Theory and Applications of Categories, Vol. 43, No. 2, 2025, pp. 23–38.

AN ELEMENTARY CHARACTERIZATION OF STABLY PRECOHESIVE GEOMETRIC MORPHISMS AS PARTICULAR PRECOHESIVE GEOMETRIC MORPHISMS

THOMAS STREICHER

Dedicated to the Memory of Bill Lawvere

Abstract. We consider various characterizations of stably precohesive geometric morphisms which are elementary in the sense that they avoid reference to concepts of relative category theory.

1. Introduction

Bill Lawvere to whom we owe among many other things the quintessential notion of elementary topos in his last decades has spent quite some effort on investigating a notion of *cohesive geometric morphism*. The aim of this concept is to axiomatize what is a "topos of spaces" over some base topos, see e.g. [LM15]. Cohesive geometric morphisms $F \dashv U : \mathcal{E} \to \mathcal{S}$ are defined as *precohesive* geometric morphims satisfying the further requirement that the canonical morphism $L(A^{FI}) \to L(A)^I$ is an isomorphism for all $I \in \mathcal{S}$ and $A \in \mathcal{E}$ where $L \dashv F$.

A geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is precohesive iff it is local and hyperconnected and F has a left adjoint L which preserves binary products. This is not verbatim Lawvere's original definition but equivalent to it as shown in [Jo11]. Recall that hyperconnected means that U preserves subobject classifiers and local means that F is full and faithful and U has a right adjoint R. As shown in [Jo11] a hyperconnected and local geometric morphism is precohesive iff F preserves exponentials (which as shown in Theorem 2 of [BP80] is equivalent to F having an $\mathcal{S}$-strong left adjoint L).

As shown in [LM15] a precohesive geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is *stably precohesive*, i.e. all its slices over some $I \in \mathcal{S}$ are precohesive, iff it is *molecular* (aka *locally connected*). This is obvious from the fact that a geometric morphism $F \dashv U$ is molecular iff one of the following equivalent conditions holds

(1) F has a left adjoint L fibered (or indexed) over $\mathcal{S}$

Managing Editor's note: unfortunately, Dr. Streicher passed away after this paper was accepted, but before it could be published in this special volume.

Received by the editors 2023-11-22 and, in final form, 2024-03-22.

Published on 2025-02-27 in the Lawvere Festschrift.

2020 Mathematics Subject Classification: 18B25.

Key words and phrases: toposes, geometric morphisms, locally connected, hyperconnected, local.

© Thomas Streicher, 2025. Permission to copy for private use granted.

(2) all slices of F preserve exponentials

(3) F preserves dependent products (i.e. right adjoints to pullback functors)

as shown in C3.3.1 of [Jo02]. Notice that condition (1) can be explicitated as the requirement that

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\ FJ \arrow[r, "Fu"'] & FI \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} LP \arrow[r, "Lq"] \arrow[d, "\widehat{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & LA \arrow[d, "\widehat{a}"] \\ J \arrow[r, "u"'] & I \end{tikzcd}$$

where we write $\widehat{a}$ for the upper transpose $\varepsilon_I \circ La : LA \to I$ of a (and similarly for p).

As shown by J.-L. Moens and M. Jibladze and explained in [Str23] geometric morphisms to $\mathcal{S}$ correspond to toposes fibered over $\mathcal{S}$ which have (internal) sums and are locally small from which it follows that the geometric morphism extends to a fibered one of the kind $\Delta \dashv \Gamma$. From this point of view it appears as most natural to require that the left adjoint of the inverse image part of the geometric morphism extends to a fibered left adjoint of Δ and, accordingly, stably precohesive appears as the more natural notion.

But in [LM15] the authors raise the question whether precohesive and stably precohesive coincide or there is a counterexample separating these two notions. Though we cannot answer this question in this paper we instead will provide alternative characterizations of stably precohesive geometric morphisms in terms of conditions which are "elementary" in the sense that they avoid any direct reference to concepts of the theory of fibered categories and thus might be easier to check.

2. The case of bireflective exponential ideals

Suppose $F \dashv U : \mathcal{E} \to \mathcal{S}$ is a geometric morphism which is connected, i.e. F is full and faithful, and which has a left adjoint L preserving binary products, i.e. $\mathcal{S}$ is a bireflective exponential ideal within $\mathcal{E}$ via F.

We will identify various conditions equivalent to $F \dashv U$ being molecular.

2.1. Theorem. *The geometric morphism $F \dashv U$ as above is molecular iff*

$$\begin{tikzcd} P \arrow[r, tail, "m"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\ FJ \arrow[r, tail, "Fn"'] & FI \end{tikzcd} \quad \textit{implies} \quad \begin{tikzcd} LP \arrow[r, tail, "Lm"] \arrow[d, "\widehat{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & LA \arrow[d, "\widehat{a}"] \\ J \arrow[r, tail, "n"'] & I \end{tikzcd}$$

for all monos $n : J \rightarrowtail I$ in $\mathcal{S}$.

PROOF. The condition given is an immediate consequence of the (explicitation of the) requirement that L is a fibered left adjoint of F as described in the introduction.

