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INNER AUTOMORPHISMS OF GROUPOIDS

In memory of Pieter Hofstra

RICHARD GARNER

Abstract. Bergman has given the following abstract characterisation of the inner automorphisms of a group G: they are exactly those automorphisms of G which can be extended functorially along any homomorphism $G \to H$ to an automorphism of H. This leads naturally to a definition of "inner automorphism" applicable to the objects of any category. Bergman and Hofstra–Parker–Scott have computed these inner automorphisms for various structures including k-algebras, monoids, lattices, unital rings, and quandles—showing that, in each case, they are given by an obvious notion of conjugation.

In this paper, we compute the inner automorphisms of groupoids, showing that they are exactly the automorphisms induced by conjugation by a bisection. The twist is that this result is *false* in the category of groupoids and functors; to make it true, we must instead work with the less familiar category of groupoids and *cofunctors* in the sense of Higgins and Mackenzie. Besides our main result, we also discuss generalisations to topological and Lie groupoids, to categories and to partial automorphisms, and examine the link with the theory of inverse semigroups.

1. Introduction and background

This article revolves around mathematical topics which I will always associate closely with Pieter Hofstra, and which formed one of the more prominent themes in his later research. The story begins with the extremely pretty article [6], a collaboration of Pieter with Jon Funk and Ben Steinberg which introduced and studied the *isotropy group* of a topos. This is closely related to what Freyd called the *core* of a topos [4] and to what algebraists term *Tannaka–Krein reconstruction.* The idea is to find an object internal to a topos $\mathcal{E}$ which in a suitable sense classifies the automorphism group of the identity functor $1_{\mathcal{E}} \colon \mathcal{E} \to \mathcal{E}$. Since any topos is cartesian closed, and so enriched over itself, one can do this using the enriched functor category. When you work it through, the isotropy group turns out to be a group object $\mathcal{Z}$ in $\mathcal{E}$ which acts on every object of $\mathcal{E}$, and acts on itself by conjugation; this is what [6] terms a crossed topos. The original motivation for these ideas came from inverse semigroup theory, where one wants to understand how effectivity of a semigroup

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is manifested in the associated topos of sheaves. The answer turns out to be: through triviality of the isotropy group.

During 2015 I had an ongoing email exchange with Pieter, Jon and Ben about ideas relating to isotropy, during which discussion Jon raised the very natural question: is the isotropy group of the classifying topos for groups given by the generic group? It turned out that the answer was "yes", and that the problem had already been solved in another extremely pretty article [2] of George Bergman, which by pure serendipity I came across not long after Jon had raised the question.

In this article, Bergman was not wittingly calculating the isotropy group of a topos; rather, his goal was to give an element-free characterisation of the inner automorphisms of a group G. He showed, following [15], that they are exactly those automorphisms of G which can be extended in a functorial way along any group homomorphism $f\colon G \to H$ to an automorphism of H. More precisely, Bergman defines an *extended inner automorphism* β of G to be a family of group automorphisms $(\beta_f\colon H \to H)$, one for each group H and homomorphism $f\colon G \to H$, with the property that, for all such f and all $g\colon H \to K$, the following square commutes:

$$\begin{tikzcd} H \arrow[r, "\beta_f"] \arrow[d, "g"'] & H \arrow[d, "g"] \\ K \arrow[r, "\beta_{gf}"] & K \end{tikzcd} \;. \tag{1.1}$$

It is easy to see each $a \in G$ gives an extended inner automorphism by taking $\beta_f(x) = f(a)xf(a)^{-1}$ for all $f\colon G \to H$; the main contribution of [2] is to prove the less obvious fact that every extended inner automorphism of G is of this form for a *unique* $a \in G$.

We can draw the link with [6] as follows. It is direct from the definitions that the isotropy group $\mathcal{Z}$ of the classifying topos $[\mathbf{Grp}_{\mathrm{fp}}, \mathbf{Set}]$ has its value at a finitely presentable group G given by the group of all extended inner automorphisms of G. Since Bergman's result identifies this group with G itself, we can identify $\mathcal{Z}\colon \mathbf{Grp}_{\mathrm{fp}} \to \mathbf{Grp}$ with the inclusion functor: that is, with the *generic group* in $[\mathbf{Grp}_{\mathrm{fp}}, \mathbf{Set}]$. This, then, is the positive answer to Jon's question. The discovery of this interesting link between topos theory and universal algebra led to a number of further papers involving Pieter, Phil Scott and Jason Parker, which sought to find the isotropy groups of other presheaf toposes $[\mathcal{C}, \mathbf{Set}]$, but now couched in terms of Bergman's language of calculating the "extended inner automorphisms" of objects in $\mathcal{C}$—by which, to be clear, we mean the following:

1.1. Definition. [Extended inner automorphism, [2]] An *extended inner automorphism* of an object G of a category $\mathcal{C}$ is a family of automorphisms $(\beta_f\colon H \to H)$, one for each map $f\colon G \to H$ in $\mathcal{C}$, which render commutative the square (1.1) for each $f\colon G \to H$ and $g\colon H \to K$.

Bergman himself performed these calculations in [2] for the extended inner automorphisms of k-algebras over a field k, as well as the extended inner *endomorphisms* (dropping the requirement of invertibility of the maps β_f) for groups and k-algebras, along with further variants such as "inner derivations". The contributions of Hofstra–Parker–Scott can be found in [8, 9] with further contributions by Parker alone in [12, 13]; these papers in particular calculate the extended inner automorphism groups in full for examples such as monoids, abelian groups, lattices, unital rings, racks, quandles, presheaf categories and monoidal categories. In each case, the extended inner automorphisms capture, as we might hope, a natural notion of conjugation.

In this article, I make a further contribution in this spirit, by showing that, in the context of *groupoids*, extended inner automorphisms are once again given by conjugation, not now by a single element, but rather by a suitable family of elements of a kind which is well-known from the study of Lie and topological groupoids. (Here, and subsequently, we write $\mathbb{G}_0$ and $\mathbb{G}_1$ for the sets of objects and morphisms of a groupoid.)

1.2. Definition. [Bisection] A *bisection* α of a groupoid $\mathbb{G}$ comprises a bijective function $\tilde{\alpha}\colon \mathbb{G}_0 \to \mathbb{G}_0$ together with a $\mathbb{G}_0$-indexed family of morphisms $(\alpha_x\colon x \to \tilde{\alpha}(x))$.

Now, each bisection induces a conjugation automorphism $c_\alpha\colon \mathbb{G} \to \mathbb{G}$ with action on objects $\tilde{\alpha}$ and action on morphisms

$$f\colon x \to y \qquad \mapsto \qquad \alpha_y \circ f \circ \alpha_x^{-1}\colon \tilde{\alpha}(x) \to \tilde{\alpha}(y)\ , \tag{1.2}$$

and this led me, in a 2017 continuation of my email exchange with Pieter, Ben and Jon, to conjecture rather breezily that the isotropy group of the classifying topos for groupoids should take a finitely presentable groupoid $\mathbb{G}$ to its group of bisections. I got no further with this, but when I next saw Pieter in 2019, he patiently and diplomatically explained to me that my conjecture was nonsense.

Indeed, it is *not* true that conjugation by a bisection $c_\alpha\colon \mathbb{G} \to \mathbb{G}$ is the $1_\mathbb{G}$-component of an extended inner automorphism of $\mathbb{G}$ in the category **Grpd** of small groupoids and functors. There is an intuitively clear explanation for this fact: given a functor $F\colon \mathbb{G} \to \mathbb{H}$, we should like to define $\beta_F = c_{F\alpha}$, as in the case of groups, but there is no obvious way of defining the pushforward $F\alpha$ of the bisection α along F. While this does not exclude the possibility that there is a *non*-obvious way of defining the pushfoward, we have, in fact:

1.3. Proposition. *There are no non-trivial extended inner automorphisms in* **Grpd**.

Proof. Let β be an extended inner automorphism of the small groupoid $\mathbb{G}$. Consider the coproduct $\mathbb{G} + 1$ of $\mathbb{G}$ with the terminal groupoid, and $\iota\colon \mathbb{G} \to \mathbb{G} + 1$ the coproduct injection. By (1.1), we have $\beta_\iota \circ \iota = \iota \circ \beta_{1_\mathbb{G}}$, and so the automorphism $\beta_\iota\colon \mathbb{G} + 1 \to \mathbb{G} + 1$ must map the full subcategory $\mathbb{G}$ of $\mathbb{G} + 1$ into itself; thus, to be bijective on objects, it must map the remaining object $\star$ of $\mathbb{G} + 1$ to itself.

