A SYNTACTIC CHARACTERIZATION OF WEAKLY MAL'TSEV VARIETIES

In memory of Pieter Hofstra

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ABSTRACT. The notion of a weakly Mal'tsev category, as it was introduced in 2008 by the third author, is a generalization of the classical notion of a Mal'tsev category. It is well-known that a variety of universal algebras is a Mal'tsev category if and only if its theory admits a Mal'tsev term. In the main theorem of this paper, we prove a syntactic characterization of the varieties that are weakly Mal'tsev categories. We apply our result to the variety of distributive lattices which was known to be a weakly Mal'tsev category before. By a result of Z. Janelidze and the third author, a finitely complete category is weakly Mal'tsev if and only if any internal strong reflexive relation is an equivalence relation. In the last part of this paper, we give a syntactic characterization of those varieties in which any regular reflexive relation is an equivalence relation.

1. Introduction

The study of Mal'tsev categories originates with Mal'tsev's paper [19] from 1954 where he showed that, for a variety $\mathbb{V}$ of (finitary one-sorted) universal algebras, the composition of congruences on a fixed algebra is commutative if and only if the algebraic theory of $\mathbb{V}$ contains a ternary term $p(x, y, z)$ such that the two identities $p(x, x, y) = y$ and $p(x, y, y) = x$ are satisfied. Such varieties are nowadays called Mal'tsev varieties (or 2-permutable varieties) and a term $p(x, y, z)$ as described above a Mal'tsev term. Examples of Mal'tsev varieties are given by the varieties of groups, of rings, of Lie algebras and of Heyting algebras. In [6] from 1990, Carboni, Lambek and Pedicchio introduced the notion of a Mal'tsev category in the context of (Barr-)exact categories via the former condition, and developed some aspects of non-abelian homological algebra. In this setting, the commutativity of the composition of internal equivalence relations on a fixed object is equivalent to any reflexive relation being an equivalence relation or any relation being difunctional. In [7] from 1992, Carboni, Pedicchio and Pirovano defined Mal'tsev categories in the finitely complete setting via the latter two, still equivalent, conditions. In addition to the varietal examples given above, one can cite as examples of Mal'tsev

categories the category of topological groups, any abelian category and the dual of any elementary topos. We refer the reader to [4] for further details on the history of the development of Mal'tsev categories.

Mal'tsev categories turned out to be a central concept in categorical algebra, especially in the development of the notion of centrality of equivalence relations [26, 3]. Further results in Mal'tsev categories were proved in the study of central extensions [16, 10, 9] and homological lemmas such as the denormalized 3×3-lemma [2] which is, in the regular context, equivalent to the weaker Goursat property (also known as 3-permutability). More recently, some embedding theorems have been established for Mal'tsev categories [12, 13], similar to the Freyd-Mitchell embedding theorem for abelian categories.

Weakly Mal'tsev categories were introduced by the third author in [21] as a generalization of the notion of a Mal'tsev category. It was shown in [1] that a Mal'tsev category is exactly a finitely complete category $\mathfrak{C}$ such that for any pullback diagram

$$
\begin{array}{ccc}
X \times_Z Y & \xrightarrow[\ e_2\ ]{\ p_2\ } & Y \\
{\scriptstyle p_1} \big\updownarrow {\scriptstyle e_1} & \lrcorner & {\scriptstyle s} \big\updownarrow {\scriptstyle g} \\
X & \xrightarrow[\ f\ ]{\ r\ } & Z
\end{array}
\tag{1}
$$

in $\mathfrak{C}$, where f and g are two split epimorphisms with respective splittings r and s, and p_1 and p_2 are the pullback projections, the canonical pullback injections e_1 and e_2 induced by r and s, respectively, are jointly strongly epimorphic. In the finitely complete context, this is equivalent to say that e_1 and e_2 are jointly extremally epimorphic. In [21], a category $\mathfrak{C}$ is called weakly Mal'tsev if it admits pullbacks of split epimorphisms along split epimorphisms, and the pullback injections e_1 and e_2 in a diagram as (1) are jointly epimorphic. A direct consequence of this definition is that a reflexive graph

$$
C_1 \underset{c}{\overset{d}{\underset{\longleftarrow}{\rightrightarrows}}} \; e \; \longleftarrow \; C_0
$$

in a weakly Mal'tsev category admits at most one multiplicative graph structure. This means that there exists at most one "composition" map $m \colon C_2 \to C_1$, where C_2 is the pullback

$$
\begin{array}{ccc}
C_2 & \xrightarrow[\ \overline{e_2}\ ]{\ \overline{p_2}\ } & C_1 \\
{\scriptstyle \overline{p_1}} \big\updownarrow {\scriptstyle \overline{e_1}} & \lrcorner & {\scriptstyle e} \big\updownarrow {\scriptstyle c} \\
C_1 & \xrightarrow[\ d\ ]{\ e\ } & C_0
\end{array}
$$

of d along c, such that

$$
m\overline{e_1} = 1_{C_1} = m\overline{e_2}
$$

holds, where $\overline{e_1}$, $\overline{e_2}$ are the pullback injections induced by the common splitting e of d and c. Furthermore, one can show that every multiplicative graph in a weakly Mal'tsev category yields automatically an internal category, i.e., m satisfies the usual identity and associativity axioms internally. However, an internal category in a weakly Mal'tsev category can fail to yield an internal groupoid as it is the case for Mal'tsev categories [21, 25]. This means that there are internal categories

$$C_2 \xrightarrow{m} C_1 \underset{\xrightarrow{c}}{\overset{\xrightarrow{d}}{\xleftarrow{e}}} C_0$$

in certain weakly Mal'tsev categories that do not allow for an "inverse" map $i\colon C_1 \to C_1$.

A (finitary one-sorted) variety $\mathbb{V}$ of universal algebras is a Mal'tsev category if and only if its theory contains a ternary term $p(x, y, z)$ such that the equations $p(x, x, y) = y$ and $p(x, y, y) = x$ are satisfied in $\mathbb{V}$ [19, 20]. Surprisingly, no similar syntactic characterization of weakly Mal'tsev varieties was proved. The main purpose of this paper is to establish such a syntactic characterization (Theorem 3.5). In contrast to Mal'tsev varieties where we have one ternary term $p(x, y, z)$ which fulfills certain identities, we get that a variety $\mathbb{V}$ is a weakly Mal'tsev category if and only if there exist integers $k, m, N \geqslant 0$, binary terms $f_1, g_1, \ldots, f_k, g_k$, ternary terms $p_1, \ldots, p_m$, $(2(k + 2m + 1))$-ary terms $s_1, \ldots, s_N$, $(2(k + m + 2))$-ary terms $\sigma_1, \ldots, \sigma_{N+1}$ and, for all $i \in \{1, \ldots, N + 1\}$, $(k + m + 1)$-ary terms $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ that satisfy certain equations. This phenomenon of having the number of terms or their arities not being fixed in a syntactic characterization also occurs for congruence distributive [18], congruence modular [8, 11] and protomodular varieties [5].

Two (quasi-)algebraic examples of weakly Mal'tsev categories that are not Mal'tsev are given by the category of commutative monoids with cancellation [21] and the category of distributive lattices [22], see also [23] for examples of co-weakly Mal'tsev categories, i.e., categories whose dual category is weakly Mal'tsev. We will apply our main result to the variety of distributive lattices.

Let us briefly describe the strategy we used to find the syntactic characterization of weakly Mal'tsev varieties. The first step was to look for the right formulation of the property of being a weakly Mal'tsev category and apply it to the right diagram made from free algebras in the variety. In order to do so, we expressed the property, in the finitely complete and cocomplete context, as the property that for each pair of split epimorphisms f and g with common codomain and respective sections r and s, considering the pullback diagram (1) and the cokernel pair

$$X + Y \xrightarrow{[e_1, e_2]} X \times_Z Y \underset{q_2}{\overset{q_1}{\rightrightarrows}} Q$$

of the induced morphism $[e_1, e_2]$ from the coproduct $X + Y$, one has $q_1 = q_2$. This property is of the type studied in [14] (generalizing in some context the type of properties studied in [15] of which the Mal'tsev property is an example). Using the results from [14], we immediately get that, for a variety $\mathbb{V}$, it is equivalent to only consider the particular case

of the pullback

$$
\begin{array}{ccc}
P & \xrightarrow{\;\;p_2\;\;} \xleftarrow{\;\;e_2\;\;} & \mathsf{F}(x,y) \\
\big\downarrow\big\uparrow {\scriptstyle p_1}\;{\scriptstyle e_1} & \lrcorner & \big\uparrow {\scriptstyle s}\;{\scriptstyle f} \\
\mathsf{F}(x,y) & \xleftarrow{\;\;r\;\;} \xrightarrow{\;\;f\;\;} & \mathsf{F}(x),
\end{array}
$$

where f is the unique morphism from the free algebra on two generators to the free algebra on one generator such that $f(x) = x = f(y)$, r is the unique morphism such that $r(x) = x$ and s is the unique morphism such that $s(x) = y$. Furthermore, Theorem 4.1 in [14] tells us that the variety $\mathbb{V}$ is weakly Mal'tsev if and only if $q_1(y, x) = q_2(y, x)$, where (q_1, q_2) is the cokernel pair of $[e_1, e_2] \colon \mathsf{F}(x, y) + \mathsf{F}(x, y) \to P$

$$
\mathsf{F}(x,y) + \mathsf{F}(x,y) \xrightarrow{\;\;[e_1,e_2]\;\;} P \xrightarrow[q_2]{q_1} Q
$$

and where $(y, x) \in P$ is the unique element such that $p_1(y, x) = y$ and $p_2(y, x) = x$. For the sake of completeness, we reprove this result here for this specific property instead of applying the results of [14] (Lemma 3.4).

The second part of the proof consists in identifying the terms whose existence is equivalent to the equality $q_1(y, x) = q_2(y, x)$. In order to do so, we use the description of q_1 and q_2 by means of the coequalizer q of the two maps $\iota_1[e_1, e_2]$ and $\iota_2[e_1, e_2]$, where ι_1, ι_2 are the two coproduct inclusions from P to $P + P$:

$$
\begin{array}{ccc}
\mathsf{F}(x,y) + \mathsf{F}(x,y) & \xrightarrow{\;\;[e_1,e_2]\;\;} & P \\
{\scriptstyle [e_1,e_2]}\big\downarrow & & \big\downarrow{\scriptstyle \iota_2} \searrow{\scriptstyle q_2} \\
P & \xrightarrow[\iota_1]{} & P + P \searrow{\scriptstyle q} \\
& \searrow{\scriptstyle q_1} & Q
\end{array}
$$

We thus think of Q as a quotient of $P + P$ which itself can be constructed as a quotient of the free algebra $\mathsf{F}(\mathsf{U}P + \mathsf{U}P)$ on the disjoint union of two copies of the underlying set $\mathsf{U}P$ of P.

The term condition we obtain by 'brute-force description' of the equality $q_1(y, x) = q_2(y, x)$ is unfortunately very long and complex. The final step of the proof is to simplify this characterization in order to get an equivalent formulation of it which is easier to get intuition of.

