

$(\infty, 1)$ -CATEGORICAL COMPREHENSION SCHEMES

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ABSTRACT. We define and study notions of comprehension in $(\infty, 1)$ -category theory. In essence, we do so by implementing Bénabou’s foundations of naive category theory in a univalent meta-theory. In particular, we develop natural generalizations of smallness and relative definability in this context, and show for instance that the universal cartesian fibration is small. Furthermore, by building on Johnstone’s notion of comprehension schemes for ordinary fibered categories, we characterize and relate numerous higher categorical properties and structures such as left exactness, local cartesian closedness, univalent morphisms and internal $(\infty, 1)$ -categories in terms of comprehension schemes.

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1. Introduction

COMPREHENSION À LA BÉNABOU. Comprehension schemes arose as crucial notions in the early work on the foundations of set theory, and hence found expression in a large variety of foundational settings for mathematics. Particularly, they have been introduced to the context of categorical logic first by Lawvere and then by Bénabou in the 1970s. Since, they have been studied in different forms and have been applied to many examples throughout the literature of category theory. The notion of a comprehension scheme as used in this paper is ultimately rooted in the Axiom scheme of Restricted Comprehension

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(often referred to as the Axiom scheme of Separation) which is part of the Zermelo-Fraenkel axiomatization of set theory. The scheme states that every definable subclass of a set is again a set. Or in other words, that all sets in the set theoretic universe satisfy all set theoretical comprehension schemes.

In his critique of the foundations of naive category theory [8], Bénabou provided an intuition to define the notion of a comprehension scheme not only in category theory (over the topos of sets), but in category theory over any other category in a syntax-free way. In this generality, comprehension schemes become properties of Grothendieck fibrations over arbitrary categories. Which particular comprehension schemes are satisfied by a given Grothendieck fibration $\mathcal{E} \rightarrow \mathcal{C}$ then depends on the categorical constructions available in \mathcal{E} over \mathcal{C} and in \mathcal{C} itself. We point out that Bénabou in his work purposefully did not even specify what exactly a “category” is [8, Paragraph 0.5], and that, on the basis of his effectively meta-meta-mathematical analysis, the framework of $(\infty, 1)$ -category theory over the base ∞ -topos of spaces playing the role of “sets” appears to yield a suitable notion of such a category theory in his sense. This will be justified implicitly by the notions recalled in Section 2 together with the constructions and results of this paper. In this reading of “sets” as spaces one then is to replace the occurrences of “ZF” in [8] by a suitable univalent type theory. By virtue of univalence of the universes \mathbb{U} in the meta-theory of spaces ([24] and [8, Paragraph 6.2]), we argue that the theory of fibered $(\infty, 1)$ -categories is in fact a model of Bénabou’s theory of “categories” where equality of objects is well-behaved and intrinsic (rather than “extra structure, which a “category” p may or may not admit” [8, Paragraph 8.9.1]). This is instantiated by the fact that the difference between functors and pseudo-functors and hence the difference between cartesian fibrations and split cartesian fibrations vanishes. We will explain this in more detail in the coming paragraphs.

A style of definition for (what we will call “diagrammatic”) comprehension schemes which tie together the elementary examples given in the glossary of [8] has been introduced by Johnstone in [21, Section B1.3]. Subject to a few technical adjustments, these comprehension schemes give rise to a very well-behaved theory in the $(\infty, 1)$ -categorical context which we will introduce in Section 3. For instance, just as all sets in the set theoretic universe under ZF satisfy all set theoretic comprehension schemes, we will see that all $(\infty, 1)$ -categories internal to any complete (left exact) universe \mathcal{C} of discrete $(\infty, 1)$ -categories satisfy all (finite) diagrammatic comprehension schemes (see, Corollary 5.26 and Corollary 5.18). Among all fibered $(\infty, 1)$ -categories over \mathcal{C} they in fact are characterized by this property, as internalizability (i.e. “smallness”) is a set of comprehension schemes. While Bénabou himself derived his motivation of the notion of comprehension at least in part directly from set theory (i.e. in the case of “relative definability” [8, Paragraphs 6.2, 7.1, 7.4]), we will give a motivation of the notion via the categorical semantics of type theory and its underlying Lawvere-style notion of comprehension [26] instead. This is not due to personal preference, but, first, due to the fact that a categorical model of type theory is virtually the same thing as a general comprehension scheme on a category in the role of a universe as far as this paper is concerned, and second, due to the

fact that the meta-theory of spaces from an $(\infty, 1)$ -categorical point of view is currently best expressed in type theoretic terms [24].

Therefore, first, let us recall some of the intuition about “categories” laid out in [8] which we will apply to the theories of both ordinary and higher (fibered) categories. Thus, we think of a “category” \mathcal{C} as an $((\infty, 1)$ -category fibered over the $((\infty, 1)$ -category Cat of small $((\infty, 1)$ -categories by way of its associated fibration $\text{Diag}(\mathcal{C}) \twoheadrightarrow \text{Cat}$ of diagrams. This fibration is defined as the Grothendieck construction of the exponential $\text{Fun}(\cdot, \mathcal{C}): \text{Cat}^{op} \rightarrow \text{Cat}$. When restricted to the category $\iota: \mathcal{S} \hookrightarrow \text{Cat}$ of discrete categories, it returns the classic fibration-of-families construction $\text{Fam}(\mathcal{C}) \rightarrow \mathcal{S}$. When postcomposed with the underlying discrete category construction $(\cdot)^{\simeq}: \text{Cat} \rightarrow \mathcal{S}$, it returns the representable presheaf of diagrams $y(\mathcal{C}): \text{Cat}^{op} \rightarrow \mathcal{S}$. Both functors $\text{Fam} := \text{Fun}(\iota(\cdot), -): \text{Cat} \rightarrow \text{Fun}(\mathcal{S}^{op}, \text{Cat})$ and $y = \text{Fun}(\cdot, -)^{\simeq}: \text{Cat} \rightarrow \text{Fun}(\text{Cat}^{op}, \mathcal{S})$ are fully faithful and preserve both limits and exponentials. Indeed, for fully faithfulness of the former in the $(\infty, 1)$ -categorical case see Example 5.12; for continuity, exponentials and $(\infty, 2)$ -categorical aspects see [46]. In this sense, both category theory indexed over discrete categories as well as discrete category theory indexed over categories are faithful generalizations of category theory, where we consider \mathcal{S} as a universe for discrete category theory and Cat as a universe for category theory. Indeed, category theory is always a theory of categories in context (of some meta-theory). This can be exemplified by changing the base of the indexing we consider from \mathcal{S} or Cat to any category \mathcal{C} in the role of a universe for category theory. Thereby, if we think of a given category \mathcal{C} as a synthetic universe of discrete categories, we think of fibrations over \mathcal{C} – or \mathcal{C} -indexed categories, respectively – as categories in context of \mathcal{C} . The indexed category $\text{Fam}(\mathcal{C}): \mathcal{S}^{op} \rightarrow \text{Cat}$ for a category \mathcal{C} (internal to \mathcal{S}) is replaced by the externalization $\text{Ext}(X): \mathcal{C}^{op} \rightarrow \text{Cat}$ of a category X internal to \mathcal{C} ([20, Section 7.3] and Section 5). The (large) indexed category of families $\text{Fam}(\mathcal{S}) \simeq \mathcal{S}_{/(\cdot)}: \mathcal{S}^{op} \rightarrow \text{Cat}$ associated to the large category \mathcal{S} itself is replaced by the canonical indexing $\mathcal{C}_{/(\cdot)}: \mathcal{C}^{op} \rightarrow \text{Cat}$ whenever \mathcal{C} has pullbacks. If we think of a given category \mathcal{C} as a synthetic universe of categories instead, then the presheaf of diagrams $\text{Fun}(\cdot, \mathcal{C})^{\simeq}: \text{Cat}^{op} \rightarrow \mathcal{S}$ associated to a category $\mathcal{C} \in \text{Cat}$ is replaced by the representable presheaves of the form $y(C): \mathcal{C}^{op} \rightarrow \mathcal{S}$ for objects $C \in \mathcal{C}$.

Furthermore, it is useful to think of the objects in the base category \mathcal{C} themselves as contexts, too, as well as of a general indexed category $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}$ as a similarly generalized doctrine of types in varying contexts in \mathcal{C} . Therefore, we recall that to make a category of contexts \mathcal{C} together with a family $\text{Ty}: \mathcal{C}^{op} \rightarrow \mathcal{S}$ of types in context into an actual categorical model of type theory, one essentially only further requires a notion of context extension in \mathcal{C} with respect to Ty . Such an associated notion of context extension then automatically gives rise to a notion of typed terms in context as well by considering sections of context extensions in \mathcal{C} . In ordinary category theory, a notion of context extension can be expressed in various (but essentially equivalent) terms, e.g. by way of full comprehension categories [19], categories with attributes [13, 29] or categories with families [15]. One further such notion in indexed terms is given by Awodey’s natural models [4]; that is, a category \mathcal{C} together with a representable natural transformation $\text{Tm} \rightarrow \text{Ty}$

in $\hat{\mathcal{C}} = \text{Fun}(\mathcal{C}^{op}, \mathcal{S})$ as originally introduced by Grothendieck (see Definition 2.7). Such a natural transformation equips \mathcal{C} for every $C \in \mathcal{C}$ and every type $A \in \text{Ty}(C)$ with a context $C.A \in \mathcal{C}_{/C}$ which represents the type A in context C in a suitable fashion. This particular abstract expression of a “categorical type theory” has been studied axiomatically in ($\infty, 1$)-category theory by Nguyen and Uemura in [31].

1.1. DEFINITION. *Let \mathcal{C} be a category. A general comprehension scheme on \mathcal{C} is a natural transformation $f: X \rightarrow Y$ in $\hat{\mathcal{C}}$. We say that \mathcal{C} has f -comprehension if f is representable.*

In this paper we study various classes of instances of this particular notion of comprehension. It may be worth to point out here however that, in contrast to the ordinary categorical situation, the structure of a general comprehension scheme over an ($\infty, 1$)-category \mathcal{C} is provably richer than that of a full comprehension ($\infty, 1$)-category over \mathcal{C} (when defined accordingly, see Proposition 6.6 and Remark 6.7).

Now, every indexed category $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}$ naturally induces a Cat -indexed collection of general comprehension schemes on \mathcal{C} by way of its accordingly indexed presheaf of diagrams. We will call these the diagrammatic comprehension schemes associated to \mathcal{E} . More precisely, the \mathcal{C} -indexed category $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}$ (after a choice of an equivalent splitting in the ordinary case) induces the \mathcal{C} -indexed “generalized category”

$$\begin{aligned} y \circ \mathcal{E}: \mathcal{C}^{op} &\rightarrow \text{Fun}(\text{Cat}^{op}, \mathcal{S}) \\ C &\mapsto \text{Fun}(\cdot, \mathcal{E}(C))^{\simeq}, \end{aligned}$$

which we will consider as a functor $[[\cdot, \mathcal{E}]]: \text{Cat}^{op} \rightarrow \text{Fun}(\mathcal{C}^{op}, \mathcal{S})$ by Currying.

1.2. DEFINITION. *Let \mathcal{C} be a category and let \mathcal{E} be a \mathcal{C} -indexed category. A (standard) diagrammatic comprehension scheme on \mathcal{E} over \mathcal{C} is a functor $G: I \rightarrow J$ of categories together with the induced general comprehension scheme*

$$G^*: [[J, \mathcal{E}]] \rightarrow [[I, \mathcal{E}]]$$

in $\hat{\mathcal{C}}$. We say that \mathcal{E} has G -comprehension if \mathcal{C} has G^* -comprehension as in Definition 1.1.

With Definition 1.2 in mind, we think of each presheaf $[[I, \mathcal{E}]] = \text{Cat}(I, \mathcal{E}(\cdot))$ as a family of types over \mathcal{C} . Among the most basic such families will be the type family $\mathcal{E}(\cdot)^{\simeq}$ of objects in \mathcal{E} and the type family $\text{Cat}(\Delta^1, \mathcal{E}(\cdot))$ of morphisms in \mathcal{E} . For every functor $G: I \rightarrow J$ of categories – which we think of as a morphism of categorical shapes and which often will be an extension of such – satisfaction of the according diagrammatic comprehension scheme requires the existence of a context extension operator in \mathcal{C} which for a context $C \in \mathcal{C}$ and a type $F: I \rightarrow \mathcal{E}(C)$ assigns an object $C.F \in \mathcal{C}_{/C}$ which represents diagram extensions of F along G in \mathcal{E} in a suitable way. In most of the paper we will be interested in specifically these kinds of diagrammatic comprehension schemes. However we will define diagrammatic comprehension schemes more generally for contravariant functors of the form $\mathcal{E}: \mathcal{C}^{op} \rightarrow K$ valued in categories K potentially different from Cat (Definition 3.4). This additional generality ensures that diagrammatic

comprehension schemes are capable to capture additional algebraic structures which a given \mathcal{C} -indexed category \mathcal{E} may come equipped with. This for example subsumes all instances of Johnstone’s “generalized” diagrammatic comprehension schemes [21, Section B1.3]. This additional generality also has been considered conceptually at least in part already by Bénabou himself in his definition of K -fibrations respective a corpus K [6, 7]. Apart from a few examples however, we will consider general comprehension schemes which go beyond Definition 1.2 only in Section 6 to briefly discuss their general theory and address definability in this context.

The expressive power of (generalized) diagrammatic comprehension schemes encompasses important examples in ordinary fibered category theory, most notably necessary and sufficient conditions to characterize elementary toposes (over other elementary toposes), see [8, Paragraph 11.6.(iii)] and [50, Theorem 11.1]. We will see in the course of the paper that it does so in the context of fibered $(\infty, 1)$ -category theory as well (see the outline of main results below). In fact, we will see that diagrammatic comprehension schemes are a more expressive and more natural notion in $(\infty, 1)$ -category theory than they are in ordinary category theory, for the reason that meta-theoretical equalities which exceed the language of formal category theory – and which yet still naturally arise in the context of ordinary category theory all the time – will intrinsically be replaced by instances of equivalences between $(\infty, 1)$ -categories. For instance, commutativity of squares in $(\infty, 1)$ -categories is again a matter of equivalence between $(\infty, 1)$ -categories rather than of set-theoretical equality. This eliminates the gymnastics with elementary fibrations in [50], and also affects the remarks on the “strangeness” of equality in [8, Section 8]. In this sense, the study of equality becomes a study of equivalence, which is exactly due to the fact that the meta-theory satisfies the Univalence Axiom. This difference to ordinary category theory is in practice exemplified by Bénabou’s comments about “equality of objects” which in ordinary fibered category theory is instantiated by the generally non-existent, non-canonical and in particular non-unique choice of a splitting of a fibration [8, Paragraph 9.4]. In $(\infty, 1)$ -category theory, as composition on all levels itself is only defined up to equivalence, a splitting only can be defined in homotopy-coherent terms. However, every cartesian fibration is split homotopy-coherently in an essentially unique way. This means that $(\infty, 1)$ -categories in fact do have equality (i.e. equivalence) of objects in the sense of [8, Paragraphs 8.6 and 8.7]. Indeed we will see that all locally small fibered $(\infty, 1)$ -categories over a left exact base satisfy all of the identity principles in [8, Paragraph 8.7] which are there said to be unnatural and hence implausible for a general “category” (in a non-univalent meta-theory) to satisfy (Corollary 3.34).

The main sources for the ordinary categorical constructions and definitions we have adapted are Johnstone [21], Jacobs [19, 20] and Streicher [50]. The results presented in this paper build to a large extent on the fundamental work on quasi-categories provided by Lurie in [27] and, ultimately, by Joyal in [22]. An exhaustive treatment of fibered $(\infty, 1)$ -categories with a scope comparable to [21, Chapter B1] or [50] in the ordinary categorical context would naturally require $(\infty, 2)$ -categorical considerations. These however are barely touched upon in this paper because the notions studied here are properties of one

indexed $(\infty, 1)$ -category at a time. The collection of all \mathcal{C} -indexed $(\infty, 1)$ -categories is only introduced in Section 2 to state its equivalence to the collection of fibered $(\infty, 1)$ -categories over \mathcal{C} via Lurie’s unstraightening construction, and therefore it is defined as an $(\infty, 1)$ -category itself. Partial treatments of fibered $(\infty, 1)$ -category theory in such generality can be found e.g. in [5, 27, 39].

OUTLINE OF THE PAPER AND MAIN RESULTS. The results of this paper can be understood as a proof that fibered $(\infty, 1)$ -category theory is a (univalent model of) “category” theory in the sense of Bénabou [8], and that it in particular subsumes all positive statements about definability discussed in the paper and more.

We will develop the notion of comprehension schemes in the language of indexed quasi-categories.¹ We do so because the basic technical notions underlying the definition of comprehension schemes are more straightforward here than they are for fibered quasi-categories, and the proofs are easier to read (following suggestions of a referee). We will however give an equivalent formulation of comprehension entirely in the language of fibered quasi-categories in Section 4. This latter formulation is not only more faithful to Lawvere’s, Bénabou’s and Johnstone’s original frameworks, but it will also provide an environment which allows to generalize the notion to ∞ -cosmoses of not necessarily $(\infty, 1)$ -categories in the sense of [39], as well as to give a straightforward proof of the fact that the notions and results in this paper are independent of the specific choice of model for $(\infty, 1)$ -category theory (Section 8).

In Section 2 we recall the necessary material about cartesian fibrations from Lurie’s book [27]. We introduce the basic notions relevant to define and study comprehension schemes for fibered $(\infty, 1)$ -categories, give reference to the equivalence between indexed $(\infty, 1)$ -category theory and fibered $(\infty, 1)$ -category theory via the corresponding Grothendieck construction (“unstraightening”), and discuss the most essential examples of indexed $(\infty, 1)$ -categories and their fibered counterparts. We recall the notion of a representable natural transformation and prove all statements about such that will be used in our applications in the subsequent sections.

Section 3 introduces the definition of comprehension schemes and discusses many examples like global smallness, local smallness and definability of equivalences between objects and parallel pairs of n -morphisms for different cartesian fibrations. We provide tools for mutual reduction and verification of various instances (Lemma 3.28, Lemma 3.35), and apply them to prove an interplay between the most fundamental examples (Proposition 3.31). Some of these instances are generalizations of notions and results from [21, Section B1.3], some are entirely new. In Section 4 we give an equivalent formulation of diagrammatic comprehension schemes (for the base case of $K = \text{Cat}_\infty$) entirely in the language of fibered $(\infty, 1)$ -category theory.

4.6. PROPOSITION. *Let $G: I \rightarrow J$ be a map of simplicial sets and $p: \mathcal{E} \rightarrow \mathcal{C}$ be a cartesian*

¹Possibly much to the dismay of Bénabou himself according to [8, Paragraph 12]. One may argue however that the coherence issues he lamented to arise in indexed category theory do not arise in the same way in $(\infty, 1)$ -category theory.

fibration. The fibration p has G -comprehension if and only if for every vertical diagram $X \in \llbracket I, \mathcal{E} \rrbracket$ the $(\infty, 1)$ -category $G^ \downarrow X$ of vertical J -structures extending X horizontally has a terminal object.*

The following are some of the fundamental instances of comprehension that are introduced in Section 3 when formulated for various fibrations via Section 4.

1.3. EXAMPLES.

- A cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ over an $(\infty, 1)$ -category \mathcal{C} with finite products is globally small if and only if its core \mathcal{E}^\times has a terminal object (Example 3.19).
- The canonical fibration $t: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$ associated to a small $(\infty, 1)$ -category \mathcal{C} with pullbacks is locally small if and only if \mathcal{C} is locally cartesian closed (Proposition 3.24).
- The representable right fibrations $\mathcal{C}/_C \rightarrow \mathcal{C}$ are locally small if and only if \mathcal{C} has equalizers. They are always globally small whenever \mathcal{C} has finite products (Example 3.25).
- Any $(\infty, 1)$ -category \mathcal{C} considered as a cartesian fibration over the point is globally small if and only if its core $\mathcal{C}^\times \simeq \mathcal{C}^\simeq$ is contractible. It is locally small if and only if its hom-spaces are contractible (Example 3.26).
- Given a (potentially large and locally large) $(\infty, 1)$ -category \mathcal{C} , its fibration of families $\text{Fam}(\mathcal{C}) \rightarrow \mathcal{S}$ is (locally) small if and only if the $(\infty, 1)$ -category \mathcal{C} is a (locally) small $(\infty, 1)$ -category (Example 3.27).
- Given a cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ and an $(\infty, 1)$ -category \mathcal{D} with pullbacks together with a left adjoint functor $F: \mathcal{D} \rightarrow \mathcal{C}$, we have $\text{Comp}(F^*p) \subseteq \text{Comp}(p)$ (Lemma 3.35).
- The universal cartesian fibration $\pi^{op}: \text{Dat}_\infty^{op} \rightarrow \text{Cat}_\infty^{op}$ and the universal right fibration $\mathcal{S}_*^{op} \rightarrow \mathcal{S}^{op}$ have G -comprehension for every functor $G: \mathcal{I} \rightarrow \mathcal{J}$ between small $(\infty, 1)$ -categories \mathcal{I}, \mathcal{J} (Example 3.22).

In Section 5 we define the externalization construction

$$\text{Ext}: \text{CS}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty) \tag{1}$$

of $(\infty, 1)$ -categories internal to an $(\infty, 1)$ -category \mathcal{C} with pullbacks. The functor (1) is essentially given as a pushforward of the Yoneda embedding of $y: \mathcal{C} \rightarrow \hat{\mathcal{C}}$, and is hence a direct generalization of the synonymous ordinary categorical construction e.g. given in [20, Section 7.3]. It hence shares many properties of the Yoneda embedding; in fact when restricted to the $(\infty, 1)$ -category of internal ∞ -groupoids it recovers y up to equivalence. The functor Ext furthermore is fully faithful as well, and allows the detection of the according “representables” via a smallness criterion: namely, we will see that the indexed $(\infty, 1)$ -categories arising from internal $(\infty, 1)$ -categories in this way can be characterized as exactly the globally small and locally small ones.

5.15. THEOREM. *Let \mathcal{C} be an $(\infty, 1)$ -category with finite limits and \mathcal{E} be a \mathcal{C} -indexed $(\infty, 1)$ -category. Then \mathcal{E} is represented by an internal $(\infty, 1)$ -category if and only if it is both globally small and locally small.*

Given the chosen definition of global smallness in terms of comprehension, this is a result which holds in ordinary category theory only with some caveats, see Remark 3.20 for more details.

Two fundamental examples of such representable cartesian fibrations are given by the universal (κ -small) cartesian fibration $\text{Dat}_\infty^{\text{op}} \rightarrow \text{Cat}_\infty^{\text{op}}$ and the universal (κ -small) right fibration $\mathcal{S}_*^{\text{op}} \rightarrow \mathcal{S}^{\text{op}}$ as constructed in [27, Section 3.3.2].² There, Lurie showed that the universal right fibration is represented by the terminal object in the $(\infty, 1)$ -category \mathcal{S} of spaces. The universal cartesian fibration on the other hand cannot be representable by an object in Cat_∞ in the same sense, but we will see that it is representable in terms of an internal $(\infty, 1)$ -category via the externalization functor (Example 5.21). The associated internal $(\infty, 1)$ -category is given by the interval object $\Delta^\bullet \in \text{Fun}(N(\Delta), \text{Cat}_\infty)$. In fact, it will follow from general considerations about comprehension schemes that the universal right fibration must then be represented by the pointwise invertible interval $I\Delta^\bullet$ in \mathcal{S} in the same way (Example 4.7, Remark 5.21). But this object is contractible in $\text{Fun}(N(\Delta), \mathcal{S})$, which recovers Lurie’s original result by other means. These results have various applications. For instance, they allow for the definition of cotensors X^J of internal $(\infty, 1)$ -categories $X \in \text{CS}(\mathcal{C})$ with finite $(\infty, 1)$ -categories J , and furthermore with arbitrarily sized $(\infty, 1)$ -categories J whenever \mathcal{C} is complete (Remark 5.20). They also allow us to prove a generalization of the well-known fact that the limit preserving presheaves over a presentable $(\infty, 1)$ -category are exactly the representable ones.

5.23. PROPOSITION. *Suppose \mathcal{C} is a complete $(\infty, 1)$ -category. Then a \mathcal{C} -indexed $(\infty, 1)$ -category \mathcal{E} is small if and only if the functor $\mathcal{E}: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_\infty$ has a left adjoint.*

In Section 6 we use the interplay between indexed and fibered structures to discuss various alternative characterizations of general comprehension schemes (Proposition 6.6, Remark 6.7), to show the existence of a universal such comprehension scheme whenever \mathcal{C} is small and has pullbacks (Proposition 6.10), and to discuss definability and smallness in the context of full subfibrations. For instance, we show the following.

6.16. PROPOSITION. *Let \mathcal{C} be an $(\infty, 1)$ -category with finite limits, let Y be an internal $(\infty, 1)$ -category in \mathcal{C} , and let $f: p \rightarrow \text{Ext}(Y)$ be a cartesian functor over \mathcal{C} . Then p is represented by an internal $(\infty, 1)$ -category if and only if \mathcal{C} has f -comprehension. In particular, \mathcal{C} has $\text{Ext}(f)$ -comprehension for all internal functors $f: X \rightarrow Y$ in $\text{CS}(\mathcal{C})$.*

In Section 7 we apply our results about the externalization construction from Section 5 to full and replete comprehension $(\infty, 1)$ -categories over a fixed $(\infty, 1)$ -category \mathcal{C} with

²We will eventually show that the domain Dat_∞ is the $(\infty, 1)$ -category $(\text{Cat}_\infty)_{\Delta^\bullet/}$ of “oplax-pointed” $(\infty, 1)$ -categories. At least until then an abstract notation is necessary to refer to it given that it is constructed by entirely abstract means. We chose $\text{Dat}_\infty \rightarrow \text{Cat}_\infty$ for this purpose simply because the fibration functorially associates to an $(\infty, 1)$ -category \mathcal{C} its higher categorical data $\Delta^n \rightarrow \mathcal{C}$.

pullbacks in the sense of Jacobs [19] as introduced in Section 6. Here we show that global smallness characterizes univalent morphisms in \mathcal{C} whenever \mathcal{C} has enough structure to define univalence in the first place.

7.2. PROPOSITION. *Suppose \mathcal{C} is left exact and locally cartesian closed. Given a full and replete comprehension $(\infty, 1)$ -category J over \mathcal{C} , the following are equivalent.*

1. $J \rightarrow \mathcal{C}$ is globally small.
2. J is represented by an internal $(\infty, 1)$ -category.
3. There is a univalent morphism $q: E \rightarrow B$ in \mathcal{C} that classifies the morphisms in J .

Thus, first, via Proposition 5.23 above this generalizes Gepner and Kock’s characterization of univalent morphisms in presentable $(\infty, 1)$ -categories in terms of an associated sheaf property ([17, Proposition 3.8], Corollary 7.3). And second, it allows us to characterize (elementary) ∞ -toposes entirely in terms of comprehension schemes: they are the finitely bicomplete ∞ -categories which – while generally not globally small over themselves – can instead be covered by a collection of small “neighbourhoods” (Corollary 7.8). This entails the following characterization of (Grothendieck) ∞ -toposes.

7.6. COROLLARY. *A presentable $(\infty, 1)$ -category \mathcal{C} is an ∞ -topos if and only if the following two conditions hold.*

1. The canonical fibration $t: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$ is locally small.
2. For all sufficiently large regular cardinals κ , the full comprehension $(\infty, 1)$ -category $t_\kappa: \text{Fun}(\Delta^1, \mathcal{C})_\kappa \rightarrow \mathcal{C}$ is small.

Lastly, in Section 8 we use the fibered formulations of both general and diagrammatic comprehension to give a simple proof of model independence of our results and constructions.

1.4. NOTATION. The notation will follow Lurie [27] in large. In particular, for the sake of readability, whenever we say “ ∞ -category” we mean $(\infty, 1)$ -category. The exponential $\mathcal{D}^{\mathcal{C}}$ of two quasi-categories \mathcal{C} , \mathcal{D} in the category \mathbf{S} of simplicial sets will be denoted by $\text{Fun}(\mathcal{C}, \mathcal{D})$. The large quasi-category of small quasi-categories as defined and studied in [27, Chapter 3] will be denoted by Cat_∞ . Its (small) hom-spaces $\text{Cat}_\infty(\mathcal{C}, \mathcal{D})$ are given by the core $\text{Fun}(\mathcal{C}, \mathcal{D})^\simeq$, that is, the largest Kan complex contained in the quasi-category $\text{Fun}(\mathcal{C}, \mathcal{D})$. The quasi-category of spaces will be denoted by \mathcal{S} , the (super-large) quasi-category of large quasi-categories will be denoted by CAT_∞ . A quasi-category will be said to be small precisely if it is contained in Cat_∞ . Smallness, largeness (and super-largeness) are relative notions; thus, to not carry around cardinals whose absolute size does not matter after all as all results apply to all regular cardinals polymorphically, “smallness” may refer to κ -smallness for some regular cardinal κ . In particular, we will work with a single universal fibration $\pi: \text{Dat}_\infty \rightarrow \text{Cat}_\infty$ which is to be thought of as a place-holder

for the universal κ -small fibration $\pi_\kappa: (\text{Dat}_\infty)_\kappa \twoheadrightarrow (\text{Cat}_\infty)_\kappa$ for any given regular cardinal κ (as introduced in [27, Section 3.3.2]). Furthermore, we will implicitly assume that a general base quasi-category is locally (essentially) small by definition.

Given a quasi-category \mathcal{C} and an object $C \in \mathcal{C}$, we denote the associated slice quasi-category (called *overcategory* in [27]) by $\mathcal{C}_{/C}$. To avoid awkward iterated hyphenations, we will refer to quasi-categories \mathcal{C} together with an inclusion of simplicial sets into another quasi-category \mathcal{D} as quasi-subcategories of \mathcal{D} . A full quasi-subcategory is hence such where the respective inclusion of simplicial sets is the identity on the sets of edges.

The standard n -simplex in the category of simplicial sets will be denoted by Δ^n .

There are two model structures on the category \mathbf{S} of particular importance for this paper. These are the Quillen model structure (\mathbf{S}, Kan) for Kan complexes and the Joyal model structure $(\mathbf{S}, \text{QCat})$ for quasi-categories.

While the main results of this paper are invariant under the choice of any common model of $(\infty, 1)$ -category theory (see Section 8), the author chose to work with the analytical presentation of quasi-categories for the sake of computations, argumentation and constructions by virtue of the extensive use of the results in the literature. As far as the results are concerned, the terms “quasi-category” and “ ∞ -category” will be used interchangeably.

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2. Preliminaries on indexed and fibered $(\infty, 1)$ -category theory

In order to define and study comprehension schemes for cartesian fibrations of ∞ -categories, in this section we cover, first, the underlying basic definitions and examples of indexed ∞ -categories and cartesian fibrations, second, the ∞ -categorical Grothendieck construction in form of Lurie’s straightening and unstraightening pair, and, lastly, a few definitions and statements regarding representable natural transformations. The presentation of the material up to the representable natural transformations is largely non-technical and mainly serves the purpose of introducing and organizing the notions and examples to be studied in later sections. The underlying calculations and proofs are contained in [27] where all given ∞ -categorical concepts are developed in detail.

2.1. DEFINITION. *Let \mathcal{C} be an ∞ -category. A \mathcal{C} -indexed ∞ -category is a functor $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$. If \mathcal{C} has a terminal object $1 \in \mathcal{C}$, we say that $\mathcal{E}(1)$ is the underlying ∞ -category*

of \mathcal{E} . A functor of \mathcal{C} -indexed ∞ -categories \mathcal{E}, \mathcal{F} is simply a morphism in the ∞ -category $\text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$ of \mathcal{C} -indexed ∞ -categories, that is, a natural transformation from \mathcal{E} to \mathcal{F} . Likewise, a \mathcal{C} -indexed ∞ -groupoid is a functor $X: \mathcal{C}^{op} \rightarrow \mathcal{S}$. The ∞ -category of \mathcal{C} -indexed ∞ -groupoids is the ∞ -category $\hat{\mathcal{C}} \simeq \text{Fun}(\mathcal{C}^{op}, \mathcal{S})$ of presheaves on \mathcal{C} .

The following examples are generalizations of some of the 1-categorical examples considered in [21, Section B1.2].

2.2. EXAMPLES.

1. For $1 \in \text{Cat}_\infty$ the terminal ∞ -category, the ∞ -category $\text{Fun}(1^{op}, \text{Cat}_\infty)$ of 1-indexed ∞ -categories is just the ∞ -category Cat_∞ of small ∞ -categories itself.

