg}}\mathsf{M}_{n+1}\Leftrightarrow_{\mathsf{reg}}\mathsf{M}_n\Leftrightarrow_{\mathsf{reg}}\cdots\Leftrightarrow_{\mathsf{reg}}\mathsf{M}_5\Leftrightarrow_{\mathsf{reg}}\mathsf{M}_4\Leftrightarrow_{\mathsf{reg}}\mathsf{M}_3$$

i.e., for any integer $n\geqslant 3$, in the regular context, the property (M_n) is equivalent to the majority property (M_3).

Proof. In view of Proposition 2.2 and since $\mathsf{M}_3=\mathsf{Maj}$, it is enough to prove that any regular majority category $\mathbb{C}$ satisfies (M_n) for each $n\geqslant 3$. Given such $\mathbb{C}$ and n, let $r\colon R\rightarrowtail X_1\times\cdots\times X_n$ be any monomorphism and let $y\colon S\to X_1\times\cdots\times X_n$ be any morphism in $\mathbb{C}$ such that, for any $i,j\in\{1,\ldots,n\}$, $\pi_{i,j}y\sqsubset\pi_{i,j}r$. We must show that $y\sqsubset r$. For all $i,j\in\{1,\ldots,n\}$, let

$$R\xrightarrow{e_{i,j}}R_{i,j}\xrightarrow{m_{i,j}}X_i\times X_j$$

be the (regular epimorphism, monomorphism)-factorisation of the composite

$$R\xrightarrow{r}X_1\times\cdots\times X_n\xrightarrow{\pi_{i,j}}X_i\times X_j.$$

By the equivalence of (i) and (v) of Theorem 5.1 in [8], and Proposition 4.1 in [8], the diagram

$$\begin{array}{ccc}
R & \xrightarrow{(e_{i,j})_{(i,j)\in\{1,\ldots,n\}^2}} & \prod_{i,j=1}^{n} R_{i,j}\\
{\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle \prod_{i,j=1}^{n} m_{i,j}}\\
\prod_{l=1}^{n} X_l & \xrightarrow[(\pi_{i,j})_{(i,j)\in\{1,\ldots,n\}^2}]{} & \prod_{i,j=1}^{n} X_i\times X_j
\end{array}$$

is a pullback. Since, for all $i, j \in \{1, \dots, n\}$, we have supposed $\pi_{i,j} y \sqsubset \pi_{i,j} r$, we know that $\pi_{i,j} y \sqsubset m_{i,j}$; producing a morphism $S \to \prod_{i,j=1}^{n} R_{i,j}$ so that, by the universal property of the above pullback, we get $y \sqsubset r$. ∎

3.2. Corollary. *For any matrix* M *in* matr*, we have that either* $\mathsf{M} \Rightarrow_{\mathsf{reg}} \mathsf{Mal}$ *or* $\mathsf{Maj} \Rightarrow_{\mathsf{reg}} \mathsf{M}$*, and these two implications cannot occur simultaneously.*

Proof. If these two implications occur simultaneously, we would have $\mathsf{Maj} \Rightarrow_{\mathsf{reg}} \mathsf{Mal}$, which, by Theorem 2.4 in [10], is equivalent to $\mathsf{Maj} \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$. But this last implication does not hold, for instance by Proposition 2.4.

Let us now prove that $\mathsf{M} \Rightarrow_{\mathsf{reg}} \mathsf{Mal}$ or $\mathsf{Maj} \Rightarrow_{\mathsf{reg}} \mathsf{M}$. If M is trivial, then $\mathsf{M} \Rightarrow_{\mathsf{reg}} \mathsf{Mal}$; and if M is anti-trivial, we have $\mathsf{Maj} \Rightarrow_{\mathsf{reg}} \mathsf{M}$. It remains to treat the case where M is non-degenerate. Suppose that M has n rows. If $n \leqslant 2$, by Proposition 2.3, $\mathsf{M} \Leftrightarrow_{\mathsf{lex}} \mathsf{Mal}$ and so in particular $\mathsf{M} \Rightarrow_{\mathsf{reg}} \mathsf{Mal}$. Let us now suppose that $n \geqslant 3$. Then, we have that either $\mathsf{M} \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$ or $\mathsf{M}_n \Rightarrow_{\mathsf{lex}} \mathsf{M}$ by Proposition 2.5. The result then follows since $\mathsf{Maj} \Leftrightarrow_{\mathsf{reg}} \mathsf{M}_n$ for all $n \geqslant 3$ by Theorem 3.1. ∎