The paper is structured as follows. In Section 2, we recall the necessary material for our main theorem. In particular, we recall the description of the coproduct of two

non-empty algebras A and B in a variety $\mathbb{V}$ of universal algebras (Section 2.1) and some characterizations of Mal'tsev and weakly Mal'tsev categories in the finitely complete context (Section 2.3). Section 3 proves our syntactic characterization of weakly Mal'tsev varieties (Theorem 3.5). Furthermore, in Example 3.7, we give the weakly Mal'tsev terms for the variety of distributive lattices. Finally, we show that a slight variation of Theorem 3.5 yields a syntactic characterization of the varieties in which any reflexive regular relation is an equivalence relation (Theorem 4.6).

Terminology: In this paper, all varieties of universal algebras are understood to be finitary and one-sorted.

2. Preliminaries

2.1. COPRODUCT OF TWO ALGEBRAS. As we will need it in the proof of our main result, we recall a description of the coproduct of two non-empty algebras A and B in a variety $\mathbb{V}$ of universal algebras. We denote by $\mathsf{U}\colon \mathbb{V} \to \mathsf{Set}$ the forgetful functor and $\mathsf{F}\colon \mathsf{Set} \to \mathbb{V}$ the free functor. Then the coproduct $A + B$ is given by the quotient of the free algebra $\mathsf{F}(\mathsf{U}A + \mathsf{U}B)$ on the disjoint union of the underlying sets of A and B with respect to the smallest congruence C that turns the two set-theoretic functions $i_1\colon A \to \mathsf{F}(\mathsf{U}A + \mathsf{U}B)$ and $i_2\colon B \to \mathsf{F}(\mathsf{U}A + \mathsf{U}B)$, that interpret elements of A and B as variables, into morphisms in $\mathbb{V}$. More precisely, C is the congruence generated by all pairs of the form $(i_1(\omega^A(a_1, \ldots, a_k)), \omega(i_1(a_1), \ldots, i_1(a_k)))$ or $(i_2(\omega^B(b_1, \ldots, b_k)), \omega(i_2(b_1), \ldots, i_2(b_k)))$, where $k \geqslant 0$ is an integer, $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_k \in B$ are elements, and ω is a k-ary operation of $\mathbb{V}$. Here ω^A represents the realization of ω as a function $\omega^A\colon A^k \to A$ and similarly for ω^B. In the following, we will omit the functions i_1 and i_2. Given integers $k, \ell \geqslant 0$, elements $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_\ell \in B$, and a $(k + \ell)$-ary term s, we have that

$$s(\iota_1(a_1), \ldots, \iota_1(a_k), \iota_2(b_1), \ldots, \iota_2(b_\ell)) = [s(a_1, \ldots, a_k, b_1, \ldots, b_\ell)], \tag{2}$$

where $\iota_1\colon A \to \mathsf{F}(\mathsf{U}A + \mathsf{U}B)/C$ and $\iota_2\colon B \to \mathsf{F}(\mathsf{U}A + \mathsf{U}B)/C$ are the coproduct inclusions induced by i_1 and i_2 respectively, and $[\cdot]$ denotes the equivalence class of an element in $\mathsf{F}(\mathsf{U}A + \mathsf{U}B)$ with respect to C. Given another algebra D, and morphisms $f\colon A \to D$ and

$g\colon B \to D$, the unique morphism $\varphi\colon \mathsf{F}(\mathsf{U}A + \mathsf{U}B)/C \to D$ such that $\varphi\iota_1 = f$ and $\varphi\iota_2 = g$ is given by

$$\varphi([s(a_1, \ldots, a_k, b_1, \ldots, b_\ell)]) := s^D(f(a_1), \ldots, f(a_k), g(b_1), \ldots, g(b_\ell)).$$

Alternatively, we consider the congruence C' on $\mathsf{F}(\mathsf{U}A + \mathsf{U}B)$ generated by the relation $\sim$ given by all pairs of the form

$$(\tau(a_1, \ldots, a_m, b_1, \ldots, b_n, \mu_1(a_1, \ldots, a_m), \mu_2(b_1, \ldots, b_n)),$$
$$\tau(a_1, \ldots, a_m, b_1, \ldots, b_n, \lambda_1(a_1, \ldots, a_m), \lambda_2(b_1, \ldots, b_n))),$$

where $m, n \geqslant 0$ are integers, $a_1, \ldots, a_m \in A$ and $b_1 \ldots, b_n \in B$ are elements, μ_1, λ_1 are m-ary terms, μ_2, λ_2 are n-ary terms and τ is an $(m + n + 2)$-ary term with

$$\mu_1^A(a_1, \ldots, a_m) = \lambda_1^A(a_1, \ldots, a_m),$$
$$\mu_2^B(b_1, \ldots, b_n) = \lambda_2^B(b_1, \ldots, b_n).$$

Since A and B are supposed to be non-empty, the relation $\sim$ is reflexive. Furthermore, it is symmetric. We state the following classical results.

2.2. PROPOSITION. *Let A and B be two non-empty algebras in $\mathbb{V}$.*

1. *The two congruences C and C' on $\mathsf{F}(\mathsf{U}A + \mathsf{U}B)$ are equal.*

2. *The congruence C' is given by the transitive closure of $\sim$.*

2.3. (WEAKLY) MAL'TSEV CATEGORIES. Let us recall some characterizations of Mal'tsev categories in the finitely complete context.

2.4. DEFINITION. [Mal'tsev category [7]] *Let $\mathfrak{C}$ be a finitely complete category. We call $\mathfrak{C}$ a Mal'tsev category if any reflexive relation $r\colon R \rightarrowtail X \times X$ in $\mathfrak{C}$ is an equivalence relation.*

We recall that a relation $R \subseteq X \times Y$ between two sets X and Y is called *difunctional* [27] if, for given elements $x, x' \in X$ and $y, y' \in Y$, the conditions xRy, $x'Ry$, $x'Ry'$ imply that xRy'. This can be immediately generalized to relations in an arbitrary category using generalized elements.

2.5. PROPOSITION. *[7] Let $\mathfrak{C}$ be a finitely complete category. Then the following conditions are equivalent:*

1. *The category $\mathfrak{C}$ is a Mal'tsev category.*

2. *Any reflexive relation $r\colon R \rightarrowtail X \times X$ in $\mathfrak{C}$ is symmetric.*

3. *Any reflexive relation $r\colon R \rightarrowtail X \times X$ in $\mathfrak{C}$ is transitive.*

4. *Any relation $r\colon R \rightarrowtail X \times X$ in $\mathfrak{C}$ is difunctional.*

In the following, we will also use another characterization of those finitely complete categories which are Mal'tsev categories. Let $f\colon X \to Z$ and $g\colon Y \to Z$ be two split epimorphisms in a finitely complete category $\mathfrak{C}$ with splittings $r\colon Z \to X$ and $s\colon Z \to Y$ respectively, i.e., $fr = 1_Z = gs$. We consider the pullback

$$
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{\ \ p_2\ \ } & Y \\[2pt]
\Big\uparrow{\scriptstyle p_1}\ {\scriptstyle e_1} & \lrcorner & \Big\uparrow{\scriptstyle s}\ {\scriptstyle g} \\[2pt]
X & \xrightarrow[\ \ f\ \ ]{r} & Z
\end{array}
\tag{3}
$$

of f along g. Let $e_1\colon X \to X \times_Z Y$ and $e_2\colon Y \to X \times_Z Y$ be the unique morphisms such that

$$
\begin{aligned}
p_1 e_1 &= 1_X, & p_2 e_1 &= sf, \\
p_1 e_2 &= rg, & p_2 e_2 &= 1_Y.
\end{aligned}
$$

2.6. PROPOSITION. *[1] Let $\mathfrak{C}$ be a finitely complete category. Then the following conditions are equivalent:*

1. *The category $\mathfrak{C}$ is a Mal'tsev category.*

2. *For any Diagram (3) in $\mathfrak{C}$, the morphisms $e_1\colon X \to X \times_Z Y$ and $e_2\colon Y \to X \times_Z Y$ are jointly extremally epimorphic, i.e., if there exist an object M and morphisms $f_1\colon X \to M$, $f_2\colon Y \to M$ and a monomorphism $m\colon M \rightarrowtail X \times_Z Y$ such that $mf_i = e_i$ for each $i \in \{1,2\}$, then m is an isomorphism.*

A direct generalization of this characterization of Mal'tsev categories gives rise to the notion of a weakly Mal'tsev category.

2.7. DEFINITION. [Weakly Mal'tsev category [21]] *Let $\mathfrak{C}$ be a category. We call $\mathfrak{C}$ a weakly Mal'tsev category if the following conditions hold:*

1. *The category $\mathfrak{C}$ possesses all pullbacks of split epimorphisms along split epimorphisms.*

2. *For any Diagram (3) in $\mathfrak{C}$, the morphisms $e_1\colon X \to X \times_Z Y$ and $e_2\colon Y \to X \times_Z Y$ are jointly epimorphic, i.e, given an object A and morphisms $u, v\colon X \times_Z Y \to A$ such that $ue_1 = ve_1$ and $ue_2 = ve_2$, then $u = v$.*

We call a relation $r\colon R \rightarrowtail X \times Y$ *strong* if it is a strong monomorphism, i.e., for any epimorphism $e\colon A \twoheadrightarrow B$ and morphisms $a\colon A \to R$ and $b\colon B \to X \times Y$ such that the outer part of the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ e\ \ } & B \\[2pt]
{\scriptstyle a}\Big\downarrow & \ \ {}^{\displaystyle d}\nearrow & \Big\downarrow{\scriptstyle b} \\[2pt]
R & \xrightarrow[\ \ r\ \ ]{} & X \times Y
\end{array}
$$

commutes, there exists a unique morphism $d\colon B \to R$ such that $de = a$ and $rd = b$.

2.8. PROPOSITION. *[17] Let $\mathfrak{C}$ be a finitely complete category. Then the following conditions are equivalent:*

1. *The category $\mathfrak{C}$ is a weakly Mal'tsev category.*

2. *Any reflexive strong relation $r\colon R \rightarrowtail X \times X$ in $\mathfrak{C}$ is an equivalence relation.*

3. *Any reflexive strong relation $r\colon R \rightarrowtail X \times X$ in $\mathfrak{C}$ is symmetric.*

4. *Any reflexive strong relation $r\colon R \rightarrowtail X \times X$ in $\mathfrak{C}$ is transitive.*

5. *Any strong relation $r\colon R \rightarrowtail X \times X$ in $\mathfrak{C}$ is difunctional.*

3. Weakly Mal'tsev varieties

Our aim is to find a syntactic characterization of the varieties which are weakly Mal'tsev categories. Let us recall the classical theorem of Mal'tsev which identifies all the varieties that are Mal'tsev categories.

