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NAIVE HOMOTOPY THEORIES IN CARTESIAN CLOSED CATEGORIES

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Abstract. An elementary notion of homotopy can be introduced between arrows in a cartesian closed category $\mathcal{E}$. The input is henceforth called *connectedness structure*: a finite-product-preserving endofunctor Π_0 with a natural transformation p from the identity which is surjective on global elements. As expected, the output is a new category $\mathcal{E}_p$ with objects the same objects as $\mathcal{E}$.

Further assumptions on $\mathcal{E}$ provide a finer description of $\mathcal{E}_p$ that relates it to the classical homotopy theory where Π_0 could be interpreted as the "path-connected components" functor on convenient categories of topological spaces.

If $\mathcal{E}$ is a topos such that any non initial object has points and is furthermore assumed to be precohesive over a boolean base (as is the case for some classical models of Synthetic Differential Geometry), then there is an obvious choice of connectedness structure p. In this case, the passage from $\mathcal{E}$ to $\mathcal{E}_p$ is naturally described in terms of explicit homotopies—and so is the internal notion of contractible space. Furthermore, they coincide with the suggestions of Lawvere in his proposal for Axiomatic Cohesion.

1. Introduction

Lawvere [6] discusses the notion of contractible space in the process of deriving some consequences of his Axiomatic Cohesion. Therein, he suggests the existence of a homotopy category within the definition of his notion of extensive quality.

A guiding idea is definitely that of homotopies between continuous maps. A topological space is *contractible* if its identity map is homotopic to a constant map. Intuitively, the category of homotopy classes is thus defined to have the same objects and as arrows functions modulo homotopy, $[X, Y]$. For well-behaved topological spaces X, regarding the set $\pi_0(X)$ of path-connected components as a topological space with the discrete topology produces an endofunctor $X \mapsto \pi_0(X)$ together with a natural transformation $X \to \pi_0(X)$. One can obtain the following identification

$$[X, Y] \cong \pi_0(Y^X), \tag{1}$$

once Y^X is endowed with a canonical topology (e.g. the compact-open topology on sufficiently nice spaces). Conversely, one could begin with the natural transformation

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$1 \Rightarrow \pi_0$, and use (1) as the definition of the equivalence classes of arrows that are *homotopic* in some abstract sense.

Define a *connectedness structure* on a cartesian closed category $\mathcal{E}$ to be a natural transformation $p : 1_{\mathcal{E}} \Rightarrow \Pi_0 : \mathcal{E} \to \mathcal{E}$ such that the endofunctor Π_0 preserves products and the function

$$\mathcal{E}(1, p_X) : \mathcal{E}(1, X) \Rightarrow \mathcal{E}(1, \Pi_0(X))$$

is surjective for every $X \in \mathcal{E}$. In this case, let an object X be *connected* whenever $\Pi_0(X) = 1$ and *discrete* when p_X is an isomorphism. This notion of connectedness differs from another very natural one: in the context of an extensive category an object is connected if it has exactly two complemented subobjects. They do agree sometimes, e.g. when the category is (1) cartesian closed and extensive, (2) such that any non initial object has points, (3) the subcategory of decidable objects is an exponential ideal, and (4) the connectedness structure is given by the corresponding unit.

For every cartesian closed category, the identity functor is clearly a connectedness structure. If the category is further such that any non initial object has points, p is a connectedness structure as soon as it is epic (e.g. the trivial $!_X : X \to 1$ is a connectedness structure in this case).

Local operators j on toposes such that any non initial object has points produce more examples by considering the unit of the adjunction given by the exponential ideal of j-separated objects (see 3.12).

Associate the following homotopy theory to a connectedness structure p: First recall that the name of an arrow $f : X \to Y$ is the transpose 'f': $1 \to Y^X$ under the adjunction $(-) \times A \dashv (-)^A$ of the arrow $f \circ \pi_X : 1 \times X \to Y$. Now, any two arrows $f, g : X \to Y$ in $\mathcal{E}$ are *p-homotopic* to each other, and denoted by $f \sim_p g$, if and only if their names 'f','g' $: 1 \to Y^X$ satisfy

$$p_{Y^X} \circ \text{`}f\text{'} = p_{Y^X} \circ \text{`}g\text{'}.$$

1.1. THEOREM. *For a cartesian closed category $\mathcal{E}$ with a connectedness structure $p : 1_{\mathcal{E}} \Rightarrow \Pi_0 : \mathcal{E} \to \mathcal{E}$, there is a cartesian closed homotopy category $\mathcal{E}_p$ for $\mathcal{E}$, with an evident functor H_p from $\mathcal{E}$ to $\mathcal{E}_p$, as follows:* $\mathrm{Ob}(\mathcal{E}_p) := \mathrm{Ob}(\mathcal{E})$ *and*

$$\begin{aligned}\mathcal{E}_p(X, Y) &:= \mathcal{E}(X, Y)/{\sim_p} \\ &\cong \mathcal{E}(1, \Pi_0(Y^X)).\end{aligned}$$

If $\mathcal{E}$ is further assumed to be distributive and extensive, then so is $\mathcal{E}_p$.

The proof of this result is the content of Sections 3 and 4. By considering enriched categories, it is possible to obtain $\mathcal{E}_p$ directly, without passing through the homotopy equivalence relation between arrows: Every cartesian closed category—being symmetric monoidal closed—is canonically enriched over itself (see Borceux [2]).

Also, for any symmetric monoidal functor $F : V \to W$ and a V-category C, by considering the objects of C, and the arrows $F(C(a, b))$ one obtains a W-category $F_\bullet(C)$. This is in fact a 2-functor $F_\bullet : V\text{-}\mathbf{Cat} \to W\text{-}\mathbf{Cat}$.

In particular, for a cartesian closed category $\mathcal{E}$ with a connectedness structure $p : 1 \Rightarrow \Pi_0$, since the functor $\mathcal{E}(1, \Pi_0-)$ is symmetric monoidal, it follows that

$$\mathcal{E}_p = (\mathcal{E}(1, \Pi_0-))_\bullet(\mathcal{E}).$$

However, since we are interested in the homotopy equivalence relation between arrows and the structure induced in $\mathcal{E}_p$ by that of $\mathcal{E}$, the aforementioned method for producing $\mathcal{E}_p$ is not manifestly sufficient.

An object A in $\mathcal{E}$ is decidable if its diagonal is complemented in $A \times A$ and let $\mathrm{Dec}(\mathcal{E})$ be the full subcategory of decidable objects of $\mathcal{E}$. Whenever $\mathrm{Dec}(\mathcal{E})$ is an exponential ideal, its inclusion $\mathcal{I}$ has a left adjoint $\Pi \dashv \mathcal{I}$ that preserves products and the unit p of the adjunction is a good candidate for a connectedness structure. The surjectivity of $\mathcal{E}(1, p_X)$ would remain to be satisfied. Under the assumption that any non-initial object has points this is indeed the case.

1.2. Theorem. *For a cartesian closed category $\mathcal{E}$ in which any non-initial object has points and such that $\Pi \dashv \mathcal{I} : \mathrm{Dec}(\mathcal{E}) \to \mathcal{E}$ is an exponential ideal with epic unit p, there is an adjunction*

$$\begin{array}{c} \mathcal{E}_p \\ q_! \downarrow \dashv \uparrow q^* \\ \mathrm{Dec}(\mathcal{E}), \end{array} \tag{2}$$

where $q^* = H_p\mathcal{I}$ is fully faithful. Furthermore, $q_! H_p = \Pi$. In particular, $\mathrm{Dec}(\mathcal{E})$ is also an exponential ideal of $\mathcal{E}_p$.

The proof of this result is given in Section 5. If $\mathcal{E}$ is a topos, the epic quality of p in the previous theorem is automatic.