Now for any functor $F\colon \mathbb{G} \to \mathbb{H}$ and any $x \in \mathbb{H}$, there is a functor $\langle F, x\rangle\colon \mathbb{G} + 1 \to \mathbb{H}$ such that $\langle F, x\rangle \circ \iota = F$ and $\langle F, x\rangle(\star) = x$. The first condition implies using (1.1) that $\beta_F \circ \langle F, x\rangle = \langle F, x\rangle \circ \beta_\iota$, whence

$$\beta_F(x) = \beta_F(\langle F, x\rangle(\star)) = \langle F, x\rangle(\beta_\iota(\star)) = \langle F, x\rangle(\star) = x \tag{1.3}$$

so that each β_F is the identity on objects.

Consider now the groupoid $\mathbb{G} + \mathcal{J}$ and coproduct injection $\jmath\colon \mathbb{G} \to \mathbb{G} + \mathcal{J}$, where here $\mathcal{J}$ is the groupoid with two objects and a single isomorphism between them. Writing $\varphi\colon \xi \to \gamma$ for the image of this isomorphism in $\mathbb{G} + \mathcal{J}$, we conclude from the fact that $\beta_\jmath\colon \mathbb{G} + \mathcal{J} \to \mathbb{G} + \mathcal{J}$ is the identity on objects that $\beta_\jmath(\varphi) = \varphi$.

Now for any functor $F\colon \mathbb{G} \to \mathbb{H}$ and any arrow $f\colon x \to y \in \mathbb{H}$, there is a (unique) functor $\langle F, f\rangle\colon \mathbb{G} + \mathcal{J} \to \mathcal{H}$ with $\langle F, f\rangle \circ \iota = F$ and $\langle F, x\rangle(\alpha) = f$. Repeating the calculation (1.3), *mutatis mutandis*, we conclude that $\beta_F(f) = f$ so that each β_F is also the identity on morphisms. ∎

This would seem to dash any hope of characterising conjugation by bisections as extended inner automorphisms; but it turns out that, despite the preceding negative result, we *can* do this: we simply need to alter the kind of morphism that we consider between groupoids. Rather than the usual functors, we must instead consider the *cofunctors* of Higgins and Mackenzie [7], whose definition we recall in Definition 2.1 below. A cofunctor $F\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ between groupoids involves a mapping *backwards* on objects, but a mapping *forwards* on morphisms. Most importantly for us, and as in [1], it turns out that bisections *do* transport along cofunctors; this rectifies the problem we observed earlier, allowing us to show that:

1.4. Theorem. *The extended inner automorphisms of an object $\mathbb{G}$ of the category of small groupoids and cofunctors are in bijection with the bisections of $\mathbb{G}$. The extended inner automorphism corresponding to a bisection α is given by the family of congjuation automorphisms $(c_{F\alpha} \mid F\colon \mathbb{G} \rightsquigarrow \mathbb{H})$.*

This is our main result, and will be proven in Section 3 below. Preceding this is Section 2, which sets up the necessary background on [7]'s notion of cofunctor, the relation to functors, and the link with bisections. Finally, after proving our main result, we describe in Section 4 various natural generalisations—to topological and Lie groupoids, and to categories—and finally, to close the loop and bring things back round to the original motivations for [6], we discuss an alternative perspective involving inverse semigroups.

2. Cofunctors and bisections

The definition of cofunctor we give here is not the original one of [7], but a reformulation due to [1], from where we also take the name.

2.1. Definition. [Cofunctor] A *cofunctor* $F\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ between small groupoids comprises a function $F\colon \mathbb{H}_0 \to \mathbb{G}_0$ together with, for each object x of $\mathbb{H}$ and morphism $f\colon F(x) \to y$ of $\mathbb{G}$, an object $f_!(x)$ of $\mathbb{H}$ and morphism $F_x(f)\colon x \to f_!(x)$. These data are subject to the following axioms:

(i) $F(f_! x) = y$ for all $x \in \mathbb{H}$ and $f\colon F(x) \to y$ in $\mathbb{G}$;

(ii) $(1_{Fx})_!(x) = x$ and $F_x(1_{Fx}) = 1_x$ for all $x \in \mathbb{H}$;

(iii) $g_!(f_!(x)) = (gf)_!(x)$ and $F_x(gf) = F_{f_!x}(g) \circ F_x(f)$ for all $x \in \mathbb{H}$ and $f\colon F(x) \to y$ and $g\colon y \to z$ in $\mathbb{G}$.

Cofunctors may be composed in the evident manner, and in this way we obtain a category $\mathbf{Grpd}_{\mathrm{co}}$ of small groupoids and cofunctors.

There are two ways in which a functor can give rise to a cofunctor. On the one hand, any bijective-on-objects functor $F\colon \mathbb{G} \to \mathbb{H}$ induces a cofunctor $F_*\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ which on objects acts as the inverse to $F\colon \mathbb{G}_0 \to \mathbb{H}_0$, and which assigns to the object $x \in \mathbb{H}$ and arrow $f\colon F^{-1}(x) \to y$ in $\mathbb{G}$ the object $F(y) \in \mathbb{H}$ and arrow $F(f)\colon x \to F(y)$.

On the other hand, we can obtain a cofunctor from any *discrete opfibration*. Recall that a functor $F\colon \mathbb{G} \to \mathbb{H}$ is a discrete opfibration if, for each $x \in \mathbb{G}$ and map $f\colon Fx \to y$ in $\mathbb{H}$, there is a *unique* pair of an object $f_!(x) \in \mathbb{G}$ and map $F_x(f)\colon x \to f_!(x)$ such that $F(f_!(x)) = y$ and $F(F_x(f)) = f$. In this situation, the action on objects of F and the unique liftings of arrows provide the data of a cofunctor $F^*\colon \mathbb{H} \rightsquigarrow \mathbb{G}$.

In fact, as explained in [1, Section 4.4], all cofunctors are generated from those in the image of $(–)_*$ and $(–)^*$:

2.2. Proposition. *Any cofunctor $F\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ can be decomposed as*

$$F = \mathbb{G} \xrightarrow{(F_1)^*} \mathbb{K} \xrightarrow{(F_2)_*} \mathbb{H}\ . \tag{2.1}$$

Proof. Consider the directed graph $\mathbb{K}$ whose vertices are the objects of $\mathbb{H}$ and whose edges $x \twoheadrightarrow y$ are maps $f\colon F(x) \to F(y)$ of $\mathbb{G}$ such that $y = f_!(x)$. We claim that $\mathbb{K}$ is a groupoid under the composition inherited from $\mathbb{G}$. On the one hand, by cofunctor axioms (ii) and (iii), the condition $y = f_!(x)$ is stable under binary and nullary composition, so that $\mathbb{K}$ is a category. On the other hand, the cofunctor axioms easily imply that $(F_x(f))^{-1} = F_{f_!(x)}(f^{-1})$; so if $f\colon F(x) \to F(y)$ satisfies $y = f_!(x)$, then $f^{-1}\colon F(y) \to F(x)$ satisfies $x = (f^{-1})_!(y)$. So the category $\mathbb{K}$ is closed under inverses in $\mathbb{G}$, whence a groupoid.