3.1. THEOREM. [Mal'tsev's theorem [19, 20]] *Let $\mathbb{V}$ be a variety. The category $\mathbb{V}$ is Mal'tsev if and only if there exists a ternary term $p \in \mathsf{F}(x,y,z)$ such that $p(x,x,y) = y$ and $p(x,y,y) = x$.*

We recall a proof of Mal'tsev's theorem which relies on the characterization of Mal'tsev categories as those finitely complete categories in which the canonical inclusions into a pullback of split epimorphisms are jointly extremally epimorphic that we recalled in Proposition 2.6. We will then adapt it for the case of weakly Mal'tsev categories. Let us consider the pullback

$$
\begin{array}{ccc}
P & \xrightarrow{\ \ p_2\ \ } & \mathsf{F}(x,y) \\[-2pt]
& \xleftarrow{e_2} & \\
{\scriptstyle p_1}\Big\updownarrow{\scriptstyle e_1} \quad \lrcorner & & {\scriptstyle s}\Big\updownarrow{\scriptstyle f} \\
\mathsf{F}(x,y) & \xrightarrow[f]{\ \ r\ \ } & \mathsf{F}(x),
\end{array}
$$

where $\mathsf{F}(x)$ and $\mathsf{F}(x,y)$ are the free algebras on the variables x and x,y respectively, f is the unique morphism such that $f(x) = x = f(y)$, r is the unique morphism such that $r(x) = x$ and s is the unique morphism such that $s(x) = y$. It is clear that both r and s yield splittings of f. The arrows p_1, p_2 are the pullback projections and e_1, e_2 are the canonical pullback injections. The algebra P is given by all the pairs $(t_1, t_2) \in \mathsf{F}(x,y) \times \mathsf{F}(x,y)$ of binary terms such that the equation $t_1(x,x) = t_2(x,x)$ holds in $\mathbb{V}$. We note that, since varieties are cocomplete regular categories, the condition that e_1, e_2 are jointly extremally epimorphic is equivalent to the condition that their induced morphism $[e_1, e_2]\colon \mathsf{F}(x,y) + \mathsf{F}(x,y) \to P$ is a regular epimorphism, i.e., a surjective homomorphism. We split the proof of Mal'tsev's theorem into two parts (Lemmas 3.2 and 3.3).

3.2. LEMMA. *The variety $\mathbb{V}$ is a Mal'tsev category if and only if $(y, x) \in \mathrm{Im}([e_1, e_2])$.*

PROOF. "$\Rightarrow$": If $\mathbb{V}$ is a Mal'tsev category, the map $[e_1, e_2]$ is a surjection. The claim follows since clearly $(y, x) \in P$.

"$\Leftarrow$": Let

$$
\begin{array}{ccc}
P' & \xrightarrow{p'_2} & B \\
{\scriptstyle p'_1} \Big\uparrow {\scriptstyle e'_1} & \lrcorner \quad {\scriptstyle e'_2} & {\scriptstyle s'} \Big\uparrow {\scriptstyle g'} \\
A & \xleftarrow[f']{r'} & C
\end{array}
\tag{4}
$$

be a pullback of split epimorphisms f', g' in $\mathbb{V}$ with r', s' the respective splittings and e'_1, e'_2 the canonical inclusions. We show that $[e'_1, e'_2] \colon A + B \to P'$ is surjective. To this end, let $(a, b) \in P' = \{(a', b') \in A \times B \mid f'(a') = g'(b')\}$ be arbitrary. It is easy to see that the unique morphism $\alpha \colon \mathsf{F}(x, y) \to A$ such that $\alpha(x) = r'f'(a)$ and $\alpha(y) = a$, and the unique morphism $\beta \colon \mathsf{F}(x, y) \to B$ such that $\beta(x) = b$ and $\beta(y) = s'g'(b)$ induce a unique morphism $\delta \colon P \to P'$ such that the cube

'reasonably' commutes, where $\gamma \colon \mathsf{F}(x) \to C$ is the unique morphism such that $\gamma(x) = f'(a) = g'(b)$. Hence we get the commutative diagram

$$
\begin{array}{ccc}
\mathsf{F}(x, y) + \mathsf{F}(x, y) & \xrightarrow{[e_1, e_2]} & P \\
{\scriptstyle \alpha + \beta} \Big\downarrow & & \Big\downarrow {\scriptstyle \delta} \\
A + B & \xrightarrow[{[e'_1, e'_2]}]{} & P'.
\end{array}
\tag{5}
$$

By assumption, there exists $X \in \mathsf{F}(x, y) + \mathsf{F}(x, y)$ with $[e_1, e_2](X) = (y, x)$. Thus,

$$
(a, b) = \delta(y, x) = \delta[e_1, e_2](X) = [e'_1, e'_2](\alpha + \beta)(X) \in \mathrm{Im}([e'_1, e'_2]),
$$

which proves that $[e'_1, e'_2]$ is surjective since (a, b) was arbitrary in P'. ∎

3.3. LEMMA. *The image of* $[e_1, e_2] \colon \mathsf{F}(x, y) + \mathsf{F}(x, y) \to P$ *contains* (y, x) *if and only if there exists a ternary term* $p \in \mathsf{F}(x, y, z)$ *such that* $p(x, x, y) = y$ *and* $p(x, y, y) = x$.

PROOF. We show that

$$\mathrm{Im}([e_1, e_2]) = \{(p(x, x, y), p(x, y, y)) \mid p \in \mathsf{F}(x, y, z)\}, \tag{6}$$

which implies the claim.

"$\supseteq$": We know that $(x, x) = e_2(x)$, $(x, y) = e_1(x) = e_2(y)$ and $(y, y) = e_1(y)$ are in $\mathrm{Im}([e_1, e_2])$. Hence, given a ternary term $p \in \mathsf{F}(x, y, z)$,

$$(p(x, x, y), p(x, y, y)) = p((x, x), (x, y), (y, y))$$

lies also in the subalgebra $\mathrm{Im}([e_1, e_2]) \subseteq P$.

"$\subseteq$": By Equation (2), we know that we can write an arbitrary element of $\mathsf{F}(x, y) + \mathsf{F}(x, y)$ as

$$\tau(\iota_1(s_1), \ldots, \iota_1(s_k), \iota_2(t_1), \ldots, \iota_2(t_\ell))$$

where $k, \ell \geqslant 0$ are integers, $s_1, \ldots, s_k, t_1, \ldots, t_\ell \in \mathsf{F}(x, y)$, $\iota_1, \iota_2 \colon \mathsf{F}(x, y) \to \mathsf{F}(x, y) + \mathsf{F}(x, y)$ are the coproduct inclusions of $\mathsf{F}(x, y) + \mathsf{F}(x, y)$ and τ is a $(k + \ell)$-ary term. We compute

$$[e_1, e_2]\Big(\tau\big(\iota_1(s_1), \ldots, \iota_1(s_k), \iota_2(t_1), \ldots, \iota_2(t_\ell)\big)\Big)$$

$$= \tau\Big([e_1, e_2]\big(\iota_1(s_1)\big), \ldots, [e_1, e_2]\big(\iota_1(s_k)\big), [e_1, e_2]\big(\iota_2(t_1)\big), \ldots, [e_1, e_2]\big(\iota_2(t_\ell)\big)\Big)$$

$$= \tau\big(e_1(s_1), \ldots, e_1(s_k), e_2(t_1), \ldots, e_2(t_\ell)\big)$$

$$= \tau\Big(s_1\big(e_1(x), e_1(y)\big), \ldots, s_k\big(e_1(x), e_1(y)\big), t_1(e_2(x), e_2(y)), \ldots, t_\ell(e_2(x), e_2(y))\Big)$$

$$= \tau\Big(s_1\big((x, y), (y, y)\big), \ldots, s_k\big((x, y), (y, y)\big), t_1\big((x, x), (x, y)\big), \ldots, t_\ell\big((x, x), (x, y)\big)\Big)$$

$$= \tau\Big(\big(s_1(x, y), s_1(y, y)\big), \ldots, \big(s_k(x, y), s_k(y, y)\big), \big(t_1(x, x), t_1(x, y)\big), \ldots, \big(t_\ell(x, x), t_\ell(x, y)\big)\Big)$$

$$= \Big(\tau\big(s_1(x, y), \ldots, s_k(x, y), t_1(x, x), \ldots, t_\ell(x, x)\big),$$

$$\tau\big(s_1(y, y), \ldots, s_k(y, y), t_1(x, y), \ldots, t_\ell(x, y)\big)\Big)$$

$$= (p(x, x, y), p(x, y, y)),$$

where we set $p(x, y, z) := \tau(s_1(y, z), \ldots, s_k(y, z), t_1(x, y), \ldots, t_\ell(x, y))$. Hence the claim is proven. ∎

If the variety $\mathbb{V}$ is a weakly Mal'tsev category, the morphism $[e_1, e_2] \colon \mathsf{F}(x, y) + \mathsf{F}(x, y) \to P$ is only an epimorphism. Let us consider its cokernel pair

$$\mathsf{F}(x, y) + \mathsf{F}(x, y) \xrightarrow{\ [e_1, e_2]\ } P \xrightarrow[\ q_2\ ]{\ q_1\ } Q.$$

3.4. LEMMA. *The variety* $\mathbb{V}$ *is a weakly Mal'tsev category if and only if* $q_1(y, x) = q_2(y, x)$.

PROOF. "$\Rightarrow$": If $\mathbb{V}$ is a weakly Mal'tsev category, the arrow $[e_1, e_2]$ is an epimorphism. Hence $q_1 = q_2$.

"$\Leftarrow$": As in the proof of Lemma 3.2, we consider a general pullback of split epimorphisms in $\mathbb{V}$ as in Diagram (4). We show that $[e'_1, e'_2]$ is an epimorphism. Let $(a, b) \in P'$ be arbitrary. We consider the extended version

$$
\begin{array}{ccccc}
\mathsf{F}(x,y) + \mathsf{F}(x,y) & \xrightarrow{\;[e_1,e_2]\;} & P & \overset{q_1}{\underset{q_2}{\rightrightarrows}} & Q \\
{\scriptstyle \alpha+\beta} \downarrow & & {\scriptstyle \delta}\downarrow & & \vdots\,{\scriptstyle \rho} \\
A + B & \xrightarrow[\;[e'_1,e'_2]\;]{} & P' & \overset{q'_1}{\underset{q'_2}{\rightrightarrows}} & Q'
\end{array}
$$

of Diagram (5), where (Q', q'_1, q'_2) is the cokernel pair of $[e'_1, e'_2]$ and $\rho \colon Q \to Q'$ is the unique morphism such that $\rho q_i = q'_i \delta$ for each $i \in \{1, 2\}$. We have that

$$q'_1(a, b) = q'_1 \delta(y, x) = \rho q_1(y, x) = \rho q_2(y, x) = q'_2 \delta(y, x) = q'_2(a, b).$$

Hence $q'_1 = q'_2$ and the claim is proven. ∎

We are now ready to state and prove our main theorem.