Marmolejo and Menni [8] provide a construction for a homotopy category in the context of precohesion. Lawvere [6] defines a cartesian closed category $\mathcal{E}$ to be (pre)-cohesive over a cartesian closed category $\mathcal{S}$ provided that there is a 4-string of adjunctions

$$\begin{array}{c} \mathcal{E} \\ f_! \dashv f^* \dashv f_* \dashv f^! \\ \mathcal{S} \end{array}$$

such that f^* is fully faithful, $f_!$ preserves finite products, and the counit of $f^* \dashv f_*$ is monic. Considering $f_!$ and f_* as symmetric monoidal functors, [8] observe that $(f_*)_\bullet(\mathcal{E})$ is simply $\mathcal{E}$—as an $\mathcal{S}$-category— and regard $(f_!)_\bullet(\mathcal{E})$ as a homotopy category with $(f_!)_\bullet(\mathcal{E})(X, Y) = f_!(Y^X)$. They prove among many other things that there is an adjunction

$$\begin{array}{c} (f_!)_\bullet(\mathcal{E}) \\ h_! \downarrow \dashv \uparrow h^* \\ \mathcal{S}, \end{array} \tag{3}$$

such that $h_!\theta = f_!$, where θ is induced by the natural transformation from f_* to $f_!$.

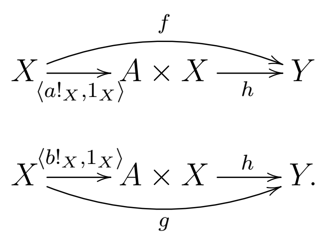
In the case when $\mathcal{E}$ is such that any non initial object has points and it is precohesive over $\mathrm{Dec}(\mathcal{E})$, setting $\Pi_0 = f^* f_!$ and p the unit of $f_! \dashv f^*$ yields that

$$\mathcal{E}_p(X, Y) = \mathrm{Dec}(\mathcal{E})(1, f_!(Y^X)).$$

From this it follows that in the case $\mathcal{S} = \mathrm{Dec}(\mathcal{E}) = \mathbf{Set}$, adjunctions (2) and (3) are equivalent. However, the fact that $\mathrm{Dec}(\mathcal{E})$ need not be $\mathbf{Set}$, nor the functor $\mathcal{E}(1, \Pi_0-)$ be part of the adjunction $f_! \dashv f^*$—or not in principle—, further establishes a difference of goals and proceedings from those of Marmolejo and Menni [8, §9].

In the topological context, being homotopic is given explicitly by way of homotopies, i.e. continuous maps from the domain times an interval to the codomain. In general, the relationship $\sim_p$ does not explicitly have such a description. One direction is always true, provided that one substitutes "an interval" with "a connected object". This replacement is intuitively natural recalling that Π_0 is meant to abstract "path-connected components".

1.3. Theorem. *For any cartesian closed $\mathcal{E}$ with a connectedness structure p, two arrows $f, g : X \to Y$ are p-homotopic if there is a connected object A with two global elements $a, b : 1 \to A$ and an arrow $h : A \times X \to Y$ such that the following diagrams commute:*



Furthermore, in the case when $\mathcal{E}$ is a topos such that non initial objects have points and $\mathrm{Dec}(\mathcal{E})$ is an exponential ideal of $\mathcal{E}$, the converse holds for the induced connectedness structure.

The proof of this result is given in 6.3 and 7.1.

For a connectedness structure p, an object A is *p-contractible* whenever the identity map 1_A is p-homotopic to the constant map $a!$ for some point $a : 1 \to A$.

1.4. Theorem. *Let p be a connectedness structure on a cartesian closed category $\mathcal{E}$. Let $A \in \mathcal{E}$ and for any X let $\sigma_X^A : X \to X^A$ be the transpose under the adjunction $(-) \times A \dashv (-)^A$ of the projection $X \times A \to X$. The following are equivalent.*

1. *The object A is p-contractible.*
2. *For every object $X \in \mathcal{E}$, $\Pi_0(\sigma_X^A) : \Pi_0(X) \to \Pi_0(X^A)$ is an isomorphism.*

Moreover, they are also equivalent to the following when $\mathcal{E}$ is a topos such that non initial objects have points, $\mathrm{Dec}(\mathcal{E})$ is an exponential ideal that is also a topos, and p is the associated connectedness structure.

3. *For every* $X \in \mathcal{E}$, A^X *is connected.*

4. *The object* A *has a point and* A^A *is connected.*

This is proved in 6.4, 7.2 and 7.4. Lawvere [6] defines an object to be *contractible* provided it satisfies the property (3) in the previous theorem. Therefore, in said context, an object is *Lawvere contractible* if and only if it is p-contractible.

In Section 8, we further explore the application of these constructions in the context of specific cases of precohesion.

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2. Basic definitions and preliminaries

A category $\mathcal{E}$ is *cartesian closed* if it has finite products $\Pi_i Y_i$ and for every object A in $\mathcal{E}$ there is an adjunction

$$(\,\cdot\,) \times A \dashv (\,\cdot\,)^A.$$

Both transpositions will be denoted by the same symbol, i.e. if $f : X \times A \to Y$ and $g : X \to Y^A$ are transposes of each other under the bijection, then both g and f can be denoted by $\widehat{f}$ or $\widehat{g}$, respectively. The evaluation map

$$\mathrm{ev}_X^A := \widehat{1_{X^A}} : X^A \times A \to X$$

is the counit of this adjunction. Angle brackets $\langle f_1, \ldots, f_n\rangle$ denote the unique arrow induced into a finite product given arrows $f_1, \ldots, f_n$ from a common domain X, and when $n = 0$ the unique arrow into the terminal object 1 is denoted by $! =!_X$.

Given an arrow $\varphi : A \to B$ there is a natural transformation $(\,\cdot\,)^\varphi$ from $(\,\cdot\,)^B$ to $(\,\cdot\,)^A$ such that for every object X,

$$\mathrm{ev}_X^A \circ (X^\varphi \times 1_A) = \mathrm{ev}_X^B(1_{X^B} \times \varphi).$$

Let $\sigma_X^A : X \to X^A$ denote the transpose of $\pi_X : X \times A \to X$. It is natural in X. In the case $A = 1$, $\sigma_X^1 : X \to X^1$ is an isomorphism, and its inverse is

$$X^1 \xrightarrow[\pi_{X^1}^{-1}]{} X^1 \times 1 \xrightarrow[\mathrm{ev}_X^1]{} X.$$

If $X, A \in \mathcal{E}$, then $\sigma_X^A : X \to X^A$ is naturally isomorphic to $X^{!_A} : X^1 \to X^A$ in the sense that

$$X^{!_A} \circ \sigma_X^1 = \sigma_X^A. \tag{4}$$

The name 'f': $1 \to Y^X$ of an arrow $f : X \to Y$ is the transpose of the arrow $f \circ \pi_X : 1 \times X \to Y$. The maps σ assign to any point the name of the corresponding constant function: If $a : 1 \to A$ is a point of $A \in \mathcal{E}$ and B is arbitrary, then

$$\sigma_A^B \circ a = \text{`}a\circ !_B\text{'}. \tag{5}$$

For a point $t : 1 \to T$ and an object $X \in \mathcal{E}$, define $\mathrm{ev}_X^t : X^T \to X$ as the composite

$$X^T \xrightarrow{\langle 1_{X^T}, !_{X^T}\rangle} X^T \times 1 \xrightarrow{1\times t} X^T \times T \xrightarrow{\mathrm{ev}_X^T} X. \tag{6}$$

It is naturally isomorphic to $X^t : X^T \to X^1$:

$$X^t = \sigma_X^1 \circ \mathrm{ev}_X^t, \tag{7}$$

which justifies X^t being considered as—or even called—evaluation at t. Also, since $1 = ! \circ t$,

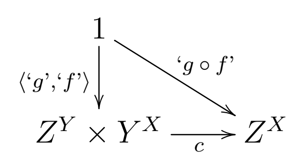
$$\mathrm{ev}_X^t \circ \sigma_X^T = 1, \tag{8}$$

which proves that σ_X^T is split monic.