There is an identity-on-objects functor $F_2\colon \mathbb{K} \to \mathbb{H}$ whose action on morphisms is given by $F_2(f\colon x \twoheadrightarrow y) = F_x(f)\colon x \to y$—where functoriality follows from axioms (ii) and (iii) for a cofunctor—and so we can form $(F_2)_*\colon \mathbb{K} \rightsquigarrow \mathbb{H}$. There is also a functor $F_1\colon \mathbb{K} \to \mathbb{G}$ with the same action as F on objects, and action on morphisms $F_1(f\colon x \twoheadrightarrow y) = f\colon F(x) \to F(y)$. Note that, for any map $f\colon F(x) \to y$ in $\mathbb{G}$, the unique map of $\mathbb{K}$ with domain x whose F_1-image is f is $f\colon x \twoheadrightarrow f_!(x)$. So F_1 is a discrete opfibration, and we can form $(F_1)^*\colon \mathbb{G} \rightsquigarrow \mathbb{K}$. It is now direct from the definitions that $f = (F_2)_*(F_1)^*$ as in (2.1). ∎

In fact, we can equally define cofunctors $\mathbb{G} \rightsquigarrow \mathbb{H}$ as (equivalence classes of) spans $\mathbb{G} \leftarrow \mathbb{K} \to \mathbb{H}$ with left leg a discrete opfibration and right leg bijective-on-objects; this is essentially the original definition of [7]. The following "Beck–Chevalley lemma" shows that composition of cofunctors corresponds to the composition of the representing spans by pullback.

2.3. Lemma. *Given a commuting square of functors*

$$\begin{tikzcd} \mathbb{G} \arrow[r, "F"] \arrow[d, "H"'] & \mathbb{H} \arrow[d, "K"] \\ \mathbb{K} \arrow[r, "G"] & \mathbb{L} \end{tikzcd}$$

where both F and G are bijective on objects and both H and K are discrete opfibrations (it is easy to see that any such square is necessarily a pullback), we have $F_*H^* = K^*G_* \colon \mathbb{K} \rightsquigarrow \mathbb{H}$.

Proof. On objects, the two composites act by $x \mapsto G^{-1}K(x)$ and $x \mapsto HF^{-1}(x)$; these coincide since $GH = KF$. For a map $f \colon G^{-1}K(x) = HF^{-1}(x) \to y$ in $\mathbb{K}$, the map $(K^*G_*)_x(f)$ is the unique map of $\mathbb{H}$ with domain x and K-image $G(f)$. On the other hand, $(F_*H^*)_x(f) = F(g)$, where g is the unique map of $\mathbb{G}$ with domain $F^{-1}(x)$ and H-image f. It follows that $F(g)$ has domain x and K-image $KF(g) = GH(g) = G(f)$ whence $(F_*H^*)_x(f) = (K^*G_*)_x(f)$ as desired. ∎

Since we will be interested in automorphisms in the category $\mathbf{Grpd}_{\mathrm{co}}$, the following lemma will be useful; its straightforward proof is left to the reader.

2.4. Lemma. *Any invertible map $\mathbb{G} \rightsquigarrow \mathbb{H}$ in $\mathbf{Grpd}_{\mathrm{co}}$ is of the form F_* for a unique invertible functor $F \colon \mathbb{G} \to \mathbb{H}$; moreover, we have $(F_*)^{-1} = F^* \colon \mathbb{H} \rightsquigarrow \mathbb{G}$.*

Finally in this section, we discuss the relationship between cofunctors and bisections. This is most clearly expressed in the terms of the fully faithful functor

$$\Sigma \colon \mathbf{Grp} \to \mathbf{Grpd}_{\mathrm{co}}$$

from the category of groups which on objects takes G to the corresponding one-object groupoid ΣG. We will show that taking bisections provides a right adjoint to this functor.

First observe that the set $\mathrm{Bis}(\mathbb{G})$ of bisections of a groupoid $\mathbb{G}$ is indeed a group under the operation on bisections $\beta, \alpha \mapsto \beta \cdot \alpha$ given by

$$(\beta \cdot \alpha)_x = x \xrightarrow{\alpha_x} \tilde{\alpha}(x) \xrightarrow{\beta_{\tilde{\alpha}(x)}} \tilde{\beta}(\tilde{\alpha}(x)) \ . \tag{2.2}$$

The identity element is the bisection $\mathbf{1}$ with $(\mathbf{1})_u = 1_u$. The inverse of the bisection α is the bisection α^{-1} determined by $(\alpha^{-1})_{\tilde{\alpha}(u)} = (\alpha_u)^{-1}$.

2.5. Proposition. *The full embedding $\Sigma \colon \mathbf{Grp} \to \mathbf{Grpd}_{\mathrm{co}}$ has a right adjoint whose value at a groupoid $\mathbb{G}$ is given by the group of bisections* $\mathrm{Bis}(\mathbb{G})$.

Proof. Let H be a group and $\mathbb{G}$ a groupoid, and consider what it is to give a cofunctor $F \colon \Sigma H \rightsquigarrow \mathbb{G}$. On objects, F must send each object $x \in \mathbb{G}$ to the unique object $*$ of ΣH. On morphisms, we must provide for each object $x \in \mathbb{G}$ and each morphism $h \colon F(x) \to y$ in ΣH—which is simply an element $h \in H$—an object $h_!(x) \in \mathbb{G}$ and morphism $F_x(h) \colon x \to h_!(x)$, subject to the axioms (i)–(iii) of Definition 2.1. Now, axiom (i) is trivial as ΣH has only one object. Axiom (ii) says that for all $x \in \mathbb{G}$ we have

$$e_!(x) = x \text{ and } F_x(e) = 1_x \tag{2.3}$$

where e is the identity element of H. Finally, axiom (iii) says that for all $x \in \mathbb{G}$ and $h, k \in H$ we have:

$$k_!(h_!(x)) = (kh)_!(x) \text{ and } F_x(kh) = F_{h_!x}(k) \circ F_x(h) \ . \tag{2.4}$$

The left equations in (2.3) and (2.4) say that the assignment $h, x \mapsto h_!(x)$ gives a left H-action on $\mathbb{G}_0$; in particular, each function $h_! \colon \mathbb{G}_0 \to \mathbb{G}_0$ is invertible. It follows that for each $h \in H$, the collection of maps $(F_x(h) \colon x \to h_!x)$ constitutes a bisection $\bar{F}(h) \in \mathrm{Bis}(\mathbb{G})$. Now comparing the right equations in (2.3) and (2.4) with (2.2), we see that they assert precisely that the mapping $\bar{F} \colon H \to \mathrm{Bis}(\mathbb{G})$ so obtained is a group homomorphism. In this way, we have produced bijections

$$(\overline{\ \ }) \colon \mathbf{Grpd}_{\mathrm{co}}(\Sigma H, \mathbb{G}) \to \mathbf{Gpd}(H, \mathrm{Bis}(\mathbb{G})) \tag{2.5}$$

which are easily seen to be natural in H. This shows, as claimed, that Σ has a right adjoint whose value at the groupoid $\mathbb{G}$ is given by $\mathrm{Bis}(\mathbb{G})$. ∎

Due to the adjointness exhibited above, the assignment $\mathbb{G} \mapsto \mathrm{Bis}(\mathbb{G})$ extends uniquely to a functor $\mathrm{Bis} \colon \mathbf{Grpd}_{\mathrm{co}} \to \mathbf{Grp}$ making the bijections (2.5) natural in $\mathbb{G}$ as well as H. In other words, there is a canonical way of transporting bisections along cofunctors. To read off an explicit formula, note that, since bisections of $\mathbb{G}$ correspond bijectively to group homomorphisms $\mathbb{Z} \to \mathrm{Bis}(\mathbb{G})$, they also correspond bijectively to cofunctors $\Sigma\mathbb{Z} \rightsquigarrow \mathbb{G}$, and in these terms, the transport of a bisection along a cofunctor $\mathbb{G} \rightsquigarrow \mathbb{H}$ is given simply by postcomposition. Spelling this out, we obtain:

2.6. Definition. [Pushforward bisection] Given a a cofunctor $F \colon \mathbb{G} \rightsquigarrow \mathbb{H}$ and a bisection α of $\mathbb{G}$, the *pushforward bisection* $F\alpha$ of $\mathbb{H}$ has components

$$(F\alpha)_x = F_x(\alpha_{Fx}) \colon x \to (\alpha_{Fx})_!(x) \ .$$

In particular, if $F \colon \mathbb{G} \to \mathbb{H}$ is a bijective-on-objects functor, then pushing forward the bisection α of $\mathbb{G}$ along F_* yields the bisection $F_*\alpha$ of $\mathbb{H}$ with components determined by

$$(F_*\alpha)_{F(x)} = F(\alpha_x) \ . \tag{2.6}$$

On the other hand, if $F \colon \mathbb{H} \to \mathbb{G}$ is a discrete opfibration, then we can push forward α along F^* to obtain the bisection $F^*\alpha$ of $\mathbb{H}$ uniquely determined by

$$F((F^*\alpha)_x) = \alpha_{F(x)} \ . \tag{2.7}$$

3. Inner automorphisms of groupoids

We now prove our main result. We begin with the easier direction.

