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# TOPOLOGICAL ENDOMORPHISM MONOIDS OF MODELS OF GEOMETRIC THEORIES

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Abstract. A classical model theory result states that for a (set-theoretic) model of a first-order theory, there is a Galois connection between subgroups of the automorphism group of the model and 'relational extensions' of the model, and the subgroups which are fixed by this connection are precisely the closed subgroups for the 'pointwise convergence topology' on the automorphism group. We prove an analogous result for endomorphism monoids of models, grounded in the theory of classifying toposes. In particular, we show that the topos of continuous actions of the endomorphism monoid with respect to the pointwise convergence topology classifies a natural theory associated to the model.

## 1. Introduction

I had the good fortune of having Pieter Hofstra on my PhD thesis committee at the end of 2021. Before then, he had already influenced my work indirectly through his collaboration with Jonathon Funk on toposes and actions of inverse semigroups [5]. This paper is based on a question which he asked me in my thesis defence about the connection between my work on actions of topological monoids and a classical result in model theory regarding the topological automorphism group of a model.

Many connections between model theory and category theory are well-established. For example, the theories of locally presentable and accessible categories [1] straddle set-theoretic and category-theoretic foundations to the end of capturing the global structure of categories of models. Meanwhile, Caramello [3] and Kubiś [12] have provided categorical abstractions of the Fraïssé construction, which originates in model theory. The present paper is a further contribution to the broader effort to lift celebrated results from model theory to a wider categorical context.

1.1. Motivation. The following result appears in Wilfrid Hodges' *Model Theory* [7].

1.1.1. Proposition. [7, Theorem 4.1.4] *Let $\Omega$ be a set and $G \leq \mathrm{Aut}(\Omega)$ a subgroup of its topological automorphism group, with the* ***pointwise convergence topology*** *whose opens are unions of cosets of stabilizers of finite subsets. Let $H \leq G$ be a subgroup. Then the following are equivalent:*

- *$H$ is closed in $G$.*

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- *There is a structure* $A$ *with* $\mathrm{dom}(A) = \Omega$ *such that* $H = \mathrm{Aut}(A) \cap G$.

Hodges uses *structure* to refer to a set (the *domain*, $\mathrm{dom}(A)$) equipped with relations which automorphisms are required to preserve; our terminology in the remainder of the article will more closely align with the point of view that such a structure is a **model of a theory** in some fragment of logic. We can, for instance, choose $G$ to be the automorphism group of a model $M$ (in **Set**) of a (single-sorted) theory $\mathbb{T}$ whose basic sort is interpreted as the set $\Omega$; Proposition 1.1.1 says that the closed subgroups of $G$ are automorphism groups of models obtained by adding relations to $M$, and hence to $\mathbb{T}$.

Underlying this result is a Galois connection whose construction requires no advanced theory:

$$\{\text{subgroups } H \leq G\} \leftrightarrows \{\text{families of relations on } \Omega\}.$$

Given a subgroup $H \leq G$, we can construct the set of relations on $\Omega$ which are preserved by all elements of $H$, and given a family of relations we can extract the subgroup of automorphisms preserving all relations in the family. As usual, each operation is order-reversing and passing back and forth across the Galois connection produces an idempotent *closure* operation on each side. It is possible to directly extend the above Galois connection by putting submonoids of the endomorphism monoid of $M$ on the left-hand side. Both Galois connections and some further related ones were developed in the 1930s by Marc Krasner [11] (this a reference to Krasner's expository work in French; he gives several references to his original work at the end of the introduction there), and have re-emerged several times since, such as in the work of Rosenburg [16].

On the subgroup side, the closure operation preserves the bottom element and (much less obviously) binary joins of subgroups, so it determines a topology on $G$, which is precisely the pointwise convergence topology of Proposition 1.1.1. Hodges provides an explicit description of a topology and then proves that closure with respect to this topology coincides with the closure operator induced by the Galois connection. While Hodges' exposition is transparent, one cannot help but wonder if there is more to this story. As such, we seek to address the following question in the present article via topos theory:

**What exactly is the pointwise convergence topology capturing, and what does the corresponding notion of continuity have to do with model theory?**

For reasons that will become apparent later on, we begin from the more general setting of *topological endomorphism monoids* of models of geometric theories. By the end of the article we will have recovered the equivalences of posets featuring in Proposition 1.1.1.

1.2. Summary. We begin in Section 2 with some pointers to topos theory background. We lean heavily on [15], but since that work is not widely known we reproduce many of the results we need here. We have not made the same level of effort for the much better-established edifice of categorical logic, but we at least provide references for the necessary foundational aspects.

In Section 3 we take a model $M$ of a geometric theory $\mathbb{T}$ and factorize a corresponding geometric morphism (a point of the classifying topos for $\mathbb{T}$). In our view the most significant result of this article is Theorem 3.1.6, where we demonstrate that an intermediate topos in this factorization is both the topos of continuous actions for the topological endomorphism monoid of the model and the classifying topos of a natural extension of $\mathbb{T}$ determined by $M$. This is our answer to the question posed above. Drawing some further results from [15], we go from there to recover a monoid-theoretic analogue of the Galois connection described above, Theorem 3.2.1.

Finally, in Section 4 we recover the group-theoretic versions of the preceding results (Corollaries 4.2.4 and 4.2.5). We exhibit some interesting auxiliary results along the way, such as an example of a 'pseudomonic' geometric morphism which is not an inclusion (Remark 4.1.5).

## 2. Background

Throughout, 'topos' is shorthand for *Grothendieck* topos. We freely use the fact that every such category can be presented as the category of sheaves on a *site*; see [13, Chapter III].

### 2.1. Properties of geometric morphisms.

2.1.1. Definition. *Let $\mathcal{E}, \mathcal{F}$ be toposes. Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ (consisting of an adjunction*

$$\begin{tikzcd} \mathcal{F} \arrow[r, shift right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

*in which the left adjoint, called the inverse image functor, preserves finite limits) is called:*

- *an **inclusion** if $f_*$ is full and faithful;*
- *a **surjection** if $f^*$ is faithful;*
- ***localic** if each object of $\mathcal{F}$ is a subquotient of one in the essential image of $f^*$;*
- ***hyperconnected** if $f^*$ is full and faithful and its essential image is closed under subquotients;*
- ***essential** if $f^*$ has an extra left adjoint (denoted $f_!$).*

Recall that toposes and geometric morphisms assemble into a *bicategory*, where the 2-morphisms are natural transformations between inverse image functors. We write $\mathrm{Geom}(\mathcal{F}, \mathcal{E})$ for the category of geometric morphisms from $\mathcal{F}$ to $\mathcal{E}$. This bicategory admits several well-known orthogonal factorization systems; we shall exploit the *surjection–inclusion* and *hyperconnected–localic* factorization systems (see [9, §A4]) in the present paper. We can also examine the representable properties of geometric morphisms in this bicategory.

