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CATEGORIES WHICH ARE VARIETIES OF CLASSICAL OR ORDERED ALGEBRAS

Dedicated to the memory of Bill Lawvere

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ABSTRACT. Following ideas of Lawvere and Linton we prove that classical varieties are precisely the exact categories with a varietal generator. This means a strong generator which is abstractly finite and regularly projective.

An analogous characterization of varieties of ordered algebras is also presented. We work with order-enriched categories, and introduce the concept of subcongruence (corresponding to congruence in ordinary categories): it is a relation which is order-reflexive and transitive. Varieties of ordered algebras are precisely the categories with effective subcongruences and a subvarietal generator. This means a strong generator which is abstractly finite and subregularly projective.

1. Introduction

One of the fundamental achievements of the thesis of Bill Lawvere was a characterization of categories equivalent to varieties of (finitary, one-sorted) algebras. He introduced the concept of an abstractly finite object G (weaker than the concept of a finitely generated object, later used by Gabriel and Ulmer): every morphism from G to its copower factorizes through a finite subcopower. Lawvere formulated a theorem stating that a category is equivalent to a variety iff it has

(1) Finite limits.

(2) Effective congruences.

(3) A generator with copowers which is abstractly finite and regularly projective (its hom-functor preserves regular epimorphisms).

Unfortunately, a small correction is needed: in (1) the existence of coequalizers should be added (since Lawvere uses them twice in his proof), and the generator in (3) needs to be regular (also used in that proof). A category satisfying the three conditions above which, however, is not equivalent to a variety is presented below (Example 2.14).

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Thus Lawvere's, very elegant, proof is a verification of the following theorem.

1.1. Theorem. *A category is equivalent to a variety iff it has*

(1) Finite limits and coequalizers.

(2) Effective congruences.

(3) A regular generator G with copowers which is an abstractly finite regular projective.

Actually, Lawvere worked in (2) with effectivity of congruences with respect to G, but we prove in Proposition 2.12 that this makes no difference in case G is a regular generator. An improved version of the above theorem was presented in [ARV]: kernel pairs and reflexive coequalizers are sufficient in (1), and strong (rather than regular) generator in (3). This leads us to the following concept: A *varietal generator* is a strong generator with copowers which is an abstractly finite regular projective.

By applying Linton's characterization of monadicity over **Set**, we present a shorter proof and make one further simplification step: in (1) coequalizers of kernel pairs are sufficient. This corresponds well to Barr's concept of an exact category (Def. 2.13 below): he only assumed that kernel pairs and their coequalizers exist. We obtain the following result (Theorem 5.8 below).

1.2. Theorem. *A category is equivalent to a variety iff it is exact and has a varietal generator.*

Our second topic is a characterization of varieties of ordered algebras. Here one works with algebras acting on posets so that the operations are monotone. A variety is a full subcategory presented by inequations between terms. Varieties are enriched categories over the cartesian closed category **Pos** of posets. In [ARO] a characterization of varieties of ordered algebra has been presented, and our purpose is to sharpen and correct that result slightly. For that we introduce the concept of a subcongruence. In ordinary categories a congruence is a reflexive, symmetric and transitive relation. In order-enriched categories, we lose the symmetry, but gain a stronger property than reflexivity – we call it order-reflexivity (Definiton 4.8). A *subcongruence* is an order-reflexive and transitive relation. (Example: in **Pos** itself a subcongruene on A is a transitive relation containing the order of A.) Given a morphism $f\colon X \longrightarrow Y$, its subkernel pair r_0, $r_1\colon R \longrightarrow X$ (universal with respect to $f \cdot r_0 \leq f \cdot r_1$) is a subcongruence. We prove that every variety of ordered algebras has *effective subcongruences*: each subcongruence is the subkernel pair of a morphism.

Whereas reflexive coequalizers play an important role for classical varieties (because they are preserved by the forgetful functors to **Set**), for ordered varieties reflexive coinserters play the analogous role. A *subregular epimorphism* is a morphism which is a coinserter of a reflexive pair. In varieties these are precisley the surjective homomorphisms. An object whose hom-functor to **Pos** preserves subregular epimorphisms is a *subregular projective.* The concept corresponding to the varietal generators in ordinary categories is a *subvarietal generator*: a strong generator with copowers which is an abstractly finite subregular projective. Example: in every variety the free algebra on one generator is a varietal generator. (Sorry about the double meaning of the word generator...)

The following result (see Theorem 5.8 below) slightly improves and corrects the characterization presented in [ARO]:

1.3. Theorem. *An order-enriched category is equivalent to a variety of ordered algebras iff it has (1) a subvarietal generator, (2) effective subcongruences, and (3) subkernel pairs and reflexive coinserters.*

Related Work Vitale characterized monadic categories over **Set** as precisely the finitely complete, exact categories with a regularly projective regular generator ([V, Prop. 3.2]). He does not assume finite limits, but uses them all over his proof. Our proof, based on Linton's theorem 2.18, shows that finite limits (beyond kernel pairs) need not be assumed, and a regularly projective strong generator is sufficient to characterize varieties.

Similarly, Rosický and the author characterized classical varieties using the existence of reflexive coequalizers (rather than just coequalizers of kernel pairs) – otherwise Corollary 3.6 in [ARV] is the same as Corollary 5.10 below. Our Theorem 2.19 is just a small improvement, however, precisely that needed for getting the characterization using Barr's exactness. Moreover, the proof we present is simpler than that in [ARV].

A closely related result is a recent characterization of varieties of ordered algebras due to Rosický and the author: in [ARO] subregular epimorphisms and subregular projectives have been introduced, and a characterization theorem was proved that differs from Theorem 2.19 essentially by not working with suncongruences, and by assuming the generator to be subregular. Since small gaps appear in op. cit., we present a corrected version.

Our concept of subcongruence is new. It is related to congruences due to Kurz and Velebil [KV] for poset-enriched categories and Bourke and Garner [BG] in general enriched categories, but those definitions are quite more technical. See Section 6 for details.

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Note Added in Proof The author has found out too late that the concept of subcongruence was already introduced (under the name congruence) by Vasileios Aravantinos-Sotiropoulos in his PhD thesis 'Aspects of Regular and Exact Completions', Univeritsité Catolique de Louvain, 2001.

2. Varieties

Classical (finitary, one-sorted) varieties were characterized by Lawvere. We explain why a small correction is needed, and present some simplifications.

Throughout the paper we work with algebras of a finitary signature $\Sigma = (\Sigma_n)_n \in \Bbbk N$, where Σ_n is the set of all n-ary operation symbols. For an object G in a category $\mathcal{K}$ with copowers (denoted by $M \cdot G$ for all sets M) we obtain the *canonical morphisms*

$$[f]\colon \mathcal{K}(G,X)\cdot G = \coprod_{f\colon G \longrightarrow X} G \longrightarrow X \qquad (X \in \operatorname{obj}\mathcal{K})\,.$$

Recall that G is a *generator* if all canonical morphisms are epic, a *strong generator* if they are extremally epic (do not factorize through a proper subobject of X), and a *regular generator* if they are regular epimorphisms. Recall further that G is a *regular projective* if for each regular epimorphism $e\colon X \longrightarrow Y$ all morphisms from G to Y factorize through e. Shortly: $\mathcal{K}(G,-)$ preserves regular epimorphisms.

Lawvere introduced the following concept; he attributed it to Freyd.

2.1. Definition. [L] An object G is *abstractly finite* if every morphism from G to a copower $M \cdot G$ factorizes through a finite subcopower.

That is, for every set M and every morphism $f\colon G \longrightarrow M\cdot G$ there exists a finite subset $u\colon M_0 \hookrightarrow M$ such that f factorizes through $u \cdot G\colon M_0 \cdot G \longrightarrow M \cdot G$ (the morphism induced by u).

2.2. Example. (1) In **Set** this means that G is finite, in the category of vector spaces that G is finite-dimensional.

(2) Every finite poset is abstractly finite in **Pos**. But also the linearly ordered set $\mathbb{R}$ is. In fact, every poset with finitely many connected components is abstractly finite.

(3) The free algebra on one generator is abstractly finite in the category of Σ-algebras. This fololows from the fact that the copower indexed by a set M is the free algebra on M.

2.3. Definition. A *varietal generator* is a strong generator with copowers which is an abstractly finite regular projective.

2.4. Lemma. *Let $\mathcal{K}$ have kernel pairs and their coequalizers. Every regularly projective strong generator G with copowers is a regular generator.*

Proof. For every object X let us prove that the morphism

$$[h]\colon \coprod_{h\colon G \longrightarrow X} G \longrightarrow X$$

is the coequalizer of its kernel pair r_0, r_1. Let e be the coequalizer of that pair, and m the unique factorization:

$$\begin{tikzcd}
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_0"'] & \coprod\limits_{h\colon G \longrightarrow X} G \arrow[r, "{[h]}"] \arrow[d, "e"] & X \\
G \arrow[u, "v"] \arrow[ur, shift left, "u'_0"] \arrow[ur, shift right, "u'_1"'] \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_0"'] & Y \arrow[ur, "m"'] &
\end{tikzcd}$$

Since $[h]$ is a strong epimorphism, so is m. Thus it sufficient to prove that m is monic: then it is invertible. Since G is a generator, we can restrict ourselves to parallel pairs with domain G.

Given $u_0,\ u_1\colon G \longrightarrow Y$ with $m \cdot u_0 = m \cdot u_1$, we verify $u_0 = u_1$. Since G is a regular projective, we have u_i' with $u_i = e \cdot u_i'$ $(i = 0, 1)$. From $m \cdot u_0 = m \cdot u_1$ we get $[h] \cdot u_0' = [h] \cdot u_1'$. Since $r_0,\ r_1$ is the kernel pair of $[h]$, there is $v\colon G \longrightarrow R$ with $r_i \cdot v = u_i'$ $(i = 0, 1)$. Thus

$$u_0 = e \cdot r_0 \cdot v = e \cdot r_1 \cdot v = u_1 \,.$$

■

Recall that an object is *finitely generated* if irs hom-hom-functor preserves directed colimits of monomorphisms.

2.5. Lemma. *Let $\mathcal{K}$ be a cocomplete category with kernel pairs.*

(1) *Every finitely generated object is abstractly finite.*

(2) *Every abstractly finite regular generator is finitely generated.*

Proof. (1) Just use that for M infinite the copower $M \cdot G$ is the directed colimit of all $M_0 \cdot G$ for $\phi \neq M_0 \subseteq M$ finite. The connecting morphisms are split monomorphisms.