For the reverse direction we show that the condition implies that for all $I \in \mathcal{S}$ the left adjoint to $F/I : \mathcal{S}/I \to \mathcal{E}/FI$ preserves binary products. For this purpose suppose that

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & FI \end{tikzcd}$$

i.e.

$$\begin{tikzcd} P \arrow[r, tail, "\langle p,q \rangle"] \arrow[d, "d"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \times B \arrow[d, "f \times g"] \\ FI \arrow[r, tail, "F\delta_I"'] & FI \times FI \end{tikzcd}$$

where $pf = d = qg$. Then by the assumed condition it follows that

$$\begin{tikzcd} LP \arrow[r, tail, "\langle Lp, Lq \rangle"] \arrow[d, "\widehat{d}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & LA \times LB \arrow[d, "\widehat{f} \times \widehat{g}"] \\ I \arrow[r, tail, "\delta_I"'] & I \times I \end{tikzcd}$$

and thus

$$\begin{tikzcd} LP \arrow[r, "Lq"] \arrow[d, "Lp"'] \arrow[dr, phantom, very near start, "\lrcorner"] & LB \arrow[d, "Lg"] \\ LA \arrow[r, "Lf"'] & LFI \end{tikzcd}$$

since $\widehat{f} = \varepsilon_I \circ Lf$ and $\widehat{g} = \varepsilon_I \circ Lg$ and the counit ε of $L \dashv F$ is a natural isomorphism because F is full and faithful as follows from the assumption that the geometric morphism $F \dashv U$ is connected. ∎

Next we show that the previous theorem holds already when requiring the condition given there only for the cases where n is $\top_{\mathcal{S}}$.

2.2. Theorem. *The geometric morphism $F \dashv U$ is molecular iff*

$$\begin{tikzcd} P \arrow[r, tail, "m"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\ F1_{\mathcal{S}} \arrow[r, tail, "F\top_{\mathcal{S}}"'] & F\Omega_{\mathcal{S}} \end{tikzcd} \quad \textit{implies} \quad \begin{tikzcd} LP \arrow[r, tail, "Lm"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & LA \arrow[d, "\widehat{p}"] \\ 1_{\mathcal{S}} \arrow[r, tail, "\top_{\mathcal{S}}"'] & \Omega_{\mathcal{S}} \end{tikzcd}$$

Proof. The necessity of this condition follows from Theorem 2.1. The reverse direction follows since

$$\begin{tikzcd} P \arrow[r, tail, "m"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\ FJ \arrow[r, tail, "Fn"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & FI \arrow[d, "Fq"] \\ F1_{\mathcal{S}} \arrow[r, tail, "F\top_{\mathcal{S}}"'] & F\Omega_{\mathcal{S}} \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} LP \arrow[r, tail, "Lm"] \arrow[d, "\widehat{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & LA \arrow[d, "\widehat{a}"] \\ J \arrow[r, tail, "n"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "q"] \\ 1_{\mathcal{S}} \arrow[r, tail, "\top_{\mathcal{S}}"'] & \Omega_{\mathcal{S}} \end{tikzcd}$$

because our assumption guarantees the required implication for the outer rectangles, the lower squares are pullbacks anyway and thus the required implication holds for the upper squares by the usual pullback lemma. ∎

Recall from [BP80] that a morphism in $\mathcal{E}$ is called *$\mathcal{S}$-definable* if it can be obtained as pullback of Fu for some morphism u in $\mathcal{S}$. It is straightforward to check that a monomorphism $m : P \rightarrowtail A$ in $\mathcal{E}$ is $\mathcal{S}$-definable iff for some mono $n : J \rightarrowtail I$ in $\mathcal{S}$ it can be obtained as pullback of $F(n : J \rightarrowtail I)$ along some $a : A \to FI$. As already observed in the proof of the previous theorem m can be obtained as pullback of $F\top_{\mathcal{S}}$ along $Fq \circ a$ where $q : I \to \Omega_{\mathcal{S}}$ classifies n. Thus, a monomorphism in $\mathcal{E}$ is $\mathcal{S}$-definable iff can be obtained as pullback of $F\top_{\mathcal{S}}$.

Recall from [Jo80] that $F \dashv U$ is called *subopen* iff the classifying map $\tau : F\Omega_{\mathcal{S}} \to \Omega_{\mathcal{E}}$ for $F\top_{\mathcal{S}}$ is monic. Thus, if $F \dashv U$ is subopen then $F\top_{\mathcal{S}}$ classifies $\mathcal{S}$-definable monos in $\mathcal{E}$.