3.5. THEOREM. *A finitary one-sorted variety $\mathbb{V}$ of universal algebras is a weakly Mal'tsev category if and only if there exist integers $k, m, N \geqslant 0$, binary terms $f_1, g_1, \ldots, f_k, g_k \in \mathsf{F}(x, y)$, ternary terms $p_1, \ldots, p_m \in \mathsf{F}(x, y, z)$, $(2(k + 2m + 1))$-ary terms $s_1, \ldots, s_N$, $(2(k + m + 2))$-ary terms $\sigma_1, \ldots, \sigma_{N+1}$ and, for all $i \in \{1, \ldots, N + 1\}$, $(k + m + 1)$-ary terms $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ such that the following identities (on variables $x, y, u, u', v_1, \ldots, v_k$, $v'_1, \ldots, v'_k$, $w_1, \ldots, w_m$ and $w'_1, \ldots, w'_m$) hold, where we write $\vec{v}$ for $v_1, \ldots, v_k$ and $\vec{w}$ for $w_1, \ldots, w_m$, and analogously for $\vec{v}'$ and $\vec{w}'$:*

$$f_i(x, x) = g_i(x, x) \tag{7}$$

for all $i \in \{1, \ldots, k\}$;

$$
\begin{aligned}
&\eta_\alpha^{(i)}(y, f_1(x, y), \ldots, f_k(x, y), p_1(x, x, y), \ldots, p_m(x, x, y)) \\
=&\epsilon_\alpha^{(i)}(y, f_1(x, y), \ldots, f_k(x, y), p_1(x, x, y), \ldots, p_m(x, x, y)),
\end{aligned} \tag{8a}
$$

$$
\begin{aligned}
&\eta_\alpha^{(i)}(x, g_1(x, y), \ldots, g_k(x, y), p_1(x, y, y), \ldots, p_m(x, y, y)) \\
=&\epsilon_\alpha^{(i)}(x, g_1(x, y), \ldots, g_k(x, y), p_1(x, y, y), \ldots, p_m(x, y, y)),
\end{aligned} \tag{8b}
$$

for all $i \in \{1, \ldots, N + 1\}$ and $\alpha \in \{1, 2\}$;

$$
\begin{aligned}
&\sigma_i(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(i)}(u, \vec{v}, \vec{w}), \epsilon_2^{(i)}(u', \vec{v}', \vec{w}')) \\
=&s_i(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}'),
\end{aligned} \tag{9a}
$$

$$s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w})$$
$$= \sigma_{i+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(i+1)}(u, \vec{v}, \vec{w}), \eta_2^{(i+1)}(u', \vec{v}', \vec{w}')), \tag{9b}$$

for all $i \in \{1, \ldots, N\}$ and

$$u = \sigma_1(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(1)}(u, \vec{v}, \vec{w}), \eta_2^{(1)}(u', \vec{v}', \vec{w}')), \tag{10a}$$
$$u' = \sigma_{N+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(N+1)}(u, \vec{v}, \vec{w}), \epsilon_2^{(N+1)}(u', \vec{v}', \vec{w}')). \tag{10b}$$

Note that, in comparison to Equation (9a), the second occurrences of $\vec{w}$ and $\vec{w}'$ in the term s_i are swapped in Equation (9b).

PROOF. We start by showing that if $\mathbb{V}$ is a weakly Mal'tsev category, then the terms mentioned in the statement exist. We first treat the case where $\mathbb{V}$ is a Mal'tsev category separately. If a Mal'tsev term $p \in \mathsf{F}(x, y, z)$, which fulfills $p(x, x, y) = y$ and $p(x, y, y) = x$, exists, we can easily construct terms that satisfy the requirements of the theorem. We can choose for instance k to be 0 and m, N to be 1, and we can set

$$p_1(x, y, z) := p(x, y, z),$$
$$s_1(u, w, \tilde{w}, u', w', \tilde{w}') := \tilde{w},$$
$$\sigma_1(u, w, u', w', a, b) := a,$$
$$\sigma_2(u, w, u', w', a, b) := b,$$
$$\eta_1^{(1)}(u, w) := \epsilon_2^{(2)}(u, w) := u,$$
$$\epsilon_1^{(1)}(u, w) := \eta_2^{(2)}(u, w) := w.$$

Then the binary terms $\eta_2^{(1)}, \epsilon_2^{(1)}, \eta_1^{(2)}, \epsilon_1^{(2)}$ have only to satisfy the Equations (8), which can easily be attained.

Let us now suppose that $\mathbb{V}$ is a weakly Mal'tsev variety but not a Mal'tsev variety. By Lemma 3.4, we know that $\mathbb{V}$ is weakly Mal'tsev if and only if $q_1(y, x) = q_2(y, x)$. We construct the cokernel pair $(q_1, q_2 \colon P \to Q)$ of the map $[e_1, e_2] \colon \mathsf{F}(x, y) + \mathsf{F}(x, y) \to P$ by means of the coequalizer q of the maps $\iota_1[e_1, e_2], \iota_2[e_1, e_2]$, where ι_1, ι_2 are the coproduct inclusions into $P + P$, as in the following diagram:
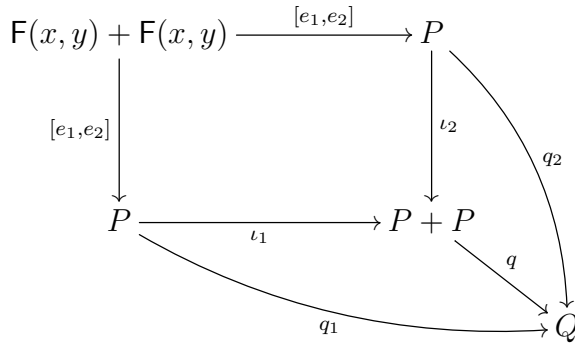
The algebra Q is given by the quotient of $P + P$ with respect to, due to Equation (6), the smallest congruence $C \subseteq (P + P)^2$ containing the subset

$$S := \left\{ \Big(\iota_1\big(p(x, x, y), p(x, y, y)\big), \iota_2\big(p(x, x, y), p(x, y, y)\big) \Big) \in (P + P)^2 \mid p \in \mathsf{F}(x, y, z) \right\}.$$

More precisely, we first define the subset

$$
\begin{aligned}
S_{\mathrm{S}} := & \left\{ \Big(\iota_1\big(p(x, x, y), p(x, y, y)\big), \iota_2\big(p(x, x, y), p(x, y, y)\big) \Big) \in (P + P)^2 \mid p \in \mathsf{F}(x, y, z) \right\} \\
& \cup \left\{ \Big(\iota_2\big(p(x, x, y), p(x, y, y)\big), \iota_1\big(p(x, x, y), p(x, y, y)\big) \Big) \in (P + P)^2 \mid p \in \mathsf{F}(x, y, z) \right\},
\end{aligned}
$$

which yields the smallest symmetric relation on $P + P$ containing S. Secondly, we define

$$S_{\mathrm{SR}} := S_{\mathrm{S}} \cup \{ (X, X) \in (P + P)^2 \mid X \in P + P \},$$

which yields the smallest reflexive symmetric relation on $P + P$ containing S. Thirdly, we define

$$
\begin{aligned}
S_{\mathrm{SRO}} := \{ t((X_1, Y_1), \ldots, (X_n, Y_n)) \in (P + P)^2 \mid & \, n \geqslant 0 \text{ is an integer,} \\
& (X_1, Y_1), \ldots, (X_n, Y_n) \in S_{\mathrm{SR}} \\
& \text{and } t \text{ is an } n\text{-ary term} \},
\end{aligned}
$$

yielding the smallest symmetric reflexive relation on $P + P$ containing S and being closed under operations. Fourthly, we define

$$
\begin{aligned}
S_{\mathrm{SROT}} := \{ (X, Y) \in (P + P)^2 \mid & \text{ there exists an integer } n \geqslant 1 \text{ and } X_1, \ldots, X_{n+1} \in P + P \\
& \text{such that } (X_1, X_2), (X_2, X_3), \ldots, (X_n, X_{n+1}) \in S_{\mathrm{SRO}} \\
& \text{and } X = X_1 \text{ and } X_{n+1} = Y \}
\end{aligned}
$$

to be the transitive closure of S_{SRO}. It is well-known and easy to prove that C coincides with S_{SROT}. Thus, the condition $q_1(y, x) = q_2(y, x)$ is equivalent to $\big(\iota_1(y, x), \iota_2(y, x) \big) \in S_{\mathrm{SROT}}$. This means that there exist an integer $n \geqslant 1$ and elements $X_1, \ldots, X_{n+1} \in P + P$ such that

$$(X_1, X_2), (X_2, X_3), \ldots, (X_{n-1}, X_n), (X_n, X_{n+1}) \in S_{\mathrm{SRO}},$$

$\iota_1(y, x) = X_1$ and $X_{n+1} = \iota_2(y, x)$. For $i \in \{1, \ldots, n\}$, the condition $(X_i, X_{i+1}) \in S_{\mathrm{SRO}}$ means that there exist an integer $a_i \geqslant 0$, elements $(X_1^{(i)}, Y_1^{(i)}), \ldots, (X_{a_i}^{(i)}, Y_{a_i}^{(i)}) \in S_{\mathrm{SR}}$ and an a_i-ary term t_i such that

$$(X_i, X_{i+1}) = (t_i(X_1^{(i)}, \ldots, X_{a_i}^{(i)}), t_i(Y_1^{(i)}, \ldots, Y_{a_i}^{(i)})).$$

By definition of S_{SR}, each pair $(X_j^{(i)}, Y_j^{(i)})$ is either of the form

$$\Big(\iota_1\big(p_j^{(i)}(x, x, y), p_j^{(i)}(x, y, y)\big), \iota_2\big(p_j^{(i)}(x, x, y), p_j^{(i)}(x, y, y)\big) \Big),$$

where $p_j^{(i)} \in \mathsf{F}(x, y, z)$, of the form

$$\left(\iota_2\big(q_j^{(i)}(x, x, y), q_j^{(i)}(x, y, y)\big), \iota_1\big(q_j^{(i)}(x, x, y), q_j^{(i)}(x, y, y)\big) \right),$$

where $q_j^{(i)} \in \mathsf{F}(x, y, z)$, or of the form

$$(Z_j^{(i)}, Z_j^{(i)}),$$

where $Z_j^{(i)} = X_j^{(i)} = Y_j^{(i)} \in P + P$. Hence we see that, by possibly permuting the variables in the term t_i, the condition $(X_i, X_{i+1}) \in S_{\mathrm{SRO}}$ is equivalent to the existence of integers $m_i, m_i', m_i'' \geqslant 0$, ternary terms $p_1^{(i)}, \ldots, p_{m_i}^{(i)}$ and $q_1^{(i)}, \ldots, q_{m_i'}^{(i)}$, elements $Z_1^{(i)}, \ldots, Z_{m_i''}^{(i)} \in P + P$ and an $(m_i + m_i' + m_i'')$-ary term t_i such that the identities

$$\begin{aligned}
X_i = t_i\Big(&Z_1^{(i)}, \ldots, Z_{m_i''}^{(i)}, \\
&\iota_1\big(p_1^{(i)}(x, x, y), p_1^{(i)}(x, y, y)\big), \ldots, \iota_1\big(p_{m_i}^{(i)}(x, x, y), p_{m_i}^{(i)}(x, y, y)\big), \\
&\iota_2\big(q_1^{(i)}(x, x, y), q_1^{(i)}(x, y, y)\big), \ldots, \iota_2\big(q_{m_i'}^{(i)}(x, x, y), q_{m_i'}^{(i)}(x, y, y)\big) \Big)
\end{aligned}$$