(2) Let G be an abstractly finite regular generator. We first prove an auxilliary fact:

(a) Every morphism $f\colon G \longrightarrow \coprod_{i \in I} A_i$ factorizes through a finite subcopower of $\coprod_{i \in I} A_i$. Indeed, each of the canonical morphisms

$$c_i = [h]\colon \coprod_{h\colon G \longrightarrow A_i} G \longrightarrow A_i$$

is a regular epimorphism . Thus the morphism

$$c = \coprod_{i \in I} c_i\colon \coprod_{i \in I} \coprod_{h\colon G \longrightarrow A_i} G \longrightarrow \coprod_{i \in I} A_i$$

is also a regular epimorphism. As G is a regular projective, there exists a factorization of f as $f = c \cdot g$ for some $g\colon G \longrightarrow \coprod_{i \in I} \coprod_{h\colon G \longrightarrow A_i} G$. Since g factorizes through a finite subcoproduct, we have a finite subset $J \subseteq I$ such that g factorizes through the subcopower $\coprod_{i \in J} \coprod_{h\colon G \longrightarrow A_i} G$. Consequently $f = c \cdot g$ factorizes through $\coprod_{i \in J} A_i$, as claimed.

(b) We prove that G is finitely generated. Given a colimit $a_i\colon A_i \longrightarrow A$ $(i \in I)$ of a directed diagram D of monomorphisms, our task is to prove that $\mathcal{K}(G, -)$ preserves it. In other words: every morphism $f\colon G \longrightarrow A$ factorizes through some a_i. The standard construction of colimits via coproducts and coequalizers proves that $[a_i]\colon \coprod_{i \in I} A_i \longrightarrow A$ is a regular epimorphism. Thus f factorizes through it. By (a), f factorizes through $[a_j]_{j \in J}\colon \coprod_{j \in J} A_j \longrightarrow A$ for some finite subset $J \subseteq I$. The diagram D is directed, so we can find an upper bound $i \in I$ of J. Then $[a_j]_{j \in J}$ factorizes through a_i, thus so does f. ■

Recall that regular epimorphisms are *stable under pullback* if in every pullback

$$\begin{tikzcd} & P \arrow[dl, "f'"'] \arrow[dr, "e'"] & \\ A \arrow[dr, "e"'] & & B \arrow[dl, "f"] \\ & Q & \end{tikzcd}$$

with e a regular epimorphism, so is e'.

2.6. Lemma. *Let G be a regularly projective strong generator with copowers. If $\mathcal{K}$ has kernel pairs and their coequalizers, then*

(1) *It has (Regular Epi, Mono)-factorizations;*

(2) *Regular epimorphisms are stable under pullback.*

Proof. (1) Every morphism $f\colon A \longrightarrow B$ with kernel pair r_0, r_1 factorizes as $f = m \cdot c$, where c is the coequalizer of r_0, r_1. The proof that m is monic is analogous to the proof of Lemma 2.4.

(2) In the above pullback, where e is a regular epimorphism, we observe that every morphism $g\colon G \longrightarrow B$ factorizes through e':

$$\begin{tikzcd} & G \arrow[ddl, "h"'] \arrow[d, dashed] \arrow[ddr, "g"] & \\ & P \arrow[dl, "f'"'] \arrow[dr, "e'"] & \\ A \arrow[dr, "e"'] & & B \arrow[dl, "f"] \\ & Q & \end{tikzcd}$$

Indeed, since G is a regular projective, for the composite $f \cdot g\colon G \longrightarrow C$ there is a factorization (say, h) through e. The universal property of the pullback yields the desired factorization of g.

Let $e' = m \cdot c$ be the factorization of e' as in (1). Then every morphism $g\colon G \longrightarrow B$ factorizes also through m, and since G is a strong generator, this proves that m is invertible. Thus e' is a regular epimorphism. ∎

2.7. Remark. We recall that in a (not necessarily finitely complete) category a *relation* on an object A is represented by a collectively monic (ordered) pair of morphisms

$$l, r \colon R \longrightarrow A\,.$$

We say that a parallel pair s, $s' \colon S \longrightarrow A$ *factorizes* through the relation if there is $f \colon S \longrightarrow R$ with $s = l \cdot f$ and $s' = r \cdot f$.

The concept of congruence is usually defined in categories with pullbacks. However, in general categories there is also a natural definition of congruence: a parallel pair that every hom-functor $\mathcal{K}(S, -)$ takes to a set-theoretical eqivalence relation on the set $\mathcal{K}(S, R)$. Here are the details:

2.8. Definition. A *congruence* is a relation $l, r \colon R \longrightarrow A$ which is

(i) *Reflexive:* l, r are split epimorphisms with a joint splitting. Equivalently: for every morphism $s \colon S \longrightarrow A$ the pair s, s factorizes through l, r.

(ii) *Symmetric:* the pair r, l factorizes through l, r. Equivalently: whenever a pair s, $s' \colon S \longrightarrow A$ factorizes through l, r, then so does s', s.

(iii) *Transitive:* given morphisms $s, s', s'' \colon S \longrightarrow A$ such that both pairs s, s' and s', s'' factorize through l, r, then s, s'' also factorizes through l, r.

2.9. Remark. In finitely complete categories a relation represents a subobject of X^2. Reflexivity means that the diagonal is contained in that subobject. And transitivity simplifies as follows: given the pullback P of r and l (on the left), the pair $l \cdot \bar{l}$, $r \cdot \bar{r}$ factorizes through l, r via a morphism p:

$$\begin{tikzcd}
& & P \arrow[dl, "\bar{l}"'] \arrow[dr, "\bar{r}"] \arrow[d, phantom, "\vee"] & & \\
& R \arrow[dl, "l"'] \arrow[dr, "r"'] & {} & R \arrow[dl, "l"] \arrow[dr, "r"] & \\
X & & X & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
& & P \arrow[dl, "\bar{l}"'] \arrow[dr, "\bar{r}"] \arrow[dd, "p"'] & & \\
& R \arrow[dl, "l"'] & & R \arrow[dr, "r"] & \\
X & & R \arrow[ll, "l"] \arrow[rr, "r"'] & & X
\end{tikzcd}$$

2.10. Example. Let $f \colon A \longrightarrow B$ be a morphism. Its *kernel pair* (which is a universal pair r_0, $r_1 \colon R \longrightarrow A$ with $f \cdot r_0 = f \cdot r_1$) is a congruence. A category has *effective congruences* if every congruence is a kernel pair of some morphism.

2.11. Remark. (1) A parallel pair $l, r \colon R \longrightarrow A$ is a congruence iff for every object S the relation

$$\{(l \cdot f, r \cdot f); f \colon S \longrightarrow R\}$$

on the set $\mathcal{K}(S, A)$ is reflexive, symmetric, and transitive.

(2) Lawvere worked, for a given object G, with a relative concept of reflexivity, symmetry and transitivity: instead of taking an arbitrary object S as above, he restricted it to $S = G$. He then called the relation a *congruence with respect to* G if the set-theoretical relation on $\mathcal{K}(G, A)$ is an equivalence relation. However, this makes no difference in case G is a regular generator:

2.12. Proposition. *In any category, if G is a regular generator with copowers, then every congruence with respect to G is a congruence.*

Proof. Let $r_0,\ r_1\colon R \longrightarrow A$ be a congruence with respect to G. We prove that it is a congruence.

(1) Reflexivity. Let $u_0,\ u_1\colon U \longrightarrow A$ be a pair with the coequalizer $[h]\colon \coprod_{h\colon G \longrightarrow A} G \longrightarrow A$. Since $r_0,\ r_1$ is reflexive with respect to G, each pair h, h factorizes through r_0, r_1: there exists $h'\colon G \longrightarrow R$ with $h = r_0 \cdot h' = r_1 \cdot h'$. The morphism $[h']\colon \coprod_{h\colon G \longrightarrow A} G \longrightarrow R$ merges u_0 and u_1:

$$\begin{tikzcd} U \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_0"'] & \coprod_{h\colon G \to A} G \arrow[r, "{[h]}"] \arrow[d, "{[h']}"'] & A \\ & R \arrow[ur, shift left, "r_1"] \arrow[ur, shift right, "r_0"'] & \end{tikzcd}$$

Indeed, we use that the pair $r_0,\ r_1$ is collectively monic. For r_0 we have

$$r_0 \cdot [h'] \cdot u_i = [r_0 \cdot h'] \cdot u_i = [h] \cdot u_i$$

which is independent of $i = 0, 1$. The same holds for r_1. Consequently, $[h'] \cdot u_0 = [h'] \cdot u_1$. Therefore $[h']$ factorizes through $[h]$: we have $d\colon A \longrightarrow R$ with $[h'] = d \cdot [h]$. This is a joint splitting of r_0 and r_1. Indeed, $r_0 \cdot d = \mathrm{id}$ because $[h]$ is epic and

$$r_0 \cdot d \cdot [h] = r_0 \cdot [d \cdot h] = r_0 \cdot [h'] = [r_0 \cdot h'] = [h]\,.$$

Analogously for r_1.

(2) Symmetry. Let $u_0,\ u_1\colon U \longrightarrow \coprod_{h\colon G \longrightarrow R} G$ be a pair with coequalizer

$$[h]\colon \coprod_{h\colon G \longrightarrow R} G \longrightarrow R.$$

Then symmetry with respect to G implies that given $h\colon G \longrightarrow R$ (which is a factorization of the pair $r_0 \cdot h,\ r_1 \cdot h$ through r_0, r_1), there exists $h'\colon G \longrightarrow R$ factorizing $r_1 \cdot h,\ r_0 \cdot h$ through $r_0,\ r_1$. Thus we have the following commutative squares

$$\begin{tikzcd} G \arrow[r, "h'"] \arrow[d, "h"'] & R \arrow[d, "r_0"] \\ R \arrow[r, "r_1"'] & A \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "h'"] \arrow[d, "h"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_0"'] & A \end{tikzcd}$$

The morphism $[h']\colon \coprod_{h\colon G \longrightarrow R} G \longrightarrow R$ merges u_0 and u_1. This is analogous to (1): for r_0 we have

$$r_0 \cdot [h'] \cdot u_i = [r_0 \cdot h'] \cdot u_i = [r_1 \cdot h] \cdot u_i = r_1 \cdot [h] \cdot u_i$$

which is independent of $i = 0, 1$. The same holds for r_1.

The morphism $d \colon R \longrightarrow R$ defined by $[h'] = d \cdot [h]$ is the desired factorization of r_1, r_0 through r_0, r_1. Indeed, $r_0 = r_1 \cdot d$ because $[h]$ is epic and

$$r_0 \cdot [h] = [r_0 \cdot h] = [r_1 \cdot h'] = r_1 \cdot [h'] = r_1 \cdot d \cdot [h] .$$

Analogously for $r_1 = r_0 \cdot d$.