Let \mathcal{C} be an ∞ -category.

2. Whenever \mathcal{C} is small, the exponential

$$\text{Fun}(\cdot, \mathcal{C}): \text{Cat}_\infty^{op} \rightarrow \text{Cat}_\infty$$

is a Cat_∞ -indexed ∞ -category and is represented by \mathcal{C} in an $(\infty, 2)$ -categorical sense. Its restriction to the category \mathcal{S} of spaces is the “naive” \mathcal{S} -indexed ∞ -category $\text{Fun}(\cdot, \mathcal{C}): \mathcal{S}^{op} \rightarrow \text{Cat}_\infty$. Its underlying ∞ -category is \mathcal{C} itself.

3. If \mathcal{C} is small and has pullbacks, the slice functor $\mathcal{C}_{/(\cdot)}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ is the “canonical indexing of \mathcal{C} over itself”. It takes an object $C \in \mathcal{C}$ to the sliced ∞ -category $\mathcal{C}_{/C}$ and a morphism $f: C \rightarrow D$ in \mathcal{C} to the pullback functor $f^*: \mathcal{C}_{/D} \rightarrow \mathcal{C}_{/C}$. Technically, it is defined as the unstraightening of the target fibration over \mathcal{C} (Example 2.4.2). Again, its underlying ∞ -category is \mathcal{C} whenever \mathcal{C} is left exact. When applied to $\mathcal{C} \simeq \mathcal{S}$, the naive indexing $\text{Fun}(\cdot, \mathcal{S}): \mathcal{S}^{op} \rightarrow \text{CAT}_\infty$ and the canonical indexing $\mathcal{S}_{/(\cdot)}: \mathcal{S}^{op} \rightarrow \text{CAT}_\infty$ are pointwise equivalent by the (unmarked) straightening construction over Kan complexes [27, Theorem 2.2.1.2]. The straightening construction will be discussed below in more generality.
4. Given a \mathcal{C} -indexed ∞ -category $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$, each functor $F: \mathcal{D} \rightarrow \mathcal{C}$ induces a change of base $F^*\mathcal{E}: \mathcal{D}^{op} \rightarrow \text{Cat}_\infty$ via precomposition with F^{op} . It defines a functor

$$F^*: \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty) \rightarrow \text{Fun}(\mathcal{D}^{op}, \text{Cat}_\infty).$$

5. The most fundamental examples of indexed ∞ -groupoids are the representable presheaves $y(C) = \mathcal{C}(\cdot, C)$. Whenever \mathcal{C} has finite limits, they are the archetypes of “small” indexed ∞ -groupoids over \mathcal{C} and as such they will serve as reference structures in the definition of comprehension schemes in Section 3.
6. If \mathcal{C} has finite products, we obtain an action

$$(-)^{(\cdot)}: \mathcal{C}^{op} \times \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty) \rightarrow \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$$

given on objects by $\mathcal{E}^C(C') = \mathcal{E}(C' \times C)$. It is the Currying of the change of base along the product $\times : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$. Whenever \mathcal{C} is small, this action induces a functor

$$[\cdot, \cdot]: \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)^{op} \times \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty) \rightarrow \hat{\mathcal{C}}$$

as the curried adjoint of the composition of

$$\text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)^{op} \times \mathcal{C}^{op} \times \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty) \xrightarrow{1 \times (-)^{(\cdot)}} \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)^{op} \times \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$$

with the hom-functor $\text{Hom}: \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)^{op} \times \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty) \rightarrow \mathcal{S}$. For \mathcal{C} -indexed ∞ -categories \mathcal{E}, \mathcal{F} one may thus form the \mathcal{C} -indexed ∞ -groupoid $[\mathcal{E}, \mathcal{F}]$ of \mathcal{C} -indexed functors between them. It is given pointwise by the mapping spaces

$$[\mathcal{E}, \mathcal{F}](C) = \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)(\mathcal{E}, \mathcal{F}^C),$$

its underlying ∞ -category is the mapping space $\text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)(\mathcal{E}, \mathcal{F})$.

Generally, given that \mathcal{C} is small, it follows for instance from the Yoneda lemma for indexed ∞ -categories [39, Theorem 5.7.3] that the ∞ -category $\text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$ is cartesian closed (see e.g. [5, Section 9]; to make sense of this however requires to consider $\text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$ together with its canonical $(\infty, 2)$ -categorical structure). Then, whenever \mathcal{C} has finite products, for $\mathcal{F} \in \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$ and $C \in \mathcal{C}$ the \mathcal{C} -indexed ∞ -category \mathcal{F}^C computes the exponential $\mathcal{F}^{y(C)} \in \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$, and $[\mathcal{E}, \mathcal{F}]$ computes pointwise the maximal ∞ -subgroupoids contained in the exponential $\mathcal{F}^\mathcal{E} \in \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$.

In ordinary category theory – that is, the 2-category theory of 1-categories – the Grothendieck construction of pseudo-functors into the category Cat of small categories yields a 2-categorical equivalence between indexed category theory and fibered category theory, see e.g. [21, Theorem 1.3.6]. An $(\infty, 1)$ -categorical analogue of the Grothendieck construction was first given by Lurie’s (un)straightening construction [27, Section 3.2], which provides a very general framework to pass between cartesian fibrations over a simplicial set S and functors from S^{op} into the ∞ -category Cat_∞ . This construction yields an equivalence

$$\text{St}_\mathcal{C}: \text{Cart}(\mathcal{C}) \xrightleftharpoons[\simeq]{} \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty): \text{Un}_\mathcal{C}$$

between the ∞ -category of cartesian fibrations and cartesian functors over a small ∞ -category \mathcal{C} [27, Definition 2.4.2.1] and the ∞ -category of \mathcal{C} -indexed small ∞ -categories. The fibers of the fibration $\text{Un}_\mathcal{C}(\mathcal{E}) \in \text{Cart}(\mathcal{C})$ are equivalent to the values of the functor $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ on the objects of \mathcal{C} . Furthermore, the construction yields an equivalence

$$\text{St}_\mathcal{C}: \text{RFib}(\mathcal{C}) \xrightleftharpoons[\simeq]{} \text{Fun}(\mathcal{C}^{op}, \mathcal{S}): \text{Un}_\mathcal{C}$$

between the ∞ -category of right fibrations over \mathcal{C} [27, Section 2] and the ∞ -category of \mathcal{C} -indexed ∞ -groupoids (presheaves over \mathcal{C} , that is). We will omit the index \mathcal{C} and simply

write St and Un to refer to the respective straightening and unstraightening functors whenever the base \mathcal{C} is clear from context. In this sense, indexed ∞ -category theory and fibered ∞ -category theory are equivalent notions for all purposes of this paper.³

2.3. DEFINITION. *Let $(\cdot)^\simeq: \text{Cat}_\infty \rightarrow \mathcal{S}$ be the functor that assigns to each ∞ -category its core, i.e. \mathcal{C}^\simeq is the largest ∞ -groupoid contained in \mathcal{C} . Then postcomposition of an indexed ∞ -category $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ with this core functor yields a presheaf $\mathcal{E}^\simeq: \mathcal{C}^{op} \rightarrow \mathcal{S}$.*

Given a cartesian fibration $p: \mathcal{E} \twoheadrightarrow \mathcal{C}$, we denote the unstraightening of $\text{St}(p)^\simeq$ by $p^\times: \mathcal{E}^\times \twoheadrightarrow \mathcal{C}$ and call it the core of p .

Given a cartesian fibration $p: \mathcal{E} \twoheadrightarrow \mathcal{C}$, the domain \mathcal{E}^\times is an ∞ -category itself. It can be constructed as the wide ∞ -subcategory $\mathcal{E}^\times \subseteq \mathcal{E}$ spanned by the p -cartesian morphisms in \mathcal{E} . The restriction $p|_{\mathcal{E}^\times}$ is a right fibration by [27, Corollary 2.4.2.5] and is equivalent to the right fibration p^\times . Thus, the p^\times -cartesian morphisms in \mathcal{E}^\times are exactly the p -cartesian morphisms in \mathcal{E} , and the vertical morphisms in \mathcal{E}^\times are exactly the vertical equivalences in \mathcal{E} .

2.4. EXAMPLES. Let \mathcal{C} be an ∞ -category.

1. Whenever \mathcal{C} is small, the forgetful functor $s: (\text{Cat}_\infty)_{/\mathcal{C}} \twoheadrightarrow \text{Cat}_\infty$ is the unstraightening of the representable functor $\text{Cat}_\infty(\cdot, \mathcal{C}): \text{Cat}_\infty^{op} \rightarrow \mathcal{S}$. It therefore gives an example of a representable fibration as to be defined below. Similarly, the forgetful functor $s: \mathcal{S}_{/\mathcal{C}} \twoheadrightarrow \mathcal{S}$ is the unstraightening of the restriction $\text{Cat}_\infty(\cdot, \mathcal{C}): \mathcal{S}^{op} \rightarrow \mathcal{S}$. More generally, we may consider the unstraightening of the exponential

$$\text{Fun}(\cdot, \mathcal{C}): \text{Cat}_\infty^{op} \rightarrow \text{Cat}_\infty$$

and the unstraightening of the naive indexing

$$\text{Fun}(\cdot, \mathcal{C}): \mathcal{S}^{op} \rightarrow \text{Cat}_\infty$$

of \mathcal{C} . These are the fibration of diagrams and the fibration of families of \mathcal{C} , respectively, which generalize the classic construction of the 1-category $\text{Fam}(\mathbb{C})$ of families over a 1-category \mathbb{C} . By construction, the core of the fibration of diagrams of \mathcal{C} over Cat_∞ is exactly the representable fibration $s: (\text{Cat}_\infty)_{/\mathcal{C}} \twoheadrightarrow \text{Cat}_\infty$.

2. The codomain functor $t: \text{Fun}(\Delta^1, \mathcal{C}) \twoheadrightarrow \mathcal{C}$ is a cartesian fibration whenever \mathcal{C} has pullbacks. If \mathcal{C} is also small, its unstraightening is the canonical indexing of \mathcal{C} over itself (by definition). It therefore will be called the canonical fibration over \mathcal{C} .
3. Change of base of an indexed ∞ -category $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ along a functor $F: \mathcal{D} \rightarrow \mathcal{C}$ unstraightens to the pullback of fibrations

$$\begin{array}{ccc} \text{Un}(F^*\mathcal{I}) & \longrightarrow & \text{Un}(\mathcal{I}) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{D} & \xrightarrow{F} & \mathcal{C}. \end{array}$$

³In Section 4 we will implicitly need that unstraightening in fact constitutes an equivalence of $(\infty, 2)$ -categories; this will be addressed in due course however.

This follows directly from [27, Proposition 3.2.1.4].

Up to cartesian equivalence, every cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ with essentially small fibers is the change of base of the universal cartesian fibration

$$\pi^{op}: \text{Dat}_\infty^{op} \twoheadrightarrow \text{Cat}_\infty^{op}$$

along its straightening $\text{St}(p)^{op}: \mathcal{C} \rightarrow \text{Cat}_\infty^{op}$. The universal cocartesian fibration $\pi: \text{Dat}_\infty \twoheadrightarrow \text{Cat}_\infty$ itself is the unstraightening of the identity $\text{id}: \text{Cat}_\infty \rightarrow \text{Cat}_\infty$ [27, Section 3.3.2].

In particular, whenever \mathcal{C} is small and has pullbacks, the canonical fibration

$$t: \text{Fun}(\Delta^1, \mathcal{C}) \twoheadrightarrow \mathcal{C}$$

is equivalent to the change of base of the universal cartesian fibration along the canonical indexing of \mathcal{C} . In particular, we obtain a homotopy-cartesian square

$$\begin{array}{ccc} \text{Fun}(\Delta^1, \mathcal{C}) & \longrightarrow & \text{Dat}_\infty^{op} \\ \downarrow t & & \downarrow \pi^{op} \\ \mathcal{C} & \xrightarrow{\mathcal{C}/(\cdot)} & \text{Cat}_\infty^{op} \end{array}$$

in the Joyal model structure $(\mathbf{S}, \text{QCat})$.

Change of base along some functor $F: \mathcal{D} \rightarrow \mathcal{C}$ of the canonical indexing of \mathcal{C} over itself gives rise to the ‘‘Artin gluing’’ of F :

$$\begin{array}{ccc} \mathcal{C} \downarrow F & \longrightarrow & \text{Fun}(\Delta^1, \mathcal{C}) \\ \text{gl}(F) \downarrow & \lrcorner & \downarrow t \\ \mathcal{D} & \xrightarrow{F} & \mathcal{C}. \end{array}$$

One computes that the ∞ -category of cartesian sections of the Artin gluing $\text{gl}(F)$ is the ∞ -category of cartesian natural transformations over F , and as such it is equivalent to the limit of the functor $F^*(\mathcal{C}/(\cdot)): \mathcal{D}^{op} \rightarrow \text{Cat}_\infty$ [27, Section 3.3.3]. The localization of the ∞ -category $\mathcal{C} \downarrow F$ at the class of $\text{gl}(F)$ -cartesian morphisms is equivalent to the colimit of the very same functor [27, Section 3.3.4].

4. Similarly, up to cartesian equivalence, every right fibration with small fibers is the change of base of the universal right fibration $\mathcal{S}_*^{op} \twoheadrightarrow \mathcal{S}^{op}$ along its straightening.

For example, every object $C \in \mathcal{C}$ yields a functor $\text{ev}(C): \hat{\mathcal{C}} \rightarrow \mathcal{S}$ by evaluation of presheaves over \mathcal{C} at C . If by $y: \mathcal{C} \rightarrow \hat{\mathcal{C}}$ we denote the Yoneda embedding, the associated right fibration is given by the forgetful functor $s: \hat{\mathcal{C}}_{/y(C)}^{op} \twoheadrightarrow \hat{\mathcal{C}}^{op}$. That

means the square

$$\begin{array}{ccc} \hat{\mathcal{C}}_{y(C)/} & \longrightarrow & \mathcal{S}_* \\ \downarrow & & \downarrow \\ \hat{\mathcal{C}} & \xrightarrow{\text{ev}(C)} & \mathcal{S} \end{array}$$

is homotopy-cartesian in the Joyal model structure. This is shown in [27, Theorem 5.1.5.2], but also follows from Corollary 5.22 and the fact that the evaluation functor at C has a left adjoint.

2.5. DEFINITION. *A right fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ is representable if there is a representable presheaf over \mathcal{C} whose unstraightening is equivalent to p over \mathcal{C} .*

Thus, up to equivalence, the representable right fibrations over an ∞ -category \mathcal{C} are exactly the right fibrations of the form $s: \mathcal{C}_{/C} \rightarrow \mathcal{C}$ for objects C in \mathcal{C} . To verify whether a given right fibration is representable without having to compute its straightening, we may apply the following lemma.

2.6. LEMMA. [27, Proposition 4.4.4.5] *A right fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ is representable if and only if the domain \mathcal{E} has a terminal object $t \in \mathcal{E}(C)$; then $\mathcal{E} \simeq \mathcal{C}_{/C}$ over \mathcal{C} . \blacksquare*

2.7. DEFINITION. *Let \mathcal{C} be an ∞ -category. A natural transformation $f: X \rightarrow Y$ in $\hat{\mathcal{C}}$ is representable if for every object $C \in \mathcal{C}$ and every natural transformation $v: y(C) \rightarrow Y$, the pullback $v^*X \in \hat{\mathcal{C}}$ is a representable presheaf.*

Given an ∞ -category \mathcal{C} and an object $C \in \mathcal{C}$, there is a natural equivalence $\hat{\mathcal{C}}_{/y(C)} \simeq \widehat{\mathcal{C}_{/C}}$ of ∞ -categories. Under this equivalence, a natural transformation $f: X \rightarrow Y$ in $\hat{\mathcal{C}}$ is representable as in Definition 2.7 if and only if for all generalized elements $v: y(C) \rightarrow Y$, the pullback $v^*f \in \widehat{\mathcal{C}_{/C}}$ is a representable presheaf.

2.8. EXAMPLE. Let \mathcal{C} be an ∞ -category. By definition, a natural transformation $X \rightarrow *$ over the terminal presheaf in $\hat{\mathcal{C}}$ is representable if and only if for all objects $C \in \mathcal{C}$ the product $y(C) \times X$ is a representable presheaf. Whenever \mathcal{C} has finite products, this holds if and only if the presheaf X is representable itself. Indeed, the Yoneda embedding is limit preserving and so if every $y(C) \times X$ is representable, then in particular so is $y(*) \times X \simeq * \times X \simeq X$. And if $X \simeq y(D)$ for some $D \in \mathcal{C}$, then $y(C) \times X \simeq y(C) \times y(D) \simeq y(C \times D)$ is representable as well.

2.9. PROPOSITION. *Let \mathcal{C} be an ∞ -category. A natural transformation $f: X \rightarrow Y$ in $\hat{\mathcal{C}}$ is representable if and only if its unstraightening*

$$\text{Un}(f): \text{Un}(X) \rightarrow \text{Un}(Y)$$

over \mathcal{C} has a (generally not fibered) right adjoint.

PROOF. The functor $\mathrm{Un}(f): \mathrm{Un}(X) \rightarrow \mathrm{Un}(Y)$ has a right adjoint (as a functor of total ∞ -categories) if and only if for every vertex $v \in \mathrm{Un}(Y)$, the comma ∞ -category $\mathrm{Un}(f) \downarrow v$ defined as the pullback

$$\begin{array}{ccc}
 \mathrm{Un}(f) \downarrow v & \longrightarrow & \mathrm{Un}(Y)_{/v} \\
 \downarrow & \lrcorner & \downarrow s \\
 \mathrm{Un}(X) & \xrightarrow{\mathrm{Un}(f)} & \mathrm{Un}(Y) \\
 & \searrow p & \swarrow q \\
 & & \mathcal{C}
 \end{array} \tag{2}$$

has a terminal object ([30, Proposition 3.1.2] and [14, Proposition 6.1.11]). The projection $s: \mathrm{Un}(Y)_{/v} \rightarrow \mathrm{Un}(Y)$ is a right fibration by [27, Corollary 2.1.2.2], and hence so is the composite

$$\mathrm{Un}(f) \downarrow v \rightarrow \mathrm{Un}(X) \rightarrow \mathcal{C}.$$

Thus, by Lemma 2.6, the ∞ -category $\mathrm{Un}(f) \downarrow v$ has a terminal object if and only if this composite right fibration is representable. We hence want to show that f is a representable natural transformation if and only if the right fibration $\mathrm{Un}(f) \downarrow v \rightarrow \mathcal{C}$ is representable for all $v \in Y$.

For every $v \in \mathrm{Un}(Y)$, the cartesian square (2) is homotopy-cartesian in the contravariant model structure on $\mathbf{S}_{/\mathcal{C}}$ essentially by an application of [27, Proposition 4.1.2.18]. Furthermore, the right fibration $\mathrm{Un}(Y) \rightarrow \mathcal{C}$ induces a trivial Kan fibration

$$\begin{array}{ccc}
 \mathrm{Un}(Y)_{/v} & \xrightarrow{\sim} & \mathcal{C}_{/q(v)} \\
 s \downarrow & \dashrightarrow^{\lrcorner v^\lrcorner} & \downarrow s \\
 \mathrm{Un}(Y) & \longrightarrow & \mathcal{C}
 \end{array} \tag{3}$$

of slice ∞ -categories such that the outer square commutes. A dotted natural transformation $\lrcorner v^\lrcorner: \mathcal{C}_{/q(v)} \rightarrow \mathrm{Un}(Y)$ over \mathcal{C} as depicted in (3) such that the top triangle commutes up to homotopy is hence given by any section of the trivial fibration on the top. As straightening over \mathcal{C} is an equivalence of ∞ -categories and hence preserves pullback squares, it follows from (2) and (3) that the induced square

$$\begin{array}{ccc}
 \mathrm{St}(\mathrm{Un}(f) \downarrow v) & \longrightarrow & y(q(v)) \\
 \downarrow & & \downarrow \mathrm{St}(\lrcorner v^\lrcorner) \\
 X & \xrightarrow{f} & Y
 \end{array}$$

is cartesian in $\hat{\mathcal{C}}$.

Thus, whenever f is a representable natural transformation, it follows that for all $v \in \mathrm{Un}(Y)$ the right fibration $\mathrm{Un}(f) \downarrow v \rightarrow \mathcal{C}$ is representable, and so the functor $\mathrm{Un}(f): \mathrm{Un}(X) \rightarrow \mathrm{Un}(Y)$ has a right adjoint as stated above. Vice versa, suppose $\mathrm{Un}(f)$

has a right adjoint, and let $\ulcorner v \urcorner: y(C) \rightarrow Y$ be some natural transformation in $\hat{\mathcal{C}}$ for $C \in \mathcal{C}$. We obtain a functor $\text{Un}(\ulcorner v \urcorner): \mathcal{C}/C \rightarrow \text{Un}(Y)$ of right fibrations over \mathcal{C} and hence a vertex $v := \text{Un}(\ulcorner v \urcorner)(\text{id}_C)$ in the fiber over C together with a trivial Kan fibration as in (3). It follows that $\text{St}(\text{Un}(f) \downarrow v) \simeq (\ulcorner v \urcorner)^*X$ in $\hat{\mathcal{C}}$. But the right fibration $\text{Un}(f) \downarrow v \rightarrow \mathcal{C}$ is representable since $\text{Un}(f)$ is assumed to have a right adjoint, and hence so is the pullback $(\ulcorner v \urcorner)^*X$ in $\hat{\mathcal{C}}$. \blacksquare

2.10. **REMARK.** To elaborate on the interpretation of representable natural transformations as general comprehension schemes in Definition 1.1, let \mathcal{C} be an ∞ -category and $f: X \rightarrow Y$ be a representable natural transformation in $\hat{\mathcal{C}}$. Given objects $C \in \mathcal{C}$ and $v \in Y(C)$ represented by some $\ulcorner v \urcorner: y(C) \rightarrow Y$, we obtain a pullback square of the form

$$\begin{array}{ccc} y(C.(f, v)) & \xrightarrow{\ulcorner x.v \urcorner} & X \\ \varepsilon_{(f, v)} \downarrow & \lrcorner & \downarrow f \\ y(C) & \xrightarrow{\ulcorner v \urcorner} & Y \end{array}$$

in $\hat{\mathcal{C}}$. Following the proof of Lemma 2.9, the object $x.v \in X(C.(f, v))$ together with the morphism $\varepsilon_{(f, v)}: C.(f, v) \rightarrow C$ gives rise to a terminal object in the comma ∞ -category $\text{Un}(f) \downarrow v$. Algebraically, this pair can be understood as the universal partial X -structure on C which lifts the Y -structure v on C along f . That is to say that whenever some X -structure $w \in X(D)$ on some object $D \in \mathcal{C}$ is such that the projection $f(w) \in Y(D)$ is (equivalent to) the restriction g^*v for some morphism $g: D \rightarrow C$, then it already is the restriction of the X -structure $x.v \in X(C.(f, v))$ along some essentially uniquely assigned morphism $D \rightarrow C.(f, v)$ in \mathcal{C}/C . In type theoretic terms, the morphism $\varepsilon_{(f, v)}: C.(f, v) \rightarrow C$ can be understood as the extension of the context C by the type v in as much as it is the universal context morphism along which the reindexing of the type $v \in Y(C)$ has a term $x.v \in X$ (given syntactically by the associated Variable Introduction rule $c: C, x: v \vdash x: v$).

2.11. **LEMMA.** *Let \mathcal{C} be an ∞ -category.*

1. *The class of representable natural transformations in $\hat{\mathcal{C}}$ is closed under composition, is stable under pullbacks and contains all equivalences.*
2. *Whenever \mathcal{C} has pullbacks, the class of representable natural transformations in $\hat{\mathcal{C}}$ (considered as a full ∞ -subcategory of $\text{Fun}(\Delta^1, \hat{\mathcal{C}})$) furthermore is closed under finite limits. In particular, it has the left cancellation property.*
3. *Suppose \mathcal{D} is an ∞ -category with pullbacks and $F: \mathcal{D} \rightarrow \mathcal{C}$ is a functor with a right adjoint. Then the restriction functor $F^*: \hat{\mathcal{C}} \rightarrow \hat{\mathcal{D}}$ preserves representability of natural transformations.*

PROOF. Part 1 is straightforward. For Part 2, given a presheaf X and a span of representable natural transformations $f_1 \rightarrow f_3 \leftarrow f_2$ in $\hat{\mathcal{C}}_{/X}$, the pullback of the limit $f_1 \times_{f_3} f_2$ in $\hat{\mathcal{C}}_{/X}$ along an element $x: y(C) \rightarrow X$ is represented by the according fiber product of the representatives in $\mathcal{C}_{/C}$ of the elements $x^*f_i \in \hat{\mathcal{C}}_{/y(C)}$. Since equivalences are representable natural transformations, it follows that the class of representable natural transformations is fiberwise closed under finite limits. As it is stable under pullbacks by Part 1 as well, it thus is closed under all finite limits (as in [18, Proposition 3.3]). The left cancellation property follows formally, see e.g. [2, Lemma 3.1.15].

For Part 3, we first note that $F: \mathcal{D} \rightarrow \mathcal{C}$ has a right adjoint if and only if the restriction functor $F^*: \hat{\mathcal{C}} \rightarrow \hat{\mathcal{D}}$ preserves representables. This is essentially [30, Proposition 3.1.2] together with Lemma 2.6 and Example 2.4.3. Now suppose \mathcal{D} has pullbacks and $g: X \rightarrow Y$ is a representable natural transformation in $\hat{\mathcal{C}}$. Let $D \in \mathcal{D}$ and $f: y(D) \rightarrow F^*Y$ be a natural transformation in $\hat{\mathcal{D}}$. By the Yoneda lemma we obtain a natural transformation $\bar{f}: y(FD) \rightarrow Y$ together with a diagram

$$\begin{array}{ccc}
 \bullet & \xrightarrow{\quad} & F^*X \\
 \downarrow & \lrcorner & \downarrow F^*g \\
 & \bullet & \\
 \downarrow & \lrcorner & \downarrow \\
 y(D) & \xrightarrow{f} & F^*Y \\
 \downarrow & \lrcorner & \downarrow \\
 & \bullet & \\
 \downarrow \lrcorner & & \downarrow \\
 \lrcorner 1_{FD} \lrcorner & \xrightarrow{F^*\bar{f}} & F^*y(FD)
 \end{array} \tag{4}$$

in $\hat{\mathcal{D}}$, where the bullet points denote the obvious fiber products. The factorization of f on the bottom always exists, but one may note that it also follows directly from the existence of the left adjoint $F_!: \hat{\mathcal{D}} \rightarrow \hat{\mathcal{C}}$ of F^* whenever \mathcal{D} is small (as $F_!$ automatically preserves representables by construction). As the natural transformation g is representable, the pullback \bar{f}^*X is represented by some object $C \in \mathcal{C}$. And as F has a right adjoint, both restrictions $F^*y(FD)$ and $F^*y(C)$ are representable over \mathcal{D} . Thus, the top left corner in Diagram (4) is a fiber product of representable presheaves and hence is representable itself whenever \mathcal{D} has pullbacks. ■

This subsumes the basic notions of indexed and fibered ∞ -category theory necessary for the subsequent sections to follow. Further constructions, e.g. concerning limits and colimits in this context can be found in [27, Section 4.3.1] and in more detail in [27, Section 6.1.1] in the case \mathcal{C} is presentable. Other constructions have been studied in [5].

We end this section with a short recap of the category \mathbf{S}^+ of marked simplicial sets and the cartesian model structure as defined in [27, Section 3.1]. A marked simplicial set is a pair $X = (I, A)$ where I is a simplicial set and $A \subseteq I_1$ is a subset which contains all degenerate edges. We say that A is the set of marked edges in X . A map $f: (I, A) \rightarrow (J, B)$ of marked simplicial sets is a functor $f: I \rightarrow J$ of simplicial sets such that $f[A] \subseteq B$.

For every simplicial set I we obtain two cases of special interest. One, the minimally marked simplicial set $I^b := (I, s_0[I_0])$ and, two, the maximally marked simplicial set $I^\sharp := (I, I)$. The category \mathbf{S}^+ of marked simplicial sets is cartesian closed, the underlying simplicial set of its internal hom-objects Y^X is denoted by $\mathrm{Map}^b(X, Y)$. The simplicial subset $\mathrm{Map}^\sharp(X, Y) \subseteq \mathrm{Map}^b(X, Y)$ consists of the simplices whose 1-boundaries are all marked in Y^X . These two simplicial sets are characterized by the formulas

$$\mathbf{S}(I, \mathrm{Map}^b(X, Y)) \cong \mathbf{S}^+(I^b \times X, Y), \quad (5)$$

$$\mathbf{S}(I, \mathrm{Map}^\sharp(X, Y)) \cong \mathbf{S}^+(I^\sharp \times X, Y). \quad (6)$$

Whenever $p: \mathcal{E} \rightarrow \mathcal{C}$ is a cartesian fibration, the object $\mathcal{E}^\natural \in \mathbf{S}^+$ denotes the marked simplicial set $(\mathcal{E}, \{p\text{-cartesian edges}\})$. Thus, \mathcal{E}^\times is the largest subsimplicial set $S \subseteq \mathcal{E}$ such that $p|_S: S \rightarrow \mathcal{C}$ is a subfibration of p and $S^\natural = S^\sharp$. It follows that a cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ is a right fibration if and only if $\mathcal{E}^\natural = \mathcal{E}^\sharp$.

As a special case of [27, Section 2.1.4, Section 3.1.3], given an ∞ -category \mathcal{C} , Lurie constructs two model structures on the slice category $\mathbf{S}_{/\mathcal{C}^\sharp}^+$. These are the contravariant model structure $\mathrm{RFib}(\mathcal{C})$, whose fibrant objects are exactly maps of the form $p: \mathcal{E}^\natural \rightarrow \mathcal{C}^\sharp$ such that p is a right fibration of underlying simplicial sets, and the cartesian model structure $\mathrm{Cart}(\mathcal{C})$, whose fibrant objects are exactly of the form $p: \mathcal{E}^\natural \rightarrow \mathcal{C}^\sharp$ such that p is a cartesian fibration on underlying simplicial sets. The induced simplicial enrichment of the slice-category $\mathbf{S}_{/\mathcal{C}^\sharp}^+$ via the “sharp” hom-object construction (6) endows both model categories with a simplicial enrichment over the model category $(\mathbf{S}, \mathrm{Kan})$ for Kan complexes. The induced simplicial enrichment of the slice-category $\mathbf{S}_{/\mathcal{C}^\sharp}^+$ via the “flat” hom-object construction (5) endows both model categories with a simplicial enrichment over the model category $(\mathbf{S}, \mathrm{QCat})$ for quasi-categories.

3. Diagrammatic $(\infty, 1)$ -comprehension schemes

In this section we generalize the notion of comprehension for fibered categories defined in [21, Section B1.3] to notions of comprehension for indexed ∞ -categories, and study the most fundamental examples. We will do so first in terms of indexed ∞ -categories over the base \mathcal{C} rather than in terms of fibered ∞ -categories over \mathcal{C} , because some of the structural arguments are more straightforward in the indexed language. Additionally, this allows us to define comprehension schemes with respect to a further parameter in form of an abstract ∞ -category K which determines the sort of diagrammatic shapes we want to consider comprehension for. Non-standard choices of such ∞ -categories K will capture instances of “generalized” comprehension schemes such as well-poweredness or structured comprehension schemes which for example can capture the interpretation of (essentially) algebraic theories. As concrete examples of indexed ∞ -categories often have to be constructed in terms of their fibrational counterparts however, we will give an equivalent characterization of comprehension schemes for cartesian fibrations in Section 4 and relate it to Johnstone’s original notion in Remark 4.2.

3.1. NOTATION. We fix an ∞ -category \mathcal{C} for the entire section. It can be assumed to be large (and locally small) unless explicitly stated to be small otherwise. We recall that the latter means that \mathcal{C} is an object of the large ∞ -category Cat_∞ .

3.2. REMARK. As long as a given (large) ∞ -category \mathcal{C} can be assumed to be an object of some ∞ -category CAT_∞ of (large) ∞ -categories, all results apply to such (large) ∞ -categories \mathcal{C} together with Cat_∞ replaced by CAT_∞ accordingly as well. This is just the common kind of universe polymorphism also referred to in Notation 1.4 and usually left implicit.