and

$$\begin{aligned}
X_{i+1} = t_i\Big(&Z_1^{(i)}, \ldots, Z_{m_i''}^{(i)}, \\
&\iota_2\big(p_1^{(i)}(x, x, y), p_1^{(i)}(x, y, y)\big), \ldots, \iota_2\big(p_{m_i}^{(i)}(x, x, y), p_{m_i}^{(i)}(x, y, y)\big), \\
&\iota_1\big(q_1^{(i)}(x, x, y), q_1^{(i)}(x, y, y)\big), \ldots, \iota_1\big(q_{m_i'}^{(i)}(x, x, y), q_{m_i'}^{(i)}(x, y, y)\big) \Big)
\end{aligned}$$

hold in $P + P$. Given a term $t \in \mathsf{F}(x_1, \ldots, x_v)$, where $v \geqslant 0$ is an integer, we can add a variable x_{v+1} to t in the sense that we define a term $t' \in \mathsf{F}(x_1, \ldots, x_{v+1})$ by setting $t'(x_1, \ldots, x_{v+1}) := t(x_1, \ldots, x_v)$. In this spirit, we can consider the unions

$$\bigcup_{i \in \{1, \ldots, n\}} \{p_j^{(i)} \mid j \in \{1, \ldots, m_i\}\},$$

$$\bigcup_{i \in \{1, \ldots, n\}} \{q_j^{(i)} \mid j \in \{1, \ldots, m_i'\}\}$$

and

$$\bigcup_{i \in \{1, \ldots, n\}} \{Z_j^{(i)} \mid j \in \{1, \ldots, m_i''\}\},$$

and assume, without loss of generality, that $m_1 = \cdots = m_n =: m$, $m_1' = \cdots = m_n' =: m'$ and $m_1'' = \cdots = m_n'' =: m''$. Hence we can omit the upper index from the elements $p_j^{(i)}, q_j^{(i)}, Z_j^{(i)}$. Furthermore, by looking at the union

$$\{p_j \mid j \in \{1, \ldots, m\}\} \cup \{q_j \mid j \in \{1, \ldots, m'\}\},$$

we can assume, without loss of generality, that $m = m'$ and $p_j = q_j$ for all $j \in \{1, \ldots, m\}$. Thus, the condition that $(X_i, X_{i+1}) \in S_{\mathrm{SRO}}$ for all $i \in \{1, \ldots, n\}$ is equivalent to the existence of integers $m, m'' \geqslant 0$, terms $p_1, \ldots, p_m \in \mathsf{F}(x, y, z)$, elements $Z_1, \ldots, Z_{m''} \in P + P$ and $(2m + m'')$-ary terms $t_1, \ldots, t_n$ such that the identities

$$X_i = t_i \Big(Z_1, \ldots, Z_{m''}, \tag{11}$$
$$\iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big)$$
$$\iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big) \Big)$$

and

$$X_{i+1} = t_i \Big(Z_1, \ldots, Z_{m''}, \tag{12}$$
$$\iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big)$$
$$\iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big) \Big)$$

hold in $P + P$ for all $i \in \{1, \ldots, n\}$. By Equation (2), there exist, for all $i \in \{1, \ldots, m''\}$, integers $\hat{k}_i, \tilde{k}_i \geqslant 0$, elements $(\hat{f}_1^{(i)}, \hat{g}_1^{(i)}), \ldots, (\hat{f}_{\hat{k}_i}^{(i)}, \hat{g}_{\hat{k}_i}^{(i)})$ and $(\tilde{f}_1^{(i)}, \tilde{g}_1^{(i)}), \ldots, (\tilde{f}_{\tilde{k}_i}^{(i)}, \tilde{g}_{\tilde{k}_i}^{(i)})$ in P, and a $(\hat{k}_i + \tilde{k}_i)$-ary term s_i such that

$$Z_i = s_i\big(\iota_1(\hat{f}_1^{(i)}, \hat{g}_1^{(i)}), \ldots, \iota_1(\hat{f}_{\hat{k}_i}^{(i)}, \hat{g}_{\hat{k}_i}^{(i)}), \iota_2(\tilde{f}_1^{(i)}, \tilde{g}_1^{(i)}), \ldots, \iota_2(\tilde{f}_{\tilde{k}_i}^{(i)}, \tilde{g}_{\tilde{k}_i}^{(i)})\big).$$

Proceeding analogously as we did for the $p_1^{(i)}, \ldots, p_{m_i}^{(i)}$ and $q_1^{(i)}, \ldots, q_{m_i'}^{(i)}$ from above, we can first assume, without loss of generality, that $\hat{k}_1 = \cdots = \hat{k}_{m''} =: \hat{k}$ and $\tilde{k}_1 = \cdots = \tilde{k}_{m''} =: \tilde{k}$ and omit the upper index from the elements $\hat{f}_j^{(i)}, \hat{g}_j^{(i)}, \tilde{f}_j^{(i)}, \tilde{g}_j^{(i)}$, and then that $\hat{k} = \tilde{k} =: k$, and $\hat{f}_j = \tilde{f}_j$ and $\hat{g}_j = \tilde{g}_j$ for all $j \in \{1, \ldots, k\}$. In a next step, we see that we can assume without loss of generality that we have an integer $k \geqslant 0$, binary terms $f_1, g_1, \ldots, f_k, g_k$ such that $(f_j, g_j) \in P$, i.e., $f_j(x, x) = g_j(x, x)$, for all $j \in \{1, \ldots, k\}$, and $(2(k+m+1))$-ary terms $s_1, \ldots, s_{m''}$ such that the $k + m + 1$ elements

$$(y, x), (f_1, g_1) \ldots, (f_k, g_k), (p_1(x, x, y), p_1(x, y, y)) \ldots, (p_m(x, x, y), p_m(x, y, y))$$

of P are pairwise distinct and

$$Z_i = s_i(\iota_1(y, x), \iota_1(f_1, g_1), \ldots, \iota_1(f_k, g_k),$$
$$\iota_1(p_1(x, x, y), p_1(x, y, y)), \ldots, \iota_1(p_m(x, x, y), p_m(x, y, y)),$$
$$\iota_2(y, x), \iota_2(f_1, g_1), \ldots, \iota_2(f_k, g_k),$$
$$\iota_2(p_1(x, x, y), p_1(x, y, y)), \ldots, \iota_2(p_m(x, x, y), p_m(x, y, y)))$$

for all $i \in \{1, \ldots, m''\}$. Notice that (y, x) cannot be of the form $(p_j(x, x, y), p_j(x, y, y))$ since we supposed that $\mathbb{V}$ is not a Mal'tsev variety. By setting, for all $i \in \{1, \ldots, n\}$, the

$(2(k + 2m + 1))$-ary term

$$t_i'(u, v_1, \ldots, v_k, w_1, \ldots, w_m, \widetilde{w}_1, \ldots, \widetilde{w}_m, u', v_1', \ldots, v_k', w_1', \ldots, w_m', \widetilde{w}_1', \ldots, \widetilde{w}_m')$$
$$:= t_i\big(s_1(u, v_1, \ldots, v_k, w_1, \ldots, w_m, u', v_1', \ldots, v_k', w_1', \ldots, w_m') \ldots,$$
$$s_{m''}(u, v_1, \ldots, v_k, w_1, \ldots, w_m, u', v_1', \ldots, v_k', w_1', \ldots, w_m'), \widetilde{w}_1, \ldots, \widetilde{w}_m, \widetilde{w}_1', \ldots, \widetilde{w}_m'\big),$$

Equations (11) and (12) translate into

$$X_i = t_i'\Big(\iota_1(y, x), \iota_1(f_1, g_1), \ldots, \iota_1(f_k, g_k),$$
$$\iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big),$$
$$\iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big),$$
$$\iota_2(y, x), \iota_2(f_1, g_1), \ldots, \iota_2(f_k, g_k),$$
$$\iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big),$$
$$\iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big)\Big)$$
$$=: t_i'\{1, 2\}$$

and

$$X_{i+1} = t_i'\Big(\iota_1(y, x), \iota_1(f_1, g_1), \ldots, \iota_1(f_k, g_k),$$
$$\iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big),$$
$$\iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big),$$
$$\iota_2(y, x), \iota_2(f_1, g_1), \ldots, \iota_2(f_k, g_k),$$
$$\iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big),$$
$$\iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big)\Big)$$
$$=: t_i'\{2, 1\}$$

respectively. In the notation $t_i'\{1, 2\}$ and $t_i'\{2, 1\}$, the first (respectively the second) entry inside the brackets corresponds to which of ι_1 or ι_2 is used in the third (respectively the sixth) row in the above expressions. Hence we see that the condition $q_1(y, x) = q_2(y, x)$ implies the existence of integers $k, m \geqslant 0$ and $n \geqslant 1$, binary terms $f_1, g_1, \ldots, f_k, g_k$ with $f_i(x, x) = g_i(x, x)$ for all $i \in \{1, \ldots, k\}$, ternary terms $p_1, \ldots, p_m$ and $(2(k + 2m + 1))$-ary

terms $t'_1, \ldots, t'_n$ such that the equations

$$
\begin{aligned}
&\iota_1(y, x) \\
&= t'_1\Big(\iota_1(y, x), \iota_1(f_1, g_1), \ldots, \iota_1(f_k, g_k), \\
&\qquad \iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_2(y, x), \iota_2(f_1, g_1), \ldots, \iota_2(f_k, g_k), \\
&\qquad \iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big)\Big)
\end{aligned}
\tag{a}
$$

and, for all $i \in \{1, \ldots, n-1\}$,

$$
\begin{aligned}
&t'_i\Big(\iota_1(y, x), \iota_1(f_1, g_1), \ldots, \iota_1(f_k, g_k), \\
&\qquad \iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_2(y, x), \iota_2(f_1, g_1), \ldots, \iota_2(f_k, g_k), \\
&\qquad \iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big)\Big) \\
&= t'_{i+1}\Big(\iota_1(y, x), \iota_1(f_1, g_1), \ldots, \iota_1(f_k, g_k), \\
&\qquad \iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_2(y, x), \iota_2(f_1, g_1), \ldots, \iota_2(f_k, g_k), \\
&\qquad \iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big)\Big)
\end{aligned}
\tag{b(i)}
$$

and

$$
\begin{aligned}
&t'_n\Big(\iota_1(y, x), \iota_1(f_1, g_1), \ldots, \iota_1(f_k, g_k), \\
&\qquad \iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_2(y, x), \iota_2(f_1, g_1), \ldots, \iota_2(f_k, g_k), \\
&\qquad \iota_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_2\big(p_m(x, x, y), p_m(x, y, y)\big), \\
&\qquad \iota_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, \iota_1\big(p_m(x, x, y), p_m(x, y, y)\big)\Big) \\
&= \iota_2(y, x)
\end{aligned}
\tag{c}
$$

hold in $P + P$. We display the Equations (a), (b(i)) for all $i \in \{1, \ldots, n-1\}$ and (c) in the diagram

$$\iota_1(y,x) \overset{\text{(a)}}{=\!=} t_1'\{1,2\} \cdots\cdots t_1'\{2,1\} \overset{\text{(b(1))}}{=\!=} t_2'\{1,2\} \cdots\cdots \cdots \cdots t_{n-1}'\{2,1\} \overset{\text{(b($n-1$))}}{=\!=} t_n'\{1,2\} \cdots\cdots t_n'\{2,1\} \overset{\text{(c)}}{=\!=} \iota_2(y,x). \tag{13}$$