(3) Transitivity. We are given morphisms $s, s', s'' \colon S \longrightarrow A$ for which factorizations t and t' through r_0, r_1 below exist:

$$\begin{tikzcd}
 & S \arrow[dl, "s"'] \arrow[d, "t"] \arrow[dr, "s'"] & \\
A & R \arrow[l, "r_0"'] \arrow[r, "r_1"] & A \\
 & S \arrow[ul, "s'"] \arrow[u, "t'"'] \arrow[ur, "s''"'] &
\end{tikzcd}$$

Our task is to find $t'' \colon S \longrightarrow R$ with

$$s = r_0 \cdot t'' \quad \text{and} \quad s'' = r_1 \cdot t'' .$$

Since r_0, r_1 is a transitive relation with respect to G, the set-theoretical relation $\hat{R}$ on $\mathcal{K}(G, R)$ consisting of all pairs $(r_0 \cdot h, r_1 \cdot h)$ for $h \colon G \longrightarrow R$ is transitive. Consider an arbitrary morphism $g \colon G \longrightarrow S$. Due to t, the pair $(s \cdot g, s' \cdot g)$ lies in $\hat{R}$; due to t' the pair $(s' \cdot g, s'' \cdot g)$ also lies there. Thus, $(s \cdot g, s'' \cdot g) \in \hat{R}$. Hence for each $g \colon G \longrightarrow S$ there exists $\bar{g} \colon G \longrightarrow R$ with

$$s \cdot g = r_0 \cdot \bar{g} \quad \text{and} \quad s'' \cdot g = r_1 \cdot \bar{g} .$$

Let u_0, $u_1 \colon U \longrightarrow \coprod_{g \colon G \longrightarrow S} G$ be a pair with coequalizer $[g] \colon \coprod_{g \colon G \longrightarrow S} G \longrightarrow S$. The morphism $[\bar{g}] \colon \coprod_{g \colon G \longrightarrow S} G \longrightarrow R$ merges u_0, u_1. Indeed, for r_0 we have

$$r_0 \cdot [\bar{g}] \cdot u_i = [r_0 \cdot \bar{g}] \cdot u_i = [s \cdot g] \cdot u_i = s \cdot [g] \cdot u_i$$

which is independent of $i = 0, 1$. The same holds for r_1. We thus get a morphism

$$t'' \colon S \longrightarrow R \quad \text{with} \quad [\bar{g}] = t'' \cdot [g] .$$

It has the desired properties: $s = r_0 \cdot t''$ follows from

$$s \cdot [g] = [s \cdot g] = [r_0 \cdot \bar{g}] = r_0 \cdot [\bar{g}] = r_0 \cdot t'' \cdot [g] .$$

Analogously for $s'' = r_1 \cdot t''$. ■

We now recall Barr-exactness. In his paper [B] Barr does not require finite limits: only kernel pairs are included in his definition. By the way, congruences are called equivalences in op. cit.

2.13. Definition. [B] A category is *exact* if

(1) Kernel pairs and their coequalizers exist.
(2) Congruences are effective.
(3) Regular epimorphisms are stable under pullback.

We have mentioned in the Introduction the claim in Lawvere's thesis ([L, Thm. 3.2.1]) that varieties are characterized by having finite limits, effective congruences and a generator with copowers which is an abstractly finite regular projective. Here is a counter-example.

2.14. Example. The following category $\mathbf{Set}^*$ is not equivalent to a variety: we add to $\mathbf{Set}$ a formal terminal object $*$ (with $\mathbf{Set}(*, X) = \emptyset$ for all sets X). Then for the terminal set 1 we have the monomorphism $1 \longrightarrow *$ demonstrating that no object of $\mathbf{Set}^*$ is a strong generator. In contrast, free algebras in varieties are strong generators.

The category $\mathbf{Set}^*$ has finite limits: $\mathbf{Set}$ is closed under nonempty limits in $\mathbf{Set}^*$. A product $X \times *$ where X is a set is X itself, and there are no new parallel pairs of distinct morphism in $\mathbf{Set}^*$. Effectivity of congruences in $\mathbf{Set}^*$ also follows from this fact. Finally, 1 is an abstractly finite, regularly projective generator of $\mathbf{Set}^*$.

2.15. Remark. As observed in [A] another source of counter-examples are non-complete lattices with a top element.

As mentioned in the Introduction, Lawvere proved in [L] Theorem 1.1. Several authors presented various simplifications. For example Pedicchio and Wood [PW] showed that effective congruences can be deleted in case the hom-functor of the generator in (3) is assumed to preserve reflexive coequalizers. This has led to the following

2.16. Definition. [ARV] An object is *effective* if its hom-functor preserves coequalizers of congruences.

Explicitly, this means that every coequalizer $c\colon A \longrightarrow C$ of a congruence $r_0, r_1\colon R \longrightarrow A$ has the following properties:

(1) G is projective with respect to c.

(2) Given morphisms $u, v\colon G \longrightarrow A$ with $c \cdot u = c \cdot v$, then they are connected in $\mathcal{K}(G, A)$ by a ziz-zag of composites with r_i for $i = 0, 1$.

2.17. Proposition. *Let $\mathcal{K}$ be a category with kernel pairs and their coequalizers. For every regularly projective strong generator G we have the equivalence*

$$G \text{ effective} \Leftrightarrow \mathcal{K} \text{ has effective congruences.}$$

Proof. (1) Let $\mathcal{K}$ have effective congruences. Given a regular epimorphism $c\colon A \longrightarrow C$ and its kernel pair $r_0, r_1\colon R \longrightarrow A$, our task is to prove that the map

$$c \cdot (-)\colon \mathcal{K}(G, A) \longrightarrow \mathcal{K}(G, C)$$

is a coequalizer of $r_i \cdot (-)$ for $i = 0, 1$. Since G is a regular generator (Proposition 2.4), the map $c \cdot (-)$ is surjective. Thus, we only need to verify that it has the kernel pair $r_i \cdot (-)$. Indeed, let $c \cdot (-)$ merge a pair in $\mathcal{K}(G, A)$, say, $c \cdot f_0 = c \cdot f_1$. Then there is a unique $f'\colon G \longrightarrow R$ with $f_i = r_i \cdot f'$ $(i = 0, 1)$.

(2) Suppose that $\mathcal{K}(G, -)$ is effective. Let $r_0, r_1\colon R \longrightarrow A$ be a congruence. Since $\mathcal{K}(G, -)$ is faithful and preserves pullbacks, the pair $\mathcal{K}(G, r_0), \mathcal{K}(G, r_1)\colon \mathcal{K}(G, R) \longrightarrow \mathcal{K}(G, A)$ is a congruence in **Set**. We know that the coequalizer $c\colon A \longrightarrow C$ of r_0, r_1 yields a coequalizer $\mathcal{K}(G, c)$ of $\mathcal{K}(G, r_i)$. It follows that the above pair is a kernel pair of $\mathcal{K}(G, c)$.

To verify that r_0, r_1 is the kernel pair of c, be u_0, $u_1 \in \mathcal{K}(G, A)$ fulfil $c \cdot u_0 = c \cdot u_1$. Since the relation of all $(r_0 \cdot v, r_1 \cdot v)$ with $v\colon G \longrightarrow R$ is an equivalence on $\mathcal{K}(G, A)$, and $c \cdot (-)$ is its quotient map, there is a unique $v \in \mathcal{K}(G, R)$ with $u_i = r_i \cdot v$ $(i = 0, 1)$. ∎

We now prove the main result of the present section. We use the monadicity theorem of Linton:

2.18. Theorem. [Li, Prop. 3] *A functor $U\colon \mathcal{K} \longrightarrow$ **Set** is monadic iff*

(a) *U is right adjoint.*

(b) *$\mathcal{K}$ has kernel pairs and coequalizers of congruences.*

(c) *U preserves and reflects congruences.*

(d) *U preserves and reflects regular epimorphisms.*

2.19. Theorem. *A category is equivalent to a variety iff it is exact and has a varietal generator.*

Proof. Necessity. Every variety $\mathcal{V}$ is well known to be a cocomplete and exact category. Its free algebra G on one generator is a regular projective: regular epimorphisms are precisely the surjective homomorphisms and $\mathcal{V}(G, -)$ is naturally isomorphic to the forgetful functor. Further, it is an abstractly finite object since it is finitely generated (Lemma 2.5). Finally, G is a strong (indeed, regular) generator since its copowers are the free algebras of $\mathcal{V}$.

Sufficiency. Let $\mathcal{K}$ be an exact category and G a varietal generator. For the hom-functor

$$U = \mathcal{K}(G, -)\colon \mathcal{K} \longrightarrow Set$$

we prove that it is monadic, and the corresponding monad is finitary. Consequently, $\mathcal{K}$ is equivalent to a variety (as proved by Linton [Li]).

(U is monadic. Indeed, U has the left adjoint $M \mapsto M \cdot G$, and Condition (b) in Linton's theorem is a part of exactness.

We thus only need to verify (c) and (d) .

(c1) U preserves congruences. In fact, let $r_0,\ r_1\colon R \longrightarrow A$ be a congruence. Since U is faithful, $Ur_0,\ Ur_1$ is collectively monic. The relation $Ur_0,\ Ur_1$ in **Set** represents the set-theoretical relation $\hat{R}$ on $\mathcal{K}(G,R)$ defined by

$$\hat{R} = \big\{(r_0 \cdot g, r_1 \cdot g); g\colon G \longrightarrow R\big\}\,.$$

Since $r_0,\ r_1$ is reflexive, so is $\hat{R}$: given $d\colon A \longrightarrow R$ with $r_0 \cdot d = \mathrm{id} = r_1 \cdot d$, we have, for each $h\colon G \longrightarrow A$

$$(h,h) = (r_0 \cdot d \cdot h, r_1 \cdot d \cdot h) \in \hat{R}\,.$$

Analogously, $\hat{R}$ is symmetric. To verify transitivity, let $(r_0 \cdot g, r_1 \cdot g)$ and $(r_0 \cdot g', r_1 \cdot g')$ be members of $\hat{R}$ with $r_1 \cdot g = r_0 \cdot g'$. The pair $r_0 \cdot g,\ r_1 \cdot g$ also factorizes through r_0, r_1 via g, and the pair $r_0 \cdot g, r_1 \cdot g'$ factorizes via g'. Since r_0, r_1 is transitive, the pair $r_0 \cdot g, r_1 \cdot g'$ factorizes through r_0, r_1: we have g'' with

$$r_0 \cdot g = r_0 \cdot g'' \quad \text{and} \quad r_1 \cdot g' = r_1 \cdot g''\,.$$

This proves $(r_0 \cdot g, r_1 \cdot g') \in \hat{R}$, as desired.

(c2) U reflects congruences. Let $r_0, r_1\colon R \longrightarrow A$ be a pair such that Ur_0, Ur_1 is a congruence. Since G is a generator, the fact that $Ur_i = r_i \cdot (-)$ is a collectively monic pair for $i = 0,1$ implies that r_0, r_1 is collectively monic. To say that Ur_0, Ur_1 is a congruence means that r_0, r_1 is a congruence with respect to G (Remark 2.11).