3.3. DEFINITION. *Let K be an ∞ -category, and let $\mathcal{E}: \mathcal{C}^{op} \rightarrow K$ be a functor. For an object $k \in K$, let $K(k, \mathcal{E})$ be the presheaf defined by the composition*

$$\mathcal{C}^{op} \xrightarrow{\mathcal{E}} K \xrightarrow{K(k, \cdot)} \mathcal{S}. \quad (7)$$

Given a morphism $G: k \rightarrow l$ in K , the restriction $G^*: K(l, \cdot) \rightarrow K(k, \cdot)$ in $\text{Fun}(K, \mathcal{S})$ induces for every functor $\mathcal{E}: \mathcal{C}^{op} \rightarrow K$ a natural transformation $G^*: K(l, \mathcal{E}) \rightarrow K(k, \mathcal{E})$ in $\hat{\mathcal{C}}$ by precomposition with \mathcal{E} .

3.4. DEFINITION. *Let K be an ∞ -category and $G: k \rightarrow l$ be a morphism in K . Say a functor $\mathcal{E}: \mathcal{C}^{op} \rightarrow K$ has (K, G) -comprehension if the natural transformation*

$$G^*: K(l, \mathcal{E}) \rightarrow K(k, \mathcal{E})$$

in $\hat{\mathcal{C}}$ is representable.

3.5. LEMMA. [Homotopy invariance I] *Let K be an ∞ -category and $\mathcal{E}: \mathcal{C}^{op} \rightarrow K$ be a functor. Suppose $G: k \rightarrow l$ and $G': k' \rightarrow l'$ are equivalent morphisms in K . Then \mathcal{E} has (K, G) -comprehension if and only if it has (K, G') -comprehension.*

PROOF. Every set Σ of 2-cells in K which witnesses that G and G' are equivalent morphisms in K induces a set $y(\Sigma)$ of 2-cells in $\text{Fun}(K, \mathcal{S})$ which witnesses that $G^* = yG$ and $(G')^* = yG'$ are equivalent. Precomposition with \mathcal{E} is functorial and so we obtain a set $\mathcal{E}^*y(\Sigma)$ of 2-cells in $\hat{\mathcal{C}}$ which witnesses that the natural transformations G^* and $(G')^*$ are equivalent. Hence, one is representable if and only if the other is. \blacksquare

Comprehension schemes defined in this generality encompass a wide variety of examples. We will give three such general examples to illustrate the ubiquity, but will then restrict our attention to comprehension schemes with respect to a (not necessarily full) ∞ -subcategory $K \subseteq \text{Cat}_\infty$. The standard case is formed by $K = \text{Cat}_\infty$ itself, which will be studied in most detail.

3.6. EXAMPLE. Let K be an ∞ -category and $\mathcal{E}: \mathcal{C}^{op} \rightarrow K$ be a functor. Then \mathcal{E} has (K, G) -comprehension for every equivalence $G: k \rightarrow l$ in K . Indeed, for every such G the restriction

$$G^*: K(l, \mathcal{E}) \rightarrow K(k, \mathcal{E}) \quad (8)$$

is again an equivalence in $\hat{\mathcal{C}}$, and thus is representable by Lemma 2.11.1.

3.7. EXAMPLE. Let \mathcal{C} be an ∞ -category with finite products and let $C \in \mathcal{C}$ be an object. Via Example 2.8 the functor

$$C \times (\cdot): \mathcal{C}^{op} \rightarrow \mathcal{C}^{op}$$

has $(\mathcal{C}^{op}, \emptyset \rightarrow D)$ -comprehension for some object $D \in \mathcal{C}$ if and only if the presheaf $\mathcal{C}(C \times (\cdot), D) \in \hat{\mathcal{C}}$ is representable. That is, if and only if the exponential D^C exists in \mathcal{C} . One computes that this in turn implies (and hence is equivalent to) $(\mathcal{C}^{op}, \iota_2: E \rightarrow D \sqcup E)$ -comprehension for all objects $E \in \mathcal{C}$.

3.8. EXAMPLE. More generally, a functor $\mathcal{E}: \mathcal{C}^{op} \rightarrow K$ into an ∞ -category K with an initial object has $(K, \emptyset \rightarrow k)$ -comprehension for all objects k (contained in a full ∞ -subcategory $K' \subseteq K$) if and only if the functor \mathcal{E} has a (partial) left adjoint (with respect to the inclusion $K' \subseteq K$). Accordingly, the “syntax-to-semantics” functors $\text{pt}: \text{Thy}^{op} \rightarrow \text{Mod}$ arising in the common Stone duality type theorems all have $(\text{Mod}, \emptyset \rightarrow M)$ -comprehension for all objects $M \in \text{Mod}$.

Let $\text{Cat}_\infty^{\text{pb}} \subseteq \text{Cat}_\infty$ denote the ∞ -category of small ∞ -categories with pullbacks and pullback-preserving functors. Let $\text{Cat}_\infty^{\text{lex}} \subseteq \text{Cat}_\infty$ denote the ∞ -category of small left exact ∞ -categories and left exact functors. Note that for example the ∞ -categories Δ^0 and Δ^1 are both left exact, and that the endpoint-inclusion $d^0: \Delta^0 \rightarrow \Delta^1$ is left exact as well. A pullback-preserving functor $\Delta^0 \rightarrow \mathcal{D}$ into an ∞ -category \mathcal{D} with pullbacks is determined by an object of \mathcal{D} . The space of left exact functors $\Delta^0 \rightarrow \mathcal{D}$ into a (left exact) ∞ -category \mathcal{D} with terminal object $*$ is contractible – with point of contraction the morphism $* \mapsto *$. A pullback-preserving functor $\Delta^1 \rightarrow \mathcal{D}$ into an ∞ -category \mathcal{D} (with pullbacks) is determined by a monomorphism in \mathcal{D} . A left exact functor $\Delta^1 \rightarrow \mathcal{D}$ into a (left exact) ∞ -category \mathcal{D} with terminal object is determined by a subterminal object in \mathcal{D} .

3.9. EXAMPLE. [Well-poweredness] Let $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty^{\text{pb}}$ be an indexed ∞ -category with pullbacks. Say that \mathcal{E} is *well-powered* if it has $(\text{Cat}_\infty^{\text{pb}}, d^0: \Delta^0 \rightarrow \Delta^1)$ -comprehension. Thus, \mathcal{E} is well-powered if for every pair of objects $C \in \mathcal{C}$, $E \in \mathcal{E}(C)$ there is a morphism $\varepsilon_E: P_C(E) \rightarrow C$ in \mathcal{C} together with a monomorphism $\top_E: \text{el}_E \hookrightarrow \varepsilon_E^*(E)$ in $\mathcal{E}(P_C(E))$ which for all $f: D \rightarrow C$ classifies the subobjects of $f^*E \in \mathcal{E}(D)$ in the sense that the natural transformation

$$((\varepsilon_E)_*, \lceil \top_E \rceil): y(P_C(E)) \rightarrow yC \times_{\mathcal{E}} \text{Cat}_\infty^{\text{pb}}(\Delta^1, \mathcal{E})$$

is an equivalence (following the blueprint in Remark 2.10). The given comprehension scheme corresponds exactly to the “generalized comprehension scheme” for pointwise monomorphic functors from [21, Example B1.3.14] in the ordinary categorical case. In particular, if \mathcal{C} is small and has pullbacks itself, the canonical indexing $\mathcal{C}_{/(\cdot)}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty^{\text{pb}}$ is well-powered if and only if all slices $\mathcal{C}_{/C}$ have (pullback-stable) power objects.

3.10. REMARK. One can define well-poweredness more generally for arbitrary indexed ∞ -categories $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ as $((\text{Cat}_\infty, \mathcal{M}), d_0)$ -comprehension, where $(\text{Cat}_\infty, \mathcal{M}) \subset \text{Cat}_\infty$ denotes the wide ∞ -subcategory spanned by the monomorphism-preserving functors.

3.11. **EXAMPLE.** [Subterminal object classifiers] Let $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty^{\text{lex}}$ be an indexed left exact ∞ -category. Let us consider $(\text{Cat}_\infty^{\text{lex}}, d^0: \Delta^0 \rightarrow \Delta^1)$ -comprehension. It holds if for every object $C \in \mathcal{C}$ there is a morphism $\varepsilon: \Omega_C \rightarrow C$ in \mathcal{C} together with a subterminal object $\top: \text{el} \hookrightarrow *$ in $\mathcal{E}(\Omega_C)$ which classifies the subterminal objects in the fibers of \mathcal{E} . In that case let's say that \mathcal{E} has a subterminal object classifier. In particular, if \mathcal{C} is small and has pullbacks itself, the canonical indexing $\mathcal{C}_{/\cdot}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty^{\text{lex}}$ has a subterminal object classifier if and only if all slices $\mathcal{C}_{/C}$ have (pullback-stable) subobject classifiers.

Even in this generality, for every indexed left exact ∞ -category $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty^{\text{lex}}$, one can show that well-poweredness of the pushforward $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty^{\text{pb}}$ implies the existence of subterminal object classifiers in \mathcal{E} . Indeed, under the given assumptions there is a cartesian square

$$\begin{array}{ccc}
 \text{Cat}_\infty^{\text{lex}}(\Delta^1, \mathcal{E}) & \longrightarrow & \text{Cat}_\infty^{\text{pb}}(\Delta^1, \mathcal{E}) \\
 \downarrow & & \downarrow d_0^* \\
 \Delta^0 & \xrightarrow{\lceil *_{\mathcal{E}} \rceil} & \mathcal{E}^\simeq
 \end{array} \tag{9}$$

in $\hat{\mathcal{C}}$. Here, the transformation on the top is given by the natural forgetful functor. We identify the constant terminal presheaf Δ^0 in the lower left corner with $\text{Cat}_\infty^{\text{lex}}(\Delta^0, \mathcal{E})$, and the lower right corner with $\text{Cat}_\infty^{\text{pb}}(\Delta^0, \mathcal{E})$. The terminal objects in the fibers $\mathcal{E}(C)$ assemble to a natural transformation $\lceil *_{\mathcal{E}} \rceil: \Delta^0 \rightarrow \mathcal{E}^\simeq$ precisely because the transition functors of \mathcal{E} are assumed to preserve terminal objects (see e.g. [39, Proposition 12.2.6]). The fact that the square (9) is cartesian follows from the characterization of the upper right corner as the presheaf of fiberwise monomorphisms, and the upper left corner as the presheaf of fiberwise subterminal objects. Thus, representability of the right vertical transformation in (9) implies representability of the left vertical transformation by Lemma 2.11.1.

3.12. **REMARK.** One might expect that the notion of elementary ∞ -topos, however defined, should be recoverable from suitable comprehension schemes. Such a presentation would be useful to define and study elementary ∞ -toposes over arbitrary base ∞ -toposes. For example, in 1-category theory, a left exact 1-category \mathcal{C} is an elementary 1-topos if and only if the (large) canonical indexing $\mathcal{C}_{/\cdot}$ is well-powered, see [50, Theorem 11.1]. But well-poweredness of the canonical indexing $\mathcal{C}_{/\cdot}: \mathcal{C}^{op} \rightarrow \text{CAT}_\infty^{\text{lex}}$ for left exact ∞ -categories does not characterize ∞ -toposes, as such require (among others) the existence of object classifiers rather than of subobject classifiers only. ‘‘Super-poweredness’’ in form of unrestricted $(\text{Cat}_\infty, d^0: \Delta^0 \rightarrow \Delta^1)$ -comprehension is however too strong, since it would require each slice of \mathcal{C} to have an object classifier with no size restriction whatsoever.

Whenever \mathcal{C} is presentable however, for any regular cardinal κ one may consider the comprehension scheme of super-poweredness restricted to pointwise relatively κ -compact diagrams. To make this precise we may consider the full ∞ -subcategory $\text{Pres} \subset \text{CAT}_\infty$ spanned by the presentable ∞ -categories, and the wide ∞ -subcategory $\text{Pres}^{\rightarrow \kappa} \subset \text{Pres}$ spanned by those functors which preserve relatively κ -compact morphisms. For instance, for every regular cardinal κ , the endpoint-inclusion $d_0: \Delta^0 \rightarrow \Delta^1$ is a functor in $\text{Pres}^{\rightarrow \kappa}$,

and the canonical indexing $\mathcal{C}_{/(\cdot)}: \mathcal{C}^{op} \rightarrow \text{CAT}_\infty$ factors through $\text{Pres}^{\rightarrow\kappa}$ as well. Then one can show that the canonical indexing $\mathcal{C}_{/(\cdot)}$ has $(\text{Pres}^{\rightarrow\kappa}, d_0)$ -comprehension if and only if all slices of \mathcal{C} have pullback-stable object classifiers for relatively κ -compact morphisms (via [27, Theorem 6.1.6.8] and [27, Proposition 6.3.5.1.(1)]). One may say in this case that each $\mathcal{C}_{/(\cdot)}: \mathcal{C}^{op} \rightarrow \text{Pres}^{\rightarrow\kappa}$ is “ κ -super powered”. We will give a more elegant characterization of ∞ -toposes and elementary ∞ -toposes in the sense of [34, Section 3] in terms of comprehension schemes in the end of Section 7.

3.13. EXAMPLE. [Generic models of theories] Let T be a Lawvere theory – such as, say, the theory of monoids, the theory of (abelian) groups or the theory of (commutative) rings – and let $\text{Cat}_\infty^\Pi \subset \text{Cat}_\infty$ be the ∞ -category of small ∞ -categories with finite products and finite product preserving functors. Suppose \mathcal{C} has finite products itself. An indexed ∞ -category $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty^\Pi$ has $(\text{Cat}_\infty^\Pi, * \rightarrow T)$ -comprehension if the natural transformation $\text{Fun}^\Pi(T, \mathcal{E})^\simeq \rightarrow *$ is representable. By Example 2.8 this holds if and only if the presheaf $\text{Fun}^\Pi(T, \mathcal{E})^\simeq$ of T -models in \mathcal{E} is representable. Informally, that means there is an object $M \in \mathcal{C}$ together with a T -model $m_{\text{gen}}: T \rightarrow \mathcal{E}(M)$ which is a generic such model in the sense that for every T -model of the form $m: T \rightarrow \mathcal{E}(C)$ for some $C \in \mathcal{C}$ there is an essentially unique morphism $\ulcorner m \urcorner: C \rightarrow M$ in \mathcal{C} such that $\mathcal{E}(\ulcorner m \urcorner) \circ m_{\text{gen}} \simeq m$. In the case \mathcal{C} is small and has all finite limits, and $\mathcal{E} = \mathcal{C}_{/(\cdot)}$ is the canonical indexing for example, this means that there is a universal “ T -bundle” in \mathcal{C} .

Example 3.13 also applies to all sorts of other theories such as, say, finite limit theories together with indexed ∞ -categories $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty^{\text{lex}}$, or geometric theories as follows.

3.14. EXAMPLE. [Classifying toposes] Let RTop be the very large ∞ -category of ∞ -toposes and geometric morphisms. In [44, Section 7] the author defines an ∞ -category $\text{GeoCAT}_\infty^\kappa \subset \text{CAT}_\infty$ of large higher κ -geometric ∞ -categories and higher κ -geometric functors for any regular cardinal κ . Its objects are the (large) ∞ -categories with finite limits and universal and effective κ -small colimits. Its morphisms are the finite limit preserving and κ -small colimit preserving functors. The forgetful functor

$$U: \text{RTop}^{op} \rightarrow \text{GeoCAT}_\infty^\kappa$$

has $(\text{GeoCAT}_\infty^\kappa, * \rightarrow \mathcal{C})$ -comprehension for every small higher κ -geometric ∞ -category \mathcal{C} precisely because every such \mathcal{C} has a classifying ∞ -topos $\text{Sh}_{\text{geo}, \kappa}(\mathcal{C})$ such that the Yoneda embedding of \mathcal{C} factors through a higher κ -geometric functor $y: \mathcal{C} \rightarrow \text{Sh}_{\text{Geo}, \kappa}(\mathcal{C})$. The latter is the generic \mathcal{C} -model associated to U .

In the ordinary categorical case, the same holds for the $(2, 1)$ -category GeoCAT_1^κ of large κ -geometric categories, the $(2, 1)$ -category RTop_1 of ordinary Grothendieck toposes, and the according forgetful 2-functor $U: \text{RTop}_1^{op} \rightarrow \text{GeoCAT}_1^\kappa$. The fact that U has $(\text{GeoCAT}_1^\kappa, * \rightarrow \mathcal{C})$ -comprehension for all small κ -geometric categories \mathcal{C} corresponds exactly to the fact that for any given small κ -geometric theory T the composite pseudo-functor

$$\text{RTop}_1^{op} \xrightarrow{T\text{-Mod}(\cdot)} \text{CAT} \xrightarrow{(\cdot)^\simeq} \text{GRPD} \quad (10)$$

of T -models is representable [12, Remark 2.1.5]. This means that the indexed ∞ -category $T\text{-Mod}(\cdot): \text{RTop}_1^{op} \rightarrow \text{CAT}$ is globally small in the sense of Example 3.19. The sequence (10) is part of a very classic construction in case of the geometric theory T_{BG} of G -torsors for a discrete group G . Here, precomposition of (10) with the functor $\text{Sh}: \text{Top} \rightarrow \text{RTop}_1$ which assigns to a topological space its associated localic topos of sheaves over it is naturally equivalent to the functor $PB_G: \text{Top}^{op} \rightarrow \text{GRPD}$ which assigns to a space the groupoid of principle G -bundles over it, see e.g. [25, Section VIII.2].⁴

3.15. EXAMPLE. [Generic models of theory extensions] To elaborate further on Example 3.13, we also may consider a functor $F: T \rightarrow T'$ of Lawvere theories [11, Section 3.7]. Informally, an indexed ∞ -category $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty^\Pi$ has $(\text{Cat}_\infty^\Pi, F)$ -comprehension if for every $C \in \mathcal{C}$ and every T -model $M: T \rightarrow \mathcal{E}(C)$ there is an object $\varepsilon_{(F,M)} \in \mathcal{C}_{/C}$ together with a T' -model $M': T' \rightarrow \mathcal{E}(\text{dom}(\varepsilon_{(F,M)}))$ which is the generic F -extension of M in the sense that it is the universal such model such that $\mathcal{E}(\varepsilon_{(F,M)}) \circ M \simeq M' \circ F$. In the case of the canonical indexing $\mathcal{E} = \mathcal{C}_{/(\cdot)}$ over a small ∞ -category \mathcal{C} with pullbacks, and, say, for the canonical functor $F: \text{Mnd} \rightarrow \text{Grp}$ from the theory of monoids to the theory of groups, this means that for every monoid object $M \in \mathcal{C}_{/C}$ for some $C \in \mathcal{C}$ there is a universal pair consisting of an object $\varepsilon_M \in \mathcal{C}_{/C}$ and a group object $G_M \in \mathcal{C}_{/\text{dom}(\varepsilon_M)}$ such that the underlying monoid of G is equivalent to the restriction of M along ε_M . In that sense, the object $\text{dom}(\varepsilon_M) \in \mathcal{C}$ classifies pullbacks of the monoid M which happen to be groups already. For example, the scheme holds whenever \mathcal{C} is an ∞ -topos, as for a given monoid M in $\mathcal{C}_{/C}$ it is witnessed by the colimit $\varepsilon_M \in \mathcal{C}_{/C}$ of all elements $g: G \rightarrow C$ for a generator G such that the monoid $g^*M \in \mathcal{C}_{/G}$ is a group.

3.16. NOTATION. Whenever $K \subseteq \text{Cat}_\infty$ is a full ∞ -subcategory and $G: k \rightarrow l$ is a morphism in K , a functor $\mathcal{E}: \mathcal{C}^{op} \rightarrow K$ has (K, G) -comprehension if and only if its post-composition $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ has (Cat_∞, G) -comprehension. In this case we will suppress the parameter K , and simply say that an indexed ∞ -category $\mathcal{E}: \mathcal{C}^{op} \rightarrow K \subseteq \text{Cat}_\infty$ has G -comprehension whenever it has (Cat_∞, G) -comprehension. This is exactly standard diagrammatic G -comprehension in the sense of Definition 1.2.

In the following we focus on the standard case of Notation 3.16. Recall that the ∞ -category Cat_∞ can be described as the homotopy-coherent nerve of the full simplicial category $\mathbf{QCat} \subset \mathbf{S}$ spanned by the quasi-categories. Every simplicial set I induces a simplicial functor

$$(\cdot)^I: \mathbf{QCat} \rightarrow \mathbf{QCat}$$

by exponentiation. Whenever I itself is a quasi-category and \mathcal{E} is a \mathcal{C} -indexed ∞ -category, the presheaf $\text{Cat}_\infty(I, \mathcal{E}) \in \hat{\mathcal{C}}$ is by construction the core $(\mathcal{E}^I)^\simeq$ of the pointwise induced

⁴Presumably the picture can be completed ∞ -categorically with respect to a notion of principle ∞ -bundle [32]. This would mean that there is a geometric ∞ -category T_{BG_∞} for every ∞ -group G such that $EG \in \mathcal{S}_{/BG}$ is the universal T_{BG_∞} -bundle in the sense of Example 3.13. Or in other words, that the canonical indexing $\mathcal{S}_{/(\cdot)}: \mathcal{S}^{op} \rightarrow \text{CAT}_\infty$ has $(\text{GeoCAT}^{|G|}, * \rightarrow T_{BG_\infty})$ -comprehension.

exponential

$$\mathcal{E}^I: \mathcal{C}^{op} \xrightarrow{\mathcal{E}} \text{Cat}_\infty \xrightarrow{(\cdot)^I} \text{Cat}_\infty.$$

In this case, we therefore can define comprehension more generally for all maps of simplicial sets as follows.

3.17. NOTATION. Let $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ be an indexed ∞ -category, and let $G: I \rightarrow J$ be a map of simplicial sets. Say that \mathcal{E} has G -comprehension if the natural transformation

$$G^*: (\mathcal{E}^J)^\simeq \rightarrow (\mathcal{E}^I)^\simeq \tag{11}$$

in $\hat{\mathcal{C}}$ is representable.

3.18. LEMMA. [Homotopy Invariance II] *Let $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ be an indexed ∞ -category. If $G: I \rightarrow J$ and $G': I' \rightarrow J'$ in \mathbf{S} are weakly equivalent maps in the Joyal model structure, then an indexed ∞ -category $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ has G -comprehension if and only if it has G' -comprehension. In particular, if $\mathbb{R}G: \mathbb{R}I \rightarrow \mathbb{R}J$ denotes a fibrant replacement of $G: I \rightarrow J$ in the Joyal model structure, then \mathcal{E} has G -comprehension in the sense of Notation 3.17 if and only if it has $\mathbb{R}G$ -comprehension in the sense of Definition 3.4.*

PROOF. Suppose we are given a commutative square

$$\begin{array}{ccc} I & \xrightarrow{G} & J \\ \downarrow & & \downarrow \\ I' & \xrightarrow{G'} & J' \end{array}$$

in \mathbf{S} such that the two vertical maps are weak equivalences in the Joyal model structure. This induces a square

$$\begin{array}{ccc} (\mathcal{E}^{J'})^\simeq & \longrightarrow & (\mathcal{E}^J)^\simeq \\ (G')^* \downarrow & & \downarrow G^* \\ (\mathcal{E}^{I'})^\simeq & \longrightarrow & (\mathcal{E}^I)^\simeq \end{array}$$

in $\hat{\mathcal{C}}$ where the horizontal natural transformations are equivalences (given that the Joyal model structure is cartesian closed). It follows that G^* is representable if and only if $(G')^*$ is representable, and so the two corresponding instances of comprehension are equivalent. ■

Therefore, in the following we will not further distinguish whether we consider a given functor $G: I \rightarrow J$ of ∞ -categories as a morphism in Cat_∞ or as a map of underlying simplicial sets.

3.19. **EXAMPLE.** [Global smallness] Let $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ be an indexed ∞ -category. Say p is *globally small* if it has $(\emptyset \rightarrow \Delta^0)$ -comprehension, i.e. if the natural transformation $\mathcal{E}^\simeq \rightarrow *$ in $\hat{\mathcal{C}}$ is representable. Whenever \mathcal{C} has finite products, that is if and only if the core $\mathcal{E}^\simeq \in \hat{\mathcal{C}}$ is representable by Example 2.8.

The choice of terminology in Example 3.19 follows [20, Section 5.2] (and hence ultimately [33]) where the term is used to denote the existence of (split) generic objects in a (split) fibration of 1-categories.

3.20. **REMARK.** Let $\mathcal{E}: \mathcal{C} \rightarrow \text{Cat}$ be a pseudo-functor over an ordinary category \mathcal{C} with products. By definition, \mathcal{E} is globally small if and only if the \mathcal{C} -indexed groupoid \mathcal{E}^\simeq is representable. As \mathcal{C} is a 1-category however, its Yoneda embedding factors through the category $\text{Fun}(\mathcal{C}^{op}, \text{Set})$ of set-valued presheaves. Thus, the existence of a natural equivalence $\mathcal{E}^\simeq \simeq y(C)$ for some $C \in \mathcal{C}$ implies that the groupoids $\mathcal{E}(D)$ for $D \in \mathcal{C}$ are all essentially discrete. This for instance implies that the automorphism groups of all objects $E \in \mathcal{E}(D)$ for $D \in \mathcal{C}$ are trivial. This latter implication is exactly the observation in [21, Example B1.3.13] as $(\emptyset \rightarrow \Delta^1)$ -comprehension here really is a $(2, 1)$ -categorical comprehension scheme in the sense that it requires representability of the indexed groupoid \mathcal{E}^\simeq up to equivalence. This observation leads to the introduction of a variety of generic objects [20, 48] in the ordinary categorical context to replace global smallness, and so to capture essential examples with non-trivial automorphism groups. To be more precise, whenever \mathcal{E} is a strict functor, and comprehension is defined in terms of the underlying-set functor $\text{Cat} \rightarrow \text{Set}$ as discussed in the introduction, then global smallness corresponds exactly to the existence of a split generic object. This notion is not equivalence-invariant however, and to remedy the fact that the \mathcal{C} -indexed underlying set of a general pseudo-functor does not even exist, global smallness for a general pseudo-functor (or a general fibration respectively) has to be defined in other terms. Such are given for example by the existence of (weak) generic objects [20, 48]. These however go beyond the framework of diagrammatic comprehension schemes.

In that sense, global smallness is a much more natural notion in ∞ -category theory. We will further elaborate on this in Remark 5.17 in the context of the externalization functor. Indeed, in Section 5 we will see examples of globally small indexed ∞ -categories with highly non-trivial vertical automorphism spaces in their fibers. For instance, such an example is given by the identity $\mathcal{S} \rightarrow \mathcal{S}$ considered as an \mathcal{S}^{op} -indexed ∞ -category. It is represented by the terminal object $*$ in \mathcal{S} (considered as an object of \mathcal{S}^{op}) and hence it is globally small. The automorphism space of a point $x \in X$ for a space $X \in \mathcal{S}$ is the (generally non-contractible) loop space $\Omega_X(x)$.

3.21. **EXAMPLE.** More generally, given an ∞ -category \mathcal{C} with finite products, for any given simplicial set I , a \mathcal{C} -indexed ∞ -category \mathcal{E} has $(\emptyset \rightarrow I)$ -comprehension if and only if the presheaf $(\mathcal{E}^I)^\simeq$ of fiberwise I -indexed diagrams in \mathcal{E} is representable. For instance, the identity $1: \text{Cat}_\infty \rightarrow \text{Cat}_\infty$ considered as a Cat_∞^{op} -indexed ∞ -category has $(\emptyset \rightarrow I)$ -comprehension in trivial fashion. Thus, just as global smallness of \mathcal{E} expresses representability of the presheaf \mathcal{E}^\simeq of objects in \mathcal{E} , for instance $(\emptyset \rightarrow \Delta^0 \sqcup \Delta^0)$ -comprehension

expresses representability of the presheaf $\mathcal{E}^{\simeq} \times \mathcal{E}^{\simeq}$ of pairs of objects in \mathcal{E} .

3.22. EXAMPLE. In fact, the identity $1: (\text{Cat}_{\infty}^{\text{op}})^{\text{op}} \rightarrow \text{Cat}_{\infty}$ has G -comprehension for every functor $G: \mathcal{I} \rightarrow \mathcal{J}$ between small ∞ -categories \mathcal{I}, \mathcal{J} . Indeed, for a given span $\text{Cat}_{\infty}(\mathcal{C}, \cdot) \rightarrow \text{Cat}_{\infty}(\mathcal{I}, \cdot) \xleftarrow{G^*} \text{Cat}_{\infty}(\mathcal{J}, \cdot)$ in $\widehat{\text{Cat}_{\infty}^{\text{op}}}$, the pullback is represented by the pushout of the associated cospan $\mathcal{C} \leftarrow \mathcal{I} \xrightarrow{G} \mathcal{J}$ in Cat_{∞} .

3.23. EXAMPLE. [Local smallness] Say $\mathcal{E}: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_{\infty}$ is *locally small* if it has δ^1 -comprehension, where $\delta^1: \partial\Delta^1 \rightarrow \Delta^1$ is the boundary inclusion. That is, if for all $C \in \mathcal{C}$ and all $e_1, e_2: y(C) \rightarrow \mathcal{E}^{\simeq}$ in $\hat{\mathcal{C}}$, the pullback

$$\begin{array}{ccc} \bullet & \longrightarrow & (\mathcal{E}^{\Delta^1})^{\simeq} \\ \downarrow & \lrcorner & \downarrow (\delta^1)^* \\ y(C) & \xrightarrow{(e_1, e_2)} & \mathcal{E}^{\simeq} \times \mathcal{E}^{\simeq} \end{array}$$

is representable.

Recall that an ∞ -category \mathcal{C} is defined to be locally cartesian closed if and only if for every object $C \in \mathcal{C}$, the slice $\mathcal{C}_{/C}$ is cartesian closed. That means that for every pair $x, y \in \mathcal{C}_{/C}$, the presheaf

$$\mathcal{C}_{/C}(x \times_C (\cdot), y): (\mathcal{C}_{/C})^{\text{op}} \rightarrow \mathcal{S}$$

is representable. We have the following characterization of local cartesian closedness, generalizing the corresponding result for 1-categories in [50, Theorem 10.2].

3.24. PROPOSITION. *Suppose \mathcal{C} is a small ∞ -category and has pullbacks. Then the canonical indexing $\mathcal{C}_{/(\cdot)}: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_{\infty}$ is locally small if and only if \mathcal{C} is locally cartesian closed.*

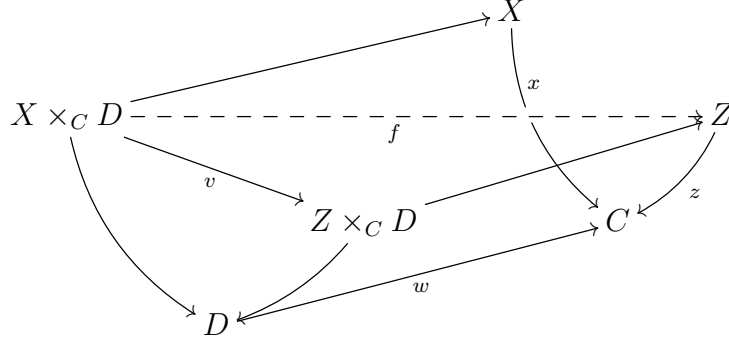
PROOF. We show that under the equivalence $\hat{\mathcal{C}}_{/y(C)} \simeq \widehat{\mathcal{C}_{/C}}$, the pullback

$$\begin{array}{ccc} P(x, z) & \longrightarrow & ((\mathcal{C}_{/(\cdot)})^{\Delta^1})^{\simeq} \\ \downarrow & \lrcorner & \downarrow (d_1^*, d_0^*) \\ y(C) & \xrightarrow{(x, z)} & (\mathcal{C}_{/(\cdot)})^{\simeq} \times (\mathcal{C}_{/(\cdot)})^{\simeq} \end{array} \tag{12}$$

is equivalent to the presheaf $\mathcal{C}_{/C}(x \times_C (\cdot), z): (\mathcal{C}_{/C})^{\text{op}} \rightarrow \mathcal{S}$ for all $C \in \mathcal{C}$ and all morphisms $x, z \in \mathcal{C}_{/C}$. Since under this equivalence a morphism over $y(C)$ with representable domain corresponds exactly to a representable presheaf in $\widehat{\mathcal{C}_{/C}}$, it follows that representability of one implies representability of the other. In particular, the slices $\mathcal{C}_{/C}$ are cartesian closed if and only if the canonical indexing of \mathcal{C} is locally small.