2.1.2. Definition. *A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is called* ***representably faithful*** *(resp.* ***representably full and faithful****) if composition with $f$ is faithful (resp. full and faithful) as a functor:*

$$f \circ - : \mathrm{Geom}(\mathcal{G}, \mathcal{F}) \to \mathrm{Geom}(\mathcal{G}, \mathcal{E})$$

*for each topos $\mathcal{G}$.*

Such representable properties (and related ones) were important in [15], and we shall examine some related properties in Lemma 3.1.2 and Lemma 4.2.1 below. Recall the following relationships between Definitions 2.1.1 and 2.1.2.

2.1.3. Lemma. [15, Propositions 4.6, 4.10] *Localic morphisms are representably faithful. Inclusions are representably full and faithful.*

2.2. Continuous actions of a topological monoid. In this article, we will denote monoids by $N$ or $N'$, reserving $M$ and variants thereof for models of theories. The unit element of a monoid will be denoted by 1 (adding a superscript as in $1_N$ if we fear ambiguity).

Recall that a **right action** of a monoid $N$ on a set $X$ consists of a function $\alpha : X \times N \to X$ such that for all $x \in X$ and $m, n \in N$,

$$\alpha(x, 1) = x \text{ and } \alpha(x, mn) = \alpha(\alpha(x, m), n).$$

The pair $(X, \alpha)$ is an **act** (sometimes called *module*) of $N$, and these form a category with the evident notion of homomorphism. This category is equivalent to the category of presheaves on $N$ when the latter is viewed as a one-object category; as such, we conflate the two and denote the category of acts by $\mathrm{PSh}(N)$.

If $N$ is equipped with a topology $\tau$, we can construct the full subcategory of $\mathrm{PSh}(N)$ for which the action $\alpha$ is continuous when $X$ is equipped with the discrete topology and $X \times N$ equipped with the product of $\tau$ with the discrete topology. The resulting category, denoted $\mathrm{Cont}(N, \tau)$, is coreflective in $\mathrm{PSh}(N)$ and closed under finite limits and subquotients (see [15, Proposition 1.4]). In fact, both $\mathrm{PSh}(N)$ and $\mathrm{Cont}(N, \tau)$ are toposes (see [15, Corollary 1.19 of Proposition 1.12] for the latter), and we have geometric morphisms:

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left=50, "- \times N"] \arrow[r, bend right=50, "\mathrm{Hom}_{\mathbf{Set}}(N,-)"'] \arrow[r, phantom, bend left=25, "\perp"] \arrow[r, phantom, bend right=25, "\perp"] & \mathrm{PSh}(N) \arrow[l, "U"'] \arrow[r, shift right=1.5, "h_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Cont}(N, \tau), \arrow[l, shift right=1.5, "h^*"'] \end{tikzcd} \tag{1}$$

where $U$ is the forgetful functor sending an act $(X, \alpha)$ to $X$ and $V$ is the inclusion of $\mathrm{Cont}(N, \tau)$ into $\mathrm{PSh}(N)$. The geometric morphism on the left is an essential surjection and that on the right is hyperconnected.

A geometric morphism $\mathbf{Set} \to \mathcal{E}$ is called a **point** of $\mathcal{E}$, so the composite of the morphisms in (1) provides a point of $\mathrm{Cont}(N, \tau)$, which we call its **canonical point**; beware that this point is canonical for this presentation of the topos of actions, rather than being intrinsically determined by the categorical structure of the topos.

Conversely, suppose we are given a point $p : \mathbf{Set} \to \mathcal{E}$ and let $N := \mathrm{End}(p)^{\mathrm{op}}$ be (the opposite of) the monoid of endomorphisms of $p$ in $\mathrm{Geom}(\mathbf{Set}, \mathcal{E})$. Then we can construct a canonical factorization of $p$ through the essential surjection $\mathbf{Set} \to \mathrm{PSh}(N)$. Indeed, for each object $X$ of $\mathcal{E}$, the set $p^*(X)$ is canonically equipped with a right action of $N$: an element $\alpha \in N$ acts on $X$ by the component at $X$ of the natural endomorphism of $p$ corresponding to $\alpha$; naturality ensures that $p^*$ sends morphisms of $\mathcal{E}$ to act homomorphisms for these actions. Thanks to the fact that the forgetful functor $\mathrm{PSh}(N) \to \mathbf{Set}$ creates all limits and colimits, the resulting functor $q^* : \mathcal{E} \to \mathrm{PSh}(N)$ still preserves finite limits and inherits a right adjoint $q_*$, as required. This argument appears in [14, Proposition 6.1.1].

We can further factor the morphism $q$ just constructed. Let $\tau$ be the coarsest topology on $N$ making all $N$-acts of the form $q^*(X)$ continuous. Then we end up with a diagram extending (1):

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left=60, "-\times N"] \arrow[r, bend right=60, "\mathrm{Hom}_{\mathbf{Set}}(N,-)"'] & \mathrm{PSh}(N) \arrow[l, "U"'] \arrow[r, shift right, "h_*"'] & \mathrm{Cont}(N,\tau) \arrow[l, shift right, "h^*"'] \arrow[r, shift right, "l_*"'] & \mathcal{E}, \arrow[l, shift right, "l^*"']
\end{tikzcd} \tag{2}$$

where $l_*$ is simply $q_* l^*$. We call $(N, \tau)$ the **topological endomorphism monoid** of $p$. Observe that when $l$ is an equivalence, this gives us a presentation of $\mathcal{E}$ as a topos of continuous actions of a monoid.

2.2.1. THEOREM. [15, Theorem 3.20] *Suppose $p$ is a point of $\mathcal{E}$ which decomposes as the canonical point of* $\mathrm{PSh}(N')$ *followed by a hyperconnected morphism for some monoid $N'$. Then the morphism $l$ in* (2) *is an equivalence. That is, $\mathcal{E} \simeq$* $\mathrm{Cont}(N, \tau)$ *(although the monoid $N$ may be distinct from $N'$ in general).*

In [15], we called a topological monoid *complete* if it was isomorphic to (the opposite of) the topological automorphism monoid of the canonical point of its topos of actions.

2.2.2. EXAMPLE. Observe that $\mathrm{PSh}(N) = \mathrm{Cont}(N, \tau_{\mathrm{disc}})$, where $\tau_{\mathrm{disc}}$ is the discrete topology. The monoid of endomorphisms of the canonical point of $\mathrm{PSh}(N)$ is isomorphic to $N^{\mathrm{op}}$. In other words, discrete monoids are complete in the above sense. We will use this fact later on.

