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WEIL 2-RIGS

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Abstract. Among commutative unital semirings (rigs, for short), we call Weil the ones that have a unique homomorphism into the initial algebra. Weil rigs can be thought of as coordinate algebras of spaces with a single point. In the category of additively idempotent rigs (2-rigs, for short) 2 is the initial algebra. We characterize Weil 2-rigs as those that have a unique saturated prime ideal and provide an axiomatization thereof in geometric logic. We further prove that the category of Weil 2-rigs is a co-reflective full subcategory of the category of 2-rigs. Finally, we show that both the varieties of rigs, 2-rigs and integral rigs are generated by finite rigs with a unique homomorphism into 2.

1. Introduction

The age-old quest for a suitable mathematical notion of "space" has seen William Lawvere among its finest modern contributors. After providing an elementary characterization of Grothendieck's notion of a category "of spaces", Lawvere proceeded to develop an axiomatic theory of these categories that could accommodate the different models of dynamical mathematical theories [6]. A fundamental property in this framework is *extensivity*. A category C with finite coproducts is **extensive** [1] if the canonical functor $\mathsf{C}/X \times \mathsf{C}/Y \to \mathsf{C}/(X+Y)$ is an equivalence for every pair of objects X, Y in C. For instance, the category of topological spaces and continuous functions between them is extensive; the category of groups is not. Extensivity describes a basic property of coproducts in categories "of spaces", with many interesting consequences.

In [7] Lawvere notices that the category Rig of *rigs* (i.e., commutative and unital semirings) is coextensive (i.e., dual to an extensive category, see also [2]). In light of the above considerations, one can expect that the category of rigs should have many things in common with the category of k-algebras for an algebraically closed field k and that it can be a useful conceptual guide to think of the category $\mathsf{Rig}^{\mathrm{op}}$ as a category "of spaces". This leads us to the concept of *Weil rigs*, which we are about to introduce.

Let C be a category with a terminal object 1. If X is an object of C, a *point* of X is an arrow $1 \to X$. It is well known that there may exist several non-isomorphic objects with only one point. As proved in [8], the finitely generated local complex algebras that have a unique homomorphism into the (initial algebra) $\mathbb{C}$ are the so-called *Weil algebras*; intuitively, these are function algebras of spaces with a single point (see also [4, p. 260]).

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In this paper, we present two characterizations of Weil 2-rigs: the first is in terms of its prime ideals, the second is an explicit description in geometric logic (see Theorem 3.5).

In a category of algebras in a finitary language, Birkhoff's subdirect representation theorem ensures that the maps into subdirectly irreducible objects are jointly monic. Dually, in a category of spaces, points are often not jointly epic; yet it may happen that maps whose domains have exactly one point are jointly epic. It would then be natural to expect a strong connection between Weil and subdirectly irreducible algebras. Surprisingly, the two classes are different as we show in Remark 3.2. However, in Theorem 5.7 we show that if an equation fails in the variety of 2-rigs, then there exists a finite Weil 2-rig in which it fails; a similar result also holds for rigs (see Theorem 4.10).

As a further example of a fruitful geometry-inspired concept, consider the following one. An arrow $f\colon X \to Y$ in a category C is called **constant** if it factors through the terminal object 1. More generally, an arrow $f\colon X \to Y$ is called **pseudo-constant** if it coequalizes all the points of X. That is, for every pair of points $a, b\colon 1 \to X$,

$$1 \underset{a}{\overset{b}{\rightrightarrows}} X \xrightarrow{\ f\ } D$$

one has $f(a) = f(b)$. Of course, every constant arrow is a pseudo-constant and in the category of sets the two concepts are equivalent. However, in $2\mathsf{Rig}^{\mathrm{op}}$ this equivalence fails. Yet, in Corollary 3.9 we show that the homomorphisms in $2\mathsf{Rig}$ that are dual to pseudo-constant functions always factor through a Weil 2-rig.

2. Preliminaries

For the purposes of this article, we understand a **semiring** to be an algebra $\langle A, +, \cdot, 0\rangle$ such that $\langle A, +, 0\rangle$ is a commutative monoid, $\cdot$ is associative and distributes over $+$, i.e., $x \cdot (y + z) \approx (x \cdot y) + (x \cdot z)$ and $(y + z) \cdot x \approx (y \cdot x) + (z \cdot x)$ hold, and 0 is multiplicatively absorbing, i.e., $0 \cdot x \approx 0$ holds.

A semiring is called **commutative** if $\cdot$ is commutative, **unital** if it has a neutral element 1 for $\cdot$, **idempotent** if $+$ is idempotent ($x + x \approx x$), and **integral** if it satisfies the identity $(x \cdot y) + y \approx y$. We call **rig** any commutative unital semiring and **2-rig** any idempotent rig.[1] By distributivity, idempotency is equivalent to satisfying $1 + 1 \approx 1$. An **irig** is an integral rig, which is equivalent to stipulating that 1 be additively absorbing: $1 + x \approx 1$. It is therefore immediate that every irig is a 2-rig.

2.1. Notation. We suppress multiplication and always assume that concatenation binds tighter than sum, i.e., $xy + z = (x \cdot y) + z$. We recursively define $\underline{n}x$ by setting $\underline{0}x := 0$ and $\underline{n+1}x := x + \underline{n}x$; when no confusion may arise, we simply write nx. Similarly, x^n is recursively defined by $x^0 := 1$ and $x^{n+1} := x \cdot x^n$.

[1] Similar to the pun that a *rng* is "a **r**ing without **i**dentity", the nomenclature *rig* serves to evoke "a **ri**ng without **n**egatives" (cf., [2, §1]).

2.2. Example. The natural numbers with the usual operations $\langle \mathbb{N}, +, \cdot, 0, 1\rangle$ forms a rig, which we call the **standard rig** and denote it simply by $\mathbb{N}$. The two-element distributive lattice $\langle \{0,1\}, \max, \min, 0, 1\rangle$ is an example of a 2-rig, which we denote by $\mathsf{2}$. Notice that $\mathbb{N}$ is the initial object in Rig and $\mathsf{2}$ is the initial object in $\mathsf{2Rig}$.

Recall that a class of algebras sharing the same finitary signature is called a **variety** if it is closed under direct products, subalgebras, and homomorphic images. By a celebrated result of Birkhoff, varieties are exactly the equationally axiomatizable classes of algebras. Therefore, the classes of rigs, 2-rigs and irigs are varieties, also denoted by Rig, $\mathsf{2Rig}$, and iRig, respectively.

2.3. Definition. *In every monoid $\langle M, \circ, 0\rangle$ it remains defined a (*natural*) preorder:*

$$a \preceq b \iff \exists m \in M \text{ such that } a \circ m = b.$$

When we use this relation on rigs, we always assume $\circ = +$.

Distributivity implies that, in a rig R, $\preceq$ is compatible with $+$ and $\cdot$, and thus homomorphisms preserve $\preceq$. In any 2-rig $\preceq$ is a partial order, as $x + r = y$ if and only if $x + y = y$. In other words, $\langle R, \preceq, 0\rangle$ is a join-semilattice with least element 0, for any 2-rig R. Notice that in any irig 1 is the top of this order.

If $\star$ is a binary operation on a set X and $S, T \subseteq X$, we define $S \star T := \{s \star t \mid s \in S, t \in T\}$, called the **complex** $\star$ of S and T. For $x \in X$, we abbreviate: $S \star x := S \star \{x\}$. In accordance with our previous conventions, we write Rx for $R \cdot \{x\}$.

2.4. Definition. *Let R be a rig. An* ***ideal*** *of R is a subset $I \subseteq R$ that satisfies the following conditions.*

(I1) $0 \in I$.

(I2) $RI \subseteq I$; *i.e., $r \in R$ and $i \in I$ implies $ri \in I$.*

(I3) $I + I \subseteq I$; *i.e., $i \in I$ and $i' \in I$ implies $i + i' \in I$.*

Dually, a subset $F \subseteq R$ is called a ***filter*** *if it satisfies the following conditions.*

(F1) $1 \in F$.

(F2) $R + F \subseteq F$; *i.e., $r \in R$ and $a \in F$ implies $r + a \in F$.*

(F3) $F \cdot F \subseteq F$; *i.e., $a \in F$ and $b \in F$ implies $ab \in F$.*

It is easy to see that the set $\{0\}$ is always an ideal, and the set $\uparrow 1 := R + 1$ is always a filter. Clearly R is itself both an ideal and a filter. Ideals and filters are called **proper** if they are not equal to R.

Of course, ideals and filters are closed under arbitrary intersections, so it makes sense to define for any $S \subseteq R$ the **ideal generated by** S as the smallest ideal containing S, denoted $\mathrm{Id}(S)$; similarly, the **filter generated by** S is the smallest filter containing S, denoted $\mathrm{Fi}(S)$. The proof of the following lemma is routine.

