# THE EVENTUAL IMAGE

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ABSTRACT.   In a category with enough limits and colimits, one can form the universal automorphism on an endomorphism in two dual senses.  Sometimes these dual constructions coincide, including in the categories of finite sets, finite-dimensional vector spaces, and compact metric spaces.  There, beginning with an endomorphism $f$, there is a doubly-universal automorphism on $f$ whose underlying object is the eventual image $\bigcap_{n \geq 0} \operatorname{im}(f^n)$.  Our main theorem unifies these examples, stating that in any category with a factorization system satisfying certain axioms, the eventual image has two dual universal properties.  A further theorem characterizes the eventual image as a terminal coalgebra.  In all, nine characterizations of the eventual image are given, valid at different levels of generality.

## 1. Introduction

Any endomorphism in a suitably complete category $\mathcal{C}$ gives rise to an automorphism in $\mathcal{C}$ in two dual universal ways.  Indeed, let $X^{\circlearrowleft f}$ be an endomorphism in $\mathcal{C}$.  There is an automorphism $L^{\circlearrowleft u}$ in $\mathcal{C}$ together with a map $L^{\circlearrowleft u} \to X^{\circlearrowleft f}$ that is terminal among all maps from an automorphism to $X^{\circlearrowleft f}$.  Dually, there is another automorphism $M^{\circlearrowleft v}$ with a map $X^{\circlearrowleft f} \to M^{\circlearrowleft v}$ that is initial as such.

This much is a categorical triviality, following from the existence of Kan extensions. Less trivial is the observation that in categories whose objects are sufficiently finite in nature, these two dual universal constructions coincide.  For example, they coincide in the categories of finite sets, of finite-dimensional vector spaces, and of compact metric spaces.

Every endomorphism $X^{\circlearrowleft f}$ in such a category $\mathcal{C}$ therefore gives rise to a single object equipped with an automorphism, with two dual universal properties.  In the examples just mentioned, this object can be constructed as $\bigcap_{n \in \mathbb{N}} \operatorname{im}(f^n)$, the *eventual image* of $f$.  It can also be characterized as the space of points $x \in X$ that are periodic in a category-sensitive sense: $x$ belongs to the set $\{f(x), f^2(x), \ldots\}$ in the set case, or its span in the linear case, or its closure in the metric case.

This work is intended as a small step towards a categorical treatment of dynamical systems.  An endomorphism $X^{\circlearrowleft f}$ can be seen as a discrete time dynamical system in which $f$ is performed once with every tick of the clock.  The dynamical viewpoint is

appropriate when $f$ is to be iterated indefinitely. For example, Devaney's introductory dynamics text ([8], p. 17) states:

> The basic goal of the theory of dynamical systems is to understand the eventual or asymptotic behaviour of an iterative process.

Compared to the dynamical systems studied by practitioners of the subject, the ones considered here are very primitive. But if there is to be any hope of developing a helpful categorical approach to the theory of dynamical systems in all its subtlety and complexity, we must first learn to handle the most basic situations.

We begin with the definitions (Section 2). Following the principle that Kan extensions are best done pointwise, the eventual image of $X^{\circlearrowright f}$ is defined via (co)limits: if the diagram

$$\cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots \tag{1}$$

has both a limit and a colimit, and the canonical map between them is an isomorphism, we say that $f$ *has eventual image duality* and define the *eventual image* $\mathrm{im}^\infty(f)$ to be that (co)limit. We show that $\mathrm{im}^\infty(f)$ carries a canonical automorphism $\widetilde{f}$ and that $\mathrm{im}^\infty(f)^{\circlearrowright \widetilde{f}}$ has the requisite universal properties. The 0th limit projection and colimit coprojection are maps

$$\mathrm{im}^\infty(f)^{\circlearrowright \widetilde{f}} \underset{\pi_f}{\overset{\iota_f}{\rightleftarrows}} X^{\circlearrowright f}$$

satisfying $\pi_f \iota_f = 1_{\mathrm{im}^\infty(f)}$. Hence $\iota_f \pi_f$ is an idempotent on $X$, called $f^\infty$, whose image is $\mathrm{im}^\infty(f)$. Thus, the original dynamical system $X^{\circlearrowright f}$ gives rise not only to the reversible system $\mathrm{im}^\infty(f)^{\circlearrowright \widetilde{f}}$, but also to a system $X^{\circlearrowright f^\infty}$ that stabilizes in a single step.

With the definitions made, we prove some general results (Section 3). For example, $\mathrm{im}^\infty(f^n) = \mathrm{im}^\infty(f)$ and $(f^n)^\infty = f^\infty$ for every $n \geq 1$, meaning that the eventual image construction is independent of timescale. We also relate eventual images to shift equivalence, a standard relation in symbolic dynamics.

The main theorem (Section 4) states that if $\mathcal{C}$ admits a factorization system of 'finite type' (defined there), then every endomorphism in $\mathcal{C}$ has eventual image duality. The theorem not only proves the existence of eventual images, but also provides two dual explicit constructions. Indeed, $\mathrm{im}^\infty(f)$ is constructed as $\bigcap_{n \in \mathbb{N}} \mathrm{im}(f^n)$, or more precisely, as the limit of the diagram

$$\cdots \rightarrowtail \mathrm{im}(f^2) \rightarrowtail \mathrm{im}(f) \rightarrowtail X.$$

It is also the colimit of the dual diagram. The main theorem applies to all three categories mentioned: finite sets, finite-dimensional vector spaces, and compact metric spaces with distance-decreasing maps.

In the definition of factorization system of finite type, the main condition is that an endomorphism belonging to either the left or the right class must be invertible. This is a

Dedekind finiteness condition, together with its dual. Many categories of interest satisfy one condition or the other [23], but categories satisfying both are rarer.

In our three main examples, the eventual image of $X^{\circlearrowleft f}$ is the largest subspace $A$ of $X$ satisfying $A \subseteq fA$. Generally, we prove that in a category with a factorization system of finite type, the eventual image is the terminal coalgebra for an endofunctor $A \mapsto fA$ (Section 5).

Sections 6, 7 and 8 analyse the eventual image in the categories of finite sets, finite-dimensional vector spaces and compact metric spaces. In all three cases, we find explicit descriptions of the idempotent $f^{\infty}$ on $X$. In the first, $f^{\infty}$ belongs to the set $\{1, f, f^2, \ldots\}$, in the second, it is in its linear span, and in the third, it is in its closure. Finally, Section 9 gathers further examples of the eventual image. For instance, in a Cauchy-complete category whose hom-sets are finite, every endomorphism has eventual image duality.

In summary, we describe the eventual image of an endomorphism $X^{\circlearrowleft f}$ in nine equivalent ways:

  i. as the universal automorphism equipped with a map into $X^{\circlearrowleft f}$;

  ii. as the limit of the diagram $\cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots$;

  iii. as the limit of the diagram $\cdots \rightarrowtail \mathrm{im}(f^2) \rightarrowtail \mathrm{im}(f) \rightarrowtail X$;

  iv. as the terminal coalgebra for the endofunctor $A \mapsto fA$ on subobjects $A \rightarrowtail X$;

  v. as the space of periodic points of $f$,

together with the duals of (i)–(iv). These descriptions are valid at different levels of generality: (i) and (ii) whenever $f$ has eventual image duality, (iii) and (iv) when $\mathcal{C}$ has a factorization system of finite type, and (v) for the three leading examples, interpreting 'periodic' appropriately.

**Related and further work** The eventual image appears in both symbolic dynamics and semigroup theory (for instance, Definition 7.4.2 of [19] and p. 79 of [22]). There, it is more often called the eventual range, although 'eventual image' has been used (e.g. [14], p. 53). The idempotent $f^{\infty}$ is sometimes written as $f^{\omega}$, evoking the countable ordinal $\omega$. While the eventual image $\mathrm{im}^{\infty}(f) = \mathrm{im}(f^{\infty})$ is indeed the countable intersection $\bigcap_{n \in \mathbb{N}} \mathrm{im}(f^n)$ in the cases studied here, that is only because of their finite character. There are other settings in which the restriction of $f$ to $\bigcap_{n \in \mathbb{N}} \mathrm{im}(f^n)$ need not be surjective, and one has to iterate further through the ordinals to obtain an automorphism. Notation aside, semigroup theory provides a lens through which to view this work (Steinberg [17, 18]).

In topological dynamics, limits of diagrams of the form (1) are known as generalized solenoids ([25], p. 341).

Limit-colimit coincidences are closely related to absolute (co)limits: consider direct sums in **Ab**-categories and idempotent splittings, for instance, or see Section 6 of Kelly

and Schmitt [13]. But some simultaneous limits and colimits are not absolute. The eventual image is one case (Example 8.4); another can be found in the representation theory of finite groups, where induced and coinduced representations coincide.

Sections 6–8 reveal many commonalities between the three principal examples (sets, vector spaces and metric spaces). Some of those commonalities are accounted for by the results of Sections 4 and 5 on factorization systems. Others, such as the description of $\mathrm{im}^\infty(f)$ in terms of periodic points, are not. These merit further investigation.

One can also seek to generalize the examples given. The vector space case can perhaps be extended to more general categories of modules. In the metric case, our maps are distance-decreasing, but the crucial feature of distance-decreasing endomorphisms $f$ is that $\{f^n : n \in \mathbb{N}\}$ is equicontinuous—a condition that refers only to the uniform structure, not the metric (as noted by Steinberg [18]).

Many of the results on sets, vector spaces and metric spaces in Sections 6–8 are elementary; they are not claimed to be original. They are assembled in this way in order to reveal the common patterns and show their place in the theory of the eventual image.

Our focus is on the very special categories in which the left and right universal methods for converting an endomorphism into an automorphism coincide. We leave open many questions about more general categories. For example, there is a sense in which the eventual image of the continuous map $z \mapsto z^2$ on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ ought to be $\{z \in \mathbb{C} : |z| = 1\} \cup \{0, \infty\}$, even though this is not what is given by any of the nine characterizations listed above. Developing a general theory of the eventual image that covers such examples remains a challenge.
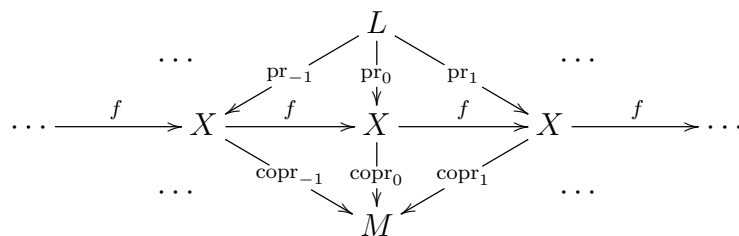
This paper is dedicated to the memory of Pieter Hofstra, whose death is such a terrible loss.

## 2. Definitions

Let $\mathcal{C}$ be a category and let $f\colon X \to X$ be an endomorphism in $\mathcal{C}$, which we write as $X^{\circlearrowright f}$. Suppose that the diagram

$$\cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots \tag{2}$$

has both a limit cone $\left(L \xrightarrow{\mathrm{pr}_n} X\right)_{n\in\mathbb{Z}}$ and a colimit cone $\left(X \xrightarrow{\mathrm{copr}_n} M\right)_{n\in\mathbb{Z}}$:

Then there is a canonical map $L \to M$, defined as the composite

$$L \xrightarrow{\mathrm{pr}_n} X \xrightarrow{\mathrm{copr}_n} M,$$

which is independent of $n \in \mathbb{Z}$.

2.1. DEFINITION. An endomorphism $X^{\circlearrowleft f}$ in a category **has eventual image duality** if the diagram (2) has both a limit $L$ and a colimit $M$, and the canonical map $L \to M$ is an isomorphism. In that case, the object $L \cong M$, together with the limit projections and colimit coprojections, is an **eventual image** of $f$.

2.2. EXAMPLE. In **Set**, most endomorphisms $X^{\circlearrowleft f}$ do not have eventual image duality.

The limit $L$ of (2) is the set of all double sequences $(x_n)_{n \in \mathbb{Z}}$ such that $x_n \in X$ and $f(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$.

The colimit $M$ is the set of equivalence classes of pairs $(n, x)$ with $n \in \mathbb{Z}$ and $x \in X$, where $(n, x) \sim (m, y)$ if there is some $p \geq m, n$ such that $f^{p-n}(x) = f^{p-m}(y)$. Alternatively, it is the set of equivalence classes of tails $(x_n)_{n \geq N}$ with $N \in \mathbb{Z}$ and $f(x_n) = x_{n+1}$ for all $n \geq N$, where two tails $(x_n)_{n \geq N}$ and $(y_n)_{n \geq P}$ are equivalent if $x_n = y_n$ for all sufficiently large $n$.

The canonical map $L \to M$ sends $(x_n)_{n \in \mathbb{Z}}$ to the equivalence class of $(0, x_0)$ in the first description of the colimit, or the equivalence class of $(x_n)_{n \geq 0}$ in the second. It is typically not bijective. For example, it is not surjective when $f$ is the squaring map on the real interval $[2, \infty)$, and not injective when $f$ is the squaring map on $\mathbb{C}$.

When $f$ has eventual image duality, we identify the limit and colimit objects via the canonical isomorphism, writing $\mathrm{im}^\infty(f)$ for both. Thus, we have limit and colimit cones

$$\left( \mathrm{im}^\infty(f) \xrightarrow{\mathrm{pr}_n} X \right)_{n \in \mathbb{Z}}, \qquad \left( X \xrightarrow{\mathrm{copr}_n} \mathrm{im}^\infty(f) \right)_{n \in \mathbb{Z}}.$$

Write

$$\iota_f = \mathrm{pr}_0, \qquad \pi_f = \mathrm{copr}_0.$$

Then the composite

$$\mathrm{im}^\infty(f) \xrightarrow{\iota_f} X \xrightarrow{\pi_f} \mathrm{im}^\infty(f)$$

is the identity, so there is an idempotent $X^{\circlearrowleft f^\infty}$ defined by

$$f^\infty = \left( X \xrightarrow{\pi_f} \mathrm{im}^\infty(f) \xrightarrow{\iota_f} X \right).$$

The map of diagrams



induces an endomorphism $\mathrm{im}^\infty(f)^{\circlearrowleft \widetilde{f}}$. In principle it induces two such endomorphisms, depending on whether $\mathrm{im}^\infty(f)$ is viewed as a limit or a colimit, but they are equal.

**2.3. LEMMA.** *Let $X^{\circlearrowleft f}$ be an endomorphism with eventual image duality, in any category. Then $\mathrm{im}^\infty(f)^{\circlearrowleft \tilde{f}}$ is invertible.*

**PROOF.** There is a cone $\left(\mathrm{im}^\infty(f) \xrightarrow{\mathrm{pr}_n^-} X\right)_{n\in\mathbb{Z}}$ on (2) in which $\mathrm{pr}_n^- = \mathrm{pr}_{n-1}$. Since $\left(\mathrm{im}^\infty(f) \xrightarrow{\mathrm{pr}_n} X\right)_{n\in\mathbb{Z}}$ is a limit cone, the cone $(\mathrm{pr}_n^-)$ induces an endomorphism $\mathrm{im}^\infty(f)^{\circlearrowleft}$, which one can check is inverse to $\tilde{f}$. ∎

**2.4. EXAMPLE.** Every automorphism $X^{\circlearrowleft f}$ has eventual image duality. The eventual image of $f$ is $X$ itself, with $\iota_f = \pi_f = f^\infty = 1_X$ and $\tilde{f} = f$.

**2.5. EXAMPLE.** Every split idempotent has eventual image duality. Indeed, let $e\colon X \to X$ be an idempotent in $\mathcal{C}$ splitting as

$$\begin{array}{c} X \\ {\scriptstyle i}\uparrow\downarrow{\scriptstyle p} \\ I. \end{array}$$

Then $I$ is both the limit and colimit of $\cdots \xrightarrow{e} X \xrightarrow{e} \cdots$, with projections $i$ and coprojections $p$. So $\mathrm{im}^\infty(e) = I$ with $\iota_e = i$, $\pi_e = p$, $e^\infty = e$ and $\tilde{e} = 1_I$.