3.1. Proposition. *Each bisection α of the groupoid $\mathbb{G}$ gives an extended inner automorphism of $\mathbb{G}$ in $\mathbf{Grpd}_{\mathrm{co}}$ whose component at $F \colon \mathbb{G} \rightsquigarrow \mathbb{H}$ is the conjugation isomorphism $(c_{F\alpha})_* \colon \mathbb{H} \rightsquigarrow \mathbb{H}$.*

PROOF. We must check that, for each bisection α of a groupoid $\mathbb{G}$, and each $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ and $g\colon \mathbb{H} \rightsquigarrow \mathbb{K}$, the square of cofunctors left below commutes:

$$\begin{tikzcd} \mathbb{H} \arrow[r, rightsquigarrow, "(c_{F\alpha})_*"] \arrow[d, rightsquigarrow, "G"'] & \mathbb{H} \arrow[d, rightsquigarrow, "G"] \\ \mathbb{K} \arrow[r, rightsquigarrow, "(c_{GF\alpha})_*"] & \mathbb{K} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{G} \arrow[r, rightsquigarrow, "(c_{\alpha})_*"] \arrow[d, rightsquigarrow, "G"'] & \mathbb{G} \arrow[d, rightsquigarrow, "G"] \\ \mathbb{K} \arrow[r, rightsquigarrow, "(c_{G\alpha})_*"] & \mathbb{K}\,. \end{tikzcd} \tag{3.1}$$

Since $F\alpha$ is a bisection of $\mathbb{H}$, we can without loss of generality assume that $\mathbb{H} = \mathbb{G}$ and $F = 1$, and so reduce to checking commutativity as right above. By Proposition 2.2 we can in turn reduce to the cases where $G = F_*$ or where $G = F^*$. If $G = F_*$ for a bijective-on-objects F, then it is sufficent to check commutativity to the left in:

$$\begin{tikzcd} \mathbb{G} \arrow[r, "c_\alpha"] \arrow[d, "F"'] & \mathbb{G} \arrow[d, "F"] \\ \mathbb{K} \arrow[r, "c_{F_*\alpha}"] & \mathbb{K} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K} \arrow[r, "c_{F^*\alpha}"] \arrow[d, "F"'] & \mathbb{K} \arrow[d, "F"] \\ \mathbb{G} \arrow[r, "c_{\alpha}"] & \mathbb{G}\,; \end{tikzcd}$$

and this holds at a map $f\colon x \to y$ of $\mathbb{G}$ since, by functoriality of F and (2.6),

$$F(\alpha_y \circ f \circ \alpha_x^{-1}) = F(\alpha_y) \circ Ff \circ F(\alpha_x)^{-1} = (F_*\alpha)_y \circ Ff \circ (F_*\alpha)_x^{-1}\ .$$

On the other hand, if $G = F^*$ for a discrete opfibration F, then on replacing the horizontal maps $(c_\alpha)_*$ and $(c_{F^*\alpha})_*$ in (3.1) by their inverses $(c_\alpha)^*$ and $(c_{F^*\alpha})^*$, we may reduce to checking commutativity of the square right above. This equality is verified at $f\colon x \to y$ in $\mathbb{K}$ since, by functoriality of F and (2.7), $F\Big((F^*\alpha)_y \circ f \circ (F^*\alpha)_x^{-1}\Big) = F((F^*\alpha)_y) \circ Ff \circ F((F^*\alpha)_x^{-1}) = \alpha_{Fy} \circ Ff \circ \alpha_{Fx}^{-1}$. ∎

It remains to show that:

3.2. PROPOSITION. *Each extended inner automorphism of $\mathbb{G}$ in $\mathbf{Grpd}_{\mathrm{co}}$ is induced in the manner of Proposition 3.1 from a* unique *bisection α of $\mathbb{G}$.*

PROOF. Suppose we are given an extended inner automorphism β of $\mathbb{G}$ with components $(\beta_F)_*\colon \mathbb{H} \rightsquigarrow \mathbb{H}$. To prove the result, we must exhibit a unique bisection α of $\mathbb{G}$ such that $\beta_F = c_{F\alpha}$ for each F.

We first construct α. For each $x \in \mathbb{G}$, consider the coslice groupoid $x/\mathbb{G}$, whose objects are arrows $f\colon x \to y$ of $\mathbb{G}$ with domain x, and whose morphisms are commuting triangles under x. The obvious codomain projection $\pi_x\colon x/\mathbb{G} \to \mathbb{G}$ is a discrete opfibration, and so among the data of β is an automorphism $\beta_{\pi_x^*}\colon x/\mathbb{G} \to x/\mathbb{G}$. Let $\alpha_x\colon x \to \tilde{\alpha}(x)$ be the image of $1_x \in x/\mathbb{G}$ under $\beta_{\pi_x^*}$.

Now as β is an extended inner automorphism, the square of cofunctors left below commutes. Replacing $(\beta_{1_\mathbb{G}})_*$ and $(\beta_{\pi_x^*})_*$ by their inverses $(\beta_{1_\mathbb{G}})^*$ and $(\beta_{\pi_x^*})^*$, this is to say that the square of functors to the right below commutes. (Henceforth we will make such reductions to functors without comment.)

$$\begin{tikzcd} x/\mathbb{G} \arrow[r, rightsquigarrow, "(\beta_{\pi_x^*})_*"] & x/\mathbb{G} \\ \mathbb{G} \arrow[u, rightsquigarrow, "(\pi_x)^*"] \arrow[r, rightsquigarrow, "(\beta_{1_\mathbb{G}})_*"] & \mathbb{G} \arrow[u, rightsquigarrow, "(\pi_x)^*"'] \end{tikzcd} \qquad \begin{tikzcd} x/\mathbb{G} \arrow[r, "\beta_{\pi_x^*}"] \arrow[d, "\pi_x"'] & x/\mathbb{G} \arrow[d, "\pi_x"] \\ \mathbb{G} \arrow[r, "\beta_{1_\mathbb{G}}"] & \mathbb{G} \end{tikzcd} \qquad (3.2)$$

Tracing 1_x around this square yields $\beta_{1_\mathbb{G}}(x) = \tilde{\alpha}(x)$. Thus, since $\beta_{1_\mathbb{G}}$ is invertible, so is the function $x \mapsto \tilde{\alpha}(x)$; whence $(\alpha_x)_{x \in \mathbb{G}}$ is a bisection of $\mathbb{G}$.

We now show that $\beta_{1_\mathbb{G}} = c_\alpha \colon \mathbb{G} \to \mathbb{G}$. Consider a map $f \colon x \to y$ of $\mathbb{G}$. This induces by precomposition a functor $(-) \circ f \colon y/\mathbb{G} \to x/\mathbb{G}$, which fits into a commuting triangle of discrete opfibrations as left below. Since β is an extended inner automorphism, this implies the commutativity of the square of functors to the right.

$$\begin{tikzcd} y/\mathbb{G} \arrow[rr, "(-)\circ f"] \arrow[dr, "\pi_y"'] & & x/\mathbb{G} \arrow[dl, "\pi_x"] \\ & \mathbb{G} & \end{tikzcd} \qquad \begin{tikzcd} y/\mathbb{G} \arrow[r, "\beta_{\pi_y^*}"] \arrow[d, "(-)\circ f"'] & y/\mathbb{G} \arrow[d, "(-)\circ f"] \\ x/\mathbb{G} \arrow[r, "\beta_{\pi_x^*}"] & x/\mathbb{G} \end{tikzcd}$$

Tracing 1_y around this square, we find that $\beta_{\pi_x^*}$ sends the object $f \in x/\mathbb{G}$ to $\alpha_y \circ f \in x/\mathbb{G}$. Thus $\beta_{\pi_x^*}$ sends the *map* $f \colon 1_x \to f$ of $x/\mathbb{G}$ to a map $\alpha_x \to \alpha_y \circ f$ of $x/\mathbb{G}$. Such a map is equally well a map $h \colon \tilde{\alpha}(x) \to \tilde{\alpha}(y)$ of $\mathbb{G}$ satisfying $h \circ \alpha_x = \alpha_y \circ f$, and so necessarily $h = \alpha_y \circ f \circ \alpha_x^{-1}$. Thus, tracing the map $f \colon 1_x \to f$ around the right square of (3.2), we see that $\beta_{1_\mathbb{G}}(f \colon x \to y) = \alpha_y \circ f \circ \alpha_x^{-1}$, and so $\beta_{1_\mathbb{G}} = c_\alpha$ as claimed.