Since G is a regular generator (Proposition 2.4) the proof follows from Lemma 2.12.

(d1) U preserves regular epimorphisms because G is a regular projective.

(d2) U reflects regular epimorphisms. That is, given a morphism $e\colon A \longrightarrow B$ such that every morphism $g\colon G \longrightarrow B$ factorizes through it, we verify that e is a coequalizer of its kernel pair $r_0, r_1\colon R \longrightarrow A$.

Let $c\colon A \longrightarrow C$ be a coequalizer of r_0, r_1, and let h make the triangle below commutative:

$$\begin{tikzcd}
 & G \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_0"'] \arrow[dl, "v"'] \arrow[d, shift right, "v_0"'] \arrow[d, shift left, "v_1"] & C \arrow[d, "h"] \\
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_0"'] & A \arrow[ur, "c"] \arrow[r, "e"'] & B
\end{tikzcd}$$

We prove that h is an isomorphism, thus, $e = \mathrm{coeq}(r_0, r_1)$. Every morphism $g\colon G \longrightarrow B$ factorizes through e, hence also through h. Hence, to verify that h is invertible, it is sufficient to prove that it is monic (using that G is a strong generator). Indeed, for every pair $u_0, u_1\colon G \longrightarrow C$ with

$$h \cdot u_0 = h \cdot u_1$$

we derive $u_0 = u_1$ Since c is a regular epimorphism, we have v_i with $u_i = c \cdot v_i$. We deduce that

$$e \cdot v_0 = h \cdot c \cdot v_0 = h \cdot c \cdot v_1 = e \cdot v_1\,.$$

Therefore there is $v\colon G \longrightarrow R$ with $v_i = r_i \cdot v$. Thus

$$u_i = c \cdot v_i = c \cdot r_i \cdot v$$

is independent of $i = 0, 1$.

(ii) The functor $T = UF$, where F is the left adjoint of U, is finitary because G is finitely generated (Lemma 2.5). Indeed, F preserves directed colimits of nonempty monomorphisms and these monomorphisms split. Consequently, $T = \mathcal{K}(G, -) \cdot F$ preserves these colimits, too. Given an infinite set X, express it as the directed colimit of all of its finite nonempty subsets. Since T preserves this colimit, for every element $x \in TX$ there exists a finite subset $m\colon M \hookrightarrow X$ such that x lies in the image of Tm. By [AMSW], Thm 3.4, this implies that T is finitary. ∎

Observe that we have not used the stability of regular epimorphisms under pullback in the above proof. (No surprise – see Lemma 2.6.) We thus get, using Proposition 2.17, the following statement slightly improving Corollary 36 of [ARV].

2.20. Corollary. *A category is equivalent to a variety iff it has*

(1) *Kernel pairs and their coequalizers.*
(2) *An effective varietal generator.*

For an analogous result about many-sorted algebras see [ARV].

3. Reflexive Coequalizers

Before turning to order-enriched varieties in Section 4, we prove an auxiliary proposition for enriched categories in general. In the present section we assume that a symmetric monoidal closed category

$$(\mathcal{V}, \otimes, I)$$

is given (which in Sections 4 and 5 will be the cartesian closed category of posets).

Throughout this section let $\mathcal{K}$ be an enriched category over $\mathcal{V}$.

3.1. Remark. When speaking about ordinary colimits (coproducts, coequalizers, etc.) we always mean the conical ones: weighted colimits with the weight constant with value I.

Reflexive coequalizers are (conical) coequalizers of pairs $r_0, r_1\colon R \longrightarrow X$ that are reflexive: there is $d\colon X \longrightarrow R$ with $r_i \cdot d = \mathrm{id}_X$.

Recall that an object G *has tensors* if the hom-functor $\mathcal{K}(G, -)\colon \mathcal{K} \longrightarrow \mathcal{V}$ has a left adjoint F. The notation is $V \otimes G$ for FV. That is, we have an isomorphism between $\mathcal{K}(V \otimes G, X)$ and $[V, \mathcal{K}(G, X)]$ natural in $X \in \mathcal{K}$.

3.2. Definition. [K] A full subcategory $\mathcal{A}$ of $\mathcal{K}$ is *dense* if the functor

$$E\colon \mathcal{K} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$$

assigning to $K \in \mathcal{K}$ the restriction of $\mathcal{K}(-, A)$ to $\mathcal{A}^{\mathrm{op}}$ is fully faithful.

3.3. Example. In the category of Σ-algebras the free algebras $T_\Sigma n$ on $n = \{0, ..., n-1\}$ form a dense subcategory for $n \in \mathbb{N}$. Indeed, given algebras $A = (X, (\sigma_A))$ and $B = (Y, (\sigma_B))$, then a function $f\colon X \longrightarrow Y$ is a homomorphism whenever for all n its composites with all homomorphisms $h\colon T_\Sigma n \longrightarrow A$ are homomorphisms. Indeed, given an n-ary symbol σ, we verify

$$f \cdot \sigma_A = \sigma_B \cdot f^n \colon A^n \longrightarrow B.$$

For every n-tuple $h_0\colon n \longrightarrow A$ let $h\colon T_\Sigma n \longrightarrow A$ be the corresponding homomorphism. By assumption we have

$$(f \cdot h) \cdot \sigma_{T_\Sigma n} = \sigma_B \cdot (f \cdot h)^n.$$

We apply this to $\eta_n \in (T_\Sigma n)^n$ for which

$$h \cdot \sigma_{T_\Sigma n}(\eta_n) = \sigma_A \cdot h^n(\eta_n) = \sigma_A(h_0).$$

This proves the desired equality: we have $f \cdot \sigma_A(h_0) = (f \cdot h) \cdot \sigma_{T_\Sigma n}(\eta_n)$, as well as $\sigma_B \cdot f^n(h_0) = \sigma_B \cdot f^n(h \cdot \eta_n) = (f \cdot h) \cdot \sigma_{T_\Sigma n}(\eta_n)$.

3.4. Proposition. *Let $\mathcal{A}$ be a small dense subcategory such that $\mathcal{K}$ has (1) reflexive coequalizers, (2) tensors for all objects of $\mathcal{A}$, and (3) coproducts of such tensors.*

Then $\mathcal{K}$ it is equivalent to a full reflective subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$.

Proof. Since the functor $E\colon \mathcal{K} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is fully faithful, we only need to prove that it has an enriched left adjoint. That is, $E[\mathcal{K}]$ is a reflective subcategory. For that we verify that the inclusion functor $J\colon \mathcal{A} \longrightarrow \mathcal{K}$ has weighted colimits

$$colim_H J$$

for all weights $H\colon \mathcal{A}^{op} \longrightarrow \mathcal{V}$. Then the functor assigning to each H the above colimit is an enriched left adjoint of E.

We first form coproducts

$$X = \coprod_{A \in \mathrm{obj}\,\mathcal{A}} HA \otimes A \quad \text{and} \quad Y = \coprod_{f\colon B \longrightarrow A} HA \otimes B$$

with injections

$$i(A)\colon HA \otimes A \longrightarrow X \quad \text{and} \quad j(f)\colon HA \otimes B \longrightarrow Y\,.$$

Every morphism $f\colon B \longrightarrow A$ of $\mathcal{A}$ yields a parallel pair $p_f, q_f\colon HA \otimes B \longrightarrow X$ as follows

$$\begin{tikzcd}
 & HA \otimes B \arrow[dl, "HA\otimes f"'] \arrow[d, shift right, "p_f"'] \arrow[d, shift left, "q_f"] \arrow[dr, "Hf\otimes B"] & \\
HA \otimes A \arrow[r, "i(A)"'] & X & HB \otimes B \arrow[l, "i(B)"]
\end{tikzcd}$$

The resulting pair $p = [p_f]$ and $q = [q_f]$ of morphisms from Y to X is reflexive: both morphisms are split by the morphism

$$\big[j(\mathrm{id}_{\mathcal{A}})\big] : X \longrightarrow Y\,.$$

The desired colimit $C = colim_H J$ is given by the following reflexive coequalizer

$$Y \underset{p}{\overset{q}{\rightrightarrows}} X \xrightarrow{\ c\ } C$$

That is, morphisms from C to Z correspond under a bijection (natural in Z) to natural transformations from H to $\mathcal{K}(J-, Z)$. This follows from the argument (in dual form) presented by Kelly [K] in Section 3.3.10, see the formulas 3.68 and 3.70.

■

Recall that an enriched category is (co)complete if it has all small weighted (co)limits.

3.5. Corollary. *Let $\mathcal{K}$ be a $\mathcal{V}$-category with reflexive coequalizers. If an object G with tensors exists whose finite copowers form a dense subcategory then $\mathcal{K}$ is complete and cocomplete.*

Proof. The full subcategory $\mathcal{A}$ of all finite copowers of G (which is essentially small) satisfies (2) and (3) of the above proposition: for (2) use $V \otimes \big(\coprod_n G\big) = \big(\coprod_n V\big) \otimes G$. Analogously for (3). Since $\mathcal{V}$ is (co)complete, so is $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ ([K], Section 3.3.3), and since $\mathcal{K}$ is equivalent to a full reflective subcategory, it is also (co)complete ([K], Section 3.3.5). ■

3.6. Remark. In the proof of Proposition 3.4 the object Y is a tensor of G, whenever $B = n.G$: we have $Y = V \otimes G$ for $n \cdot \mathcal{K}(G, B) \times HA$. Thus, in the above corollary it is sufficient to assume the existence of reflexive coequalizers for parallel pairs whose domains are tensors of G.

4. Subcongruences

We now introduce subcongruences in order-enriched categories. In the subsequent section we will apply them for a characterization of varieties of ordered algebras. Throughout the rest of our paper we work with categories enriched over the cartesian closed category **Pos** of posets (shortly: order-enriched categories). Thus, each hom-set carries a partial order such that composition is monotone. Enriched functors are functors which are locally monotone. Enriched natural transformations are just the usual natural transformations between the underlying ordinary functors. Here is an important example of an enriched category:

4.1. Notation. Let $\Sigma = (\Sigma_n)_{n<\in\mathbb{N}}$ be a signature, where Σ_n is the set of all n-ary operation symbols. We denote by

$$\Sigma\text{-}\mathbf{Pos}$$

the category of ordered Σ-algebras with monotone operations as objects, and monotone homomorphisms as morphisms. This is an order-enriched category with the pointwise ordering of parallel homomorphisms.

Whereas coequalizers and regular epimorhisms play a central role in the characterization of classical varieties, the corresponding role is taken by coinserters and subregular epimorphisms in Σ-**Pos**.