To show equivalence of the two presheaves over $\mathcal{C}_{/C}$, it suffices to show that their associated right fibrations over $\mathcal{C}_{/C}$ are equivalent to one another. Indeed, as the canonical indexing of \mathcal{C} is only indirectly defined as the straightening of the target fibration over \mathcal{C} ,

it is much easier to work in the cartesian model structure over \mathcal{C} using the observations made in Section 2. In essence, this amounts to showing that the space of morphisms $f: x \times_{\mathcal{C}} w \rightarrow z$ in the slice $\mathcal{C}/_{\mathcal{C}}$ is equivalent to the space of squares considered as morphisms $w^*x \rightarrow z$ in the ∞ -category $\text{Fun}(\Delta^1, \mathcal{C})$. Such in turn are essentially given by tuples (w, v) in the ∞ -category $\text{Un}((P(x, z)))$ by factorization through the pullback w^*z .



Hence, the statement follows from the fact that the class of squares in \mathcal{C} which are vertical up to equivalence, together with the class of cartesian squares in \mathcal{C} , yields a factorization system on the domain $\text{Fun}(\Delta^1, \mathcal{C})$.

Now to the implementation of the proof. The unstraightening of the pullback square (12) yields a pullback square of the form

$$\begin{array}{ccc}
 \text{Un}_{\mathcal{C}}(P(x, z)) & \longrightarrow & (\text{Fun}(\Delta^1, \mathcal{C})^{\Delta^1})^{\times} \\
 p(x, z) \downarrow & \lrcorner & \downarrow (d_1, d_0) \\
 \mathcal{C}/_{\mathcal{C}} & \xrightarrow{(x, z)} & \text{Fun}(\Delta^1, \mathcal{C})^{\times} \times_{\mathcal{C}} \text{Fun}(\Delta^1, \mathcal{C})^{\times}
 \end{array} \tag{13}$$

in the contravariant model category $\text{RFib}(\mathcal{C})$ over \mathcal{C} . Here, the upper right corner is the core of the cartesian fibration t^{Δ^1} of vertical morphisms in $t: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$. It is computed as the according simplicial cotensor in the cartesian model category $\text{Cart}(\mathcal{C})$. The fact that the vertical fibration on the right hand side of (13) is equivalent to the unstraightening of the according natural transformation in (12) is an instance of Proposition 4.4.

We hence want to show that the fibration $p(x, z)$ is equivalent to the unstraightening $\text{Un}_{\mathcal{C}/_{\mathcal{C}}}(\mathcal{C}/_{\mathcal{C}}(x \times_{\mathcal{C}} (\cdot), z)) \rightarrow \mathcal{C}/_{\mathcal{C}}$. Using that every right fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ induces trivial fibrations $p/e: \mathcal{E}/_e \rightarrow \mathcal{C}/_{p(e)}$ on overcategories for every $e \in \mathcal{E}$, we obtain a sequence of categorical equivalences over \mathcal{C} as follows.

$$\begin{aligned}
 \text{Un}_{\mathcal{C}}(P(x, z)) &\simeq \left(\mathcal{C}/_{\mathcal{C}} \times_{\mathcal{C}/_{\mathcal{C}}} \mathcal{C}/_{\mathcal{C}} \right) \times_{(\text{Fun}(\Delta^1, \mathcal{C}))^{\times} \times_{\mathcal{C}} (\text{Fun}(\Delta^1, \mathcal{C}))^{\times}} (\text{Fun}(\Delta^1, \mathcal{C})^{\Delta^1})^{\times} \\
 &\simeq \left(\text{Fun}(\Delta^1, \mathcal{C})^{\times}_{/x} \times_{\mathcal{C}/_{\mathcal{C}}} \text{Fun}(\Delta^1, \mathcal{C})^{\times}_{/z} \right) \times_{(\text{Fun}(\Delta^1, \mathcal{C}))^{\times} \times_{\mathcal{C}} (\text{Fun}(\Delta^1, \mathcal{C}))^{\times}} (\text{Fun}(\Delta^1, \mathcal{C})^{\Delta^1})^{\times} \\
 &\simeq \text{Fun}(\Delta^1, \mathcal{C})^{\times}_{/x} \times_{(\mathcal{C}/_{\mathcal{C}} \times_{\mathcal{C}} (\text{Fun}(\Delta^1, \mathcal{C}))^{\times})} \underbrace{\left(\text{Fun}(\Delta^1, \mathcal{C})^{\times}_{/z} \times_{\text{Fun}(\Delta^1, \mathcal{C})^{\times}} (\text{Fun}(\Delta^1, \mathcal{C})^{\Delta^1})^{\times} \right)}_{(*)}
 \end{aligned} \tag{14}$$

Here, the last equivalence is simply given by a permutation of components. Using Joyal's alternative join construction and its according alternative overcategories ([27, Section 4.2.1, Proposition 4.2.1.2]), the right component of the fiber product (14) can be in turn expressed over the base $\mathcal{C}/\mathcal{C} \times_{\mathcal{C}} (\text{Fun}(\Delta^1, \mathcal{C}))^\times$ as follows.

$$\begin{aligned}
(*) &\simeq (\{z\} \times_{\text{Fun}(\Delta^1, \mathcal{C})^\times} \text{Fun}(\Delta^1, \text{Fun}(\Delta^1, \mathcal{C})^\times)) \times_{\text{Fun}(\Delta^1, \mathcal{C})^\times} (\text{Fun}(\Delta^1, \mathcal{C})^{\Delta^1})^\times \\
&\simeq (\{z\} \times_{\text{Fun}(\Delta^1, \mathcal{C})^\times} \text{Fun}^\sharp((\Delta^1)^\sharp, \text{Fun}(\Delta^1, \mathcal{C})^\natural)) \times_{\text{Fun}(\Delta^1, \mathcal{C})^\times} \text{Fun}_v^\sharp((\Delta^1)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural) \\
&\simeq \{z\} \times_{\text{Fun}(\Delta^1, \mathcal{C})^\times} (\text{Fun}^\sharp((\Delta^1)^\sharp, \text{Fun}(\Delta^1, \mathcal{C})^\natural) \times_{\text{Fun}(\Delta^1, \mathcal{C})^\times} \text{Fun}_v^\sharp((\Delta^1)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural)) \\
&\simeq \{z\} \times_{\text{Fun}(\Delta^1, \mathcal{C})^\times} \text{Fun}_{hv}^\sharp((\Lambda_1^2)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural) \\
&\simeq \{z\} \times_{\text{Fun}(\Delta^1, \mathcal{C})^\times} \text{Fun}_{hv}^\sharp((\Delta^2)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural) \\
&\simeq \{z\} \times_{\text{Fun}(\Delta^1, \mathcal{C})^\times} \text{Fun}^\sharp((\Delta^1)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural) \\
&\simeq (\text{Fun}(\Delta^1, \mathcal{C})/z)^{\text{cart.}}.
\end{aligned} \tag{15}$$

Here, $\text{Fun}_v^\sharp((\Delta^1)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural) \subset \text{Fun}^\sharp((\Delta^1)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural)$ denotes the full ∞ -subcategory spanned by the vertical squares, i.e. those whose target morphism in \mathcal{C} is an equivalence. It is the essential image of the inclusion $(\text{Fun}(\Delta^1, \mathcal{C})^{\Delta^1})^\times \hookrightarrow \text{Fun}^\sharp((\Delta^1)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural)$. The ∞ -category $\text{Fun}_{hv}^\sharp((\Lambda_1^2)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural)$ denotes the full subcategory of $\text{Fun}^\sharp((\Lambda_1^2)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural)$ given by those spans whose first leg is a vertical square and whose second leg is a cartesian square. The ∞ -category $\text{Fun}_{hv}^\sharp((\Delta^2)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural)$ is defined accordingly as the fiber of $\text{Fun}_{hv}^\sharp((\Lambda_1^2)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural)$ of the obvious restriction functor along the inner horn inclusion $h_1^2: (\Lambda_1^2)^\flat \hookrightarrow (\Delta^2)^\flat$. Then the equivalence (15) follows directly from the fact that the pair of vertical and cartesian morphisms in the domain $\text{Fun}(\Delta^1, \mathcal{C})$ form a factorization system, see specifically [27, Example 5.2.8.15] and [27, Proposition 5.2.8.17]. Lastly, the ∞ -category $(\text{Fun}(\Delta^1, \mathcal{C})/z)^{\text{cart.}} := \{z\} \times_{\text{Fun}(\Delta^1, \mathcal{C})^\times} \text{Fun}^\sharp((\Delta^1)^\flat, \text{Fun}(\Delta^1, \mathcal{C})^\natural)$ denotes the wide ∞ -subcategory of the slice $\text{Fun}(\Delta^1, \mathcal{C})/z$ spanned by the cartesian squares. Hence, via (14), the unstraightening $\text{Un}_{\mathcal{C}}(P(x, z))$ fits into a homotopy-cartesian square of ∞ -categories over \mathcal{C} as follows.

$$\begin{array}{ccc}
\text{Un}_{\mathcal{C}}(P(x, z)) & \longrightarrow & (\text{Fun}(\Delta^1, \mathcal{C})/z)^{\text{cart.}} \\
\downarrow & & \downarrow (t/z, s) \\
\text{Fun}(\Delta^1, \mathcal{C})/x & \xrightarrow{(t/x, s)} & \mathcal{C}/\mathcal{C} \times_{\mathcal{C}} \text{Fun}(\Delta^1, \mathcal{C})^\times
\end{array}$$

The base $\mathcal{C}/\mathcal{C} \times_{\mathcal{C}} \text{Fun}(\Delta^1, \mathcal{C})^\times$ is isomorphic to the ∞ -category $\text{Fun}_{\text{hall}}(\Lambda_1^2, \mathcal{C}) \times_{\mathcal{C}} \{C\}$, where $\text{Fun}_{\text{hall}}(\Lambda_1^2, \mathcal{C})$ is defined as the wide ∞ -subcategory of $\text{Fun}(\Lambda_1^2, \mathcal{C})$ given by those spans of squares whose $d^{\{0,1\}}$ -face square is cartesian. The fiber $\text{Fun}_{\text{hall}}(\Lambda_1^2, \mathcal{C}) \times_{\mathcal{C}} \{C\}$ at $C \in \mathcal{C}$ is taken with respect to the restriction of \mathcal{C} -valued functors along the end-point inclusion

$\{2\}: \Delta^0 \hookrightarrow \Lambda_1^2$. We obtain a composite diagram as follows.

$$\begin{array}{ccccc}
 \mathrm{Un}_{\mathcal{C}}(P(x, z)) & \longrightarrow & (\mathrm{Fun}(\Delta^1, \mathcal{C})_{/z})^{\mathrm{cart.}} & \xlongequal{\quad} & (\mathrm{Fun}(\Delta^1, \mathcal{C})_{/z})^{\mathrm{cart.}} & \longrightarrow & (\mathcal{C}_{/C})_{/z} \\
 \downarrow & & \downarrow & & \downarrow & \lrcorner & \downarrow s \\
 p(x, z) \left(\mathrm{Fun}(\Delta^1, \mathcal{C})_{/x}^{\times} \right. & \longrightarrow & \mathrm{Fun}_{\mathrm{hall}}(\Lambda_1^2, \mathcal{C}) \times_{\mathcal{C}} \{C\} & \xleftarrow[\sim]{(h_1^2)^*} & \mathrm{Fun}_{\mathrm{hall}}(\Delta^2, \mathcal{C}) \times_{\mathcal{C}} \{C\} & \xrightarrow{d_1} & \mathcal{C}_{/C} \\
 \downarrow \wr & & \downarrow & \dashrightarrow & \downarrow & \dashrightarrow & \downarrow \\
 \mathcal{C}_{/C} & & & & & & \mathcal{C}_{/C} \\
 & & & \dashrightarrow & & & \\
 & & & x \times_{\mathcal{C}} (\cdot) & & &
 \end{array}$$

Here, the ∞ -category $\mathrm{Fun}_{\mathrm{hall}}(\Delta^2, \mathcal{C})$ is again the wide subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$ given by those prisms $\Delta^2 \times \Delta^1$ whose $(d_2 \times 1)$ -face is a cartesian square. The fact that the upper right hand square yields a cartesian square in Cat_{∞} can be computed combinatorially, using that any strict pullback of the right fibration $s: (\mathcal{C}_{/C})_{/z} \rightarrow \mathcal{C}_{/C}$ represents the according homotopy-pullback. The dotted functor $x \times_{\mathcal{C}} (\cdot): \mathcal{C}_{/C} \rightarrow \mathcal{C}_{/C}$ is the essentially unique functor making the resulting triangle for any given section of the trivial fibration $(h_1^2)^*$ commute up to homotopy. Since all three squares are homotopy-cartesian (for any given section of $(h_1^2)^*$), it follows that the composite rectangle is homotopy-cartesian. Hence, as the right fibration $\mathrm{Un}_{\mathcal{C}_{/C}}(\mathcal{C}_{/C}(x \times_{\mathcal{C}} (\cdot), z)) \rightarrow \mathcal{C}_{/C}$ is the homotopy-pullback of the representable $(\mathcal{C}_{/C})_{/z} \rightarrow \mathcal{C}_{/C}$ along the fiber-product $x \times_{\mathcal{C}} (\cdot): \mathcal{C}_{/C} \rightarrow \mathcal{C}_{/C}$ as well, we obtain an equivalence

$$p(x, z) \simeq \mathrm{Un}(\mathcal{C}_{/C}(x \times_{\mathcal{C}} (\cdot), z))$$

over $\mathcal{C}_{/C}$. ■

3.25. EXAMPLE. [Smallness of representables] Let $C \in \mathcal{C}$ be an object and consider its associated representable presheaf $y(C) \in \hat{\mathcal{C}}$. On the one hand, the indexed ∞ -category $y(C): \mathcal{C}^{op} \rightarrow \mathcal{S} \hookrightarrow \mathrm{Cat}_{\infty}$ satisfies $(\partial\Delta^1 \rightarrow \Delta^1)$ -comprehension if and only if the natural transformation $y(C)^{\Delta^1} \rightarrow y(C) \times y(C)$ is representable. This natural transformation however is equivalent to the diagonal $\Delta: y(C) \rightarrow y(C) \times y(C)$. Thus, $y(C)$ has $(\partial\Delta^1 \rightarrow \Delta^1)$ -comprehension if and only if for every object $D \in \mathcal{C}$ and every pair of morphisms $f, g: D \rightarrow C$, the pullback

$$\begin{array}{ccc}
 \mathrm{Eq}(y(f), y(g)) & \longrightarrow & y(C) \\
 \downarrow & \lrcorner & \downarrow \Delta \\
 y(D) & \xrightarrow{(y(f), y(g))} & y(C) \times y(C)
 \end{array}$$

is representable. This pullback computes the equalizer of the pair $y(f), y(g): y(D) \rightarrow y(C)$ and hence the limit of the diagram

$$y(C) \begin{array}{c} \xrightarrow{y(f)} \\ \xrightarrow{y(g)} \end{array} y(D)$$

in $\hat{\mathcal{C}}$. As the Yoneda embedding reflects limits, we see that all such pullbacks $\text{Eq}(y(f), y(g))$ are representable if and only if \mathcal{C} has all equalizers of pairs of morphisms with codomain C . Thus, all representable presheaves of \mathcal{C} are locally small if and only if the ∞ -category \mathcal{C} has all equalizers. On the other hand, suppose that \mathcal{C} has a terminal object. Then $y(C)$ is globally small if and only if products with C in \mathcal{C} exist (via Example 2.8 and the fact that the Yoneda embedding reflects limits). Thus, all representable presheaves of \mathcal{C} are globally small if and only if the ∞ -category \mathcal{C} has all finite products. In particular, assuming \mathcal{C} has a terminal object, it has all finite limits if and only if its representable presheaves are both globally and locally small over \mathcal{C} . In turn, evidently, \mathcal{C} has a terminal object if and only if the terminal presheaf $* \in \hat{\mathcal{C}}$ is representable.

3.26. EXAMPLE. Consider the indexed ∞ -category $\{\mathcal{C}\}: \Delta^0 \rightarrow \text{Cat}_\infty$ given by the value $\mathcal{C} \in \text{Cat}_\infty$. By Example 3.19, as the ∞ -category Δ^0 has products, the indexed ∞ -category $\{\mathcal{C}\}$ is globally small if and only if the presheaf $\{\mathcal{C}\}^\simeq$ is representable, or in other words, if and only if the core \mathcal{C}^\simeq is a contractible space. It is locally small if and only if all hom-spaces of \mathcal{C} are contractible.

The two characterizations in Example 3.26 allow easy constructions of cartesian fibrations which are globally small but not locally small. For instance, we may take the free monoid on one generator and consider it as an ordinary category \mathcal{M} with one object. Its nerve yields a cartesian fibration $N(\mathcal{M}) \rightarrow \Delta^0$ which is globally small but not locally small. Vice versa, for instance the ∞ -category \mathcal{S} of spaces is locally small over itself (i.e. its canonical indexing is locally small). Since it is a locally small ∞ -category as well, but its slice ∞ -categories are generally large, one can show that it cannot be globally small over itself. Hence, Proposition 3.24 and Example 3.26 together show that, in general, global smallness and local smallness are mutually independent properties. This mutual independence can also directly be seen via the following example.

3.27. EXAMPLE. [Smallness and size] The ∞ -category Cat_∞ as introduced in Section 2 is the ∞ -category of *small* ∞ -categories. Accordingly, CAT_∞ denotes the ∞ -category of large ∞ -categories (via assumption of an inaccessible cardinal for example). Consider the fully faithful inclusion $\iota: \text{Cat}_\infty \hookrightarrow \text{CAT}_\infty$. Given a large ∞ -category \mathcal{C} , we can consider its associated large indexing from Example 2.2.1 restricted to small ∞ -categories given as follows.

$$\text{Cat}_\infty \xrightarrow{\iota} \text{CAT}_\infty \xrightarrow{\text{Fun}(\cdot, \mathcal{C})} \text{CAT}_\infty \quad (16)$$

The core of the indexing $\text{Fun}(\cdot, \mathcal{C})$ over CAT_∞ is representable as explained in Example 2.4.1. That means, it is a globally small CAT_∞ -indexed ∞ -category. Let us show that the ∞ -category \mathcal{C} is essentially small if and only if the restricted indexing (16) is a globally small Cat_∞ -indexed ∞ -category. Therefore, recall that \mathcal{C} is essentially small if and only if there is an ∞ -category $\mathcal{C}' \in \text{Cat}_\infty$ together with an equivalence $\mathcal{C}' \simeq \mathcal{C}$. Now, every such equivalence between \mathcal{C} and some $\mathcal{C}' \in \text{Cat}_\infty$ induces an equivalence between the presheaf $\text{CAT}_\infty(\iota(\cdot), \mathcal{C})$ and the representable presheaf $\text{Cat}_\infty(\cdot, \mathcal{C}') \simeq$

$\text{CAT}_\infty(\iota(\cdot), \iota(\mathcal{C}'))$. Vice versa, if $\text{CAT}_\infty(\iota(\cdot), \mathcal{C})$ is represented by some $\mathcal{C}' \in \text{Cat}_\infty$, the equivalence $\text{Cat}_\infty(\cdot, \mathcal{C}) \xrightarrow{\cong} \text{CAT}_\infty(\iota(\cdot), \mathcal{C})$ induces an equivalence

$$\text{Fun}(\Delta^\bullet, \mathcal{C}') \simeq \xrightarrow{\cong} \text{Fun}(\Delta^\bullet, \mathcal{C}) \simeq$$

of simplicial spaces via precomposition with the canonical inclusion $\Delta \hookrightarrow \text{Cat}_\infty$. This equivalence is a functor between the complete Segal spaces associated to the quasi-categories \mathcal{C} and \mathcal{C}' (constructed in [23, Section 4], see Section 5). Thus, it in turn yields an equivalence $\mathcal{C}' \xrightarrow{\cong} \mathcal{C}$.

Similarly, recall from [27, Section 5.4.1] that an ∞ -category \mathcal{C} is *locally small* if all its associated hom-spaces are essentially small. For example, the large ∞ -category Cat_∞ itself is locally small. Then one can show that a large ∞ -category \mathcal{C} is locally small if and only if its restricted indexing (16) is a locally small Cat_∞ -indexed ∞ -category. In order to show this, one computes that the latter holds if and only if for all small ∞ -categories \mathcal{D} and all pairs of functors $F, G: \mathcal{D} \rightarrow \mathcal{C}$ the pullback

$$\begin{array}{ccc} \text{Fun}(\iota(\cdot), F \downarrow G) \simeq & \longrightarrow & \text{Fun}(\iota(\cdot), \text{Fun}(\Delta^1, \mathcal{C})) \simeq \\ \downarrow & & \downarrow (s,t)_* \\ \text{Fun}(\cdot, \mathcal{D}) \simeq & \xrightarrow{(F,G)} & \text{Fun}(\iota(\cdot), \mathcal{C} \times \mathcal{C}) \simeq \end{array}$$

of presheaves over Cat_∞ is representable. This in turn holds for a given such pair $F, G: \mathcal{D} \rightarrow \mathcal{C}$ if and only if the comma- ∞ -category $F \downarrow G$ is essentially small (via the first part of this example). Essential smallness of these comma ∞ -categories directly implies essential smallness of the hom-space $\mathcal{C}(C, D) \simeq C \downarrow D$ for all pairs of objects $C, D: \Delta^0 \rightarrow \mathcal{C}$. For the converse it is easy to see that local smallness of \mathcal{C} and smallness of \mathcal{D} implies essential smallness of the comma ∞ -category $F \downarrow G$ for all functors $F, G: \mathcal{D} \rightarrow \mathcal{C}$.

Alternatively, one can consider the naive large indexing

$$\mathcal{S} \hookrightarrow \mathcal{S}^+ \xrightarrow{\text{Fun}(\cdot, \mathcal{C})} \text{CAT}_\infty \tag{17}$$

where \mathcal{S}^+ denotes the ∞ -category of large ∞ -groupoids accordingly. Then one computes that the \mathcal{S} -indexed large ∞ -category (17) is globally small if and only if the core \mathcal{C}^\simeq is a small space. It is a locally small \mathcal{S} -indexed large ∞ -category again if and only if \mathcal{C} is a locally small ∞ -category.

In order to deduce various comprehension schemes from a given set of comprehension schemes for a fixed \mathcal{C} -indexed ∞ -category \mathcal{E} (as stated to be possible in principle in [8, Paragraph 8.5]), we observe that the class of functors

$$\text{Comp}(\mathcal{E}) := \{(G: I \rightarrow J) \in \mathbf{S} \mid \mathcal{E} \text{ satisfies } G\text{-comprehension}\}$$

satisfies the following stability properties, where G -comprehension for a map $G: I \rightarrow J$ of simplicial sets was defined in Notation 3.17.

3.28. LEMMA. *Let \mathcal{E} be a \mathcal{C} -indexed ∞ -category.*

1. *The class $\text{Comp}(\mathcal{E})$ contains all weak categorical equivalences. It is closed under composition and under homotopy-pushouts (in the Joyal model structure). In particular, it is closed under pushouts along monomorphisms.*
2. *The class of monomorphisms in $\text{Comp}(\mathcal{E})$ contains all trivial cofibrations in the Joyal model structure and is closed under compositions and arbitrary pushouts.*
3. *If \mathcal{C} has pullbacks, the class $\text{Comp}(\mathcal{E})$ is furthermore closed under finite homotopy-colimits in the Joyal model structure and has the right cancellation property.*

PROOF. Part 1 basically follows directly from Lemma 2.11.1. More precisely, the class $\text{Comp}(\mathcal{E})$ contains the weak categorical equivalences by Example 3.6 and Lemma 3.18. Furthermore, as the class of representable natural transformations is stable under pullbacks, and the Yoneda embedding $\text{Cat}_\infty^{\text{op}} \rightarrow \text{Fun}(\text{Cat}_\infty, \mathcal{S})$ preserves small limits, it follows that the class $\text{Comp}(\mathcal{E})$ is closed under homotopy-pushouts. As the Joyal model structure is left proper, it follows that $\text{Comp}(\mathcal{E})$ is closed under pushouts along monomorphisms as well.

For Part 2 we only are left to show that arbitrary pushouts of monomorphisms in $\text{Comp}(\mathcal{E})$ are contained in $\text{Comp}(\mathcal{E})$ again. Therefore, let $j: A \hookrightarrow B$ be a monomorphism in $\text{Comp}(\mathcal{E})$ and consider a pushout square of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ j \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & D \end{array}$$

in \mathbf{S} . We can factor the map $f: A \rightarrow C$ into a monomorphism $i: A \rightarrow C'$ followed by a trivial isofibration $q: C' \rightarrow C$, and consider the induced factorization of pushouts

$$\begin{array}{ccccc} A & \xrightarrow{i} & C' & \xrightarrow{q} & C \\ j \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & D' & \longrightarrow & D. \end{array}$$

Then the map $C' \rightarrow D'$ is contained in $\text{Comp}(\mathcal{E})$ by Part 2. The map $D' \rightarrow D$ is a weak categorical equivalence, again because the Joyal model structure is left proper. By Example 3.6, it follows that the map $C' \rightarrow D'$ is contained in $\text{Comp}(\mathcal{E})$ if and only if $C \rightarrow D$ is contained in $\text{Comp}(\mathcal{E})$. Part 3 follows directly from Lemma 2.11.2. \blacksquare

3.29. EXAMPLE. Following [21, Lemma B1.3.15], we say that a \mathcal{C} -indexed ∞ -category \mathcal{E} satisfies *definability of invertibility (for morphisms)* if it has *inv*-comprehension, where $\text{inv}: \Delta^1 \rightarrow I\Delta^1$ is the embedding of Δ^1 into the free groupoid generated by it.

It has *definability of identity (of parallel morphisms)* if it has ∇_1 -comprehension, where the map $\nabla_1: (\Delta^1 \cup_{\partial\Delta^1} \Delta^1) \rightarrow \Delta^1$ identifies the two parallel non-degenerate morphisms. The domain can be described as the nerve of the parallel morphisms category $\bullet \rightrightarrows \bullet$.

Furthermore, say \mathcal{E} has *definability of identity of parallel n -morphisms* if it has ∇_n -comprehension, where the map $\nabla_n: (\Delta^n \cup_{\partial\Delta^n} \Delta^n) \rightarrow \Delta^n$ is the codiagonal identifying the two parallel non-degenerate n -morphisms.

3.30. **REMARK.** Definability of identity of n -morphisms for $n \geq 1$ is phrased as a principle of strict identity rather than of equivalence. However, it can equivalently be expressed as comprehension for the canonical inclusion $\nabla_n^{\simeq}: (\Delta^n \cup_{\partial\Delta^n} \Delta^n) \hookrightarrow B^n$ where B^n denotes the pushout

$$\begin{array}{ccc} \partial\Delta^n \times I\Delta^1 & \xrightarrow{\delta^n \times 1} & \Delta^n \times I\Delta^1 \\ \pi_1 \downarrow & & \downarrow \\ \partial\Delta^n & \xrightarrow{\quad \Gamma \quad} & B^n. \end{array} \tag{18}$$

The simplicial set B^n essentially consists of two n -cells glued together along their boundary and a homotopy pasted in between. Since the space $I\Delta^1$ is contractible as a quasi-category and the Joyal model structure is left proper, it follows that the canonical map $B^n \rightarrow \Delta^n$ induced by the projection $\pi_1: \Delta^n \times I\Delta^1 \rightarrow \Delta^n$ is a categorical equivalence. The composition $(\Delta^n \cup_{\partial\Delta^n} \Delta^n) \hookrightarrow B^n \xrightarrow{\simeq} \Delta^n$ is exactly ∇_n , and thus it follows that ∇_n -comprehension is equivalent to ∇_n^{\simeq} -comprehension via Lemma 3.28.1.

Note that in the case $n = 1$, the pushout (18) is in fact a pushout of (nerves of) 1-categories. However, the according pushout computed in \mathbf{Cat} (instead of \mathbf{S}) is just Δ^1 itself. We hence obtain no such inclusion of 1-categories which represents definability of identity of parallel morphisms in an indexed (∞ -)category.

The following lemma can be thought of as an ∞ -categorical generalization of [21, Lemma 1.3.15] extended by its natural higher analogues.

3.31. **PROPOSITION.** *Let \mathcal{E} be a \mathcal{C} -indexed ∞ -category and $\delta^n: \partial\Delta^n \hookrightarrow \Delta^n$ be the standard boundary inclusion for $n \geq 0$.*

1. *Let $n \geq 1$. If \mathcal{E} has definable identity of parallel n -morphisms, it has δ^{n+1} -comprehension.*
2. *Suppose all slice ∞ -categories of \mathcal{C} have equalizers of global sections.*
 - (a) *Let $n \geq 1$. If \mathcal{E} has δ^n -comprehension, it has definable identity for parallel n -morphisms.*
 - (b) *If \mathcal{E} is locally small, it has δ^n -comprehension for all $n \geq 1$.*
 - (c) *If \mathcal{E} is locally small, it has definability of invertibility.*

PROOF. For Part 1. let $n \geq 1$ and $0 < i < n$. We may consider a factorization of the n -th boundary inclusion given by the bottom left square in the following diagram.

$$\begin{array}{ccccc}
\Lambda_i^{n+1} & \xrightarrow{h_i^{n+1}} & \Delta^{n+1} & & \\
\downarrow \wr_2 & & \downarrow \wr_2 & & \\
\Delta^n \cup_{\partial \Delta^n} \Lambda_i^{n+1} & \xrightarrow{(id, h_i^{n+1})} & \Delta^n \cup_{\partial \Delta^n} \Delta^{n+1} & \xrightarrow{(id, d^i)} & \Delta^n \cup_{\partial \Delta^n} \Delta^n \\
\uparrow \wr_1 & & \downarrow (d^i, id) & & \downarrow \nabla_n \\
\partial \Delta^{n+1} & \xrightarrow{\delta^{n+1}} & \Delta^{n+1} & \xleftarrow{d_i} & \Delta^n
\end{array}$$

The inner horn inclusion $h_i^{n+1}: \Lambda_i^{n+1} \rightarrow \Delta^{n+1}$ is contained in $\text{Comp}(\mathcal{E})$ by Example 3.6, the codiagonal $\nabla_n: \Delta^n \cup_{\partial \Delta^n} \Delta^n \rightarrow \Delta^n$ is contained in the class $\text{Comp}(\mathcal{E})$ by assumption. It thus follows from Lemma 3.28.2 that the composition $\delta^{n+1}: \partial \Delta^{n+1} \rightarrow \Delta^{n+1}$ is contained in the class $\text{Comp}(\mathcal{E})$ as well.