The internal composition $c : Z^Y \times Y^X \to Z^X$ is the transpose of

$$(Z^Y \times Y^X) \times X \xrightarrow{\cong} Z^Y \times (Y^X \times X) \xrightarrow{1\times \mathrm{ev}} Z^Y \times Y \xrightarrow{\mathrm{ev}} Z,$$

and makes the following diagram commute:



(9)

for every arrow $f : X \to Y$ and every arrow $g : Y \to Z$ in $\mathcal{E}$ (see McLarty [10, Section 6.3]). If $f : X \to Y$ and $g : Y \to Z$ are arrows in $\mathcal{E}$, then

$$g^X \circ \text{`}f\text{'} = \text{`}g \circ f\text{'}, \tag{10}$$

and

$$Z^f \circ \text{`}g\text{'} = \text{`}g \circ f\text{'}. \tag{11}$$

For a category with finite coproducts $\Sigma_i X_i$, curly brackets $\{f_1, \dots, f_n\}$ denote the unique arrow induced from a finite sum given arrows $f_1, \dots, f_n$ into a common codomain Y, and when $n = 0$ the unique arrow from the initial 0 is also denoted by $! = !_Y$.

The following two definitions could be studied independently, yet for the purposes of this report they will be considered within the context of cartesian closed categories.

A category $\mathcal{E}$ is *distributive* if it has both finite products and finite coproducts and the natural map

$$X \times Y + X \times Z \longrightarrow X \times (Y + Z)$$

is an isomorphism. Evidently, a cartesian closed category $\mathcal{E}$ is distributive as soon as it has finite coproducts. Furthermore, in that case the natural $\alpha : Z^{X+Y} \to Z^X \times Z^Y$ is an isomorphism for every $X, Y, Z \in \mathcal{E}$, with

$$\alpha \circ \hat{f} = \langle \widehat{f_1}, \widehat{f_2} \rangle \tag{12}$$

for any $f : A \times (X + Y) \to Z$, where the arrows f_1 and f_2 are defined by the following commutative diagram:

$$\begin{array}{ccc} A \times X + A \times Y & & \\ \cong\downarrow & \searrow^{\{f_1, f_2\}} & \\ A \times (X + Y) & \xrightarrow[f]{} & Z. \end{array}$$

A category $\mathcal{E}$ is *extensive* if it has finite coproducts and the canonical functor

$$+ : \mathcal{E}/X \times \mathcal{E}/Y \to \mathcal{E}/(X + Y)$$

$$\begin{array}{ccccc} A & B & & A + B \\ f\downarrow & g\downarrow & \mapsto & \downarrow f+g \\ X, & Y & & X + Y \end{array} \tag{13}$$

is an equivalence for every pair of objects $X, Y \in \mathcal{E}$ (See the work of Carboni, Lack, and Walters [3] for further equivalent definitions).

3. Main definitions, first consequences and examples

3.1. Definition. *A* connectedness structure *on a cartesian closed category $\mathcal{E}$ is a natural transformation $p : 1_{\mathcal{E}} \Rightarrow \Pi_0 : \mathcal{E} \to \mathcal{E}$ such that the functor Π_0 preserves finite products and the function $\mathcal{E}(1, p_X) : \mathcal{E}(1, X) \Rightarrow \mathcal{E}(1, \Pi_0(X))$ is surjective for every $X \in \mathcal{E}$.*

In particular, it follows that the next diagram commutes:

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_{X \times Y}} & \Pi_0(X \times Y) \\ & \searrow_{p_X \times p_Y} & \downarrow\cong \\ & & \Pi_0(X) \times \Pi_0(Y). \end{array} \tag{14}$$

3.2. Definition. *A* connected object *X is one for which $\Pi_0(X)$ is terminal. An object X is* discrete *if p_X is an isomorphism.*

3.3. Remark. *Epic images of connected objects are connected. Indeed, if $f : A \to B$ is an epic arrow in $\mathcal{E}$ with $\Pi_0(A) = 1$, then, as Π_0 preserves epics since it is left adjoint, $\Pi_0 f$ is epic and monic and, accordingly, iso; that is, in that case $\Pi_0(B) \cong 1$.*

3.4. Definition. *Two arrows $f, g : X \to Y$ in $\mathcal{E}$ are* homotopic *to each other if and only if their names ‘f’, ‘g’ $: 1 \to Y^X$ satisfy*

$$p_{Y^X} \circ \text{‘}f\text{’} = p_{Y^X} \circ \text{‘}g\text{’}.$$

Denote this equivalence relation by $\sim$.

3.5. Remark. *Notice that for a connected object, any two points are homotopic.*

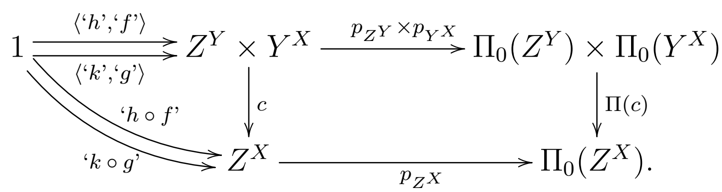
3.6. Remark. *Notice that if the category is such that any non initial object has points and if p is epic, then $\mathcal{E}(1, p)$ is automatically epic.*

3.7. Definition. *Define the homotopy category $\mathcal{E}_p$ for $\mathcal{E}$ as follows:* $\mathrm{Ob}(\mathcal{E}_p) := \mathrm{Ob}(\mathcal{E})$ *and*

$$\mathcal{E}_p(X, Y) := \mathcal{E}(X, Y)/\sim \tag{15}$$

Composition is defined as the equivalence class of the composition of representatives.

To see that this is indeed well defined, let $f, g \in \mathcal{E}(X, Y)$ with $f \sim g$ and $h, k \in \mathcal{E}(Y, Z)$ with $h \sim k$. Then, by (9) and (14), the following diagram commutes:



Hence $h \circ f \sim k \circ g$.

Notice that $\mathcal{E}(X, Y)/\sim\ \cong \mathcal{E}(1, Y^X)/\sim'$, with ‘$f$’ $\sim'$ ‘g’ if and only if $p_{Y^X} \circ$‘f’ $= p_{Y^X} \circ$‘g’. So one has a bijective correspondence

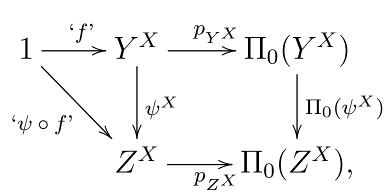
$$\mathcal{E}_p(X, Y) \cong \mathcal{E}(1, \Pi_0(Y^X)). \tag{16}$$

Under this identification, the class $[f]$ of an arrow f corresponds to p_{Y^X}(‘f’).

3.8. Proposition. *Let $\mathcal{E}$ be a cartesian closed category with a connectedness structure $p : 1 \Rightarrow \Pi_0$. Under the identification given by (16), the following natural transformations agree for arbitrary arrows φ and ψ in $\mathcal{E}$:*

$$\mathcal{E}_p(\varphi, Y) \cong \mathcal{E}(1, \Pi_0(Y^\varphi)) \qquad \mathcal{E}_p(X, \psi) \cong \mathcal{E}(1, \Pi_0(\psi^X)), \tag{17}$$

Proof. Fix $\psi : Y \to Z$ and let $f : X \to Y$ be an arbitrary arrow in $\mathcal{E}$. By equation (10), ‘$\psi \circ f$’ $= \psi^X \circ$ ‘f’, so that the following diagram commutes by the naturality of p:



and thus

$$[\psi \circ f] = \Pi_0(\psi^X) \circ [f]. \tag{18}$$

Analogously, fix $\varphi : W \to X$ and let $f : X \to Y$ be an arbitrary arrow in $\mathcal{E}$. By equation (11), '$f \circ \varphi$' $= Y^\varphi \circ$ 'f' and thus, by naturality of p,

$$[f \circ \varphi] = \Pi_0(Y^\varphi) \circ [f]. \tag{19}$$

Naturality follows from $Z^\varphi \circ \psi^X = \psi^W \circ Y^\varphi$ together with the functoriality of Π_0. ■