4.2. Definition. Let $f_0,\ f_1\colon X \longrightarrow Y$ be morphisms of an order-enriched category $\mathcal{K}$. (We use indices 0, 1 to indicate that f_0 comes first and f_1 second. We do *not* assume $f_0 \le f_1$ in $\mathcal{K}(X,Y)$.)

Their *coinserter* is the universal morphism $c\colon Y \longrightarrow Z$ with respect to $c \cdot f_0 \le c \cdot f_1$ in $\mathcal{K}(X,Y)$. That is:

(1) Every morphism $c'\colon Y \longrightarrow Z'$ with $c' \cdot f_0 \le c' \cdot f_1$ factorizes through c.

(2) Given $u_0,\ u_1\colon Z \longrightarrow U$ with $u_0 \cdot c \le u_1 \cdot c$, it follows that $u_0 \le u_1$.

Observe that (2) implies that the factorization in (1) is unique.

4.3. Example. In **Pos** the coinserter of $f_0,\ f_1\colon X \longrightarrow Y$ is given as follows. Recall that a *preorder* is a reflexive and transitive relation. The *posetal reflection* of a preordered set $(Y, \sqsubseteq)$ is the quotient modulo the equivalence $\sim$ with $y \sim y'$ iff $y \sqsubseteq y' \sqsubseteq y$.

Let $\sqsubseteq$ be the least preorder on Y with $f_0(x) \sqsubseteq f_1(x)$ (for all $x \in X$) and containing the order $\le$ of Y. The coinserter $c\colon (Y, \le) \longrightarrow Z$ of f_0, f_1 is given by the posetal reflection of that preorder

$$c\colon (Y, \sqsubseteq) \longrightarrow (Y, \sqsubseteq)/\sim = Z.$$

4.4. Definition. [ARO] A morphism $c\colon Y \longrightarrow Z$ in an order-enriched category is a *subregular epimorphism* if it is a coinserter of a reflexive parallel pair:

$$\begin{tikzcd} X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_0"'] & Y \arrow[l, bend right=60, "d"'] \arrow[r, "c"] & Z \end{tikzcd} \qquad f_0 \cdot d = \mathrm{id}_Y = f_1 \cdot d\,.$$

4.5. Example. (1) Subregular epimorphisms in **Pos**, and more generally in Σ-**Pos**, are precisely the surjective homomorphisms ([ARO], Proposition 4.4).

(2) If an order-enriched category has finite coproducts, then we have

$$\text{regular epi} \Rightarrow \text{subregular epi} \Rightarrow \text{epi}$$

([ARO], Example 3.4).

4.6. Definition. Let $\mathcal{K}$ be an order-enriched category.

(1) A morphism $r\colon R \longrightarrow A$ is an *embedding* if for all pairs $f,\ f'\colon X \longrightarrow R$ with $r\cdot f \le r\cdot f'$ we have $f \le f'$.

(2) A parallel pair $r_0,\ r_1\colon R \longrightarrow A$ is a *collective embedding* provided that whenever morphisms $f,\ f'\colon X \longrightarrow R$ fulfil $r_0 \cdot f \le r_0 \cdot f'$ and $r_1 \cdot f \le r_1 \cdot f'$, then $f \le f'$. (If the power A^2 exists, this means precisely that $< r_0, r_1 >\colon R \longrightarrow A^2$ is an embedding.)

Such a parallel pair represents a *relation* on the object A.

(3) A *subkernel pair* of a morphism $h\colon A \longrightarrow B$ is a universal parallel pair $r_0, r_1\colon R \longrightarrow A$ with respect to $h \cdot r_0 \le h \cdot r_1$.

That is, a relation on A such that every pair $v_0,\ v_1\colon V \longrightarrow A$ with $h \cdot v_0 \le h \cdot v_1$ factorizes through $r_0,\ r_1$.

(4) A morphism $c\colon C \longrightarrow D$ is a *quotient* if for every pair $u, v\colon A \longrightarrow C$ with $u \cdot c \le c \cdot v$ we have $u \cdot v$.

4.7. Example. (1) Every coinserter is a quotient.

(2) In **Pos** embedings represent precisely the subposets of the poset A, and relations represent the subposets of the power A^2. The subkernel pair of $h\colon A \longrightarrow B$ is the subposet of A^2 on all pairs x, y with $h(x) \le h(y)$. Quotients are precisely the surjective morphisms.

In ordinary category theory the concept of congruence is an abstraction of kernel pair: every kernel pair is a congruence, and the opposite implication holds in **Set** (and other categories, e.g. varieties). In order-enriched categories we introduce subcongruences which are abstractions of subkernel pairs. Each subkernel pair is reflexive (even order-reflexive, see below) and transitive. It is, of course, usually not symmetric.

Recall from Remark 2.7 that the reflexivity of a pair $r_0,\ r_1\colon R \longrightarrow A$ means that of all $s\colon S \longrightarrow A$ the pair $s,\ s$ factorizes through $r_0,\ r_1$. Here is a stronger property:

4.8. Definition. A relation $r_0,\ r_1\colon R \longrightarrow A$ in an order-enriched category is *order-reflexive* if every comparable pair $s_0 \le s_1\colon S \longrightarrow A$ factorizes through $r_0,\ r_1$.

4.9. Example. In **Pos** a relation on a poset $A = (X, \le)$ is a relation R on X (endowed with the coordinate-vise order). It is order-reflexive iff it contains the relation $\le$. Indeed, for the necessity choose $S = 1$ in the above definition: we conclude for every pair $s_0 \le s_1$ in A that $s_0 R s_1$. Sufficiency is clear.

4.10. Proposition. *Given an order-enriched category, every subkernel pair is order-reflexive and transitive (Definition 2.8).*

Proof. Let $r_0, r_1\colon R \longrightarrow A$ be the subkernel pair of $f\colon A \longrightarrow B$.

(1) Order-reflexivity: from $s_0 \le s_1\colon S \longrightarrow A$ it follows that $f \cdot s_0 \le f \cdot s_1$, thus the pair s_0, s_1 factorizes through the subkernel pair of f.

(2) Transitivity: let $s, s', s''\colon S \longrightarrow A$ be morphisms such that both s, s' and s', s'' factorize through r_0, r_1. Then f merges both s, s' and s', s'', consequently, it merges s, s'' which implies that that pair also factorizes through r_0, r_1.

∎

4.11. Definition. In an order-enriched category a *subcongruence* is an order-reflexive and transitive relation.

Thus every subkernel pair is a subcongruence.

4.12. Example. (1) In **Pos** a subcongruence on a poset is a preorder containing the order-relation.

(2) In Σ-**Pos** a subcongruence on an algebra is a preorder containing the order-relation which is compatible with all operations.

(3) In an arbitrary order-enriched category a subcongruence is precisely a parallel pair that every hom-functor takes to a subcongruence in **Pos**.

4.13. Construction. For every subcongruence $r_0, r_1 \colon R \longrightarrow A$ in **Pos** the following relation $\sqsubseteq$ on A is a preorder:

$$x \sqsubseteq y \quad \text{iff} \quad x = r_0(z) \quad \text{and} \quad y = r_1(z) \quad \text{for some} \quad z \in R.$$

The posetal reflection $c \colon (A, \sqsubseteq) \longrightarrow C$ yields the coinserter $c \colon A \longrightarrow C$ of r_0 and r_1.

In all varieties of ordered algebras we will prove in Section 5 the that every subcongruence is a subkernel pair. To achieve this, we first show how coinserters of subcongruences are constructed in **Pos**.

Proof. Let $\leq$ denote the given partial order on A. We verify that $\sqsubseteq$ is indeed a preorder, and that it contains $\leq$. Thus c is a monotone map from $(A, \leq)$ to C. It then easily follows that c is the coinserter of r_0 and r_1.

(1) The relation $\sqsubseteq$ is reflexive because r_0, r_1 is order-reflexive. We verify the transitivity:

$$\text{if} \;\; x \sqsubseteq x' \sqsubseteq x'' \quad \text{, then} \quad x \sqsubseteq x''.$$

We are given z, $z' \in R$ with

$$x = r_0(z)\,, \;\; x' = r_1(z) = r_0(z') \quad \text{and} \quad x'' = r_1(z')\,.$$

Let $s, s', s'' \colon 1 \longrightarrow A$ be the morphisms representing x, x' and x'', resp. Then the pair s, s' factorizes through r_0, r_1: use the morphism $1 \longrightarrow R$ representing z. Analogously, s', s'' factorizes through r_0, r_1. Since the relation r_0, r_1 is transitive, the pair s, s'' also factorizes through r_0, r_1. The factorizing morphism represents an element $z'' \in R$ such that $x = r_0(z) = r_0(z'')$ and $x'' = r_1(z') = r_1(z'')$. This verifies that $x \sqsubseteq x''$.

(2) We show that

$$x_0 \leq x_1 \quad \text{implies} \quad x_0 \sqsubseteq x_1\,.$$

We have morphisms $q_i \colon 1 \longrightarrow A$ which represent x_i ($i = 0, 1$). Then $q_0 \leq q_1$, thus by order-reflexivity there exists $k \colon 1 \longrightarrow R$ with $q_0 = r_0 \cdot k$ and $q_1 = r_1 \cdot k$. In other words, the element $z \in R$ represented by k fulfils $x_0 = r_0(z)$ and $x_1 = r_1(z)$; hence $x_0 \sqsubseteq x_1$.

(3) The monotone map $c \colon (A, \leq) \longrightarrow C$ is a coinserter of r_0 and r_1. In fact, $c \cdot r_0 \leq c \cdot r_1$: for every $z \in R$ we have $r_0(z) \sqsubseteq r_1(z)$, thus $c \cdot r_0(z) \leq c \cdot r_1(z)$.

Let $c'\colon A \longrightarrow C'$ fulfill $c' \cdot r_0 \leq c' \cdot r_1$. To prove that c' factorizes through the posetal reflection c of $(A, \sqsubseteq)$, we just need to observe that

$$x_0 \sqsubseteq x_1 \quad \text{implies} \quad c'(x_0) \leq c'(x_1) \quad \text{in} \quad C'.$$

But this follows trivially from $c' \cdot r_0 \leq c' \cdot r_1$.

Since c is surjective, Property (2) of Definition 4.4 is clear. ∎

4.14. Definition. An order-enriched category has *effective subcongruences* if every subcongruence is the subkernel pair of some morphism.

4.15. Notation. (1) The forgetful functor of Σ-**Pos** is denoted by $U\colon \Sigma\text{-}\mathbf{Pos} \longrightarrow \mathbf{Pos}$.

(2) Every set is considered as a (discretely ordered) poset.