2.3. Lemma. *If $F \dashv U$ is molecular then it is subopen.*

Proof. Suppose $p_1, p_2 : A \to \Omega_{\mathcal{S}}$ with $\tau \circ p_1 = \tau \circ p_2$. Let $m : P \rightarrowtail A$ be the mono classified by this morphism. Then for $i = 1, 2$ we have

$$\begin{tikzcd}
P \arrow[r, tail, "m"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p_i"] \\
F1_{\mathcal{S}} \arrow[r, tail, "F\top_{\mathcal{S}}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F\Omega_{\mathcal{S}} \arrow[d, "\tau"] \\
1_{\mathcal{E}} \arrow[r, "\top_{\mathcal{E}}"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

from which it follows by Theorem 2.2 that $\widehat{p_1}$ and $\widehat{p_2}$ both classify the mono Lm. Thus $\widehat{p_1} = \widehat{p_2}$ from which it follows that $p_1 = p_2$. ∎

Next we give a further characterization of $\mathcal{S}$-definable monos under the assumption that the geometric morphism under consideration is essential.

2.4. Lemma. *Suppose $F \dashv U : \mathcal{E} \to \mathcal{S}$ is an essential geometric morphism, i.e. F has a left adjoint L. Then a mono $m : P \rightarrowtail A$ in $\mathcal{E}$ is $\mathcal{S}$-definable iff it appears as pullback along $\eta_A : A \to FLA$ of Fn for some mono $n : Q \rightarrowtail LA$ in $\mathcal{S}$.*

Proof. Obviously, the condition is sufficient. For the reverse direction suppose that m appears as pullback of $F\top_{\mathcal{S}}$ along some $p : A \to F\Omega_{\mathcal{S}}$. Let $n : Q \rightarrowtail LA$ be the mono classified by $\widehat{p} : LA \to \Omega_{\mathcal{S}}$. Then we have

$$\begin{tikzcd}
P \arrow[r, tail, "m"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\eta_A"] \\
FQ \arrow[r, "Fn"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & FLA \arrow[d, "F\widehat{p}"] \\
F1_{\mathcal{S}} \arrow[r, "F\top_{\mathcal{S}}"'] & F\Omega_{\mathcal{S}}
\end{tikzcd}$$

where the lower square is a pullback since F preserves pullbacks and the outer rectangle is a pulback since m arises as pullback of $F\top_{\mathcal{S}}$ along $p = F\widehat{p} \circ \eta_A$ and thus the upper square is a pullback as desired. ∎

For the rest of this section we resume the assumption that $F \dashv U$ is connected and F has a left adjoint preserving binary products.

2.5. Theorem. *The geometric morphism $F \dashv U$ is molecular iff it is subopen and for all $\mathcal{S}$-definable monos $m : P \rightarrowtail A$ in $\mathcal{E}$ the map Lm is a mono in $\mathcal{S}$ and*

$$\begin{tikzcd}
P \arrow[r, "m"] \arrow[d, "\eta_P"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\eta_A"] \\
FLP \arrow[r, "FLm"'] & FLA
\end{tikzcd}$$

is a pullback in $\mathcal{E}$.

Proof. For the forward direction suppose $F \dashv U$ is molecular. Then it is also subopen by Lemma 2.3. Suppose $m : P \rightarrowtail A$ is $\mathcal{S}$-definable. Then m appears as pullback of $F\top_{\mathcal{S}}$ along a unique map $p : A \to F\Omega_{\mathcal{S}}$. Thus, by Theorem 2.2 it holds that

$$\begin{tikzcd}
LP \arrow[r, tail, "Lm"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & LA \arrow[d, "\widehat{p}"] \\
1_{\mathcal{S}} \arrow[r, "\top_{\mathcal{S}}"'] & \Omega_{\mathcal{S}}
\end{tikzcd}$$

from which it follows that in

$$\begin{tikzcd}
P \arrow[r, tail, "m"] \arrow[d, "\eta_P"'] & A \arrow[d, "\eta_A"] \\
FLP \arrow[r, tail, "FLm"] \arrow[d] & FLA \arrow[d, "F\widehat{p}"] \\
F1_{\mathcal{S}} \arrow[r, tail, "F\top_{\mathcal{S}}"'] & F\Omega_{\mathcal{S}}
\end{tikzcd}$$

the lower square is a pullback since F preserves pullbacks and the outer rectangle is a pulback since $p = F\widehat{p} \circ \eta_A$ classifies m. Accordingly, the upper square is a pullback square as well.

For the backwards direction suppose $F \dashv U$ is subopen and L sends $\mathcal{S}$-definable monos in $\mathcal{E}$ to monos in $\mathcal{S}$ such that the naturality square for η is a pullback square for $\mathcal{S}$-definable monos m. For showing that $F \dashv U$ is molecular we verify the criterion provided by Theorem 2.2.

For this purpose suppose $m : P \rightarrowtail A$ appears as pullback of $F\top_{\mathcal{S}}$ along some (necessarily unique) map $p : A \to F\Omega_{\mathcal{S}}$. Then Lm is monic. Thus, there exists a unique map $q : LA \to \Omega_{\mathcal{S}}$ classifying Lm. Then we have

$$\begin{tikzcd}
P \arrow[r, tail, "m"] \arrow[d, "\eta_P"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\eta_A"] \\
FLP \arrow[r, tail, "FLm"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & FLA \arrow[d, "Fq"] \\
F1_{\mathcal{S}} \arrow[r, tail, "F\top_{\mathcal{S}}"'] & F\Omega_{\mathcal{S}}
\end{tikzcd}$$

from which it follows that $p = Fq \circ \eta_A$ and thus $q = \widehat{p}$. Thus Lm appears as pullback of $\top_{\mathcal{S}}$ along $\widehat{p}$ as desired. ∎