It remains to show that $\beta_F = c_{F\alpha}$ for *all* cofunctors $F \colon \mathbb{G} \rightsquigarrow \mathbb{H}$. For this, it suffices to show that the bisection associated to the extended inner automorphism $\beta_{(-)F}$ of $\mathbb{H}$ is $F\alpha$, since then $\beta_F = \beta_{(1_\mathbb{H})F} = c_{F\alpha}$ as desired. That is, we must prove:

$$\beta_{\pi_x^* F}(1_x) = (F\alpha)_x \qquad \text{for all } F \colon \mathbb{G} \rightsquigarrow \mathbb{H} \text{ and } x \in \mathbb{H}\ . \qquad (3.3)$$

Step 1. Suppose first that $F = G^*$ for some discrete opfibration $G \colon \mathbb{H} \to \mathbb{G}$. We then have a commuting square of discrete opfibrations as to the left in:

$$\begin{tikzcd} x/\mathbb{H} \arrow[r, "x/G"] \arrow[d, "\pi_x"'] & Gx/\mathbb{G} \arrow[d, "\pi_{Gx}"] \\ \mathbb{H} \arrow[r, "G"] & \mathbb{G} \end{tikzcd} \qquad \begin{tikzcd} x/\mathbb{H} \arrow[r, "\beta_{\pi_x^* G^*}"] \arrow[d, "x/G"'] & x/\mathbb{H} \arrow[d, "x/G"] \\ Gx/\mathbb{G} \arrow[r, "\beta_{\pi_{Gx}^*}"] & Gx/\mathbb{G} \end{tikzcd} ,$$

and so, since β is an extended inner automorphism, a commuting square of functors as to the right. Tracing 1_x around this square yields $G(\beta_{\pi_x^* G^*}(1_x)) = \beta_{\pi_{Gx}^*}(1_{Gx}) = \alpha_{Gx}$, and so by (2.7) that $\beta_{\pi_x^* G^*}(1_x) = (G^*\alpha)_x$ as required for (3.3).

Step 2. Now suppose that $F = H_*$ for some bijective-on-objects functor $H\colon \mathbb{G} \to \mathbb{H}$. We form the outer square, the pullback and the induced comparison map as to the left in:

$$
\begin{tikzcd}
x/\mathbb{G} \arrow[rr, "x/H"] \arrow[dr, dotted, "R"] \arrow[dd, "\pi_x"'] & & Hx/\mathbb{H} \arrow[dd, "\pi_{Hx}"] \\
& \mathbb{P} \arrow[ur, "Q"'] \arrow[dl, "P"] \arrow[dr, phantom, "\lrcorner", very near start] & \\
\mathbb{G} \arrow[rr, "H"] & & \mathbb{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
x/\mathbb{G} \arrow[r, "\beta_{\pi_x *}"] \arrow[d, "R"'] & x/\mathbb{G} \arrow[d, "R"] \\
\mathbb{P} \arrow[r, "\beta_{P^*}"] \arrow[d, "Q"'] & \mathbb{P} \arrow[d, "Q"] \\
Hx/\mathbb{G} \arrow[r, "\beta_{Q_* P^*}"] & Hx/\mathbb{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
x/\mathbb{G} \arrow[r, "\beta_{\pi_x *}"] \arrow[d, "x/H"'] & x/\mathbb{G} \arrow[d, "x/H"] \\
Hx/\mathbb{G} \arrow[r, "\beta_{\pi^*_{Hx} H_*}"] & Hx/\mathbb{G}
\end{tikzcd}
$$

In this square, π_x and π_{Hx} are discrete opfibrations, and so is P, since it is a pullback of π_{Hx}. It follows that the comparison map R is also a discrete opfibration. Since $PR = \pi_x$ and β is an extended inner automorphism, we have that the top square centre above commutes. On the other hand, Q is bijective-on-objects as a pullback of F, and so, since β is an extended inner automorphism, the bottom square centre above commutes.

By applying Lemma 2.3 to the pullback square left above, we have $Q_*P^* = \pi^*_{Hx}H_*$, and so the composite of the two centre squares is equally the square right above. Tracing the object 1_x around both sides yields $\beta_{\pi^*_{Hx}H_*}(1_{Hx}) = H(\alpha_x)$, and so by (2.6) we conclude that $\beta_{\pi^*_{Hx}H_*}(1_x) = (H_*\alpha)_x$ as required for (3.3).

Step 3. We now prove for a general $F\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ that the bisection associated to the inner automorphism $\beta_{(-)F}$ of $\mathbb{H}$ is $F\alpha$. We first apply Proposition 2.2 to decompose F as $H_*G^*\colon \mathbb{G} \rightsquigarrow \mathbb{K} \rightsquigarrow \mathbb{H}$. By Step 1 applied to β and G, the bisection associated to the inner automorphism $\beta_{(-)G^*}$ of $\mathbb{K}$ is $G^*\alpha$. Now by Step 2 applied to $\beta_{(-)G^*}$ and H, the bisection associated to the inner automorphism $\beta_{(-)H_*G^*} = \beta_{(-)F}$ of $\mathbb{K}$ is $H_*G^*\alpha = F\alpha$, as required. ∎

We have thus proved the theorem stated in the introduction. In fact, we can do slightly better. The extended inner automorphisms of any object in any category form a group under the operation of composition. We noted above that the bisections of a groupoid also form a group. It is easily seen that these two group structures are related by the equation $c_\beta \circ c_\alpha = c_{\beta\cdot\alpha}$, and so we have:

3.3. Theorem. *The group of extended inner automorphisms of* $\mathbb{G} \in \mathbf{Grpd}_{\mathrm{co}}$ *is isomorphic to the group* $\mathrm{Bis}(\mathbb{G})$. *The extended inner automorphism corresponding to* $\alpha \in \mathrm{Bis}(\mathbb{G})$ *is given by the family of automorphisms* $((c_{F\alpha})_*\colon \mathbb{H} \rightsquigarrow \mathbb{H})$ *as* F *ranges over cofunctors* $\mathbb{G} \rightsquigarrow \mathbb{H}$.

4. Generalisations and further perspectives

4.1. Topological and Lie groupoids. As mentioned in the introduction, bisections show up frequently in the study of Lie and topological groupoids. It is therefore natural to ask if our results generalise to those settings. The answer is yes. As the adaptations in the two cases are so similar, we concentrate on the topological one.

First we must adapt the basic notions. For a topological groupoid $\mathbb{G}$, we restrict attention to *continuous* bisections α: those for which the assignment $x \mapsto \alpha_x$ is continuous as a map $\mathbb{G}_0 \to \mathbb{G}_1$. This implies, easily, that the associated conjugation homomorphism $c_\alpha \colon \mathbb{G} \to \mathbb{G}$ is a continuous map of topological groupoids. We should like to identify these c_α's as the extended inner automorphisms of $\mathbb{G}$ in a suitable category.