(3) The classical free Σ-algebra $T_\Sigma X$ on a set X (of all terms in variables from X), discretely ordered, is also a free ordered Σ-algebra on X. Given an ordered Σ-algebra A and a map $h\colon X \longrightarrow U_\Sigma A$, we denote by

$$h^\#\colon T_\Sigma X \longrightarrow A$$

the corresponding homomorphism.

4.16. Lemma. *Let A be an ordered algebra. Given a set X and maps $h \leq k\colon X \longrightarrow UA$ in $[X, UA]$, it follows that $h^\# \leq k^\#$.*

Proof. By structural induction on the complexity of terms $t \in T_\Sigma X$ we prove that $h^\#(t) \leq k^\#(t)$. For variables in X this is our assumption. In case $t = \sigma(t_1, ..., t_n)$ for some $\sigma \in \Sigma_n$, and the inequality holds for each t_i, we use the monotonicity of σ to derive the desired inequality for t. ∎

4.17. Proposition. *The category Σ-**Pos** of ordered Σ-algebras has effective subcongruences, and the forgetful functor **U** preserves their coinserters.*

Proof. (1) **Pos** has effective subcongruences. Indeed, given a subcongruence $r_0, r_1\colon R \longrightarrow A$ and the morphism $c\colon A \longrightarrow C$ of Construction 4.13, then r_0, r_1 is the subkernel pair of c: first, $c \cdot r_0 \leq c \cdot r_1$ clearly holds. Second, for every pair

$$u_0, u_1\colon S \longrightarrow A \quad \text{with} \quad c \cdot u_0 \leq c \cdot u_1$$

we show that it factorizes through r_0, r_1 which (since r_0, r_1 is a collective embedding) implies the desired universal property. Indeed, for every $z \in S$ we have $c\big(u_0(z)\big) \leq c\big(u_1(z)\big)$, which means $u_0(z) \sqsubseteq u_1(z)$. Thus there exists a (necessarily unique) element $v(z) \in R$ with

$$r_i\big(v(z)\big) = u_i(z) \qquad (i = 0, 1).$$

This defines a mapping $v\colon S \longrightarrow R$ with $u_i = r_i \cdot v$. Moreover, v is monotone. This follows from r_0, r_1 being a collective embedding: given $z \leq z'$ in S, for $i = 0, 1$ we have (using that u_i is monotone) that

$$r_i\big(v(z)\big) = u_i(z) \leq u_i(z') = r_i\big(v(z')\big) \qquad (i = 0, 1).$$

(2) For every $n \in \mathbb{N}$ the morphism $c^n \colon A^n \longrightarrow C^n$ has the subkernel pair r_0^n, r_1^n. This follows easily from Construction 4.13.

(3) The forgetful functor $U \colon \Sigma\text{-}\mathbf{Pos} \longrightarrow \mathbf{Pos}$ preserves subcongruences. In fact, let r_0, r_1 be a subcongruence in Σ-**Pos**.

a. Ur_0, Ur_1 is transitive: see Remark 2.9, and use that U preserves pullbacks (in fact, it creates limits).

b. Ur_0, Ur_1 is order-reflexive: let $s_0 \leq s_1 \colon S \longrightarrow UA$ be given. The corresponding homomorphisms $s_i^\sharp \colon T_\Sigma S \longrightarrow A$ are also comparable by the preceding Lermma, thus they factorize through r_0, r_1 in Σ-**Pos**. Consequently $s_i = Us_i^\sharp \cdot \eta_S$ implies that s_0, s_1 factorize through Ur_0, Ur_1.

(4) We are ready to prove that Σ-**Pos** has effective subcongruences. Let $r_0, r_1 \colon R \longrightarrow A$ be homomorphisms forming a subcongruence in Σ-**Pos**. By Item (1) Ur_0, Ur_1 is the kernel pair of a (surjective) morphism $c \colon A \longrightarrow C$ in **Pos**. We prove that C carries a unique structure of an algebra making c a homomorphism. In other words, for every n-ary operation symbol $\sigma \in \Sigma$ a unique morphism σ_C exists making the square below commutative:

$$\begin{tikzcd} R^n \arrow[r, shift left, dashed, "r_1^n"] \arrow[r, shift right, dashed, "r_0^n"'] & A^n \arrow[r, "\sigma_A"] \arrow[d, "c^n"'] & A \arrow[d, "c"] \\ & C^n \arrow[r, "\sigma_C"'] & C \end{tikzcd}$$

Indeed, by (2), r_0^n, r_1^n is a subcongruence on A^n, and c^n is the coinserter. Since r_0, r_1 are homomorphisms, we have $(c \cdot \sigma_A) \cdot r_0^n \leq (c \cdot \sigma_A) \cdot r_1^n$, indeed:

$$c \cdot \sigma_A \cdot r_0^n = c \cdot r_0 \cdot \sigma_R \leq c \cdot r_1 \cdot \sigma_R = c \cdot \sigma_A \cdot r_1^n .$$

Thus we get the unique σ_C as stated.

Moreover, the resulting homomorphism c is the coinserter of r_0 and r_1 in Σ-**Pos**. Indeed, given a homomorphism $c' \colon A \longrightarrow C'$ with $c' \cdot r_0 \leq c' \cdot r_1$, there is a unique monotone map h making the triangle below commutative in **Pos**:

$$\begin{tikzcd} & A \arrow[dl, "c"'] \arrow[dr, "c'"] & \\ C \arrow[rr, "h"'] & & C' \end{tikzcd}$$

Since c and c' are homomorphisms and c is surjective, it follows that h is also a homomorphism. Thus, c is the coinserter of r_0 and r_1 in Σ-**Pos**: property (2) in Definition 4.2 follows since c is surjective.

(5) From the above description it easily follows that U preserves coinserters of subcongruences.

∎

4.18. Definition. [ARO] An object G of an order-enriched category is a *subregular projective* if its hom-functor to **Pos** preserves subregular epimorphisms. That is, given a subregular epimorphism $e\colon A \longrightarrow B$, every morphism from G to B factorizes through e:

$$\begin{tikzcd} & G \arrow[dl, dashed, "\exists"'] \arrow[d, "\forall"] \\ A \arrow[r, "e"'] & B \end{tikzcd}$$

4.19. Example. (1) A poset is a subregular projective in **Pos** iff it is discrete. Indeed, sufficiency follows from subregular epimorphisms being surjective (Example 4.5). Let G be a non-discrete poset. Then for the 2-chain B with $0 < 1$ we have a monomorphism $f\colon B \longrightarrow G$. Let $id\colon A \longrightarrow B$ be the morphism from the discrete poset $A = \{0, 1\}$. This is a subregular epimorphism (Example 4.5) through which f does not factorize. Thus G is not a subgerular projective.

(2) Let X be a set. The free algebra $T_\Sigma X$ is a regular projective in Σ-**Pos**, since subregular epimorphisms are surjective.

As in Remark 3.1, colimits in order-enriched categories are understood to be conical.

4.20. Lemma. *In an order-enriched category with reflexive coinserters every object G with copowers has tensors.*

Proof. We describe, for every poset P, the tensor $C = P \otimes G$ as the following reflexive coinserter:

$$\begin{tikzcd} \coprod_R G \arrow[r, shift left, "\bar r_1"] \arrow[r, shift right, "\bar r_0"'] & \coprod_{|P|} G \arrow[r, "c"] & C \end{tikzcd}$$

Here $|P|$ is the underlying set of P and $R \subseteq |P| \times |P|$ is its order relation. The morphisms $\bar r_i = r_i \cdot G$ are induced by the projection $r_i\colon R \longrightarrow |P|$ given by $r_i(x_0, x_1) = x_i$. The diagonal $\Delta\colon |P| \longrightarrow R$ yields a joint splitting $\Delta \cdot G$ of the pair $\bar r_1, \bar r_2$, thus, the coinserter exists. Its components are denoted by $c_x\colon G \longrightarrow C$ for $x \in P$.

Our task is to find a natural isomorphism

$$\frac{C \xrightarrow{\ f\ } X}{P \xrightarrow[i(f)]{} \mathcal{K}(G, X)}$$

Given f, define $i(f)$ in $x \in P$ as $f \cdot c_x$. This map $i(f)$ is monotone since $\coprod_{|P|} G$ is a conical coproduct. The resulting map $i\colon \mathcal{K}(C, X) \longrightarrow [P, \mathcal{K}(G, X)]$ is also monotone: $f \le f'\colon C \longrightarrow G$ implies $f \cdot c_x \le f' \cdot c_x$ for all x, thus $i(f) \le i(f')$.

Conversely, given $g\colon P \longrightarrow \mathcal{K}(G, X)$ in **Pos**, then the morphism $\bar g\colon \coprod_{|P|} G \longrightarrow X$ given by $\bar g = [g(x)]_{x \in |P|}$ fulfils $\bar g \cdot \bar r_0 \le \bar g \cdot \bar r_1$ because each pair $x_0 \le x_1$ in R yields $g(x_0) \le g(x_1)$ in $\mathcal{K}(G, X)$. Let $j(g)\colon C \longrightarrow X$ be the unique morphism with

$$\bar g = j(g) \cdot c\,.$$

This defines a monotone map $j\colon [P, \mathcal{K}(G,X) \longrightarrow \mathcal{K}(C,X)$: if $g \le h\colon P \longrightarrow \mathcal{K}(G,X)$, then $\bar{g} \le \bar{h}$, thus $j(g) \le j(h)$ by the universal property of c.

It is easy to see that i and j are inverse to each other. And i is natural: given $u\colon X \longrightarrow X'$, then $i(u \cdot f)$ assigns to x the value $u \cdot f \cdot c_x$, which is what $u \cdot i(f)$ does, too. Thus $u \cdot i(-) = i(u \cdot -)$. ∎

4.21. Definition. A *subvarietal generator* in an order-enriched category is a strong generator with copowers which is an abstractly finite subregular projective.

4.22. Proposition. *Let $\mathcal{K}$ be an order-enriched category with subkernel pairs, reflexive coinserters, and a subvarietal generator G.*

(a) $\mathcal{K}$ is complete and cocomplete.

(b) A parallel pair $p, q\colon K \longrightarrow L$ fulfils $p \le q$ iff $p \cdot f \le q \cdot f$ holds for all morphisms $f\colon G \longrightarrow K$.

(c) A morphism $m\colon A \longrightarrow B$ is an embedding (Definition 4.6) iff given a pair $u_0, u_1\colon G \longrightarrow A$ with $m \cdot u_0 \le m \cdot u_1$, we have $u_0 \le u_1$.

Proof. (1) For every object K we denote the adjoint transpose of the identity morphism on $\mathcal{K}(G,K)$ by $c_K\colon \mathcal{K}(G,K) \otimes G \longrightarrow K$. It is a subgerular epimorphism: see Lemma 3.22 in [ARO], where G is thus called a subregular generator.