Write $\mathbf{En}(\mathcal{C})$ for the category whose objects $X^{\circlearrowleft f}$ are the endomorphisms in $\mathcal{C}$ and whose maps $X^{\circlearrowleft f} \to Y^{\circlearrowleft g}$ are the maps $u\colon X \to Y$ in $\mathcal{C}$ satisfying $uf = gu$. It has a full subcategory $\mathbf{Au}(\mathcal{C})$ consisting of the objects $X^{\circlearrowleft f}$ where $f$ is an automorphism.

Let $X^{\circlearrowleft f} \in \mathbf{En}(\mathcal{C})$. If $f$ has eventual image duality then we obtain an object $\mathrm{im}^\infty(f)^{\circlearrowleft \tilde{f}} \in \mathbf{Au}(\mathcal{C})$ together with maps

$$\begin{array}{c} X^{\circlearrowleft f} \\ {\scriptstyle \iota_f}\uparrow\downarrow{\scriptstyle \pi_f} \\ \mathrm{im}^\infty(f)^{\circlearrowleft \tilde{f}} \end{array} \qquad (3)$$

such that $\pi_f \iota_f = 1$. Given only the idempotent $f^\infty$ in $\mathcal{C}$, we can reconstruct $\mathrm{im}^\infty(f)$, $\iota_f$ and $\pi_f$ as the splitting data of $f^\infty$, and $\tilde{f}$ as $\pi_f \circ f \circ \iota_f$. In turn, the data (3) determines the limit and colimit cones of Definition 2.1 as follows.

**2.6. LEMMA.** *Let $\mathcal{C}$ be a category. Let $X^{\circlearrowleft f}$ be an endomorphism in $\mathcal{C}$ with eventual image duality. With notation as above, the limit and colimit cones*

$$\left(\mathrm{im}^\infty(f) \xrightarrow{\mathrm{pr}_n} X\right)_{n\in\mathbb{Z}}, \qquad \left(X \xrightarrow{\mathrm{copr}_n} \mathrm{im}^\infty(f)\right)_{n\in\mathbb{Z}}$$

*are given by*

$$\mathrm{pr}_n = \left(\mathrm{im}^\infty(f) \xrightarrow{\tilde{f}^n} \mathrm{im}^\infty(f) \xrightarrow{\iota_f} X\right),$$

$$\mathrm{copr}_n = \left(X \xrightarrow{\pi_f} \mathrm{im}^\infty(f) \xrightarrow{\tilde{f}^n} \mathrm{im}^\infty(f)\right).$$

PROOF. By duality, it suffices to prove the first statement. Recall that $\iota_f = \mathrm{pr}_0$. When $n \geq 0$, the diagram

$$
\begin{array}{ccc}
\mathrm{im}^\infty(f) & \xrightarrow{\ \iota_f\ } & X \\
\widetilde{f}^n \downarrow & & \downarrow f^n \\
\mathrm{im}^\infty(f) & \xrightarrow[\ \iota_f\ ]{} & X
\end{array}
\quad \xrightarrow{\ \mathrm{pr}_n\ }
$$

commutes, the square by definition of $\widetilde{f}$ and the triangle by the cone property of $(\mathrm{pr}_n)$. Similarly, for $n \geq 0$, the diagram

$$
\begin{array}{ccc}
\mathrm{im}^\infty(f) & \xrightarrow{\ \mathrm{pr}_{-n}\ } & X \\
\widetilde{f}^n \downarrow & & \downarrow f^n \\
\mathrm{im}^\infty(f) & \xrightarrow[\ \mathrm{pr}_{-n}\ ]{} & X
\end{array}
\quad \xrightarrow{\ \mathrm{pr}_0 = \iota_f\ }
$$

commutes, giving $\mathrm{pr}_{-n} = \iota_f \circ \widetilde{f}^{-n}$. ∎

2.7. PROPOSITION. *Let $\mathcal{C}$ be a category and let $f \colon X \to X$ be an endomorphism in $\mathcal{C}$ with eventual image duality. Then:*

*i. $\iota_f \colon \mathrm{im}^\infty(f)^{\circlearrowright \widetilde{f}} \to X^{\circlearrowright f}$ is terminal among maps to $X^{\circlearrowright f}$ from objects of $\mathbf{Au}(\mathcal{C})$;*

*ii. $\pi_f \colon X^{\circlearrowright f} \to \mathrm{im}^\infty(f)^{\circlearrowright \widetilde{f}}$ is initial among maps from $X^{\circlearrowright f}$ to objects of $\mathbf{Au}(\mathcal{C})$.*

PROOF. By duality, it is enough to prove (i). One can check terminality directly. Alternatively, denote by $B\mathbb{N}$ the one-object category corresponding to the additive monoid $\mathbb{N}$, and similarly $B\mathbb{Z}$. Then

$$
\mathbf{En}(\mathcal{C}) \simeq \mathcal{C}^{B\mathbb{N}}, \qquad \mathbf{Au}(\mathcal{C}) \simeq \mathcal{C}^{B\mathbb{Z}},
$$

and the inclusion $\mathbf{Au}(\mathcal{C}) \hookrightarrow \mathbf{En}(\mathcal{C})$ is induced by the inclusion $B\mathbb{N} \hookrightarrow B\mathbb{Z}$. The standard end or limit formula for Kan extensions reduces, in this case, to the statement that the right Kan extension of $X^{\circlearrowright f} \colon B\mathbb{N} \to \mathcal{C}$ along $B\mathbb{N} \hookrightarrow B\mathbb{Z}$ is $\mathrm{im}^\infty(f)^{\circlearrowright \widetilde{f}}$, with canonical map $\iota_f$. ∎

The universal property of the eventual image construction makes it functorial. Explicitly, let $u \colon X^{\circlearrowright f} \to Y^{\circlearrowright g}$ be a map in $\mathbf{En}(\mathcal{C})$. Assuming that both $X^{\circlearrowright f}$ and $Y^{\circlearrowright g}$ have eventual image duality, the map of diagrams

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{f} & X & \xrightarrow{f} & X & \xrightarrow{f} & X & \xrightarrow{f} & \cdots \\
& & \downarrow u & & \downarrow u & & \downarrow u & & \\
\cdots & \xrightarrow{g} & Y & \xrightarrow{g} & Y & \xrightarrow{g} & Y & \xrightarrow{g} & \cdots
\end{array}
$$

induces a map $u_* \colon \mathrm{im}^\infty(f) \to \mathrm{im}^\infty(g)$ on the limits or, equivalently, the colimits. This construction is functorial where defined: $(1_X)_* = 1_{\mathrm{im}^\infty(f)}$ and $(vu)_* = v_* u_*$.

2.8. EXAMPLE. Let $X^{\circlearrowright f}$ be an endomorphism with eventual image duality, and view $f$ as a map $X^{\circlearrowright f} \to X^{\circlearrowright f}$ in $\mathbf{En}(\mathcal{C})$. By definition,

$$f_* = \widetilde{f} \colon \operatorname{im}^\infty(f) \to \operatorname{im}^\infty(f).$$

2.9. LEMMA. *Let $X^{\circlearrowright f}$ and $Y^{\circlearrowright g}$ be endomorphisms in $\mathcal{C}$ with eventual image duality, and let $u \colon X^{\circlearrowright f} \to Y^{\circlearrowright g}$ be a map in $\mathbf{En}(\mathcal{C})$. Then:*

*i. $u_* \colon \operatorname{im}^\infty(f) \to \operatorname{im}^\infty(g)$ is a map*

$$u_* \colon \operatorname{im}^\infty(f)^{\circlearrowright \widetilde{f}} \to \operatorname{im}^\infty(g)^{\circlearrowright \widetilde{g}}$$

*in $\mathbf{Au}(\mathcal{C})$;*

*ii. $u \colon X \to Y$ is a map*

$$u \colon X^{\circlearrowright f^\infty} \to Y^{\circlearrowright g^\infty}$$

*in $\mathbf{En}(\mathcal{C})$.*

PROOF. For (i), $uf = gu$, hence $u_* f_* = g_* u_*$ by functoriality. But $f_* = \widetilde{f}$ and $g_* = \widetilde{g}$ by Example 2.8.

For (ii), the diagram



commutes by definition of $u_*$.                                                ∎

2.10. LEMMA. *Let $X^{\circlearrowright f}$ be an endomorphism in $\mathcal{C}$ with eventual image duality. Then:*

*i. the endomorphisms $f$ and $f^\infty$ commute;*

*ii. $f^\infty \colon X \to X$ is a map $X^{\circlearrowright f} \to X^{\circlearrowright f}$ in $\mathbf{En}(\mathcal{C})$;*

*iii. the induced map $(f^\infty)_* \colon \operatorname{im}^\infty(f) \to \operatorname{im}^\infty(f)$ is the identity.*

PROOF. Part (i) is the case $g = u = f$ of Lemma 2.9(ii), and part (ii) follows. For (iii), it is enough to show that for each $n \in \mathbb{Z}$, the outer triangle of

commutes. The upper and lower triangles commute by Lemma 2.6, and the right-hand triangle commutes because $f^\infty = \iota_f \pi_f$ and $\pi_f \iota_f = 1$.  ∎

Now suppose that $\mathcal{C}$ **has eventual image duality**, meaning that every endomorphism in $\mathcal{C}$ does. There is a functor

$$\mathrm{im}^\infty \colon \mathbf{En}(\mathcal{C}) \to \mathbf{Au}(\mathcal{C})$$

defined on objects by $X^{\circlearrowright f} \mapsto \mathrm{im}^\infty(f)^{\circlearrowright \tilde{f}}$ and on maps by $u \mapsto u_*$ (which is valid by Lemma 2.9(i)). There is also an inclusion functor

$$U \colon \mathbf{Au}(\mathcal{C}) \to \mathbf{En}(\mathcal{C}),$$

with $\mathrm{im}^\infty \circ U \cong 1_{\mathbf{Au}(\mathcal{C})}$ by Example 2.4, and there are natural transformations

$$\iota \colon U \circ \mathrm{im}^\infty \to 1_{\mathbf{En}(\mathcal{C})}, \qquad \pi \colon 1_{\mathbf{En}(\mathcal{C})} \to U \circ \mathrm{im}^\infty$$

whose components at $X^{\circlearrowright f} \in \mathbf{En}(\mathcal{C})$ are $\iota_f$ and $\pi_f$. They satisfy $\pi\iota = 1$. Proposition 2.7 implies:

2.11. PROPOSITION. *Let $\mathcal{C}$ be a category with eventual image duality. Then the functor $\mathrm{im}^\infty$ is both left and right adjoint to the inclusion $U \colon \mathbf{Au}(\mathcal{C}) \to \mathbf{En}(\mathcal{C})$. The units and counits of the adjunctions are $\iota$, $\pi$ and the canonical isomorphism $\mathrm{im}^\infty \circ U \cong 1_{\mathbf{Au}(\mathcal{C})}$, and the unit-counit composite*

$$U \circ \mathrm{im}^\infty \xrightarrow{\iota} 1_{\mathbf{En}(\mathcal{C})} \xrightarrow{\pi} U \circ \mathrm{im}^\infty$$

*is the identity.*  ∎

Simultaneous left and right adjunctions are sometimes called ambidextrous adjunctions or ambijunctions [16].

## 3. General properties of the eventual image

Here we establish some properties of the constructions $f \mapsto \mathrm{im}^\infty(f)$ and $f \mapsto f^\infty$. They largely concern invariance: when do two endomorphisms have the same eventual image or the same associated idempotent?

Throughout this section, let $\mathcal{C}$ be a category.

Despite the notation, $f^\infty \circ f \neq f^\infty$ in general. For example, they are not equal when $f$ is a nontrivial automorphism, by Example 2.4. But it is true that $f^\infty \circ f = f \circ f^\infty$, by Lemma 2.10(i). Moreover:

3.1. PROPOSITION. *Let $X^{\circlearrowright f}$ be an endomorphism in $\mathcal{C}$ with eventual image duality. Let $n \geq 1$. Then $\mathrm{im}^\infty(f^n) \cong \mathrm{im}^\infty(f)$ and $(f^n)^\infty = f^\infty$.*

PROOF. The inclusion $(n\mathbb{Z}, \leq) \hookrightarrow (\mathbb{Z}, \leq)$ is cofinal,[1] so the canonical map

$$\mathrm{im}^\infty(f) = \lim\left(\cdots \xrightarrow{f} X \xrightarrow{f} \cdots\right) \xrightarrow{\quad k \quad} \lim\left(\cdots \xrightarrow{f^n} X \xrightarrow{f^n} \cdots\right) = \mathrm{im}^\infty(f^n)$$

is an isomorphism. A dual statement holds for colimits, giving an isomorphism $\ell\colon \mathrm{im}^\infty(f^n) \to \mathrm{im}^\infty(f)$. The diagram



commutes, and $k$ and $\ell$ are isomorphisms, so $\ell = k^{-1}$. But now the commutative diagram



shows that $f^\infty = (f^n)^\infty$.                                      ∎

The property of the eventual image established in Proposition 3.1 is shared by other dynamical constructs. For example, every holomorphic self-map $f$ of a compact Riemann surface $X$ has a Julia set $J(f) \subseteq X$, and $J(f^n) = J(f)$ for all $n \geq 1$ (Lemma 4.2 of Milnor [20]). If our endomorphism $f$ is applied to $X$ once per second, then the equality $\mathrm{im}^\infty(f) = \mathrm{im}^\infty(f^{60})$ means that the eventual image is the same whether the process is observed every second or every minute: it is independent of timescale.

In symbolic and topological dynamics, there is a standard notion of shift equivalence (Wagoner [24]; Williams [25], p. 342). Two endomorphisms $X^{\circlearrowleft f}$ and $Y^{\circlearrowleft g}$ in $\mathcal{C}$ are **shift equivalent** if there exist $n \in \mathbb{N}$ and maps

$$X^{\circlearrowleft f} \underset{v}{\overset{u}{\rightleftarrows}} Y^{\circlearrowleft g} \tag{4}$$

in $\mathbf{En}(\mathcal{C})$ such that $vu = f^n$ and $uv = g^n$. (Then the same is true for all $N \geq n$: replace $u$ by $uf^{N-n}$.) When $X^{\circlearrowleft f}$ and $Y^{\circlearrowleft g}$ both have eventual image duality, call $X^{\circlearrowleft f}$ and $Y^{\circlearrowleft g}$ **eventually equivalent** if there exist maps (4) such that $vu = f^\infty$ and $uv = g^\infty$.

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[1]We use the terminological convention in which *cofinal* functors leave *limits* unchanged.

3.2. PROPOSITION. *Let $X^{\circlearrowright f}$ and $Y^{\circlearrowright g}$ be endomorphisms in $\mathcal{C}$, both with eventual image duality. Then*

$$X^{\circlearrowright f} \text{ and } Y^{\circlearrowright g} \text{ are shift equivalent}$$
$$\implies \operatorname{im}^\infty(f)^{\circlearrowright \widetilde{f}} \cong \operatorname{im}^\infty(g)^{\circlearrowright \widetilde{g}}$$
$$\iff X^{\circlearrowright f} \text{ and } Y^{\circlearrowright g} \text{ are eventually equivalent.}$$

PROOF. Suppose that $X^{\circlearrowright f}$ and $Y^{\circlearrowright g}$ are shift equivalent, with maps $u$ and $v$ as in (4). They induce maps

$$\operatorname{im}^\infty(f)^{\circlearrowright \widetilde{f}} \underset{v_*}{\overset{u_*}{\rightleftarrows}} \operatorname{im}^\infty(g)^{\circlearrowright \widetilde{g}}, \tag{5}$$

which satisfy

$$v_* u_* = (vu)_* = (f^n)_* = (f_*)^n = \widetilde{f}^{\,n}$$

(the last step by Example 2.8). Hence $v_* u_*$ is an isomorphism, and dually, so is $u_* v_*$. It follows that $u_*$ is an isomorphism, giving $\operatorname{im}^\infty(f)^{\circlearrowright \widetilde{f}} \cong \operatorname{im}^\infty(g)^{\circlearrowright \widetilde{g}}$.