For a less trivial example, let $\mathbb{Z}$ denote the additive group of integers and consider the coreflective subcategory $\mathcal{C}$ of $\mathrm{PSh}(\mathbb{Z})$ on the acts for which all orbits are finite. Then $\mathcal{C} \simeq \mathrm{Cont}(\mathbb{Z}, \tau)$, where $\tau$ is generated by the cosets $\{n\mathbb{Z} + k \mid n \geq 1,\, k \in \mathbb{Z}\}$. However, the above procedure yields an equivalence between $\mathcal{C}$ and the continuous actions of the profinite completion of $\mathbb{Z}$. In other words, $(\mathbb{Z}, \tau)$ is not complete. See [15, Example 3.25] for more detail on this example.

2.3. EXTENSIONS OF THEORIES. We must assume for this section that the reader is familiar with the essentials of categorical logic, as presented in [9, §D1] or [4, Chapters 1 and 2]; out of respect for the reader's sanity, we endeavour to make our notation consistent with those references. Recall that any geometric theory $\mathbb{T}$ (over a signature $\Sigma$)

has a **classifying topos**: a topos denoted $\mathbf{Set}[\mathbb{T}]$ such that for any topos $\mathcal{F}$ we have an equivalence,

$$\mathrm{Geom}(\mathcal{F}, \mathbf{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathcal{F}), \tag{3}$$

naturally in $\mathcal{F}$, where the left-hand side is the category of geometric morphisms between the respective toposes and the right-hand side is the category of $\mathbb{T}$-models in $\mathcal{F}$. We shall be concerned almost entirely with the case $\mathcal{F} = \mathbf{Set}$, but we shall speculate in Section 5.1 about how our results generalize. Explicitly, the classifying topos can be constructed as the topos of sheaves on the (geometric) *syntactic site* of $\mathbb{T}$, denoted $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$, which we shall only handle informally in the present text.

Two geometric theories are said to be **Morita-equivalent** if their syntactic sites are equivalent, which is the same as their categories of model in any (pre)topos being equivalent. The latter makes it clear that we should consider Morita-equivalent theories to be 'essentially the same' from a category theory perspective.

Besides corresponding to a model, a geometric morphism $\mathcal{E} \to \mathbf{Set}[\mathbb{T}]$ is induced by a presentation of $\mathcal{E}$ as the classifying topos for an *extension* $\mathbb{T}'$ of the theory $\mathbb{T}$: a theory obtained by adding ingredients (sorts or function symbols or relation symbols) to the signature and axioms in the resulting extended language to those of $\mathbb{T}$. While there are typically many extensions producing essentially the same geometric morphism, we may deduce the existence of an extension of a particular form from properties of the geometric morphism and vice versa. We will take advantage of such results in the present paper, so we recall them now.

2.3.1. Definition. *Let $\mathbb{T}$ be a geometric theory (over a signature $\Sigma$).*

*An **axiomatic extension** (called a quotient in [4, Definition 3.2.2]) of $\mathbb{T}$ is a theory $\mathbb{T}'$ over the same signature $\Sigma$ obtained by adding axioms (i.e. geometric sequents) to $\mathbb{T}$ (or more generally, such that every axiom of $\mathbb{T}$ is provable in $\mathbb{T}'$).*

*A **relational extension** (called a localic extension in [4, Definition 7.1.1]) of $\mathbb{T}$ is a theory $\mathbb{T}'$ obtained by adding only relation symbols to $\Sigma$ and axioms to $\mathbb{T}$.*

*In either case, the syntactic embedding of $\mathbb{T}$ into $\mathbb{T}'$ induces a morphism of sites $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \to (\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$ which induces a geometric morphism $p : \mathbf{Set}[\mathbb{T}'] \to \mathbf{Set}[\mathbb{T}]$.*

2.3.2. Theorem. [4, §3.2; Theorems 7.1.3 and 7.1.5] *Let $\mathbb{T}$ be a geometric theory over a signature $\Sigma$. If $\mathbb{T}'$ is an axiomatic extension (resp. relational extension) of $\mathbb{T}$, the induced geometric morphism $\mathbf{Set}[\mathbb{T}'] \to \mathbf{Set}[\mathbb{T}]$ is an inclusion (resp. is localic).*

*Conversely, let $p : \mathcal{E} \to \mathbf{Set}[\mathbb{T}]$ be an inclusion (resp. a localic morphism). Then $p$ factors as $\mathcal{E} \simeq \mathbf{Set}[\mathbb{T}'] \to \mathbf{Set}[\mathbb{T}]$, where $\mathbb{T}'$ is an axiomatic extension (resp. relational extension) of $\mathbb{T}$.*

Proof. We refer the reader to the above-cited parts of Caramello's book [4] for the full proofs, but we note in each case how $\mathbb{T}'$ is constructed from $p$ in the converse direction.

For $p$ an inclusion, an axiomatic extension is obtained by adding a sequent $\psi \vdash_{\vec{x}} \phi \wedge \psi$ to $\mathbb{T}$ for each morphism $\{\vec{x}.\phi \wedge \psi\} \to \{\vec{x}.\psi\}$ whose image in $\mathbf{Set}[\mathbb{T}]$ is inverted by $p^*$.

When $p$ is localic, we construct $\mathbb{T}'$ by first adding to $\Sigma$ a relation for each subobject in $\mathcal{E}$ of a finite product of objects of the form $p^*(\mathbf{y}_{\mathbb{T}}(\{\vec{x}.\top\}))$ (which form a generating set of objects for $\mathcal{E}$ when $p$ is localic), then adding to $\mathbb{T}$ all sequents valid in $\mathcal{E}$ for the extended signature. Note that in the special case that $p$ is actually an inclusion, the added relation symbols are all $\mathbb{T}$-provably equivalent to formulas expressible in $\Sigma$, so we end up with a theory Morita-equivalent to the preceding one at the end of this process. ■

In fact, any relational extension $\mathbb{T}'$ of $\mathbb{T}$ which makes all subobjects of products of objects of the form $p^*(\mathbf{y}_{\mathbb{T}}(\{\vec{x}.\top\}))$ definable and makes any two such presentations $\mathbb{T}'$-provably equivalent will be Morita-equivalent to the extension described in the proof of Theorem 2.3.2. We use such a variant in the proof of Theorem 3.1.6.