2.5. Lemma. *Let R be a rig and $S \subseteq R$. Then*

$$\mathrm{Id}(S) = \{r_1 s_1 + \cdots + r_n s_n \mid n \in \mathbb{N},\ s_1, \ldots, s_n \in S,\ r_1, \ldots, r_n \in R\}, \text{ and}$$
$$\mathrm{Fi}(S) = \{r + s_1 \ldots s_n \mid n \in \mathbb{N},\ r \in R,\ s_1, \ldots, s_n \in S\}$$

In particular, if $p \in R$, I is an ideal and F is a filter of R, then

$$\mathrm{Id}(I \cup \{p\}) = Rp + I \quad \text{and} \quad \mathrm{Fi}(F \cup \{p\}) = R + Fp.$$

As a consequence of the above lemma, for any element p in a rig R, $\mathrm{Id}(\{p\}) = Rp$ and $\mathrm{Fi}(\{p\}) = R + \{p^n \mid n \in \mathbb{N}\}$.

2.6. Definition. *An ideal I is called:*

1. ***saturated*** *if $x + y \in I$ if and only if $x \in I$ and $y \in I$.*
2. ***prime*** *if it is proper and $xy \in I$ if and only if $x \in I$ or $y \in I$.*

Similarly, a filter F is called:

3. ***saturated*** *if $x \cdot y \in F$ if and only if $x \in F$ and $y \in F$.*
4. ***prime*** *if it is proper and $x + y \in F$ if and only if $x \in F$ or $y \in F$.*

It is easy to see that our notion of saturated ideal coincides with that of [9] when restricted to 2-rigs.

2.7. Remark. For any subset S of a rig R, set:

$$\downarrow S \coloneqq \{x \in R \mid \exists r \in R,\ r + x \in S\} \quad \text{and} \quad \div S \coloneqq \{x \in R \mid \exists r \in R,\ r \cdot x \in S\}$$

It is straightforward from the definitions that a proper ideal I is saturated if and only if $I = \downarrow I$ and a filter F is saturated if and only if $F = \div F$.

The following lemma is also readily verified by applying the above definitions.

2.8. Lemma. *For a rig R and $S \subseteq R$, the set S is a saturated prime ideal if and only if its complement is a saturated prime filter.*

2.9. Lemma. [Prime ideal lemma] *Let R be a rig and $S, T \subseteq R$ be non-empty sets. If $\mathrm{Id}(S) \cap T = \emptyset$ and $T \cdot T \subseteq T$, then there exists a prime ideal extending S and disjoint from T. If in addition T is a filter, then there is a saturated prime ideal P extending S and a saturated prime filter F extending T so that P and F are disjoint.*

Proof. Let

$$\mathcal{I} \coloneqq \{I \mid I \text{ ideal of } R,\ S \subseteq I \text{ and } I \cap T = \emptyset\}.$$

As $\mathrm{Id}(S) \cap T$ is empty, $\mathcal{I}$ is non-empty. Let $\mathcal{C}$ be a chain in $(\mathcal{I}, \subseteq)$. It is elementary to verify that $\bigcup \mathcal{C}$ is a member of $\mathcal{I}$. So, Zorn's Lemma applies yielding a maximal element M of $\mathcal{I}$. We show that M is prime. Note that $M \neq R$ as T is non-empty.

Suppose $a \notin M$. Then $\mathrm{Id}(M \cup \{a\})$ is an ideal properly extending M, and therefore must have non-empty intersection with T. By Lemma 2.5, there are $t \in T$, $r \in R$, and $m \in M$ such that $t = ra + m$. Similarly, if $b \notin M$ then there is $t' \in T$, $r' \in R$, and $m' \in M$ with $t' = r'b + m'$. So,

$$tt' = (ra + m)(r'b + m') = rr'ab + mr'b + ram' + mm',$$

since $mr'b + ram' + mm' \in M$ we obtain $tt' \in \mathrm{Id}(M \cup \{ab\})$. Since $T \cdot T \subseteq T$ by assumption, it follows that $\mathrm{Id}(M \cup \{ab\})$ has non-empty intersection with T, and so this ideal properly extends M. Hence $ab \notin M$ and therefore M is a prime ideal.

For the second claim, suppose additionally that T is a filter. We claim that M is also saturated. Consider again the elements $a, b \in R \setminus M$ and $t \in T$ from above. Observe that,

$$t + rb = (ra + m) + rb = r(a + b) + m \in R\{a + b\} + M = \mathrm{Id}(M \cup \{a + b\}).$$

Now, $t + rb \in T$ since T is a filter and thus $t + rb \notin M$, so the ideal $\mathrm{Id}(M \cup \{a + b\})$ properly extends M. Hence $a + b \notin M$. Consequently, M is also a saturated ideal. Since M is a saturated prime ideal, from Lemma 2.8, it follows that $R \setminus M$ is a saturated prime filter which, moreover, contains T. Therefore, taking $P \coloneqq M$ and $F \coloneqq R \setminus M$, we have established our second claim. ■

2.10. Definition. *Let R be a rig and $S \subseteq R$. We set:*

$$\sqrt{S} \coloneqq \{x \in R \mid \exists n \in \mathbb{Z}^+, x^n \in S\},$$

$$\mathsf{Rad}(S) \coloneqq \bigcap \{I \mid I \textit{ prime ideal},\ S \subseteq I\},$$

$$\mathsf{dRad}(S) \coloneqq \bigcap \{I \mid I \textit{ saturated prime ideal},\ S \subseteq I\},$$

$$\mathsf{coRad}(S) \coloneqq \bigcap \{F \mid F \textit{ prime filter},\ S \subseteq F\},$$

$$\mathsf{codRad}(S) \coloneqq \bigcap \{F \mid F \textit{ saturated prime filter},\ S \subseteq F\}.$$

By $\mathsf{Rad}(R)$, the **radical** of R, we mean the intersection of all the prime ideals in R. Since 0 belongs to any ideal, it is clear that $\mathsf{Rad}(R) = \mathsf{Rad}(\{0\})$ and $\mathsf{dRad}(R) = \mathsf{dRad}(\{0\})$. For similar reasons, we set $\mathsf{coRad}(R) \coloneqq \mathsf{coRad}(\{1\})$ and $\mathsf{codRad}(R) \coloneqq \mathsf{codRad}(\{1\})$.

2.11. Remark. Notice that as an immediate consequence of the fact that 0 is multiplicatively absorbing, additively idempotent, and addition is commutative, we obtain $\downarrow 0$ to be a saturated ideal. Furthermore, since 0 belongs to any ideal, any saturated ideal must contain $\downarrow 0$. Similarly, the fact that $\uparrow 1$ is a filter is immediate from the fact that 1 is a multiplicative identity and R is distributive. It follows that $\uparrow 1$ is included in any filter. Thus, in any non-trivial rig $\downarrow 0 \subseteq \mathsf{Rad}(S) \subseteq \mathsf{dRad}(S)$ and $\uparrow 1 \subseteq \mathsf{coRad}(S) \subseteq \mathsf{codRad}(S)$. Therefore, none of these sets can be empty.

2.12. LEMMA. *Let R be a rig. For any proper ideal $I \subseteq R$, we have $\sqrt{I} = \mathsf{Rad}(I)$. Furthermore, if I is a saturated ideal, then $\sqrt{I} = \mathsf{dRad}(I)$.*

PROOF. Suppose that $x \in \sqrt{I}$, then there exists $n \in \mathbb{Z}^+$ such that $x^n \in I$. So x^n belongs to any prime ideal extending I and by primeness it follows that x also does. Whence $x \in \mathsf{Rad}(I)$. Vice versa, suppose $x \notin \sqrt{I}$ and set $T := \{x^n \mid n \in \mathbb{Z}^+\}$. Then $I \cap T = \emptyset$. Clearly $T \cdot T \subseteq T$, and so from Lemma 2.9 there exists a prime ideal P that extends I and does not contain any power of x. Hence $x \notin \mathsf{Rad}(I)$, completing the first claim. For the second claim in the statement, suppose further that $I = {\downarrow}I$. If $I \cap \mathrm{Fi}(T) = \emptyset$, then by Lemma 2.9 we can assume that P is also a saturated ideal that does not intersect T. Therefore, $x \notin \mathsf{dRad}(S)$ and the claim is verified since also $\mathsf{Rad}(S) \subseteq \mathsf{dRad}(S)$. So suppose towards contradiction that $a \in I \cap \mathrm{Fi}(T)$. As $a \in \mathrm{Fi}(T)$, by Lemma 2.5 and the fact that $T \cdot T \subseteq T$, we find $a = r + t$ for some $r \in R$ and $t \in T$. Since $a \in I = {\downarrow}I$, this means that $t \in I$, a contradiction. ■