Next suppose that $\operatorname{im}^\infty(f)^{\circlearrowright \widetilde{f}} \cong \operatorname{im}^\infty(g)^{\circlearrowright \widetilde{g}}$. Choose an isomorphism $k$, and define

$$u = \left( X^{\circlearrowright f} \xrightarrow{\pi_f} \operatorname{im}^\infty(f)^{\circlearrowright \widetilde{f}} \xrightarrow{k} \operatorname{im}^\infty(g)^{\circlearrowright \widetilde{g}} \xrightarrow{\iota_g} Y^{\circlearrowright g} \right),$$
$$v = \left( Y^{\circlearrowright g} \xrightarrow{\pi_g} \operatorname{im}^\infty(g)^{\circlearrowright \widetilde{g}} \xrightarrow{k^{-1}} \operatorname{im}^\infty(f)^{\circlearrowright \widetilde{f}} \xrightarrow{\iota_f} X^{\circlearrowright f} \right).$$

Then $vu = f^\infty$ and $uv = g^\infty$, so $X^{\circlearrowright f}$ and $Y^{\circlearrowright g}$ are eventually equivalent.

Finally, suppose that $X^{\circlearrowright f}$ and $Y^{\circlearrowright g}$ are eventually equivalent, and take maps $u$ and $v$ as in the definition. The induced maps (5) satisfy

$$v_* u_* = (vu)_* = (f^\infty)_* = 1_{\operatorname{im}^\infty(f)}$$

by Lemma 2.10(iii), and dually, $u_* v_* = 1_{\operatorname{im}^\infty(g)}$. Hence $u_*$ and $v_*$ are mutually inverse maps between $\operatorname{im}^\infty(f)^{\circlearrowright \widetilde{f}}$ and $\operatorname{im}^\infty(g)^{\circlearrowright \widetilde{g}}$. ∎

3.3. COROLLARY. *Let $X \underset{v}{\overset{u}{\rightleftarrows}} Y$ be maps in $\mathcal{C}$, and suppose that $vu$ and $uv$ have eventual image duality. Then $\operatorname{im}^\infty(vu)^{\circlearrowright \widetilde{vu}} \cong \operatorname{im}^\infty(uv)^{\circlearrowright \widetilde{uv}}$.*

PROOF. The maps $X^{\circlearrowright vu} \underset{v}{\overset{u}{\rightleftarrows}} Y^{\circlearrowright uv}$ in $\mathbf{En}(\mathcal{C})$ define a shift equivalence, so Proposition 3.2 applies. ∎

Finally, let $f$ and $g$ be endomorphisms of the same object of $\mathcal{C}$. In general, $(gf)^\infty \neq g^\infty f^\infty$. For example, let $f$ and $g$ be split idempotents such that $gf$ is not idempotent (such as the linear operators on $\mathbb{R}^2$ represented by $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$). By Example 2.5, $g^\infty f^\infty$ is $gf$, which is not idempotent and so not equal to $(gf)^\infty$. However:

3.4. PROPOSITION. *Let $f$ and $g$ be commuting endomorphisms of an object $X$ of $\mathcal{C}$. Suppose that $f$, $g$ and $gf$ have eventual image duality. Then $(gf)^\infty = g^\infty f^\infty$.*

PROOF. Consider the commutative diagram
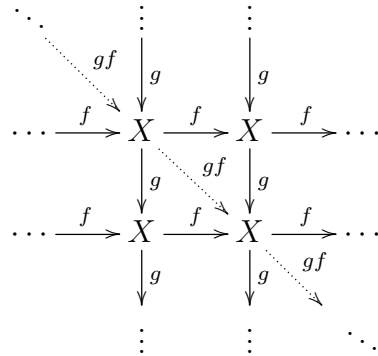


The solid part shows a functor $(\mathbb{Z}, \leq) \times (\mathbb{Z}, \leq) \to \mathcal{C}$, which restricted to the diagonal is the functor $(\mathbb{Z}, \leq) \to \mathcal{C}$ shown as the dotted part. Since the diagonal subset of $\mathbb{Z} \times \mathbb{Z}$ is cofinal, the dotted and solid parts have the same limits. Now the dotted part has limit $\text{im}^\infty(gf)$, and the limit of the solid part can be calculated by taking limits in rows and then columns:
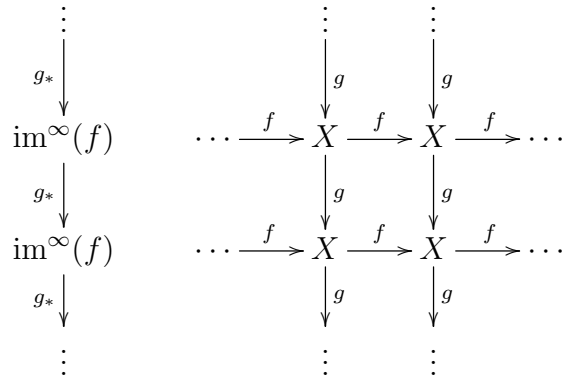


Thus, the limit of $\cdots \xrightarrow{g_*} \text{im}^\infty(f) \xrightarrow{g_*} \cdots$ is $\text{im}^\infty(gf)$, and the 0th limit projection $\iota_{g_*} \colon \text{im}^\infty(gf) \to \text{im}^\infty(f)$ makes the triangle

$$
\begin{array}{c}
\text{im}^\infty(gf) \\
{\scriptstyle \iota_{g_*}} \downarrow \qquad \searrow^{\iota_{gf}} \\
\text{im}^\infty(f) \xrightarrow[\iota_f]{} X
\end{array}
\tag{6}
$$

commute. The dual argument applies to colimits. Putting together triangle (6) with its dual gives a commutative diagram

$$
\begin{array}{c}
\mathrm{im}^\infty(gf) \\
\quad \overset{\iota_{g*}}{\Big\downarrow} \quad \overset{\iota_{gf}}{\searrow} \\
\mathrm{im}^\infty(f) \xrightarrow{\ \iota_f\ } X \xrightarrow{\ \pi_f\ } \mathrm{im}^\infty(f) \\
\qquad \overset{\pi_{gf}}{\searrow} \qquad \Big\downarrow {\scriptstyle \pi_{g*}} \\
\mathrm{im}^\infty(gf).
\end{array}
$$

Since $\pi_f \iota_f = 1$ and $\pi_{gf}\iota_{gf} = 1$, it follows that $\pi_{g*}\iota_{g*} = 1$, so that $\mathrm{im}^\infty(f)^{\circlearrowleft g*}$ has eventual image duality with $\mathrm{im}^\infty(g_*) \cong \mathrm{im}^\infty(gf)$.

Now consider the diagram

$$
\begin{array}{c}
X \xrightarrow{\ \pi_f\ } \mathrm{im}^\infty(f) \xrightarrow{\ \iota_f\ } X \\
\quad \overset{\pi_{gf}}{\searrow} \quad {\scriptstyle (A)} \ \Big\downarrow {\scriptstyle \pi_{g*}} \qquad \Big\downarrow {\scriptstyle \pi_g} \\
\mathrm{im}^\infty(gf) \xrightarrow{\ (\iota_f)_*\ } \mathrm{im}^\infty(g) \\
\quad \overset{\iota_{g*}}{\Big\downarrow} \quad {\scriptstyle (B)} \ \overset{\iota_{gf}}{\searrow} \quad \Big\downarrow {\scriptstyle \iota_g} \\
\mathrm{im}^\infty(f) \xrightarrow{\ \iota_f\ } X.
\end{array}
$$

Triangles (A) and (B) have already been shown to commute, and the two squares commute because $\iota_f$ is a map $\mathrm{im}^\infty(f)^{\circlearrowleft g*} \to X^{\circlearrowleft g}$. Hence the triangle between the three copies of $X$ commutes; that is, $(gf)^\infty = g^\infty f^\infty$. ∎

## 4. Factorization systems and the main theorem

Here we prove our main theorem: a category admitting a factorization system of a suitable kind has eventual image duality.

Recall that a **factorization system** on a category $\mathcal{C}$ consists of subcategories $\mathcal{L}$ and $\mathcal{R}$, each containing all the objects and isomorphisms, such that every map in $\mathcal{C}$ factorizes as a map in $\mathcal{L}$ followed by a map in $\mathcal{R}$ uniquely up to unique isomorphism ([10], Section 2). We call maps in $\mathcal{L}$ **coverings** and denote them by $\twoheadrightarrow$; maps in $\mathcal{R}$ are **embeddings**, $\rightarrowtail$. The uniqueness of factorization up to unique isomorphism means that for any solid commutative square

$$
\begin{array}{c}
& I & \\
\nearrow & \ {\scriptstyle \cong}\Big\downarrow{\scriptstyle k}\ & \searrow \\
X & & Y, \\
\searrow & & \nearrow \\
& I' &
\end{array}
$$

there is a unique isomorphism $k$ such that the triangles commute. When a map $f \colon X \to Y$ factorizes as $X \twoheadrightarrow I \rightarrowtail Y$, we write $I$ as $\mathrm{im}(f)$. Typically we leave the maps $X \twoheadrightarrow \mathrm{im}(f) \rightarrowtail Y$ nameless.

The axioms have some standard elementary consequences (proofs omitted).

4.1. LEMMA. *Every factorization system has the following properties.*

i. **(Isomorphisms)** *A map that is both a covering and an embedding is an isomorphism.*

ii. **(Two out of three)** *For composable maps $f$ and $g$, if $gf$ and $g$ are embeddings then so is $f$, and if $gf$ and $f$ are coverings then so is $g$;*

iii. **(Orthogonality)** *The coverings are left orthogonal to the embeddings: for any solid commutative square*

$$
\begin{array}{ccc}
W & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
Y & \longrightarrow & Z,
\end{array}
$$

*there is a unique dotted arrow such that the triangles commute.*

iv. **(Functoriality)** *For any commutative square*

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
u \downarrow & & \downarrow w \\
X' & \xrightarrow[\ f'\ ]{} & Y',
\end{array}
$$

*there is a unique map $v\colon \operatorname{im}(f) \to \operatorname{im}(f')$ such that*

$$
\begin{array}{ccccc}
X & \twoheadrightarrow & \operatorname{im}(f) & \rightarrowtail & Y \\
u \downarrow & & v \downarrow & & \downarrow w \\
X' & \twoheadrightarrow & \operatorname{im}(f') & \rightarrowtail & Y'
\end{array}
$$

*commutes.*

∎

Let $X^{\circlearrowleft f}$ be an endomorphism in a category with a factorization system. Let $n, k \geq 0$. Functoriality applied to the squares

$$
\begin{array}{ccc}
X & \xrightarrow{\ f^{n+k}\ } & X \\
f^k \downarrow & & \downarrow 1 \\
X & \xrightarrow[\ f^n\ ]{} & X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\ f^n\ } & X \\
1 \downarrow & & \downarrow f^k \\
X & \xrightarrow[\ f^{n+k}\ ]{} & X
\end{array}
$$

gives unique dotted maps such that the diagrams

$$
\begin{array}{ccc}
X \longrightarrow \mathrm{im}(f^{n+k}) \rightarrowtail X \\
f^k \downarrow \quad\quad\quad \downarrow \quad\quad\quad \downarrow 1 \\
X \longrightarrow \mathrm{im}(f^n) \rightarrowtail X
\end{array}
\qquad
\begin{array}{ccc}
X \longrightarrow \mathrm{im}(f^n) \rightarrowtail X \\
1 \downarrow \quad\quad\quad \downarrow \quad\quad\quad \downarrow f^k \\
X \longrightarrow \mathrm{im}(f^{n+k}) \rightarrowtail X
\end{array}
\tag{7}
$$

commute. By the two out of three property, the first dotted map is an embedding and the second is a covering. We leave them nameless, writing them as simply

$$
\mathrm{im}(f^{n+k}) \rightarrowtail \mathrm{im}(f^n), \qquad \mathrm{im}(f^n) \twoheadrightarrow \mathrm{im}(f^{n+k}). \tag{8}
$$

The uniqueness in (7) implies that $\mathrm{im}(f^n) \rightarrowtail \mathrm{im}(f^n)$ is the identity and that

$$
\begin{array}{ccc}
 & \mathrm{im}(f^{n+k}) & \\
 & \nearrow \qquad \searrow & \\
\mathrm{im}(f^{n+k+\ell}) \rightarrowtail & \longrightarrow & \mathrm{im}(f^n)
\end{array}
$$

commutes for all $n, k, \ell \geq 0$, and dually. The embeddings and coverings (8) are compatible in the following sense.

4.2. LEMMA. *Let $X^{\circlearrowleft f}$ be an endomorphism in a category with a factorization system. Then for all $n, k, \ell \geq 0$, the square*

$$
\begin{array}{ccc}
\mathrm{im}(f^{n+k}) & \twoheadrightarrow & \mathrm{im}(f^{n+k+\ell}) \\
\downarrow & & \downarrow \\
\mathrm{im}(f^n) & \longrightarrow & \mathrm{im}(f^{n+\ell})
\end{array}
\tag{9}
$$

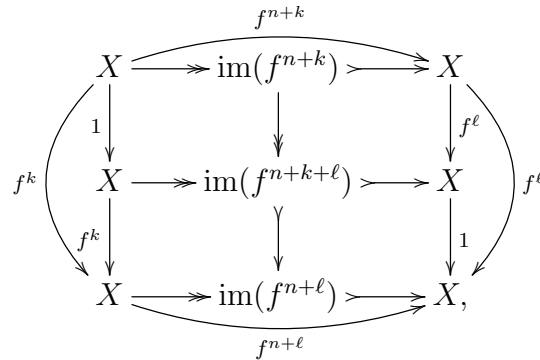*commutes.*

PROOF. By functoriality, there is a unique map $v$ such that that the diagram

$$
\begin{array}{ccc}
 & f^{n+k} & \\
X \rightrightarrows \mathrm{im}(f^{n+k}) \rightarrowtail X \\
f^k \downarrow \quad\quad v \downarrow \quad\quad \downarrow f^\ell \\
X \longrightarrow \mathrm{im}(f^{n+\ell}) \rightarrowtail X \\
 & f^{n+\ell} &
\end{array}
\tag{10}
$$

commutes. It is therefore enough to show that taking $v$ to be either composite around the square (9) makes (10) commute. That the clockwise composite does so follows from

the commutativity of the diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\hspace{1em}} & \mathrm{im}(f^{n+k}) & \rightarrowtail & X \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle f^{\ell}} \\
X & \longrightarrow & \mathrm{im}(f^{n+k+\ell}) & \longrightarrow & X \\
\downarrow{\scriptstyle f^{k}} & & \downarrow & & \downarrow{\scriptstyle 1} \\
X & \xrightarrow{\hspace{1em}} & \mathrm{im}(f^{n+\ell}) & \rightarrowtail & X,
\end{array}
$$

and a similar argument applies to the anticlockwise composite.  ∎

We now formulate conditions on a factorization system expressing the idea that the objects of the category are in some sense finite. The three main examples are as follows; details can be found in Sections 6–8.

4.3. EXAMPLES.

   i. Let **FinSet** be the category of finite sets, with the factorization system in which embeddings are injections and coverings are surjections.

  ii. Let **FDVect** be the category of finite-dimensional vector spaces over a field $k$, again with the injective and surjective maps as the embeddings and coverings.

 iii. Let **CptMet** be the category of compact metric spaces and distance-decreasing (1-Lipschitz) maps. It has a factorization system in which the embeddings are the distance-preserving maps and the coverings are the surjective maps.