3.9. Example. *Let $\mathcal{E}$ be a cartesian closed category. The identity $1 : 1_\mathcal{E} \Rightarrow 1_\mathcal{E}$ is a connectedness structure.*

3.10. Example. *If $\mathcal{E}$ has an initial object and is such that any non initial object has points, then $! : 1_\mathcal{E} \Rightarrow (-)^0$ is a connectedness structure.*

3.11. Example. *For a fixed object A, $\sigma^A : 1 \to (\cdot)^A$ might not be a connectedness structure when $A \ncong 1$.*

3.12. Example. *Each local operator in a topos such that any non initial object has points gives rise to a canonical connectedness structure: Let $\mathrm{Sep}_j(\mathcal{E})$ be the full subcategory of j-separated objects[1] of a local operator j. It is reflective, an exponential ideal (A4.3.1, A4.4.3, A4.4.4 in Johnstone [4])) and closed under subobjects (Proposition 5.9 in Bell [1]). The corresponding unit $p : 1 \to \Pi_0$ is epic and Π_0 preserves finite products. By 3.6 it is a connectedness structure.*

4. Categorical properties of the homotopy category

Now that it has been established that $\mathcal{E}_p$ is a category, the main purpose of this section is to finish the proof of 1.1: That $\mathcal{E}_p$ inherits the properties of being cartesian closed, extensive and distributive.

4.1. Proposition. *If $\mathcal{E}$ is a cartesian closed category with a connectedness structure, then $\mathcal{E}_p$ has finite products.*

Proof. The terminal object of $\mathcal{E}$ is clearly also terminal in $\mathcal{E}_p$. Now, for binary products, let $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ be a product diagram in $\mathcal{E}$. It follows that

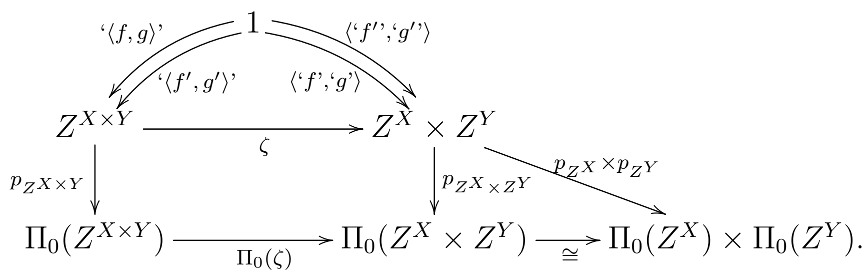
$$X \xleftarrow{[\pi_X]} X \times Y \xrightarrow{[\pi_Y]} Y$$

is a product diagram in $\mathcal{E}_p$. To see this, let $f, f' : Z \to X$ and $g, g' : Z \to Y$ and let ζ be the isomorphism $(X \times Y)^Z \cong X^Z \times Y^Z$. In general, for an arrow $\langle r_1, r_2 \rangle : A \times Z \to X \times Y$, it follows that $\langle \widehat{r_1}, \widehat{r_2} \rangle = \zeta \circ \widehat{\langle r_1, r_2 \rangle}$. In particular,

$$\langle \text{`}f\text{'}, \text{`}g\text{'} \rangle = \zeta \circ \text{`}\langle f, g \rangle\text{'}.$$

[1]An example of separated objects will be analyzed in Section 8.

Whence, the top triangles in the following diagram commute. The rest of the diagram commutes by the naturality of p and because Π_0 preserves finite products.



Therefore $\langle ‘f’, ‘g’\rangle \sim \langle ‘f'’, ‘g'’\rangle$ if and only if $‘\langle f,g\rangle’ \sim ‘\langle f',g'\rangle’$, and thus for arrows $[f] : Z \to X$ and $[g] : Z \to Y$ in $\mathcal{E}_p$, the required unique arrow is $[\langle f, g\rangle] : Z \to X \times Y$. ■

4.2. Proposition. *If $\mathcal{E}$ is a cartesian closed category with a connectedness structure, then $\mathcal{E}_p$ has exponentials.*

Proof. Fix an object A in $\mathcal{E}_p$. In order to have an adjunction

$$(\,\cdot\,) \times A \dashv (\,\cdot\,)^A : \mathcal{E}_p \to \mathcal{E}_p$$

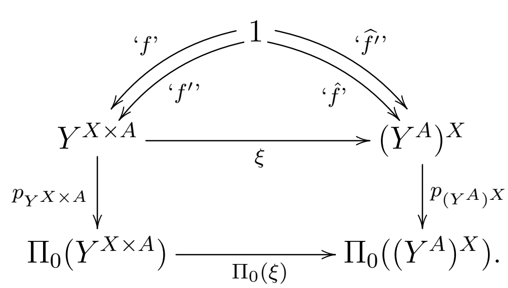
with Y^A the same object as in $\mathcal{E}$ for an arbitrary object Y in $\mathcal{E}_p$, it suffices to verify that there is a universal arrow $\varepsilon : Y^A \times A \to Y$ from $(\,\cdot\,) \times A$ to Y in $\mathcal{E}_p$.

To this effect, let $f, f' : X \times A \to Y$ in $\mathcal{E}$, and let ξ be the isomorphism $Y^{X\times A} \cong (Y^A)^X$ in $\mathcal{E}$. So

$$‘\hat{f}’ = \xi \circ ‘f’ \tag{20}$$

since in general, for an arrow $r : X \times A \times B \to Y$, one has that $\widehat{\hat{r}} = \xi \circ \widehat{r}$, where on the left-hand side of the equation the inner $\hat{-}$ is with respect to B and the exterior $\widehat{-}$ with respect to A, and on the right-hand side of the equation the $\widehat{-}$ is with respect to $A \times B$.

Thus, by (20), the top triangles in the following diagram commute. The rest of the diagram commutes by the naturality of p.



Therefore $f \sim f'$ if and only if $\hat{f} \sim \widehat{f'}$, and $\varepsilon = [ev^A_X]$ is the required universal arrow. ■

4.3. Proposition. *Let $\mathcal{E}$ be a cartesian closed category with a connectedness structure. If $\mathcal{E}$ is distributive then $\mathcal{E}_p$ is also distributive.*

Proof. Since, by 4.1 and 4.2, $\mathcal{E}_p$ is already cartesian closed, it will be distributive as soon as it has finite sums. The initial object of $\mathcal{E}$ is clearly also initial in $\mathcal{E}_p$.

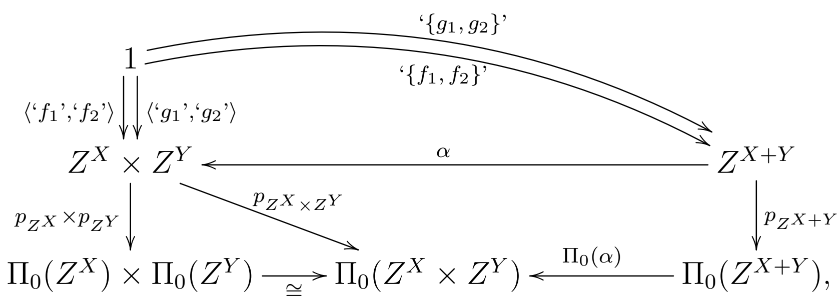
Now, for binary coproducts, let $X \xrightarrow{i_X} X+Y \xleftarrow{i_Y} Y$ be a coproduct diagram in $\mathcal{E}$. It follows that

$$X \xrightarrow{[i_X]} X+Y \xleftarrow{[i_Y]} Y$$

is a coproduct diagram in $\mathcal{E}_p$. To see this, let $f_1, g_1 : X \to Z$ and $f_2, g_2 : Y \to Z$ in $\mathcal{E}$ and α the isomorphism of (12). In particular,

$$\begin{aligned} \langle `f_1\text{'}, `f_2\text{'}\rangle &= \alpha \circ `\{f_1, f_2\}\text{'} \\ \langle `g_1\text{'}, `g_2\text{'}\rangle &= \alpha \circ `\{g_1, g_2\}\text{'}. \end{aligned} \tag{21}$$