2.6. Lemma. *If $F \dashv U$ is molecular then $m \mapsto Lm$ and $n \mapsto \eta_A^* Fn$ establishes a 1-1-correspondence between $\mathcal{S}$-definable subobjects m of A and arbitrary subobjects n of LA.*

Proof. Suppose $n : Q \rightarrowtail LA$ is classified by $q : LA \to \Omega_{\mathcal{S}}$. We have to show that n and $L(\eta_A^* Fn)$ are isomorphic as subobjects of LA. For this purpose consider

$$\begin{tikzcd}
P \arrow[r, tail, "m"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\eta_A"] \\
FQ \arrow[r, tail, "Fn"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & FLA \arrow[d, "Fq"] \\
F1_{\mathcal{S}} \arrow[r, tail, "F\top_{\mathcal{S}}"'] & F\Omega_{\mathcal{S}}
\end{tikzcd}$$

from which it follows by Theorem 2.1 that

$$\begin{tikzcd} LP \arrow[r, "Lm"] \arrow[d, "\widehat{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & LA \arrow[d, equal] \\ Q \arrow[r, tail, "n"'] & LA \end{tikzcd}$$

and thus Lm and n are isomorphic as subobjects of LA via $\widehat{p}$, which is an isomorphism since it arises as pullback of id_{LA}.

Suppose $m : P \rightarrowtail A$ is $\mathcal{S}$-definable. Then $m \cong \eta_A^* FLm$ follows from Theorem 2.5. ∎

In [GS21] one can find examples of local geometric morphism having a further left adjoint preserving finite products but nevertheless are not molecular. However, none of these counterexamples is hyperconnected for which reason the question of Lawvere and Menni still remains open.

Nevertheless assuming $F \dashv U$ to be hyperconnected or even precohesive allows us to improve some of our results as we will see in the next section.

3. Exploiting the additional assumption that $F \dashv U$ is precohesive

From now on we assume that the geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ under consideration is not only connected with F having a finite product preserving left adjoint L but also that it is hyperconnected, i.e. fullfills one of the following equivalent conditions

(1) U preserves subobject classifiers

(2) for all $I \in \mathcal{S}$ the functor $F/I : \mathcal{S}/I \to \mathcal{E}/FI$ restricts to an equivalence between $\mathsf{Sub}_{\mathcal{S}}(I)$ and $\mathsf{Sub}_{\mathcal{E}}(FI)$

(3) all units and counits of $F \dashv U$ are monic.

See A4.6.6 of [Jo02] for a proof and further characterizations. The intuition is that hyperconnected is the opposite of localic in the sense that these two classes form a factorization system on geometric morphisms in the appropriate 2-categorical sense.

Since $U\top_{\mathcal{E}} \cong \top_{\mathcal{S}}$ we have

$$\begin{tikzcd} FU1_{\mathcal{E}} \arrow[r, tail, "FU\top_{\mathcal{E}}"] \arrow[d, "\varepsilon_{1_{\mathcal{E}}}"', "\cong"] \arrow[dr, phantom, "\lrcorner", very near start] & FU\Omega_{\mathcal{E}} \arrow[d, tail, "\varepsilon_{\Omega_{\mathcal{E}}}"] \\ 1_{\mathcal{E}} \arrow[r, tail, "\top_{\mathcal{E}}"'] & \Omega_{\mathcal{E}} \end{tikzcd}$$

with $\varepsilon_{\Omega_\mathcal{E}}$ being monic because the counit of a hyperconnected geometric morphism is monic. Since $FU\top_\mathcal{E} \cong F\top_\mathcal{S}$ it follows that $\varepsilon_{\Omega_\mathcal{E}} \cong \tau$ and thus τ is monic. Thus hyperconnected geometric morphisms are necessarily subopen. Actually, by Corollary C3.1.9(i) of [Jo02] they are even open.

Thus, under the current assumptions a mono $m : P \rightarrowtail A$ is $\mathcal{S}$-definable iff its classifying map $\chi : A \to \Omega_\mathcal{E}$ factors through $\tau : F\Omega_\mathcal{S} \rightarrowtail \Omega_\mathcal{E}$ via a unique map $p : A \to F\Omega_\mathcal{S}$ whose transpose $\widehat{p} : LA \to \Omega_\mathcal{S}$ classifies the unique mono $n : Q \rightarrowtail LA$ with $m \cong \eta_A^* Fn$. Without the assumption that $F \dashv U$ is molecular it is, however, not clear at all why this n should be isomorphic to Lm nor is it clear why the additional assumption of $F \dashv U$ being local should ensure this.

For the rest of this section we further assume that $F \dashv U$ is precohesive, i.e. also hyperconnected and local.