4.4. DEFINITION. A factorization system is of **finite type** if it satisfies the following three axioms:

  **I** every endomorphism that is an embedding is an isomorphism;

 **II** every sequence $\cdots \rightarrowtail \cdot \rightarrowtail \cdot$ has a limit;

**III** for every commutative diagram

$$
\begin{array}{ccccc}
\cdots & \rightarrowtail & X_1 & \rightarrowtail & X_0 \\
 & & \downarrow & & \downarrow \\
\cdots & \rightarrowtail & Y_1 & \rightarrowtail & Y_0,
\end{array}
$$

the induced map $\lim X_n \to \lim Y_n$ is a covering;

together with their duals:

$\mathbf{I}^*$  every endomorphism that is a covering is an isomorphism;

$\mathbf{II}^*$  every sequence $\cdot \twoheadrightarrow \cdot \twoheadrightarrow \cdots$ has a colimit;

$\mathbf{III}^*$  for every commutative diagram

$$
\begin{array}{ccc}
Y_0 & \longrightarrow\!\!\!\!\twoheadrightarrow & Y_1 \longrightarrow\!\!\!\!\twoheadrightarrow \cdots \\
\downarrow & & \downarrow \\
X_0 & \longrightarrow\!\!\!\!\twoheadrightarrow & X_1 \longrightarrow\!\!\!\!\twoheadrightarrow \cdots \ ,
\end{array}
$$

the induced map $\operatorname{colim} Y_n \to \operatorname{colim} X_n$ is an embedding.

All three of Examples 4.3 are of finite type, as shown in Sections 6–8.

4.5.  REMARK. If the coverings in axiom $\mathbf{III}$ are replaced by embeddings then the induced map $\lim X_n \to \lim Y_n$ is automatically an embedding, by the two out of three property and Lemma 4.8 below. Hence axiom $\mathbf{III}$ is equivalent to the statement that factorizations are preserved by sequential limits of embeddings.

We set out some elementary consequences of the axioms.

4.6.  LEMMA. *In a factorization system satisfying axioms $\mathbf{I}$ and $\mathbf{I}^*$:*

*i.  every split monic covering is an isomorphism, and every split epic embedding is an isomorphism;*

*ii.  every split monic is an embedding, and every split epic is a covering.*

PROOF. By duality, it suffices to prove the first statement in each part. For (i), let $i\colon Y \twoheadrightarrow X$ be a split monic covering, so that $pi = 1_Y$ for some $p\colon X \to Y$. By the two out of three property, $p$ is also a covering. Hence $ip\colon X \to X$ is a covering, which by axiom $\mathbf{I}^*$ implies that $ip$ is an isomorphism. So $i$ is epic as well as split monic, and is therefore an isomorphism.

For (ii), let $i\colon Y \to X$ be a split monic. Factorize $i$ as

$$
i = \big(Y \overset{q}{\twoheadrightarrow} \operatorname{im}(i) \rightarrowtail X\big).
$$

Then $q$ is also split monic. By (i), $q$ is an isomorphism, so $i$ is an embedding.  ∎

4.7.  EXAMPLE. In a category with a factorization system satisfying $\mathbf{I}$ and $\mathbf{I}^*$, Lemma 4.6(ii) implies that the splitting object of a split idempotent $e$ is $\operatorname{im}(e)$. So when $f$ is an endomorphism with eventual image duality, $\operatorname{im}^\infty(f) = \operatorname{im}(f^\infty)$.

4.8.  LEMMA. *Let $\mathcal{C}$ be a category with a factorization system satisfying axioms $\mathbf{I}$ and $\mathbf{I}^*$. Let*

$$
\cdots \overset{f_1}{\rightarrowtail} X_1 \overset{f_0}{\rightarrowtail} X_0 \tag{11}
$$

*be a diagram in $\mathcal{C}$ with a limit cone $\big(L \overset{j_n}{\longrightarrow} X_n\big)_{n\in\mathbb{N}}$. Then $j_n$ is an embedding for all $n \in \mathbb{N}$. The dual statement holds for sequential colimits of coverings.*

PROOF. First we show that $j_0, j_1, \ldots$ all have the same image. For each $n \in \mathbb{N}$, factorize $j_n$ as

$$j_n = \left( L \xrightarrow{q_n} \mathrm{im}(j_n) \xrightarrow{k_n} X_n \right).$$

By the cone property,

$$j_n = f_n j_{n+1} = \left( L \xrightarrow{q_{n+1}} \mathrm{im}(j_{n+1}) \xrightarrow{k_{n+1}} X_{n+1} \xrightarrow{f_n} X_n \right).$$

By uniqueness of factorizations, there is a unique isomorphism $\mathrm{im}(j_n) \xrightarrow{\sim} \mathrm{im}(j_{n+1})$ compatible with these two factorizations of $j_n$. Put $I = \mathrm{im}(j_0)$ and $q = q_0 \colon L \twoheadrightarrow I$, and let $j'_n \colon I \rightarrowtail X_n$ be the composite

$$I = \mathrm{im}(j_0) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathrm{im}(j_n) \xrightarrow{k_n} X_n.$$

One easily checks that for all $n \in \mathbb{N}$,

$$j_n = \left( L \xrightarrow{q} I \xrightarrow{j'_n} X_n \right), \qquad j'_n = \left( I \xrightarrow{j'_{n+1}} X_{n+1} \xrightarrow{f_n} X_n \right).$$

Hence $\left( I \xrightarrow{j'_n} X_n \right)_{n \in \mathbb{N}}$ is a cone on the diagram (11), so there is a unique map $r \colon I \to L$ such that

$$j'_n = \left( I \xrightarrow{r} L \xrightarrow{j_n} X_n \right)$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we have $j_n r q = j'_n q = j_n$, and it follows from the limit property of $L$ that $rq = 1_L$. So $q$ is a split monic covering, hence an isomorphism by Lemma 4.6(i). But $j_n = j'_n q$, so $j_n$ is an embedding. ∎

We can now prove the main theorem.

4.9. THEOREM. *A category admitting a factorization system of finite type has eventual image duality. Moreover, in such a category:*

i. *the eventual image* $\mathrm{im}^\infty(f)$ *of an endomorphism* $X^{\circlearrowleft f}$ *is the limit of the diagram*

$$\cdots \rightarrowtail \mathrm{im}(f^2) \rightarrowtail \mathrm{im}(f) \rightarrowtail X,$$

*the map* $\iota_f \colon \mathrm{im}^\infty(f) \to X$ *is the 0th projection of the limit cone, and* $\mathrm{im}^\infty(f)^{\circlearrowleft \widetilde{f}}$ *is the map on limits induced by the map of diagrams*

$$
\begin{array}{ccccc}
\cdots \rightarrowtail & \mathrm{im}(f^2) & \longrightarrow & \mathrm{im}(f) & \longrightarrow & X \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \rightarrowtail & \mathrm{im}(f^3) & \longrightarrow & \mathrm{im}(f^2) & \longrightarrow & \mathrm{im}(f);
\end{array}
\tag{12}
$$

ii. *dually, the eventual image* $\mathrm{im}^\infty(f)$ *of* $X^{\circlearrowright f}$ *is the colimit of the diagram*

$$X \twoheadrightarrow \mathrm{im}(f) \twoheadrightarrow \mathrm{im}(f^2) \twoheadrightarrow \cdots,$$

*the map* $\pi_f\colon X \to \mathrm{im}^\infty(f)$ *is the* 0*th coprojection of the colimit cone, and* $\mathrm{im}^\infty(f)^{\circlearrowright \widetilde{f}}$ *is the map on colimits induced by the map of diagrams*

$$
\begin{array}{ccccccc}
\mathrm{im}(f) & \longrightarrow\!\!\!\!\!\rightarrow & \mathrm{im}(f^2) & \longrightarrow\!\!\!\!\!\rightarrow & \mathrm{im}(f^3) & \longrightarrow\!\!\!\!\!\rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
X & \longrightarrow\!\!\!\!\!\rightarrow & \mathrm{im}(f) & \longrightarrow\!\!\!\!\!\rightarrow & \mathrm{im}(f^2) & \longrightarrow\!\!\!\!\!\rightarrow & \cdots.
\end{array}
\tag{13}
$$

The commutativity of diagrams (12) and (13) follows from Lemma 4.2.

PROOF. Let $\mathcal{C}$ be a category with a factorization system of finite type, and let $X^{\circlearrowright f}$ in $\mathcal{C}$. The diagram

$$\cdots \rightarrowtail \mathrm{im}(f^2) \rightarrowtail \mathrm{im}(f) \rightarrowtail \mathrm{im}(f^0) = X \tag{14}$$

has a limit cone $\left(L \xrightarrow{j_n} \mathrm{im}(f^n)\right)_{n\in\mathbb{Z}}$, where $j_n$ is an embedding by Lemma 4.8. Dually,

$$X = \mathrm{im}(f^0) \twoheadrightarrow \mathrm{im}(f) \twoheadrightarrow \mathrm{im}(f^2) \twoheadrightarrow \cdots$$

has a colimit cone $\left(\mathrm{im}(f^n) \xrightarrow{k_n} M\right)_{n\in\mathbb{Z}}$.

We will show that $L$ is also a limit of $\cdots \xrightarrow{f} X \xrightarrow{f} \cdots$. It will follow by duality that $M$ is its colimit, and we then show that the canonical map $L \to M$ is an isomorphism.

First we construct an automorphism of $L$. Taking limits in diagram (12), there is a unique map $\hat{f}\colon L \to L$ such that
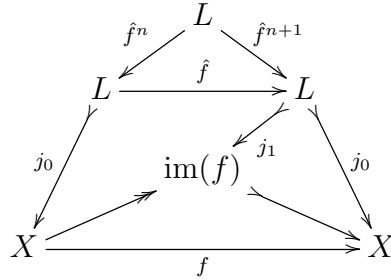
$$
\begin{array}{ccc}
L & \xrightarrow{\;\;j_n\;\;} & \mathrm{im}(f^n) \\
{\scriptstyle \hat{f}}\downarrow & & \downarrow \\
L & \xrightarrow[\;\;j_{n+1}\;\;]{} & \mathrm{im}(f^{n+1})
\end{array}
\tag{15}
$$

commutes for all $n \in \mathbb{N}$. By axiom **III**$^*$, $\hat{f}$ is a covering, which by axiom **I**$^*$ implies that $\hat{f}$ is an automorphism of $L$.

Next observe that the family of maps

$$\left(L \xrightarrow{\hat{f}^n} L \xrightarrow{j_0} X\right)_{n\in\mathbb{Z}} \tag{16}$$

is a cone on $\cdots \xrightarrow{f} X \xrightarrow{f} \cdots$, since the diagram



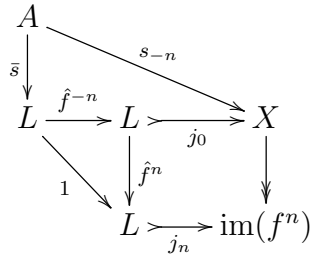commutes for each $n \in \mathbb{Z}$.

We will prove that (16) is a limit cone. Take an arbitrary cone $\left(A \xrightarrow{s_n} X\right)_{n \in \mathbb{Z}}$ on $\cdots \xrightarrow{f} X \xrightarrow{f} \cdots$. We must show there is a unique map $\bar{s} \colon A \to L$ such that the diagram
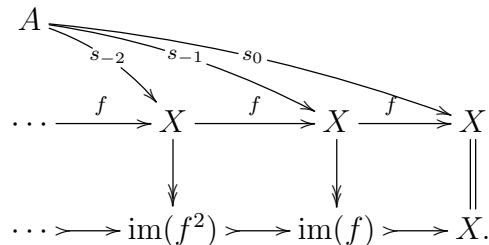
$$\begin{array}{c} A \\ {\scriptstyle\bar{s}}\downarrow \quad \searrow^{s_n} \\ L \xrightarrow{\hat{f}^n} L \rightarrowtail_{j_0} X \end{array}$$

(17)

commutes for all $n \in \mathbb{Z}$.

For uniqueness, take such a map $\bar{s}$. Then for all $n \in \mathbb{N}$, the diagram



commutes, the inner square by diagram (15) and induction. Since the cone $\left(L \xrightarrow{j_n} \operatorname{im}(f^n)\right)_{n \in \mathbb{N}}$ is a limit, this property determines $\bar{s}$ uniquely.

For existence, consider the diagram



The upper part commutes by definition of cone, and the lower part commutes by the leftmost square of diagrams (7) in the case $k = 1$. Hence

$$\left(A \xrightarrow{s_{-n}} X \twoheadrightarrow \operatorname{im}(f^n)\right)_{n \in \mathbb{N}}$$

is a cone on (14). There is, therefore, a unique map $\bar{s}\colon A \to L$ such that

$$
\begin{array}{ccc}
A & \xrightarrow{\ s_{-n}\ } & X \\
{\scriptstyle \bar{s}}\downarrow & & \downarrow \\
L & \underset{j_n}{\rightarrowtail} & \mathrm{im}(f^n)
\end{array}
\tag{18}
$$

commutes for all $n \in \mathbb{N}$. Our task is to show that diagram (17) commutes for all $n \in \mathbb{Z}$. Now for each $n \in \mathbb{Z}$, there is a cone

$$
\begin{array}{c}
A \\
\cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} X \\
\cdots \rightarrowtail \mathrm{im}(f^2) \rightarrowtail \mathrm{im}(f) \rightarrowtail X
\end{array}
$$

on $\cdots \rightarrowtail \mathrm{im}(f) \rightarrowtail X$, so there is a unique map $\bar{s}_n\colon A \to L$ such that for all $m \geq 0$,

$$
\begin{array}{ccc}
A & \xrightarrow{\ s_{n-m}\ } & X \\
{\scriptstyle \bar{s}_n}\downarrow & & \downarrow \\
L & \underset{j_m}{\rightarrowtail} & \mathrm{im}(f^m)
\end{array}
\tag{19}
$$

commutes. In particular, $\bar{s}_0 = \bar{s}$.

I claim that $\bar{s}_{n+1} = \hat{f} \circ \bar{s}_n$ for all $n \in \mathbb{Z}$. By the limit property of $L$, it is enough to prove that for each $m \geq 1$, the outside of the diagram

$$
\begin{array}{c}
A \xrightarrow{\ \bar{s}_{n+1}\ } L \\
\end{array}
$$

commutes. The inner polygons commute, the squares being cases of (19) and (15), so the claim is proved.

It follows that for all $n \in \mathbb{Z}$, the left-hand triangle of

$$
\begin{array}{c}
A \\
L \xrightarrow{\hat{f}^n} L \xrightarrow{j_0} X
\end{array}
$$

commutes. The right-hand triangle also commutes, being the case $m = 0$ of diagram (19). Hence the outside, which is diagram (17), commutes. This completes the proof that $\left(L \xrightarrow{\hat{f}^n} L \xrightarrow{j_0} X\right)_{n \in \mathbb{Z}}$ is a limit cone on $\cdots \xrightarrow{f} X \xrightarrow{f} \cdots$.

Dually, $\left(X \xrightarrow{k_0} M \xrightarrow{\hat{f}^n} M\right)_{n \in \mathbb{Z}}$ is a colimit cone on the same diagram.

Next we show that the composite $L \xrightarrow{j_0} X \xrightarrow{k_0} M$ is an isomorphism. By (15), the diagram

$$
\begin{array}{ccccccc}
L & \xrightarrow{\hat{f}}_{\sim} & L & \xrightarrow{\hat{f}}_{\sim} & L & \xrightarrow{\hat{f}}_{\sim} & \cdots \\
\scriptstyle{j_0}\downarrow & & \scriptstyle{j_1}\downarrow & & \scriptstyle{j_2}\downarrow & & \\
X & \longrightarrow\!\!\!\!\to & \mathrm{im}(f) & \longrightarrow\!\!\!\!\to & \mathrm{im}(f^2) & \longrightarrow\!\!\!\!\to & \cdots
\end{array}
,
$$

commutes. The top row has colimit $L$ with $n$th coprojection $\hat{f}^{-n}$; the bottom row has colimit $M$. Write $\phi\colon L \to M$ for the induced map, which is unique such that

$$
\begin{array}{ccc}
L & \xrightarrow{\hat{f}^{-n}} & L \\
\scriptstyle{j_n}\downarrow & & \downarrow\scriptstyle{\phi} \\
\mathrm{im}(f^n) & \xrightarrow{k_n} & M
\end{array}
$$

commutes for all $n \in \mathbb{N}$. In particular, it commutes for $n = 0$, so $\phi = k_0 j_0$. But by axiom **III***, $\phi$ is an embedding, so $k_0 j_0$ is an embedding. By duality, $k_0 j_0$ is also a covering. Hence $k_0 j_0$ is an isomorphism, as claimed.