## 3. Endomorphisms of models

3.1. Factorizing the point. Let $\mathbb{T}$ be any geometric theory. An ordinary **Set**-model $M$ of $\mathbb{T}$ corresponds via (3) to a geometric morphism $\ulcorner M \urcorner : \mathbf{Set} \to \mathbf{Set}[\mathbb{T}]$, and moreover the endomorphisms of this model correspond to the endomorphisms of that geometric morphism. Let $N := \mathrm{End}(M)^{\mathrm{op}}$ be (the opposite of) the endomorphism monoid of $M$. In this section we shall factor $\ulcorner M \urcorner$ as follows.

$$\begin{tikzcd}
\mathbf{Set} \arrow[rrrr, "\ulcorner M \urcorner"] \arrow[dr, "e"'] & & & & \mathbf{Set}[\mathbb{T}] \\
& \mathrm{PSh}(N) \arrow[rr, "s"] \arrow[dr, "h"'] & & \mathbf{Set}[\mathbb{T}_M] \arrow[ur, hook, "i"'] & \\
& & \mathcal{E} \arrow[ur, "l"'] & &
\end{tikzcd} \tag{4}$$

First of all, we consider the **surjection–inclusion** factorization of $\ulcorner M \urcorner$. The resulting subtopos of $\mathbf{Set}[\mathbb{T}]$, whose inclusion we call $i$, classifies the **theory of** $M$, the extension $\mathbb{T}_M$ of $\mathbb{T}$ obtained by adding to the axioms of $\mathbb{T}$ all geometric sequents which are valid in $M$, or equivalently by the procedure from Theorem 2.3.2.

Next, we factor the surjective part of $\ulcorner M \urcorner$ as described before (2) to obtain an essential surjection $e$ followed by a surjection $s$. By Lemma 2.1.3, these two factorizations can be performed in either order, since endomorphism monoid of the point of $\mathbf{Set}[\mathbb{T}_N]$ is isomorphic to $N$.

Let us pause to examine the relationship between the monoid $N$ and the functors thus constructed.

3.1.1. Lemma. *Let $(\mathcal{C}, J)$ be any site for $\mathbf{Set}[\mathbb{T}_M]$. Denote by $\int_{\mathcal{C}} s^*$ the category of elements of the restriction of $s^*$ to the representable sheaves. The right $N$-act $N$, as an object of* $\mathrm{PSh}(N)$*, is the limit:*[1]

$$N = \lim_{(X,x)\in\int_{\mathcal{C}} s^*} s^*(X).$$

[1] Here and elsewhere we abuse notation a little in writing $N$ for both the monoid defined previously and its action on itself by right multiplication.

Proof. Since the forgetful functor $e^*$ creates all limits and colimits, we can compute (the underlying set of) this limit in **Set**. An element of the limit is easily interpreted as an endomorphism of $\coprod_{X \in \mathrm{ob}(\mathcal{C})} s^*(X)$ which commutes with each morphism of $\mathcal{C}$, which are precisely the elements of $N$. Meanwhile, the action of $N$ on the right on $s^*(X)$ is exactly the action by application of the endomorphisms on the left, so the action of $N$ on itself is recovered as expected. ∎

From a calculation standpoint, Lemma 3.1.1 tells us nothing new. However, this result identifies a universal property relating the site $(\mathcal{C}, J)$ and a generating object in $\mathrm{PSh}(N)$, which enables us to deduce a useful property of the geometric morphism $s$.

3.1.2. Lemma. *The geometric morphism $s$ is* ***representably full and faithful on essential geometric morphisms****, in the sense that for any topos $\mathcal{F}$, the functor*

$$s \circ - : \mathrm{EssGeom}(\mathcal{F}, \mathrm{PSh}(N)) \to \mathrm{Geom}(\mathcal{F}, \mathbf{Set}[\mathbb{T}_M])$$

*is full and faithful, where* $\mathrm{EssGeom}(-,=)$ *is the full subcategory of* $\mathrm{Geom}(-,=)$ *on the essential geometric morphisms.*

Proof. Concretely, the effect of $s$ on a natural transformation $\alpha : x^* \Rightarrow y^*$ is restriction to the image of $s^*$. In other words, $\alpha \mapsto \alpha_{s^*} : x^*s^* \Rightarrow y^*s^*$. Since $y$ is essential, $y^*$ preserves small limits, so that in particular $y^*(N) = \lim_{(X,x) \in \int_{\mathcal{C}} s^*} y^*(s^*(X))$ for any site $(\mathcal{C}, J)$ presenting $\mathbf{Set}[\mathbb{T}_M]$ by Lemma 3.1.1. As such, $\alpha_N$ is forced to be the unique comparison morphism to this limit when $x^*(N)$ is viewed as the apex of a cone over the same diagram. Since $N$ is a generator of $\mathrm{PSh}(N)$ and $x^*, y^*$ preserve colimits, $\alpha$ is uniquely determined by its component at $N$ and hence by the components $\alpha_{s^*}$. It follows that $s \circ -$ is bijective on such natural transformations, as required. ∎

3.1.3. Remark. There is no reason to expect that $s$ would be representably full and faithful on a wider subcategory of $\mathrm{Geom}(\mathcal{F}, \mathrm{PSh}(N))$ in general. While we judge that the construction of an explicit counterexample would distract from the narrative of the present paper, we note in passing that the possibility of a *representably fully faithful surjection* (or more specifically one which is not an equivalence) would be of interest in its own right, since it is not known whether there are representably fully faithful geometric morphisms which are not inclusions. See also Remark 4.1.5 below.

We complete diagram (4) by taking the **hyperconnected–localic** factorization of $s$.

3.1.4. Lemma. *Let $h;l$ be the hyperconnected–localic factorization of $s$ in* (4)*. Then $h$ is representably full and faithful on essential geometric morphisms.*

Proof. Suppose we are given essential geometric morphisms $x, y : \mathcal{F} \rightrightarrows \mathrm{PSh}(N)$. Consider the maps:

$$\mathrm{Hom}(x,y) \xrightarrow{h\circ -} \mathrm{Hom}(h \circ x, h \circ y) \xrightarrow{l\circ -} \mathrm{Hom}(s \circ x, s \circ y);$$

the composite map corresponds to composition with $s$, and so is injective and surjective by Lemma 3.1.2. Injectivity ensures injectivity of the left-hand map, so $h$ is representably faithful on essential geometric morphisms. Surjectivity forces the right-hand map to be surjective, but by Lemma 2.1.3 the right-hand map is also injective and thus a bijection. It follows that the left-hand map is surjective, as required. ■

We are now ready to state the first main theorem. We first introduce some definitions to streamline the theorem statement.

3.1.5. Definition. *Let $\mathbb{T}$ be a geometric theory over a signature $\Sigma$, $M$ a* **Set***-model of $\mathbb{T}$ and $N$ the monoid of $\Sigma$-structure endomorphisms of $M$. The* ***equivariant theory of*** *$M$ is the theory obtained from $\mathbb{T}$ by adding a relation symbol $R \hookrightarrow A_1, \dots, A_n$ for each $N$-equivariant relation $R \hookrightarrow [\![A_1]\!]_M \times \dots [\![A_n]\!]_M$ and all axioms relating these and geometric formulas over $\Sigma$ to one another which are valid in $M$. This is a localic extension of the theory of $M$, $\mathbb{T}_M$, introduced earlier. We denote it by $\mathbb{T}_{\looparrowleft M}$, FWoBN*[2].