By Proposition 2.2, Equation (a) means that there exist an integer $\ell_0 \geqslant 0$ and elements $c_{(0,1)}, \ldots, c_{(0,\ell_0)} \in \mathsf{F}(\mathsf{U}P + \mathsf{U}P)$ such that

$$
\begin{aligned}
(y,x)^1 &\sim c_{(0,1)}, \\
c_{(0,1)} &\sim c_{(0,2)}, \ldots, c_{(0,\ell_0-1)} \sim c_{(0,\ell_0)}, \\
c_{(0,\ell_0)} &\sim t_1'\big((y,x)^1, (f_1,g_1)^1, \ldots, (f_k,g_k)^1, \\
&\qquad (p_1(x,x,y), p_1(x,y,y))^1, \ldots, (p_m(x,x,y), p_m(x,y,y))^1, \\
&\qquad (p_1(x,x,y), p_1(x,y,y))^1, \ldots, (p_m(x,x,y), p_m(x,y,y))^1, \\
&\qquad (y,x)^2, (f_1,g_1)^2, \ldots, (f_k,g_k)^2, \\
&\qquad (p_1(x,x,y), p_1(x,y,y))^2, \ldots, (p_m(x,x,y), p_m(x,y,y))^2, \\
&\qquad (p_1(x,x,y), p_1(x,y,y))^2, \ldots, (p_m(x,x,y), p_m(x,y,y))^2\big)
\end{aligned}
$$

where we write, for an element $X \in P$, X^1 and X^2 to indicate that we view them as variables in $\mathsf{F}(\mathsf{U}P + \mathsf{U}P)$ coming from the first and second copy of $\mathsf{U}P$ in $\mathsf{U}P + \mathsf{U}P$ respectively, and $\sim$ is the relation on $\mathsf{F}(\mathsf{U}P + \mathsf{U}P)$ from Proposition 2.2. More explicitly, this means that there exist, for all $j \in \{0, \ldots, \ell_0\}$, integers $a_{(0,j)}, b_{(0,j)} \geqslant 0$, elements $D_1^{(0,j)}, \ldots, D_{a_{(0,j)}}^{(0,j)}, E_1^{(0,j)}, \ldots, E_{b_{(0,j)}}^{(0,j)} \in P$, $a_{(0,j)}$-ary terms $\mu_1^{(0,j)}, \lambda_1^{(0,j)}$, $b_{(0,j)}$-ary terms $\mu_2^{(0,j)}, \lambda_2^{(0,j)}$ and $(a_{(0,j)} + b_{(0,j)} + 2)$-ary terms $\tau_{(0,j)}$ such that the equations

$$
\begin{aligned}
&(y,x)^1 \\
&= \tau_{(0,0)}\Big((D_1^{(0,0)})^1, \ldots, (D_{a_{(0,0)}}^{(0,0)})^1, (E_1^{(0,0)})^2, \ldots, (E_{b_{(0,0)}}^{(0,0)})^2, \\
&\qquad \mu_1^{(0,0)}\big((D_1^{(0,0)})^1, \ldots, (D_{a_{(0,0)}}^{(0,0)})^1\big), \mu_2^{(0,0)}\big((E_1^{(0,0)})^2, \ldots, (E_{b_{(0,0)}}^{(0,0)})^2\big)\Big)
\end{aligned}
\tag{B1}
$$

and

$$
\begin{aligned}
\tau_{(0,0)}\Big(&(D_1^{(0,0)})^1, \ldots, (D_{a_{(0,0)}}^{(0,0)})^1, (E_1^{(0,0)})^2, \ldots, (E_{b_{(0,0)}}^{(0,0)})^2, \\
&\lambda_1^{(0,0)}\big((D_1^{(0,0)})^1, \ldots, (D_{a_{(0,0)}}^{(0,0)})^1\big), \lambda_2^{(0,0)}\big((E_1^{(0,0)})^2, \ldots, (E_{b_{(0,0)}}^{(0,0)})^2\big)\Big) \\
= &\, c_{(0,1)}
\end{aligned}
\tag{C(0,1)}
$$

hold in $\mathsf{F}(\mathsf{U}P + \mathsf{U}P)$ and the equations

$$\mu_1^{(0,0)}\big(D_1^{(0,0)}, \ldots, D_{a_{(0,0)}}^{(0,0)}\big) = \lambda_1^{(0,0)}\big(D_1^{(0,0)}, \ldots, D_{a_{(0,0)}}^{(0,0)}\big), \tag{M(0,0)}$$

$$\mu_2^{(0,0)}\big(E_1^{(0,0)}, \ldots, E_{b_{(0,0)}}^{(0,0)}\big) = \lambda_2^{(0,0)}\big(E_1^{(0,0)}, \ldots, E_{b_{(0,0)}}^{(0,0)}\big) \tag{N(0,0)}$$

hold in P; the equations

$$c_{(0,j)}$$
$$= \tau_{(0,j)}\Big((D_1^{(0,j)})^1,\ldots,(D_{a_{(0,j)}}^{(0,j)})^1,(E_1^{(0,j)})^2,\ldots,(E_{b_{(0,j)}}^{(0,j)})^2,$$
$$\mu_1^{(0,j)}\big((D_1^{(0,j)})^1,\ldots,(D_{a_{(0,j)}}^{(0,j)})^1\big),\mu_2^{(0,j)}\big((E_1^{(0,j)})^2,\ldots,(E_{b_{(0,j)}}^{(0,j)})^2\big)\Big),$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{D}(0,j))$$

$$\tau_{(0,j)}\Big((D_1^{(0,j)})^1,\ldots,(D_{a_{(0,j)}}^{(0,j)})^1,(E_1^{(0,j)})^2,\ldots,(E_{b_{(0,j)}}^{(0,j)})^2,$$
$$\lambda_1^{(0,j)}\big((D_1^{(0,j)})^1,\ldots,(D_{a_{(0,j)}}^{(0,j)})^1\big),\lambda_2^{(0,j)}\big((E_1^{(0,j)})^2,\ldots,(E_{b_{(0,j)}}^{(0,j)})^2\big)\Big)$$
$$= c_{(0,j+1)} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{C}(0,j+1))$$

and

$$\mu_1^{(0,j)}\big(D_1^{(0,j)},\ldots,D_{a_{(0,j)}}^{(0,j)}\big) = \lambda_1^{(0,j)}\big(D_1^{(0,j)},\ldots,D_{a_{(0,j)}}^{(0,j)}\big), \qquad (\mathrm{M}(0,j))$$
$$\mu_2^{(0,j)}\big(E_1^{(0,j)},\ldots,E_{b_{(0,j)}}^{(0,j)}\big) = \lambda_2^{(0,j)}\big(E_1^{(0,j)},\ldots,E_{b_{(0,j)}}^{(0,j)}\big) \qquad (\mathrm{N}(0,j))$$

are satisfied for all $j \in \{1,\ldots,\ell_0-1\}$; and moreover the equations

$$c_{(0,\ell_0)}$$
$$= \tau_{(0,\ell_0)}\Big((D_1^{(0,\ell_0)})^1,\ldots,(D_{a_{(0,\ell_0)}}^{(0,\ell_0)})^1,(E_1^{(0,\ell_0)})^2,\ldots,(E_{b_{(0,\ell_0)}}^{(0,\ell_0)})^2,$$
$$\mu_1^{(0,\ell_0)}\big((D_1^{(0,\ell_0)})^1,\ldots,(D_{a_{(0,\ell_0)}}^{(0,\ell_0)})^1\big),\mu_2^{(0,\ell_0)}\big((E_1^{(0,\ell_0)})^2,\ldots,(E_{b_{(0,\ell_0)}}^{(0,\ell_0)})^2\big)\Big),$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{D}(0,\ell_0))$$

$$\tau_{(0,\ell_0)}\Big((D_1^{(0,\ell_0)})^1,\ldots,(D_{a_{(0,\ell_0)}}^{(0,\ell_0)})^1,(E_1^{(0,\ell_0)})^2,\ldots,(E_{b_{(0,\ell_0)}}^{(0,\ell_0)})^2,$$
$$\lambda_1^{(0,\ell_0)}\big((D_1^{(0,\ell_0)})^1,\ldots,(D_{a_{(0,\ell_0)}}^{(0,\ell_0)})^1\big),\lambda_2^{(0,\ell_0)}\big((E_1^{(0,\ell_0)})^2,\ldots,(E_{b_{(0,\ell_0)}}^{(0,\ell_0)})^2\big)\Big)$$
$$= t_1'\Big((y,x)^1,(f_1,g_1)^1,\ldots,(f_k,g_k)^1, \qquad\qquad\qquad\qquad\qquad (\mathrm{T}(1))$$
$$(p_1(x,x,y),p_1(x,y,y))^1,\ldots,(p_m(x,x,y),p_m(x,y,y))^1,$$
$$(p_1(x,x,y),p_1(x,y,y))^1,\ldots,(p_m(x,x,y),p_m(x,y,y))^1,$$
$$(y,x)^2,(f_1,g_1)^2,\ldots,(f_k,g_k)^2,$$
$$(p_1(x,x,y),p_1(x,y,y))^2,\ldots,(p_m(x,x,y),p_m(x,y,y))^2,$$
$$(p_1(x,x,y),p_1(x,y,y))^2,\ldots,(p_m(x,x,y),p_m(x,y,y))^2\Big)$$

and

$$\mu_1^{(0,\ell_0)}\big(D_1^{(0,\ell_0)},\ldots,D_{a_{(0,\ell_0)}}^{(0,\ell_0)}\big) = \lambda_1^{(0,\ell_0)}\big(D_1^{(0,\ell_0)},\ldots,D_{a_{(0,\ell_0)}}^{(0,\ell_0)}\big), \qquad (\mathrm{M}(0,\ell_0))$$
$$\mu_2^{(0,\ell_0)}\big(E_1^{(0,\ell_0)},\ldots,E_{b_{(0,\ell_0)}}^{(0,\ell_0)}\big) = \lambda_2^{(0,\ell_0)}\big(E_1^{(0,\ell_0)},\ldots,E_{b_{(0,\ell_0)}}^{(0,\ell_0)}\big) \qquad (\mathrm{N}(0,\ell_0))$$

hold. For $i \in \{1, \ldots, n-1\}$, Equation (b(i)) means that there exist an integer $\ell_i \geqslant 0$ and elements $c_{(i,1)}, \ldots, c_{(i,\ell_i)} \in \mathsf{F}(\mathsf{U}P + \mathsf{U}P)$ such that

$$
\begin{aligned}
t_i'\big(& (y,x)^1, (f_1, g_1)^1, \ldots, (f_k, g_k)^1, \\
& (p_1(x,x,y), p_1(x,y,y))^1, \ldots, (p_m(x,x,y), p_m(x,y,y))^1, \\
& (p_1(x,x,y), p_1(x,y,y))^2, \ldots, (p_m(x,x,y), p_m(x,y,y))^2, \\
& (y,x)^2, (f_1, g_1)^2, \ldots, (f_k, g_k)^2, \\
& (p_1(x,x,y), p_1(x,y,y))^2, \ldots, (p_m(x,x,y), p_m(x,y,y))^2, \\
& (p_1(x,x,y), p_1(x,y,y))^1, \ldots, (p_m(x,x,y), p_m(x,y,y))^1 \big) \sim c_{(i,1)},
\end{aligned}
$$

$$
c_{(i,1)} \sim c_{(i,2)}, \ldots, c_{(i,\ell_i-1)} \sim c_{(i,\ell_i)},
$$

$$
\begin{aligned}
c_{(i,\ell_i)} \sim t_{i+1}'\big(& (y,x)^1, (f_1, g_1)^1, \ldots, (f_k, g_k)^1, \\
& (p_1(x,x,y), p_1(x,y,y))^1, \ldots, (p_m(x,x,y), p_m(x,y,y))^1, \\
& (p_1(x,x,y), p_1(x,y,y))^1, \ldots, (p_m(x,x,y), p_m(x,y,y))^1, \\
& (y,x)^2, (f_1, g_1)^2, \ldots, (f_k, g_k)^2, \\
& (p_1(x,x,y), p_1(x,y,y))^2, \ldots, (p_m(x,x,y), p_m(x,y,y))^2, \\
& (p_1(x,x,y), p_1(x,y,y))^2, \ldots, (p_m(x,x,y), p_m(x,y,y))^2 \big).
\end{aligned}
$$