3.1. Theorem. *Let $F \dashv U : \mathcal{E} \to \mathcal{S}$ be a precohesive geometric morphism. Thus F has a left adjoint L. Then $n \mapsto \eta_A^* Fn$ establishes a 1-1-correspondence between subobjects of LA in $\mathcal{S}$ and $\mathcal{S}$-definable subobjects of A in $\mathcal{E}$. Its inverse is given by sending an $\mathcal{S}$-definable subobject m of A in $\mathcal{E}$ to the image of Lm in $\mathcal{S}$.*

Proof. Since $F \dashv U$ is precohesive it follows by Cor. 2.5 in [Jo11] that for all $A \in \mathcal{E}$ the unit map $\eta_A : A \to FLA$ is epic.

Since $F \dashv U$ is hyperconnected it is in particular subopen and thus a subobject of A is $\mathcal{S}$-definable iff its characteristic map factors through the mono $\tau : F\Omega_\mathcal{S} \rightarrowtail \Omega_\mathcal{E}$ as argued immediately before Lemma 2.3.

Suppose $p : A \to F\Omega_\mathcal{S}$. We write $n : Q \rightarrowtail LA$ for the mono classified by $\widehat{p}$ and $m : P \rightarrowtail A$ for the $\mathcal{S}$-definable mono classified by $p = F\widehat{p} \circ \eta_A$. Thus, the two squares

$$\begin{tikzcd} P \arrow[r, tail, "m"] \arrow[d] & A \arrow[d, "p"] \\ F1_\mathcal{S} \arrow[r, "F\top_\mathcal{S}"'] & F\Omega_\mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, "n"] \arrow[d] & LA \arrow[d, "\widehat{p}"] \\ 1_\mathcal{S} \arrow[r, "\top_\mathcal{S}"'] & \Omega_\mathcal{S} \end{tikzcd}$$

are both pullbacks. Since F as a right adjoint preserves pullbacks the square

$$\begin{tikzcd} FQ \arrow[r, "Fn"] \arrow[d] & FLA \arrow[d, "F\widehat{p}"] \\ F1_\mathcal{S} \arrow[r, "F\top_\mathcal{S}"'] & F\Omega_\mathcal{S} \end{tikzcd}$$

is a pullback, too. Let e be the unique arrow making the following diagram

$$
\begin{tikzcd}
P \arrow[r, tail, "m"] \arrow[d, "e"'] & A \arrow[d, "\eta_A"] \\
FQ \arrow[r, tail, "Fn"] \arrow[d] & FLA \arrow[d, "F\widehat{p}"] \\
F1_{\mathcal{S}} \arrow[r, tail, "F\top_{\mathcal{S}}"'] & F\Omega_{\mathcal{S}}
\end{tikzcd}
$$

commute. It exists since the lower square and the outer rectangle are pullbacks. Thus the upper square is a pullback as well. Thus e is epic since η_A is epic and epis are stable under pullbacks along arbitrary morphisms. Another consequence is that m is the pullback of Fn along η_A. Moreover, from $\eta_A \circ m = Fn \circ e$ it follows by taking the upper transpose of both sides w.r.t. $L \dashv F$ that $Lm = \mathrm{id}_{LA} \circ Lm = \widehat{\eta_A} \circ Lm = n \circ \widehat{e}$. Since left adjoints preserve epis Le is an epi and since ε_Q is an iso it follows that $\widehat{e} = \varepsilon_Q \circ Le$ is an epi. Thus, from $Lm = n \circ \widehat{e}$ it follows that n is the image of Lm. ■

Based on this result we can give the following characterization of stably precohesive geometric morphism as precohesive ones validating a simple preservation property.

3.2. Theorem. *A precohesive geometric morphism $F \dashv U$ is stably precohesive, i.e. also molecular, iff the left adjoint L of F sends $\mathcal{S}$-definable monos in $\mathcal{E}$ to monos in $\mathcal{S}$.*

Proof. Suppose $F \dashv U$ is precohesive.

If $F \dashv U$ is molecular then it follows from Theorem 2.1 that L sends $\mathcal{S}$-definable monos in $\mathcal{E}$ to monos in $\mathcal{S}$.

For the backward direction suppose F has a left adjoint L sending $\mathcal{S}$-definable monos in $\mathcal{E}$ to monos in $\mathcal{S}$. For showing that $F \dashv U$ is molecular we will verify the condition given in Thm 2.2. For this purpose suppose

$$
\begin{tikzcd}
P \arrow[r, tail, "m"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
F1_{\mathcal{S}} \arrow[r, tail, "F\top_{\mathcal{S}}"'] & F\Omega_{\mathcal{S}}
\end{tikzcd}
$$

for some $p : A \to F\Omega_{\mathcal{S}}$. Then we have

$$\begin{tikzcd} P \arrow[r, tail, "m"] \arrow[d, "e"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\eta_A"] \\ FQ \arrow[r, tail, "Fn"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & FLA \arrow[d, "F\widehat{p}"] \\ F1\mathcal{S} \arrow[r, tail, "F\top_{\mathcal{S}}"'] & F\Omega_{\mathcal{S}} \end{tikzcd} \quad \text{where} \quad \begin{tikzcd} Q \arrow[r, tail, "n"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & LA \arrow[d, "\widehat{p}"] \\ 1_{\mathcal{S}} \arrow[r, tail, "\top_{\mathcal{S}}"'] & \Omega_{\mathcal{S}} \end{tikzcd}$$