We have shown that $X^{\circlearrowright f}$ has eventual image duality and that $\mathrm{im}^\infty(f)$ can be constructed as either the limit $L$ of $\cdots \rightarrowtail \mathrm{im}(f) \rightarrowtail X$ (with $\iota_f$ as the 0th projection $j_0$) or the colimit $M$ of $X \twoheadrightarrow \mathrm{im}(f) \twoheadrightarrow \cdots$ (with $\pi_f = k_0$). It only remains to prove that the map $\hat{f}$ induced on limits by the map of diagrams (12) is $\widetilde{f}$; the dual statement on colimits will follow by duality. For this, we must prove that the outside of the square

$$
\begin{array}{ccc}
L & \xrightarrow{\mathrm{pr}_n} & X \\
\scriptstyle{\hat{f}}\downarrow & \searrow^{\mathrm{pr}_{n+1}} & \downarrow\scriptstyle{f} \\
L & \xrightarrow{\mathrm{pr}_n} & X
\end{array}
$$

commutes for each $n \in \mathbb{Z}$, where $\mathrm{pr}_n = j_0 \circ \hat{f}^n$ is the $n$th projection of the limit cone just constructed. The lower triangle commutes by definition of $\mathrm{pr}_n$, and the upper triangle since $(\mathrm{pr}_m)_{m \in \mathbb{Z}}$ is a cone. This completes the proof. ∎

## 5. The eventual image is a terminal coalgebra

In our three main example categories, the eventual image of an endomorphism $X^{\circlearrowright f}$ is the largest subspace $A$ of $X$ satisfying $A \subseteq fA$. Here, we generalize this statement to categories with a factorization system of finite type.

The general result will be expressed in terms of terminal coalgebras. Recall that given an endofunctor $T$ of a category $\mathcal{A}$, a $T$-**coalgebra** is a pair $(A, \alpha)$ with $A \in \mathcal{A}$ and $\alpha \colon A \to TA$. With the obvious maps, $T$-coalgebras form a category. The *terminal* $T$-coalgebra, if it exists, plays an important role, and Lambek showed that it is a fixed point: its structure map $\alpha$ is an isomorphism (Lemma 2.2 of [15]).

Coalgebras in this sense arise in many situations in mathematics and computer science, typically involving infinite iteration or coinduction. To give just two examples, bisimulation in the context of Milner's concurrency theory can be described in terms of coalgebras [1], and weak $\infty$-categories can be defined using terminal coalgebras [7]. See Adámek [3] and Rutten [21] for surveys.

Let $\mathcal{C}$ be a category with a factorization system. Let $X \in \mathcal{C}$. The slice category $\mathcal{C}/X$ has a full subcategory $\mathbf{Emb}(X)$ consisting of the embeddings into $X$. A map from $A \stackrel{j}{\rightarrowtail} X$ to $B \stackrel{k}{\rightarrowtail} X$ in $\mathbf{Emb}(X)$ is, then, a map $u \colon A \to B$ in $\mathcal{C}$ such that

$$
\begin{array}{ccc}
A & \xrightarrow{\;u\;} & B \\
& \underset{j}{\searrow} \;\; X \;\; \underset{k}{\swarrow} &
\end{array}
\tag{20}
$$

commutes, and the two out of three property implies that $u$ is also an embedding.

Given also an endomorphism $f$ of $X$, there is an endofunctor $f_!$ of $\mathbf{Emb}(X)$ defined as follows. For an object $A \stackrel{j}{\rightarrowtail} X$, take the image factorization

$$
\begin{array}{ccc}
A & \stackrel{j}{\rightarrowtail} & X \\
\downarrow & & \downarrow{\scriptstyle f} \\
fA & \underset{j\#}{\rightarrowtail} & X
\end{array}
$$

of $fj$ (where $fA$ is alternative notation for $\mathrm{im}(fj)$) and define

$$
f_! \left( A \stackrel{j}{\rightarrowtail} X \right) = \left( fA \stackrel{j\#}{\rightarrowtail} X \right).
$$

For a map (20) in $\mathbf{Emb}(X)$, the solid part of the diagram

$$
\begin{array}{ccc}
A & \stackrel{u}{\rightarrowtail} & B \\
\downarrow & \searrow_j \; X \; \swarrow^k & \downarrow \\
fA & \stackrel{f_!(u)}{\dashrightarrow} & fB \\
& \searrow_{j\#} \; \downarrow^f X \; \swarrow^{k\#} &
\end{array}
$$

commutes, so by orthogonality (Lemma 4.1(iii)), there is a unique map $f_!(u) \colon fA \to fB$ making the diagram commute. Then $f_!(u)$ is a map from $f_!\left( A \stackrel{j}{\rightarrowtail} X \right)$ to $f_!\left( B \stackrel{k}{\rightarrowtail} X \right)$ in $\mathbf{Emb}(X)$. This defines an endofunctor $f_!$ of $\mathbf{Emb}(X)$.

5.1. EXAMPLE. Let $X^{\circlearrowright f}$ be an endomorphism in $\mathcal{C}$ with eventual image duality. The diagram

$$
\begin{array}{ccc}
\mathrm{im}^\infty(f) & \xrightarrow{\ \iota_f\ } & X \\
\widetilde{f}\ \downarrow\ \cong & & \downarrow\ f \\
\mathrm{im}^\infty(f) & \xrightarrow[\ \iota_f\ ]{} & X
\end{array}
$$

commutes by definition of $\widetilde{f}$, so $f_!$ fixes the object $\mathrm{im}^\infty(f) \overset{\iota_f}{\rightarrowtail} X$ of $\mathbf{Emb}(X)$. Together with the identity, this object is a coalgebra for $f_!$, which we call just $\mathrm{im}^\infty(f)$.

We prove that $\mathrm{im}^\infty(f)$ is the terminal $f_!$-coalgebra using a standard result generally attributed to Adámek [2]; see also [3], Corollary 3.18.

5.2. THEOREM. [Adámek] *Let $T$ be an endofunctor of a category $\mathcal{A}$. Suppose that $\mathcal{A}$ has a terminal object $1$, that the diagram*

$$
\cdots \xrightarrow{T^2!} T^2 1 \xrightarrow{T!} T1 \xrightarrow{!} 1 \tag{21}
$$

*has a limit $\left(L \xrightarrow{j_n} T^n 1\right)_{n\in\mathbb{N}}$ in $\mathcal{A}$ (where ! is the unique map $T1 \to 1$), and that this limit is preserved by $T$. Write $\lambda$ for the canonical isomorphism $TL \to L$. Then $(L, \lambda^{-1})$ is the terminal $T$-coalgebra.*

Here $\lambda$ is the unique map $TL \to L$ such that

$$
\begin{array}{ccc}
TL & \xrightarrow{\ \lambda\ } & L \\
 & {\scriptstyle T(j_n)} \searrow & \ \downarrow {\scriptstyle j_{n+1}} \\
 & & T^{n+1} 1
\end{array}
$$

commutes for all $n \in \mathbb{N}$, which is an isomorphism since $T$ preserves the limit.

To apply Adámek's theorem, we use the following observation.

5.3. LEMMA. *Let $\mathcal{C}$ be a category with a factorization system of finite type, and let $X \in \mathcal{C}$. Then $\mathbf{Emb}(X)$ has, and the forgetful functor $\mathbf{Emb}(X) \to \mathcal{C}$ creates, sequential limits.*

PROOF. Take a diagram

$$
\cdots \xrightarrow{u_1} \left( \begin{array}{c} A_1 \\ \downarrow {\scriptstyle i_1} \\ X \end{array} \right) \xrightarrow{u_0} \left( \begin{array}{c} A_1 \\ \downarrow {\scriptstyle i_0} \\ X \end{array} \right) \tag{22}
$$

in $\mathbf{Emb}(X)$. We show that $\cdots \xrightarrow{u_1} A_1 \xrightarrow{u_0} A_0$ has a limit cone in $\mathcal{C}$ and that any such cone lifts uniquely to a cone in $\mathbf{Emb}(X)$, which is also a limit cone.

By the two out of three property, each $u_n$ is an embedding. Hence the diagram $\cdots \overset{u_1}{\rightarrowtail} A_1 \overset{u_0}{\rightarrowtail} A_0$ has a limit cone $\left(L \xrightarrow{j_n} A_n\right)_{n\in\mathbb{N}}$ in $\mathcal{C}$, and by Lemma 4.8, each $j_n$ is an

embedding. The forgetful functor $\mathcal{C}/X \to \mathcal{C}$ strictly creates connected limits, so there is a unique map $k\colon L \to X$ such that

$$\left(\left(\begin{array}{c} L \\ \downarrow k \\ X \end{array}\right) \;\xrightarrow{\;j_n\;}\; \left(\begin{array}{c} A_n \\ \downarrow i_n \\ X \end{array}\right)\right)_{n\in\mathbb{N}} \tag{23}$$

is a limit cone in $\mathcal{C}/X$. Then $k = i_0 j_0$, and $i_0$ and $j_0$ are embeddings, so $k$ is too. Hence (23) is a limit cone on (22) in $\mathbf{Emb}(X)$.                                   ∎

5.4. THEOREM. *Let $\mathcal{C}$ be a category with a factorization system of finite type. Let $X^{\circlearrowleft f}$ be an endomorphism in $\mathcal{C}$. Then $\mathrm{im}^\infty(f)$ is the terminal coalgebra for the endofunctor* $\mathbf{Emb}(X)^{\circlearrowleft f_!}$.

PROOF. We use Adámek's theorem, first showing that the diagram (21) is in this case

$$\cdots \;\rightarrowtail\; \left(\begin{array}{c} \mathrm{im}(f^2) \\ \downarrow \\ X \end{array}\right) \;\rightarrowtail\; \left(\begin{array}{c} \mathrm{im}(f) \\ \downarrow \\ X \end{array}\right) \;\rightarrowtail\; \left(\begin{array}{c} X \\ \downarrow 1 \\ X \end{array}\right). \tag{24}$$

The terminal object of $\mathbf{Emb}(X)$ is $(X \xrightarrow{1} X)$. That $f_!^n$ applied to the terminal object is $\mathrm{im}(f^n) \rightarrowtail X$ follows by induction from the rightmost square of diagrams (7) with $k = 1$. Now assume inductively that the map $T^n!$ of diagram (21) is the embedding

$$\left(\begin{array}{c} \mathrm{im}(f^{n+1}) \\ \downarrow \\ X \end{array}\right) \;\rightarrowtail\; \left(\begin{array}{c} \mathrm{im}(f^n) \\ \downarrow \\ X \end{array}\right). \tag{25}$$

By definition, $f_!$ applied to the map (25) is the unique dotted map making the diagram

$$
\begin{array}{ccc}
\mathrm{im}(f^{n+1}) & \rightarrowtail & \mathrm{im}(f^n) \\
\downarrow & & \downarrow \\
\mathrm{im}(f^{n+2}) & \dashrightarrow & \mathrm{im}(f^{n+1}) \\
& \searrow \quad \swarrow & \\
& X &
\end{array}
$$

commute. But by Lemma 4.2, the embedding $\mathrm{im}(f^{n+2}) \rightarrowtail \mathrm{im}(f^{n+1})$ makes this diagram commute, completing the induction.

Theorem 4.9(i) gives a limit cone

$$\left(\mathrm{im}^\infty(f) \xrightarrow{\;j_n\;} \mathrm{im}(f^n)\right)_{n\in\mathbb{N}}$$

on $\cdots \rightarrowtail \mathrm{im}(f) \rightarrowtail X$, and then

$$\left(\left(\begin{array}{c}\mathrm{im}^\infty(f) \\ \Big\downarrow {\iota_f = j_0} \\ X\end{array}\right) \xrightarrow{\;\;j_n\;\;} \left(\begin{array}{c}\mathrm{im}(f^n) \\ \Big\downarrow \\ X\end{array}\right)\right)_{n\in\mathbb{N}} \tag{26}$$

is a cone on (24). By Lemma 5.3, it is a limit cone.

It remains to show that this limit is preserved by $f_!$, and for this, it is enough to prove that $f_!$ maps the cone (26) to

$$\left(\left(\begin{array}{c}\mathrm{im}^\infty(f) \\ \Big\downarrow {\iota_f} \\ X\end{array}\right) \xrightarrow{\;\;j_{n+1}\;\;} \left(\begin{array}{c}\mathrm{im}(f^{n+1}) \\ \Big\downarrow \\ X\end{array}\right)\right)_{n\in\mathbb{N}}.$$

We have already shown that $f_!$ fixes the object $\mathrm{im}^\infty(f) \xrightarrow{\iota_f} X$ of $\mathbf{Emb}(X)$ and that it maps $\mathrm{im}(f^n) \rightarrowtail X$ to $\mathrm{im}(f^{n+1}) \rightarrowtail X$. Moreover, for each $n \in \mathbb{N}$ we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{im}^\infty(f) & \xrightarrow{\;\;j_n\;\;} & \mathrm{im}(f^n) \\
{\tilde{f}}\Big\downarrow {\cong} & & \Big\downarrow \\
\mathrm{im}^\infty(f) & \xrightarrow{\;\;j_{n+1}\;\;} & \mathrm{im}(f^{n+1}) \\
& \searrow{\iota_f} \quad \swarrow & \\
& X, &
\end{array}$$

where the square commutes by Theorem 4.9(i). By definition of $f_!$ on morphisms, this means that $f_!(j_n) = j_{n+1}$, as required. ∎

The dual result characterizes the eventual image as the initial algebra for an endofunctor on the category of covering maps out of $X$.

## 6. Finite sets

The category $\mathbf{FinSet}$ of finite sets has a factorization system consisting of injections and surjections. It is of finite type: axioms $\mathbf{I}$ and $\mathbf{I}^*$ state that any injective or surjective endomorphism of a finite set is invertible, and the rest of the axioms are trivial because any diagram

$$\cdots \rightarrowtail X_1 \rightarrowtail X_0 \qquad \text{or} \qquad X_0 \twoheadrightarrow X_1 \twoheadrightarrow \cdots$$

in $\mathbf{FinSet}$ stabilizes after a finite number of steps. So Theorems 4.9 and 5.4 apply, showing that $\mathbf{FinSet}$ has eventual image duality and providing characterizations of the eventual image, which we now study in detail.
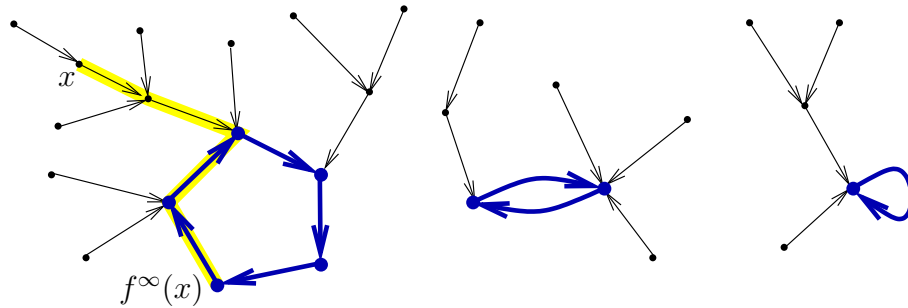
Figure 1: An endomorphism $f$ of a finite set. The eventual image and its automorphism $\widetilde{f}$ are shown in bold blue. A point $x$ and the resulting point $f^\infty(x)$ are shown, with the yellow highlighted path illustrating the back and forth description of $f^\infty$.