Hence the following diagram commutes by (21), the naturality of p, and the finite product preservation:



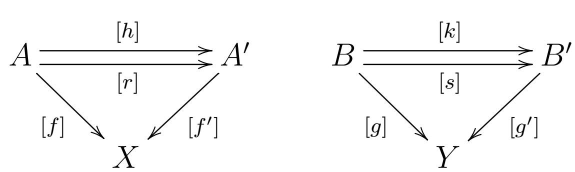
Wherefore, $f_1 \sim g_1$ and $f_2 \sim g_2$ if and only if $\{f_1, f_2\} \sim \{g_1, g_2\}$. Thus for arrows $[f] : X \to Z$ and $[g] : Y \to Z$ in $\mathcal{E}_p$, the required unique arrow is $[\{f, g\}] : X + Y \to Z$. ■

4.4. Proposition. *Let $\mathcal{E}$ be a cartesian closed category with a connectedness structure. If $\mathcal{E}$ is extensive then so is $\mathcal{E}_p$.*

Proof. Recall that $\mathcal{E}$ is extensive if and only if the functor in (13) is an equivalence for every pair of objects $X, Y \in \mathcal{E}$.

So consider the corresponding functor $+_p : \mathcal{E}_p/X \times \mathcal{E}_p/Y \to \mathcal{E}_p/(X+Y)$. To see that it is an equivalence one can verify that it is essentially surjective on objects, full, and faithful. Since $+$ is an equivalence, $+_p$ is clearly essentially surjective on objects and full.

To see that it is faithful, let $X \xrightarrow{i_X} X+Y \xleftarrow{i_Y} Y$ be a coproduct diagram in $\mathcal{E}$ and let $h, r : A \to A'$ and $k, s : B \to B'$ be arrows such that the following diagrams commute:



Suppose that $[h] +_p [k] = [r] +_p [s]$. By 4.3, this means that $[h + k] = [r + s]$ and

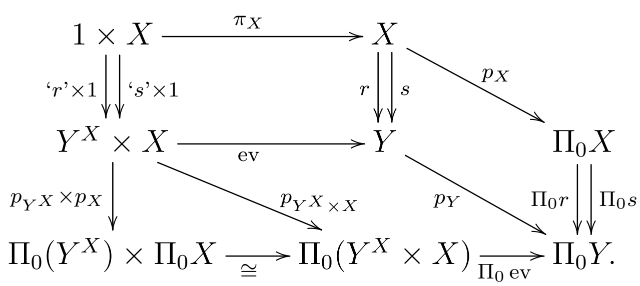
$$[i_X] \circ [h] = [i_X] \circ [r] \qquad \text{and} \qquad [i_Y] \circ [k] = [i_Y] \circ [s].$$

As $\mathcal{E}_p$ is distributive (again by 4.3), the coproduct injections are monic (see [3, Proposition 3.3]). Therefore $[h] = [r]$ and $[k] = [s]$. Thus proving that $+_p$ is faithful and that it is an equivalence of categories. ■

5. Reflectivity of the subcategory of decidables under homotopy

5.1. Proposition. *Let $\mathcal{E}$ be cartesian closed category in which any non-initial object has points and such that $\Pi \dashv \mathcal{I} : \mathrm{Dec}(\mathcal{E}) \to \mathcal{E}$ is an exponential ideal with epic unit $p : 1 \Rightarrow \Pi_0 := \mathcal{I}\Pi$. Then $\Pi_0 r = \Pi_0 s$ for any two $r, s : X \to Y$ p-homotopic arrows in $\mathcal{E}$. If, furthermore, Y is decidable, then $r = s$.*

Proof. Notice that the following diagram commutes:

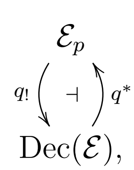


Since by assumption p_X is epic and thus so is $p_X \circ \pi_X$, it follows that $\Pi_0 r = \Pi_0 s$.

Lastly, since p_Y is an isomorphism for Y decidable, $\Pi_0 r = \Pi_0 s$ if and only if $r = s$. ■

5.2. Definition. *Let H_p be the following assignment, which is evidently functorial (see 3.7 and the following paragraph). To each object X in $\mathcal{E}$, let $H_p(X) = X$ in $\mathcal{E}_p$, and to each arrow f in $\mathcal{E}$, let $H_p(f) = [f]$.*

5.3. Theorem. [Re-statement of 1.2] *For a cartesian closed category $\mathcal{E}$ in which any non-initial object has points and such that $\Pi \dashv \mathcal{I} : \mathrm{Dec}(\mathcal{E}) \to \mathcal{E}$ is an exponential ideal with epic unit p, there is an adjunction*



(22)

where $q^* = H_p\mathcal{I}$ is fully faithful. Furthermore, $q_! H_p = \Pi$. In particular, $\mathrm{Dec}(\mathcal{E})$ is also an exponential ideal of $\mathcal{E}_p$.

Proof. Let $q^* := H_p\mathcal{I}$. By 5.1, it is full and faithful. To see that there is an adjunction

$$q_! \dashv q^* : \mathcal{E}_p \to \mathrm{Dec}(\mathcal{E})$$

it suffices to find, for each X in $\mathcal{E}_p$, a decidable object $q_!(X)$ and a universal arrow $\eta_X : X \to q^*q_!(X)$ universal from X to q^* in $\mathcal{E}_p$.

Since one must also have $\Pi = q_! H_p$ it follows that the object part of $q_!$ should be that of Π. To see that $\eta_X := [p_X] : X \to q^*\Pi(X)$ is such universal arrow, let $[f] : X \to q^*(A)$. By 5.1 it follows that $f' \sim f$ if and only if $f' = f$. Now, since p_X is universal from X to $\mathcal{I}$, there exists a unique arrow $g : \Pi(X) \to A$ in $\mathrm{Dec}(\mathcal{E})$ such that $f = \mathcal{I}(g) \circ p_X$. Therefore

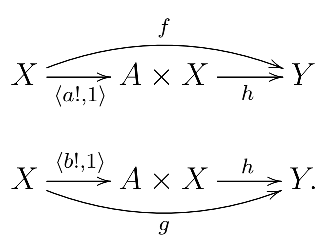
$$[f] = H_p(f) = H_P(\mathcal{I}(g) \circ p_X) = H_p(\mathcal{I}(g)) \circ H_p(p_X) = q^*(g) \circ [p_X],$$

and the conclusion follows. ∎

6. Explicit homotopies and contractibility in CCC

Two notions are fundamentally associated with the concept of homotopy theory: one is the concept of homotopy between maps and the other one is that of a space being contractible. At this generality not much more than the definition can be said, yet as advertised by the first parts of 1.3 and 1.4, they are consistent with one's intuition. This is verified in 6.3 and 6.4 below.

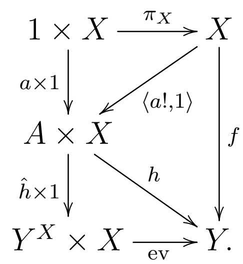
6.1. Definition. *A* p-homotopy *(or simply* a homotopy*) between two arrows f and g in a cartesian closed category $\mathcal{E}$ with a connectedness structure $p : 1 \Rightarrow \Pi_0$ is an arrow $h : A \times X \to Y$ where A is connected, $\Pi_0(A) = 1$, for which there are two points $a, b : 1 \to A$ such that*



6.2. Definition. *An object A is said to be* p-contractible *if it has a point $a : 1 \to A$ such that $a! \sim 1_A$.*

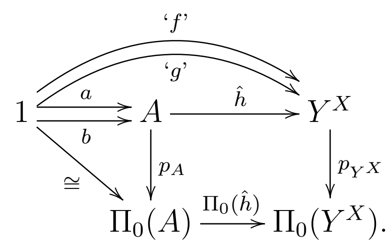
6.3. Theorem. *Let $\mathcal{E}$ be a cartesian closed category with a connectedness structure $p : 1 \Rightarrow \Pi_0$. If there is a homotopy between f and g, then f and g are homotopic.*

Proof. Let $a, b : 1 \to A$ and $h : A \times X \to Y$ be as required. The following diagram commutes by 6.1 and by the definition of transpose:



Hence $\hat{h}a = $ ‘f’. Similarly, $\hat{h}b = $ ‘g’.