*The* ***pointwise convergence topology*** *on $N$ induced by $M$ has as basis of neighbourhoods of an element $m \in N$ the sets*

$$U_{x_1,\dots,x_k}(m) = \{m' \in N \mid m'(x_1) = m(x_1), \dots, m'(x_k) = m(x_k)\},$$

*where $k$ varies over the natural numbers and $x_i \in [\![A_i]\!]_M$ for sorts $A_1, \dots, A_k$ in $\Sigma$. We denote this topology by $\tau$.*

3.1.6. Theorem. *The intermediate topos $\mathcal{E}$ in (4) can equivalently be expressed as either the topos of continuous actions of $N$ with the pointwise convergence topology, or as the classifying topos for equivariant theory of $M$. That is,*

$$\mathrm{Cont}(N, \tau) \simeq \mathbf{Set}[\mathbb{T}_{\looparrowleft M}]. \tag{5}$$

Proof. The conclusion of Lemma 3.1.4 looks a lot like [15, Proposition 4.5], which characterizes *complete monoids* mentioned before Example 2.2.2, and this is no coincidence; the current proof is comparable to the proof of that result. By Theorem 2.2.1, $\mathcal{E}$ is equivalent to $\mathrm{Cont}(N', \tau')$, where $(N', \tau')$ is the topological endomorphism monoid of $h \circ e$. But from Example 2.2.2, the endomorphism monoid of $e$ is $N^{\mathrm{op}}$, and $h$ being full and faithful by Lemma 3.1.4 means that $\mathrm{End}(h \circ e) \cong \mathrm{End}(e)$, so $N' = N$.

Meanwhile, $\tau'$ is the coarsest topology making all actions of the form $h^*(X)$, for $X$ in $\mathcal{E}$, continuous. Since a subquotient of a continuous act is continuous, this coincides with the topology making acts of the form $s^*(X)$ for $X$ in $\mathbf{Set}[\mathbb{T}_M]$ continuous. In the language of [15], the opens of the form $U_x(m)$ for $x \in s^*(X)$ and $m \in N$ are precisely the 'necessary clopens': the subsets of $N$ which must be open in order for the action of $N$ on $s^*(X)$ to be continuous at $x$. By inspection, $U_{x_1,\dots,x_k}(m) = \bigcap_{i=1}^k U_{x_i}(m)$, so these subsets are forced to be open as soon as the necessary clopens are present. This proves that $\tau' = \tau$ coincides with the pointwise convergence topology.

On the other side, by Theorem 2.3.2 $\mathcal{E}$ classifies a relational extension of $\mathbb{T}_M$; we need only show that the equivariant theory of $M$ is $\mathbb{T}_{\looparrowleft M}$ (up to Morita-equivalence). Indeed, using a further construction from [4, Theorems 7.1.3], we can construct an extension $\mathbb{T}'$

[2]For Want of Better Notation.

of $\mathbb{T}_M$ classified by $\mathrm{PSh}(N)$ by adding a (suggestively named) sort $N$ to the signature $\Sigma$, a function symbol $m : N \to N$ for each element of $N$ (or at least for the unit element and each generator in a presentation of $N$, if one is provided) and a function symbol $x : N \to A$ for each element $x \in [\![A]\!]_M$. The axioms to be added are those expressing the multiplication in $N$, the action of $N$ on the $M$-interpretations of the sorts in $\Sigma$, and axioms such as:

$$\top \vdash_{y:A} \bigvee_{x \in [\![A]\!]_M} (\exists a : N)\,(x(1) = y)$$

asserting that the function symbols corresponding to elements of $[\![A]\!]_M$ cover the interpretation of $A$. We can recover a theory classified by $\mathcal{E}$ as the fragment of this theory obtained by eliminating $N$, in the sense of retaining only relations definable in the larger theory on lists of sorts not including $N$. We only introduced function symbols *out of* $N$, so the relations resulting from this elimination are precisely disjunctions of relations of the form

$$(\exists a : N)\,(x_1(a) = y_1) \wedge \cdots \wedge (x_k(a) = y_k),$$

(up to substitution of terms $y_1, \ldots, y_k$). These constitute exactly the relations appearing in the equivariant theory of $M$. The axioms in $\mathbb{T}_{\hookrightarrow M}$ are similarly all of the axioms of this fragment, as required. ■

3.1.7. Remark. While we have been working in the language of classifying toposes, the proofs of Lemmas 3.1.2 and 3.1.4 did not rely on the logical presentation. In particular, the argument of the first part of the proof of Theorem 3.1.6 shows that the factorization $h;l$ appearing in (2) is the hyperconnected–localic factorization of the morphism $q : \mathrm{PSh}(N) \to \mathcal{E}$ constructed beforehand.

3.2. Relational extensions of a model. Let $M'$ be a relational extension of the model $M$. This corresponds not merely to a localic morphism into $\mathbf{Set}[\mathbb{T}_M]$, but more particularly to a morphism through which $\ulcorner M \urcorner$ factors, since $M'$ retains the interpretations of the signature determining $M$. If we add this to the diagram (we name it $a$) and, from there, factorize as we did in the previous section, we end up with the following diagram:

$$\begin{tikzcd}
\mathbf{Set} \arrow[rrrr, "\ulcorner M \urcorner"] \arrow[dr, "e"] \arrow[ddr, "e'"'] & & & & \mathbf{Set}[\mathbb{T}] \\
& \mathrm{PSh}(N) \arrow[rr, "s"] \arrow[dr, "h"'] & & \mathbf{Set}[\mathbb{T}_M] \arrow[ur, hook, "i"] & \\
& \mathrm{PSh}(N') \arrow[u, "b"] \arrow[dr, "h'"'] & \mathcal{E} \arrow[ur, "l"] & \mathbf{Set}[\mathbb{T}_{M'}] \arrow[u, "a"] \arrow[uur, dashed] & \\
& & \mathcal{E}' \arrow[u, "c"] \arrow[ur, "l'"'] & &
\end{tikzcd} \tag{6}$$

The morphism $b$ is induced by the inclusion of submonoids $N' \subseteq N$ and thus is an essential localic surjection; see [6, Corollary 3.4]. Indeed, $N' = \mathrm{End}(M')^{\mathrm{op}}$ is necessarily a submonoid of $N$ (this could be proved using Lemma 3.1.2, say).