For the rest of this section, let $f$ be an endomorphism of a finite set $X$ (Figure 1). The eventual image of $f$ is both the limit and colimit of the diagram

$$\cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots . \tag{27}$$

Example 2.2 gives explicit descriptions of the limit, the colimit, and the canonical map from limit to colimit. That the canonical map is bijective means that for any $N \in \mathbb{Z}$ and $x \in X$, there is a unique double sequence $(y_n)_{n \in \mathbb{Z}}$ such that $f(y_n) = y_{n+1}$ for all $n$ and $y_n = f^{n-N}(x)$ for all sufficiently large $n$. Writing $x_n = f^{n-N}(x)$, this condition can be depicted as follows:

$$\cdots \longmapsto y_N \longmapsto \cdots \longmapsto y_{p-1}$$
$$x = x_N \longmapsto \cdots \longmapsto x_{p-1} \longmapsto x_p = y_p \longmapsto x_{p+1} = y_{p+1} \longmapsto \cdots$$

Theorem 4.9(i) implies that $\operatorname{im}^\infty(f) \cong \bigcap_{n \in \mathbb{N}} \operatorname{im}(f^n)$. The chain of inclusions

$$\cdots \subseteq \operatorname{im}(f^2) \subseteq \operatorname{im}(f) \subseteq X \tag{28}$$

contains at most $|X|$ proper inclusions, and if any inclusion is an equality then so are all the inclusions to its left. Hence the sequence stabilizes after at most $|X|$ steps and $\operatorname{im}^\infty(f) = \operatorname{im}(f^{|X|})$. The canonical isomorphism from the limit of (27) to $\bigcap_{n \in \mathbb{N}} \operatorname{im}(f^n)$ is $(x_n)_{n \in \mathbb{Z}} \mapsto x_0$.

Dually, Theorem 4.9(ii) implies that $\operatorname{im}^\infty(f) \cong X/\sim$, where $x \sim y$ if $f^n(x) = f^n(y)$ for some $n \in \mathbb{N}$. By a similar stabilization argument, $x \sim y$ if and only if $f^{|X|}(x) = f^{|X|}(y)$. The canonical isomorphism from $X/\sim$ to the colimit of (27) maps the equivalence class of $x$ to the equivalence class of $(0, x)$.

Let us temporarily write $\operatorname{im}^\infty(f)$ and $\operatorname{coim}^\infty(f)$ for the limit and colimit of (27), respectively. The map $\iota_f \colon \operatorname{im}^\infty(f) \to X$ is $(x_n)_{n \in \mathbb{Z}} \mapsto x_0$, and the map $\pi_f \colon X \to \operatorname{coim}^\infty(f)$

is $x \mapsto [(0, x)]$. Their composite is the canonical map

$$\begin{array}{rcl} \text{can}: \quad \text{im}^\infty(f) & \to & \text{coim}^\infty(f) \\ (x_n)_{n\in\mathbb{Z}} & \mapsto & [(0, x_0)], \end{array}$$

which is a bijection. Now consider the idempotent $f^\infty$, which is the composite

$$X \xrightarrow{\pi_f} \text{coim}^\infty(f) \xrightarrow{\text{can}^{-1}} \text{im}^\infty(f) \xrightarrow{\iota_f} X. \tag{29}$$

6.1. PROPOSITION. *Let $x \in X$. Then for all $n \geq |X|$, we have $f^n(x) \in \text{im}^\infty(f)$ and $f^\infty(x) = \widetilde{f}^{-n}(f^n(x))$.*

Typically we treat $\text{im}^\infty(f)$ as a subset of $X$, as in this statement.

PROOF. Let $x \in X$. The effect of the maps (29) on $x$ is

$$x \mapsto [(0, x)] \mapsto (y_n)_{n\in\mathbb{Z}} \mapsto y_0 = f^\infty(x),$$

where $(y_n)_{n\in\mathbb{Z}}$ is the unique double sequence in $X$ such that $f(y_n) = y_{n+1}$ for all $n \in \mathbb{Z}$ and $y_n = f^n(x)$ for all sufficiently large $n$. Equivalently, 'for all sufficiently large $n$' can be replaced by 'for all $n \geq |X|$' (by the description of $\sim$ above). Hence for all $n \geq |X|$,

$$f^n(f^\infty(x)) = f^n(y_0) = y_n = f^n(x) \tag{30}$$

Since $f^\infty(x)$ is in the subset $\text{im}^\infty(f)$ of $X$, which is $f$-invariant, $f^n(x) \in \text{im}^\infty(f)$. Finally, applying $\widetilde{f}^{-n}$ to (30) gives $f^\infty(x) = \widetilde{f}^{-n}(f^n(x))$. ∎

This result gives a back and forth algorithm for computing $f^\infty(x)$ (Figure 1): apply $f$ to $x$ enough times to put it into the eventual image, then apply $\widetilde{f}^{-1}$ the same number of times.

The back and forth description of $f^\infty$ is well known in finite semigroup theory, and further light is shed by a standard result (Corollary 1.2 of [22]):

6.2. LEMMA. *Let $S$ be a finite semigroup and $\sigma \in S$. Then the set $\{\sigma, \sigma^2, \ldots\}$ contains exactly one idempotent.*

PROOF. Since $S$ is finite, there exist $m, k \geq 1$ such that $\sigma^m = \sigma^{m+k}$. Then $\sigma^{n+rk} = \sigma^n$ for all $n \geq m$ and $r \geq 0$. Hence $\sigma^{mk}$ is idempotent. Moreover, if both $\sigma^p$ and $\sigma^q$ are idempotent $(p, q \geq 1)$ then $\sigma^p = (\sigma^p)^q = (\sigma^q)^p = \sigma^q$. ∎

Next we show that $f^\infty$ is a finite power of $f$.

6.3. PROPOSITION. *The endomorphism $f^\infty$ is the unique idempotent element of $\{f, f^2, \ldots\}$. In fact, $f^\infty = f^{|X|!}$.*

PROOF. Since $\widetilde{f}$ is a permutation of a finite set, $\widetilde{f}^{-1} = \widetilde{f}^r$ for some $r \geq 0$. Then by Proposition 6.1, $f^\infty = f^{(r+1)|X|} \in \{f, f^2, \ldots\}$.

To prove that $f^\infty = f^{|X|!}$, we refine the proof of Lemma 6.2. Let $x \in X$. Since $x, f(x), \ldots, f^{|X|}(x)$ are not all distinct, $f^m(x) = f^{m+k}(x)$ for some $m \geq 0$ and $k \geq 1$ such that $m + k \leq |X|$. Then $f^{n+rk}(x) = f^n(x)$ for all $n \geq m$ and $r \geq 0$. In particular, $f^{2n}(x) = f^n(x)$ whenever $n \geq m$ and $k \mid n$. It follows that $f^{|X|!}$ is idempotent, so by the uniqueness part of Lemma 6.2, $f^\infty = f^{|X|!}$. ∎

Theorem 5.4 describes $\mathrm{im}^\infty(f)$ as the terminal coalgebra for the endofunctor $f_!$ of $\mathbf{Emb}(X)$. In this case, $\mathbf{Emb}(X)$ is equivalent to the power set of $X$, and Theorem 5.4 states that $\mathrm{im}^\infty(f)$ is the largest subset $A$ of $X$ satisfying $A \subseteq fA$. The dual theorem states that $\mathrm{im}^\infty(f) = X/\sim$, where $\sim$ is the finest equivalence relation on $X$ such that $f(x) \sim f(y) \implies x \sim y$ for all $x, y \in X$.

A point $x \in X$ is **periodic** for $f$ if $x \in \{f(x), f^2(x), \ldots\}$. As Figure 1 suggests:

6.4. PROPOSITION. *The set of periodic points of $f$ is $\mathrm{im}^\infty(f)$.*

PROOF. Let $A$ be the set of periodic points. By definition, $A \subseteq fA$, so Theorem 5.4 implies that $A \subseteq \mathrm{im}^\infty(f)$. Conversely, every element $x \in \mathrm{im}^\infty(f)$ is periodic: $\widetilde{f}$ is a permutation of the finite set $\mathrm{im}^\infty(f)$, so $\widetilde{f}^n = 1$ for some $n \geq 1$, and then $x = \widetilde{f}^n(x) = f^n(x)$.                    ∎

# 7. Finite-dimensional vector spaces

Let **FDVect** be the category of finite-dimensional vector spaces over a field $k$. The injective and surjective linear maps form a factorization system of finite type: axioms **I** and **I**$^*$ hold because any injective or surjective endomorphism is invertible (by the rank-nullity formula), and the other axioms hold because any nested sequence of subspaces or quotient spaces must stabilize after a finite number of steps. By Theorem 4.9, **FDVect** has eventual image duality.

For the rest of this section, let $f$ be an endomorphism of a finite-dimensional vector space $X$.

As well as the eventual image of $f$, we consider its **eventual kernel** $\ker^\infty(f)$, the union of the nested sequence

$$\{0\} = \ker(f^0) \subseteq \ker(f^1) \subseteq \ker(f^2) \subseteq \cdots. \tag{31}$$

A standard lemma states:

7.1. LEMMA. [Fitting] $X = \mathrm{im}^\infty(f) \oplus \ker^\infty(f)$. *Moreover, $f$ restricts to an automorphism of $\mathrm{im}^\infty(f)$ and a nilpotent operator on $\ker^\infty(f)$. This is the unique decomposition of $f$ as the direct sum of an automorphism and a nilpotent.*

PROOF. The first part is Theorem 8.5 of [4], and the other parts follow.                    ∎

Theorem 4.9(i) shows that $\mathrm{im}^\infty(f) = \bigcap_{n \in \mathbb{N}} \mathrm{im}(f^n)$ and that $\iota_f \colon \mathrm{im}^\infty(f) \rightarrowtail X$ is the inclusion. On the other hand, Theorem 4.9(ii) shows that $\mathrm{im}^\infty(f) = X/\sim$, where

$$x \sim y \iff f^n(x) = f^n(y) \text{ for some } n \in \mathbb{N}.$$

Equivalently, $\mathrm{im}^\infty(f) = X/\ker^\infty(f)$. Moreover, $\pi_f \colon X \twoheadrightarrow \mathrm{im}^\infty(f)$ is the quotient map $X \twoheadrightarrow X/\ker^\infty(f)$.

The canonical map from the limit to the colimit is

$$\begin{aligned} \pi_f \iota_f \colon \quad \mathrm{im}^\infty(f) \quad &\to \quad X/\ker^\infty(f) \\ y \quad &\mapsto \quad y + \ker^\infty(f). \end{aligned} \tag{32}$$

That this is an isomorphism means that for all $x \in X$, there is a unique $y \in \mathrm{im}^\infty(f)$ such that $x + \ker^\infty(f) = y + \ker^\infty(f)$. This is equivalent to the statement that $X = \mathrm{im}^\infty(f) \oplus \ker^\infty(f)$ (Lemma 7.1). By definition, $y = f^\infty(x)$. Thus, $f^\infty$ is the projection to $\mathrm{im}^\infty(f)$ associated with the decomposition $X = \ker^\infty(f) \oplus \mathrm{im}^\infty(f)$. In particular, $\ker^\infty(f) = \ker(f^\infty)$.

Much as for finite sets, the chain of inclusions (28) must stabilize after at most $\dim X$ steps, so that $\mathrm{im}^\infty(f) = \mathrm{im}(f^{\dim X})$. Similarly, the chain of inclusions (31) stabilizes after at most $\dim X$ steps, so $\ker^\infty(f) = \ker(f^{\dim X})$.

7.2. REMARK. The canonical isomorphism (32) is *not* the isomorphism $\mathrm{im}(f^{\dim X}) \to X/\ker(f^{\dim X})$ provided by the first isomorphism theorem. Any nontrivial automorphism of a one-dimensional space is a counterexample.

There is a back and forth description of $f^\infty$ analogous to Proposition 6.1:

7.3. PROPOSITION. *Let $x \in X$. Then for all $n \geq \dim X$, we have $f^n(x) \in \mathrm{im}^\infty(f)$ and $f^\infty(x) = \widetilde{f}^{-n}(f^n(x))$.*

PROOF. Let $n \geq \dim X$. Then $\mathrm{im}(f^n) = \mathrm{im}^\infty(f)$ and $\ker(f^n) = \ker^\infty(f)$. Write $x = y + z$ with $y = f^\infty(x) \in \mathrm{im}^\infty(f)$ and $z \in \ker^\infty(f)$. We have $f^n(x) \in \mathrm{im}^\infty(f)$ and

$$\widetilde{f}^{-n}(f^n(x)) = \widetilde{f}^{-n}(f^n(y) + f^n(z)) = \widetilde{f}^{-n}(f^n(y)) = y = f^\infty(x).$$

∎

The proof of Proposition 6.3 used the fact that the inverse of an automorphism $g$ of a finite set is a nonnegative power of $g$. We will need the linear analogue. Write $\chi_g(t) = \det(g - tI)$ for the characteristic polynomial of an operator $g$.

7.4. LEMMA. *Let $g$ be an automorphism of a finite-dimensional vector space. Then $g^{-1}$ is a polynomial in $g$; indeed, $g^{-1} = q(g)$ where*

$$q(t) = \frac{\det g - \chi_g(t)}{(\det g)t} \in k[t].$$

PROOF. Write $\chi_g(t) = \sum_{n \geq 0} a_n t^n$. Since $g$ is an automorphism, $0 \neq \det g = a_0$. By the Cayley–Hamilton theorem,

$$0 = a_0 + g \sum_{n \geq 1} a_n g^{n-1}.$$

Rearranging shows that $\frac{-1}{a_0} \sum_{n \geq 1} a_n g^{n-1}$ is inverse to $g$, and the result follows. ∎

7.5. PROPOSITION. *$\widetilde{f}^{-1}$ is a polynomial in $\widetilde{f}$.* ∎

7.6. PROPOSITION. *$f^\infty$ is a polynomial in $f$. Indeed,*

$$f^\infty = \left(1 - \frac{\chi_{\widetilde{f}}(f)}{\det \widetilde{f}}\right)^n$$

*whenever $n \geq \dim X$, and $f^\infty \in \mathrm{span}\{f, f^2, \ldots\}$ in $\mathrm{Hom}(X, X)$.*

PROOF. By Lemma 7.4, $\widetilde{f}^{-1} = q(\widetilde{f})$ where

$$q(t) = \frac{\det \widetilde{f} - \chi_{\widetilde{f}}(t)}{(\det \widetilde{f})t} \in k[t].$$

Write $r(t) = tq(t)$. Then by Proposition 7.3, for all $x \in X$ and $n \geq \dim X$,

$$f^{\infty}(x) = q(\widetilde{f})^n(f^n(x)) = q(f)^n(f^n(x)) = r(f)^n(x).$$

Finally, $f^{\infty} \in \mathrm{span}\{f, f^2, \ldots\}$ since $r(t)$ has constant term 0.   ∎

7.7. REMARK. The proofs of Propositions 7.3 and 7.6 can be refined to weaken the lower bound on $n$ to $\dim \ker^{\infty}(f)$.

In the case of **FDVect**, Theorem 5.4 on terminal coalgebras states that $\mathrm{im}^{\infty}(f)$ is the largest linear subspace $W$ of $X$ satisfying $W \subseteq fW$. The dual of Theorem 5.4 states that $\mathrm{im}^{\infty}(f) = X/\ker^{\infty}(f)$, with $\ker^{\infty}(f)$ characterized as the smallest linear subspace $U$ of $X$ satisfying $f^{-1}U \subseteq U$.

Call an element $x \in X$ **linearly periodic** for $f$ if $x \in \mathrm{span}\{f(x), f^2(x), \ldots\}$.