More precisely, there exist, for all $j \in \{0, \ldots, \ell_i\}$, integers $a_{(i,j)}, b_{(i,j)} \geqslant 0$, elements $D_1^{(i,j)}, \ldots, D_{a_{(i,j)}}^{(i,j)}, E_1^{(i,j)}, \ldots, E_{b_{(i,j)}}^{(i,j)} \in P$, $a_{(i,j)}$-ary terms $\mu_1^{(i,j)}, \lambda_1^{(i,j)}$, $b_{(i,j)}$-ary terms $\mu_2^{(i,j)}$, $\lambda_2^{(i,j)}$ and $(a_{(i,j)} + b_{(i,j)} + 2)$-ary terms $\tau_{(i,j)}$ such that the equations

$$
\begin{aligned}
t_i'\Big(& (y,x)^1, (f_1, g_1)^1, \ldots, (f_k, g_k)^1, \\
& (p_1(x,x,y), p_1(x,y,y))^1, \ldots, (p_m(x,x,y), p_m(x,y,y))^1, \\
& (p_1(x,x,y), p_1(x,y,y))^2, \ldots, (p_m(x,x,y), p_m(x,y,y))^2, \\
& (y,x)^2, (f_1, g_1)^2, \ldots, (f_k, g_k)^2, \\
& (p_1(x,x,y), p_1(x,y,y))^2, \ldots, (p_m(x,x,y), p_m(x,y,y))^2, \\
& (p_1(x,x,y), p_1(x,y,y))^1, \ldots, (p_m(x,x,y), p_m(x,y,y))^1 \Big) \\
=\tau_{(i,0)}\Big(& (D_1^{(i,0)})^1, \ldots, (D_{a_{(i,0)}}^{(i,0)})^1, (E_1^{(i,0)})^2, \ldots, (E_{b_{(i,0)}}^{(i,0)})^2, \\
& \mu_1^{(i,0)}\big((D_1^{(i,0)})^1, \ldots, (D_{a_{(i,0)}}^{(i,0)})^1\big), \mu_2^{(i,0)}\big((E_1^{(i,0)})^2, \ldots, (E_{b_{(i,0)}}^{(i,0)})^2\big)\Big)
\end{aligned}
\tag{S(i)}
$$

and

$$
\begin{aligned}
\tau_{(i,0)}\Big(& (D_1^{(i,0)})^1, \ldots, (D_{a_{(i,0)}}^{(i,0)})^1, (E_1^{(i,0)})^2, \ldots, (E_{b_{(i,0)}}^{(i,0)})^2, \\
& \lambda_1^{(i,0)}\big((D_1^{(i,0)})^1, \ldots, (D_{a_{(i,0)}}^{(i,0)})^1\big), \lambda_2^{(i,0)}\big((E_1^{(i,0)})^2, \ldots, (E_{b_{(i,0)}}^{(i,0)})^2\big)\Big) \\
=c_{(i,1)} &
\end{aligned}
\tag{C(i, 1)}
$$

hold in $\mathsf{F}(\mathsf{U}P + \mathsf{U}P)$ and the equations

$$\mu_1^{(i,0)}\big(D_1^{(i,0)},\ldots,D_{a_{(i,0)}}^{(i,0)}\big) = \lambda_1^{(i,0)}\big(D_1^{(i,0)},\ldots,D_{a_{(i,0)}}^{(i,0)}\big), \qquad (\mathrm{M}(i,0))$$

$$\mu_2^{(i,0)}\big(E_1^{(i,0)},\ldots,E_{b_{(i,0)}}^{(i,0)}\big) = \lambda_2^{(i,0)}\big(E_1^{(i,0)},\ldots,E_{b_{(i,0)}}^{(i,0)}\big) \qquad (\mathrm{N}(i,0))$$

hold in P; the equations

$$
\begin{aligned}
c_{(i,j)} & \\
= \tau_{(i,j)}\Big(& (D_1^{(i,j)})^1,\ldots,(D_{a_{(i,j)}}^{(i,j)})^1, (E_1^{(i,j)})^2,\ldots,(E_{b_{(i,j)}}^{(i,j)})^2, \qquad (\mathrm{D}(i,j)) \\
& \mu_1^{(i,j)}\big((D_1^{(i,j)})^1,\ldots,(D_{a_{(i,j)}}^{(i,j)})^1\big), \mu_2^{(i,j)}\big((E_1^{(i,j)})^2,\ldots,(E_{b_{(i,j)}}^{(i,j)})^2\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{(i,j)}\Big(& (D_1^{(i,j)})^1,\ldots,(D_{a_{(i,j)}}^{(i,j)})^1, (E_1^{(i,j)})^2,\ldots,(E_{b_{(i,j)}}^{(i,j)})^2, \\
& \lambda_1^{(i,j)}\big((D_1^{(i,j)})^1,\ldots,(D_{a_{(i,j)}}^{(i,j)})^1\big), \lambda_2^{(i,j)}\big((E_1^{(i,j)})^2,\ldots,(E_{b_{(i,j)}}^{(i,j)})^2\big)\Big) \\
= c_{(i,j+1)} & \qquad (\mathrm{C}(i,j+1))
\end{aligned}
$$

and

$$\mu_1^{(i,j)}\big(D_1^{(i,j)},\ldots,D_{a_{(i,j)}}^{(i,j)}\big) = \lambda_1^{(i,j)}\big(D_1^{(i,j)},\ldots,D_{a_{(i,j)}}^{(i,j)}\big), \qquad (\mathrm{M}(i,j))$$

$$\mu_2^{(i,j)}\big(E_1^{(i,j)},\ldots,E_{b_{(i,j)}}^{(i,j)}\big) = \lambda_2^{(i,j)}\big(E_1^{(i,j)},\ldots,E_{b_{(i,j)}}^{(i,j)}\big) \qquad (\mathrm{N}(i,j))$$

are satisfied for all $j \in \{1,\ldots,\ell_i - 1\}$; and moreover the equations

$$
\begin{aligned}
c_{(i,\ell_i)} & \\
= \tau_{(i,\ell_i)}\Big(& (D_1^{(i,\ell_i)})^1,\ldots,(D_{a_{(i,\ell_i)}}^{(i,\ell_i)})^1, (E_1^{(i,\ell_i)})^2,\ldots,(E_{b_{(i,\ell_i)}}^{(i,\ell_i)})^2, \qquad (\mathrm{D}(i,\ell_i)) \\
& \mu_1^{(i,\ell_i)}\big((D_1^{(i,\ell_i)})^1,\ldots,(D_{a_{(i,\ell_i)}}^{(i,\ell_i)})^1\big), \mu_2^{(i,\ell_i)}\big((E_1^{(i,\ell_i)})^2,\ldots,(E_{b_{(i,\ell_i)}}^{(i,\ell_i)})^2\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{(i,\ell_i)}\Big(& (D_1^{(i,\ell_i)})^1,\ldots,(D_{a_{(i,\ell_i)}}^{(i,\ell_i)})^1, (E_1^{(i,\ell_i)})^2,\ldots,(E_{b_{(i,\ell_i)}}^{(i,\ell_i)})^2, \\
& \lambda_1^{(i,\ell_i)}\big((D_1^{(i,\ell_i)})^1,\ldots,(D_{a_{(i,\ell_i)}}^{(i,\ell_i)})^1\big), \lambda_2^{(i,\ell_i)}\big((E_1^{(i,\ell_i)})^2,\ldots,(E_{b_{(i,\ell_i)}}^{(i,\ell_i)})^2\big)\Big) \\
= t_{i+1}'\Big(& (y,x)^1, (f_1,g_1)^1,\ldots,(f_k,g_k)^1, \qquad (\mathrm{T}(i+1)) \\
& (p_1(x,x,y),p_1(x,y,y))^1,\ldots,(p_m(x,x,y),p_m(x,y,y))^1, \\
& (p_1(x,x,y),p_1(x,y,y))^1,\ldots,(p_m(x,x,y),p_m(x,y,y))^1, \\
& (y,x)^2, (f_1,g_1)^2,\ldots,(f_k,g_k)^2, \\
& (p_1(x,x,y),p_1(x,y,y))^2,\ldots,(p_m(x,x,y),p_m(x,y,y))^2, \\
& (p_1(x,x,y),p_1(x,y,y))^2,\ldots,(p_m(x,x,y),p_m(x,y,y))^2\Big)
\end{aligned}
$$

and

$$\mu_1^{(i,\ell_i)}\big(D_1^{(i,\ell_i)},\ldots,D_{a_{(i,\ell_i)}}^{(i,\ell_i)}\big) = \lambda_1^{(i,\ell_i)}\big(D_1^{(i,\ell_i)},\ldots,D_{a_{(i,\ell_i)}}^{(i,\ell_i)}\big), \qquad (\mathrm{M}(i,\ell_i))$$

$$\mu_2^{(i,\ell_i)}\big(E_1^{(i,\ell_i)},\ldots,E_{b_{(i,\ell_i)}}^{(i,\ell_i)}\big) = \lambda_2^{(i,\ell_i)}\big(E_1^{(i,\ell_i)},\ldots,E_{b_{(i,\ell_i)}}^{(i,\ell_i)}\big) \qquad (\mathrm{N}(i,\ell_i))$$

hold. Equation (c) means that there exist an integer $\ell_n \geqslant 0$ and elements $c_{(n,1)},\ldots,c_{(n,\ell_n)}$ in $\mathsf{F}(\mathsf{U}P+\mathsf{U}P)$ such that

$$
\begin{aligned}
t'_n\big(&(y,x)^1,(f_1,g_1)^1,\ldots,(f_k,g_k)^1,\\
&(p_1(x,x,y),p_1(x,y,y))^1,\ldots,(p_m(x,x,y),p_m(x,y,y))^1,\\
&(p_1(x,x,y),p_1(x,y,y))^2,\ldots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&(y,x)^2,(f_1,g_1)^2,\ldots,(f_k,g_k)^2,\\
&(p_1(x,x,y),p_1(x,y,y))^2,\ldots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&(p_1(x,x,y),p_1(x,y,y))^1,\ldots,(p_m(x,x,y),p_m(x,y,y))^1\big) \sim c_{(n,1)},\\
&c_{(n,1)} \sim c_{(n,2)},\ldots,c_{(n,\ell_n-1)} \sim c_{(n,\ell_n)},\\
&c_{(n,\ell_n)} \sim (y,x)^2.
\end{aligned}
$$