2.13. LEMMA. *Let R be a rig. For any filter $F \subseteq R$, $\div F = \mathsf{codRad}(F)$.*

PROOF. By definition $x \in \div F$ if and only if there exists $r \in R$ such that $rx \in F$. So rx is contained in all saturated prime filters extending F. By saturation, this entails that also x belongs to all saturated prime filters extending F. Thus, $x \in \mathsf{codRad}(F)$. We conclude that $\div F \subseteq \mathsf{codRad}(F)$. Vice versa, if $x \notin \div F$, then $F \cap Rx = \emptyset$ by definition. Since Rx is an ideal and F is a filter, by Lemma 2.9, there is a saturated prime filter G extending F and disjoint from Rx. In particular, $x \notin G$, and so $x \notin \mathsf{codRad}(F)$. ■

2.14. LEMMA. *For any rig R,*

$$\mathsf{codRad}(R) = \div({\uparrow}1) = \{x \mid \exists r \in R, 1 \preceq rx\}$$
$$\mathsf{dRad}(R) = \sqrt{({\downarrow}0)} = \{x \mid \exists n \in \mathbb{N}, x^n \preceq 0\}.$$

PROOF. The left-most equalities are consequences of Remark 2.11 and Lemmas 2.12 and 2.13. The rightmost equalities are obtained simply by unfolding the definitions. ■

Of course, the category of rigs contains all commutative unital rings. However, these rigs do not have any homomorphism into $\mathsf{2}$ because -1 cannot be mapped to 0 nor to 1 in $\mathsf{2}$. We call **proper-rig** a rig that is not a ring. In the next section, we will show that proper-rigs are exactly the ones that have a homomorphism into $\mathsf{2}$. For the moment, we just show some easy equivalent conditions.

2.15. LEMMA. *Let R be a rig. The following are equivalent:*

1. *R is a proper-rig,*

2. *some element in R does not have an additive inverse,*

3. *for all $r \in R$, $1 + r \neq 0$,*

4. *$1 \not\preceq 0$ in R,*

5. $\downarrow 0$ *and* $\uparrow 1$ *are disjoint,*

6. R *has at least one saturated prime ideal.*

7. $\mathsf{dRad}(R)$ *and* $\mathsf{codRad}(R)$ *are disjoint.*

PROOF.

(1)⇔(2) Trivially.

(2)⇔(3) One direction is obvious and the other follows from distributivity: $1 + r = 0$ implies $s + rs = 0$, for any $s \in R$.

(3)⇔(4) By the definition of $\preceq$.

(4)⇔(5) By the transitivity of $\preceq$.

(5)⇒(6) Since $\downarrow 0$ is an ideal disjoint from the filter $\uparrow 1$, by Lemma 2.9 there is a prime saturated ideal P extending $\downarrow 0$.

(6)⇒(7) Recall that by Lemma 2.8, if P is a saturated prime ideal in R, then $R \setminus P$ is a saturated prime filter. Since they are obviously disjoint, so are $\mathsf{dRad}(R)$ and $\mathsf{codRad}(R)$.

(7)⇒(5) Because by Remark 2.11, $\mathsf{dRad}(R) \cap \mathsf{codRad}(R) = \emptyset$ implies $\downarrow 0 \cap \uparrow 1 = \emptyset$.

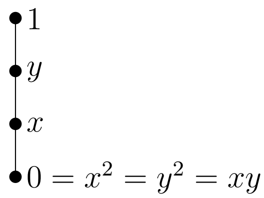
■

3. Weil rigs

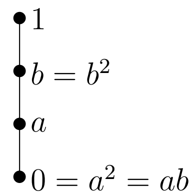
3.1. DEFINITION. *Let $\mathcal{V}$ be a variety of rigs. A rig $R \in \mathcal{V}$ is called* ***Weil in*** $\mathcal{V}$ *if there is a unique rig homomorphism from R into the initial object of $\mathcal{V}$.*

So, recalling Example 2.2, a rig is Weil in Rig if and only if it has a unique homomorphism into $\mathbb{N}$, while it is Weil in (any non-trivial subvariety of) $2\mathsf{Rig}$ if and only if it has a unique homomorphism into $\mathsf{2}$. In this paper, we are mostly interested in Weil rigs in (subvarieties of) $2\mathsf{Rig}$, but most of our results do not need the assumption that the Weil rig is itself a 2-rig. For this reason, we call **2-Weil** any rig with a unique homomorphism into $\mathsf{2}$. We also note that no finite rig is Weil in Rig as there is no homomorphism from a finite rig into $\mathbb{N}$ (since the constant 1 must always map to $1 \in \mathbb{N}$).

3.2. Remark. Notice that the class of subdirectly irreducible and Weil 2-rigs are incomparable; the 2-rigs C and D below provide counter-examples to both inclusions.



The 2-rig C.



The 2-rig D.

Indeed, as direct inspection shows (see also [5]) C is not subdirectly irreducible, but is Weil because the only homomorphism into $\mathsf{2}$ is the one sending everything in 0 but 1. The rig D is subdirectly irreducible (again, as direct inspection shows), but it has two homomorphisms into $\mathsf{2}$: the first sends everything into 0 but 1, the second sends 0 and a into 0 and the rest into 1.

3.3. Lemma. *Let R be a rig. The homomorphisms of R into $\mathsf{2}$ are in bijection with its saturated prime ideals.*

Proof. We show that if $f\colon R \to \mathsf{2}$ is a homomorphism, then $\ker f := \{x \mid f(x) = 0\}$ is a saturated prime ideal. Clearly $\ker f$ satisfies (I1) and (I2) as f is a homomorphism, i.e., $f(0) = 0$ and $f(rx) = f(r)f(x) = f(r)0 = 0$ for $x \in \ker f$. To see that it is saturated, observe that $f(x + y) = 0$ if and only if $f(x) + f(y) = 0$ if and only if $f(x) = 0$ and $f(y) = 0$. To conclude that it is a prime ideal, observe that $f(xy) = 0$ if and only if $f(x)f(y) = 0$ if and only if $f(x) = 0$ or $f(y) = 0$.

Vice versa, let P be a saturated prime ideal of R, consider the map from R into $\mathsf{2}$ defined by:

$$f_P(r) = \begin{cases} 0 & \text{if } x \in P \\ 1 & \text{if } x \notin P. \end{cases}$$

We proceed to prove by cases that f_P is a homomorphism. If $r, s \in P$, then $f_P(rs) = 0$ and $f_P(r + s) = 0$, as P is closed under product and sum by definition. The case $r, s \notin P$ is similar because the complement of P is closed under sum by upward-closure and is closed under product as a consequence of the primality of P. Finally, suppose $r \in P$ and $s \notin P$, then $rs \in P$ because P is an ideal, and $r + s \notin P$ as otherwise s would be in P because the its complement is saturated (Lemma 2.8). We deduce that $f_P(r + s) = 1 = f_P(r) + f_P(s)$ and $f_P(rs) = 0 = f_P(r)f_P(s)$. It is straightforward to observe that the above correspondence is bijective by checking that both compositions give the identity. ∎

3.4. Corollary. *A rig R is a proper-rig if and only if there exists a homomorphism from R into $\mathsf{2}$.*

Proof. By Lemma 2.15 being a proper-rig is equivalent to having at least one saturated prime ideal. The latter condition is equivalent to having a homomorphism into $\mathsf{2}$ by Lemma 3.3. ∎

3.5. THEOREM. *Let R be a rig. Then the following are equivalent.*

1. *R is 2-Weil.*
2. *R has a unique saturated prime ideal.*
3. *$R = \mathsf{dRad}(R) \cup \mathsf{codRad}(R)$ with $\mathsf{dRad}(R) \cap \mathsf{codRad}(R) = \emptyset$.*
4. *R is a proper-rig such that for every $x \in R$*

$$\exists n \in \mathbb{N}, x^n \preceq 0 \quad or \quad \exists r \in R, 1 \preceq rx. \tag{W}$$

PROOF. The fact that the first two items are equivalent is an immediate corollary of Lemma 3.3. The equivalence between item 3 and item 4 is an immediate consequence of Lemma 2.14 and Lemma 2.15. To see that item 2 implies item 3, suppose that R has a unique saturated prime ideal P. Then $P = \mathsf{dRad}(R)$ by definition. By Lemma 2.8, $R \setminus P$ is a saturated prime filter, which must be unique as well. Hence $R \setminus P = \mathsf{codRad}(R)$. So $R = \mathsf{dRad}(R) \cup \mathsf{codRad}(R)$, and clearly the intersection is empty. Finally, to obtain the implication from item 3 to item 2, observe that by Lemma 2.9, R must have a prime ideal P and by definition $\mathsf{dRad}(R) \subseteq P$. We claim that they are equal, so $\mathsf{dRad}(R)$ is the unique prime ideal. To prove the claim, it is enough to notice that under the hypothesis of item 3 $x \notin \mathsf{dRad}(R)$ if and only if $x \in \mathsf{codRad}(R)$. But then $1 \in {\downarrow}\mathrm{Id}(x)$ which implies ${\downarrow}\mathrm{Id}(x) = R$. Therefore, $x \notin P$, as the latter is proper by definition. ∎

3.6. COROLLARY. *A non-trivial 2-rig is Weil in $\mathsf{2Rig}$ if and only if it satisfies condition (W).*

PROOF. It is enough to show that any non-trivial 2-rig is a proper-rig. Notice that in any non-trivial 2-rig $1 \not\preceq 0$ because $0 \preceq 1$ in any rig and in any 2-rig $\preceq$ is a partial order. Thus, the claim can be obtained using the characterization of Lemma 2.15. ∎

It is worth remarking that, as a consequence of the argument above and Corollary 3.4, every non-trivial 2-rig has a homomorphism into $\mathsf{2}$. Note also that condition (W) reduces to $x^n = 0$ or $1 \preceq rx$ in the case of 2-rigs, since 0 is the least element, and $x^n = 0$ or $x = 1$ in the case of irigs, since there 1 is the greatest element. Therefore, an irig R is 2-Weil if and only if $\sqrt{R} = R \setminus \{1\}$.