(2) To prove (a), we form a small full subcategory $\mathcal{A}$ of $\mathcal{K}$ containing G and closed under finite coproducts. This subcategory is dense: see [ARO], Theorem 3.23. (In the formulation of that theorem $\mathcal{K}$ is assumed to also have kernel pairs. However, this assumption is not used in the proof, except for guaranteeing that G is a subregular generator– which holds by (a).) Moreover, $\mathcal{K}$ has coequalizers of reflexive pairs $p, q\colon X \longrightarrow Y$ in $\mathcal{A}$, where X is a tensor of G: the coproduct $X + X$ exists (being a tensor of G, too), and the following pair

$$X + X \underset{[q,p]}{\overset{[p,q]}{\rightrightarrows}} Y$$

is reflexive, and its coinserter is the coequalizer of p and q.

By Corollary 3.5 and Remark 3.6, $\mathcal{K}$ is complete and cocomplete.

(3) To prove (b), let $\mathcal{K}_0(G,K)$ be the underlying set of $\mathcal{K}(G,K)$, and denote by $i \cdot \mathcal{K}_0(G,K) \longrightarrow \mathcal{K}(G,K)$ the identity-carried map. This is a subregular epimorphisms (Example 4.5). The functor $(-) \otimes G$ is a left adjoint, hence, it preserves tesnors. T therefore $i \otimes G$ is also a subregular epimorphims. Thus the composite

$$c_k \cdot (i \otimes G)\colon \coprod_{f \in \mathcal{K}_0(G,K)} G \longrightarrow K$$

is a quotient (Definition 4.6). The components of this quotient are $f\colon G \longrightarrow K$. Thus from $p \cdot f \le q \cdot q$, and the fact that coproducts are conical, we conclude $p \le q$.

(4) To prove (c), let $v_0, v_1 : K \longrightarrow A$ be a parallel pair with $m \cdot v_0 \le m \cdot v_1$, then we verify that $v_0 \le v_1$. As in Item (3), we only need to verify $v_0 \cdot c_K \cdot (i \otimes G) \le v_1 \cdot c_K \cdot (i \otimes G)$.

This again follows from from coproducts being conical, since for every $f\colon G \longrightarrow K$ we have $v_0 \cdot f \le v_1 \cdot f$: apply (c) to $u_i = v_i \cdot f$.

■

5. Varieties of Ordered Algebras

Here we present a characterization of varieties of ordered algebras analogous to Theorem 2.19. This follows ideas of [ARO], which we slightly correct and improve, endowed with the concept of a subexact category introduced below.

5.1. Definition. A *variety of ordered algebras* is a full subcategory of Σ-**Pos** presented by inequations $t \le s$ between terms $t,\ s \in T_\Sigma X$ for finite sets X. It consists of algebras A such that $h^\#(t) \le h^\#(s)$ holds for each of the inequations and each interpretation $h\colon X \longrightarrow A$ of the variables.

In the classical universal algebra Birkhoff's Variety Theorem states that a full subcategorry of Σ-**Alg** is a variety iff if is closed under products, subalgebras, and quotients (= homomorphic images). For ordered algebras we have the analogous three constructions:

(1) A product of algebras A_i $(i \in I)$ is their cartesian product with both operations and order given coordinate-wise.

(2) By a *subalgebra* of an ordered algebra A is meant a subposet closed under the operations. Thus subalgebras are represented by homomorphisms $m\colon B \longrightarrow A$ carried by embeddings (Definition 4.8).

(3) By a *homomorphic image* of an algebra we mean a quotient represented by a subregular epimorphism $e\colon A \longrightarrow B$. (That is, e is surjective, see Example 4.5.)

5.2. Birkhoff Variety Theorem. [ADV] *A full subcategory of* Σ-**Pos** *is a variety of ordered algebras iff it is closed under products, subalgebras, and homomorphic images.*

5.3. Remark. The category Σ-**Pos** has weighted limits and colimits. Indeed, it has coequalizers of pairs $f, g\colon A \longrightarrow B$ obtained by forming the wide pushout of all quotients of B carried by surjective morphisms and merging f and g. The free algebra $G = T_\Sigma 1$ on one generator has the property that its finite copowers form the full subcategory of all finitely generated free algebras. This subcategory is dense: this is completely analogous ton Example 3.3, we just need to add that given homomorphisms $f, f'\colon A \longrightarrow B$, then $f \le f'$ iff $f \cdot h \le f' \cdot h$ holds for all homomorphisms $h\colon T_\Sigma 1 \longrightarrow A$. Thus we can apply Corollary 3.5.

5.4. Corollary. *Every variety of ordered algebras has*

(1) Weighted limits and colimits.

(2) Effective subcongruences.

Proof. (1) The category Σ-**Pos** is complete and has the factorization system with $\mathcal{E}$=surjective homomorphisms, and $\mathcal{M}$=subalgebra embeddings. It is clearly co-well-powered with respect to $\mathcal{E}$. Given a variety $\mathcal{V} \subseteq \Sigma$-**Pos**, it is a reflective subcategory with reflection maps in $\mathcal{E}$. This follows since it is closed under products and subalgebras ([AHS], Theorem 16.8). The reflector $R\colon \Sigma\text{-}\mathbf{Pos} \longrightarrow \mathcal{V}$ is enriched: given morphisms $f \leq g\colon A_1 \longrightarrow A_2$, for the reflections maps $r_i\colon A_i \longrightarrow RA_i$ we have

$$Rf \cdot r_1 = r_2 \cdot f \leq r_2 \cdot g = Rg \cdot r_1.$$

Since r_1 is surjective, this implies $Rf \leq Rg$.

Thus the preceding Remark implies that $\mathcal{V}$ has weighted limits and colimits ([K], Section 3.3.8).

(2) Since $\mathcal{V}$ is closed under pullbacks (in fact, under limits), every subcongruence $r_0, r_1\colon R \longrightarrow A$ in $\mathcal{V}$ is also a subcongruence in Σ-**Pos** (Remark 2.9). By Proposition 4.17 there is a homomorphism $h\colon A \longrightarrow B$ in Σ-**Pos** with subkernel pair r_0, r_1. From the proof of that proposition we know that h is surjective. Hence B is a homomorphic image of $A \in \mathcal{V}$. Consequently, $B \in \mathcal{V}$ and h is a morphism of $\mathcal{V}$ with the subkernel pair r_0, r_1. ∎

Recall that effective objects (in ordinary categories) are those with hom-functor preserving coequalizers of congruences. Here is the enriched variant:

5.5. Definition. [ARO] An object G of an order-enriched category is *subeffective* if its hom-functor to **Pos** preserves coinserters of subcongruences.

In a category with subkernel pairs every subeffective object is, of course, a subeffective projective.

5.6. Example. The free algebra G on one generator in a variety $\mathcal{V}$ of ordered algebras is subeffective; moreover it is a subvarietal generator. Indeed, the forgetful functor of $\mathcal{V}$ has a left adjoint $F\colon \mathbf{Pos} \longrightarrow \mathcal{V}$, and the free algebra $G = F1$ is an abstractly finite subregular projective by [ARO], Example 4.6. The hom-functor of G is naturally isomorphic to the forgetful functor $U\colon \mathcal{V} \longrightarrow \mathbf{Pos}$. By Proposition 4.17 in case $\mathcal{V} = \Sigma$-**Pos**, the functor U preserves coinserters of subcongruences. For general varieties $\mathcal{V}$ the same is true since $\mathcal{V}$ is closed under subgerular quotients.

The following proposition has a completely analogous proof to that of Proposition 2.17:

5.7. Proposition. *In an order-enriched category $\mathcal{K}$ with subkernel pairs and their coinserters, let G be a subregularly projective strong generator. Then we have the following equivalence:*

$$G \textit{ subeffective} \Leftrightarrow \mathcal{K} \textit{ has effective subcongruences.}$$

5.8. Theorem. *An order-enriched category is equivalent to a variety iff it has*

(1) Effective subcongruences.
(2) A subvarietal generator.
(3) Subkernel pairs and reflexive coinserters.

Proof. Every variety $\mathcal{V}$ satisfies the above conditions by Corollary 5.4 and Example 5.6.

Let $\mathcal{K}$ be a category as above with a subvarietal generator G. We first verify some properties of $\mathcal{K}$.

(a) $\mathcal{K}$ is complete and cocomplete by Proposition 4.22.

(b) $\mathcal{K}$ has factorizations of morphisms as a subregular epimorphism followed by an embedding (Definitoins 4.4 and4 4.6). (The proof presented in [ARO] is incomplete.)

Given a morphism $f\colon A \longrightarrow B$, form the subkernel pair $r_0,\, r_1\colon R \longrightarrow A$ of f. This is a reflexive pair, thus, a coinserter $c\colon A \longrightarrow C$ exists. We have the unique morphism m with $f = m \cdot c$

$$\begin{tikzcd}
R \arrow[dr, shift left, "r_1"] \arrow[dr, shift right, "r_0"'] & & \\
& A \arrow[r, "f"] \arrow[d, "c"] & B \\
G \arrow[uu, dashed, "v"] \arrow[ur, dashed, shift left, "v_1"] \arrow[ur, dashed, shift right, "v_0"'] \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_0"'] & C \arrow[ur, "m"'] &
\end{tikzcd}$$

We prove that it is an embedding. By Proposition 4.22 this is equivalent to proving for all $u_0,\, u_1\colon G \longrightarrow C$ that

$$m \cdot u_0 \leq m \cdot u_1 \quad \text{implies} \quad u_0 \leq u_1\,.$$

As G is a subregular projective, there exist morphisms $v_i\colon G \longrightarrow A$ with $u_i = c \cdot v_i$. Then

$$f \cdot v_0 = m \cdot u_0 \leq m \cdot u_1 = f \cdot v_1.$$

This implies that we have $v\colon G \longrightarrow R$ with $v_i = r_i \cdot v$. This proves the desired inequality:

$$u_0 = c \cdot r_0 \cdot v \leq c \cdot r_1 \cdot v = u_1\,.$$

(c) In [ARO] the following signature Σ is used: its n-ary operations are the morphisms from G to $n \cdot G$:

$$\Sigma_n = \mathcal{K}(G, n \cdot G) \qquad (n \in \mathbb{N})\,.$$

As proved in Item(2a) of Thm. 4.8 in loc. cit., we obtain a full embedding

$$E\colon \mathcal{K} \longrightarrow \Sigma\text{-}\,\mathbf{Pos}$$

as follows. The algebra EK for an object K of $\mathcal{K}$ has the underlying poset $\mathcal{K}(G, K)$. Given an n-ary operation symbol $\sigma\colon G \longrightarrow n \cdot G$, to every n-tuple $f_i\colon G \longrightarrow K (i < n)$ the map σ_{EK} assigns the following composite

$$\sigma_{EK}(f_i) \equiv G \xrightarrow{\ \sigma\ } n \cdot G \xrightarrow{\ [f_i]\ } K\,.$$

To a morphism $h\colon K \longrightarrow L$ the functor E assigns the homomorphism

$$Eh = h \cdot (-)\colon \mathcal{K}(G, K) \longrightarrow \mathcal{K}(G, L)\,.$$

Let $\bar{\mathcal{K}}$ be the closure of $E[\mathcal{K}]$ under isomorphism in Σ-**Pos**. Then $\mathcal{K}$ is equivalent to $\bar{\mathcal{K}}$, and we use the Birkhoff Variety Theorem to verify that $\bar{\mathcal{K}}$ is a variety, thus finishing our proof.