There are several ways to construct $c$. Most abstractly, we can use orthogonality of hyperconnected morphisms against localic morphisms. From here, we conclude that $c$ is a localic surjection because it is a left factor of the localic composite $a \circ l'$ and a right factor of the surjective composite $h \circ b$. Alternatively, one can verify that $b^*$ maps continuous $(N, \tau)$-acts to continuous $(N', \tau')$-acts, which guarantees that the inclusion $N' \hookrightarrow N$ is continuous by [15, Lemma 4.2]. But $(N, \tau)$ and $(N', \tau')$ are complete monoids by construction, and a geometric morphism $\mathrm{Cont}(N', \tau') \to \mathrm{Cont}(N, \tau)$ induced by a continuous monoid homomorphism is localic if and only if it is a **closed** inclusion, by [15, Theorem 4.24].

In the opposite direction, suppose we are given a submonoid $N'$ of $N$, and hence a geometric morphism $b : \mathrm{PSh}(N') \to \mathrm{PSh}(N)$. Taking the hyperconnected–localic factorization of $h \circ b$, we get a diagram:

$$\begin{tikzcd}
\mathbf{Set} \arrow[rrrr, "\ulcorner M \urcorner"] \arrow[dr, "e"] \arrow[ddr, "e'"'] & & & & \mathbf{Set}[\mathbb{T}] \\
& \mathrm{PSh}(N) \arrow[rr, "s"] \arrow[dr, "h"] & & \mathbf{Set}[\mathbb{T}_M] \arrow[ur, hook, "i"] & \\
& \mathrm{PSh}(N') \arrow[u, "b"] \arrow[dr, "h'"'] & \mathcal{E} \arrow[ur, "l"] & \mathbf{Set}[\mathbb{T}_{M'}] \arrow[u, dashed] & \\
& & \mathcal{E}' \arrow[u, "c"] \arrow[ur, dashed, equal] & &
\end{tikzcd} \tag{7}$$

From the classifying topos point of view, we recover a relational extension of the theory of $M$ (indeed, of the equivariant theory of $M$) from the composite $l \circ c$. Practically speaking, this operation boils down to the corresponding direction of the Galois connection described in Section 1.1: $M'$ is obtained by adding to $M$ all of the relations preserved by endomorphisms in $N'$. Applying [15, Theorem 4.19 and Corollary 4.23], if $(L, \rho)$ is the closure of $N'$ in $(N, \tau)$ (with the restriction topology) then $(L, \rho)$ is the complete monoid such that $\mathcal{E}' \simeq \mathrm{Cont}(L, \rho)$, which provides another means of showing that the closed submonoids are the fixed points on the left-hand side of the correspondence.

In summary, we have proved the following result.

3.2.1. Theorem. *Let $\mathbb{T}$ be a geometric theory (over a signature $\Sigma$) and $M$ a model of $\mathbb{T}$ in* **Set**. *Let $(N, \tau)$ be (the opposite of the) topological endomorphism monoid of $M$. There is a Galois connection,*

$$\{\textit{submonoids } N' \leq N\} \leftrightarrows \{\textit{relational extensions of } M\},$$

*whose fixed points on the left are closed submonoids, and whose fixed points on the right are the maximal equivariant extensions.*

# 4. Automorphisms of models

Observe that a submonoid (and even a closed submonoid) of a topological group need not be a subgroup: consider $\mathbb{N}$ in $\mathbb{Z}$, for example. At first glance, Theorem 3.2.1 would therefore seem to be in conflict with the theorem for groups quoted in Section 1.1! To resolve this, we remind the reader that our 'relational extensions' have always implicitly meant 'relational *geometric* extension'. That is, since geometric logic does not include negation as a logical connective, when we add a relation to the model, we need not also add its complement, and the endomorphisms of a relational extension need not preserve the complement of a relation. By restricting our attention to relational extensions where relations are complemented, we can recover the group-theoretic version of the result.

Before getting there, though, we need a few more auxiliary results about invertible elements of monoids which are interesting in their own right.

## 4.1. Invertible elements of a monoid.

4.1.1. Definition. *Given a monoid $N$, we write $N^{\rtimes}$ for the submonoid of* ***right invertible elements****. That is,*

$$N^{\rtimes} := \{u \in N : \exists v \in N, uv = 1\}.$$

*We similarly write $N^{\times}$ for the elements invertible on both sides.*

4.1.2. Lemma. *For any monoid $N$, $(N^{\rtimes})^{\rtimes} = N^{\times}$.*

We leave the proof of this amusing result to the reader.

4.1.3. Lemma. *The essential localic geometric morphism $r : \mathrm{PSh}(N^{\rtimes}) \to \mathrm{PSh}(N)$ induced by the inclusion $N^{\rtimes} \subseteq N$ is representably faithful and representably full* ***on isomorphisms****.*

Proof. Being representably faithful follows from Lemma 2.1.3, so we need only show representable fullness on isomorphisms. Observe that the complement $\overline{N^{\rtimes}}$ of $N^{\rtimes} \subseteq N$ is a sub-$N$-set of $N$. Indeed, if $n \in N$ is not right invertible, then the same is true of $nm$ for any $m \in N$. As such, we can take the cokernel in $\mathrm{PSh}(N)$ of the inclusion of $\overline{N^{\rtimes}}$ into $N$, which is the pushout:

$$\begin{tikzcd}
\overline{N^{\rtimes}} \arrow[r, hook] \arrow[d, "!"'] & N \arrow[d] \\
1 \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & P.
\end{tikzcd}$$

By construction, $r^*(P) \cong N^{\rtimes} + 1$ decomposes as a coproduct, and the first inclusion is in the image of $r^*$.

Now suppose we are given geometric morphisms $x, y : \mathcal{F} \to \mathrm{PSh}(N^{\rtimes})$ and a natural isomorphism $\alpha : x^* r^* \Rightarrow y^* r^*$. Since $x^*, y^*$ preserve the pushout above, we have that $\alpha_P$ defines an isomorphism $x^*(N^{\rtimes}+1) \to y^*(N^{\rtimes}+1)$ which restricts to an isomorphism on the 1 component. As such, we can extract a complementary isomorphism $\beta_{N^{\rtimes}} : x^*(N^{\rtimes}) \to$

$y^*(N^{\rtimes})$ which is a restriction of $\alpha_N$. Just as in Lemma 3.1.2, $\beta_{N^{\rtimes}}$ determines a natural isomorphism $\beta : x^* \Rightarrow y^*$ by colimits which must agree with $\alpha$ on the image of $r^*$. That is, $\alpha$ is the restriction along $r$ of $\beta$, which demonstrates fullness. ∎

4.1.4. Remark. The fact that $N^{\rtimes}$ is a complemented subset of $N$ was important in the proof. Should one wish to study actions of monoids in/over a non-Boolean topos, Lemma 4.1.3 may fail (for a monoid where $N^{\rtimes}$ is not complemented in the object underlying $N$). In particular, this is a hint that the monoid-theoretic result developed in the last section may be more robust to generalization than its group-theoretic counterpart to follow.