3.7. LEMMA. *If R is a proper-rig, then $W_R \coloneqq \mathsf{dRad}(R) \cup \mathsf{codRad}(R)$ is the largest subalgebra of R with a unique homomorphism into $\mathsf{2}$.*

PROOF. It is straightforward to check that W_R is a subalgebra of R, as any subset of a rig obtained from the union of an ideal and a filter is again a rig. Clearly, if $1 \not\preceq 0$ holds in a rig then it also holds in all its subrigs. Thus, by Lemma 2.15 any subrig of a proper-rig is a proper-rig. We now prove that $W_R = \mathsf{dRad}(W_R) \cup \mathsf{codRad}(W_R)$. Using Lemma 2.14 it is straightforward to check that $\mathsf{dRad}(W_R) = \mathsf{dRad}(R)$. Next, we prove that $\mathsf{codRad}(W_R) = \mathsf{codRad}(R)$. Using the fact that $\mathsf{codRad}(R) = \{x \in R \mid \exists r \in R,\, 1 \preceq rx\}$ (cf. Lemma 2.14) it is clear that $\mathsf{codRad}(W_R) \subseteq \mathsf{codRad}(R)$. Now, suppose $x \in \mathsf{codRad}(R)$, again by Lemma 2.14 there exists $r \in R$ such that $1 \preceq rx$, but for the same reason also $r \in \mathsf{codRad}(R)$. Therefore, $r \in W_R$ and thus $x \in \mathsf{codRad}(W_R)$. Summing up, we have proved that W_R is a proper-rig such that $W_R = \mathsf{dRad}(W_R) \cup \mathsf{codRad}(W_R)$, combining Lemma 2.15 and Theorem 3.5 we conclude that W_R is 2-Weil. ■

3.8. Corollary. *The inclusion functor from the category of 2-Weil rigs into the category of proper-rigs has a right adjoint that assigns to any proper-rig R the 2-Weil rig W_R.*

Proof. It is enough to check that any homomorphism of rigs $f\colon W \to R$, with a 2-Weil domain and a proper-rig codomain factors through W_R. Now, by Theorem 3.5 $W = \mathsf{dRad}(W) \cup \mathsf{codRad}(W)$. If $x \in \mathsf{dRad}(W)$ then by Lemma 2.14 there exists $n \in \mathbb{Z}^+$ such that $x^n \preceq 0$, since f is a homomorphism $f(x)^n = f(x^n) \preceq 0$, thus $f(x) \in \mathsf{dRad}(R)$. If $x \in \mathsf{codRad}(W)$, then again by Lemma 2.14 there exists $w \in W$ such that $1 \preceq wx$, thus $1 \preceq f(w)f(x)$. Therefore, $f(x) \in \mathsf{codRad}(R)$. ■

As an immediate consequence, it follows that the rig W_R of Lemma 3.7 is the largest 2-Weil subrig of R. A further consequence is that the dual of a pseudo-constant function factors through a 2-Weil rig, as shown in the next corollary.

3.9. Corollary. *If $f : R \to S$ is a rig homomorphism such that $g \circ f = h \circ f : R \to \mathsf{2}$ for every $g, h : S \to \mathsf{2}$, then f factors through W_S.*

Proof. Note that, if S is not a proper-rig then there are no homomorphisms to $\mathsf{2}$, and the claim is vacuously satisfied. So, assume S is a proper-rig and notice that $f[R]$ is a subrig of S. By hypothesis, there exists a unique homomorphism from $f[R]$ into $\mathsf{2}$, thus $f[R]$ is 2-Weil. But from Corollary 3.8 W_S is the largest 2-Weil subrig of S, therefore $f[R] \subseteq W_S$. ■

4. Free and generic algebras in varieties of rigs

Recall that by a fundamental result of Birkhoff any variety $\mathcal{V}$ has free algebras over any set of generators $X \coloneqq \{x_i\}_{i \in I}$, which we indicate with $F_{\mathcal{V}}(I)$. We now review some results concerning free rigs that probably belong to the folklore in semiring theory. Let A be any algebra and $R \subseteq A \times A$. We write $\mathrm{Cg}_A(R)$ for the smallest congruence containing R. If $s(x)$ and $t(x)$ are terms in a single variable, we sometimes abbreviate $\mathrm{Cg}_A(\{(s(a), t(a)) \mid a \in A\})$ by $(s \approx t)$.

4.1. Remark. It follows from Example 2.2 that $\mathbb{N}$ is the 0-generated free rig and $\mathsf{2}$ is the 0-generated free 2-rig. It is also clear that for every $n \in \mathbb{N}$ the n-generated free rig is isomorphic to the rig $\mathbb{N}[x_1, \dots, x_n]$ of polynomials with coefficients in $\mathbb{N}$ and the n-generated free 2-rig is isomorphic to $\mathsf{2}[x_1, \dots, x_n]$, i.e., the rig of polynomials with coefficients in $\mathsf{2}$ (addition is, of course, idempotent).

For any commutative monoid $\langle A, \circ, \mathsf{e} \rangle$ consider the set of *finite multisets* over A; i.e.,

$$\mathcal{M}(A) \coloneqq \big\{\sigma : A \to \mathbb{N} \mid \sigma^{-1}[\mathbb{N} \setminus \{0\}] \text{ is finite}\big\}.$$

Recall that if $\sigma, \tau \in \mathcal{M}(A)$ then their **convolution** is defined as

$$(\sigma * \tau)(a) \coloneqq \sum \{\sigma(x) \cdot \tau(y) \mid a = x \circ y, \text{ for } x, y \in A\}.$$

The set $\mathcal{M}(A)$ is a rig when considered with **multiset union** (i.e., point-wise addition) and convolution; the neutral element for the former operation is the empty multiset and for the latter is the singleton (multi)set $\{\mathsf{e}\}$. We will be interested in $\mathcal{M}(\mathbb{N}^n)$, the rig of finite multisets over the direct product $\langle \mathbb{N}^n, +, \bar{0}\rangle$ of n copies of the additive monoid of natural numbers.

It is clear that any rig term $s(x_1, \dots, x_n)$ is equivalent in $F_{\mathsf{Rig}}(n)$ to one of the form

$$\sum_{i=0}^{k} a_i x_1^{v_i(1)} \cdot \dots \cdot x_n^{v_i(n)}$$

for $a_i \in \mathbb{N}$ and $v_i \in \mathbb{N}^n$. Thus, one can associate to each term s a finite $S \subseteq \mathbb{N}^n$ and a related set $\{a_v \in \mathbb{N}^+ \mid v \in S\}$, so that $s \approx \sum\{a_v \bar{x}^v \mid v \in S\}$ holds in all rigs (recall that $0 \coloneqq \sum \emptyset$).

The set S is called the **support** of s, and the a_v are the associated **coefficients**. An element $v \in S$ is often referred to as the **Parikh vector** for the monoid term $x_1^{v(1)} \dots x_n^{v(n)}$. When n is understood, we use the abbreviation $\bar{x}^v \coloneqq x_1^{v(1)} \dots x_n^{v(n)}$.

4.2. Proposition. *For each $n \in \mathbb{N}$, the rig $\mathcal{M}(\mathbb{N}^n)$ is the n-generated free rig.*

Proof. It is enough to show that $\mathbb{N}[x_1, \dots, x_n]$ and $\mathcal{M}(\mathbb{N}^n)$ are isomorphic. Note that $\mathbb{N}^n$ is generated by its elements e_i, for $i = 1, \dots, n$, consisting of a single non-zero entry equal to 1 in the i-th coordinate. Consider the homomorphism induced by freely mapping each x_i to the singleton (multi)set $\{e_i\} \in \mathbb{N}^n$. It is easy to see that this homomorphism assigns each monomial $m = a x_1^{v(1)} \dots x_n^{v(n)}$ to the multiset containing only v with multiplicity a, i.e. to the multiset $\mu : \mathbb{N}^n \to \mathbb{N}$ defined by

$$\mu(u) = \begin{cases} a & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify this map is a rig isomorphism. ∎

Let $\langle A, \circ, \mathsf{e}\rangle$ be a commutative monoid and recall that the **complex multiplication** between subsets X and Y of A is defined as $X \circ Y \coloneqq \{x \circ y \mid x \in X, y \in Y\}$. The collection $\mathcal{S}(A)$ of finite subsets of A with union and complex multiplication is a 2-rig. It is easy to see that $\mathcal{S}(A) \cong \mathcal{M}(A)/(1 + 1 \approx 1)$.