(i) $\bar{\mathcal{K}}$ is closed under products because $\mathcal{K}$ has products by (a), and E clearly preserves limits.

(ii) $\bar{\mathcal{K}}$ is closed under subalgebras. The proof presented in [ARO] is incomplete, we present a proof now. A subalgebra of EK, for $K \in K$, is a subposet $M \subseteq \mathcal{K}(G, K)$ closed under the operations. That is, given an n-ary symbol σ, we have

$$[f_i] \cdot \sigma \in M \quad \text{for all} \quad f_0, \dots, f_{n-1} \in M .$$

We are to find an object $C \in \mathcal{K}$ with $EC \simeq M$.

The morphism $[h] \colon \coprod_{h \in M} G \longrightarrow K$ has a factorization as a subregular epimorphism c followed by an embedding m:

$$\begin{tikzcd} \coprod_{h\in M} G \arrow[rr, "{[h]}"] \arrow[dr, "c"'] & & K \\ & C \arrow[ur, "m"'] & \end{tikzcd}$$

We prove that the ordered algebras EC and M are isomorphic. For that, we verify that in **Pos** the image of Em (a subposet of EK) is M:

$$M = Em[EC] .$$

Since both the subposets M and $Em[EC]$ are closed under the operations, this implies $M = EC$ in Σ-**Pos**, as desired.

The inclusion $M \subseteq Em[EC]$ is obvious: given $h \in M$, the corresponding component $c_h \colon G \longrightarrow C$ of c above lies in EC, and fulfils $h = m \cdot c_h$.

Conversely, we prove

$$Em(g) = m \cdot g \in M \quad \text{for each} \quad g \colon G \longrightarrow C .$$

Since G is a subregular projective, $g = c \cdot g'$ for some morphism g':

$$\begin{tikzcd} n \cdot G \arrow[rr, "u \cdot G"] & & M \cdot G \arrow[rr, "{[h]}"] \arrow[dr, "c"] & & K \\ & G \arrow[ul, "\sigma"] \arrow[ur, "g'"'] \arrow[rr, "g"'] & & C \arrow[ur, "m"'] & \end{tikzcd}$$

Finite abstractness yields an injection $u \colon n \longrightarrow M$ (where n denotes the discrete poset $\{0, \dots, n-1\}$) such that g' factorizes through $u \cdot G$. We denote by σ the factorizing morphism lying in Σ_n.

Then for $h_i = u(i) \in M$ we get $[h] \cdot (u \cdot G) = [h_i]_{i<n}$. Since

$$\sigma_{EK}(h_i)_{i<n} = [h_i]_{i<n} \cdot \sigma \, ,$$

we obtain from the above diagram that

$$\sigma_{EK}(h_i)_{i<n} = m \cdot g \, .$$

This concludes the proof of $m \cdot g \in Em[EC]$ since $h_i \in M$ and M is closed under σ_{EK}.

(iii) $\bar{\mathcal{K}}$ is closed under homomorphic images. See [ARO], Item (3c) of the proof of Theorem 4.8. ■

5.9. Definition. An order-enriched category is *subexact* if it has

(1) Subkernel pairs and reflexive coinserters.
(2) Effective subcongruences.

5.10. Corollary. *An order-enriched category is equivalent to a variety of ordered algebras iff it is subexact, and has a subeffetive subvarietal generator.*

This follows from Example 5.6, Proposition 5.7, and the above theorem.

We have not included stability of subregular epimorphisms under pullback in the definition of exact category. The reason is that in the presence of a subvarietal generator this property follows from the existence of subkernel pairs and reflexive coinserters. This is a consequence of the above theorem, a direct proof is analogous to that of Lemma 2.6.

6. Congruences Versus Subcongruences

We now compare the concepts of congruence studied in [KV] and [BG] with our concept of subcongruence. In case of congruences in **Pos** we show that these concepts are equivalent to that of subcongruence. It folllows that they are also equivalent for arbitrary poset-enriched categories because a parallel pair is a congruence in op. cit. iff every hom-functor takes it to a congruence in **Pos** (compare with Example 4.12(3)).

6.1 Congruences due to Kurz and Velebil

Congruences in order-enriched categories $\mathcal{A}$ are defined by Kurz and Velebil [KV] as follows. A *category object in* $\mathcal{A}$ is a diagram

$$\begin{tikzcd} A_2 \arrow[r, shift left=2, "d_2^2"] \arrow[r, "d_1^2" description] \arrow[r, shift right=2, "d_0^2"'] & A_1 \arrow[r, shift left=2, "d_1^1"] \arrow[r, shift right=2, "d_0^1"'] & A_0 \arrow[l, "i_0^0" description] \end{tikzcd}$$

such that

(1) The square $d_!^1 \cdot d_0^2 = d_0^1 \cdot d_2^2$ is a pullback.
(2) $d_1^1 \cdot d_2^2 = d_1^1 \cdot d_1^2$ and $d_0^1 \cdot d_0^2 = d_0^1 \cdot d_1^2$.

(3) $d^1_1 \cdot i^0_0 = id = d^1_0 \cdot i^0_0$.

A *congruence* on an object A_0 is a category object such that (a) the pair d^1_0, d^1_1 is a collective embeddindg (Definition 4.6), and (b) the span d^1_0, d^1_1 is a two-sided discrete fibration. Now (a) means that we have a relation A_1 on the object A_0. This relation is transitive. Indeed, let us use Remark 2.8 with $l = d^1_0$ and $r = d^1_1$. From (1) we then get $\bar{l} = d^2_0$ and $\bar{r} = d^2_2$. Thus the desired factorization of $l \cdot \bar{l}, r \cdot \bar{r}$ through l, r is $p = d^2_1$: use (2).

We verify order-reflexivity of the relation A_1 in case $\mathcal{A} = \mathbf{Pos}$. Given a poset $A_0 = (X, \leq)$, the relation A_1 is reflexive due to (3). As explained in [KV], Example 4.5, the fact that A_1 is a discrete fibration means that it contains both of its composite with $\leq$. Since A_1 is reflexive, this implies that it contains $\leq$, hence it is order-reflexive by Example 4.9.

Conversely, given a subcongruence on A_0 which we now denote by $d^1_0, d^1_1 \colon A_1 \longrightarrow A_0$, then Condition (1) above tells us how to construct the appropriate category object. It clearly satisfies (a), and order-reflexivity implies (b).

6.2 Congruences due to Bourke and Garner

Bourke and Garner [BG] work with general congruences, called $\mathcal{F}$-cogruences, in $\mathcal{V}$-categories, where $\mathcal{V}$ is a symmetric monoidal closed category (here **Pos**). Their concept depends on the choice of a $\mathcal{V}$-category $\mathcal{F}$ containing the free $\mathcal{V}$-category $\mathbf{2}$ on a single arrow $1 \longrightarrow 0$. The full subcategory of $\mathcal{F}$ on all objects but 0 is denoted by $\mathcal{K}$. We thus have full embeddings $I \colon \mathcal{K} \longrightarrow \mathcal{F}$ and $J \colon \mathbf{2} \longrightarrow \mathcal{F}$.

Let $\mathcal{A}$ be a finitely complete $\mathcal{V}$-category. Then right Kan-extensions along J yield a functor from $[\mathbf{2}, \mathcal{A}]$ to $[\mathcal{F}, \mathcal{A}]$. We compose it with $(-) \cdot I \colon [\mathcal{F}, \mathcal{A}] \longrightarrow [\mathcal{K}, \mathcal{A}]$ to get a a functor K with a left adjoint Q:

$$Q \vdash K \colon [\mathbf{2}, \mathcal{A}] \longrightarrow [\mathcal{K}, \mathcal{A}].$$

An $\mathcal{F}$-*kernel* in $\mathcal{A}$ is an object of $[\mathcal{K}, \mathcal{A}]$ in the image of K, and an $\mathcal{F}$-*quotient* is a morphism of $[\mathbf{2}, \mathcal{A}]$ in the essential image of Q.

Let us choose $\mathcal{V} = \mathbf{Pos}$ and as $\mathcal{F}$ the enriched category obtained from $\mathbf{2}$ by adding an object X and a parallel pair of morphisms from it to 1 such that $\mathcal{K}(X, 1)$ is discretely ordered, whereas $\mathcal{K}(X, 0)$ is a two chain. Then $[\mathcal{K}, \mathcal{A}]$ is the category of all parallel pairs in $\mathcal{A}$. The functor K assings to every morphism of $\mathcal{A}$ its subkernel pair, whereas Q assigns to every parallel pair its coinserter. Thus $\mathcal{F}$-quotients are precisely the subregular epimorphisms in $\mathcal{A}$, and $\mathcal{F}$-kernels are the subkernel pairs.

Bourke and Garner call a morphism between finitely presentable objects of $[\mathcal{K}, \mathcal{V}]$ an $\mathcal{F}$-*congruence axiom* provided that it is orthogonal to every $\mathcal{F}$-kernel. Then an $\mathcal{F}$-*congruence* in $\mathcal{V}$ is an object of $[\mathcal{K}, \mathcal{V}]$ orthogonal to every $\mathcal{F}$-congruence axiom. Finally, an $\mathcal{F}$-*congruence in* $\mathcal{A}$ is an object of $[\mathcal{K}, \mathcal{A}]$ such that every hom-functor maps it to an $\mathcal{F}$-congruence in $\mathcal{V}$.

With our choice of $\mathcal{V}$ and $\mathcal{F}$ above, an $\mathcal{F}$-congruence in **Pos** is a parallel pair of morphisms of **Pos** whose orthogonality to all $\mathcal{F}$-congruence axioms means precisely that it is an order-reflexive and transitive relation. This can be verified analogously to Section

5.2 in [BG] where the special case that $\mathcal{V}$ is the category of small categories, and $\mathcal{F}$ is chosen so that $\mathcal{F}$-quotients are precisely the functors surjective on objects is presented. By using a completely analogous argument for our choice of $\mathcal{V}$ and $\mathcal{F}$ above, it follows that an $\mathcal{F}$-congruence in **Pos** is precisely a congruence in the sense of Kurz and Velebil.

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