7.8. PROPOSITION. *The set of linearly periodic points for $f$ is $\mathrm{im}^{\infty}(f)$. In particular, the set of linearly periodic points is a linear subspace of $X$.*

PROOF. Let $x$ be a linearly periodic point. Then $x = (p(f) \circ f)(x)$ for some $p(t) \in k[t]$. Hence for all $n \in \mathbb{N}$,

$$x = (p(f) \circ f)^n(x) = f^n(p(f)^n(x)) \in \mathrm{im}(f^n),$$

giving $x \in \mathrm{im}^{\infty}(f)$. The converse follows from the last part of Proposition 7.6.   ∎

Recall the notion of shift equivalence from Section 3. In the paper by Williams in which it was first introduced ([25], p. 342), the following result was proved using zeta functions. Here we give a different proof.

7.9. PROPOSITION. *Let $X^{\circlearrowright f}$ and $Y^{\circlearrowright g}$ be shift equivalent endomorphisms in **FDVect**. Then the characteristic polynomials of $f$ and $g$ are equal up to a factor of $\pm t^p$, for some $p \in \mathbb{Z}$.*

PROOF. By the decomposition in Lemma 7.1, $\chi_f = \chi_{\widetilde{f}} \cdot \chi_{f_0}$, where $f_0$ is the operator $f$ restricted to $\ker^{\infty}(f)$. Since $f_0$ is nilpotent, $\chi_{f_0}(t) = \pm t^i$ for some $i \geq 0$. Hence $\chi_f(t) = \pm t^i \chi_{\widetilde{f}}(t)$. Similarly, $\chi_g(t) = \pm t^j \chi_{\widetilde{g}}(t)$ for some $j \geq 0$.

By shift equivalence and Proposition 3.2, $\mathrm{im}^{\infty}(f)^{\circlearrowright \widetilde{f}} \cong \mathrm{im}^{\infty}(g)^{\circlearrowright \widetilde{g}}$. In particular, $\chi_{\widetilde{f}} = \chi_{\widetilde{g}}$, giving $\chi_f(t) = \pm t^{i-j}\chi_g(t)$.   ∎

When $k$ is algebraically closed, $X$ decomposes canonically into its **generalized eigenspaces** $\ker^\infty(f - \lambda)$:

$$X = \bigoplus_{\lambda \in k} \ker^\infty(f - \lambda)$$

([4], Theorem 8.21). The dimension of $\ker^\infty(f - \lambda)$ is the algebraic multiplicity of $\lambda$, taken to be 0 unless $\lambda$ is an eigenvalue. This decomposition refines the earlier decomposition $X = \mathrm{im}^\infty(f) \oplus \ker^\infty(f)$:

$$\mathrm{im}^\infty(f) = \bigoplus_{0 \neq \lambda \in k} \ker^\infty(f - \lambda),$$

providing yet another description of the eventual image.

## 8. Compact metric spaces

Here we study the category **CptMet** of compact metric spaces $X = (X, d)$. Its maps $X \to Y$ are the functions $f \colon X \to Y$ that are **distance-decreasing**: $d(f(x), f(x')) \leq d(x, x')$ for all $x, x' \in X$. Such a map is an **isometry** if it is distance-preserving: $d(f(x), f(x')) = d(x, x')$. The isomorphisms in **CptMet** are the surjective isometries.

The isometries and surjections define a factorization system on **CptMet**. We will prove that it is of finite type.

8.1. LEMMA. *A self-isometry of a compact metric space is surjective.*

This result is classical (Theorem 1.6.14 of [6]), but we give the short proof.

PROOF. For $\varepsilon > 0$ and compact $X$, let $N_\varepsilon(X)$ be the maximal cardinality of a subset of $X$ that is $\varepsilon$-**separated**: distinct points are at least $\varepsilon$ apart. We show that whenever $Y$ is a compact proper subspace of $X$, there is some $\varepsilon > 0$ such that $N_\varepsilon(Y) < N_\varepsilon(X)$. The result follows: for if $X^{\circlearrowright f}$ is an isometry then $X \cong fX$, so $N_\varepsilon(X) = N_\varepsilon(fX)$ for all $\varepsilon$, so $fX = X$.

Choose $x \in X$ and $\varepsilon > 0$ such that the ball $B(x, \varepsilon)$ is disjoint from $Y$. Choose an $\varepsilon$-separated set $S$ in $Y$ of cardinality $N_\varepsilon(Y)$. Then $S \cup \{x\}$ is an $\varepsilon$-separated set in $X$, proving that $N_\varepsilon(Y) < N_\varepsilon(X)$. ∎

This proves axiom **I**. Axiom **I**$^*$ is also standard (Theorem 1.6.15(1) of [6]), but we give a categorical proof that may have further applications.

8.2. LEMMA. *Let $\mathcal{C}$ be a category with a factorization system and a closed structure $([-, -], I)$. Suppose that for all coverings $f \colon X \twoheadrightarrow Y$ and objects $Z$, the map $[f, Z] \colon [Y, Z] \to [X, Z]$ is an embedding. Suppose also that every split monic covering is an isomorphism. Then axiom **I** implies axiom **I**$^*$.*

As in Eilenberg and Kelly [9], a **closed structure** on $\mathcal{C}$ consists of a functor $[-, -] \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ and an object $I \in \mathcal{C}$ satisfying axioms. For example, $\mathcal{C}$ carries a closed structure if it is monoidal closed. The axioms on a closed structure imply that

$$\mathcal{C}(X, Y) \cong \mathcal{C}(I, [X, Y]) \tag{33}$$

naturally in $X, Y \in \mathcal{C}$. We write this isomorphism as $f \mapsto \ulcorner f \urcorner$.

PROOF. Suppose that axiom **I** holds, and let $f \colon X \to X$ be a covering. Then the endo-morphism $[f, X]$ of $[X, X]$ is an embedding, hence, by axiom **I**, an isomorphism. There is a unique map $g \colon X \to X$ such that

$$\ulcorner g \urcorner = \left( I \xrightarrow{\ulcorner 1_X \urcorner} [X, X] \xrightarrow{[f,X]^{-1}} [X, X] \right).$$

The naturality of the isomorphism (33) and the definition of $g$ give

$$\ulcorner g \circ f \urcorner = [f, X] \circ \ulcorner g \urcorner = \ulcorner 1_X \urcorner,$$

so $g \circ f = 1_X$. But then $f$ is a split monic covering, and, therefore, an isomorphism. ∎

As is well known, the category of metric spaces (allowing $\infty$ as a distance) and distance-decreasing maps has the following symmetric monoidal closed structure. The tensor prod-uct $X \otimes Y$ is the cartesian product with distances defined by adding the distances in $X$ and $Y$. The unit object $I$ is the one-point space. The function space $[X, Y]$ is the set of distance-decreasing maps $X \to Y$ with metric $d_\infty(f, g) = \sup_{x \in X} d(f(x), g(x))$. (Its underlying topology is that of uniform convergence.) Moreover, this symmetric monoidal closed structure restricts to one on **CptMet**.

The hypotheses of Lemma 8.2 are easily verified, so axiom **I**$^*$ holds in **CptMet**. For future use, we also note that the monoidal closed structure gives a composition map

$$[Y, Z] \otimes [X, Y] \to [X, Z] \tag{34}$$

in **CptMet** for each $X$, $Y$ and $Z$. In particular, composition is continuous with respect to the product topology on the domain of (34).

8.3. LEMMA. *The isometries and surjections in* **CptMet** *define a factorization system of finite type.*

PROOF. We have already proved axioms **I** and **I**$^*$. For axiom **II**, a diagram

$$\cdots \rightarrowtail X_1 \rightarrowtail X_0$$

in **CptMet** is essentially a nested sequence of closed subspaces $X_n$ of $X_0$, and the limit is $\bigcap_{n \in \mathbb{N}} X_n$.

For axiom **III**, consider a map of diagrams

$$
\begin{array}{ccc}
\cdots \rightarrowtail & X_1 \rightarrowtail & X_0 \\
& \downarrow{\scriptstyle u_1} & \downarrow{\scriptstyle u_0} \\
\cdots \rightarrowtail & Y_1 \rightarrowtail & Y_0
\end{array}
$$

in **CptMet**. Regarding $X_n$ and $Y_n$ as subspaces of $X_0$ and $Y_0$ respectively, the induced map $u \colon \bigcap X_n \to \bigcap Y_n$ on limits is the restriction of $u_0$. To show that $u$ is surjective, let

$y \in \bigcap Y_n$. For each $n \in \mathbb{N}$, choose $x_n \in u_n^{-1}(y)$, as we may since $u_n$ is surjective. The sequence $(x_n)$ in $X_0$ has a subsequence converging to $x$, say, and then $x \in \bigcap X_n$ with $u(x) = y$.

For axiom $\mathbf{II}^*$, consider a diagram

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots$$

in $\mathbf{CptMet}$, and write $f^n = f_n \circ \cdots \circ f_1$. The colimit of the diagram is $X_0/\sim$, where $x \sim x'$ if $\inf_n d(f^n(x), f^n(x')) = 0$. The metric on $X_0/\sim$ is given by

$$d([x], [x']) = \inf_n d(f^n(x), f^n(x')), \tag{35}$$

where $x, x' \in X_0$ and $[\cdot]$ denotes equivalence class. The coprojection $X_n \to X_0/\sim$ is determined by $f^n(x) \mapsto [x]$ for all $x \in X_0$.

Finally, for axiom $\mathbf{III}^*$, consider a map of diagrams

$$
\begin{array}{ccccc}
Y_0 & \xrightarrow{g_1} & Y_1 & \xrightarrow{g_2} & \cdots \\
\downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle u_1} & & \\
X_0 & \xrightarrow[f_1]{} & X_1 & \xrightarrow[f_2]{} & \cdots \ .
\end{array}
$$

Write $u \colon Y_0/\sim \to X_0/\sim$ for the induced map on colimits, given on $y \in Y_0$ by $u([y]) = [u_0(y)]$. Then for all $y, y' \in Y_0$,

$$
\begin{aligned}
d(u([y]), u([y'])) &= d([u_0(y)], [u_0(y')]) \\
&= \inf_n d(f^n u_0(y), f^n u_0(y')) \\
&= \inf_n d(u_n g^n(y), u_n g^n(y')) \\
&= \inf_n d(g^n(y), g^n(y')) \\
&= d([y], [y']),
\end{aligned}
$$

as required. ∎

It follows from Theorem 4.9 that every endomorphism of a compact metric space has eventual image duality.

8.4. EXAMPLE. Here we show that eventual image duality is not absolute. That is, we construct a functor $F \colon \mathcal{C} \to \mathcal{D}$ and an endomorphism $X^{\circlearrowleft f}$ in $\mathcal{C}$ such that $f$ has eventual image duality but $F(f)$ does not.

Define $[0,1]^{\circlearrowleft f}$ in $\mathbf{CptMet}$ by $f(x) = x/2$. Let $\mathbf{Met}$ be the category of all metric spaces and distance-decreasing maps. Define $C \colon \mathbf{CptMet}^{\mathrm{op}} \to \mathbf{Met}$ on objects by taking $C(X)$ to be the set of continuous functions $X \to \mathbb{R}$ with the sup metric, and on maps

by composition. Certainly $f$ has eventual image duality, but we show that the canonical map from the limit of the diagram

$$\cdots \xrightarrow{C(f)} C[0,1] \xrightarrow{C(f)} C[0,1] \xrightarrow{C(f)} \cdots \tag{36}$$

to its colimit is not injective, so that $C(f)$ does not have eventual image duality.

For $t \in \mathbb{R}$, let $\theta_t \in C[0,1]$ denote the function $x \mapsto tx$. Then $(\theta_{2^{-n}})_{n \in \mathbb{Z}}$ and $(\theta_0)_{n \in \mathbb{Z}}$ are both elements of the limit of (36). Their 0th projections are $\theta_1, \theta_0 \in C[0,1]$. On the other hand, it is straightforward to show that two elements $\phi, \psi \in C[0,1]$ represent the same element of the colimit if and only if $\phi(0) = \psi(0)$. Since $\theta_1(0) = 0 = \theta_0(0)$, the two elements $(\theta_{2^{-n}})$, $(\theta_0)$ of the limit map to the same element of the colimit.

For the rest of this section, let $X^{\circlearrowright f}$ be an endomorphism in **CptMet**.

Theorem 4.9(i) shows that $\mathrm{im}^\infty(f) = \bigcap_{n \in \mathbb{N}} \mathrm{im}(f^n)$ and that $\iota_f \colon \mathrm{im}^\infty(f) \rightarrowtail X$ is the inclusion. On the other hand, Theorem 4.9(ii) shows that $\mathrm{im}^\infty(f) = X/\sim$, where

$$x \sim x' \iff \inf_{n \in \mathbb{N}} d(f^n(x), f^n(x')) = 0$$

and the metric on $X/\sim$ is defined as in (35), and that $\pi_f \colon X \twoheadrightarrow \mathrm{im}^\infty(f)$ is the quotient map $X \to X/\sim$.

By Theorem 4.9, the map

$$\pi_f \iota_f \colon \bigcap_{n \in \mathbb{N}} \mathrm{im}(f^n) \to X/\sim$$

is an isomorphism. For $\pi_f \iota_f$ to be a bijection means that for all $x \in X$, there is a unique $y \in \bigcap \mathrm{im}(f^n)$ such that $\inf_n d(f^n(x), f^n(y)) = 0$. Identifying $\bigcap \mathrm{im}(f^n)$ with $X/\sim$ via $\pi_f \iota_f$, this element $y$ is $\iota_f \pi_f(x) = f^\infty(x)$. Thus, $f^\infty(x)$ is the unique element of $\bigcap \mathrm{im}(f^n)$ satisfying

$$\inf_{n \in \mathbb{N}} d\big(f^n(x), f^n(f^\infty(x))\big) = 0,$$

or equivalently

$$\lim_{n \to \infty} d\big(f^n(x), f^n(f^\infty(x))\big) = 0. \tag{37}$$

The back and forth description of $f^\infty(x)$ for sets and vector spaces (Propositions 6.1 and 7.3) has the following metric analogue.

8.5. PROPOSITION. *Let $x \in X$. Let $(f^{n_i}(x))$ be a convergent subsequence of $(f^n(x))$, with limit $z$. Then $z \in \mathrm{im}^\infty(f)$ and $f^\infty(x) = \lim_{i \to \infty} \widetilde{f}^{-n_i}(z)$.*

PROOF. First, $z \in \bigcap_{i \in \mathbb{N}} \mathrm{im}(f^{n_i}) = \mathrm{im}^\infty(f)$. Now for all $i \in \mathbb{N}$,

$$d\big(\widetilde{f}^{-n_i}(z), f^\infty(x)\big) = d\big(z, f^{n_i}(f^\infty(x))\big)$$
$$\leq d\big(z, f^{n_i}(x)\big) + d\big(f^{n_i}(x), f^{n_i}(f^\infty(x))\big).$$

From the fact that $z = \lim_{i \to \infty} f^{n_i}(x)$ and equation (37), it follows that $\widetilde{f}^{-n_i}(z) \to f^\infty(x)$ as $i \to \infty$. ∎

In Proposition 8.5, $f^\infty$ is constructed as a *pointwise* limit. We now construct $f^\infty$ as a *uniform* limit. Recall that our function spaces $[X, Y]$ have the topology of uniform convergence.

8.6. PROPOSITION. *Let $(f^{n_i})$ be a convergent subsequence of $(f^n)$ in $[X, X]$, with limit $h$. Then $\mathrm{im}(h) \subseteq \mathrm{im}^\infty(f)$ and, writing $h = \big(X \xrightarrow{h'} \mathrm{im}^\infty(f) \overset{\iota_f}{\rightarrowtail} X\big)$, we have $f^\infty = \lim_{i \to \infty} \iota_f \circ \widetilde{f}^{-n_i} \circ h'$ in $[X, X]$.*

Since $[X, X]$ is compact, $(f^n)$ does have a convergent subsequence.