4.3. Proposition. *For each $n \in \mathbb{N}$, the 2-rig $\mathcal{S}(\mathbb{N}^n)$ is the n-generated free 2-rig.*

Proof. This follows by observing that $F_{\mathsf{2Rig}}(n) \cong \mathcal{M}(\mathbb{N}^n)/(1+1 \approx 1) \cong \mathcal{S}(\mathbb{N}^n)$, or simply verifying that the map $2[x_1, \dots, x_n] \to \mathcal{S}(\mathbb{N}^n)$ induced by $x_i \mapsto \{e_i\}$ is an isomorphism. ∎

Let $(A, \circ, \mathsf{e})$ be a monoid. Recall from Definition 2.3 that $a \preceq b$ if and only if $a \circ c = b$ for some $c \in A$. Define $\uparrow X \coloneqq \{a \in A \mid \exists x \in X, x \preceq a\}$. The following lemma has a straightforward proof.

4.4. Lemma. *Let $\langle A, \circ, \mathsf{e}\rangle$ be a monoid. For any $X, Y \subseteq A$, the following properties hold*

1. $\uparrow X = X \circ A$, $A = \uparrow\{\mathsf{e}\}$, *and* $\emptyset = \uparrow\emptyset$.
2. $\uparrow(X \cup Y) = \uparrow X \cup \uparrow Y$,
3. $\uparrow(X \circ Y) = \uparrow X \circ \uparrow Y$,
4. $\uparrow X \circ A = \uparrow X$.

It follows that if $\langle A, \circ, \mathsf{e}\rangle$ is a commutative monoid, the set $\mathcal{I}(A) \coloneqq \{U \subseteq A \mid U = \uparrow U\}$ with union and complex multiplication is an irig; notice that in this case the neutral element of the complex multiplication is A. The following lemma is immediate from Lemma 4.4.

4.5. Lemma. *Let $\langle A, \circ, \mathsf{e}\rangle$ be a commutative monoid. Then $\Uparrow \coloneqq \{(S, T) \mid \uparrow S = \uparrow T\}$ is a congruence on the 2-rig $\mathcal{S}(A)$ and its quotient $\mathcal{S}(A)/\Uparrow$ embeds into $\mathcal{I}(A)$ via the map $[S]_{\Uparrow} \mapsto \uparrow S$.*

For $X \subseteq \mathbb{N}^n$, let $X_{\min}$ be the set of $\leq$-minimal elements of X. Notice that $X_{\min}$ is an antichain with respect to order $\leq$. Now, Dickson's lemma [3] ensures that $\mathbb{N}^n$ does not contain any infinite antichains with respect to $\leq$. Consequently, any upwards closed $U \subseteq \mathbb{N}^n$, i.e., $U = \uparrow U \coloneqq U + \mathbb{N}^n$, is generated as the upward closure of a unique *finite* sub-antichain which is exactly $U_{\min}$, i.e., $U = \uparrow(U_{\min})$. It is also clear that $(\uparrow S)_{\min} = S_{\min}$ for any $S \in \mathcal{S}(\mathbb{N}^n)$.

4.6. Proposition. *For each $n \in \mathbb{N}$, the irig $\mathcal{I}(\mathbb{N}^n)$ is the n-generated free irig.*

Proof. Recall from Proposition 4.3, that the n-generated free 2-rig is isomorphic to $\mathcal{S}(\mathbb{N}^n)$. As iRig is the subvariety of $\mathsf{2Rig}$ axiomatized by $x + 1 = 1$, the n-generated free irig is $F_{\mathsf{2Rig}}(n)/(x + 1 \approx 1)$. So

$$F_{\mathsf{iRig}}(n) \cong \mathcal{S}(\mathbb{N}^n)/{\equiv} \quad \text{where} \quad {\equiv} \coloneqq \mathrm{Cg}\{(X \cup \{\bar{0}\}, \{\bar{0}\}) \mid X \in \mathcal{S}(\mathbb{N}^n)\}.$$

We now prove that the congruences $\equiv$ and $\Uparrow$ (introduced in Lemma 4.5) are identical. We start by noticing that setting $X = \mathbb{N}^n$ in the definition of $\equiv$ gives $\mathbb{N}^n \equiv \{\bar{0}\}$. Since $\equiv$ is a congruence, this entails $\uparrow Y = Y + \mathbb{N}^n \equiv Y + \{\bar{0}\} = Y$ for any $Y \in \mathcal{S}(\mathbb{N}^n)$. Thus, $\uparrow S = \uparrow T$ implies $S \equiv T$ and so $\Uparrow \subseteq \equiv$. For the converse, it suffices to show that each generator of $\equiv$ is a member of $\Uparrow$. Indeed, for any set $X \in \mathcal{S}(\mathbb{N}^n)$, from Lemma 4.4 we find $\uparrow(X \cup \{\bar{0}\}) = \uparrow X \cup \uparrow\{\bar{0}\} = \uparrow X \cup \mathbb{N}^n = \mathbb{N}^n = \uparrow\{\bar{0}\}$, hence $X \cup \{\bar{0}\} \Uparrow \{\bar{0}\}$. Since $\equiv$ is defined as the smallest congruence containing the generators, we obtain the other inclusion and conclude that $\Uparrow = \equiv$. Since these congruences are identical, so are their respective quotients of $\mathcal{S}(\mathbb{N}^n)$. Thus, from Lemma 4.5 it follows that the map $[S]_{\equiv} \mapsto \uparrow S$

is a well-defined injective homomorphism from $\mathcal{S}(\mathbb{N}^n)/\equiv$ into $\mathcal{I}(\mathbb{N}^n)$. Moreover, this map is surjective as a consequence of Dickson's lemma; i.e., for any $U \in \mathcal{I}(\mathbb{N}^n)$, $U_{\min} \in \mathcal{S}(\mathbb{N}^n)$ and $[U_{\min}]_{\equiv} \mapsto U$. Therefore, the map is an isomorphism, completing our claim. ∎

Let s be a rig term, and let $S \subseteq \mathbb{N}^n$ be the support of s. We define the following elements of $2[x_1, \dots, x_n]$:

$$[s] := \sum \{\bar{x}^v \mid v \in S\} \quad \text{and} \quad [s]_{\min} := \sum \{\bar{x}^v \mid v \in S_{\min}\}. \tag{1}$$

4.7. Corollary. *Let s, t be rig terms.*

1. *The following are equivalent.*
   - *(i) The identity $s \approx t$ is satisfied in all 2-rigs.*
   - *(ii) The identity $[s] \approx [t]$ is satisfied in all rigs.*
   - *(iii) The supporting sets of s and t are identical.*
2. *The following are equivalent.*
   - *(i) The identity $s \approx t$ is satisfied in all irigs.*
   - *(ii) The identity $[s]_{\min} \approx [t]_{\min}$ is satisfied in all 2-rigs.*
   - *(iii) The identity $[s]_{\min} \approx [t]_{\min}$ is satisfied in all rigs.*
   - *(iv) The minimal elements of the supporting sets of s and t are identical.*

Proof. Let n be the total number of distinct variables occurring in the identity $s \approx t$, and let $S, T \in \mathcal{S}(\mathbb{N}^n)$ be the supporting sets for the terms s, t, respectively. For item 1, observe that $[s] \approx [t]$ holds in Rig if and only if $S = T$. Since $\mathcal{S}(\mathbb{N}^n)$ is an n-generated free 2-rig, this occurs if and only if $s \approx t$ holds in $\mathcal{S}(\mathbb{N}^n)$; equivalently, $s \approx t$ holds in $2\mathsf{Rig}$.

For item 2 we proceed in a similar manner. Notice that $[s]_{\min} \approx [t]_{\min}$ holds in $2\mathsf{Rig}$ if and only if $S_{\min} = T_{\min}$. Since $\uparrow S = \uparrow S_{\min}$ and $\uparrow T = \uparrow T_{\min}$, the latter is equivalent to $\uparrow S = \uparrow T$. But $\uparrow S = \uparrow T$ in the n-generated free irig $\mathcal{I}(\mathbb{N}^n)$ iff the identity $s \approx t$ holds in $\mathcal{I}(\mathbb{N}^n)$, which is equivalent to iRig satisfying $s \approx t$. ∎

4.8. Proposition. *The initial rig $\mathbb{N}$ generates the variety of rigs.*

Proof. Let $p(x_1, \dots, x_n) \approx q(x_1, \dots, x_n)$ be an equation in the language of rigs. Clearly, if $p(x_1, \dots, x_n) \approx q(x_1, \dots, x_n)$ holds in all rigs, then it holds also in $\mathbb{N}$. The other implication is proved by induction on the number of variables in p and q. Let $n = 1$ and suppose that $p(x) = q(x)$ for all $v \in \mathbb{N}$. Then $p(v) - g(v)$ is a polynomial in $\mathbb{Z}[x]$ with infinitely many roots. Since the ring $\mathbb{Z}$ is an integral domain, by a classical result in algebra, $p(x) - q(x)$ is the 0 polynomial. Hence, $p(x) = q(x)$.