The morphisms of this category will be cofunctors $F \colon \mathbb{G} \rightsquigarrow \mathbb{H}$ between topological groupoids which are *continuous*, in the sense of rendering continuous the following maps:

$$\begin{aligned} \mathbb{H}_0 &\to \mathbb{G}_0 \\ x &\mapsto Fx \end{aligned} \qquad\qquad \begin{aligned} \mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{H}_0 &\to \mathbb{H}_1 \\ (x, f) &\mapsto F_x(f) \ ; \end{aligned}$$

here, the fibre product $\mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{H}_0$ is taken along the source map $s \colon \mathbb{G}_1 \to \mathbb{G}_0$ and the action on objects $F \colon \mathbb{H}_0 \to \mathbb{G}_0$. Much like before, we obtain continuous cofunctors $\mathbb{G} \rightsquigarrow \mathbb{H}$ from continuous functors $\mathbb{G} \to \mathbb{H}$ which are homeomorphic-on-objects; and from functors $\mathbb{H} \to \mathbb{G}$ which are *continuous discrete opfibrations*, meaning that the operation of forming the unique lifting $F_x(f) \colon x \to f_! x$ of a map $f \colon F(x) \to y$ is a continuous map $\mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{H}_0 \to \mathbb{H}_1$. As in Proposition 2.2, every continuous cofunctor arises by composing ones of these two special kinds.

We also have an analogue of Proposition 2.5: the functor $\Sigma \colon \mathbf{Grp} \to \mathbf{TopGrpd}_{\mathrm{co}}$ embedding each discrete group G as a one-object discrete topological groupoid has a right adjoint, sending a topological groupoid $\mathbb{G}$ to its discrete group of continuous bisections $\mathrm{Bis}(\mathbb{G})$. In particular, continuous bisections of a topological groupoid can be transported along continuous cofunctors, with the same formulae as before. Using this Proposition 3.1 carries over, *mutatis mutandis*, showing that every continuous bisection of $\mathbb{G} \in \mathbf{TopGrpd}_{\mathrm{co}}$ induces an extended inner automorphism.

All that remains is to adapt the proof of Proposition 3.2, showing that every extended inner automorphism β of $\mathbb{G}$ arises in this manner. All of the constructions in this proof continue to work in the topological context, and so we can conclude immediately that β must be of the form $\beta_F = c_{F\alpha}$ for a unique, but not necessarily continuous, bisection α of $\mathbb{G}$. To prove continuity, we consider the *décalage* [10] of $\mathbb{G}$. This is the topological groupoid $\mathrm{Dec}(\mathbb{G})$ whose underlying topological graph is given by

$$\mathbb{G}_1 {}_t\!\times_s \mathbb{G}_1 \overset{\pi_1}{\underset{\mu}{\rightrightarrows}} \mathbb{G}_1 \quad ,$$

where μ is the composition map of $\mathbb{G}$. The composition and units of $\mathrm{Dec}(\mathbb{G})$ itself are determined by requiring that its underlying discrete groupoid be the disjoint union of the coslice categories $x/\mathbb{G}$. There is a continuous discrete opfibration $\pi \colon \mathrm{Dec}(\mathbb{G}) \to \mathbb{G}$ which projects onto the codomain; and for each $x \in \mathbb{G}$ this fits into a commuting triangle as to the left below, where ι_x is the obvious inclusion functor:

$$\begin{tikzcd} x/\mathbb{G} \arrow[rr, "\iota_x"] \arrow[dr, "\pi_x"'] & & \mathrm{Dec}(\mathbb{G}) \arrow[dl, "\pi"] \\ & \mathbb{G} & \end{tikzcd} \qquad \begin{tikzcd} x/\mathbb{G} \arrow[r, "\beta_{\pi_x^*}"] \arrow[d, "\iota_x"'] & x/\mathbb{G} \arrow[d, "\iota_x"] \\ \mathrm{Dec}(\mathbb{G}) \arrow[r, "\beta_{\pi^*}"] & \mathrm{Dec}(\mathbb{G}) \ . \end{tikzcd}$$

It follows that each square as right above is commutative in **TopGrpd**; so in particular, $\beta_{\pi^*}(1_x) = \beta_{\pi_x^*}(1_x) = \alpha_x$ for each $x \in \mathbb{G}$. This shows that composing the continuous identities map $1_{(-)} \colon \mathbb{G}_0 \to \mathbb{G}_1$ with the continuous map $\mathbb{G}_1 \to \mathbb{G}_1$ giving the action on objects of β_{π^*} yields the assignment $x \mapsto \alpha_x$—which is thus continuous, as desired. We thus obtain:

4.2. Theorem. *The group of extended inner automorphisms of* $\mathbb{G} \in \mathbf{TopGrpd}_{\mathrm{co}}$ *is isomorphic to the group of continuous bisections* $\mathrm{Bis}(\mathbb{G})$*, under the same correspondence as in Theorem 3.3.*

4.3. Internal groupoids. Topological and Lie groupoids are particular examples of *internal groupoids* in a category $\mathcal{C}$. It is therefore natural to ask if our results generalise further to groupoids internal to any category $\mathcal{C}$. The answer is no.

To see this, consider the category $\mathbf{Set}^{\mathbb{Z}_2}$ whose objects are sets X endowed with an involution $\tau \colon X \to X$, and whose maps are equivariant functions (i.e., ones commuting with the involutions). A groupoid internal to $\mathbf{Set}^{\mathbb{Z}_2}$ is an ordinary groupoid $\mathbb{G}$ with a (strict) involution $\tau \colon \mathbb{G} \to \mathbb{G}$; internal functors and cofunctors are just ordinary functors and cofunctors which commute with the involutions.

Now if $(\mathbb{G}, \tau)$ is an involutive groupoid, then the functor τ is easily seen to be equivariant $(\mathbb{G}, \tau) \to (\mathbb{G}, \tau)$; it follows that $(\mathbb{G}, \tau)$ has an extended inner automorphism β whose component at any $(\mathbb{G}, \tau) \rightsquigarrow (\mathbb{H}, \sigma)$ is $\sigma_* \colon (\mathbb{H}, \sigma) \rightsquigarrow (\mathbb{H}, \sigma)$. However, this β need not arise from any bisection α of $\mathbb{G}$. For example, if $\mathbb{G}$ is the discrete groupoid on two objects, and τ is the swap map, then $(\mathbb{G}, \tau)$ has no non-identity bisections, and yet the β defined above is not the identity.

The reason that things work differently in this case is really that objects in the indexing category $\mathbb{Z}_2$ can have their own non-trivial extended inner automorphisms. A more general formulation of our results would have to take this into account—but, lacking as we do any compelling reasons for developing such a generalisation, we have not pursued this further.

4.4. Categories. Another obvious direction of generalisation involves replacing groupoids everywhere by categories. There is not so much to say here; everything works without fuss. Cofunctors are defined exactly as before, and factorise in exactly the same way. For bisections, we must add the requirement that each map $\alpha_x \colon x \to \tilde{\alpha}(x)$ is invertible, and can then induce conjugation automorphisms in exactly the same way. Once again, bisections transport along cofunctors, with this now being evidenced by an an adjunction $\Sigma \colon \mathbf{Grp} \leftrightarrows \mathbf{Cat}_{\mathrm{co}} \colon \mathrm{Bis}$.