4.1.5. Remark. A geometric morphism which is representably faithful and representably full on isomorphisms is called **pseudomonic**, for example by Bunge and Lack in [2]. Lemma 4.1.3 provides an example of a pseudomonic geometric morphism which is not an inclusion. Bunge and Lack did not know of any such example, and we similarly are unaware of other examples (although it seems likely that some further related examples between presheaf toposes can be constructed from this one).

It is worth noting that the proof can be extended to conclude that $r$ is full on split monomorphisms, although the splittings are not necessarily reflected, so this stronger property is not compositional!

4.2. Recovering Hodges' result. Let $\mathbb{T}$, $\Sigma$, $M$ be as in Section 3, but now consider the group $G := \mathrm{Aut}(M)^{\mathrm{op}}$ dual to the group of automorphisms of $M$. We can perform a similar factorization of the geometric morphism $\ulcorner M \urcorner$:

$$\begin{tikzcd}
\mathbf{Set} \arrow[rrrr, "\ulcorner M \urcorner"] \arrow[dr, "e'"'] & & & & \mathbf{Set}[\mathbb{T}] \\
& \mathrm{PSh}(G) \arrow[rr, "s'"] \arrow[dr, "h'"'] & & \mathbf{Set}[\mathbb{T}_M] \arrow[ur, hook, "i"'] & \\
& & \mathcal{E}. \arrow[ur, "l'"'] & &
\end{tikzcd} \tag{8}$$

To construct $s'$, we could directly adapt the construction of $s$ by observing that $G$ has a canonical action on $\ulcorner M \urcorner^*(X)$ for each $X$ in $\mathbf{Set}[\mathbb{T}]$. Alternatively, we can identify $G$ with $(N^{\rtimes})^{\rtimes}$ thanks to Lemma 4.1.2, then compose $s$ with two instances of the geometric morphism considered in Lemma 4.2.1 to obtain $s'$ as the composite:

$$\mathrm{PSh}(G) \xrightarrow{r'} \mathrm{PSh}(N^{\rtimes}) \xrightarrow{r} \mathrm{PSh}(N) \xrightarrow{s} \mathbf{Set}[\mathbb{T}_M]. \tag{9}$$

We take the hyperconnected–localic factorization of $s'$, as before, to obtain $h'; l'$.

4.2.1. Lemma. *The geometric morphism $s'$ is representably faithful and representably full* ***on isomorphisms*** *between essential geometric morphisms, in the sense that for any (Grothendieck) topos $\mathcal{F}$, the functor*

$$s' \circ - : \mathrm{EssGeom}(\mathcal{F}, \mathrm{PSh}(G)) \to \mathrm{Geom}(\mathcal{F}, \mathbf{Set}[\mathbb{T}_M])$$

*has these properties.*

Proof. Consider the decomposition of $s'$ in (9). By Lemma 4.1.3, $r$ and $r'$ are representably faithful and full on isomorphisms between arbitrary morphisms. Moreover, they are essential, so the properties of $s$ from Lemma 3.1.2 imply the desired property of $s'$. ∎

4.2.2. Lemma. *Let $h'; l'$ be the hyperconnected–localic factorization of $s'$ in (8). Then $h$ is representably faithful and full on isomorphisms between essential geometric morphisms.*

Proof. Faithfulness can be proved via the first argument of the proof of Lemma 3.1.4. For fullness on isomorphisms, we restrict composition with $l'$ and then composition with $h'$ along the inclusion of the isomorphisms into $\mathrm{Hom}(s' \circ x, s' \circ y)$, for essential geometric morphisms $x, y : \mathcal{F} \rightrightarrows \mathrm{PSh}(G)$. By Lemma 4.2.1 this restriction is bijective, so the latter argument of Lemma 3.1.4 shows that $h'$ is full on natural transformations $h' \circ x \Rightarrow h' \circ y$ which are mapped by composition with $l'$ to isomorphisms; this includes all isomorphisms, as required. ∎

4.2.3. Definition. *Let $\mathbb{T}$ be a geometric theory over a signature $\Sigma$, $M$ a $\mathbf{Set}$-model of $\mathbb{T}$ and $G$ its group of automorphisms. The **decidable equivariant theory of** $M$ is the theory obtained from $\mathbb{T}$ by adding a relation symbol $R \hookrightarrow A_1, \dots, A_n$ for each $G$-equivariant relation $R \hookrightarrow [\![A_1]\!]_M \times \dots [\![A_n]\!]_M$ and all axioms relating these and geometric formulas over $\Sigma$ to one another which are valid in $M$. We denote it by $\mathbb{T}_{\simeq M}$, FWoBN.*

Implicit in Definition 4.2.3 is the fact that each relation in the decidable equivariant theory of $M$ is complemented: if a relation is preserved by all automorphisms of $M$ then its complement is too.

4.2.4. Corollary. *The intermediate topos $\mathcal{E}$ in (8) can equivalently be expressed as either the topos of continuous actions of $G$ with the pointwise convergence topology, or as the classifying topos for the decidable equivariant theory of $M$. That is,*

$$\mathrm{Cont}(G, \tau) \simeq \mathbf{Set}[\mathbb{T}_{\simeq M}]. \tag{10}$$

Proof. Applying Lemmas 4.2.1 and 4.2.2, we deduce that $\mathcal{E} \simeq \mathrm{Cont}(G, \tau')$ for the coarsest topology $\tau'$ making all actions in the essential image of $s^*$ continuous; the argument that this coincides with the pointwise convergence topology is identical to that in the proof of Theorem 3.1.6. The reconstruction of $\mathbb{T}_{\simeq M}$ from $l'$ is also almost identical. ∎

Finally, we define a **complemented relational extension** of $M$ to be an extension obtained by adding relations and their complements to $M$. With this notion, we can close the circle back to our motivating result.

4.2.5. Corollary. *Let $\mathbb{T}$ be a geometric theory (over a signature $\Sigma$) and $M$ a model of $\mathbb{T}$ in* $\mathbf{Set}$. *Let $(G, \tau)$ be (the opposite of the) topological automorphism group of $M$. There is a Galois connection,*

$$\{\textit{subgroups } G' \leq G\} \leftrightarrows \{\textit{complemented relational extensions of } M\},$$

*whose fixed points on the left are closed subgroups, and whose fixed points on the right are the maximal (complemented) equivariant extensions.*

Proof. As we said in the Introduction, this 'high-tech' construction is not needed to produce the Galois connection once we have explicit descriptions of the respective sides thanks to Corollary 4.2.4, so we provide only a sketch.