For the inductive step let $k \geq 0$ be the largest exponent of x_n in p and q. The polynomials $p(x_1, \dots, x_n)$ and $q(x_1, \dots, x_n)$ can be written as a sum of the polynomials $p'_j(x_1, \dots, x_{n-1})x_n^j$ and $q'_j(x_1, \dots, x_{n-1})x_n^j$, for $j = 0, \dots, k$. Since by hypothesis the equation $p(x_1, \dots, x_n) \approx q(x_1, \dots, x_n)$ holds in $\mathbb{N}$, for all $v_1, \dots, v_n \in \mathbb{N}$ we have

$$\sum_{j \le k} p'_j(v_1, \dots, v_{n-1}) v_n^j = \sum_{j \le k} q'_j(v_1, \dots, v_{n-1}) v_n^j.$$

It follows from the argument above that the 1-variable polynomials

$$\sum_{j \le k} p'_j(v_1, \dots, v_{n-1}) x_n^j \text{ and } \sum_{j \le k} q'_j(v_1, \dots, v_{n-1}) x_n^j$$

must be equal. This means that $p'_j(v_1, \dots, v_{n-1}) = q'_j(v_1, \dots, v_{n-1})$ for any $j \le k$. The induction hypothesis gives that $p'_j(x_1, \dots, x_{n-1}) = q'_j(x_1, \dots, x_{n-1})$. Thus we can conclude that $p(x_1, \dots, x_n) = q(x_1, \dots, x_n)$ in $\mathbb{N}[x_1, \dots, x_n]$. ∎

4.9. Remark. Notice that for any rig R there is a homomorphism $\mathcal{F} \colon \mathcal{M}(\mathbb{N}^k) \to R^{R^k}$ defined for any $\bar{r} \in R^k$ by

$$\mathcal{F}(\mu)(\bar{r}) := \sum_{w \in \mathbb{N}^k} \mu(w) \cdot \bar{r}^w.$$

The homomorphism $\mathcal{F}$ is a generalization of the *Fourier transform*, sending the convolution of $\mathcal{M}(\mathbb{N}^k)$ into the (pointwise) product of R^{R^k}. Since $\mathcal{M}(\mathbb{N}^k)$ is the free rig on k generators, $\mathcal{F}$ is injective precisely when R generates the variety. Proposition 4.8 above states that $\mathcal{F}$ is injective when $R = \mathbb{N}$.

The following two theorems are, to the best of our knowledge, new.

4.10. Theorem. *The variety of rigs is generated by a class of finite 2-Weil rigs.*

Proof. Since $\mathbb{N}$ generates Rig by Proposition 4.8, it suffices to show that $\mathbb{N}$ is a subdirect product of 2-Weil rigs. For $k \ge 1$, let

$$W_k := \mathbb{N}/(k \approx k+1). \tag{2}$$

It is easy to see that W_k is isomorphic to the rig over the set $\{0, 1, \dots, k\}$ with addition and multiplication truncated at k. Note that W_k is generated by 1. Therefore, the assignment $0 \mapsto 0$ and $1 \mapsto 1$ from W_k into $\mathsf{2}$ uniquely extends to a homomorphism. Consequently, W_k is a finite 2-Weil algebra, for any $k \ge 1$. Now, consider the algebra $\prod_{k=1}^{\infty} W_k$, and let $\bar{0}$ and $\bar{1}$ denote its additive and multiplicative identities, respectively. Set $\bar{\mathbb{N}}$ to be the subalgebra generated by $\bar{1}$. The members of $\bar{\mathbb{N}}$ different from $\bar{0}$ have the form:

$$\bar{n}(k) = \begin{cases} k & \text{if } k \le n \\ n & \text{if } k > n. \end{cases}$$

Thus, the unique homomorphism $\mathbb{N} \to \bar{\mathbb{N}}$ is injective. Consequently, the standard rig $\mathbb{N}$ is a subdirect product of the rigs W_k, for $k \in \mathbb{N}$. ∎

The 0-generated free 2-rig $\mathsf{2}$ obviously does not generate the whole variety $2\mathsf{Rig}$. Instead, $\mathsf{2}$ generates the variety of bounded distributive lattices; a proper subvariety of iRig. However, as we show in the next theorem, $2\mathsf{Rig}$ is generated by $F_{2\mathsf{Rig}}(1)$. Before entering into the details of the proof, we need to establish an elementary geometric fact.

For $T \in \mathcal{S}(\mathbb{N}^n)$ and $u \in \mathbb{N}^n$ let us call **adequate** a vector $\hat{x} \in \mathbb{R}^n$ with positive entries such that $\langle \hat{x}, u - v \rangle \neq 0$ for every $v \in T$, where $\langle\ ,\ \rangle$ indicates the scalar product of $\mathbb{R}^n$. Geometrically, an adequate vector is one that points in the positive orthant of $\mathbb{R}^n$ and, when applied to the point u, lies normal to a hyperplane disjoint from T.

4.11. Lemma. *Let $T \in \mathcal{S}(\mathbb{N}^n)$ and $u \in \mathbb{N}^n$. If $u \notin T$ then there exists an adequate vector with integer coordinates.*

Proof. We start by providing an adequate vector $\hat{x} \in \mathbb{R}^n$. Let us call V the $\mathbb{Q}$-vector space over $\mathbb{R}$. Observe that V is infinite dimensional, so there exists a vector $\hat{x} \in \mathbb{R}^n$ whose entries are positive and linearly independent in V (e.g., take $\hat{x}(i) := \log p_i$ for p_i the i-th prime). For any $v \in T$, the entries of $u - v$ are rational (in fact, integers) and the entries of $\hat{x}$ are linearly independent in V, it follows that $\langle \hat{x}, u - v \rangle = 0$ if and only if $u - v = 0$. In turn, this is equivalent to $u = v$, but $u \notin T$, so $\hat{x}$ is adequate.

Now, recalling that the set T is finite, consider the function $\delta \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$\delta(x) \coloneqq \min\{|\langle x, u - v \rangle| \mid v \in T\}.$$

As a composition of continuous maps, δ is continuous. Since $\hat{x}$ is adequate, $\delta(\hat{x}) > 0$. Thus, by the continuity of δ there exists a neighborhood of $\hat{x}$ on which the function is positive. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, there exists $\hat{r} \in \mathbb{Q}^n$ with all positive entries and such that $\delta(\hat{r}) > 0$. Thus, $\hat{r}$ is also adequate.

Finally, upon noticing that any positive multiple of an adequate vector is still adequate, we obtain an adequate vector with (positive) integer coordinates from $\hat{r}$ simply by multiplying $\hat{r}$ by the least common denominator of its entries. ∎

4.12. Theorem. *The variety of 2-rigs is generated by its free algebra over a single generator.*

Proof. We tacitly use Proposition 4.3, identifying $F_{2\mathsf{Rig}}(n)$ with $\mathcal{S}(\mathbb{N}^n)$. We prove that if an equation fails in the variety, then it fails in the 1-generated free algebra. Let $s \approx t$ be an equation in n variables that fails in some 2-rig, then it fails in the n-generated free algebra $F_{2\mathsf{Rig}}(n)$. Thus, by Corollary 4.7(1) the supporting sets S and T of the terms s and t must be different.

Since the evaluations of terms with n variables in an arbitrary 2-rig R are in bijection with the homomorphisms from $F_{2\mathsf{Rig}}(n)$ into R, it is enough to provide a homomorphism $\sigma \colon \mathcal{S}(\mathbb{N}^n) \to \mathcal{S}(\mathbb{N})$ such that $\sigma(S) \neq \sigma(T)$. To this end, observe that each $x \in \mathbb{N}^n$ induces a map

$$\sigma_x \colon \mathcal{S}(\mathbb{N}^n) \to \mathcal{S}(\mathbb{N}) \text{ such that } Y \in \mathcal{S}(\mathbb{N}^n) \mapsto \sigma_x(Y) \coloneqq \{\langle x, y \rangle \mid y \in Y\}.$$

It is easy to verify that σ_x is a rig homomorphism for any $x \in \mathbb{N}^n$. Indeed, it is clear that σ_x preserves addition (i.e., set-theoretic union), while the preservation of multiplication (i.e., complex-+) is garnered from the fact that the map $v \mapsto \langle x, v\rangle$ is linear.