Proposition 3.1 continues to work; and the only adaptation required in Proposition 3.2 is at the very start. Given an extended inner automorphism β of a category $\mathbb{C}$, with components $(\beta_f)_*$, we have as before the collection of maps $\alpha_x = \beta_{\pi_x^*}(1_x) \colon x \to \tilde{\alpha}(x)$. The argument showing the assignment $x \mapsto \tilde{\alpha}(x)$ is invertible still holds; but we must now also show that each α_x is invertible. For this, we first show as before that the automorphism $\beta_{\pi_x^*} \colon x/\mathbb{C} \to x/\mathbb{C}$ is given on objects by $\beta_{\pi_x^*}(f \colon x \to y) = \alpha_y \circ f$. Being an automorphism, there is in particular some such f for which $\alpha_y \circ f = 1_x$. So we have a commuting triangle as to the left below in $x/\mathbb{C}$. Applying $\beta_{\pi_x^*}$ yields a commuting triangle as to the right.

$$\begin{tikzcd} & f \arrow[dr, "\alpha_y"] & \\ 1_x \arrow[ur, "f"] \arrow[rr, "1_x"'] & & 1_x \end{tikzcd} \qquad \begin{tikzcd} & 1_x \arrow[dr, "k"] & \\ \alpha_x \arrow[ur, "h"] \arrow[rr, "1_{\tilde{\alpha}(x)}"'] & & \alpha_x \end{tikzcd}$$

Since $k\colon 1_x \to \alpha_x$ we have $k = k \circ 1_x = \alpha_x$; since $h\colon \alpha_x \to 1_x$ we have $h \circ \alpha_x = 1_x$; and since the triangle commutes we have $\alpha_x \circ h = 1_x$. So h is an inverse for α_x. The remainder of the argument now follows exactly as before, and so we have:

4.5. THEOREM. *The group of extended inner automorphisms of* $\mathbb{C} \in \mathbf{Cat}_{\mathrm{co}}$ *is isomorphic to the group of bisections* $\mathrm{Bis}(\mathbb{C})$*, under the same correspondence as in Theorem 3.3.*

4.6. INVERSE SEMIGROUPS. Our main results seem to diverge from the pattern for the computation of extended inner automorphism groups in [2, 8]. In this prior work, the categories under consideration have as objects, the models of an equational algebraic theory $\mathbb{T}$, and as morphisms, the obvious structure-preserving maps. This allows the extended inner automorphisms of a $\mathbb{T}$-model X to be characterised via universal algebra: they correspond to those invertible unary operations of the diagram theory $\mathbb{T}_X$[1] which *commute* with each $\mathbb{T}$-operation.

By contrast, our main result concerns the category of groupoids and cofunctors; and while the objects of this category are algebraic in nature—they are the models of an *essentially-algebraic* theory in the sense of [5]—the morphisms are *not* the obvious structure-preserving ones (which led only to our negative Proposition 1.3). This means that our argument for computing the extended inner automorphisms is necessarily different in nature. In fact, there is a way of reconciling our results with those of [2, 8]: we adopt a different perspective on groupoid structure in which the cofunctors *are* the natural structure-preserving maps. More precisely, we take as the basic data of a groupoid not its objects and morphisms, but its *partial* bisections:

4.7. DEFINITION. [Partial bisection] A *partial bisection* α of a groupoid $\mathbb{G}$ comprises subsets $s(\alpha)$ and $t(\alpha)$ of $\mathbb{G}_0$; a bijection $\tilde{\alpha}\colon s(\alpha) \to t(\alpha)$; and an $s(\alpha)$-indexed family of morphisms $(\alpha_x\colon x \to \tilde{\alpha}(x))$.

The set $\mathrm{PBis}(\mathbb{G})$ of partial bisections of a groupoid $\mathbb{G}$ can be endowed with the structure of a *pseudogroup*—a special kind of inverse semigroup—and this structure allows $\mathrm{PBis}(\mathbb{G})$ to represent $\mathbb{G}$ faithfully. This fits into the pattern of a well-known correspondence between étale topological groupoids and pseudogroups, detailed, for example, in [14, 11]. As explained in [3], the most general form of this correspondence equates the natural structure-preserving maps of pseudogroups with the *cofunctors* between the corresponding groupoids. Thus, we may consider our result about groupoids and cofunctors instead as a

[1] i.e., the theory obtained by extending $\mathbb{T}$ with new constants for each element of X, and new equations describing the value of each $\mathbb{T}$-operation on those constants.

result about pseudogroups and their structure-preserving morphisms, so fitting it in to the general pattern established in [2, 8].

To make the preceding claims more precise, we now define the category **PsGrp** of pseudogroups, and give a direct proof following [3] that the assignment $\mathbb{G} \mapsto \mathrm{PBis}(\mathbb{G})$ yields a full embedding of $\mathbf{Grpd}_{\mathrm{co}}$ into **PsGrp**.

4.8. Definition. [Pseudogroup] An *inverse monoid* is a unital semigroup M such that, for every $m \in M$, there is a unique $m^* \in M$ with $mm^*m = m$ and $m^*mm^* = m^*$. The *natural partial order* $\leqslant$ and the *compatibility relation* $\sim$ on M are given by

$$\begin{array}{lll} m \leqslant n & \text{iff} & mn^*n = n \\ m \sim n & \text{iff} & mn^* \text{ and } n^*m \text{ are idempotent.} \end{array}$$

A (abstract) *pseudogroup* is an inverse monoid M such that any family $S \subseteq M$ of pairwise-compatible elements admits a join $\bigvee S$ (with respect to $\leqslant$) which is preserved by each function $m \cdot (-) \colon M \to M$ and $(-) \cdot m \colon M \to M$. Pseudogroups form a category **PsGrp** wherein maps are monoid homomorphisms that preserve joins of compatible families.

4.9. Example. For any groupoid $\mathbb{G}$, the set of partial bisections $\mathrm{PBis}(\mathbb{G})$ is a pseudogroup under the binary operation $\beta, \alpha \mapsto \beta \cdot \alpha$, where $\beta \cdot \alpha$ has

$$s(\beta \cdot \alpha) = s(\alpha) \cap \tilde{\alpha}^{-1}(s(\beta)) \qquad t(\beta \cdot \alpha) = \tilde{\beta}(t(\alpha)) \cap t(\beta)$$

and components $(\beta \cdot \alpha)_x = \beta_{\tilde{\alpha}(x)} \circ \alpha_x \colon x \to \tilde{\beta}(\tilde{\alpha}(x))$. The unit for this operation is the identity bisection $\mathbf{1}$, and the partial inverse α^* of α has $s(\alpha^*) = t(\alpha)$, $t(\alpha^*) = s(\alpha)$ and components determined by $(\alpha^*)_{\tilde{\alpha}(x)} = (\alpha_x)^{-1}$.

Two partial bisections α, β are compatible if $\alpha_x = \beta_x$ for all $x \in s(\alpha) \cap s(\beta)$, while $\alpha \leqslant \beta$ if $\alpha \sim \beta$ and $s(\alpha) \subseteq s(\beta)$. The join α of a pairwise-compatible family of partial bisections $(\alpha^i : i \in S)$ has $s(\alpha) = \bigcup_i s(\alpha^i)$, $t(\alpha) = \bigcup_i t(\alpha^i)$ and components $\alpha_x = \alpha^i_x$, for any $i \in S$ with $x \in s(\alpha^i)$.

4.10. Proposition. *The assignment $\mathbb{G} \mapsto \mathrm{PBis}(\mathbb{G})$ is the action on objects of a full embedding of categories $\mathbf{Grpd}_{\mathrm{co}} \to \mathbf{PsGrp}$.*

Proof. Let $F \colon \mathbb{G} \rightsquigarrow \mathbb{H}$ be a cofunctor of groupoids, and $\alpha \in \mathrm{PBis}(\mathbb{G})$. Generalising Definition 2.6, we can define a pushforward partial bisection $F\alpha \in \mathrm{PBis}(\mathbb{H})$ by taking $s(F\alpha)$ and $t(F\alpha)$ to be the inverse images of $s(\alpha)$ and $t(\alpha)$ under the function $F \colon \mathbb{H}_0 \to \mathbb{G}_0$, and with components given like before by

$$(F\alpha)_x = F_x(\alpha_{Fx}) \colon x \to (\alpha_{Fx})_!(x) \ .$$

Straightforward checking shows that the assignment $\alpha \mapsto F\alpha$ is a pseudogroup morphism $\mathrm{PBis}(F) \colon \mathrm{PBis}(\mathbb{G}) \to \mathrm{PBis}(\mathbb{H})$ and that the assignment $F \mapsto \mathrm{PBis}(F)$ is functorial; so we have a functor $\mathrm{PBis} \colon \mathbf{Grpd}_{\mathrm{co}} \to \mathbf{PsGrp}$.