For Part 2.(a) assume that \mathcal{E} has δ^n -comprehension. The natural transformation $\nabla_n^*: (\mathcal{E}^{\Delta^n})^\simeq \rightarrow (\mathcal{E}^{\Delta^n \cup_{\partial \Delta^n} \Delta^n})^\simeq$ is equivalent to the diagonal

$$\Delta_{(\delta^n)^*}: (\mathcal{E}^{\Delta^n})^\simeq \rightarrow (\mathcal{E}^{\Delta^n})^\simeq \times_{(\mathcal{E}^{\partial \Delta^n})^\simeq} (\mathcal{E}^{\Delta^n})^\simeq.$$

Let $C \in \mathcal{C}$ and consider a natural transformation $(\ulcorner \alpha^\ulcorner, \ulcorner \beta^\ulcorner): y(C) \rightarrow (\mathcal{E}^{\Delta^n})^\simeq \times_{(\mathcal{E}^{\partial \Delta^n})^\simeq} (\mathcal{E}^{\Delta^n})^\simeq$ which represents n -simplices $\alpha, \beta \in \mathcal{E}(C)^{\Delta^n}$ together with an equivalence $e: (\delta^n)^*(\alpha) \simeq (\delta^n)^*(\beta)$ between their respective boundaries. By assumption there is a pullback square of the form

$$\begin{array}{ccc}
y\bar{C} & \xrightarrow{\ulcorner \gamma^\ulcorner} & (\mathcal{E}^{\Delta^n})^\simeq \\
y(\varepsilon) \downarrow & \lrcorner & \downarrow (\delta^n)^* \\
y(C) & \xrightarrow{(\delta^n)^* \ulcorner \alpha^\ulcorner} & (\mathcal{E}^{\partial \Delta^n})^\simeq
\end{array} \tag{19}$$

in $\hat{\mathcal{C}}$. As diagonals are stable under pullback ([3, Lemma 3.4.12]), the square

$$\begin{array}{ccc}
y\bar{C} & \xrightarrow{\ulcorner \gamma^\ulcorner} & (\mathcal{E}^{\Delta^n})^\simeq \\
\Delta_{y(\varepsilon)} \downarrow & \lrcorner & \downarrow \Delta_{(\delta)^*} \\
y\bar{C} \times_{y(C)} y\bar{C} & \xrightarrow{(\ulcorner \gamma^\ulcorner, \ulcorner \gamma^\ulcorner)} & (\mathcal{E}^{\Delta^n})^\simeq \times_{(\mathcal{E}^{\partial \Delta^n})^\simeq} (\mathcal{E}^{\Delta^n})^\simeq
\end{array} \tag{20}$$

is cartesian as well. The natural transformation $\ulcorner \alpha^\ulcorner: y(C) \rightarrow (\mathcal{E}^{\Delta^n})^\simeq$ factors via some

$$y(C) \xrightarrow{y(f_\alpha)} y\bar{C} \xrightarrow{\ulcorner \gamma^\ulcorner} (\mathcal{E}^{\Delta^n})^\simeq$$

by (19), and by virtue of the equivalence $e: (\delta^n)^*(\alpha) \simeq (\delta^n)^*(\beta)$ so does $\ulcorner \beta^\ulcorner: y(C) \rightarrow (\mathcal{E}^{\Delta^n})^\simeq$ via some

$$y(C) \xrightarrow{y(f_\beta)} y\bar{C} \xrightarrow{\ulcorner \gamma^\ulcorner} (\mathcal{E}^{\Delta^n})^\simeq.$$

We obtain a factorization of the pair $(\Gamma\alpha^\ulcorner, \Gamma\beta^\urcorner): y(C) \rightarrow (\mathcal{E}^{\Delta^n})^\simeq \times_{(\mathcal{E}^{\partial\Delta^n})^\simeq} (\mathcal{E}^{\Delta^n})^\simeq$ via

$$y(C) \xrightarrow{(f_\alpha, f_\beta)} y\bar{C} \times_{y(C)} y\bar{C} \xrightarrow{(\Gamma\gamma^\ulcorner, \Gamma\gamma^\urcorner)} (\mathcal{E}^{\Delta^n})^\simeq \times_{(\mathcal{E}^{\partial\Delta^n})^\simeq} (\mathcal{E}^{\Delta^n})^\simeq.$$

Thus, as (20) is cartesian as well, the pullback

$$\begin{array}{ccc} \bullet & \longrightarrow & (\mathcal{E}^{\Delta^n})^\simeq \\ \downarrow & \lrcorner & \downarrow \Delta_{(\delta)^*} \\ y(C) & \xrightarrow{(\Gamma\alpha^\ulcorner, \Gamma\beta^\urcorner)} & (\mathcal{E}^{\Delta^n})^\simeq \times_{(\mathcal{E}^{\partial\Delta^n})^\simeq} (\mathcal{E}^{\Delta^n})^\simeq \end{array}$$

can be computed as the pullback

$$\begin{array}{ccc} \bullet & \longrightarrow & y\bar{C} \\ \downarrow & \lrcorner & \downarrow \Delta_\varepsilon \\ y(C) & \xrightarrow{(f_\alpha, f_\beta)} & y\bar{C} \times_{y(C)} y\bar{C}. \end{array}$$

This pullback however computes the equalizer of the pair of global sections $f_\alpha, f_\beta: y(*) \rightarrow y(\varepsilon)$ in the presheaf ∞ -category $\hat{\mathcal{C}}_{/y(C)} \simeq \widehat{\mathcal{C}}_{/C}$. Thus, this pullback is representable whenever the equalizers of f_α and f_β exists in the slice $\mathcal{C}_{/C}$. This finishes Part (a).

Part 2.(b) follows immediately from Parts 1. and 2.(a) since local smallness is δ^1 -comprehension.

For Part 2.(c) let $J^{(2)} \subset I\Delta^1$ be the subsimplicial set given by exactly one of the two non-degenerate 2-simplices. It can be thought of as the free left (or right) invertible edge, depicted as follows.

$$\begin{array}{ccc} & [0] & \\ f^{-1} \nearrow & & \searrow f \\ [1] & \xrightarrow{s_0([1])} & [1] \end{array}$$

The pushout $K := J^{(2)} \cup_{\Delta^1} J^{(2)}$ along the boundaries d^0 and d^2 is the free biinvertible map, the inclusion $\text{inv}: \Delta^1 \rightarrow I\Delta^1$ factors through K . The natural map $K \rightarrow I\Delta^1$ is a weak categorical equivalence (both are interval objects in the Joyal model structure), and hence by Example 3.6 and Lemma 3.28 we can reduce inv -comprehension to $(\Delta^1 \rightarrow K)$ -comprehension. Since the boundaries $d^i: \Delta^1 \rightarrow J^{(2)}$ for $i = 0, 2$ are monomorphisms, we can reduce $(\Delta^1 \rightarrow K)$ -comprehension further to $(d^0: \Delta^1 \rightarrow J^{(2)})$ -comprehension again by Lemma 3.28.

Therefore, we factor the inclusion $d^0: \Delta^1 \rightarrow J^{(2)}$ through subobjects

$$\Delta^1 \rightarrow \partial J^{(2)} \rightarrow (\partial J^{(2)})_+ \rightarrow J^{(2)}$$

such that each component is the pushout of a map in $\text{Comp}(\mathcal{E})$ along a monomorphism. Then we can use Lemma 3.28 once more to finish the proof. First, consider the circle given by the pushout

$$\begin{array}{ccc} \partial\Delta^1 & \xrightarrow{\delta^1} & \Delta^1 \\ \bar{\delta}^1 \downarrow & & \downarrow \\ \Delta^1 & \longrightarrow & \partial J^{(2)} \end{array} \quad (21)$$

where $\bar{\delta}^1: \partial\Delta^1 \rightarrow \Delta^1$ swaps the endpoints. Second, consider the pushout

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{h_1^2} & \Delta^2 \\ \downarrow & & \downarrow \\ \partial J^{(2)} & \longrightarrow & (\partial J^{(2)})_+ \end{array} \quad (22)$$

where the left vertical map picks out the two non-degenerate edges added in (21) concatenated at $[0]$. Third, we obtain a pushout

$$\begin{array}{ccc} \Delta^1 \cup_{\partial\Delta^1} \Delta^1 & \xrightarrow{\nabla_1} & \Delta^1 \\ \downarrow & & \downarrow \\ (\partial J^{(2)})_+ & \longrightarrow & J^{(2)} \end{array}$$

where the left vertical map picks out the pair consisting of the morphism $[1] \rightarrow [1]$ freely added in (22) and the identity on $[1]$.

The composition of the bottom maps of the three pushout squares is exactly the inclusion $d^0: \Delta^1 \rightarrow J^{(2)}$. The \mathcal{C} -indexed ∞ -category \mathcal{E} has δ^1 -comprehension by assumption, and thus it has ∇_1 -comprehension by Part 2.(a). It has h_1^2 -comprehension by Example 3.6 and thus the statement follows by Lemma 3.28 as claimed. \blacksquare

3.32. COROLLARY. *Suppose \mathcal{C} has pullbacks. Then for a \mathcal{C} -indexed ∞ -category \mathcal{E} the following are equivalent.*

1. \mathcal{E} is both globally small and locally small.
2. \mathcal{E} has both $(\emptyset \rightarrow \Delta^0)$ -comprehension and $(\emptyset \rightarrow \Delta^1)$ -comprehension.
3. \mathcal{E} has j -comprehension for every monomorphism j between simplicial sets with finitely many non-degenerate simplices.
4. \mathcal{E} has j -comprehension for all finite extensions of simplicial sets, i.e for all monomorphisms $j: I \rightarrow J$ of simplicial sets such that the complement $J \setminus I$ (considered as an \mathbb{N} -indexed collection of sets) contains only finitely many non-degenerate simplices of J .

PROOF. The monomorphisms in \mathbf{S} are exactly the free cofibrations generated by the class of boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 0$. In particular, every monomorphism between simplicial sets that adds only finitely many non-degenerate simplices is the composition of pushouts of boundary inclusions. Thus, the equivalence of 1 – 4 follows from Lemma 3.28.3 and Proposition 3.31. \blacksquare

3.33. REMARK. Recall that a quasi-category is finite if it is weakly categorically equivalent to a simplicial set with finitely many non-degenerate simplices. It thus follows from Corollary 3.32 that an ∞ -category \mathcal{E} indexed over an ∞ -category \mathcal{C} with pullbacks is globally and locally small if and only if it has $(\emptyset \rightarrow \mathcal{I})$ -comprehension for all finite quasi-categories \mathcal{I} . We will see in Corollary 5.26 that the finiteness condition vanishes in case the base \mathcal{C} has all small limits. We will furthermore provide an internal characterization of the globally small and locally small indexed ∞ -categories over left exact bases in Section 5.

The following corollary addresses the three examples of comprehension schemes listed in [8, Paragraph 8.5], there considered to be implausible for a general category (in a non-univalent meta-theory) to satisfy. It shows that all three schemes hold for example for the canonical indexing over any locally cartesian closed and left exact ∞ -category.

3.34. COROLLARY. *Let \mathcal{C} be an ∞ -category and \mathcal{E} be a \mathcal{C} -indexed ∞ -category.*

1. *Whenever \mathcal{E} is locally small, it has $(\partial\Delta^1 \rightarrow \Delta^0)$ -comprehension. That means identity of objects in \mathcal{E} is definable.*
2. *\mathcal{E} has $(\text{inv}: \Delta^1 \rightarrow I\Delta^1)$ -comprehension if and only if it has $(\Delta^1 \rightarrow \Delta^0)$ -comprehension. That means invertibility of morphisms in \mathcal{E} is definable (Examples 3.29) if and only if identities in \mathcal{E} are definable (in all morphisms in \mathcal{E}).*
3. *\mathcal{E} has $(\nabla: \Delta^1 \sqcup \Delta^1 \rightarrow \Delta^1)$ -comprehension if and only if it has $((\text{id}, \{0\}), (\text{id}, \{1\})) : \Delta^1 \sqcup \Delta^1 \rightarrow \Delta^1 \times I\Delta^1$ -comprehension. In either case we say that identity of morphisms in \mathcal{E} is definable. Suppose \mathcal{C} has finite limits, and suppose that identity of objects in \mathcal{E} is definable. Then identity of parallel morphisms in \mathcal{E} is definable (Examples 3.29) if and only if identity of morphisms in \mathcal{E} is definable.*

In particular, whenever \mathcal{E} is locally small and \mathcal{C} has all finite limits, it follows that equality of objects, equality of morphisms and identities in \mathcal{E} are definable altogether.

PROOF. For Part 1 consider the diagram

$$\begin{array}{ccc} \partial\Delta^1 & \xrightarrow{\delta^1} & \Delta^1 & \xrightarrow{\text{inv}} & I\Delta^1 \\ & \searrow & & & \swarrow \\ & & \Delta^0 & & \end{array}$$

of simplicial sets. If \mathcal{E} is locally small, the map δ^1 is contained in the class $\text{Comp}(\mathcal{E})$ by definition, and so is the map inv by Proposition 3.31.2.(c). Now by Lemma 3.28.1, since

every equivalence of quasi-categories is contained in $\text{Comp}(\mathcal{E})$ as well, so is the composition of Part 1. For Part 2 we just note that $\Delta^1 \rightarrow \Delta^0$ factors through $\Delta^1 \xrightarrow{\text{inv}} I\Delta^1 \xrightarrow{\simeq} \Delta^0$.

In Part 3, the two comprehension schemes are equivalent as the two respective maps commute over the categorical equivalence $\pi_1: \Delta^1 \times I\Delta^1 \rightarrow \Delta^1$. Whenever \mathcal{E} has $(\nabla: \partial\Delta^1 \rightarrow \Delta^0)$ -comprehension, it has $(\nabla \sqcup \nabla: \partial\Delta^1 \sqcup \partial\Delta^1 \rightarrow \Delta^0 \sqcup \Delta^0)$ -comprehension by Lemma 3.28.3. It follows that the right vertical map in the pushout

$$\begin{array}{ccc} \partial\Delta^1 \sqcup \partial\Delta^1 & \xrightarrow{((0_1, 0_2), (1_1, 1_2))} & \Delta^1 \sqcup \Delta^1 \\ \nabla \sqcup \nabla \downarrow & & \downarrow \\ \Delta^0 \sqcup \Delta^0 & \xrightarrow{\Gamma} & \Delta^1 \cup_{\partial\Delta^1} \Delta^1 \end{array}$$

is contained in the class $\text{Comp}(\mathcal{E})$ as well by Lemma 3.28.1. Lastly, the triangle

$$\begin{array}{ccc} \Delta^1 \cup_{\partial\Delta^1} \Delta^1 & \longrightarrow & \Delta^1 \sqcup \Delta^1 \\ & \searrow \nabla_1 & \downarrow \nabla \\ & & \Delta^1 \end{array}$$

of simplicial sets commutes. As we just have seen that the top map is contained in the class $\text{Comp}(\mathcal{E})$, it follows again from Lemma 3.28 that ∇ is contained in $\text{Comp}(\mathcal{E})$ if and only if ∇_1 is so. ■

We end this section with the following useful transition result for comprehension schemes which is an ∞ -categorical generalization of [21, Proposition 1.3.17].

3.35. LEMMA. *Let \mathcal{E} be a \mathcal{C} -indexed ∞ -category and suppose \mathcal{D} is an ∞ -category with pullbacks. If $F: \mathcal{D} \rightarrow \mathcal{C}$ is a functor with a right adjoint, then $\text{Comp}(F^*\mathcal{E}) \subseteq \text{Comp}(\mathcal{E})$.*

PROOF. This follows from Lemma 2.11.3 by virtue of the natural equivalence

$$F^*(\text{Cat}_\infty(\cdot, \mathcal{E})) \simeq \text{Cat}_\infty(\cdot, F^*\mathcal{E})$$

simply given by associativity of pre- and postcomposition of functors. ■

3.36. REMARK. Lemma 3.35 applies more generally to all comprehension schemes in the generality discussed in the introduction. In particular, it applies to all (K, G) -comprehension schemes for any ∞ -category K and any morphism G in K . The same remark applies to Lemma 3.28.

We will see applications of Lemma 3.35 in the coming sections.

4. Standard diagrammatic comprehension for cartesian fibrations

For various reasons it may be useful at times to express comprehension schemes for a given cartesian fibration directly without having to compute its straightening first. One

reason to do so is that often examples of indexed ∞ -categories are only implicitly given as the straightening of certain cartesian fibrations as the latter are much easier to construct in practice (take the canonical indexing of an ∞ -category with pullbacks for instance as in the proof of Proposition 3.24). Another reason is that theories of fibrations exist in many different contexts and can easily be translated from one to another, while the theory of indexed ∞ -categories does not do so to the same extent. For instance, a theory of cartesian fibrations exists in every ∞ -cosmos as defined by Riehl and Verity [39] and it is furthermore preserved along cosmological functors. This enables us to give a simple proof of independence of our results about comprehension from the specific choice of model of $(\infty, 1)$ -category theory, and even allows us to express the notion of comprehension in other ∞ -cosmoses (of not necessarily $(\infty, 1)$ -categories).

Therefore, recall the flat simplicially enriched model categories $(\mathbf{S}_{/\mathcal{C}^\sharp}^+, \mathbf{RFib}(\mathcal{C}))$ for right fibrations and $(\mathbf{S}_{/\mathcal{C}^\sharp}^+, \mathbf{Cart}(\mathcal{C}))$ for cartesian fibrations over \mathcal{C} from Section 2. The associated full simplicial subcategories of fibrant objects yield ∞ -cosmoses $\mathbf{RFib}(\mathcal{C})$ and $\mathbf{Cart}(\mathcal{C})$ in the sense of [39], respectively. The following definition introduces a pinched variation of Johnstone’s rectangular diagram categories [21, Section B1.3] that he uses to define comprehension schemes of Grothendieck fibrations.

4.1. DEFINITION. *Let $p: \mathcal{E} \rightarrow \mathcal{C}$ be a cartesian fibration. For a simplicial set $I \in \mathbf{S}$, let p^I be the simplicial cotensor of p with I in the ∞ -cosmos $\mathbf{Cart}(\mathcal{C})$. The right fibration of pinched rectangular I -indexed diagrams in p is given by the core $(p^I)^\times \in \mathbf{RFib}(\mathcal{C})$ in the sense of Definition 2.3. We will denote the domain of $(p^I)^\times$ by $\llbracket I, \mathcal{E} \rrbracket$.*

The right fibration $(p^I)^\times: \llbracket I, \mathcal{E} \rrbracket \rightarrow \mathcal{C}$ can explicitly be described as the composite pullback

$$\begin{array}{ccc}
 \llbracket I, \mathcal{E} \rrbracket & \hookrightarrow & (\mathcal{E}^I)^\times \\
 \downarrow & \lrcorner & \downarrow \\
 \Delta^*(\mathcal{E}^I) & \hookrightarrow & \mathcal{E}^I \\
 p^I \downarrow & \lrcorner & \downarrow p^I \\
 \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C}^I
 \end{array} \tag{23}$$

in the simplicial category $\mathbf{QC}at$ of quasi-categories. Here, the exponential on the right hand side is computed in $\mathbf{QC}at$ and the exponential on the left hand side is computed in $\mathbf{Cart}(\mathcal{C})$. Indeed, the right vertical map $p^I: \mathbf{Map}^b(I^b, \mathcal{E}^\sharp) \rightarrow \mathbf{Map}^b(I^b, \mathcal{C}^\sharp)$ is a cartesian fibration by [27, Remark 3.1.1.10, Proposition 3.1.2.3], and hence so is its pullback $p^I: \Delta^*(\mathcal{E}^I) \rightarrow \mathcal{C}$. Its cartesian morphisms are exactly the morphisms in the wide ∞ -subcategory $(\mathcal{E}^I)^\times = \mathbf{Map}^\sharp(I^b, \mathcal{E}^\sharp)$ by [27, Proposition 3.1.2.1]. It follows that the vertical composite map $(\mathcal{E}^I)^\times \rightarrow \mathcal{C}^I$ in (23) is a right fibration, and so the pullback $(p^I)^\times: \llbracket I, \mathcal{E} \rrbracket \rightarrow \mathcal{C}$ is a right fibration as well. In particular, $\llbracket I, \mathcal{E} \rrbracket = \Delta^*(\mathcal{E}^I)^\times$ is an ∞ -category. Both pullbacks in (23) are homotopy pullbacks in the Joyal model structure for

quasi-categories (because, for instance, the pullback of a span in which all vertices are fibrant and one leg is a fibration yields a homotopy pullback in any model category).

In less formal terms, $\llbracket I, \mathcal{E} \rrbracket \subseteq \text{Fun}(I, \mathcal{E})$ is the ∞ -category of vertical diagrams and jointly horizontal natural transformations over $p: \mathcal{E} \rightarrow \mathcal{C}$. Its objects are the vertical diagrams contained in the ∞ -category $\coprod_{C \in \mathcal{C}} \text{Fun}(I, \mathcal{E}(C))$, its morphisms between two vertical diagrams $X: I \rightarrow \mathcal{E}(C)$ and $Y: I \rightarrow \mathcal{E}(C')$ are the natural transformations $\alpha: X \rightarrow Y$ in \mathcal{E} which are pointwise cartesian over one single morphism $f_\alpha: C \rightarrow C'$ in \mathcal{C} . By virtue of this single apex morphism in \mathcal{C} they can be thought as pinched rectangular or triangular prismatic.

4.2. REMARK. The reason for the departure from rectangular shaped diagrams is the correspondence in Proposition 4.4. Following [21] on the nose, one would define $\text{Rect}(I, \mathcal{E}) \subseteq (\mathcal{E}^I)^\times$ to be the full ∞ -subcategory generated by the diagrams which are vertices in $\mathbf{S}^+(I^\sharp, (\mathcal{E}, \{\text{vertical edges in } \mathcal{E}\}))$. This notion of rectangular diagram comes with a less strict notion of both vertical diagram and horizontal natural transformation than pinched rectangular diagrams do. Yet, the two derived notions of G -comprehension for a map $G: I \rightarrow J$ of simplicial sets are equivalent whenever the codomain J is connected and the ∞ -category \mathcal{C} has $\pi_0(I)$ -sized products. This in fact holds in all of the basic examples which are considered both here and in [21]. Further elaborations on this divergence will be omitted, however more details including a proof of this equivalence can be found in an earlier draft of this paper [42, Remark 3.3].

The pinched rectangular diagram construction associated to a cartesian fibration is pullback-stable in the following sense.

4.3. LEMMA. *Let $p: \mathcal{E} \rightarrow \mathcal{C}$ be a cartesian fibration, I be a simplicial set and suppose $F: \mathcal{D} \rightarrow \mathcal{C}$ is a functor of ∞ -categories. Then the right fibrations $((F^*p)^I)^\times: \llbracket I, F^*\mathcal{E} \rrbracket \rightarrow \mathcal{D}$ and $F^*((p^I)^\times): F^*(\llbracket I, \mathcal{E} \rrbracket) \rightarrow \mathcal{D}$ are isomorphic. Furthermore, for every map $G: I \rightarrow J$ of simplicial sets, the according functors $((F^*p)^G)^\times$ and $F^*((p^G)^\times)$ are isomorphic as well.*

PROOF. The pullback functor $F^*: \mathbf{Cart}(\mathcal{C}) \rightarrow \mathbf{Cart}(\mathcal{D})$ is cosmological and hence preserves simplicial cotensors naturally. It furthermore commutes with the core construction as for instance can be seen from the explicit description of $\mathcal{E}^\times \subseteq \mathcal{E}$ as the wide ∞ -subcategory spanned by the p -cartesian morphisms in Section 2. Indeed, the F^*p -cartesian morphisms in the pullback $F^*\mathcal{E}$ are exactly the morphisms mapped to a p -cartesian morphism in \mathcal{E} . ■

4.4. PROPOSITION. *Let $p: \mathcal{E} \rightarrow \mathcal{C}$ be a cartesian fibration with essentially small fibers.*

1. *Let I be a simplicial set. Then the right fibration $(p^I)^\times: \llbracket I, \mathcal{E} \rrbracket \rightarrow \mathcal{C}$ is (equivalent to) the unstraightening of the presheaf*

$$\mathcal{C}^{op} \xrightarrow{\text{St}(p)} \text{Cat}_\infty \xrightarrow{(\cdot)^I} \text{Cat}_\infty \xrightarrow{(\cdot)^\simeq} \mathcal{S}. \quad (24)$$

2. Let $G: I \rightarrow J$ be a map of simplicial sets. Then the natural restriction functor

$$G^*: \llbracket J, \mathcal{E} \rrbracket \rightarrow \llbracket I, \mathcal{E} \rrbracket \quad (25)$$

over \mathcal{C} is (equivalent to) the unstraightening of the natural transformation (11). In particular, the \mathcal{C} -indexed ∞ -category $\text{St}(p)$ has G -comprehension if and only if the restriction (25) has a (not necessarily fibered) right adjoint.

PROOF. For every \mathcal{C} -indexed ∞ -category $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ and every simplicial set I , there is a natural equivalence

$$\text{Un}(\mathcal{E}^I) \simeq \text{Un}(\mathcal{E})^I \quad (26)$$

of cartesian fibrations over \mathcal{C} . Here, the left hand side is (up to equivalence) the unstraightening of the simplicial cotensor of (a projectively fibrant model of) \mathcal{E} with I in the simplicial category $\mathbf{Fun}(\mathfrak{C}(\mathcal{C})^{op}, \mathbf{QCat})$. It is computed as the pointwise exponential of simplicial sets. The right hand side is the simplicial cotensor of the unstraightening $\text{Un}(\mathcal{E})$ computed in $\mathbf{Cart}(\mathcal{C})$. The existence of the natural equivalence (26) hence follows from the fact that the unstraightening construction preserves simplicial cotensors up to binatural equivalence. A direct proof of this can be found in [47, Theorem 1.1], but it essentially follows from the fact that unstraightening induces a biequivalence of according $(\infty, 2)$ -categories as first explicitly shown in [16, Lemma 1.4.3].

In particular, for every cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ there is an equivalence

$$\text{Un}(\text{St}(p)^I) \simeq \text{Un}(\text{St}(p))^I \simeq p^I.$$

By definition of the core construction in Definition 2.3, this induces an equivalence

$$\text{Un}((\text{St}(p)^I)^{\simeq}) \simeq (p^I)^\times$$

as stated in Part 1. Part 2 follows from the same argument by virtue of binaturality of the equivalence (26). ■

4.5. NOTATION. Given Proposition 4.4, we say that a cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ has G -comprehension for a map $G: I \rightarrow J$ of simplicial sets whenever the functor $G^*: \llbracket J, \mathcal{E} \rrbracket \rightarrow \llbracket I, \mathcal{E} \rrbracket$ of ∞ -categories has a right adjoint.

For a given map $G: I \rightarrow J$ of simplicial sets, a cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ and a vertical diagram $X \in \llbracket I, \mathcal{E} \rrbracket$, the associated comma ∞ -category $G^* \downarrow X$ given by the pullback

$$\begin{array}{ccc} G^* \downarrow X & \longrightarrow & \llbracket I, \mathcal{E} \rrbracket /_X \\ \downarrow & \lrcorner & \downarrow s \\ \llbracket J, \mathcal{E} \rrbracket & \xrightarrow{G^*} & \llbracket I, \mathcal{E} \rrbracket \end{array} \quad (27)$$

may be thought of as the right fibration of horizontal J -extensions of X . Thus, Proposition 4.4 allows us to characterize G -comprehension directly in terms of these comma ∞ -categories as follows.

4.6. PROPOSITION. *Let $G: I \rightarrow J$ be a map of simplicial sets and $p: \mathcal{E} \rightarrow \mathcal{C}$ be a cartesian fibration. The fibration p has G -comprehension if and only if for every vertical diagram $X \in \llbracket I, \mathcal{E} \rrbracket$ the ∞ -category $G^* \downarrow X$ of vertical J -structures extending X horizontally has a terminal object.*

PROOF. The right fibration $G^* \downarrow X \rightarrow \mathcal{C}$ is exactly the unstraightening of the pullback of $G^*: (\mathrm{St}(p)^J)^{\simeq} \rightarrow (\mathrm{St}(p)^I)^{\simeq}$ along $\lrcorner X \lrcorner: y(p(X)) \rightarrow (\mathrm{St}(p)^I)^{\simeq}$ in $\hat{\mathcal{C}}$. ■

4.7. EXAMPLES. The following are translations of some of the examples in Section 3 to their fibrational counterparts.

1. A cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ over an ∞ -category \mathcal{C} with finite products is globally small if and only if its core \mathcal{E}^\times has a terminal object (Example 3.19).
2. The canonical fibration $t: \mathrm{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$ associated to a small ∞ -category \mathcal{C} with pullbacks is locally small if and only if \mathcal{C} is locally cartesian closed (Proposition 3.24).
3. The representable right fibrations $\mathcal{C}/_{\mathcal{C}} \rightarrow \mathcal{C}$ are locally small if and only if \mathcal{C} has equalizers. They are always globally small whenever \mathcal{C} has finite products (Example 3.25).
4. Any ∞ -category \mathcal{C} considered as a cartesian fibration over the point is globally small if and only if its core $\mathcal{C}^\times \simeq \mathcal{C}^{\simeq}$ is contractible. It is locally small if and only if its hom-spaces are contractible (Example 3.26).
5. Given a cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ and an ∞ -category \mathcal{D} with pullbacks together with a left adjoint functor $F: \mathcal{D} \rightarrow \mathcal{C}$, we have $\mathrm{Comp}(F^*p) \subseteq \mathrm{Comp}(p)$ (Lemma 3.35).
6. The universal cartesian fibration $\pi^{op}: \mathrm{Dat}_\infty^{op} \rightarrow \mathrm{Cat}_\infty^{op}$ has G -comprehension for every functor $G: \mathcal{I} \rightarrow \mathcal{J}$ between small ∞ -categories \mathcal{I}, \mathcal{J} by Example 3.22. In particular, the left fibration $(\pi^I)^\times: \llbracket I, \mathrm{Dat}_\infty \rrbracket \rightarrow \mathrm{Cat}_\infty$ is equivalent to the corepresentable $(\mathrm{Cat}_\infty)_{I/} \rightarrow \mathrm{Cat}_\infty$ for every quasi-category I . Furthermore, the universal left fibration $\mathcal{S}_* \rightarrow \mathcal{S}$ is the pullback of the universal cocartesian fibration $\pi: \mathrm{Dat}_\infty \rightarrow \mathrm{Cat}_\infty$ along the canonical inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ [27, Section 3.3.2]. This inclusion comes with a left adjoint $F: \mathrm{Cat}_\infty \rightarrow \mathcal{S}$ which assigns to every ∞ -category the free ∞ -groupoid generated by it. Taking opposites all over this pullback square, it follows from Example 4.7.5 that the universal right fibration $\mathcal{S}_*^{op} \rightarrow \mathcal{S}^{op}$ satisfies all comprehension schemes satisfied by the universal cartesian fibration π^{op} .

5. Externalization of internal $(\infty, 1)$ -categories

In this section we generalize the externalization construction of [21] (and [20, Section 7.2]) to ∞ -category theory. Given an ∞ -category \mathcal{C} with pullbacks, we show that externalization yields an equivalence between internal ∞ -categories in \mathcal{C} and the globally and

locally small indexed ∞ -categories over \mathcal{C} . As an example we will see that the universal cartesian fibration $\pi^{op}: \text{Dat}_\infty^{op} \rightarrow \text{Cat}_\infty^{op}$ is the (unstraightening of the) externalization of the internal ∞ -category Δ^\bullet in Cat_∞^{op} .

5.1. NOTATION. For an ∞ -category \mathcal{C} , we denote the ∞ -category $\text{Fun}(N(\Delta^{op}), \mathcal{C})$ of simplicial objects in \mathcal{C} by $s\mathcal{C}$. For $n \geq 0$ and a subset $J \subseteq [n]$ of cardinality j , we denote by $d^J: [j] \rightarrow [n]$ the according inclusion of linear orders with image J , and for a simplicial object $X \in s\mathcal{C}$, by $d_J: X_n \rightarrow X_j$ the according simplicial operator.

We recall the definition of internal ∞ -categories in an ∞ -category \mathcal{C} with pullbacks from [36, Section 3], going only into as much technical detail as necessary to define and study their externalization.

5.2. DEFINITION. A Segal object in an ∞ -category \mathcal{C} with pullbacks is a simplicial object $X \in s\mathcal{C}$ such that its associated Segal morphisms

$$\xi_n: X_n \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are equivalences in \mathcal{C} [28, Definition 1.1.1].

To every Segal object X in \mathcal{C} (in fact to every $X \in s\mathcal{C}$) we may associate, first, the object $\text{Zig-zag}(X) \simeq X_1 \times_{X_0}^{d_1} X_1 \times_{X_0}^{d_0} X_1$ of zig-zags in X , and second, the object $\text{Equiv}(X) \subseteq X_3$ of internal equivalences in X (or more precisely the object of edges together with a left and a right inverse in X) defined as the pullback

$$\begin{array}{ccc} \text{Equiv}(X) & \longrightarrow & X_3 \\ \downarrow & \lrcorner & \downarrow (d_{\{0,2\}}, d_{\{1,2\}}, d_{\{1,3\}}) \\ X_1 & \xrightarrow{(s_0 d_0, 1, s_0 d_1)} & \text{Zig-zag}(X). \end{array}$$

5.3. DEFINITION. A Segal object X in \mathcal{C} is complete if the factorized degeneracy

$$\begin{array}{ccc} & & \text{Equiv}(X) \\ & \nearrow s_0 & \downarrow \\ X_0 & \xrightarrow{s_0} & X_1 \end{array}$$

induced by the degenerated 3-simplex $X(!_{[3]}): X_0 \rightarrow X_3$ is an equivalence in \mathcal{C} [36, Definition 3.3].

The definition of completeness in Definition 5.2 is directly derived from Rezk’s original definition of completeness of Segal spaces in [38]. As complete Segal spaces are exactly the internal ∞ -categories in the ∞ -category \mathcal{S} of spaces, the complete Segal objects in an ∞ -category \mathcal{C} with pullbacks amount exactly to the internal ∞ -categories in \mathcal{C} .

We obtain full ∞ -subcategories $\text{CS}(\mathcal{C}) \subseteq \text{S}(\mathcal{C}) \subseteq s\mathcal{C}$ of Segal objects and complete Segal objects in \mathcal{C} , respectively.

5.4. DEFINITION. A Segal groupoid in an ∞ -category \mathcal{C} with pullbacks is a Segal object X in \mathcal{C} such that the natural morphism $d_{\{1,2\}}: \text{Equiv}(X) \rightarrow X_1$ is an equivalence in \mathcal{C} . The full ∞ -subcategory of Segal groupoids in $\mathcal{S}(\mathcal{C})$ is denoted by $\text{SGpd}(\mathcal{C})$.