since $p = F\widehat{p} \circ \eta_A$ and F as a right adjoint preserves pullbacks. Transposing the left rectangle according to $L \dashv F$ we obtain

$$\begin{tikzcd} LP \arrow[r, tail, "Lm"] \arrow[d, "\widehat{e}"'] & LA \arrow[d, equal] \\ Q \arrow[r, tail, "n"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & LA \arrow[d, "\widehat{p}"] \\ 1_{\mathcal{S}} \arrow[r, tail, "\top_{\mathcal{S}}"'] & \Omega_{\mathcal{S}} \end{tikzcd}$$

since $\widehat{e} = \varepsilon_Q \circ Le$ is epic because Le is epic since L as a left adjoint preserves epis and ε_Q is an iso. But $\widehat{e}$ is also monic since $Lm = n \circ \widehat{e}$ is monic due to the assumption that L sends $\mathcal{S}$-definable monos in $\mathcal{E}$ to monos in $\mathcal{S}$. Thus $\widehat{e}$ is an isomorphism from which it follows by the previous diagram that

$$\begin{tikzcd} LP \arrow[r, "Lm"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & LA \arrow[d, "\widehat{p}"] \\ 1_{\mathcal{S}} \arrow[r, tail, "\top_{\mathcal{S}}"'] & \Omega_{\mathcal{S}} \end{tikzcd}$$

is a pullback as required by the condition given in Theorem 2.2. ∎

One calls a geometric morphism stably precohesive iff it is precohesive and molecular. For precohesive geometric morphisms by Theorem 3.1 for $\mathcal{S}$-definable subobjects m of A the (unique up to iso) subobject n of LA with m isomorphic to $\eta_A^* Fn$ is obtained as the image of Lm whereas for stably precohesive geometric morphism by Theorem 3.2 this n is isomorphic to Lm in $\mathcal{S}/LA$. Thus, if one does not adopt the fibered point of view the requirement of being stably precohesive may appear as somewhat *ad hoc* and thus presumably will not hold automatically for all precohesive geometric morphisms!

4. Relation to an alternative characterization of being molecular

First we give an alternative characterization of being molecular.

4.1. Theorem. *A geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is molecular iff F has a left adjoint L such that*

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & FI \arrow[d, "Fu"] \\ A \arrow[r, "\eta_A"'] & FLA \end{tikzcd} \quad \textit{implies} \quad \begin{tikzcd} LP \arrow[r, "\widehat{q}", "\cong"'] \arrow[d, "Lp"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "u"] \\ LA \arrow[r, equal] & LA \end{tikzcd}$$

for all $u : I \to LA$ in $\mathcal{S}$.

Proof. Obviously the condition is necessary.

For showing it is also sufficient suppose

$$\begin{tikzcd} P \arrow[r, "b"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & FK \arrow[d, "Fv"] \\ A \arrow[r, "a"'] & FJ \end{tikzcd}$$

for some $v : K \to J$. There is a pullback

$$\begin{tikzcd} I \arrow[r, "w"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "v"] \\ LA \arrow[r, "\widehat{a}"'] & J \end{tikzcd}$$

in $\mathcal{S}$. Since F is a right adjoint it sends this pullback to a pullback in $\mathcal{E}$. Let $q : P \to FI$ be the unique mediating arrow with $Fu \circ q = \eta_A \circ p$ and $Fw \circ q = b$. Then we have

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & FI \arrow[r, "Fw"] \arrow[d, "Fu"'] \arrow[dr, phantom, "\lrcorner", very near start] & FK \arrow[d, "Fv"] \\ A \arrow[r, "\eta_A"'] & FLA \arrow[r, "F\widehat{a}"'] & FJ \end{tikzcd}$$

since F preserves pullbacks and the rectangle is a pullback by assumption from which it follows by the usual pullback lemma that the left square is a pullback, too. Transposing the previous diagram w.r.t. the adjunction $L \dashv F$ we obtain

$$\begin{tikzcd} LP \arrow[r, "\widehat{q}"] \arrow[d, "Lp"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[r, "w"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "v"] \\ LA \arrow[r, equal] & LA \arrow[r, "\widehat{a}"'] & J \end{tikzcd}$$

where $w \circ \widehat{q} = \widehat{b}$. Thus

$$\begin{tikzcd} LP \arrow[r, "\widehat{b}"] \arrow[d, "Lp"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "v"] \\ LA \arrow[r, "\widehat{a}"'] & J \end{tikzcd}$$

as required. ∎

Notice that for an essential geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ with $L \dashv F$ we have $L/A \dashv \eta_A^* \circ F/LA$ for all objects A in $\mathcal{E}$ (see Theorem 16.4 of [Str23] for an explicitation of the units and counits of this adjunction which also works when the left adjoint is not required to preserve finite limits). The right adjoint is full and faithful iff the counit of this adjunction is an iso. That the latter holds for all objects A in $\mathcal{E}$ is obviously equivalent to the condition given in Theorem 4.1 since the square on the right is the counit of $L/A \dashv \eta_A^* \circ F/LA$ at u. Thus $F \dashv U$ is molecular iff $\eta_A^* \circ F/LA$ is full and faithful for all A in $\mathcal{E}$. This condition is given in C3.3.5(ii) of [Jo02] but we think our proof is more transparent.