3.3. Remark. Let us make clear here that, although Corollary 3.2 takes place in the regular context, it holds for the matrix properties in the sense of [13] but *not* for the matrix properties in the sense of [15]. Indeed, an example of these latter matrix properties is the property of being a Goursat category [3]. An example of a (regular) Goursat category which is not Mal'tsev is given by the category of implication algebras [17] while an example of a majority category which is not Goursat is given by the category of lattices [3, 7].

The proof of the theorem below makes use of the fact that $\mathsf{Ari} \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$ and $\mathsf{Ari} \Rightarrow_{\mathsf{lex}} \mathsf{Maj}$. These implications already appear in Figure 1. For a proof of them, we refer the reader to Section 5 of [11]. There, it is actually shown that $\mathsf{mclex}\{\mathsf{Ari}\} = \mathsf{mclex}\{\mathsf{Mal}\} \cap \mathsf{mclex}\{\mathsf{Maj}\}$, i.e., a finitely complete category is arithmetical if and only if it is both Mal'tsev and majority.

3.4. Theorem. *For any non-trivial matrix* M *in* matr*, we have* $\mathsf{Ari} \Rightarrow_{\mathsf{reg}} \mathsf{M}$*.*

Proof. Suppose that M is any non-trivial matrix with $n > 0$ rows. If $n \leqslant 2$, then $\mathsf{Mal} \Rightarrow_{\mathsf{lex}} \mathsf{M}$ by Proposition 2.3 and therefore $\mathsf{Ari} \Rightarrow_{\mathsf{lex}} \mathsf{M}$ since $\mathsf{Ari} \Rightarrow_{\mathsf{lex}} \mathsf{Mal}$. We can therefore suppose without loss of generality that $n \geqslant 3$. Let $\mathbb{C}$ be a regular arithmetical category, $r\colon R \rightarrowtail X_1 \times \cdots \times X_n$ be any monomorphism in $\mathbb{C}$ and the matrix

$$M' = \left[\begin{array}{ccc|c} x_{11} & \cdots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nm} & y_n \end{array}\right]$$

be a row-wise interpretation of the matrix M of type $(\hom(S, X_1), \dots, \hom(S, X_n))$. Viewing each of the left columns $c_1, \dots, c_m$ of M' and its right column y as morphisms $S \to X_1 \times \cdots \times X_n$, we suppose that $c_l \sqsubset r$ for each $l \in \{1, \dots, m\}$ and we must show

$y \sqsubset r$. For any $i, j \in \{1, \ldots, n\}$, consider the (regular epimorphism, monomorphism)-factorisation

$$R \xrightarrow{e_{i,j}} R_{i,j} \xrightarrow{m_{i,j}} X_i \times X_j$$

of the composite

$$R \xrightarrow{r} X_1 \times \cdots \times X_n \xrightarrow{\pi_{i,j}} X_i \times X_j.$$