PROOF. By Proposition 8.5, $\mathrm{im}(h) \subseteq \mathrm{im}^\infty(f)$ and $\iota_f \circ \widetilde{f}^{-n_i} \circ h'$ converges to $f^\infty$ pointwise. So it suffices to show that $\iota_f \circ \widetilde{f}^{-n_i} \circ h'$ converges uniformly, that is, in $[X, X]$. Since $[X, X]$ is compact, it is complete, so we need only show that this sequence is Cauchy. And indeed, for all $i \geq j \geq 0$,

$$
\begin{aligned}
d_\infty\big(\iota_f \circ \widetilde{f}^{-n_i} \circ h', \iota_f \circ \widetilde{f}^{-n_j} \circ h'\big) &= d_\infty\big(\widetilde{f}^{-n_i} \circ h', \widetilde{f}^{-n_j} \circ h'\big) \\
&= d_\infty\big(h', \widetilde{f}^{n_i - n_j} \circ h'\big) \\
&= d_\infty(h, f^{n_i - n_j} \circ h) \\
&\leq d_\infty(h, f^{n_i}) + d_\infty(f^{n_i - n_j} \circ f^{n_j}, f^{n_i - n_j} \circ h) \\
&\leq d_\infty(h, f^{n_i}) + d_\infty(f^{n_j}, h),
\end{aligned}
$$

giving the result. ∎

Lemma 7.4 implies that the inverse of a linear automorphism $g$ belongs to the set $\mathrm{span}\{1, g, g^2, \ldots\}$. The metric analogue is as follows.

8.7. LEMMA. *Let $g$ be an automorphism of a compact metric space $Y$. Then $g^{-1} \in \mathrm{Cl}\{1_Y, g, g^2, \ldots\}$, where $\mathrm{Cl}$ is the closure operator on $[Y, Y]$.*

PROOF. Write $\langle g \rangle = \mathrm{Cl}\{1_Y, g, g^2, \ldots\}$. Since $[Y, Y]$ is compact, so is $\langle g \rangle$. The automorphism $g$ of $Y$ induces an automorphism $g \circ -$ of $[Y, Y]$, which restricts to an endomorphism of $\langle g \rangle$. Then $g \circ -$ is a self-isometry of the compact metric space $\langle g \rangle$, so $g \circ -$ is an automorphism of $\langle g \rangle$. Since $1_Y \in \langle g \rangle$, there is some $g' \in \langle g \rangle$ such that $g \circ g' = 1_Y$. But $g$ is invertible, so $g^{-1} = g' \in \langle g \rangle$. ∎

8.8. PROPOSITION. *$\widetilde{f}^{-1} \in \mathrm{Cl}\{1_{\mathrm{im}^\infty(f)}, \widetilde{f}, \widetilde{f}^2, \ldots\}$, where $\mathrm{Cl}$ is the closure operator on $[\mathrm{im}^\infty(f), \mathrm{im}^\infty(f)]$.* ∎

In the next two results, we use Proposition 8.8 to give a further characterization of $f^\infty$. The first is a variant of Lemma 1.3 of Borges [5].

8.9. LEMMA. *The idempotent $f^\infty$ on $X$ belongs to $\mathrm{Cl}\{f, f^2, \ldots\}$.*

PROOF. Since $[X, X]$ is compact, the sequence $(f^n)_{n \geq 1}$ has a convergent subsequence $(f^{n_i})_{i \geq 0}$. Write $h$ for its limit. By Proposition 8.6, $\text{im}(h) \subseteq \text{im}^\infty(f)$ and, writing $h = \left( X \xrightarrow{h'} \text{im}^\infty(f) \xrightarrow{\iota_f} X \right)$, we have

$$f^\infty = \lim_{i \to \infty} \iota_f \circ \widetilde{f}^{-n_i} \circ h' \tag{38}$$

in $[X, X]$.

We now repeatedly use the continuity of composition, shown in (34). By Proposition 8.8, $\widetilde{f}^{-1} \in \text{Cl}\{\widetilde{f}^k : k \geq 0\}$. Now $\{\widetilde{f}^k : k \geq 0\}$ is closed under composition, so its closure is too, giving $\widetilde{f}^{-n_i} \in \text{Cl}\{\widetilde{f}^k : k \geq 0\}$ for each $i$. It follows that for each $i$,

$$\iota_f \circ \widetilde{f}^{-n_i} \circ h' \in \text{Cl}\{\iota_f \circ \widetilde{f}^k \circ h' : k \geq 0\} = \text{Cl}\{f^k \circ h : k \geq 0\}.$$

But $h \in \text{Cl}\{f^n : n \geq 1\}$ by definition of $h$, so for each $k \geq 0$,

$$f^k \circ h \in \text{Cl}\{f^{k+n} : n \geq 1\} \subseteq \text{Cl}\{f^n : n \geq 1\}.$$

Hence for each $i$,

$$\iota_f \circ \widetilde{f}^{-n_i} \circ h' \in \text{Cl}\{f^n : n \geq 1\},$$

and the result follows from equation (38). ∎

8.10. REMARK. Lemma 8.9 and the idempotence of $f^\infty$ imply that $f^\infty \in \text{Cl}\{f^r, f^{r+1}, \ldots\}$ for all $r \geq 0$. Analogous results hold for sets and vector spaces.

8.11. PROPOSITION. $f^\infty$ is the unique idempotent element of $\text{Cl}\{f, f^2, \ldots\}$.

This is the metric analogue of Proposition 6.3 for sets.

PROOF. Let $e$ be an idempotent in $\text{Cl}\{f, f^2, \ldots\}$. By Lemma 8.9, it suffices to prove that $e = f^\infty$. We will show that the idempotents $e$ and $f^\infty$ commute and have the same image. It will follow that $e = f^\infty$: for since $\text{im}(e) \subseteq \text{im}(f^\infty)$ and $f^\infty$ is idempotent, $f^\infty e = e$, and similarly $e f^\infty = f^\infty$, giving the result.

First, $e$ and $f^\infty$ commute. Indeed, the composition map $[X, X] \times [X, X] \to [X, X]$ is continuous and restricts to a commutative operation on $\{1, f, f^2, \ldots\}$, so it also restricts to a commutative operation on $\text{Cl}\{1, f, f^2, \ldots\}$, which contains both $e$ and $f^\infty$ (by Lemma 8.9).

Next, $\text{im}(e) \subseteq \text{im}(f^\infty)$. For let $n \geq 1$. The endomorphism of $[X, X]$ defined by $h \mapsto h^n$ restricts to a map $\{f, f^2, \ldots\} \to \{f^n, f^{n+1}, \ldots\}$, and is continuous, so it also restricts to a map

$$\text{Cl}\{f, f^2, \ldots\} \to \text{Cl}\{f^n, f^{n+1}, \ldots\}.$$

Hence $e = e^n \in \text{Cl}\{f^n, f^{n+1}, \ldots\}$, and it follows that $\text{im}(e) \subseteq \text{im}(f^n)$. This holds for all $n$, so $\text{im}(e) \subseteq \bigcap \text{im}(f^n) = \text{im}^\infty(f) = \text{im}(f^\infty)$.

Finally, $\text{im}(f^\infty) \subseteq \text{im}(e)$. Indeed, $f^n(\text{im}^\infty(f)) \subseteq \text{im}^\infty(f)$ for each $n$, so $e$ restricts to an endomorphism $\hat{e}$ of $\text{im}^\infty(f)$. But $f^n|_{\text{im}^\infty(f)}$ is an isometry for each $n$, so $\hat{e}$ is also an isometry. Hence $\hat{e}$ is a self-isometry of the compact space $\text{im}^\infty(f)$, and therefore surjective. It follows that $\text{im}(e) \supseteq \text{im}^\infty(f) = \text{im}(f^\infty)$. ∎

Thus, $\mathrm{im}^\infty(f)$ can be characterized as the image or fixed set of the unique idempotent in $\mathrm{Cl}\{f, f^2, \ldots\}$.

Theorem 5.4 provides a further characterization: $\mathrm{im}^\infty(f)$ is the largest closed subspace $V$ of $X$ such that $V \subseteq fV$.

There is yet another characterization. A point $x \in X$ is **recurrent** for $f$ if $x \in \mathrm{Cl}\{f(x), f^2(x), \ldots\}$ ([11], Section h-6.6).

**8.12. PROPOSITION.** *The set of recurrent points for $f$ is $\mathrm{im}^\infty(f)$. In particular, the set of recurrent points is closed.*

PROOF. Let $x$ be a recurrent point. We prove by induction that $x \in \mathrm{im}(f^n)$ for all $n \in \mathbb{N}$, which will imply that $x \in \mathrm{im}^\infty(f)$. The base case is trivial. For $n \geq 0$, assume inductively that $x \in \mathrm{im}(f^n)$. Then $f(x), f^2(x), \ldots$ all belong to the closed set $\mathrm{im}(f^{n+1})$, which therefore also contains $x$, completing the induction. The converse follows from Lemma 8.9. ∎

Finally, note that the endomorphism $f \mapsto f^\infty$ of $[X, X]$ is typically discontinuous. For example, let $X = [0, 1]$, and for $0 \leq t \leq 1$, define $f_t \colon X \to X$ by $f_t(x) = tx$. Then $f_t^\infty$ has constant value $0$ whenever $t < 1$, but $f_1^\infty$ is the identity. Thus, as $t \to 1^-$, we have $f_t \to f_1$ but $f_t^\infty \not\to f_1^\infty$. Even in **CptMet**, long-term dynamics are sensitive to small changes in parameters.

## 9. Further examples

We end with four further examples of categories with eventual image duality: functor categories where the codomain has eventual image duality, categories of finite models for a finitary algebraic theory, the category of finite partially ordered sets, and Cauchy-complete categories with finite hom-sets.

**9.1. PROPOSITION.** *Let $\mathbf{A}$ be a small category and $\mathcal{C}$ a category with eventual image duality. Then the functor category $\mathcal{C}^{\mathbf{A}}$ has eventual image duality, and eventual images in it are computed pointwise.*

PROOF. This follows from the fact that limits and colimits in functor categories are computed pointwise (Kelly [12], Section 3.3). ∎

**9.2. EXAMPLE.** Let $G$ be a group, let $X$ be a finite-dimensional representation of $G$, and let $f$ be a $G$-equivariant endomorphism of $X$. Then the eventual image of $f$, as an endomorphism in the category of representations of $G$, is the eventual image in **FDVect** equipped with the natural $G$-action.

If a finite set $X$ has the structure of a group, ring, etc., and if an endomorphism $f$ of $X$ preserves that structure, then $\mathrm{im}^\infty(f)$ is naturally a group, ring, etc., and the maps $\iota_f$, $\pi_f$ and $f^\infty$ are homomorphisms. In general:

9.3. PROPOSITION. *Let $T$ be a finitary algebraic theory. Write $\mathcal{C}$ for the category of $T$-algebras with finite underlying set. Then $\mathcal{C}$ has eventual image duality, and eventual images in it are computed as in* **FinSet**.

PROOF. The injections and surjections form a factorization system on $\mathcal{C}$, which we show to be of finite type. Any injective or surjective endomorphism of a finite $T$-algebra is bijective and so invertible, giving axioms **I** and **I**$^*$. The forgetful functor from the category of $T$-algebras to **Set** creates limits and filtered colimits, so the other axioms follow. The result then follows from Theorem 4.9. ∎

The theory of partially ordered sets is not algebraic. Nevertheless:

9.4. PROPOSITION. *The category of finite partially ordered sets has eventual image duality.*

PROOF. In the category $\mathcal{C}$ of finite posets, the injections and surjections form a factorization system. We show that it is of finite type. For axioms **I** and **I**$^*$, let $X^{\circlearrowright f}$ be an injection or surjection in $\mathcal{C}$. Then $f$ is a bijection. Since $X$ is finite, $f^n = 1_X$ for some $n \geq 1$; then the set-theoretic inverse $f^{-1}$ is $f^{n-1}$, which is order-preserving, so $f$ is an order-isomorphism. The other axioms follow from the fact that sequential limits and colimits in $\mathcal{C}$ are computed as in **Set**. Hence Theorem 4.9 applies. ∎

The last two propositions also follow from our final result.

9.5. THEOREM. *Let $\mathcal{C}$ be a Cauchy-complete category in which every hom-set is finite. Then $\mathcal{C}$ has eventual image duality.*

PROOF. Let $X^{\circlearrowright f}$ be an endomorphism in $\mathcal{C}$. By Lemma 6.2 applied to the finite semigroup $\{f, f^2, \ldots\}$, we can choose $N \geq 1$ such that $f^N$ is idempotent. Since $\mathcal{C}$ is Cauchy-complete, $f^N$ has a splitting

$$I \underset{p}{\overset{i}{\rightleftarrows}} X.$$

We will show that $I$ is an eventual image of $f$, with $i$ and $p$ as the maps usually called $\iota_f$ and $\pi_f$. (*If $f$ has eventual image duality then $\mathrm{im}^\infty(f)$ must be $I$: for $\mathrm{im}^\infty(f) = \mathrm{im}^\infty(f^N) = I$ by Proposition 3.1 and Example 2.5.*)

Put $\hat{f} = pfi \colon I \to I$. Then the diagram

$$
\begin{array}{ccccc}
X & \overset{p}{\longrightarrow} & I & \overset{i}{\longrightarrow} & X \\
{\scriptstyle f}\downarrow & & {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle f} \\
X & \underset{p}{\longrightarrow} & I & \underset{i}{\longrightarrow} & X
\end{array}
\tag{39}
$$

commutes, since

$$\hat{f}p = pfip = pff^N = pf^N f = pipf = pf$$

and dually for the right-hand square. So

$$\hat{f}^N = \hat{f}^N pi = pf^N i = pipi = 1_X.$$

Hence $\hat{f}$ is an automorphism of $I$.

By (39), there is a cone $\left(I \xrightarrow{\mathrm{pr}_n} X\right)_{n\in\mathbb{Z}}$ on $\cdots \xrightarrow{f} X \xrightarrow{f} \cdots$ given by

$$\mathrm{pr}_n = \left(I \xrightarrow{\hat{f}^n} I \xrightarrow{i} X\right).$$

We show that this cone is a limit. Let $\left(A \xrightarrow{q_n} X\right)_{n\in\mathbb{Z}}$ be any cone on the same diagram. We must prove that there is a unique map $\bar{q}\colon A \to I$ such that
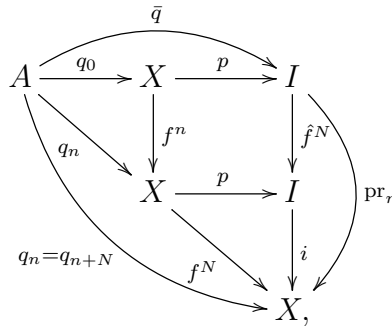
$$q_n = \left(A \xrightarrow{\bar{q}} I \xrightarrow{\mathrm{pr}_n} X\right) \tag{40}$$

for all $n \in \mathbb{Z}$.

For uniqueness, equation (40) with $n = 0$ states that $q_0 = i\bar{q}$, giving $\bar{q} = pq_0$. For existence, put $\bar{q} = pq_0$. Note that for all $n \in \mathbb{Z}$,

$$q_n = f^N \circ q_{n-N} = f^{2N} \circ q_{n-N} = q_{n+N}.$$

Now equation (40) follows from the commutativity of the diagram



where we have used the cone property of $(q_n)$ and diagram (39).

This proves that $\left(I \xrightarrow{\mathrm{pr}_n} X\right)_{n\in\mathbb{N}}$ is a limit cone. Dually, $\left(X \xrightarrow{\mathrm{copr}_n} I\right)_{n\in\mathbb{N}}$ is a colimit cone, where $\mathrm{copr}_n = \hat{f}^{-n}p$. The composite

$$I \xrightarrow{\mathrm{pr}_0} X \xrightarrow{\mathrm{copr}_0} I$$

is $pi = 1_I$, which is an isomorphism. Hence $f$ has eventual image duality. ∎

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