Thus, for molecular geometric morphisms $F \dashv U : \mathcal{E} \to \mathcal{S}$ the $\mathcal{S}$-definable maps to A form a full reflective subcategory of $\mathcal{E}/A$. Thus, a map $f : B \to A$ in $\mathcal{E}$ is $\mathcal{S}$-definable iff the unit of $L/A \dashv \eta_A^* \circ F/LA$ at f is an isomorphism, i.e.

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & FLB \arrow[d, "FLf"] \\ A \arrow[r, "\eta_A"'] & FLA \end{tikzcd}$$

is a pullback. This reflection restricts to an equivalence between $\mathcal{S}$-definable maps to A in $\mathcal{E}$ and maps to LA in $\mathcal{S}$ and thus, in particular, to an equivalence between $\mathcal{S}$-definable subobjects of A and ordinary subobjects of LA.

Accordingly, as suggested by M. Menni in private communication, showing that $F \dashv U$ is not molecular amounts to exhibiting an A in $\mathcal{E}$ such that $\eta_A^* \circ F/LA$ is not full and faithful. However, so far we have not been inventive enough to find such a thing. The main reason for this failure is that we only know examples of precohesive geometric morphisms to boolean base toposes (or whose construction is analogous to such ones) and these are all necessarily stably precohesive as shown in [Men22].

5. For precohesive geometric morphisms $F \dashv U : \mathcal{E} \to \mathcal{S}$ the $\mathcal{S}$-definable subobjects of A are a full reflective subcategory of $\mathcal{E}/A$ canonically equivalent to $\mathcal{S}/LA$

Let $F \dashv U : \mathcal{E} \to \mathcal{S}$ be a precohesive geometric morphism and $L \dashv F$. For objects A of $\mathcal{E}$ let $i_A : \mathsf{Sub}_{\mathcal{S}}(LA) \hookrightarrow \mathcal{S}/LA$ be the inclusion of subobjects of LA into the slice $\mathcal{S}/LA$. This functor has a left adjoint r_A sending maps to LA to their image in $\mathcal{S}$.

5.1. Theorem. *Let $F \dashv U : \mathcal{E} \to \mathcal{S}$ be a precohesive geometric morphism and L be a left adjoint of F. Then for every object A of $\mathcal{E}$ the functor $\eta_A^* \circ F/LA \circ i_A : \mathsf{Sub}_{\mathcal{S}}(LA) \to \mathcal{E}/A$ is full and faithful with left adjoint $r_A \circ L/A$ and thus the $\mathcal{S}$-definable subobjects of A form a full reflective subcategory of $\mathcal{E}/A$.*

Proof. First recall that the counit of $L/A \dashv \eta_A^* \circ F/LA$ at $u : I \to LA$ is given by $\widehat{q} : Lp \to u$ where

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & FI \arrow[d, "Fu"] \\ A \arrow[r, "\eta_A"'] & FLA \end{tikzcd}$$

and $\widehat{q} = \varepsilon_I \circ Lq$. As already observed at the beginning of the proof of Theorem 3.1 by Cor. 2.5 of [Jo11] the map η_A is epic since $F \dashv U$ is precohesive. Notice that $\widehat{q}$ is an epi since L as a left adjoint preserves epis and ε_I is an iso. Thus the functor r_A sends the counit $\widehat{q}$ at u in $\mathcal{S}/LA$ to an iso in $\mathsf{Sub}_{\mathcal{S}}(LA)$. Thus, since all counits of $r_A \dashv i_A$ are isos the counit of $r_A \circ L/A \dashv \eta_A^* \circ F/LA \circ i_A$ at u is an iso as desired. ∎

6. Conclusion

We have shown in Theorem 3.1 that for a precohesive geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ with $L \dashv F$ it holds that

(1) $F^*\top_{\mathcal{S}}$ classifies $\mathcal{S}$-definable monos in $\mathcal{E}$ and

(2) for $p : A \to F\Omega_{\mathcal{S}}$ from the subobject n of LA classified by the upper transpose $\widehat{p} : LA \to \Omega_{\mathcal{S}}$ the corresponding $\mathcal{S}$-definable subobject m of A classified by p may be constructed as $\eta_A^* Fn$ and

(3) n is the image of Lm.

In the subsequent Theorem 3.2 we have shown that $F \dashv U$ is stably precohesive iff L sends $\mathcal{S}$-definable monos in $\mathcal{E}$ to monos in $\mathcal{S}$. For such geometric morphisms a mono $m : P \rightarrowtail A$ is $\mathcal{S}$-definable iff

$$
\begin{tikzcd}
P \arrow[r, tail, "m"] \arrow[d, "\eta_P"'] & A \arrow[d, "\eta_A"] \\
FLP \arrow[r, "FLm"'] & FLA
\end{tikzcd}
$$

is a pullback square.