Recall that $S \neq T$. If S or T are the empty set, then any σ_x separates S and T. Otherwise, without loss of generality, we can assume that there exists $u \in S \setminus T$. Hence, Lemma 4.11 applies, yielding a vector $\bar{x} \in \mathbb{N}^n$ such that $\langle \bar{x}, u\rangle \neq \langle \bar{x}, v\rangle$ for any $v \in T$. It follows that $\sigma_{\bar{x}}(S) \neq \sigma_{\bar{x}}(T)$. ■

5. Weil determined varieties

In this section, we show that the varieties 2Rig and iRig are generated by finite 2-Weil rigs. In fact, the generating 2-Weil rigs take a special transparent form, which we proceed to describe. Fix an arbitrary $n \in \mathbb{N}$ and let $X = \{x_1, \dots, x_n\}$ be a fixed set of free generators. For each $k > 0$, let

$$S_k \coloneqq \{(x_i^k, 0) \mid 1 \leq i \leq n\} \cup \{(k, k+1)\}$$

(recall that k represents the k-fold sum of 1). When the context suffices to disambiguate, we will write $\equiv_k$ for the congruence generated by S_k either in $F_{\mathsf{Rig}}(n)$, in $F_{\mathsf{iRig}}(n)$ or in $F_{\mathsf{2Rig}}(n)$. We will prove that, for $\mathcal{V} \in \{\mathsf{Rig}, \mathsf{iRig}, \mathsf{2Rig}\}$, the rigs $F_{\mathcal{V}}(n)/{\equiv_k}$ are all 2-Weil and that, for any fixed $n \in \mathbb{N}$, the families obtained by letting k ranging among positive integers afford subdirect representations of $F_{\mathcal{V}}(n)$. We also remark that according to the above definition $F_{\mathsf{Rig}}(0)/{\equiv_k}$ coincides with the rig W_k of Equation (2).

Our first step is to provide a more concrete description of the congruences $\equiv_k$. This will be given in Lemma 5.4.

5.1. Definition. *For $v \in \mathbb{N}^n$ we denote its **magnitude** by $|v| \coloneqq \max\{v(i) \mid i = 1, \dots, n\}$. Let s be a rig term, let S be its support, and let $\{a_v \mid v \in S\}$ be the corresponding coefficients. We define*

$$s_{<k} \coloneqq \sum \{a_v \bar{x}^v \mid v \in S \text{ and } |v| < k\}$$
$$s_{\restriction k} \coloneqq \sum \{a'_v \bar{x}^v \mid v \in S,\ a'_v \coloneqq \min(k, a_v)\}.$$

It follows directly from the definition that, for $* \in \{\restriction k, {<}k\}$:

$$s_* = \sum \{(a_v \bar{x}^v)_* \mid v \in S\}. \tag{3}$$

It is also clear that the operators are idempotent, i.e., $(s_*)_* = s_*$, and commute with each other, i.e., $s_{<k' \restriction k} = s_{\restriction k < k'}$.

5.2. Lemma. *For any $a, b \in \mathbb{N}$ and any $v \in \mathbb{N}^n$,*

1. $(a\bar{x}^v + b\bar{x}^v)_{<k} = ((a\bar{x}^v)_{<k} + (b\bar{x}^v)_{<k})_{<k}$,

2. $(a\bar{x}^v + b\bar{x}^v)_{\restriction k} = ((a\bar{x}^v)_{\restriction k} + (b\bar{x}^v)_{\restriction k})_{\restriction k}$,

and for any monomials m and m',

3. $(m \cdot m')_{<k} = (m_{<k} \cdot m'_{<k})_{<k}$

4. $(m \cdot m')_{\restriction k} = (m_{\restriction k} \cdot m'_{\restriction k})_{\restriction k}$

PROOF. Item 1. Clearly, by definition $(a \cdot \bar{x}^v)_{<k} = a \cdot (\bar{x}^v)_{<k}$. Thus

$$(a\bar{x}^v + b\bar{x}^v)_{<k} = (a + b) \cdot (x^v)_{<k} = a \cdot (\bar{x}^v)_{<k} + b \cdot (\bar{x}^v)_{<k} = (a\bar{x}^v)_{<k} + (b\bar{x}^v)_{<k}.$$

The result follows from the idempotency of $(-)_{<k}$.

Item 2. Clearly, $(a\bar{x}^v + b\bar{x}^v)_{\restriction k} = (a + b)_{\restriction k} \cdot \bar{x}^v$. Observe that $(a_{\restriction k} + b_{\restriction k})_{\restriction k} = k$ if and only if $a_{\restriction k} + b_{\restriction k} \geq k$ and this inequality holds if and only if $a + b \geq k$. Finally, the latter is equivalent to $(a + b)_{\restriction k} = k$. Thus,

$$(a\bar{x}^v + b\bar{x}^v)_{\restriction k} = (a + b)_{\restriction k} \cdot \bar{x}^v = (a_{\restriction k} + b_{\restriction k})_{\restriction k} \cdot \bar{x}^v = ((a\bar{x}^v)_{\restriction k} + (b\bar{x}^v)_{\restriction k})_{\restriction k}.$$

Item 3. Note that $((\bar{x}^u)_{<k} \cdot (\bar{x}^v)_{<k})_{<k} = 0$ if and only if either $(\bar{x}^u)_{<k} \cdot (\bar{x}^v)_{<k} = 0$ or $|u + v| \geq k$. The former condition is equivalent to $|u| \geq k$ or $|v| \geq k$. Since $|u| \geq k$ or $|v| \geq k$ entails $|u + v| \geq k$, we conclude that $((\bar{x}^u)_{<k} \cdot (\bar{x}^v)_{<k})_{<k} = 0$ holds if and only if $|u + v| \geq k$. In turn, this is equivalent to $(\bar{x}^u \cdot \bar{x}^v)_{<k} = 0$. The claim follows.

Item 4. The definition of $(-)_{\restriction k}$ yields $(a \cdot \bar{x}^v)_{\restriction k} = (a_{\restriction k}) \cdot \bar{x}^v$. Now, observe that $(a_{\restriction k} \cdot b_{\restriction k})_{\restriction k} = k$ if and only if $a_{\restriction k} \cdot b_{\restriction k} \geq k$ which in turn is equivalent to $a \cdot b \geq k$ and thus also to $(a \cdot b)_{\restriction k} = k$. Using Equation (3) the claim is established. ∎

We define a further relation over terms as follows:

$$R_{<k\restriction k} \coloneqq \{(s, t) \mid s_{<k\restriction k} \approx t_{<k\restriction k} \text{ holds in } \mathsf{Rig}.\}$$

5.3. LEMMA. *The relation $R_{<k\restriction k}$ is a congruence of $F_{\mathsf{Rig}}(n)$.*

PROOF. It is obvious that $R_{<k\restriction k}$ is an equivalence relation since it is defined by an equality. Towards establishing compatibility, we first take the intermediary step of verifying, for each operator $* \in \{\restriction k, <k\}$ and operation $\# \in \{+, \cdot\}$, the following claim:

$$(s\#t)_* = (s_*\#t_*)_*. \tag{4}$$

Let terms s, t be given, with S, T being their respective supports, and $\{a_v \mid v \in S\}$, $\{b_v \mid v \in T\}$ their respective coefficients. We set $a_v \coloneqq 0$ (resp., $b_v \coloneqq 0$) whenever $v \notin S$ (resp., $v \notin T$). We start with the case $\# = +$.

$$\begin{aligned}
(s + t)_* &= \sum_{v \in S \cup T} (a_v\bar{x}^v + b_v\bar{x}^v)_* && \text{by commutativity and Equation (3)} \\
&= \sum_{v \in S \cup T} \big((a_v\bar{x}^v)_* + (b_v\bar{x}^v)_*\big)_* && \text{by items 1 and 2 in Lemma 5.2} \\
&= \left(\sum_{v \in S \cup T} (a_v\bar{x}^v)_* + (b_v\bar{x}^v)_*\right)_* && \text{by Equation (3)} \\
&= (s_* + t_*)_*
\end{aligned}$$

For the case $\# = \cdot$, let $s := \sum_{m\in I} m$ and $t := \sum_{m'\in I'} m'$. We have

$$\begin{aligned}
(s\cdot t)_* &= \Big(\sum_{m\in I}\sum_{m'\in I'} m\cdot m'\Big)_* && \text{by distributivity}\\
&= \Big(\sum_{m\in I}\sum_{m'\in I'} (m\cdot m')_*\Big)_* && \text{by repeated applications of Equation (4) for } +\\
&= \Big(\sum_{m\in I}\sum_{m'\in I'} (m_*\cdot m'_*)_*\Big)_* && \text{by items 2 and 3 in Lemma 5.2}\\
&= \Big(\sum_{m\in I}\sum_{m'\in I'} m_*\cdot m'_*\Big)_* && \text{by repeated applications of Equation (4) for } +\\
&= (s_*\cdot t_*)_*.
\end{aligned}$$