By the connectedness of A and the naturality of p, the following diagram commutes:



This proves that $f \sim g$, as required. ∎

6.4. Theorem. *Let $\mathcal{E}$ be a cartesian closed category with a connectedness structure $p : 1 \Rightarrow \Pi_0$, and let $A \in \mathcal{E}$. Then A is p-contractible if and only if, for every object $X \in \mathcal{E}$,*

$$\Pi_0(\sigma_X^A) : \Pi_0(X) \to \Pi_0(X^A)$$

is an isomorphism.

Proof. By (4), it is enough to verify the claim for $\Pi_0(X^!)$. By 3.8, $\Pi_0(X^!)$ is an isomorphism if and only if

$$\mathcal{E}_p(1, X) \xrightarrow{\mathcal{E}_p(!,X)} \mathcal{E}_p(A, X)$$

is a natural isomorphism. By Yoneda, this is equivalent to $A \cong_{\mathcal{E}_p} 1$, i.e. that there is a point $a : 1 \to A$ of A such that $a! \sim 1_A$. ∎

7. Explicit homotopies and contractibility in toposes

This section completes the proofs of 1.3 and 1.4 through the following two theorems.

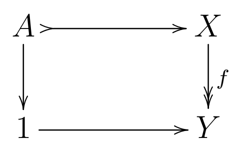
7.1. Theorem. *Let $\mathcal{E}$ be a topos in which any non-initial object has points and such that $\Pi \dashv \mathcal{I} : \mathrm{Dec}(\mathcal{E}) \to \mathcal{E}$ is an exponential ideal with unit $p : 1 \to \Pi_0 := \mathcal{I}\Pi$. If $f \sim g : X \to Y$ then there is a bipointed connected object A and a homotopy $h : A \times X \to Y$ between them.*

7.2. Theorem. *Let $\mathcal{E}$ be a topos such that non initial objects have points,* $\mathrm{Dec}(\mathcal{E})$ *is an exponential ideal that is also a topos, and p is the associated connectedness structure. Let $A \in \mathcal{E}$. Then*

$$\Pi_0(A^X) = 1$$

for every $X \in \mathcal{E}$ if and only if there is a point $a : 1 \to A$ such that $a! \sim 1_A$. That is, $\Pi_0(A^X) = 1$ for every $X \in \mathcal{E}$ if and only if A is contractible under the associated connectedness structure p for $\mathcal{E}$.

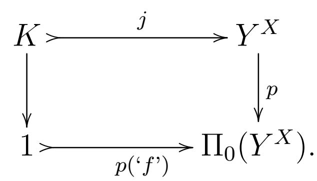
7.3. Remark. *Ruiz-Hernández and Solórzano [12] proved that if $\mathcal{E}$ is a topos in which any non-initial object has points, then for any pullback diagram of the form*



the pullback object A has only two complemented subobjects, provided the epic arrow f satisfies a technical condition—called "to have pneumoconnected fibers*"—([12, 1.4]).*

It is also proved therein that if $\mathrm{Dec}(\mathcal{E})$ *is an exponential ideal, with left adjoint Π, then A has only two complemented subobjects if and only if $\Pi(A) = 1$ ([12, 1.1]). Lastly, it is also proved that the unit p of $\Pi \dashv \mathcal{I}$ has pneumoconnected fibers ([12, 2.1]).*

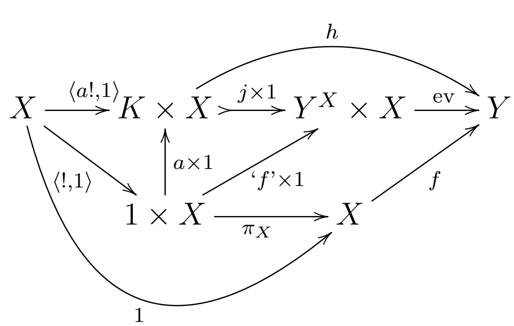
Proof of 7.1. Let $f, g : X \to Y$ be two homotopic arrows in $\mathcal{E}$. Let K be the pullback



Thus $\Pi(K) = 1$ by 1.4, 2.1, and 2.3 in [12]. Now, let $a : 1 \to K$ and $b : 1 \to K$ be the corestrictions of 'f' $: 1 \to Y^X$ and 'g' $: 1 \to Y^X$ to K, resp. Let $h : K \times X \to Y$ be the following composite:

$$K \times X \xrightarrow{j \times 1} Y^X \times X \xrightarrow{\mathrm{ev}} Y.$$

The following diagram commutes:



A similar diagram commutes for g. ∎

PROOF OF 7.2. Suppose $\Pi_0(A^X) = 1$ for every $X \in \mathcal{E}$. Hence, by definition,

$$\begin{aligned} \mathcal{E}_p(X, A) &\cong \mathcal{E}(1, \Pi_0(A^X)) \\ &\cong \mathcal{E}(1, 1) \\ &= \mathcal{E}_p(X, 1). \end{aligned}$$

Since $\mathcal{E}(X, 1) = 1$, that isomorphism is natural. Therefore, by Yoneda, $A \cong_{\mathcal{E}_p} 1$. That is, there is a point $a : 1 \to A$ of A such that $a! \sim 1_A$.

Conversely, suppose A is contractible; that is, there is a point $a : 1 \to A$ in A such that $a!_A \sim 1_A$. Hence

$$1 \xrightarrow{a} A \xrightarrow{\sigma_A^X} A^X$$

is a point of A^X. Now, let $g : X \to A$ be an arbitrary arrow in $\mathcal{E}$. So, by hypothesis, $g \sim a!_A g = a!_X$. Hence $1 = \mathcal{E}_p(X, A) = \mathcal{E}(1, \Pi_0(A^X))$. Therefore $\Pi_0(A^X)$ has just one point.

The assumption that $\mathrm{Dec}(\mathcal{E})$ be a topos has not been used thus far. The statements of (2.3) and (2.4) in [12] gives an equivalence for $\mathrm{Dec}\,\mathcal{E}$ to be a topos in this setting: the images of $\neg\neg$-dense arrows under Π are epic. In this case, this implies that the unique point of $\Pi(A^X)$ is $\neg\neg$-dense and thus already epic, i.e. $\Pi_0(A^X) = 1$. ∎

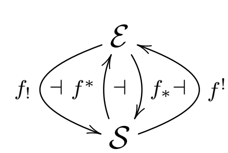
7.4. COROLLARY. *Let $\mathcal{E}$ be a topos such that non initial objects have points,* $\mathrm{Dec}(\mathcal{E})$ *is an exponential ideal that is also a topos, and p is the associated connectedness structure. Let A be an object of $\mathcal{E}$ with a point. Then* $\Pi_0(A^A) = 1$ *if and only if A has a point $a : 1 \to A$ such that $a! \sim 1_A$.*

PROOF. By 7.2, if A has a point $a : 1 \to A$ such that $a! \sim 1_A$, then $\Pi_0(A^X) = 1$; in particular, $\Pi_0(A^A) = 1$. Conversely, if $\Pi_0(A^A) = 1$ and A has a point $b : 1 \to A$, then by (16), $\mathcal{E}_p(A, A) = 1$ and thus $b! \sim 1_A$. ∎

7.5. REMARK. *For a nondegenerate topos $\mathcal{E}$,* $\Pi(0^0) = 0^0 = 1$ *yet 0 has no points. This justifies the requirement that the object have a point for it to be contractible.*

8. Applications in the setting of Axiomatic Cohesion

Lawvere [6] introduces the notion of axiomatic cohesion. A topos $\mathcal{E}$ is *precohesive* over a topos $\mathcal{S}$ if there is a string of adjunctions



(23)

such that f^* is fully faithful, $f_!$ preserves finite products, and that the counit

$$\gamma : \Gamma \equiv f^* f_* \to 1$$

is monic (See also Lawvere and Menni [5, Lemma 3.2]).