In section 4 following a suggestion of M. Menni we have observed that a precohesive geometric morphism $F \dashv U$ fails to be stably precohesive iff for some A in $\mathcal{E}$ the functor $\eta_A^* \circ F/LA : \mathcal{S}/LA \to \mathcal{E}/A$ fails to be full and faithful.

Finally, in section 5 we have shown in Theorem 5.1 that for precohesive geometric morphisms $F \dashv U : \mathcal{E} \to \mathcal{S}$ the $\mathcal{S}$-definable subobjects of A form a full reflective subcategory of $\mathcal{E}/A$ canonically equivalent to $\mathcal{S}/LA$.

Acknowledgements

I am most grateful to M. Menni for bringing the questions discussed in this paper to my attention and for the many subsequent discussions we had on this topic over the last few years.

Many thanks also to J. Funk, R. Garner and J. Hemelaer for the discussions we have had on questions related to this paper.

Finally I want to thank an anonymous referee for a most careful reading of the draft version thus helping us to eliminate many typos, other little mistakes and some infelicities of formulation.

References

[BP80] M. Barr, R. Paré *Molecular Toposes* Journal of Pure and Applied Algebra, 17(2):127-152, 1980.

[GS21] R. Garner, T. Streicher *An essential geometric morphism which is not locally connected although its inverse image is an exponential ideal* Theory Appl. Categ. 37:908–913, 2021.

[Jo80] P. T. Johnstone *Open maps of toposes* Manuscripta Math. 31:217-247, 1980.

[Jo02] P. T. Johnstone *Sketches of an Elephant. A Compendium of Topos Theory.* 2. vols. Oxford University Press (2002).

[Jo11] P. T. Johnstone *Remarks on Punctual Local Connectedness* Theory Appl. Categ. 25:51–63, 2011.

[LM15] F. W. Lawvere, M. Menni *Internal choice holds in the discrete part of any cohesive topos satisfying stable connected codiscreteness* Theory Appl. Categ. 30:909-932, 2015.

[Men22] M. Menni *Decidable Objects and Molecular Toposes* Rev. Un. Mat. Argentina (2022)

[Str23] T. Streicher *Fibered Categories à la Jean Bénabou* (2023) available from arXiv:1801.02927

This article may be accessed at http://www.tac.mta.ca/tac/

THEORY AND APPLICATIONS OF CATEGORIES will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

Subscription information Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. Full text of the journal is freely available at http://www.tac.mta.ca/tac/.

Information for authors LaTeX2e is required. Articles may be submitted in PDF by email directly to a Transmitting Editor following the author instructions at
http://www.tac.mta.ca/tac/authinfo.html.

Managing editor. Geoff Cruttwell, Mount Allison University: gcruttwell@mta.ca

TeXnical editor. Michael Barr, McGill University: michael.barr@mcgill.ca

Assistant TeX editor. Gavin Seal, Ecole Polytechnique Fédérale de Lausanne:
gavin_seal@fastmail.fm

Transmitting editors.
Clemens Berger, Université Côte d'Azur: clemens.berger@univ-cotedazur.fr
Julie Bergner, University of Virginia: jeb2md (at) virginia.edu
Richard Blute, Université d' Ottawa: rblute@uottawa.ca
John Bourke, Masaryk University: bourkej@math.muni.cz
Maria Manuel Clementino, Universidade de Coimbra: mmc@mat.uc.pt
Valeria de Paiva, Topos Institute: valeria.depaiva@gmail.com
Richard Garner, Macquarie University: richard.garner@mq.edu.au
Ezra Getzler, Northwestern University: getzler (at) northwestern(dot)edu
Rune Haugseng, Norwegian University of Science and Technology: rune.haugseng@ntnu.no
Dirk Hofmann, Universidade de Aveiro: dirk@ua.pt
Joachim Kock, Universitat Autònoma de Barcelona: Joachim.Kock (at) uab.cat
Stephen Lack, Macquarie University: steve.lack@mq.edu.au
Tom Leinster, University of Edinburgh: Tom.Leinster@ed.ac.uk
Sandra Mantovani, Università degli Studi di Milano: sandra.mantovani@unimi.it
Matias Menni, Conicet and Universidad Nacional de La Plata, Argentina: matias.menni@gmail.com
Giuseppe Metere, Università degli Studi di Palermo: giuseppe.metere (at) unipa.it
Kate Ponto, University of Kentucky: kate.ponto (at) uky.edu
Robert Rosebrugh, Mount Allison University: rrosebrugh@mta.ca
Jiri Rosický, Masaryk University: rosicky@math.muni.cz
Giuseppe Rosolini, Università di Genova: rosolini@unige.it
Michael Shulman, University of San Diego: shulman@sandiego.edu
Alex Simpson, University of Ljubljana: Alex.Simpson@fmf.uni-lj.si
James Stasheff, University of North Carolina: jds@math.upenn.edu
Tim Van der Linden, Université catholique de Louvain: tim.vanderlinden@uclouvain.be
Christina Vasilakopoulou, National Technical University of Athens: cvasilak@math.ntua.gr