We are now ready to prove the compatibility. Suppose $(s, s'), (t, t') \in R_{<k\restriction k}$, then $s_{<k\restriction k} = s'_{<k\restriction k}$ and $t_{<k\restriction k} = t'_{<k\restriction k}$. Let $\# \in \{+, \cdot\}$ and observe:

$$\begin{aligned}
(s \# t)_{<k\restriction k} &= (s_{<k} \# t_{<k})_{<k\restriction k} && \text{by (4) for } * = <k\\
&= (s_{<k} \# t_{<k})_{\restriction k<k} && \text{since the operators commute}\\
&= (s_{<k\restriction k} \# t_{<k\restriction k})_{\restriction k<k} && \text{by (4) for } * = \restriction k
\end{aligned}$$

By the same argument, it follows that $(s' \# t')_{<k\restriction k} = (s'_{<k\restriction k} \# t'_{<k\restriction k})_{\restriction k<k}$. From the assumption $(s, s'), (t, t') \in R_{<k\restriction k}$, it follows that

$$(s_{<k\restriction k} \# t_{<k\restriction k})_{\restriction k<k} = (s'_{<k\restriction k} \# t'_{<k\restriction k})_{\restriction k<k},$$

and thus $(s \# t)_{<k\restriction k} = (s' \# t')_{<k\restriction k}$. Therefore $R_{<k\restriction k}$ is compatible with each operation. ■

We remark that a similar proof shows that also the relations

$$R_{\restriction k} := \{(s,t) \mid s_{\restriction k} = t_{\restriction k}\} \text{ and } R_{<k} := \{(s,t) \mid s_{<k} = t_{<k}\}$$

are congruences on $F_{\mathsf{Rig}}(n)$. Recall that in (1) we defined for any rig term s,

$$[s] := \sum\{\bar{x}^v \mid v \in S\} \text{ and } [s]_{\min} := \sum\{\bar{x}^v \mid v \in S_{\min}\},$$

where S is the support of s and $S_{\min}$ is the antichain in S consisting of its $\leq$-minimal elements.

5.4. Lemma. *Let s, t be terms. Then for all $k \geq 1$, the following items hold:*

1. *$s \equiv_k t$ in $F_{\mathsf{Rig}}(n)$ if and only if $s_{<k\restriction k} \approx t_{<k\restriction k}$ holds.*
2. *$s \equiv_k t$ in $F_{\mathsf{2Rig}}(n)$ if and only if $[s_{<k}] \approx [t_{<k}]$ holds.*
3. *$s \equiv_k t$ in $F_{\mathsf{iRig}}(n)$ if and only if $[s_{<k}]_{\min} \approx [t_{<k}]_{\min}$ holds.*

PROOF. We only prove item 1, as the other two items follow from the first and Corollary 4.7. For the right-to-left implication of item 1, let s and t be terms such that $s_{<k\restriction k} \approx t_{<k\restriction k}$ holds. Consider the sum

$$s_{\geq k} \coloneqq \sum\{a_v \bar{x}^v \mid v \in S \text{ and } |v| \geq k\}.$$

Obviously, $s = s_{\geq k} + s_{<k}$ and since $x_i^k \equiv_k 0$, it follows that $s_{\geq k} \equiv_k 0$. Hence, $s = s_{\geq k} + s_{<k} \equiv_k s_{<k}$. Moreover, $s_{<k} \equiv_k s_{<k\restriction k}$ since $k \equiv_k k+1$ by definition. Thus, $s \equiv_k s_{<k\restriction k}$ and similarly $t \equiv_k t_{<k\restriction k}$. We conclude that $s \equiv_k t$. For the forward implication, just notice that the relation on the right is a congruence by Lemma 5.3 and obviously contains the pairs $\{(k+1, k)\} \cup \{(x^k, 0) \mid x \in X\}$; since $\equiv_k$ is the smallest such congruence, the implication must hold. ∎

5.5. LEMMA. *Let $\mathcal{V}$ be either* Rig*,* iRig *or* 2Rig*. The n-generated free algebra in $\mathcal{V}$ is a subdirect product of the rigs $F_{\mathcal{V}}(n)/{\equiv_k}$ for $k \geq 1$.*

PROOF. It is sufficient to show that the intersection of the family $\{\equiv_k \mid k \geq 1\}$ coincides with the identity congruence $\Delta_{F_{\mathcal{V}}(n)}$. If s and t are a pair of distinct elements of $F_{\mathcal{V}}(n)$, one can choose k large enough (i.e., larger than the magnitude of any Parikh vector or coefficient appearing in either s or t) so that $s = s_{<k\restriction k}$ and $t = t_{<k\restriction k}$. It follows from Lemma 5.4 that $s \not\equiv_k t$. ∎

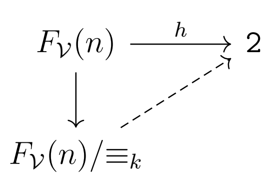
5.6. LEMMA. *Let $\mathcal{V}$ be either* Rig*,* iRig *or* 2Rig*. For every $k \geq 1$ and $n \in \mathbb{N}$ the rigs $F_{\mathcal{V}}(n)/{\equiv_k}$ are finite and have a unique homomorphism into* 2*.*

PROOF. Clearly $F_{\mathsf{Rig}}(n)/{\equiv_k}$ is finite, as there are only finitely many members $v \in \mathbb{N}^n$ with $|v| \leq k$, and every element in $F_{\mathcal{V}}(n)$ is contained in one of the equivalence classes with representative term

$$\sum_{i=0}^{\ell} a_i \bar{x}^{v_i} \text{ with } a_i \leq k \text{ and } |v_i| \leq k. \tag{5}$$

It follows that also $F_{\mathsf{2Rig}}(n)/{\equiv_k}$ and $F_{\mathsf{iRig}}(n)/{\equiv_k}$ are finite, since they are quotients of $F_{\mathsf{Rig}}(n)/{\equiv_k}$.

By the universal property of free objects, each function from X into 2 uniquely extends to a homomorphism from $F_{\mathcal{V}}(n)$ to 2.



By the First Isomorphism Theorem, there is a homomorphism along the dashed arrow in the above diagram if and only if the congruence $\equiv_k$ is contained in the kernel of h. In particular, this means that $h(x_i) = 0$ for $1 \leq i \leq n$. Thus, the only homomorphism from $F_{\mathcal{V}}(n)/{\equiv_k}$ into 2 is the one that sends the equivalence classes of the generators into 0. ∎

5.7. Theorem. *The varieties* 2Rig *and* iRig *are generated by a class of their finite Weil members.*

Proof. From Lemma 5.6 it follows that the rigs $F_{\mathsf{2Rig}}(n)/{\equiv_k}$ and $F_{\mathsf{iRig}}(n)/{\equiv_k}$ are finite and Weil algebras. Since, by Lemma 5.5 the free n-generated algebras are subdirect products of the families $\{F_{\mathcal{V}}(n)/{\equiv_k} \mid k \geq 1\}$, the claim follows. ∎

We note that a similar argument as above also establishes Theorem 4.10.

5.8. Corollary. *The variety* 2Rig *is generated by the finite 2-Weil algebras of the form* $F_{\mathsf{2Rig}}(1)/{\equiv_k}$.

Proof. By Theorem 4.12, $F_{\mathsf{2Rig}}(1)$ generates 2Rig, it follows from Lemma 5.5 that $\{W_k(1) \mid k \geq 1\}$ also generates the same variety. ∎

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References

[1] A. Carboni, S. Lack, and R. F. C. Walters. Introduction to extensive and distributive categories. *J. Pure Appl. Algebra*, 84(2):145–158, 1993.

[2] J. L. Castiglioni, M. Menni, and W. J. Zuluaga Botero. A representation theorem for integral rigs and its applications to residuated lattices. *J. Pure Appl. Algebra*, 220(10):3533–3566, 2016.

[3] L. E. Dickson. Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. *Am. J. Math.*, 35(4):413–422, 1913.

[4] E. J. Dubuc. Sur les modeles de la géométrie différentielle synthétique. *Cah. Topol. Géom. Différ. Catég.*, 20(3):231–279, 1979.

[5] P. Jipsen, L. Spada, and G. St. John. Subdirectly irreducible integral rigs. *Preprint*, 2025.

[6] F. W. Lawvere. Axiomatic cohesion. *Theory Appl. Categ.*, 19(3):41–49, 2007.

[7] F. W. Lawvere. Core varieties, extensivity, and rig geometry. *Theory Appl. Categ.*, 20(14):497–503, 2008.

[8] F. Marmolejo and M. Menni. Level ϵ. *Cah. Topol. Géom. Différ. Catég.*, 60(4):450–477, 2019.

[9] M. Menni. A basis theorem for 2-rigs and rig geometry. *Cah. Topol. Géom. Différ. Catég.*, 62(4):451–490, 2021.

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