5.5. REMARK. This definition of a Segal groupoid is equivalent to Lurie's definition of a groupoid object in an ∞ -category \mathcal{C} with pullbacks as given in [27, Definition 6.1.2.7]. This can be shown using [27, Proposition 6.2.1.6.(3)] and [43, Section 3]. Furthermore, in [28, Proposition 1.1.14], Lurie constructs the core X^\simeq of a Segal object X in \mathcal{C} whenever \mathcal{C} is left exact. That is a right adjoint $(\cdot)^\simeq: \mathcal{S}(\mathcal{C}) \rightarrow \text{SGpd}(\mathcal{C})$ to the inclusion of Segal groupoids into Segal objects in \mathcal{C} . He then defines a Segal object X to be complete if the degeneracy $s_0: X_0^\simeq \rightarrow X_1^\simeq$ is an equivalence in \mathcal{C} . That is, in Lurie's terms, if the groupoid object X^\simeq is constant [28, Remark 1.1.5]. This definition is equivalent to the one given in Definition 5.2 essentially by [28, Proposition 1.1.13.(2)].

The externalization of a complete Segal object in an ∞ -category \mathcal{C} with pullbacks can be given as follows. The Yoneda embedding $y: \mathcal{C} \rightarrow \hat{\mathcal{C}}$ induces a functor $sy: s\mathcal{C} \rightarrow s\hat{\mathcal{C}}$ by postcomposition. The ∞ -category $s\hat{\mathcal{C}}$ of simplicial objects in turn is equivalent to the ∞ -category $\text{Fun}(\mathcal{C}^{op}, s\mathcal{S})$ of \mathcal{C} -indexed simplicial spaces simply by virtue of Currying. As the Yoneda embedding is left exact, it restricts to a functor $sy: \mathcal{S}(\mathcal{C}) \rightarrow \mathcal{S}(\hat{\mathcal{C}})$ and further to a functor $sy: \text{CS}(\mathcal{C}) \rightarrow \text{CS}(\hat{\mathcal{C}})$. The equivalence $s\hat{\mathcal{C}} \simeq \text{Fun}(\mathcal{C}^{op}, s\mathcal{S})$ also restricts to equivalences $\mathcal{S}(\hat{\mathcal{C}}) \simeq \text{Fun}(\mathcal{C}^{op}, \mathcal{S}(\mathcal{S}))$ and $\text{CS}(\hat{\mathcal{C}}) \simeq \text{Fun}(\mathcal{C}^{op}, \text{CS}(\mathcal{S}))$ between (complete) Segal objects in $\hat{\mathcal{C}}$ and \mathcal{C} -indexed (complete) Segal spaces, respectively. Lastly, the ∞ -category $\text{CS}(\mathcal{S})$ of complete Segal spaces exhibits an equivalence to the ∞ -category Cat_∞ of (small) ∞ -categories given by the right derived horizontal projection $\text{Ho}_\infty(p_h): \text{CS}(\mathcal{S}) \xrightarrow{\simeq} \text{Cat}_\infty$ [23, Section 4].

5.6. DEFINITION. The externalization functor $\text{Ext}: \text{CS}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$ associated to an ∞ -category \mathcal{C} with pullbacks is given by the composition

$$\text{CS}(\mathcal{C}) \xrightarrow{sy} \text{Fun}(\mathcal{C}^{op}, \text{CS}(\mathcal{S})) \xrightarrow{\simeq} \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty).$$

The externalization functor associated to an ∞ -category \mathcal{C} with pullbacks extends the Yoneda embedding associated to \mathcal{C} in the sense that the following diagram can be shown to commute in both directions.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{y} & \text{Fun}(\mathcal{C}^{op}, \mathcal{S}) \\ \left(\downarrow \right)_{(\cdot)_0} & & \left(\downarrow \right)_{(\cdot)^\simeq} \\ \text{CS}(\mathcal{C}) & \xrightarrow{\text{Ext}} & \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty) \end{array} \quad (28)$$

Here, the left vertical inclusion assigns to an object $C \in \mathcal{C}$ the constant simplicial object in \mathcal{C} with value C , and the right vertical inclusion is the pushforward with the canonical inclusion $\mathcal{S} \hookrightarrow \text{Cat}_\infty$.

For a given complete Segal object X in \mathcal{C} , we denote the corresponding Unstraightened cartesian fibration by $\text{Ext}(X) \rightarrow \mathcal{C}$ as well. By construction, its fibers $\text{Ext}(X)(C)$ are naturally equivalent to the composition $\text{Ho}_\infty(p_h)(\text{sy}(X)(C))$. They can be described as the ∞ -categories given by the 0-th row of any Reedy fibrant representative of the complete Segal object $\text{sy}(X)(C) \simeq \mathcal{C}(C, X_{(\cdot)})$ in the category $s\mathbf{S}$ of simplicial spaces.

5.7. DEFINITION. A \mathcal{C} -indexed ∞ -category \mathcal{E} is small if there is a complete Segal object $X \in \text{CS}(\mathcal{C})$ such that $\text{Ext}(X) \simeq \mathcal{E}$.

5.8. REMARK. Under the equivalence of models of ($\infty, 1$)-category theory induced by the Quillen equivalence

$$p_h: (s\mathbf{S}, \text{CS}) \rightarrow (\mathbf{S}, \text{QCat})$$

of the Rezk model structure for complete Segal spaces on the one hand, and the Joyal model structure for quasi-categories on the other hand, the cartesian fibrations of the form $\text{Ext}(X) \rightarrow \mathcal{C}$ for a complete Segal object X in \mathcal{C} are exactly the “representable” cartesian fibrations $\mathcal{C}/_X \rightarrow \mathcal{C}$ in [36, Definition 2.1] by [36, Notation 2.5]. Note however that these are not the cartesian fibrations given by the ∞ -overcategories $\mathcal{C}/_X \rightarrow \mathcal{C}$ as defined in [27, Proposition 1.2.9.2]. In fact, whenever a cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ is presented by a “Reedy right fibration” $\mathcal{R} \rightarrow \mathcal{C}$ [37, Definition 4.2, Definition 2.2], then the right fibration $\mathcal{R}_n \rightarrow \mathcal{C}$ in $(s\mathbf{S}, \text{CS})$ translates exactly to the right fibration $[\Delta^n, \mathcal{E}] \rightarrow \mathcal{C}$ in $(\mathbf{S}, \text{QCat})$ via the Quillen equivalence p_h for every $n \geq 0$.

5.9. REMARK. Externalization can also be defined more generally for not necessarily complete Segal spaces (and even for general simplicial objects) in an ∞ -category \mathcal{C} , using that \mathcal{C} -indexed Segal spaces can be completed pointwise. More precisely, the inclusions $\text{CS}(\mathcal{S}) \hookrightarrow \text{S}(\mathcal{S}) \hookrightarrow s\mathcal{S}$ have a left adjoint each (given by the fact that the two inclusions are presented by left Bousfield localizations of the respective model structures). The left adjoint $\rho: \text{S}(\mathcal{S}) \rightarrow \text{CS}(\mathcal{S})$ maps a Segal space to its completion. We thus obtain a generalized externalization functor

$$\text{S}(\mathcal{C}) \xrightarrow{\text{sy}} \text{Fun}(\mathcal{C}^{op}, \text{S}(\mathcal{S})) \xrightarrow{\rho_*} \text{Fun}(\mathcal{C}^{op}, \text{CS}(\mathcal{S})) \xrightarrow{\simeq} \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$$

which restricts on $\text{CS}(\mathcal{C})$ (up to equivalence) exactly to the externalization functor from Definition 5.6. Note that the definition of a Segal space in an ∞ -category \mathcal{C} does not require the existence of pullbacks in \mathcal{C} , as the Segal conditions only express that a certain specified cone in \mathcal{C} is limiting.

Suppose for the moment that \mathcal{C} is a 1-category (with pullbacks). Then $\text{S}(\mathcal{C})$ is the 1-category of internal categories in \mathcal{C} , and $\text{CS}(\mathcal{C})$ is the 1-category of internal categories with a discrete core (Remark 5.5). The 1-categorical externalization functor as described in [21, Section B.2.3] and [20, Section 7.3] is a functor

$$\text{Ext}_1: \text{S}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{C}^{op}, \text{Cat}). \tag{29}$$

For an internal category $X \in \text{S}(\mathcal{C})$ and an object $C \in \mathcal{C}$, the objects of the categories $\text{Ext}_1(X)(C)$ are the morphisms $\mathcal{C}(C, X_0)$. The morphisms between vertices $x, y: C \rightarrow$

X_0 in $\text{Ext}_1(X)(C)$ are explicitly given by morphisms $f: C \rightarrow X_1$ such that $d_1 f = x$ and $d_0 f = y$. Given a morphism $f: C \rightarrow D$ in \mathcal{C} , we obtain the functorial action $f^*: \text{Ext}_1(X)(D) \rightarrow \text{Ext}_1(X)(C)$ by precomposition with f .

Although internal categories in a 1-category \mathcal{C} are generally not complete, they automatically are internal Segal *categories*. Indeed, the Yoneda embedding of \mathcal{C} factors through the ∞ -category of discrete spaces, and so the functor $sy: s\mathcal{C} \rightarrow s\mathbf{S}$ factors through the ∞ -category PCat of precategories [23, Section 5] – that is, the ∞ -category of simplicial spaces X such that X_0 is a discrete space. In fact, for any given internal category X in \mathcal{C} and any given object $C \in \mathcal{C}$, every column of the simplicial space $sy(X)(C) = \mathcal{C}(C, X_{(\cdot)})$ is discrete. It follows in particular that the simplicial spaces $sy(X)(C)$ are Reedy fibrant. If we denote the ∞ -category of Segal categories – i.e. of Segal spaces X such that X_0 is a discrete space – by $\text{SC}(\mathcal{S})$, then the right derived horizontal projection $\text{Ho}_\infty(p_h): \text{SC}(\mathcal{S}) \rightarrow \text{Cat}_\infty$ is again an equivalence [23, Theorem 5.6]. Since every object in the image of $sy: \text{S}(\mathcal{C}) \rightarrow \text{PCat}$ is already fibrant, it follows that the induced composition

$$\text{S}(\mathcal{C}) \xrightarrow{sy} \text{Fun}(\mathcal{C}^{op}, \text{SC}(\mathcal{S})) \xrightarrow{\cong} \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$$

computes exactly the 1-categorical externalization (29).

5.10. PROPOSITION. *Let \mathcal{C} be a 1-category. Then the externalization functor $\text{Ext}: \text{S}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$ from Remark 5.9 and the 1-categorical externalization functor $\text{Ext}_1: \text{S}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$ are naturally equivalent.*

PROOF. The inclusion $\iota: \text{PCat} \rightarrow s\mathbf{S}$ of precategories in simplicial spaces is part of a left Quillen equivalence $\iota: (\text{PCat}, \text{SC}) \rightarrow (s\mathbf{S}, \text{CS})$ between the Hirschowitz-Simpson model structure for Segal categories and the Rezk model structure for complete Segal spaces by [9, Section 6]. The induced (outer) triangle of equivalences

$$\begin{array}{ccc}
 \text{SC}(\mathcal{S}) & & \\
 \downarrow & \searrow^{\text{Ho}_\infty(p_h)} & \\
 \text{S}(\mathcal{C}) & & \text{Cat}_\infty \\
 \downarrow \rho & \nearrow_{\text{Ho}_\infty(p_h)} & \\
 \text{CS}(\mathcal{S}) & &
 \end{array}$$

$\text{Ho}_\infty(\iota)$ (curved arrow from $\text{SC}(\mathcal{S})$ to $\text{CS}(\mathcal{S})$)

commutes up to equivalence again by [23, Section 5]. Precomposition of this natural equivalence with the functor $sy: \text{S}(\mathcal{C}) \rightarrow \text{SC}(\mathcal{S})$ hence induces a natural equivalence between Ext_1 and Ext as stated. ■

Thus, the ∞ -categorical externalization functor is a natural generalization of its 1-categorical analogue.

5.11. PROPOSITION. *A Segal object X in an ∞ -category \mathcal{C} with pullbacks is a Segal groupoid if and only if its externalization is an indexed ∞ -groupoid.*

PROOF. Let X be a Segal object in \mathcal{C} . The Yoneda embedding is conservative and so the canonical morphism $\text{Equiv}(X) \rightarrow X_1$ is an equivalence in \mathcal{C} if and only if for every object $C \in \mathcal{C}$, the presheaf $sy(X)(C)$ is a Segal space such that $\mathcal{C}(C, \text{Equiv}(X)) \rightarrow \mathcal{C}(C, X_1)$ is an equivalence of spaces. But each of these maps is naturally equivalent to the canonical map $\text{Equiv}(sy(X)(C)) \rightarrow (sy(X)(C))_1$ associated to the Segal space $sy(X)(C)$. Hence, X is a Segal groupoid in \mathcal{C} if and only if each $sy(X)(C)$ is a Segal groupoid in \mathcal{S} . By an application of [27, Proposition 6.1.2.6.(3)], such are exactly the *Bousfield-Segal spaces* in terms of [43, Corollary 5.4]. Now, a Segal space is a Bousfield-Segal space if and only if its right derived horizontal projection is a Kan complex by [43, Remark 5.5]. In summary, it follows that a Segal object X in \mathcal{C} is a Segal object if and only if for all $C \in \mathcal{C}$, the quasi-category $\text{Ext}(X)(C) = \text{Ho}_\infty(p_h)(sy(X)(C))$ is a Kan complex. ■

For the following, recall that Joyal and Tierney constructed in fact two Quillen equivalences between the model structure for complete Segal spaces and the model structure for quasi-categories. On the one hand, we have already considered the right Quillen functor

$$p_h : (s\mathbf{S}, \text{CS}) \rightarrow (\mathbf{S}, \text{QCat}).$$

On the other hand, in [23, Section 4] the authors construct a left Quillen functor

$$t^! : (\mathbf{S}, \text{QCat}) \rightarrow (s\mathbf{S}, \text{CS})$$

which is part of a Quillen equivalence as well. If by $I\Delta^n \in \mathbf{S}$ we denote the nerve of the free groupoid on $[n]$, the functor $t^!$ is given by the formula

$$\begin{aligned} t^!(J)_{mn} &= \mathbf{S}(\Delta^m \times I[\Delta^n], J) \\ &= ((J^{\Delta^m})^\simeq)_n \end{aligned}$$

for simplicial sets J . In particular, for every quasi-category \mathcal{C} we have

$$t^!(\mathcal{C}) = \text{Cat}_\infty(\Delta^\bullet, \mathcal{C}) = sy(\Delta^\bullet)(\mathcal{C})$$

for Δ^\bullet considered as a simplicial object in Cat_∞^{op} . In the proof of [23, Theorem 4.12] the authors show that the composition $p_h \circ t^! : \mathbf{S} \rightarrow \mathbf{S}$ is isomorphic to the identity, and so it follows that the composition

$$\text{Ho}_\infty(p_h) \circ \text{Cat}_\infty(\Delta^\bullet, \cdot) : \text{Cat}_\infty \rightarrow \text{Cat}_\infty$$

is naturally equivalent to the identity on Cat_∞ .

5.12. EXAMPLE. Via the equivalence $t^! : \text{Cat}_\infty \rightarrow \text{CS}(\mathcal{S})$ we can define the externalization of a quasi-category \mathcal{C} as $\text{Ext}(t^!(\mathcal{C}))$. The fact that the composition $\text{Ho}_\infty(p_h) \circ t^! : \text{Cat}_\infty \rightarrow \text{Cat}_\infty$ recovers the identity up to equivalence induces for any given quasi-category \mathcal{C} a natural equivalence $\text{Ext}(\mathcal{C}) \simeq \text{Fun}(\cdot, \mathcal{C})$ in $\text{Fun}(\mathbf{S}^{op}, \text{Cat}_\infty)$. This means that the externalization of an ∞ -category considered as an internal category in spaces is just the naive indexing of \mathcal{C} from Example 2.2.2.

5.13. EXAMPLE. [The universal complete Segal object] The Yoneda embedding $y: \Delta \rightarrow \mathbf{S}$ yields a simplicial object Δ^\bullet in $\text{Cat}_\infty^{\text{op}}$ which is a complete Segal object precisely because the spine inclusions $S_n \rightarrow \Delta^n$ (i.e. the inclusions of the maximal chain of non-degenerate 1-simplices into Δ^n) and the endpoint inclusion $\Delta^0 \rightarrow I\Delta^1$ are weak categorical equivalences in the Joyal model structure. We therefore can construct the externalization

$$\text{Ext}(\Delta^\bullet): \text{Cat}_\infty \rightarrow \text{Cat}_\infty \tag{30}$$

given exactly by $\text{Ho}_\infty(p_h) \circ \text{Cat}_\infty(\Delta^\bullet, \cdot)$ via Definition 5.6. By the above, it follows that the given externalization is naturally equivalent to the identity on Cat_∞ . In other words, the universal cocartesian fibration $\pi: \text{Dat}_\infty \rightarrow \text{Cat}_\infty$ – defined as the unstraightening of the identity on Cat_∞ – is equivalent to the externalization of $\Delta^\bullet \in \text{Cat}_\infty^{\text{op}}$. In this sense, it is represented by the co-complete co-Segal object Δ^\bullet in Cat_∞ .

This can be understood as the rather plain and by now folklore fact that the model category $(\mathbf{S}, \text{QCat})$ together with the cosimplicial object Δ^\bullet in \mathbf{S} is a “theory of $(\infty, 1)$ -categories” in the sense of Toën [51]. Indeed, Δ^\bullet being a complete Segal object in $\text{Cat}_\infty^{\text{op}}$ means dually that Δ^\bullet considered as a “weak co-category object” in Cat_∞ is an interval [51, Definition 3.4]. The functor $t^! = \text{sy}(\Delta^\bullet): (\mathbf{S}, \text{QCat}) \rightarrow (s\mathbf{S}, \text{CS})$ is exactly the comparison functor in [51, Theorem 5.1].

Under the equivalence of models of $(\infty, 1)$ -category theory alluded to in Remark 5.8, the following proposition is equivalent to a corresponding sequence of results in [36, Section 4.1].

5.14. PROPOSITION. *Let \mathcal{C} be an ∞ -category with pullbacks.*

1. *Let X be a complete Segal object in \mathcal{C} . Then for every $n \geq 0$, the presheaf $(\text{Ext}(X)^{\Delta^n})^\simeq$ over \mathcal{C} is represented by $X_n \in \mathcal{C}$. In particular, the presheaf $\text{Ext}(X)^\simeq \in \hat{\mathcal{C}}$ is represented by $X_0 \in \mathcal{C}$.*
2. *Let \mathcal{E} be a \mathcal{C} -indexed ∞ -category such that the presheaves \mathcal{E}^\simeq and $(\mathcal{E}^{\Delta^1})^\simeq$ are represented each by some object $E_0, E_1 \in \mathcal{C}$, respectively. Then there is a complete Segal object X in \mathcal{C} such that $X_i \simeq E_i$ for $i = 0, 1$, and such that $\mathcal{E} \simeq \text{Ext}(X)$. In particular, \mathcal{E} is small.*

PROOF. For Part 1 we want to show that for all $X \in \text{CS}(\mathcal{C})$ and all $n \geq 0$ the composite

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} & \xrightarrow{\text{Ext}(X)} & \text{Cat}_\infty \\ & & \searrow \text{Fun}(\Delta^n, \cdot) \\ & & \text{Cat}_\infty \xrightarrow{(\cdot)^\simeq} \mathcal{S} \\ & & \swarrow \text{Cat}_\infty(\Delta^n, \cdot) \end{array}$$

is represented by X_n . By definition of the externalization functor, it therefore suffices to show that the following diagram of ∞ -categories commutes up to equivalence.

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} & \xrightarrow{\text{sy}(X)} & \text{CS}(\mathcal{S}) \xrightarrow{\text{Ho}_\infty(p_h)} \text{Cat}_\infty \\ & \searrow & \downarrow \text{id} \\ & & \text{CS}(\mathcal{S}) \end{array} \quad \begin{array}{c} \downarrow \text{Cat}_\infty(\Delta^\bullet, \cdot) \\ \end{array} \tag{31}$$

Towards homotopy-commutativity of the top right triangle, we have seen that both non-identity components of (31) are equivalences, and that their converse composition is equivalent to the identity. It follows that they are mutually inverse to each other, so that the composition

$$\text{Cat}_\infty(\Delta^\bullet, \cdot) \circ \text{Ho}_\infty(p_h): \text{CS}(\mathcal{S}) \rightarrow \text{CS}(\mathcal{S})$$

is equivalent to the identity on $\text{CS}(\mathcal{S})$ as well. The bottom left triangle of (31) commutes trivially as $sy(X) := y_*X := y \circ X$ is the functorial pushforward by definition.

For Part 2, consider the composition

$$\mathcal{C}^{op} \xrightarrow{\mathcal{E}} \text{Cat}_\infty \xrightarrow{\text{Cat}_\infty(\Delta^\bullet, \cdot)} \text{CS}(\mathcal{S}) \hookrightarrow s\mathcal{S}.$$

Given the canonical equivalence $\text{Fun}(\mathcal{C}^{op}, s\mathcal{S}) \simeq s\hat{\mathcal{C}}$ and its restriction $\text{Fun}(\mathcal{C}^{op}, \text{CS}(\mathcal{S})) \simeq \text{CS}(\hat{\mathcal{C}})$, we can consider the composition $\text{Cat}_\infty(\Delta^\bullet, \cdot) \circ \mathcal{E} = (\mathcal{E}^{\Delta^\bullet})^\simeq$ as a complete Segal object in $\hat{\mathcal{C}}$. We want to show that the simplicial object $(\mathcal{E}^{\Delta^\bullet})^\simeq \in s\hat{\mathcal{C}}$ is contained in the essential image of

$$sy: s\mathcal{C} \rightarrow s\hat{\mathcal{C}}.$$

Indeed, if we obtain a simplicial object $X \in s\mathcal{C}$ such that $sy(X) \simeq (\mathcal{E}^{\Delta^\bullet})^\simeq \in s\hat{\mathcal{C}}$, then X is automatically a complete Segal object in \mathcal{C} . That is, because the Yoneda embedding preserves and reflects both equivalences and all limits that exist in \mathcal{C} , and so the square

$$\begin{array}{ccc} \text{CS}(\mathcal{C}) & \xrightarrow{sy} & \text{CS}(\hat{\mathcal{C}}) \\ \downarrow & & \downarrow \\ s\mathcal{C} & \xrightarrow{sy} & s\hat{\mathcal{C}} \end{array}$$

is cartesian. Furthermore, in this case we obtain an equivalence $\text{Ho}_\infty(p_h) \circ sy(X) \simeq \text{Ho}_\infty(p_h) \circ (\text{Cat}_\infty(\Delta^\bullet, \cdot) \circ \mathcal{E})$, where the left hand side is $\text{Ext}(X)$ by definition, and the right hand side is equivalent to \mathcal{E} itself. Thus, to prove the statement we are left to construct a simplicial object X in \mathcal{C} such that $sy(X) \simeq (\mathcal{E}^{\Delta^\bullet})^\simeq$ in $s\hat{\mathcal{C}}$. Therefore in turn, it suffices to show that for all $n \geq 0$ there is an object $X_n \in \mathcal{C}$ such that X_n represents the presheaf $(\mathcal{E}^{\Delta^n})^\simeq$. That is, because any such collection of objects $(X_n)_{n \geq 0}$ in \mathcal{C} together with equivalences $e_n: y(X_n) \xrightarrow{\simeq} (\mathcal{E}^{\Delta^n})^\simeq$ in $\hat{\mathcal{C}}$ induces a homotopy-commutative square of the form

$$\begin{array}{ccc} N(\Delta^{op})_0 & \xrightarrow{(X_n)_{n \geq 0}} & \mathcal{C} \\ \downarrow & & \downarrow y \\ N(\Delta^{op}) & \xrightarrow{(\mathcal{E}^{\Delta^\bullet})^\simeq} & \hat{\mathcal{C}}. \end{array}$$

Here, the left vertical functor is the canonical inclusion of the underlying set of $N(\Delta^{op})$ into $N(\Delta^{op})$. But the class of essentially surjective functors is left orthogonal to the class

of fully faithful functors in Cat_∞ , and the Yoneda embedding is fully faithful. We thus obtain a lift $X: N(\Delta)^{op} \rightarrow \mathcal{C}$ to this square (such that both resulting triangles commute up to equivalence) which is precisely a simplicial object as required.

And indeed, such objects $X_n = E_n$ exist for $n = 0, 1$ by assumption. Since \mathcal{C} has pullbacks, for $n \geq 2$ we may simply define $X_n := X_1 \times_{X_0} \cdots \times_{X_0} X_1$, so that $y(X_n) \simeq (\mathcal{E}^{\Delta^n})^\simeq$ by virtue of the fact that $(\mathcal{E}^{\Delta^\bullet})^\simeq$ is a Segal object in $\hat{\mathcal{C}}$. ■

Generally, the existence of an internal presentation of a given \mathcal{C} -indexed ∞ -category can be captured via $(\infty, 1)$ -comprehension in the following way.

5.15. THEOREM. *Let \mathcal{C} be an ∞ -category with finite limits and \mathcal{E} be a \mathcal{C} -indexed ∞ -category. Then \mathcal{E} is small if and only if it is both globally small and locally small.*

PROOF. The indexed ∞ -category \mathcal{E} is both globally and locally small if and only if it has both $(\emptyset \rightarrow \Delta^0)$ - and $(\emptyset \rightarrow \Delta^1)$ -comprehension by Corollary 3.32. By Example 2.8, that is if and only if both the presheaves \mathcal{E}^\simeq and $(\mathcal{E}^{\Delta^1})^\simeq$ are representable. This in turn is the case if and only if \mathcal{E} is small Proposition 5.14.2. ■

5.16. REMARK. Suppose the ∞ -category \mathcal{C} has finite limits and \mathcal{E} itself is a presheaf on \mathcal{C} . In this case, by virtue of Proposition 5.11, Theorem 5.15 states that \mathcal{E} (as a \mathcal{C} -indexed ∞ -category) is small if and only if there is a complete Segal groupoid X in \mathcal{C} such that $\text{Ext}(X)$ is equivalent to \mathcal{E} over \mathcal{C} . As the square (28) commutes, and the corestriction of the inclusion $\mathcal{C} \hookrightarrow \text{CS}(\mathcal{C})$ to the full ∞ -subcategory of complete groupoid objects in \mathcal{C} is an equivalence, that means that \mathcal{E} is small if and only if the presheaf \mathcal{E} is representable. This however is a fairly trivial matter; smallness of \mathcal{E} implies representability of \mathcal{E} by definition, and local smallness of representables is for free by Example 3.25. Thus, thinking of the externalization functor as a generalization of the Yoneda embedding via Diagram (28), Theorem 5.15 can be thought of as generalization of the designating fact that the small presheaves over a left exact ∞ -category are exactly the representable ones.

5.17. REMARK. Theorem 5.15 holds in ordinary category theory if one replaces global smallness with the existence of (split) generic objects [20, 48]. To construct a split fibration with split generic object (and hence an internal category) from a fibration with generic object as in [48, Construction 39], it may be interesting to note that one uses the (identity-on-objects, fully faithful)-factorization on Cat , which falls into the category of “god-given” identity principles discussed in [8, Paragraph 8.7], and does not exist in ∞ -category theory. In Remark 6.7, we will see that this is of consequence.

We can think of global smallness as a generalization of the completeness condition in the following sense. Let $c: \mathcal{C} \rightarrow \text{S}(\mathcal{C})$ be the functor which assigns to an object C the constant simplicial object with value C . Then a Segal object in \mathcal{C} is complete exactly if the core $X^\simeq \in \text{S}(\mathcal{C})$ is contained in the essential image of the functor c (Remark 5.5). Likewise, a \mathcal{C} -indexed ∞ -category \mathcal{E} is globally small by definition exactly if its core \mathcal{E}^\simeq is contained in the essential image of the functor $y \simeq \text{Ext} \circ c$ (by commutativity of Diagram (28)). Indeed, completeness of a Segal object X implies global smallness of $\text{Ext}(X)$, and by virtue of Theorem 5.15, global smallness of a locally small \mathcal{C} -indexed ∞ -category recovers

a representing complete Segal object. Note however that for a given Segal object X in \mathcal{C} , smallness of $\text{Ext}(X)$ is equivalent to the existence of a Segal completion $X \rightarrow \rho(X)$ in \mathcal{C} rather than to completeness of X itself. Thus, the fact that the externalization of an internal category in a 1-category \mathcal{C} is generally not globally small (via Proposition 5.10) corresponds exactly to the fact that it generally is not complete and can't be completed in \mathcal{C} .

5.18. COROLLARY. *Suppose \mathcal{C} has finite limits. Then a \mathcal{C} -indexed ∞ -category \mathcal{E} has any of the equivalent classes of comprehension schemes from Corollary 3.32 if and only if it is small. In particular, for every $X \in \text{CS}(\mathcal{C})$ and for every finite ∞ -category J , there is a complete Segal object $X^J \in \text{CS}(\mathcal{C})$ which represents the \mathcal{C} -indexed ∞ -category $\text{Ext}(X)^J$ of J -indexed diagrams.*

PROOF. Given $X \in \text{CS}(\mathcal{C})$ and a finite quasi-category J , the \mathcal{C} -indexed ∞ -category $\text{Ext}(X)$ has both $(\emptyset \rightarrow J)$ -comprehension and $(\emptyset \rightarrow J \times \Delta^1)$ -comprehension by Theorem 5.15 and Corollary 3.32. This means that the \mathcal{C} -indexed ∞ -category $\text{Ext}(X)^J$ has both $(\emptyset \rightarrow \Delta^0)$ -comprehension and $(\emptyset \rightarrow \Delta^1)$ -comprehension, and hence is small again by Theorem 5.15. ■

5.19. COROLLARY. *Suppose \mathcal{C} has finite limits and let $\text{icat}(\mathcal{C}) \subset \text{Fun}(\mathcal{C}^{op}, \text{Cat}_\infty)$ be the full ∞ -subcategory of globally small and locally small indexed ∞ -categories over \mathcal{C} . Then the externalization construction*

$$\text{Ext}: \text{CS}(\mathcal{C}) \rightarrow \text{icat}(\mathcal{C}) \tag{32}$$

is an equivalence of ∞ -categories.

PROOF. The externalization construction is fully faithful as noted in [36, Theorem 2.7] (via Remark 5.8). ■

5.20. REMARK. Via Corollary 5.19 one can equip $\text{CS}(\mathcal{C})$ with a canonical $(\infty, 2)$ -categorical structure such that the externalization construction becomes an embedding of $(\infty, 2)$ -categories. The last two corollaries then show that this $(\infty, 2)$ -category $\text{CS}(\mathcal{C})$ has cotensors with finite ∞ -categories whenever \mathcal{C} has finite limits, and that the externalization functor preserves them. This generalizes [49, (4.4)] to ∞ -category theory and is developed in detail in [46]. Accordingly, it follows from Corollary 5.26 below that $\text{CS}(\mathcal{C})$ is cotensored over Cat_∞ whenever \mathcal{C} has all small limits.

5.21. EXAMPLE. [The universal cartesian fibration and the universal right fibration] We have seen in Example 5.13 that the universal cartesian fibration $\pi^{op}: \text{Dat}_\infty^{op} \twoheadrightarrow \text{Cat}_\infty^{op}$ is the externalization of the complete Segal object Δ^\bullet in Cat_∞^{op} . Unfolding the definitions, local smallness of the fibration π^{op} means that every small ∞ -category in Cat_∞ can be freely extended by an edge between any two of its objects to give another small ∞ -category (Example 3.22). Furthermore, smallness of the universal cartesian fibration recovers the fact that the universal right fibration $(\mathcal{S}_*)^{op} \twoheadrightarrow \mathcal{S}^{op}$ is represented by the terminal object in \mathcal{S} as well [27, Proposition 3.3.2.6]. Indeed, the universal right fibration

is the pullback of the universal cartesian fibration π^{op} along a left adjoint (Example 4.7.6). The following corollary then shows that the complete Segal object $I\Delta^\bullet \in \text{CS}(\mathcal{S}^{op})$ yields an equivalence $(\mathcal{S}_*)^{op} \simeq \text{Ext}(I\Delta^\bullet)$ over \mathcal{S}^{op} . But every $I\Delta^n$ is contractible in \mathcal{S} , and so $\text{Ext}(I\Delta^\bullet) \simeq \mathcal{S}_{/*}^{op} \simeq (\mathcal{S}_{*/})^{op}$.