Along with the definition and its main consequences, Lawvere [6] provides several guidelines to obtain the notions of connected space, connected components, explicit homotopies and a notion of contractible space. Once the unit

$$p : 1 \to \Pi_0 \equiv f^* f_!$$

is a connectedness structure, one recovers Lawvere's proposals—this is so in the cases studied herein. His definition of contractible space is that of a space A such that for any other space X, the space A^X is required to be connected. A context where this coincides with the definition of 6.2 is given by 7.2.

8.1. Example. *Lawvere [6] states within a proof that any object X with a pointed action of a connected monoid with zero $(M, 0, 1)$ is necessarily contractible. To make this explicit consider a pointed object X with point $x_0 : 1 \to X$ with an action*

$$M \times X \xrightarrow{\mu} X$$

such that $\mu \circ \langle 1!_X, 1_X \rangle = 1_X$ and $\mu \circ \langle 0!_X, 1_X \rangle = x_0!_X$. It is now evident that μ is the required homotopy between 1_X and $x_0!_X$. In particular, for M itself, it follows that M is connected if and only if it is contractible.

Lawvere also proposes the following two descriptions of toposes of cohesion: (1) A precohesive topos is *sufficiently cohesive* iff any object can be embedded into a contractible space; and (2) A precohesive topos is a *quality type* iff the "point-to-pieces" natural transformation $\theta : f_* \to f_!$, defined implicitly by

$$f^*\theta = p \cdot \gamma,$$

is an isomorphism (it is already an epimorphism, see [5]).

8.2. Remark. *Let $f_! \dashv f^* \dashv f_* \dashv f^! : \mathcal{E} \to \mathcal{S}$ be a quality type such that $p : 1 \to f^* f_!$ is a connectedness structure. Then two arrows are p-homotopic, if and only if they are equal. Indeed, let $h, g : X \to Y$ be two arrows in $\mathcal{E}$. The following diagram commutes:*

$$\begin{array}{ccccc}
1 & \overset{\text{‘}h\text{’}}{\underset{\text{‘}g\text{’}}{\rightrightarrows}} & Y^X & \xrightarrow{p_{Y^X}} & \Pi_0(Y^X) \\
\uparrow {\scriptstyle \gamma_1} \cong & & \uparrow {\scriptstyle \gamma_{Y^X}} & \swarrow {\scriptstyle f^*\theta^{-1}} & \\
\Gamma(1) & \longrightarrow & \Gamma(Y^X). & &
\end{array}$$

Therefore, ‘h’ $\circ\, \gamma_1$ and ‘g’ $\circ\, \gamma_1$ are both transposes of the same arrow under the adjunction $f^* \dashv f_*$, and the conclusion follows.

In Proposition 4, Lawvere [6] proves that being sufficiently cohesive is equivalent among other things to having a connected (and thus contractible by 8.1) subobject classifier or to the existence of a so-called strictly bi-pointed connected object—in the context of this report, this means simply that it has at least two global elements.

Menni shows that the negative of either being sufficiently cohesive or being a quality type imply the other for toposes of presheaves precohesive over sets, thus yielding a dichotomy in this case in the presence of a classical metalogic. The following result is a slight generalization.

8.3. THEOREM. *Let $\mathcal{E}$ be a topos such that non initial objects have points, precohesive over a boolean base $\mathcal{S}$. If $\mathcal{E}$ is not a quality type, then it is sufficiently cohesive.*

The requirement that non initial objects have points renders the topos 2-valued (the truth-value object Ω has exactly two points: $\mathrm{Sub}(1) \cong \mathcal{E}(1, \Omega)$) and being precohesive over a boolean $\mathcal{S}$ is equivalent to being precohesive over $\mathrm{Dec}(\mathcal{E})$; in this case $\mathcal{S}$ is equivalent to $\mathrm{Dec}(\mathcal{E})$ (See Menni [11], and also [12]).

PROOF OF 8.3. The "points-to-pieces" morphism $\theta_X : f_*X \to f_!X$ is monic in $\mathcal{S}$ if and only if $f^*\theta_X$ is monic in $\mathcal{E}$ since f^* is right adjoint to $f_!$. Therefore $f^*\theta_X$ is not monic in $\mathcal{E}$ if and only if θ is not monic in $\mathcal{S}$.

Suppose $\mathcal{E}$ is not a quality type; i.e., θ is not a natural isomorphism. Hence $f^*\theta_X$ is not monic in $\mathcal{E}$ for some $X \in \mathcal{E}$ and, accordingly, θ_X not monic in $\mathcal{S}$.

The internal language condition for θ_X to be monic is

$$\forall x \forall x'(x \in f_*X \wedge x' \in f_*X \wedge \theta_X(x) = \theta_X(x') \Rightarrow x = x')$$

(see Exercise VI.10 p. 344 in the textbook of Mac Lane and Moerdijk [7]).

Therefore,

$$\neg\forall x \forall x'(x \in f_*X \wedge x' \in f_*X \wedge \theta_X(x) = \theta_X(x') \Rightarrow x = x')$$

is universally valid in $\mathcal{S}$. But since $\mathcal{S}$ is boolean,

$$\exists x \exists x'(x \in f_*X \wedge x' \in f_*X \wedge \theta_X(x) = \theta_X(x') \wedge \neg(x = x'))$$

is universally valid in $\mathcal{S}$. This means that the subobject

$$\{\langle x, x' \rangle : x \in f_*X \wedge x' \in f_*X \wedge \theta_X(x) = \theta_X(x') \wedge \neg(x = x')\}$$

of $f_*X \times f_*X$ is not initial in $\mathcal{S}$. Since $\mathcal{S}$ is also such that non initial objects have points, there are global elements $a, b : 1 \to f_*X$ such that

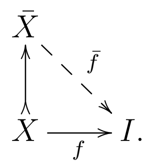
$$a \in f_*X \wedge b \in f_*X \wedge \theta_X(a) = \theta_X(b) \wedge \neg(a = b)$$

is universally valid in $\mathcal{S}$. This means that $\theta_X \circ a$ is equal to $\theta_X \circ b$. Whence, for the corresponding points $\bar{a} = \gamma a, \bar{b} = \gamma b : 1 \to X$, $p_X \circ \bar{a}$ equals $p_X \circ \bar{b}$, and $p_X(\bar{a}) = \theta_X(a)$. Thus,

$$\bar{a}, \bar{b} \in K := p_X^{-1}(p_X(\bar{a})).$$

It follows that K is connected (see 7.3) and bipointed. So $\mathcal{E}$ is sufficiently cohesive. ∎

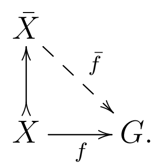
Lawvere proves that having sufficient cohesion is also equivalent to injective objects being connected. An object I is *injective* if and only if for any monic $m : X \to \bar{X}$ and any arrow $f : X \to I$ there exists $\bar{f} : \bar{X} \to I$ such that the following diagram commutes:



It is straightforward to verify that $f^!$ preserves injective objects. Therefore,

8.4. Remark. *If $\mathcal{E}$ is a topos such that non initial objects have points and is precohesive over a boolean base topos $\mathcal{S}$, sufficient cohesion is equivalent to requiring that for every object I in $\mathcal{S}$, $f^!I$ be connected as soon as it be non initial.*

An object G is a $\neg\neg$-sheaf (resp. $\neg\neg$-separated) if and only if for any $\neg\neg$-dense monic $X \to \bar{X}$ and any arrow $f : X \to G$ there exists a unique (resp. at most one) $\bar{f} : \bar{X} \to G$ such that the following diagram commutes:



Denote by $\mathcal{E}_{\neg\neg}$ the full subcategory of $\neg\neg$-sheaves with inclusion functor $J : \mathcal{E}_{\neg\neg} \to \mathcal{E}$. Since it is known to be a subtopos, let $L_0 \dashv J$ the corresponding adjunction with unit $l : 1 \Rightarrow L \equiv JL_0$. The functor L_0 is known as the $\neg\neg$-sheafification functor.