To see this functor is faithful, note that we can recover the action on objects of $F\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ from $\varphi := \mathrm{PBis}(F)\colon \mathrm{PBis}(\mathbb{G}) \to \mathrm{PBis}(\mathbb{H})$ by the formula

$$F(y) = x \qquad \text{iff} \qquad y \in s(\varphi([1_x])) \ ; \tag{4.1}$$

here, if $f\colon x \to y$ is any map of $\mathbb{G}$ then we write $[f]$ for the partial bisection whose sole component is the map f. In a similar way, we can recover the action of the cofunctor F on maps by the formula

$$F_x(f) = g \qquad \text{iff} \qquad \varphi([f])_x = g \ . \tag{4.2}$$

It remains only to show that PBis is full. So let $\varphi\colon \mathrm{PBis}(\mathbb{G}) \to \mathrm{PBis}(\mathbb{H})$ be any pseudogroup morphism. In $\mathrm{PBis}(\mathbb{G})$ we have that

$$\mathbf{1} = \textstyle\bigvee_{u \in \mathbb{G}} [1_u] \qquad \text{and} \qquad [1_u] \cdot [1_v] = \bot \text{ for } u \neq v \ ;$$

since φ is a pseudogroup morphism, it follows that in $\mathrm{PBis}(\mathbb{H})$ we have

$$\mathbf{1} = \textstyle\bigvee_{u \in \mathbb{G}} \varphi([1_u]) \qquad \text{and} \qquad \varphi([1_u]) \cdot \varphi([1_v]) = \bot \text{ for } u \neq v \ ,$$

so that the sets $s(\varphi([1_u]))$ are a partition of $\mathbb{H}_0$. We thus have a well-defined function $F\colon \mathbb{H}_0 \to \mathbb{G}_0$ determined by (4.1); whereupon we obtain the assignments on morphisms required for a cofunctor $F\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ by the formula (4.2). The cofunctor axioms now follow easily from the homomorphism axioms for φ together with the observation that $[a] \cdot [b] = [a \circ b]$ in $\mathrm{PBis}(\mathbb{G})$ whenever a and b are composable maps. Finally, to see that $\mathrm{PBis}(F) = \varphi$, we observe that $\mathrm{PBis}(F)([a]) = \varphi([a])$ by construction; now since for any $\alpha \in \mathrm{PBis}(\mathbb{G})$, we have $\alpha = \bigvee_{u \in s(\alpha)} [\alpha_u]$, and since both $\mathrm{PBis}(F)$ and φ preserve joins, it follows that $\mathrm{PBis}(F)(\alpha) = \varphi(\alpha)$ for all $\alpha \in \mathrm{PBis}(\mathbb{G})$, as desired. ∎

It is not too hard to characterise the essential image of the embedding $\mathbf{Grpd}_{\mathrm{co}} \to \mathbf{PsGrp}$; it comprises the *complete atomic* pseudogroups—those whose partially ordered set of idempotents forms a complete atomic Boolean algebra (i.e., a power-set lattice). Thus, our main result, concerning the "non-algebraic" category of groupoids and cofunctors, can be recast as one about the "algebraic" category of complete atomic pseudogroups and pseudogroup homomorphisms; and following [11], we may recast the generalisation of our main result to étale topological groupoids in terms of more general pseudogroups.

There are a couple of points worth noting here. Firstly, when translated into the language of pseudogroups, our main result states that every extended inner automorphism of a complete atomic pseudogroup M is induced by conjugation (in the usual sense) by an invertible (in the usual sense) element of the monoid M—indeed, such invertible elements correspond to *total* bisections of the corresponding groupoid. So in this sense, our result fits into the pattern established in [2, 8].

On the other hand, if we translate the proof of our main Theorem 3.3 into the language of complete atomic pseudogroups, then it is still not a proof in the same mould as [2, 8]. If it were, then the first step in determining the components of an extended inner automorphism

β of M would be to adjoin freely a new element x to M and consider the component $\beta_\iota\colon M[x] \to M[x]$ at the resulting inclusion map $\iota\colon M \to M[x]$. This is quite different from what is done in Proposition 3.2: in the language of pseudogroups, the first step there is to consider an atomic idempotent $x \in M$, and now consider the component $\beta_\jmath$ corresponding to the homomorphism

$$\begin{aligned} \jmath\colon M &\to \mathrm{PBij}(M_x) \\ m &\mapsto m \cdot (-) \end{aligned}$$

into the pseudogroup of all partial bijections of $M_x = \{m \in M : m^* m = x\}$. It may be interesting to compare these approaches more thoroughly.

4.11. Extended partial inner automorphisms. In a pseudogroup M, *any* element $a \in M$ induces a conjugation map $c_a(x) = axa^*$. However, $c_a\colon M \to M$ is not typically an automorphism of M, nor even a well-defined homomorphism, since it does not preserve the monoid unit 1 unless $a \in M$ is genuinely invertible.

Nonetheless, we would like to think of c_a as a *partial* automorphism of M, hoping for a result to the effect that every extended inner partial automorphism of a pseudogroup is induced by conjugation by an element. In the world of groupoids this would translate into the statement that every extended partial automorphism in $\mathbf{Grpd}_{\mathrm{co}}$ comes from conjugating by a partial bisection.

Making this precise is delicate because, just as conjugation on a pseudogroup does not give a pseudogroup homomorphism, so conjugation on a groupoid by a partial bisection does not give a cofunctor. Thus, much as in Section 6 of [2], we must proceed in an essentially *ad hoc* manner.

4.12. Definition. A *partial automorphism* $\varphi\colon \mathbb{G} \nrightarrow \mathbb{G}$ of a groupoid $\mathbb{G}$ is given by full subcategories $s(\varphi), t(\varphi) \subseteq \mathbb{G}$ together with an isomorphism of groupoids $\varphi\colon s(\varphi) \to t(\varphi)$. Given a cofunctor $F\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ and partial automorphisms $\varphi\colon \mathbb{G} \nrightarrow \mathbb{G}$ and $\psi\colon \mathbb{H} \nrightarrow \mathbb{H}$, we say that

$$\begin{tikzcd} \mathbb{G} \arrow[r, "\varphi", "|"] \arrow[d, rightsquigarrow, "F"'] & \mathbb{G} \arrow[d, rightsquigarrow, "F"] \\ \mathbb{H} \arrow[r, "\psi"', "|"] & \mathbb{H} \end{tikzcd} \tag{4.3}$$

is a *commuting square* if:

(i) On objects, we have $u \in s(\psi)$ if and only if $F(u) \in s(\varphi)$; and for those u where this does hold, we have that $\varphi(F(u)) = F(\psi(u))$ in $t(\varphi)$.

(ii) For all $f\colon F(x) \to y$ in $s(\varphi)$, we have $\psi(F_x(f)) = F_{\psi(x)}(\varphi(f))$ in $t(\psi)$.

Now by an *extended inner partial automorphism* of $\mathbb{G}$, we mean a family of partial automorphisms $(\beta_F\colon \mathbb{H} \nrightarrow \mathbb{H})$, one for each cofunctor $F\colon \mathbb{G} \rightsquigarrow \mathbb{H}$, such that for all cofunctors $F\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ and $G\colon \mathbb{H} \rightsquigarrow \mathbb{K}$ we have a commuting square:

$$\begin{tikzcd} \mathbb{H} \arrow[r, "\beta_F", "+" marking] \arrow[d, squiggly, "G"'] & \mathbb{H} \arrow[d, squiggly, "G"] \\ \mathbb{K} \arrow[r, "\beta_{GF}"', "+" marking] & \mathbb{K}\ . \end{tikzcd}$$

Using the fact that commuting squares of the form (4.3) stack vertically and horizontally to give commuting squares, we can now follow through the same argument as before, *mutatis mutandis*, to show that:

4.13. Theorem. *The monoid of extended partial inner automorphisms of a groupoid* $\mathbb{G}$ *is isomorphic to the monoid of partial bisections* $\mathrm{PBis}(\mathbb{G})$*. The extended inner automorphism corresponding to* $\alpha \in \mathrm{PBis}(\mathbb{G})$ *is the family of partial automorphisms* $(c_{F\alpha}\colon \mathbb{H} \rightarrowtriangle \mathbb{H})$ *as* F *ranges over cofunctors* $\mathbb{G} \rightsquigarrow \mathbb{H}$*.*

The details are sufficiently similar that we leave them to the interested reader to reconstruct.

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School of Math. & Phys. Sciences, Macquarie University, NSW 2109, Australia
Email: richard.garner@mq.edu.au

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