The fact that the identity on Cat_∞ is small has useful applications. For instance, reindexing along left adjoints preserves smallness as the following generalization of [21, Lemma 2.3.6] shows.

5.22. LEMMA. *Let $F: \mathcal{D} \rightarrow \mathcal{C}$ be a functor between ∞ -categories with pullbacks. Suppose it has a right adjoint R . Then for every complete Segal object X in \mathcal{C} , the restriction $F^*(\text{Ext}(X)) \in \text{Fun}(\mathcal{D}^{op}, \text{Cat}_\infty)$ is equivalent to the externalization of the complete Segal object $sR(X) \in \text{CS}(\mathcal{D})$.*

PROOF. We first note that $sR(X) \in s\mathcal{D}$ is a complete Segal object, because R is a right adjoint and hence preserve limits. Now consider the composition

$$F^*(sy(X)) \xrightarrow{\vec{R}} (RF)^*(sy(sR(X))) \xrightarrow{\eta^*} sy(sR(X)) \quad (33)$$

of natural transformations in $\text{Fun}(\mathcal{D}^{op}, \text{CS}(\mathcal{S}))$. Here, $\eta: 1 \rightarrow RF$ denotes the unit of the adjunction $F \dashv R$, and \vec{R} denotes the action of the right adjoint R on hom-spaces. The latter can be formally constructed via the explicit definition of the Yoneda embedding in [27, Section 5.1.3]. To show that the composition (33) is an equivalence, it suffices to do so pointwise. But for any given $D \in \mathcal{D}$ and every $n \geq 0$, the natural transformation (33) evaluated at D and at n is exactly the functor

$$\mathcal{C}(F(D), X_n) \xrightarrow{\vec{R}} \mathcal{D}(R(F(D)), R(X_n)) \xrightarrow{\eta^*} \mathcal{D}(D, R(X_n))$$

which is an equivalence by the fact that (F, R, η) forms an adjunction [27, Proposition 5.2.2.8]. In particular, the natural equivalence (33) induces an equivalence between the \mathcal{D} -indexed ∞ -categories

$$\text{Ho}_\infty(p_h) \circ (F^*(sy(X))) = F^*(\text{Ho}_\infty(p_h)(sy(X))) = F^*\text{Ext}(X)$$

and $\text{Ho}_\infty(p_h) \circ sy(sR(X)) = \text{Ext}(sR(X))$. ■

Whenever \mathcal{C} is not only left exact but complete, we thus obtain the following characterization of small indexed ∞ -categories over \mathcal{C} .

5.23. PROPOSITION. *Suppose \mathcal{C} is a complete ∞ -category. Then a \mathcal{C} -indexed ∞ -category \mathcal{E} is small if and only if the functor $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ has a left adjoint.*

PROOF. Suppose $\mathcal{E}: \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ has a left adjoint L . Then its opposite $\mathcal{E}^{op}: \mathcal{C} \rightarrow \text{Cat}_\infty^{op}$ is itself left adjoint to L^{op} . As the identity $\text{id}: (\text{Cat}_\infty^{op})^{op} \rightarrow \text{Cat}_\infty$ is the externalization of the complete Segal object $\Delta^\bullet \in \text{CS}(\text{Cat}_\infty^{op})$ (Example 5.21), it follows from Lemma 5.22

that $\mathcal{E} = \mathcal{E}^*(\text{id})$ is the externalization of the complete Segal object $L^{op}(\Delta^\bullet)$. Vice versa, let X be a complete Segal object in \mathcal{C} and consider its externalization

$$\mathcal{C}^{op} \xrightarrow{sy(X)} \text{CS}(\mathcal{S}) \xrightarrow{\text{Ho}_\infty(p_h)} \text{Cat}_\infty.$$

Since $\text{Ho}_\infty(p_h)$ has a left adjoint, it suffices to show that $sy(X) = X^* \circ y$ has a left adjoint as well. Therefore, we note that if we consider $X^{op}: \Delta \rightarrow \mathcal{C}^{op}$ as a cosimplicial object of \mathcal{C}^{op} , then $sy(X) = N(X^{op}): \mathcal{C}^{op} \rightarrow s\mathcal{S}$ is but a nerve construction of X^{op} . We therefore can show that $sy(X)$ is the right adjoint to the left Kan extension of X^{op} along the Yoneda embedding $y: \Delta \rightarrow s\mathcal{S}$. To do so explicitly, we use [30, Proposition 3.1.2] and show that for all simplicial spaces $Z \in s\mathcal{S}$, the comma ∞ -category

$$\begin{array}{ccc} Z \downarrow sy(X) & \longrightarrow & (s\mathcal{S})_{Z/} \\ \downarrow & \lrcorner & \downarrow t \\ \mathcal{C}^{op} & \xrightarrow{sy(X)} & s\mathcal{S} \end{array}$$

has an initial object (or equivalently, that it is corepresentable over \mathcal{C}^{op}). First, if Z is a representable $y([n])$, we obtain a diagram as follows.

$$\begin{array}{ccccccc} \mathcal{C}_{X_n/}^{op} & \longrightarrow & \widehat{\mathcal{C}}_{y(X_n)/}^{op} & \longrightarrow & \widehat{N(\Delta)}_{y([n])/} & \longrightarrow & \mathcal{S}_* \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}^{op} & \xrightarrow{y} & \widehat{\mathcal{C}}^{op} & \xrightarrow{X^*} & \widehat{N(\Delta)} & \xrightarrow{\text{ev}([n])} & \mathcal{S} \end{array}$$

The square on the right hand side is homotopy cartesian in the Joyal model structure by Examples 2.4.4. Since $\text{ev}([n]) \circ X^* = \text{ev}(X_n)$, the right composite rectangle is a homotopy pullback as well for the same reason. This implies that the center square is a homotopy pullback. The square on the left hand side is a homotopy pullback, because the Yoneda embedding is fully faithful. Thus, eventually, the left composite rectangle is a homotopy pullback, which yields the equivalence

$$\mathcal{C}_{X_n/}^{op} \simeq y([n]) \downarrow sy(X)$$

over \mathcal{C}^{op} . For general simplicial spaces $Z \in \widehat{N(\Delta)}$, one can now simply use that Z is the colimit of representables. Indeed, it follows that the comma ∞ -category $\widehat{N(\Delta)}_{Z/}$ is a homotopy limit of corepresentables, and hence so is the pullback along $sy(X)$ by what we just have shown, since homotopy pullbacks preserve homotopy limits. Since \mathcal{C} is assumed to be complete, this limit of corepresentables over \mathcal{C}^{op} is corepresented by the respective colimit in \mathcal{C}^{op} . It follows that $sy(X): \mathcal{C}^{op} \rightarrow \text{Fun}((\Delta)^{op}, \mathcal{S})$ has a left adjoint, and so does the corestriction $sy(X): \mathcal{C}^{op} \rightarrow \text{CS}(\mathcal{S})$ as the inclusion $\text{CS}(\mathcal{S}) \subset \text{Fun}(N(\Delta)^{op}, \mathcal{S})$ is fully faithful. ■

5.24. **REMARK.** In particular, whenever \mathcal{C} is a presentable ∞ -category, Proposition 5.23 states that a \mathcal{C} -indexed ∞ -category is the externalization of a complete Segal object if and only if it preserves limits by the Adjoint Functor Theorem [30]. In the special case of presheaves over \mathcal{C} this is exactly the basic fact that limit preserving presheaves are representable [27, Proposition 5.5.2.2].

5.25. **REMARK.** In Example 3.8 we saw that a functor $\mathcal{E}: \mathcal{C}^{op} \rightarrow K$ into an ∞ -category K with an initial object has $(K, \emptyset \rightarrow k)$ -comprehension for all objects k if and only if the functor \mathcal{E} has a left adjoint. Thus, Proposition 5.23 (together with Corollary 3.32.2) states that in the case that \mathcal{C} is complete and K is the ∞ -category Cat_∞ , this criterion can be reduced to only $(\emptyset \rightarrow \Delta^i)$ -comprehension for $i = 0, 1$.

5.26. **COROLLARY.** *Suppose \mathcal{C} is a complete ∞ -category and \mathcal{E} is a small \mathcal{C} -indexed ∞ -category. Then \mathcal{E} satisfies all comprehension schemes which are satisfied by the identity $\text{id}: (\text{Cat}_\infty^{op})^{op} \rightarrow \text{Cat}_\infty$. Thus, \mathcal{E} is small if and only if it has comprehension for all functors between small ∞ -categories. In particular, for every $X \in \text{CS}(\mathcal{C})$ and for every small ∞ -category J , there is a complete Segal object $X^J \in \text{CS}(\mathcal{C})$ which represents the \mathcal{C} -indexed ∞ -category $\text{Ext}(X)^J$ of J -indexed diagrams.*

PROOF. Whenever \mathcal{E} is small, the functor $\mathcal{E}^{op}: \mathcal{C} \rightarrow \text{Cat}_\infty^{op}$ has a right adjoint by Proposition 5.23. Thus, by Lemma 3.35, it follows that all comprehension schemes satisfied by the identity on Cat_∞ are transferred to \mathcal{E} . The second statement therefore follows from Example 3.22. The last statement follows in the same way as the according claim in Corollary 5.18. ■

6. Fibered notions of general comprehension and definability

In this section we briefly discuss the structural theory of general comprehension schemes (Definition 1.1). We further introduce the notion of relative definability of full subfibrations following [8, Paragraph 7.4] in terms of general comprehension and in context of the notions studied in Section 5 and 7.

6.1. **DEFINITION.** *Let p and q be cartesian fibrations over \mathcal{C} and $f: p \rightarrow q$ be a cartesian functor over \mathcal{C} . Say that \mathcal{C} has f -comprehension if both natural transformations $\text{St}(f)^\simeq: \text{St}(p)^\simeq \rightarrow \text{St}(q)^\simeq$ and $(\text{St}(f)^{\Delta^1})^\simeq: (\text{St}(p)^{\Delta^1})^\simeq \rightarrow (\text{St}(q)^{\Delta^1})^\simeq$ in $\hat{\mathcal{C}}$ are representable.*

In terms of Definition 1.1, an ∞ -category \mathcal{C} has f -comprehension for a cartesian functor f over \mathcal{C} exactly if \mathcal{C} has both $\text{St}(f)^\simeq$ -comprehension and $(\text{St}(f)^{\Delta^1})^\simeq$ -comprehension. Whenever \mathcal{C} has pullbacks, f -comprehension implies $(\text{St}(f)^{\Delta^n})^\simeq$ -comprehension for all $n \geq 0$. This follows directly from Lemma 2.11.2 and the fact that $(\text{St}(f)^{\Delta^\bullet})^\simeq$ is a functor between (complete) Segal objects in $\hat{\mathcal{C}}$.

6.2. DEFINITION. Let $p: \mathcal{E} \rightarrow \mathcal{C}$ be a cartesian fibration and $\mathcal{E}' \subseteq \mathcal{E}$ be a full ∞ -subcategory. If the restriction

$$\begin{array}{ccc} \mathcal{E}' & \xrightarrow{\iota} & \mathcal{E} \\ & \searrow p' & \swarrow p \\ & \mathcal{C} & \end{array}$$

is a cartesian functor of cartesian fibrations, we say that p' is a full subfibration of p . A full subfibration p' of p is definable if \mathcal{C} has $(p' \subseteq p)$ -comprehension.

Given a full subfibration $p': \mathcal{E}' \rightarrow \mathcal{C}$ of $p: \mathcal{E} \rightarrow \mathcal{C}$, it is easy to see that a morphism of \mathcal{E}' is p' -cartesian if and only if it is p -cartesian. In particular, p' is a full \mathcal{C} -subcategory of p in the sense of [5, Section 4]. We say that p' is a full and replete subfibration of p whenever the ∞ -subcategory $\mathcal{E}' \subseteq \mathcal{E}$ is also replete. By [5, Lemma 4.5], given a cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ and a full and replete ∞ -subcategory $\mathcal{E}' \subseteq \mathcal{E}$, the restriction of p to \mathcal{E}' yields a (full and replete) subfibration of p if and only if for every p -cartesian morphism f in \mathcal{E} , the morphism f is contained in \mathcal{E}' whenever its codomain is contained in \mathcal{E}' . It is easy to see that whenever $p' \subseteq p$ is full, the square

$$\begin{array}{ccc} \llbracket \Delta^1, \mathcal{E}' \rrbracket & \xrightarrow{(\delta^1)^*} & \llbracket \partial \Delta^1, \mathcal{E}' \rrbracket \\ \downarrow & \lrcorner & \downarrow \\ \llbracket \Delta^1, \mathcal{E} \rrbracket & \xrightarrow{(\delta^1)^*} & \llbracket \partial \Delta^1, \mathcal{E} \rrbracket \end{array} \tag{34}$$

is a pullback of right fibrations. We further note that inclusions of full subfibrations $p' \subseteq p$ are monomorphisms in the ∞ -category $\text{Cart}(\mathcal{C})$, and hence they induce (pointwise fully faithful) monomorphisms $\text{St}(p') \hookrightarrow \text{St}(p)$ of associated \mathcal{C} -indexed ∞ -categories as well. Thus, the term definability is to be understood as comprehension of substructures.

6.3. REMARK. In [8] (and in [50] accordingly), definability is a property of classes of objects in a cartesian fibration $\mathcal{E} \rightarrow \mathcal{C}$. The conditions thereby imposed on such classes $K \subseteq \mathcal{E}$ of objects are equivalent to the assertion that the full subcategory $\mathcal{K} \subseteq \mathcal{E}$ generated by K yields a full and replete subfibration of p . Then it is easy to show that the class K is definable in the sense of [8] if and only if the full subfibration $\mathcal{K}^\times \subseteq \mathcal{E}^\times \rightarrow \mathcal{C}$ is definable in the sense of Definition 6.2.

As alluded to in the introduction, a full subfibration of the canonical fibration over a 1-category \mathcal{C} with pullbacks is essentially the same thing as a set-valued general comprehension scheme over \mathcal{C} (Definition 1.1). With the ordinary categorical framework in mind, we make the following definitions.

6.4. DEFINITION. Let \mathcal{C} be an ∞ -category. A comprehension ∞ -category over \mathcal{C} is a cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ together with a functor

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \text{Fun}(\Delta^1, \mathcal{C}) \\ & \searrow p & \swarrow t \\ & \mathcal{C} & \end{array}$$

over \mathcal{C} which preserves cartesian morphisms. A comprehension ∞ -category is full if its underlying functor $\mathcal{E} \rightarrow \text{Fun}(\Delta^1, \mathcal{C})$ is fully faithful.

An ∞ -category with attributes over \mathcal{C} is a comprehension ∞ -category (p, f) over \mathcal{C} such that p is a right fibration.

6.5. DEFINITION. Let $\text{CmpCat}_\infty(\mathcal{C})$ be the ∞ -subcategory of the slice $((\text{Cat}_\infty)_{/\mathcal{C}})_{/t}$ spanned by the comprehension ∞ -categories over \mathcal{C} and those diagrams $F: (p, f) \rightarrow (q, g)$ such that $F: p \rightarrow q$ is a cartesian functor. Let $\text{FCmpCat}_\infty(\mathcal{C}) \subset \text{CmpCat}_\infty(\mathcal{C})$ denote the full ∞ -subcategory spanned by the full comprehension ∞ -categories.

Let $\text{CwA}_\infty(\mathcal{C}) \subset \text{CompCat}_\infty(\mathcal{C})$ be the full ∞ -subcategory spanned by the ∞ -categories with attributes over \mathcal{C} .

For a presheaf $Y \in \hat{\mathcal{C}}$, let $\text{Rep}(\mathcal{C})_{/Y} \subset \hat{\mathcal{C}}_{/Y}$ be the full ∞ -category spanned by the general comprehension schemes with base $Y \in \hat{\mathcal{C}}$ on \mathcal{C} .

Whenever \mathcal{C} has pullbacks, the target functor $t: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$ is a cartesian fibration itself, and so $\text{CmpCat}_\infty(\mathcal{C})$ is thus just the slice $\text{Cart}(\mathcal{C})_{/t}$. Analogously, in this case $\text{CwA}_\infty(\mathcal{C})$ is the slice $\text{RFib}(\mathcal{C})_{/t^\times}$.

6.6. PROPOSITION. Let \mathcal{C} be a small ∞ -category and $Y \in \hat{\mathcal{C}}$. Then the ∞ -categories $\text{Rep}(\mathcal{C})_{/Y}$ and the full ∞ -subcategory of $\text{CwA}_\infty(\mathcal{C})$ spanned by tuples (p, f) such that $p \simeq \text{Un}(Y)$ are equivalent.

PROOF. For any given presheaf $Y \in \hat{\mathcal{C}}$, the unstraightening $U: \text{Un}(Y) \rightarrow \mathcal{C}$ is the ∞ -category of elements of Y in as much as the colimit of the composition

$$\text{Un}(Y) \xrightarrow{U} \mathcal{C} \xrightarrow{y} \hat{\mathcal{C}}$$

is equivalent to Y itself [27, Lemma 5.1.5.3]. Hence, by virtue of descent of the ∞ -topos $\hat{\mathcal{C}}$, the restriction functor

$$\text{res}: \hat{\mathcal{C}}_{/Y} \rightarrow \text{Fun}(\text{Un}(Y), \hat{\mathcal{C}}) \downarrow^\times yU \tag{35}$$

given componentwise by pullback along the elements $y(C) \rightarrow Y$ is part of an equivalence (with inverse given by the colimit functor), where the right hand side denotes the ∞ -category of cartesian natural transformations over the functor yU [1, Section 3.3]. The ∞ -category $\text{Fun}(\text{Un}(Y), \hat{\mathcal{C}})_{/yU}$ of natural transformations over yU is canonically equivalent to the ∞ -category of functors from $yU: \text{Un}(Y) \rightarrow \hat{\mathcal{C}}$ to $t: \hat{\mathcal{C}}^{\Delta^1} \rightarrow \hat{\mathcal{C}}$ over $\hat{\mathcal{C}}$. Accordingly, the

codomain of (35) as a full ∞-subcategory of $\text{Fun}(\text{Un}(Y), \hat{\mathcal{C}})_{/yU}$ is in turn equivalent to the ∞-category of functors from $yU: \text{Un}(Y) \rightarrow \hat{\mathcal{C}}$ to the right fibration $t^\times: \text{Fun}(\Delta^1, \hat{\mathcal{C}})^\times \rightarrow \hat{\mathcal{C}}$ over $\hat{\mathcal{C}}$.

$$\begin{array}{ccc}
 & \text{Fun}(\Delta^1, \mathcal{C})^\times & \xrightarrow{y} & \text{Fun}(\Delta^1, \hat{\mathcal{C}})^\times \\
 & \nearrow & & \nearrow \\
 \text{Un}(Y) & \xrightarrow{\quad\quad\quad} & \text{Un}(Y) & \\
 \downarrow U & & \downarrow yU & \\
 \mathcal{C} & \xrightarrow{y} & \hat{\mathcal{C}} & \\
 & \nwarrow & \nwarrow & \\
 & \text{Fun}(\Delta^1, \mathcal{C})^\times & & \text{Fun}(\Delta^1, \hat{\mathcal{C}})^\times
 \end{array}
 \tag{36}$$

A natural transformation $f: X \rightarrow Y$ in $\hat{\mathcal{C}}$ is representable if and only if the associated functor $\text{Un}(Y) \rightarrow \text{Fun}(\Delta^1, \mathcal{C})^\times$ as depicted by a dotted arrow on the right of diagram (36) lifts along $y: \text{Fun}(\Delta^1, \mathcal{C})^\times \rightarrow \text{Fun}(\Delta^1, \hat{\mathcal{C}})^\times$ to a functor over \mathcal{C} as depicted by a dotted arrow on the left of the diagram. We hence obtain an equivalence between the ∞-category of general comprehension schemes with base Y on \mathcal{C} on the one hand, and ∞-categories with attributes of the form $(\text{Un}(Y), f)$ over \mathcal{C} on the other. ■

Clearly, we furthermore have a functor $(\cdot)^\times: \text{CmpCat}_\infty(\mathcal{C}) \rightarrow \text{CwA}_\infty(\mathcal{C})$. Its restriction to $\text{FCmpCat}_\infty(\mathcal{C})$ comes with a retraction r given by factorization of an underlying functor $\mathcal{E} \rightarrow \text{Fun}(\Delta^1, \mathcal{C})$ through an essentially surjective functor followed by a fully faithful and replete functor. The pair

$$(\cdot)^\times: \text{FCmpCat}_\infty(\mathcal{C}) \overset{\hookrightarrow}{\underset{\longleftarrow}{\rightleftarrows}} \text{CwA}_\infty(\mathcal{C}): r \tag{37}$$

however does not form an equivalence of ∞-categories; rather, full comprehension ∞-categories are particularly simple ∞-categories with attributes in which the entire higher structure of the homotopy types of the attributes are fully determined by \mathcal{C} . We elaborate on this in the following remark and the subsequent proposition.

6.7. REMARK. In 1-category theory, the abstract equivalence of categories with attributes over a category \mathcal{C} (together with a fixed discrete fibration $p: \mathcal{E} \rightarrow \mathcal{C}$) and full comprehension categories over \mathcal{C} (together with a fixed cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$) is a formal consequence of the existence of the (bijective-on-objects, fully faithful)-factorization system on Cat . See e.g. [19, Lemma 4.9, Example 4.10]. Such a factorization system (with respect to an accordingly univalent notion of “identity-on-objects”) does not exist in Cat_∞ ⁵; and in fact it is not hard to show that ∞-categories with attributes and full comprehension ∞-categories over a given ∞-category \mathcal{C} are generally non-equivalent structures.

6.8. PROPOSITION. *The ∞-categories $\text{FCmpCat}_\infty(\mathcal{C})$ and $\text{CwA}_\infty(\mathcal{C})$ are generally not equivalent.*

⁵As already noted in Remark 3.20, this factorization system on Cat relies exactly on the very specific kind of equality principle which in [8, Paragraph 8.7] is considered to be entirely meta-theoretical rather than category theoretical.

PROOF. Consider the left exact ∞ -category $\mathcal{C} = *$. The ∞ -category $\text{CwA}_\infty(*) \simeq \text{RFib}(*)_{/1*}$ is just the ∞ -category \mathcal{S} of spaces. The full ∞ -subcategory of $\text{CompCat}_\infty(*) \simeq (\text{Cat}_\infty)_{/1*}$ spanned by the fully faithful functors however is the ∞ -category of locally contractible ∞ -categories (which are exactly the locally small cartesian fibrations over the point by Example 3.26). But a locally contractible ∞ -category is automatically either empty or a contractible space, and so the homotopy category of locally contractible ∞ -categories has only two objects up to isomorphism. Hence, the two structures over a point are not equivalent. \blacksquare

Proposition 6.8 can be shown even more concretely over a fixed base $Y \in \hat{\mathcal{C}}$: let $\mathcal{C} = *$ and Y be any non-empty and non-contractible space considered as an ∞ -category with attributes over the point. Then the full subcategory of $\text{CwA}_\infty(*)$ spanned by pairs (X, f) such that $X \simeq Y$ is contractible. But the ∞ -category of locally contractible ∞ -categories such that the core $\mathcal{C}^\simeq = \mathcal{C}$ is equivalent to Y is empty.

6.9. REMARK. The proof of Proposition 6.8 in ordinary category theory computes up to equivalence the category of sets on the one hand, and the category of complete graphs on the other hand. These however are indeed isomorphic.

In fact, via Proposition 6.6 one can characterize the full comprehension ∞ -categories among all ∞ -categories with attributes exactly as those representable natural transformations that are univalent (as discussed in Section 7). Therefore, we note that the proof of Proposition 6.6 in fact shows that the “universe of sections” associated to a small ∞ -category \mathcal{C} with pullbacks is the universal comprehension scheme over \mathcal{C} as we show in the following ∞ -categorical version of [41, Proposition 2.7].

6.10. PROPOSITION. *Suppose \mathcal{C} is a small ∞ -category with pullbacks. Then there is a representable natural transformation $\pi_{\mathcal{C}}: (\mathcal{C}_{/(\cdot)})_*^\simeq \rightarrow (\mathcal{C}_{/(\cdot)})^\simeq$ in $\hat{\mathcal{C}}$ such that for every $Y \in \hat{\mathcal{C}}$, the functor*

$$(\cdot)^* \pi_{\mathcal{C}}: \hat{\mathcal{C}}(Y, (\mathcal{C}_{/(\cdot)})^\simeq) \rightarrow (\text{Rep}(\mathcal{C})_{/Y})^\simeq$$

which maps a natural transformation $f: Y \rightarrow (\mathcal{C}_{/(\cdot)})^\simeq$ to the pullback $f^ \pi_{\mathcal{C}} \in \hat{\mathcal{C}}_{/Y}$ is an equivalence of ∞ -categories.*

PROOF. By definition, the canonical indexing $(\mathcal{C}_{/(\cdot)})^\simeq \in \hat{\mathcal{C}}$ is the straightening of the right fibration $t^\times: (\text{Fun}(\Delta^1, \mathcal{C})^\times \twoheadrightarrow \mathcal{C})$. It thus follows again from [27, Lemma 5.1.5.3] that the colimit of the composition

$$\text{Fun}(\Delta^1, \mathcal{C})^\times \xrightarrow{t^\times} \mathcal{C} \xrightarrow{y} \hat{\mathcal{C}}$$

is equivalent to the presheaf $(\mathcal{C}_{/(\cdot)})^\simeq$ in $\hat{\mathcal{C}}$. In particular, the colimit of the functor

$$\text{Fun}(\Delta^1, y): \text{Fun}(\Delta^1, \mathcal{C})^\times \rightarrow \text{Fun}(\Delta^1, \hat{\mathcal{C}})^\times \quad (38)$$

is a natural transformation over $(\mathcal{C}_{/(\cdot)})^\simeq$. We denote this colimit by $\pi_{\mathcal{C}}: (\mathcal{C}_{/(\cdot)})_*^\simeq \rightarrow (\mathcal{C}_{/(\cdot)})^\simeq$. Verification of all the claims is now a routine exercise in the application of descent properties of ∞ -toposes. Indeed, we first observe that the functor

$$(\cdot)^* \pi_{\mathcal{C}}: \hat{\mathcal{C}}(Y, (\mathcal{C}_{/(\cdot)})^\simeq) \rightarrow (\hat{\mathcal{C}}_{/Y})^\simeq \quad (39)$$

induced by pullback of the natural transformation $\pi_{\mathcal{C}}$ is fully faithful (i.e. $\pi_{\mathcal{C}}$ is a univalent morphism in $\hat{\mathcal{C}}$ as discussed in Section 7). By virtue of descent of $\hat{\mathcal{C}}$ and the fact that the class of fully faithful functors between spaces is closed under small limits, it suffices to show this for $Y = y(C)$ a representable presheaf. In this case the functor (39) is simply the (core of the) Yoneda embedding

$$y: (\mathcal{C}_{/C})^{\simeq} \rightarrow (\widehat{\mathcal{C}}_{/C})^{\simeq}$$

under the canonical equivalence $\hat{\mathcal{C}}_{/y(C)} \simeq \widehat{\mathcal{C}}_{/C}$ given by the Yoneda lemma. In particular, it is fully faithful.

Second, the natural transformation $\pi_{\mathcal{C}}$ is representable as for every $C \in \mathcal{C}$ and every natural transformation $\ulcorner f \urcorner: y(C) \rightarrow (\mathcal{C}_{/(\cdot)})^{\simeq}$, the canonical square

$$\begin{array}{ccc} y(\text{dom } f) & \longrightarrow & (\mathcal{C}_{/(\cdot)})^{\simeq}_* \\ yf \downarrow & & \downarrow \pi_{\mathcal{C}} \\ y(C) & \xrightarrow{\ulcorner f \urcorner} & (\mathcal{C}_{/(\cdot)})^{\simeq} \end{array}$$

given by the cocone that defines $\pi_{\mathcal{C}}$ as the colimit of (38) is cartesian by virtue of the fact that (38) is a cartesian natural transformation in $\hat{\mathcal{C}}$, and by the fact that colimits in $\hat{\mathcal{C}}$ are effective. Thus, since representability of natural transformations is a pullback-stable property (Lemma 2.11.1), the fully faithful functor (39) factors through the full subspace $(\text{Rep}(\mathcal{C})_{/Y})^{\simeq}$. We are left to show that this functor is essentially surjective; but this is exactly the content of Proposition 6.6. More precisely, given a representable natural transformation $f: X \rightarrow Y$ in $\hat{\mathcal{C}}$, let $\text{res}(f): \text{Un}(Y) \rightarrow \text{Fun}(\Delta^1, \mathcal{C})^{\times}$ over \mathcal{C} be its associated ∞ -category with attributes from Proposition 6.6 defined via the equivalence (35). Diagram (36) induces a square in $\hat{\mathcal{C}}$ from the colimit of the composition $\text{res}(f): \text{Un}(Y) \rightarrow \text{Fun}(\Delta^1, \hat{\mathcal{C}})^{\times}$ to the colimit of the inclusion $\text{Fun}(\Delta^1, y): \text{Fun}(\Delta^1, \mathcal{C})^{\times} \rightarrow \text{Fun}(\Delta^1, \hat{\mathcal{C}})^{\times}$. That is, a square in $\hat{\mathcal{C}}$ from f to $\pi_{\mathcal{C}}$. This square is cartesian again by virtue of descent of $\hat{\mathcal{C}}$. ■

6.11. COROLLARY. ⁶ *Let \mathcal{C} be a small ∞ -category and $Y \in \hat{\mathcal{C}}$. Then the equivalence of Proposition 6.6 restricts to an equivalence between the full ∞ -subcategory spanned by the full comprehension ∞ -categories via (37) on the one hand, and the full ∞ -subcategory spanned by the univalent representable natural transformations over Y on the other hand.*

PROOF. For any given representable natural transformation $f: X \rightarrow Y$ in $\hat{\mathcal{C}}$ there is an essentially unique classifying morphism $\ulcorner f \urcorner: Y \rightarrow (\mathcal{C}_{/(\cdot)})^{\simeq}$ by Proposition 6.10. This classifying morphism is by construction the value of the colimit functor applied to the morphism

$$\text{res}_f: yU \rightarrow yt$$

⁶The observation of this corollary is due to Jonas Frey; I personally had missed this.

of functors over $\hat{\mathcal{C}}$, where $U: \text{Un}(Y) \twoheadrightarrow \mathcal{C}$ denotes the unstraightening of Y . This colimit computes the straightening $\text{St}(\text{res}_f): Y \rightarrow (\mathcal{C}_{/(\cdot)})^\simeq$, essentially because the Yoneda embedding left Kan extends to the identity along itself [27, Lemma 5.1.5.3]. It follows in turn that the fibered functor $\text{res}_f: U \rightarrow t$ over \mathcal{C} is the unstraightening of the classifying morphism $\ulcorner f \urcorner$.

Now, as the universal representable natural transformation $\pi_{\mathcal{C}}$ is univalent, the transformation f itself is univalent if and only if its classifying morphism $\ulcorner f \urcorner$ is monic [35, Proposition 2.5]. This in turn holds if and only if its unstraightening $\text{res}_f: U \rightarrow t$ is monic, which is to say that the associated ∞ -category with attributes $\text{res}_f: U \rightarrow t$ is (the core of) a full comprehension ∞ -category. ■

We return to the notion of definability introduced in Definition 6.2. We first note that definability of a full subfibration can be reduced to definability of its associated right fibration of objects in the following sense.