For every object I in $\mathcal{S}$, it is proved in McLarty [9] (see also [5]) that

$$f^! \cong Lf^*.$$

8.5. Proposition. *Let $\mathcal{E}$ be a topos such that non initial objects have points and is precohesive over a boolean base topos $\mathcal{S}$. There is sufficient cohesion in $\mathcal{E}$ if and only if every non initial $\neg\neg$-sheaf is connected.*

Proof. Since for every X, γ_X is $\neg\neg$-dense and monic,

$$L\gamma_X : L\Gamma X \to LX$$

is an isomorphism (see the remark in [1, p. 188] before Corollary 5.27). Therefore,

$$L \cong f^! f_*,$$

wherefrom every $\neg\neg$-sheaf is isomorphic to $f^!I$ for an object I in $\mathcal{S}$. By 8.4 the promised equivalence is obtained. ∎

The following result gives a sufficient condition for a sheaf to be furthermore contractible in this context.

8.6. Theorem. *Let $\mathcal{E}$ be a topos such that non initial objects have points and precohesive over a boolean topos $\mathcal{S}$. If $X \in \mathcal{E}$ is contractible, then so is its sheafification LX.*

8.7. Lemma. *Let $\mathcal{E}$ be a topos such that non initial objects have points and precohesive over a boolean topos $\mathcal{S}$. Given arrows $h : K \times X \to Y$ and $f : Y \to LX$, there is a unique arrow $h' : K \times LX \to LX$ making the following diagram commute:*

$$\begin{array}{ccc} K \times X & \xrightarrow{\ h\ } & Y \\ {\scriptstyle 1\times l_X}\downarrow & & \downarrow{\scriptstyle f} \\ K \times LX & \overset{h'}{\dashrightarrow} & LX. \end{array} \tag{24}$$

Proof. By the naturality of l, the following diagram commutes:

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\ \gamma_X\ } & X \\ {\scriptstyle l_{\Gamma X}}\downarrow & & \downarrow{\scriptstyle l_X} \\ L\Gamma X & \xrightarrow[L\gamma_X]{} & LX. \end{array}$$

As already argued in the proof of 8.5, $L\gamma_X$ is an isomorphism. Now, as ΓX is decidable, it is also $\neg\neg$-separated and thus $l_{\Gamma X}$ is $\neg\neg$-dense monic (see [1, Proposition 5.20]). Therefore $l_X \circ \gamma_X$ is $\neg\neg$-dense monic. Hence, since LX is a $\neg\neg$-sheaf and $(1 \times l_X) \circ (\gamma_K \times \gamma_X) : \Gamma K \times \Gamma X \to K \times LX$ is $\neg\neg$-dense monic, there is a unique $h' : K \times LX \to LX$ making the following diagram commute:

$$\begin{array}{ccc} \Gamma K \times \Gamma X & & \\ {\scriptstyle \gamma_K\times\gamma_X}\downarrow & & \\ K \times X & \xrightarrow{\ h\ } & Y \\ {\scriptstyle 1\times l_X}\downarrow & & \downarrow{\scriptstyle f} \\ K \times LX & \underset{h'}{\dashrightarrow} & LX \end{array}$$

Lastly, since $\gamma_K \times \gamma_X$ is $\neg\neg$-dense monic and LX is a $\neg\neg$-sheaf, (24) commutes. ∎

Proof of 8.6. Let $X \in \mathcal{E}$ be a contractible object. So, by 7.1, there is a connected object K with two points a, b and a homotopy $h : K \times X \to X$ such that

$$X \xrightarrow[\langle a!,1\rangle]{} K \times X \xrightarrow[h]{} X \quad \text{equals} \quad X \xrightarrow{\ 1\ } X$$

$$X \xrightarrow{\langle b!,1\rangle} K \times X \xrightarrow{\ h\ } X \quad \text{equals} \quad X \xrightarrow[c!]{} X.$$

for some point c of X. Now, applying 8.7 to h and l_X, there is a unique $h' : K \times LX \to LX$ such that the following diagram commutes:

$$\begin{array}{ccc} K\times X & \xrightarrow{h} & X \\ {\scriptstyle 1\times l_X}\big\downarrow & & \big\downarrow{\scriptstyle l_X} \\ K\times LX & \xrightarrow{h'} & LX. \end{array} \tag{25}$$

Whence the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\langle a!_X,1\rangle"'] \arrow[rr, bend left, "1"] \arrow[d, "l_X"'] & K\times X \arrow[r, "h"'] \arrow[d, "1\times l_X"'] & X \arrow[d, "l_X"] \\ LX \arrow[r, "\langle a!_{LX},1\rangle"] \arrow[rr, bend right, "1"'] & K\times LX \arrow[r, "h'"] & LX. \end{tikzcd}$$

By the universality of l_X,

$$\begin{tikzcd} LX \arrow[r, "\langle a!,1\rangle"] \arrow[rr, bend right, "1"'] & K\times LX \arrow[r, "h'"] & LX \end{tikzcd}$$

commutes.

Now, again, by the commutativity of (25), the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\langle b!,1\rangle"'] \arrow[rr, bend left, "c!"] \arrow[d, "l_X"'] & K\times X \arrow[r, "h"'] \arrow[d, "1\times l_X"'] & X \arrow[d, "l_X"] \\ LX \arrow[r, "\langle b!,1\rangle"] \arrow[rr, bend right, "L(c!)=L(c)!"'] & K\times LX \arrow[r, "h'"] & LX. \end{tikzcd}$$

By the universality of l_X,

$$\begin{tikzcd} LX \arrow[r, "\langle b!,1\rangle"] \arrow[rr, bend right, "L(c)!"'] & K\times LX \arrow[r, "h'"] & LX \end{tikzcd}$$

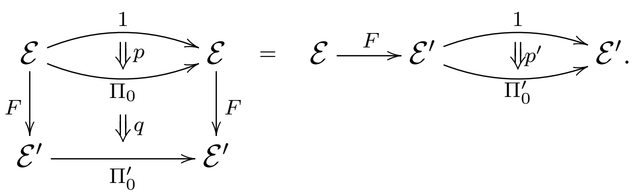
commutes. Therefore, by 6.3, $L(c)! \sim 1_{LX}$ and by 6.2 LX is contractible. ∎

9. Final Thoughts

Even though actual examples of CCC of topological spaces are frustratingly hard to come by, the results proved herein suggest that the topological notions postulated within the context of Axiomatic Cohesion (even without the Axiom of Continuity) do recover some deep truth about cohesion, albeit reasoning within an intuitionistic logical framework.

The notion of homotopy theory postulated in this report differs from the classical axioms in that no reference is made towards its construction—it is thus a synthetic investigation. However, the fact that for any local operator on a topos that further non initial objects have points there is a canonical homotopy theory associated with it suggests yet another connection to classical theories. This direction might still provide useful insights.

From the purely categorical viewpoint, and for the sake of completeness, the following definition is proposed: Given two categories $(\mathcal{E}, p), (\mathcal{E}', p')$ with corresponding connectedness structures, a morphism $(\mathcal{E}, p) \to (\mathcal{E}', p')$ between them is a pair (F, q) where $F : \mathcal{E} \to \mathcal{E}'$ is a cartesian closed functor and $q : F\Pi_0 \Rightarrow \Pi_0' F$ a natural transformation such that



That is, $q \cdot Fp = p'F$. It is clear that the identity morphism is $(1, 1)$.

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