The proof strategy mirrors that of Section 3.2. The major difference is that we must exploit the fact that the relational extension is complemented to conclude that the classifying topos of the resulting extension of $\mathbb{T}_{\simeq M}$ is atomic, so that the topological automorphism group of $M'$ is dense in its topological endomorphism monoid, and the respective monoids have equivalent categories of actions by [15, Proposition 5.1]. ∎

## 5. Closing Remarks

5.1. Different toposes. We promised earlier that we would consider how this theory generalizes. In particular, from a categorical logic perspective, it would be interesting to remove the dependence on the category of sets. There are two places that **Set** appears in this work.

First, we can consider models in a topos other than **Set**, say $\mathcal{G}$. The factorization of Section 3 can be adapted to this setting, as follows. For a geometric morphism $\ulcorner M \urcorner : \mathcal{G} \to \mathbf{Set}[\mathbb{T}]$ corresponding to a model $M$ of $\mathbb{T}$ in $\mathcal{G}$, we can still define the monoid $N := \mathrm{End}(\ulcorner M \urcorner^*)^{\mathrm{op}}$; the category of $N$-acts in $\mathcal{G}$ can be presented as a topos of internal presheaves, $\mathrm{PSh}_{\mathcal{G}}(N) := [\gamma^*(N)^{\mathrm{op}}, \mathcal{G}]$ (where $\gamma : \mathcal{G} \to \mathbf{Set}$ is the global sections morphism of $\mathcal{G}$). There is still an essential surjection $e : \mathcal{G} \to [\gamma^*(N)^{\mathrm{op}}, \mathcal{G}]$. The surjection-inclusion and hyperconnected-localic factorizations need no modification, so we get a diagram:

$$
\begin{tikzcd}
\mathcal{G} \arrow[rrr, "\ulcorner M \urcorner"] \arrow[dr, "e"'] & & & \mathbf{Set}[\mathbb{T}] \\
& \mathrm{PSh}_{\mathcal{G}}(N) \arrow[r, "s"] \arrow[dr, "h"'] & \mathbf{Set}[\mathbb{T}_M] \arrow[ur, hook, "i"'] & \\
& & \mathcal{E} \arrow[u, "l"'] &
\end{tikzcd}
\tag{11}
$$

To recover the analogue of Theorem 3.1.6, we would on the one hand need to extract a presentation of the extension of $\mathbb{T}_M$ classified by $\mathcal{E}$ using Theorem 2.3.2. On the other hand, we would need to identify the correct variant of topology on $N$ (or $\gamma^*(N)$) which corresponds to hyperconnected morphisms out of $\mathcal{G}$; the recent work of Hora [8] parametrizing hyperconnected quotients of toposes should facilitate this. The argument for Theorem 3.2.1 similarly should lift without major structural changes.

However, we shall not attempt to explicitly derive these results here, since there are many subtleties to consider. When discussing topologies, will it be sufficient to work with ordinary topologies on $N$, or will some internal notion of topology in $\mathcal{G}$ be needed? What is the right notion of continuity for topologies in this setting? Similarly, when considering submonoids in the Galois correspondence, there may be strictly more internal monoids than external ones; is there a compatible notion of closure leading to a more refined Galois connection? These questions will form the basis of future work.

Second, **Set** is implicitly playing the role of base topos. Generally speaking, we can replace **Set** with a general (elementary) topos $\mathcal{S}$ and work in the bicategory $\mathfrak{TOP}/\mathcal{S}$, and all constructively valid results will continue to hold in this setting. The theory of classifying toposes lifts to this setting, for instance, subject to the caveat that we extend to $\mathcal{S}$-geometric theories, in which disjunctions may be indexed by objects of $\mathcal{S}$ rather than mere sets. Thus if we replace **Set** with $\mathcal{S}$ in everywhere, all of the results in Section 3 will apply. Most notably, the monoid of endomorphisms of $\ulcorner M \urcorner : \mathcal{S} \to \mathcal{S}[\mathbb{T}]$ will be an internal monoid in $\mathcal{S}$. Putting these two generalizations together raises the question of whether we can obtain a more refined factorization of any $\ulcorner M \urcorner : \mathcal{G} \to \mathcal{S}[\mathbb{T}]$ through the category of actions of an internal monoid of endomorphisms in $\mathcal{G}$ rather than the 'external' $\mathcal{S}$-monoid of endomorphisms.

5.2. Duality for clones. For single-sorted relational theories, an important extension of the Galois correspondences discussed in this article is that for *clones*; we refer to the concise introduction in [10] for this section. The idea is that rather than mere endomorphisms of a model $M$, one can consider *polymorphisms*.

For context, suppose $\mathbb{T}$ is a single-sorted relational theory and that $M$ is a model where that sort is interpreted as a set $\Omega$ (the *domain* of $M$). A **polymorphism** of $M$ is a finitary operation $f : \Omega^k \to \Omega^n$ which respects the relations of $M$, in the sense that for each relation symbol $R$ definable in $\mathbb{T}$ (of arity $m$, say), if $(x_{1,1}, \dots, x_{1,m}), \dots, (x_{k,1}, \dots, x_{k,m})$ are elements of the relation $[\![R]\!]_M$ then so are the $n$ rows of

$$(f(x_{1,1}, \dots, x_{1,m}), \dots, f(x_{k,1}, \dots, x_{k,m}))\,.$$

Polymorphisms assemble into a category $\mathcal{P}_M$ called the **clone** of $M$ whose objects are indexed by natural numbers, such that a polymorphism as above becomes a morphism $k \to n$. $\mathcal{P}_M$ has finite products, since the diagonals and projection morphisms are polymorphisms by inspection. This means that $\mathcal{P}_M$ is a **cartesian operad** or **Lawvere theory**, and that we can reconstruct polymorphisms $\Omega^k \to \Omega^n$ from polymorphisms $\Omega^k \to \Omega$.

5.2.1. Remark. As presented in [10], to formally view $\mathcal{P}$ as a clone we should ignore the nullary operations; we will not go into enough detail here for this distinction to matter.

Clearly, any endomorphism of $M$ is a polymorphism, and we have an inclusion of the monoid $N$ constructed earlier as a full subcategory of $\mathcal{P}^{\mathrm{op}}$. It should come as no surprise that a central result in the theory of clones is the extension of the Galois connection we have explored here to a connection between subclones of $\mathcal{P}$ and relational extensions of $M$. This raises a number of questions. Can the factorization (4) be extended to capture this Galois connection? Does this factorization remain expressible in terms of topos theory? If so, does a notion of topology on operads (several are possible) capture the closure and continuity conditions relevant for reconstructing the intermediate topos in the factorization? It will be exciting to explore these possibilities in future work.

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