6.12. LEMMA. *Let $p: \mathcal{E} \twoheadrightarrow \mathcal{C}$ and $p': \mathcal{E}' \twoheadrightarrow \mathcal{C}$ be cartesian fibrations and let $f: p' \rightarrow p$ be a cartesian functor over \mathcal{C} .*

1. *Suppose \mathcal{C} has pullbacks and $f: p' \rightarrow p$ is the inclusion of a full subfibration. Then p' is definable if and only if the inclusion $\text{St}(p') \hookrightarrow \text{St}(p)$ is representable.*
2. *Suppose p and p' are right fibrations. Then \mathcal{C} has f -comprehension if and only if the natural transformation $\text{St}(p') \rightarrow \text{St}(p)$ is representable.*

PROOF. For Part 1 we note that the pullback (34) induces a pullback

$$\begin{array}{ccc} (\text{St}(p')^{\Delta^1})^\simeq & \xrightarrow{(d_1, d_0)^*} & (\text{St}(p'))^\simeq \times (\text{St}(p'))^\simeq \\ \downarrow & \lrcorner & \downarrow \\ (\text{St}(p)^{\Delta^1})^\simeq & \xrightarrow{(d_1, d_0)^*} & (\text{St}(p))^\simeq \times (\text{St}(p))^\simeq \end{array}$$

of presheaves in $\hat{\mathcal{C}}$. Thus, to show that the vertical natural transformation on the left hand side is representable, it suffices to show that the right hand side is representable. By assumption however the natural transformation $(\text{St}(p'))^\simeq \rightarrow (\text{St}(p))^\simeq$ is representable indeed. It follows that its product is representable whenever \mathcal{C} has pullbacks by Lemma 2.11.2. For Part 2, if p and p' are right fibrations the vertical morphisms in the diagram

$$\begin{array}{ccc} \text{St}(p')^{\Delta^1} & \longrightarrow & \text{St}(p)^{\Delta^1} \\ (s^0)^* \uparrow & & \uparrow (s^0)^* \\ \text{St}(p') & \longrightarrow & \text{St}(p) \end{array}$$

are natural equivalences, and it follows that the top horizontal natural transformation is representable if and only if the bottom one is. ■

In Example 2.8 we stated that an ∞ -category \mathcal{C} has finite products if and only if representability of a presheaf $X \in \hat{\mathcal{C}}$ is equivalent to representability of the natural transformation $X \rightarrow *$. This is a special case of the equally straightforward fact that \mathcal{C} has pullbacks (with base $C \in \mathcal{C}$) if and only if representability of a natural transformation in $\hat{\mathcal{C}}$ over a representable presheaf (represented by C) is equivalent to representability of its domain. In the rest of this section, we prove a direct generalization of (one half of) this observation to \mathcal{C} -indexed ∞ -categories: if \mathcal{C} has finite limits, then a comprehensible cartesian functor over a small \mathcal{C} -indexed ∞ -category of the form $\text{Ext}(Y)$ is the same thing as an internal functor over Y . This will be formally stated in Proposition 6.16.

6.13. LEMMA. *Let \mathcal{C} be an ∞ -category. Let $p: \mathcal{E} \twoheadrightarrow \mathcal{C}$ and $p': \mathcal{E}' \twoheadrightarrow \mathcal{C}$ be cartesian fibrations and let $f: p' \rightarrow p$ be a cartesian functor over \mathcal{C} such that \mathcal{C} has f -comprehension.*

1. *If p is a globally small fibration, then so is p' .*
2. *Suppose \mathcal{C} has pullbacks. If p is a small fibration, then so is p' .*

PROOF. In Part 1 the natural transformations $\text{St}(p)^{\simeq} \rightarrow *$ and $\text{St}(p')^{\simeq} \rightarrow \text{St}(p)^{\simeq}$ in $\hat{\mathcal{C}}$ are representable by assumption. This implies that the composition $\text{St}(p')^{\simeq} \rightarrow *$ is representable by Lemma 2.11.1. Hence, the fibration p' is globally small by definition. For Part 2 consider the square

$$\begin{array}{ccc}
 (\text{St}(p')^{\Delta^1})^{\simeq} & \longrightarrow & (\text{St}(p)^{\Delta^1})^{\simeq} \\
 \delta^* \downarrow & & \downarrow \delta^* \\
 (\text{St}(p')^{\partial\Delta^1})^{\simeq} & \longrightarrow & (\text{St}(p)^{\partial\Delta^1})^{\simeq}
 \end{array} \tag{40}$$

in $\hat{\mathcal{C}}$. The right vertical natural transformation and the top horizontal natural transformation are representable by assumption. The bottom horizontal natural transformation is representable by the assumption of f -comprehension and by the fact the class of representable natural transformation in $\hat{\mathcal{C}}$ is closed under finite limits (Lemma 2.11.2). Since representable natural transformations are furthermore left cancellable, it follows that the left vertical natural transformation is representable as well. ■

In fact, whenever $p' \subseteq p$ is full, the assumption of comprehension can be dropped in the case of local smallness alone.

6.14. LEMMA. *Let \mathcal{C} be an ∞ -category. A full subfibration of a locally small fibration over \mathcal{C} is again locally small.*

PROOF. Given a fibration p over \mathcal{C} together with some full subfibration $p' \subseteq p$, the statement follows directly from Lemma 2.11.1 and the fact that the square (40) is the straightening of the cartesian square (34) by Proposition 4.4 and hence is cartesian itself. ■

In the converse direction, we have the following.

6.15. LEMMA. *Let \mathcal{C} be an ∞ -category with pullbacks. Suppose $f: p' \rightarrow p$ is a cartesian functor between small cartesian fibrations over \mathcal{C} . Then \mathcal{C} has f -comprehension.*

PROOF. First, consider the sequence

$$\mathrm{St}(p')^{\simeq} \rightarrow \mathrm{St}(p)^{\simeq} \rightarrow *$$

of natural transformations in $\hat{\mathcal{C}}$. The composition and the second natural transformation are representable by assumption. It follows that so is the first by Lemma 2.11.2. Second, consider the square (40) in $\hat{\mathcal{C}}$ again. The two vertical natural transformations are representable by assumption. By virtue of global smallness, we have seen that the natural transformation $\mathrm{St}(p') \rightarrow \mathrm{St}(p)$ is representable as well. It follows that the bottom vertical transformation is representable, too, since the class of representable natural transformation in $\hat{\mathcal{C}}$ is closed under finite limits (again by Lemma 2.11.2). Thus, the top vertical morphism in the square is representable by the left cancellation property stated in the same lemma. ■

6.16. PROPOSITION. *Let \mathcal{C} be an ∞ -category with finite limits, let Y be a complete Segal object in \mathcal{C} , and let $f: p \rightarrow \mathrm{Ext}(Y)$ be a cartesian functor over \mathcal{C} . Then p is small if and only if \mathcal{C} has f -comprehension. In particular, \mathcal{C} has $\mathrm{Ext}(f)$ -comprehension for all internal functors $f: X \rightarrow Y$ in $\mathrm{CS}(\mathcal{C})$.*

PROOF. Immediate by Lemma 6.13, Lemma 6.15 and Theorem 5.15. ■

7. Univalence and smallness

We apply the results of the last two sections to the special case of full comprehension ∞ -categories over locally cartesian closed ∞ -categories \mathcal{C} , and find that such are globally small if and only if they are the externalization of the nerve of a univalent morphism in \mathcal{C} . Against this background, we will end the section with a short discussion of higher elementary toposes in terms of comprehension schemes.

For the rest of this section suppose \mathcal{C} is a left exact locally cartesian closed ∞ -category. For every morphism $q: E \rightarrow B$ in \mathcal{C} we can associate the full and replete comprehension ∞ -category $F_q \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{t} \mathcal{C}$ where the fiber $F_q(C) \subseteq \mathcal{C}_{/C}$ consists of those morphisms with codomain C which arise as pullback of q along some morphism $C \rightarrow B$.

7.1. LEMMA. *For every morphism $q \in \mathcal{C}$, the cartesian fibration $F_q \rightarrow \mathcal{C}$ is locally small.*

PROOF. Immediate by Proposition 3.24 and Proposition 6.13.2. ■

A morphism $q: E \rightarrow B$ in \mathcal{C} is *univalent* if it is a (-1) -truncated object in the core $\text{Fun}(\Delta^1, \mathcal{C})^\times$ [35, Definition 2.1]. Equivalently, that is whenever the object $q \in (F_q)^\times$ is terminal. By Lemma 2.6 in turn, this means that q is univalent if and only if the functor $\ulcorner q \urcorner: \mathcal{C}_{/B} \rightarrow (F_q)^\times$ of right fibrations given by $1_B \mapsto q$ via the Yoneda lemma is an equivalence.

7.2. PROPOSITION. *Given a full and replete comprehension ∞ -category J over \mathcal{C} , the following are equivalent.*

1. $J \rightarrow \mathcal{C}$ is globally small.
2. There is a complete Segal space $\mathcal{N}(q) \in \text{CS}(\mathcal{C})$ such that $\text{Ext}(\mathcal{N}(q)) \simeq J$.
3. There is a univalent morphism $q: E \rightarrow B$ in \mathcal{C} such that $J = F_q$.

PROOF. Parts 1 and 2 are equivalent by Theorem 5.15 and Lemma 7.1. If J is globally small, its core $J^\times \rightarrow \mathcal{C}$ is representable by definition and hence has a terminal object $(q: E \rightarrow B) \in J^\times$. It follows that $J = F_q$ since $J \subseteq \text{Fun}(\Delta^1, \mathcal{C})$ is full and replete. In particular, q is terminal in $(F_q)^\times$ and so q is univalent. Vice versa, whenever $J = F_q$ for a univalent morphism q in \mathcal{C} , then $q \in J^\times$ is terminal by definition, and so $\text{St}(J^\times)$ is representable. That means J is globally small. ■

The equivalence of Parts 2 and 3 has a non-univalent 1-categorical analogon as well, see [49, Section 6] and [20, Proposition 7.3.6]. In the case that \mathcal{C} is a presentable ∞ -category, Gepner and Kock have characterized univalence of morphisms q in \mathcal{C} in terms of a sheaf property of the \mathcal{C} -indexed ∞ -category $\text{St}(F_q)$ [17, Proposition 3.8]. This can be directly generalized as follows.

7.3. COROLLARY. *Suppose \mathcal{C} is locally cartesian closed and complete and $q: E \rightarrow B$ is a morphism in \mathcal{C} . Then $q \in \text{Fun}(\Delta^1, \mathcal{C})^\times$ has a retract p which is a univalent morphism in \mathcal{C} if and only if the indexed ∞ -category $\text{St}(F_q): \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ has a left adjoint. In particular, the functor $\text{St}(F_q): \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ preserves limits whenever q has a univalent retract in $\text{Fun}(\Delta^1, \mathcal{C})^\times$.*

PROOF. If q has a retract p in $\text{Fun}(\Delta^1, \mathcal{C})^\times$, then $F_q = F_p$. Furthermore, whenever p is univalent, the straightening $\text{St}(F_p): \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ has a left adjoint by Proposition 7.2 and Proposition 5.23. Vice versa, whenever $\text{St}(F_q): \mathcal{C}^{op} \rightarrow \text{Cat}_\infty$ has a left adjoint, it is globally small again by Proposition 5.23. By Proposition 7.2 it follows that there is a univalent morphism p in \mathcal{C} such that $F_q = F_p$. Thus, both morphisms p and q are mutually pullbacks of one another, which yield morphisms $q \rightarrow p$ and $p \rightarrow q$ in $(F_p)^\times$. As $p \in (F_p)^\times$ is terminal, it follows that p is a retract of q in $(F_q)^\times \subset \text{Fun}(\Delta^1, \mathcal{C})^\times$. ■

7.4. REMARK. Given a cartesian fibration $\mathcal{E} \twoheadrightarrow \mathcal{C}$, one may define in this generality an object $x \in \mathcal{E}$ to be univalent if it is (-1) -truncated in the core \mathcal{E}^\times . Accordingly, one can generalize various related constructions. For instance, the notion of univalent completion as defined in [10] and studied in more generality in [45, Section 5] can be defined for any object $x \in \mathcal{E}$ as its (-1) -truncation in the core \mathcal{E}^\times whenever it exists.

7.5. **REMARK.** The fact that smallness of F_q only implies the existence of a univalent completion $q \rightarrow p$ (with a section) in \mathcal{C} rather than univalence of q itself is a special case of the fact that smallness of $\text{Ext}(X)$ for a given Segal object X in \mathcal{C} only implies the existence of a Segal completion $X \rightarrow \rho(X)$ in \mathcal{C} rather than completeness of X itself (Remark 5.17). Indeed, for every morphism q in \mathcal{C} one can construct a Segal object $\mathcal{N}(q)$ in \mathcal{C} which is complete if and only if q is univalent [35].

In Remark 3.12 we discussed criteria towards a potential classification of elementary ∞ -toposes via comprehension schemes. Against the background of the results in this section, let us finish this paper with a review of this discussion. In Remark 3.12 we argued that a presentable ∞ -category is an ∞ -topos if and only if its associated canonical indexing is locally small and κ -super powered for all sufficiently large regular cardinals κ . To generalize this criterion to non-presentable ∞ -categories as well, we can re-parametrize the associated class of comprehension schemes imposed on the canonical indexing as a single comprehension scheme imposed on a class of canonically associated full comprehension ∞ -categories as follows. Assume for the moment being again that \mathcal{C} is presentable. Then \mathcal{C} is an ∞ -topos if and only if colimits in \mathcal{C} are universal and its canonical indexing

$$\mathcal{C}_{/(\cdot)}: \mathcal{C}^{op} \rightarrow \text{CAT}_\infty$$

preserves limits [27, Theorem 6.1.3.9]. If this canonical indexing in fact had a left adjoint, it would be small by Proposition 5.23. Vice versa, every presentable ∞ -category \mathcal{C} with a small canonical indexing is an ∞ -topos. However, although the canonical indexing may be limit preserving, it cannot have a left adjoint if \mathcal{C} is large as this would generate a single univalent universe in \mathcal{C} via Proposition 7.2 (given that such a \mathcal{C} is automatically locally cartesian closed). Yet, the underlying idea that an ∞ -topos is a presentable ∞ -category which is small over itself holds up to local approximations in the following sense.

Given a regular cardinal κ , let $\text{Fun}(\Delta^1, \mathcal{C})_\kappa \subset \text{Fun}(\Delta^1, \mathcal{C})$ be the full and replete ∞ -subcategory generated by the relative κ -compact morphisms in \mathcal{C} and consider its associated full comprehension ∞ -category $t_\kappa: \text{Fun}(\Delta^1, \mathcal{C})_\kappa \rightarrow \mathcal{C}$.

7.6. **COROLLARY.** *A presentable ∞ -category \mathcal{C} is an ∞ -topos if and only if the following two conditions hold.*

1. *The canonical fibration $t: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$ is locally small.*
2. *For all sufficiently large regular cardinals κ , the full comprehension ∞ -category $t_\kappa: \text{Fun}(\Delta^1, \mathcal{C})_\kappa \rightarrow \mathcal{C}$ is small.*

PROOF. Immediate by Proposition 7.2 and [27, Theorem 6.1.6.8]. ■

Whenever \mathcal{C} is an ∞ -topos, we can furthermore show relative smallness of all its ∞ -subtoposes (in fact, of all its modalities in the sense of [2, Definition 3.2.1]). A version for 1-toposes of this fact can be found in [20, Proposition 9.6.9]. Therefore, we note that whenever \mathcal{C} is an ∞ -category with pullbacks, then pullback-stable classes S of morphisms in \mathcal{C} stand in 1-1 correspondence with full comprehension ∞ -categories $s \subseteq t$ over \mathcal{C} . A

pullback-stable class S is said to be *local* if the induced \mathcal{C} -indexed ∞ -category $S_{/(\cdot)} := \text{St}(s): \mathcal{C}^{op} \rightarrow \text{CAT}_\infty$ preserves small limits [27, Definition 6.1.3.8].

7.7. COROLLARY. *Suppose \mathcal{C} is an ∞ -topos and S is a pullback-stable class of morphisms in \mathcal{C} . Then its associated full comprehension ∞ -category $s \subseteq t$ is definable if and only if S is a local class. In particular, every ∞ -subtopos of \mathcal{C} (in fact every modality) considered as a subfibration of $t: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$ is definable.*

PROOF. On the one hand, the full subfibration $s: S \rightarrow \mathcal{C}$ of $t: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$ is definable if and only if the full inclusion

$$S^\times \hookrightarrow \text{Fun}(\Delta^1, \mathcal{C})^\times \tag{41}$$

fibered over \mathcal{C} has a (non-fibered) right adjoint by Lemma 6.12. On the other hand, the class S is local if and only if the inclusion $S^\times \hookrightarrow \text{Fun}(\Delta^1, \mathcal{C})^\times$ creates colimits via [27, Lemma 6.1.3.7]. As the class of all morphisms in \mathcal{C} is local by assumption, this holds if and only if the full inclusion (41) creates colimits. Now, if $s \subseteq t$ is definable, then the full inclusion (41) is a left adjoint and hence creates colimits. In particular, definability of $s \subseteq t$ implies locality of S . Vice versa, suppose S is local. To show that the inclusion (41) has a right adjoint, we are to show for any given $f \in \text{Fun}(\Delta^1, \mathcal{C})^\times$ that the comma ∞ -category $S^\times \downarrow f$ has a terminal object. The morphism f however is relative κ -compact for all regular cardinals κ large enough. If we denote by S_κ the intersection of S and $\text{Fun}(\Delta^1, \mathcal{C})_\kappa$, we observe that $S_\kappa^\times \downarrow f = S^\times \downarrow f$ for any such κ . Hence, the latter has a terminal object if the inclusion

$$\iota_\kappa: S_\kappa^\times \hookrightarrow \text{Fun}(\Delta^1, \mathcal{C})_\kappa^\times$$

has a right adjoint. If we denote by $s_\kappa \subseteq s$ the respective full subfibration, this holds whenever \mathcal{C} has $(s_\kappa \subseteq t_\kappa)$ -comprehension. The class S_κ is local again, and the continuous functor $(S_\kappa)_{/(\cdot)}: \mathcal{C}^{op} \rightarrow \text{CAT}_\infty$ factors through Cat_∞ . It follows that the \mathcal{C} -indexed ∞ -category $(S_\kappa)_{/(\cdot)}$ is small by Remark 5.24. Thus, the inclusion $s_\kappa \subseteq t_\kappa$ is a cartesian functor between small fibrations, and so \mathcal{C} has $(s_\kappa \subseteq t_\kappa)$ -comprehension by Lemma 6.15.

Lastly, any given ∞ -subtopos (or any modality) of \mathcal{C} can be presented as a full subfibration of t over \mathcal{C} , see e.g. [40, Theorem A.7]. The fact that the domain of this fibration is a local class of morphisms in \mathcal{C} is shown for instance in [2, Proposition 3.2.7]. \blacksquare

For general ∞ -categories \mathcal{C} with pullbacks, a pullback-stable class S of morphisms in \mathcal{C} is “closed” [34, Definition 3.4] if for all $C \in \mathcal{C}$ the fiber $s(C)$ of its associated full comprehension ∞ -category $s \subseteq t$ is closed under all finite limits and colimits. We obtain a characterization of elementary ∞ -toposes entirely in terms of comprehension schemes as follows.

7.8. COROLLARY. *An ∞ -category \mathcal{C} has finite limits and colimits if and only if*

1. *the identity over \mathcal{C} and the identity over \mathcal{C}^{op} are both representable right fibrations,*

2. and all representable right fibrations over \mathcal{C} and all representable right fibrations over \mathcal{C}^{op} are small.

Furthermore, an ∞ -category \mathcal{C} with finite limits and colimits is an elementary higher topos in the sense of [34, Definition 3.5] if and only if the following two conditions hold.

3. The canonical fibration $t: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$ has a subterminal object classifier (Example 3.11).
4. There is a (generally proper class-sized) collection $\mathcal{A} = \{S_i \mid i \in I\}$ of closed subclasses $S_i \subset \text{Fun}(\Delta^1, \mathcal{C})$ such that $\bigcup_{i \in I} S_i = \text{Fun}(\Delta^1, \mathcal{C})$ and such that the associated full comprehension ∞ -categories $s_i: S_i \rightarrow \mathcal{C}$ are small for every $i \in I$.

PROOF. The identity over \mathcal{C} is the unstraightening of the terminal presheaf. It is representable if and only if \mathcal{C} has a terminal object. Given a terminal object in \mathcal{C} , we have seen in Example 3.25 that smallness of all representables over \mathcal{C} is equivalent to the existence of all finite limits in \mathcal{C} . The same applies to \mathcal{C}^{op} . Given Proposition 7.2 and the fact that the slice of an elementary ∞ -topos is again an elementary ∞ -topos [34, Theorem 3.10], the second part of the corollary is a mere reformulation of the axioms of an elementary ∞ -topos. ■

The first part of Corollary 7.8 can be formulated entirely in terms fibered over \mathcal{C} by characterizing (representable) right fibrations over \mathcal{C}^{op} as (corepresentable) left fibrations over \mathcal{C} . That means, \mathcal{C} has finite colimits if and only if the identity on \mathcal{C} is a corepresentable left fibration and all corepresentable left fibrations over \mathcal{C} are small in the dual sense that they are equivalent to the nerve of an interval object in \mathcal{C} in the sense of Toën [51].

The second part of the corollary can be interpreted to state that while a finitely bi-complete (and large) ∞ -category \mathcal{C} with a well-powered canonical indexing generally cannot be globally small over itself, it instead can be covered by an atlas \mathcal{A} of small neighbourhoods in \mathcal{C} exactly if it is an elementary ∞ -topos. Local smallness of an elementary ∞ -topos over itself then follows from the definition via [34, Theorem 3.11] and Proposition 3.24. Vice versa, if on top of well-poweredness also local smallness of \mathcal{C} over itself is assumed explicitly, then the existence of a cover of small neighbourhoods is equivalent to a cover of globally small neighbourhoods by Proposition 7.2. Whenever the ∞ -category \mathcal{C} is presentable as well, Condition 2 in Corollary 7.6 is exactly Condition 4 in Corollary 7.8 for the atlas $\mathcal{A} = \{S_\kappa \mid \kappa \in \text{Card sufficiently large}\}$.

8. Model independence

Lastly, in this section we show that our constructions and main results regarding comprehension are invariant under the choice of any of the common models of $(\infty, 1)$ -category theory that the constructions and results are formulated in. As announced in the beginning of Section 4, we therefore use Riehl and Verity's framework of ∞ -cosmoses [39]. Here, an ∞ -cosmos \mathbf{V} is said to be an ∞ -cosmos of $(\infty, 1)$ -categories whenever the global

sections functor $(\cdot)_0: \mathbf{V} \rightarrow \mathbf{QCat}$ into the ∞ -cosmos of quasi-categories is a cosmological biequivalence [39, Definition 1.3.10, Proposition 10.2.1]. Every known model of $(\infty, 1)$ -category theory exhibits the structure of an ∞ -cosmos together with such a cosmological biequivalence. In this section, we will stay faithful to the terminology used in the book [39] for referential purposes.

In the following we will define comprehension schemes for fibrations and functors in general ∞ -cosmoses \mathbf{V} (with additional structure) which generalize the notions introduced in Section 3 as well as in the introduction to this framework. Technically, we will prove that, first, all (suitable) cosmological functors between (suitable) ∞ -cosmoses preserve satisfaction of all such comprehension schemes⁷, and second, that cosmological biequivalences both preserve and reflect satisfaction of all such comprehension schemes. We will state our results for the cosmological global sections functors $(\cdot)_0: \mathbf{V} \rightarrow \mathbf{QCat}$ only however to not divert too far from the subject at hand. It follows that the notions of comprehension in this paper are invariant under the choice of model of $(\infty, 1)$ -category theory. It further iteratively follows that all results and constructions of this paper which are otherwise comprised of formal ∞ -cosmological structures only are model independent.

8.1. DEFINITION. *Let \mathbf{V} be an ∞ -cosmos and $p: E \rightarrow C, q: F \rightarrow C$ be a pair of discrete fibrations in \mathbf{V} . Say that a functor $f: p \rightarrow q$ in $\mathbf{V}_{/C}$ is representable in \mathbf{V} if $f: E \rightarrow F$ as a functor in \mathbf{V} has a right adjoint.*

8.2. PROPOSITION. [Invariance of representability] *Let \mathbf{V} be an ∞ -cosmos of $(\infty, 1)$ -categories and $C \in \mathbf{V}$ and object. Then the cosmological biequivalence $(\cdot)_0: \mathbf{V} \rightarrow \mathbf{QCat}$ induces a cosmological biequivalence*

$$\mathbf{DiscCart}(\mathbf{V})_{/C} \rightarrow \mathbf{DiscCart}(\mathbf{QCat})_{/C_0} \tag{42}$$

between the corresponding ∞ -cosmoses of discrete fibrations [39, Proposition 6.3.15]. In particular,

1. *we obtain a 1-1 correspondence between homotopy classes of discrete fibrations in \mathbf{V} over C , and presheaves over the quasi-category C_0 .*
2. *given two discrete fibrations $p: E \rightarrow C$ and $q: F \rightarrow C$ in \mathbf{V} over C , we obtain a 1-1 correspondence between homotopy classes of functors between p and q in $\mathbf{V}_{/C}$, and natural transformations between the presheaves $\mathbf{St}(p_0)$ and $\mathbf{St}(q_0)$ over C_0 .*

Furthermore, a functor $f: p \rightarrow q$ of right fibrations in $\mathbf{V}_{/C}$ is representable in \mathbf{V} if and only if the induced natural transformation $\mathbf{St}(f_0): \mathbf{St}(p_0) \rightarrow \mathbf{St}(q_0)$ of presheaves over C_0 is representable.

⁷The condition of being “suitable” will only be necessary for the case of basic diagrammatic comprehension. It means that the ∞ -cosmos \mathbf{V} has a fibered core construction and that the cosmological functor preserves fibered core constructions, see Definition 8.6.

PROOF. The existence of the induced cosmological biequivalence between the two respective ∞ -cosmoi of discrete fibrations is proven in [39, Corollary 10.3.7]. The ∞ -cosmos of discrete fibrations in \mathbf{QCat} over C_0 is exactly the underlying ∞ -cosmos of the model category for right fibrations over C_0 via [39, Proposition F.4.9]. The cosmological biequivalence (42) hence induces a composite equivalence of underlying ∞ -categories

$$\mathrm{DiscCart}(\mathbf{V})_{/C} \simeq \mathrm{RFib}(C_0) \simeq \widehat{C}_0$$

via the straightening/unstraightening construction for quasi-categories from Section 2. Parts 1 and 2 are hence immediate. The fact that a functor f of discrete fibrations in $\mathbf{V}_{/C}$ is representable if and only if f_0 in $\mathbf{QCat}_{/C_0}$ is so follows from [39, Proposition 10.3.6]. Equivalence of this notion of representability of f_0 to representability of its associated natural transformation of presheaves in turn follows from Proposition 2.9. ■

8.3. COROLLARY. *A general comprehension scheme over an ∞ -category C in form of a representable functor of discrete fibrations over C is a model independent notion.* ■

Let us move on to the more specific diagrammatic comprehension schemes of Section 3. A general ∞ -cosmos \mathbf{V} does not necessarily come equipped with a $(\cdot)^{op}$ -construction. As this however is the only obstruction to define (K, G) -comprehension of C -indexed objects in K internal to \mathbf{V} , we instead define (K^{op}, G) -comprehension formally as follows.

8.4. DEFINITION. *Let \mathbf{V} be an ∞ -cosmos and $E: C \rightarrow K$ be a functor in \mathbf{V} . Every morphism $\{G\}: * \rightarrow K^{\Delta^1}$, depicted as $G: l \rightarrow k$ in K , induces a functor $G_*: K_{/l} \rightarrow K_{/k}$ of representable discrete fibrations over K in \mathbf{V} . We say that E has (K^{op}, G^{op}) -comprehension if the functor $E^*(G_*): E^*K_{/l} \rightarrow E^*K_{/k}$ of discrete cartesian fibrations over C has a right adjoint in \mathbf{V} .*

An ∞ -cosmos of $(\infty, 1)$ -categories more specifically however does automatically come equipped with a suitable $(\cdot)^{op}$ -construction [39, Definition 12.1.4] for its objects. In this case we have the following characterization.

8.5. PROPOSITION. [Invariance of (K, G) -comprehension] *Let \mathbf{V} be an ∞ -cosmos of $(\infty, 1)$ -categories, $E: C \rightarrow K$ be a functor in \mathbf{V} , and $G: l \rightarrow k$ be a morphism in K . Then E has (K^{op}, G^{op}) -comprehension if and only if the associated functor $E_0^{op}: C_0^{op} \rightarrow K_0^{op}$ of quasi-categories has (K_0^{op}, G^{op}) -comprehension in the sense of Definition 3.4.*

In particular, a functor $E: C \rightarrow K^{op}$ in \mathbf{V} has $((K^{op})^{op}, (G^{op})^{op})$ -comprehension if and only if the functor $E_0^{op}: C_0^{op} \rightarrow K_0$ of quasi-categories has (K, G) -comprehension.

PROOF. Let $E: C \rightarrow K$ and $G: l \rightarrow k$ be as above. Then, again by [39, Proposition 10.3.6], the functor $E^*(G_*): E^*K_{/l} \rightarrow E^*K_{/k}$ of discrete cartesian fibrations over C has a right adjoint in \mathbf{V} if and only if the functor

$$E_0^*(G_*): E_0^*((K_0)_{/l}) \rightarrow E_0^*((K_0)_{/k}) \tag{43}$$

of right fibrations over C_0 has a right adjoint in \mathbf{QCat} . The latter however is the unstraightening of the natural transformation

$$(E_0^{op})^*(G_*) : K_0(E_0^{op}, l) \rightarrow K_0(E_0^{op}, k) \quad (44)$$

in $\widehat{C_0}$ essentially by Example 2.4.3. Thus, the functor (43) has a right adjoint if and only if the natural transformation (44) is representable by Proposition 2.9. This natural transformation is canonically equivalent to

$$(E_0^{op})^*((G^{op})^*) : K_0^{op}(l, E_0^{op}) \rightarrow K_0^{op}(k, E_0^{op}),$$

which is representable if and only if E_0^{op} has (K_0^{op}, G^{op}) -comprehension by definition. The second part is a trivial consequence of idempotency of taking opposites. \blacksquare

Lastly, we consider the special case of G -comprehension for a map $G: I \rightarrow J$ between simplicial sets. To define it, we require a fibered core construction over every base of our ∞ -cosmos \mathbf{V} so to be able to construct the discrete fibrations of pinched rectangular diagrams. Thus, we say that an ∞ -cosmos \mathbf{V} has a fibered core construction whenever for every $C \in \mathbf{V}$, every object p in the ∞ -cosmos $\mathbf{Cart}(\mathbf{V})$ has an ∞ -groupoid core $\iota: p^\times \rightarrow p$ in the sense of [39, Definition 12.1.14]. That means each p^\times is a discrete object in $\mathbf{Cart}(\mathbf{V})$ such that for any other discrete object $q \in \mathbf{Cart}(\mathbf{V})$ the pushforward

$$\iota_* : \mathbf{Cart}(\mathbf{V})(q, p^\times) \rightarrow \mathbf{Cart}(\mathbf{V})(q, p) \quad (45)$$

is an equivalence of quasi-categories. Here, note that the discrete objects in $\mathbf{Cart}(\mathbf{V})$ are exactly the discrete cartesian fibrations by definition [39, Definition 5.5.3].

8.6. DEFINITION. *Let \mathbf{V} be an ∞ -cosmos with a fibered core construction. Let $p: E \rightarrow C$ be a cartesian fibration in \mathbf{V} and $G: I \rightarrow J$ be a map of simplicial sets. Say that p has G -comprehension if the functor of ∞ -categories underlying the (essentially unique) restriction $(p^G)^\times: (p^J)^\times \rightarrow (p^I)^\times$ obtained from (45) has a right adjoint in \mathbf{V} .*

Unsurprisingly, for instance, every ∞ -cosmos \mathbf{V} of $(\infty, 1)$ -categories has a fibered core construction, as can be seen simply by reflecting the core construction from Definition 2.3 along the induced cosmological biequivalence $\mathbf{Cart}(\mathbf{V}) \xrightarrow{\sim} \mathbf{Cart}(\mathbf{QCat})$.

8.7. PROPOSITION. [Invariance of G -comprehension] *Let \mathbf{V} be an ∞ -cosmos of $(\infty, 1)$ -categories, $p: E \rightarrow C$ be a cartesian fibration in \mathbf{V} and $G: I \rightarrow J$ be a map of simplicial sets. Then p has G -comprehension if and only if the associated C_0 -indexed quasi-category $\text{St}(p_0)$ has G -comprehension in the sense of Notation 3.17.*

PROOF. The induced cosmological functor $\mathbf{Cart}(\mathbf{V}) \xrightarrow{\sim} \mathbf{Cart}(\mathbf{QCat})$ preserves both simplicial cotensors and the fibered core construction. The cosmological biequivalence $(\cdot)_0: \mathbf{V} \rightarrow \mathbf{QCat}$ preserves and reflects right adjoints, and so p has G -comprehension in \mathbf{V} if and only if the functor $(p_0^G)^\times: \llbracket J, E_0 \rrbracket \rightarrow \llbracket I, E_0 \rrbracket$ of quasi-categories has a right adjoint. This in turn holds if and only if the presheaf $\text{St}(p_0)$ has G -comprehension by Proposition 4.4. \blacksquare

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