For any such i, j, by Proposition 1.7 in [14], since M is non-trivial, the matrix $\mathsf{M}[i:, j:]$ formed from selecting the i^{th} and the j^{th} row of M is non-trivial. Hence, $\mathsf{Mal} \Rightarrow_{\mathsf{lex}} \mathsf{M}[i:, j:]$ by Proposition 2.3 and so $\mathsf{Ari} \Rightarrow_{\mathsf{lex}} \mathsf{M}[i:, j:]$. Since $\pi_{i,j}c_l \sqsubset m_{i,j}$ for each $l \in \{1, \ldots, m\}$, this implies that for any i, j there is a morphism $f_{i,j} \colon S \to R_{i,j}$ such that $m_{i,j}f_{i,j} = \pi_{i,j}y$. Considering the pullback

$$\begin{array}{ccc} Q_{i,j} & \xrightarrow{\beta_{i,j}} & R \\ {\scriptstyle \alpha_{i,j}}\downarrow & & \downarrow{\scriptstyle e_{i,j}} \\ S & \xrightarrow[f_{i,j}]{} & R_{i,j} \end{array}$$

we know that $\pi_{i,j}y\alpha_{i,j} \sqsubset \pi_{i,j}r$. Taking the limit of the diagram formed by the $\alpha_{i,j}$ produces a regular epimorphism $\alpha \colon Q \twoheadrightarrow S$ such that, for each $i, j \in \{1, \ldots, n\}$, we have $\pi_{i,j}y\alpha \sqsubset \pi_{i,j}r$. By Theorem 3.1 we have $\mathsf{Maj} \Leftrightarrow_{\mathsf{reg}} \mathsf{M}_n$, so that $\mathsf{Ari} \Rightarrow_{\mathsf{reg}} \mathsf{M}_n$. It follows that $y\alpha \sqsubset r$, and since α is a regular epimorphism and r a monomorphism, we have that $y \sqsubset r$. ∎

A regular Mal'tsev category $\mathbb{C}$ has been shown to have distributive lattices of equivalence relations if and only if it is a majority category [8]. Various alternative characterisations of *equivalence distributive* regular Mal'tsev categories have been given in [6], as well as equivalence distributive Goursat categories. By the theorem above, every regular equivalence distributive Mal'tsev category satisfies each non-trivial matrix property from Mclex.

3.5. Remark. Using Bourn localisations as in [9], one can easily extend Theorem 3.4 to the pointed context. That is, using the terminology of [9], for any non-trivial matrix $\mathsf{M} \in \mathsf{matr}_*$, one has $\mathsf{Ari} \Rightarrow_{\mathsf{reg}_*} \mathsf{M}$.

By definition, a matrix $\mathsf{M} \in \mathsf{matr}$ is trivial if any finitely complete category with M-closed relations is a preorder. In [9], it is proved that we can equivalently consider only finitely complete *pointed* categories to decide whether such a matrix is trivial, i.e., a matrix $\mathsf{M} \in \mathsf{matr}$ is trivial if and only if any finitely complete pointed category with M-closed relations is a preorder. As a corollary of Theorem 3.4, we prove that we can equivalently only consider varieties of universal algebras.

3.6. Corollary. *For a matrix* $\mathsf{M} \in \mathsf{matr}$*, the following statements are equivalent.*

(i) M *is trivial, i.e., any finitely complete category with* M*-closed relations is a preorder.*

(ii) Any pointed finitely complete category with M*-closed relations is a preorder.*

(iii) Any regular category with M*-closed relations is a preorder.*

(iv) Any variety with M*-closed relations is a preorder, i.e., the equation* $x = y$ *holds in its theory.*

(v) The dual of the category of sets $\mathsf{Set}^{\mathsf{op}}$ *does not have* M*-closed relations.*

(vi) The dual of the category of pointed sets $\mathsf{Set}_*^{\mathsf{op}}$ *does not have* M*-closed relations.*

(vii) The category of Boolean algebras Bool *does not have* M*-closed relations.*

PROOF. The equivalence (i)⇔(v) appears in [11] while the equivalences (i)⇔(ii)⇔(vi) appear in [9]. The implications (i)⇒(iii)⇒(iv) are obvious and the implication (iv)⇒(vii) follows from the fact that Bool is a variety of universal algebras which is not a preorder. Finally, the implication (vii)⇒(i) follows immediately from Theorem 3.4 since Bool is a regular arithmetical category. ■

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