More explicitly, this means that there exist, for all $j \in \{0,\ldots,\ell_n\}$, integers $a_{(n,j)}, b_{(n,j)} \geqslant 0$, elements $D_1^{(n,j)},\ldots,D_{a_{(n,j)}}^{(n,j)}, E_1^{(n,j)},\ldots,E_{b_{(n,j)}}^{(n,j)} \in P$, $a_{(n,j)}$-ary terms $\mu_1^{(n,j)}, \lambda_1^{(n,j)}$, $b_{(n,j)}$-ary terms $\mu_2^{(n,j)}, \lambda_2^{(n,j)}$ and $(a_{(n,j)}+b_{(n,j)}+2)$-ary terms $\tau_{(n,j)}$ such that the equations

$$
\begin{aligned}
t'_n\big(&(y,x)^1,(f_1,g_1)^1,\ldots,(f_k,g_k)^1,\\
&(p_1(x,x,y),p_1(x,y,y))^1,\ldots,(p_m(x,x,y),p_m(x,y,y))^1,\\
&(p_1(x,x,y),p_1(x,y,y))^2,\ldots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&(y,x)^2,(f_1,g_1)^2,\ldots,(f_k,g_k)^2,\\
&(p_1(x,x,y),p_1(x,y,y))^2,\ldots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&(p_1(x,x,y),p_1(x,y,y))^1,\ldots,(p_m(x,x,y),p_m(x,y,y))^1\big)\\
=\tau_{(n,0)}\big(&(D_1^{(n,0)})^1,\ldots,(D_{a_{(n,0)}}^{(n,0)})^1,(E_1^{(n,0)})^2,\ldots,(E_{b_{(n,0)}}^{(n,0)})^2,\\
&\mu_1^{(n,0)}\big((D_1^{(n,0)})^1,\ldots,(D_{a_{(n,0)}}^{(n,0)})^1\big),\mu_2^{(n,0)}\big((E_1^{(n,0)})^2,\ldots,(E_{b_{(n,0)}}^{(n,0)})^2\big)\big)
\end{aligned}
\qquad (\mathrm{S}(n))
$$

and

$$
\begin{aligned}
\tau_{(n,0)}\big(&(D_1^{(n,0)})^1,\ldots,(D_{a_{(n,0)}}^{(n,0)})^1,(E_1^{(n,0)})^2,\ldots,(E_{b_{(n,0)}}^{(n,0)})^2,\\
&\lambda_1^{(n,0)}\big((D_1^{(n,0)})^1,\ldots,(D_{a_{(n,0)}}^{(n,0)})^1\big),\lambda_2^{(n,0)}\big((E_1^{(n,0)})^2,\ldots,(E_{b_{(n,0)}}^{(n,0)})^2\big)\big)\\
=&c_{(n,1)}
\end{aligned}
\qquad (\mathrm{C}(n,1))
$$

hold in $\mathsf{F}(\mathsf{U}P + \mathsf{U}P)$ and the equations

$$\mu_1^{(n,0)}\big(D_1^{(n,0)}, \ldots, D_{a_{(n,0)}}^{(n,0)}\big) = \lambda_1^{(n,0)}\big(D_1^{(n,0)}, \ldots, D_{a_{(n,0)}}^{(n,0)}\big), \qquad \text{(M(n, 0))}$$

$$\mu_2^{(n,0)}\big(E_1^{(n,0)}, \ldots, E_{b_{(n,0)}}^{(n,0)}\big) = \lambda_2^{(n,0)}\big(E_1^{(n,0)}, \ldots, E_{b_{(n,0)}}^{(n,0)}\big) \qquad \text{(N(n, 0))}$$

hold in P; the equations

$$
\begin{aligned}
&c_{(n,j)}\\
&=\tau_{(n,j)}\Big((D_1^{(n,j)})^1, \ldots, (D_{a_{(n,j)}}^{(n,j)})^1, (E_1^{(n,j)})^2, \ldots, (E_{b_{(n,j)}}^{(n,j)})^2,\\
&\qquad \mu_1^{(n,j)}\big((D_1^{(n,j)})^1, \ldots, (D_{a_{(n,j)}}^{(n,j)})^1\big), \mu_2^{(n,j)}\big((E_1^{(n,j)})^2, \ldots, (E_{b_{(n,j)}}^{(n,j)})^2\big)\Big),
\end{aligned}
\qquad \text{(D(n, j))}
$$

$$
\begin{aligned}
&\tau_{(n,j)}\Big((D_1^{(n,j)})^1, \ldots, (D_{a_{(n,j)}}^{(n,j)})^1, (E_1^{(n,j)})^2, \ldots, (E_{b_{(n,j)}}^{(n,j)})^2,\\
&\qquad \lambda_1^{(n,j)}\big((D_1^{(n,j)})^1, \ldots, (D_{a_{(n,j)}}^{(n,j)})^1\big), \lambda_2^{(n,j)}\big((E_1^{(n,j)})^2, \ldots, (E_{b_{(n,j)}}^{(n,j)})^2\big)\Big)\\
&=c_{(n,j+1)}
\end{aligned}
\qquad \text{(C(n, $j+1$))}
$$

and

$$\mu_1^{(n,j)}\big(D_1^{(n,j)}, \ldots, D_{a_{(n,j)}}^{(n,j)}\big) = \lambda_1^{(n,j)}\big(D_1^{(n,j)}, \ldots, D_{a_{(n,j)}}^{(n,j)}\big), \qquad \text{(M(n, j))}$$

$$\mu_2^{(n,j)}\big(E_1^{(n,j)}, \ldots, E_{b_{(n,j)}}^{(n,j)}\big) = \lambda_2^{(n,j)}\big(E_1^{(n,j)}, \ldots, E_{b_{(n,j)}}^{(n,j)}\big) \qquad \text{(N(n, j))}$$

are satisfied for all $j \in \{1, \ldots, \ell_n - 1\}$; and moreover the equations

$$
\begin{aligned}
&c_{(n,\ell_n)}\\
&=\tau_{(n,\ell_n)}\Big((D_1^{(n,\ell_n)})^1, \ldots, (D_{a_{(n,\ell_n)}}^{(n,\ell_n)})^1, (E_1^{(n,\ell_n)})^2, \ldots, (E_{b_{(n,\ell_n)}}^{(n,\ell_n)})^2,\\
&\qquad \mu_1^{(n,\ell_n)}\big((D_1^{(n,\ell_n)})^1, \ldots, (D_{a_{(n,\ell_n)}}^{(n,\ell_n)})^1\big), \mu_2^{(n,\ell_n)}\big((E_1^{(n,\ell_n)})^2, \ldots, (E_{b_{(n,\ell_n)}}^{(n,\ell_n)})^2\big)\Big),
\end{aligned}
\qquad \text{(D(n, ℓ_n))}
$$

$$
\begin{aligned}
&\tau_{(n,\ell_n)}\Big((D_1^{(n,\ell_n)})^1, \ldots, (D_{a_{(n,\ell_n)}}^{(n,\ell_n)})^1, (E_1^{(n,\ell_n)})^2, \ldots, (E_{b_{(n,\ell_n)}}^{(n,\ell_n)})^2,\\
&\qquad \lambda_1^{(n,\ell_n)}\big((D_1^{(n,\ell_n)})^1, \ldots, (D_{a_{(n,\ell_n)}}^{(n,\ell_n)})^1\big), \lambda_2^{(n,\ell_n)}\big((E_1^{(n,\ell_n)})^2, \ldots, (E_{b_{(n,\ell_n)}}^{(n,\ell_n)})^2\big)\Big)\\
&=(y, x)^2
\end{aligned}
\qquad \text{(B2)}
$$

and

$$\mu_1^{(n,\ell_n)}\big(D_1^{(n,\ell_n)}, \ldots, D_{a_{(n,\ell_n)}}^{(n,\ell_n)}\big) = \lambda_1^{(n,\ell_n)}\big(D_1^{(n,\ell_n)}, \ldots, D_{a_{(n,\ell_n)}}^{(n,\ell_n)}\big), \qquad \text{(M(n, ℓ_n))}$$

$$\mu_2^{(n,\ell_n)}\big(E_1^{(n,\ell_n)}, \ldots, E_{b_{(n,\ell_n)}}^{(n,\ell_n)}\big) = \lambda_2^{(n,\ell_n)}\big(E_1^{(n,\ell_n)}, \ldots, E_{b_{(n,\ell_n)}}^{(n,\ell_n)}\big) \qquad \text{(N(n, ℓ_n))}$$

hold. Similar to Diagram (13), we display the Equations (B1), (B2), (T(i)), (S(i)), (C(i,j)) and (D(i,j)) in the diagram

$$
\text{(a)}\begin{cases}
(y,x)^1 \overset{\text{(B1)}}{=\!=\!=} \tau_{(0,0)}[\mu] \cdots\!\cdots \tau_{(0,0)}[\lambda] \overset{\text{(C(0,1))}}{=\!=\!=} c_{(0,1)} \overset{\text{(D(0,1))}}{=\!=\!=} \tau_{(0,1)}[\mu] \cdots\!\cdots \tau_{(0,1)}[\lambda] \overset{\text{(C(0,2))}}{=\!=\!=} c_{(0,2)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots \\
t_1'[1,2] \overset{\text{(T(1))}}{=\!=\!=} \tau_{(0,\ell_0)}[\lambda] \cdots\! \tau_{(0,\ell_0)}[\mu] \overset{\text{(D(0,\ell_0))}}{=\!=\!=} c_{(0,\ell_0)} \overset{\text{(C(0,\ell_0))}}{=\!=\!=} \tau_{(0,\ell_0-1)}[\lambda] \cdots\! \tau_{(0,\ell_0-1)}[\mu] \overset{\text{(D(0,\ell_0-1))}}{=\!=\!=} c_{(0,\ell_0-1)}
\end{cases}
$$

$$
\text{(b(i))}\begin{cases}
t_i'[2,1] \overset{\text{(S(i))}}{=\!=\!=} \tau_{(i,0)}[\mu] \cdots\!\cdots \tau_{(i,0)}[\lambda] \overset{\text{(C(i,1))}}{=\!=\!=} c_{(i,1)} \overset{\text{(D(i,1))}}{=\!=\!=} \tau_{(i,1)}[\mu] \cdots\!\cdots \tau_{(i,1)}[\lambda] \overset{\text{(C(i,2))}}{=\!=\!=} c_{(i,2)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots \\
t_{i+1}'[1,2] \overset{\text{(T(i+1))}}{=\!=\!=} \tau_{(i,\ell_i)}[\lambda] \cdots\! \tau_{(i,\ell_i)}[\mu] \overset{\text{(D(i,ℓ_i))}}{=\!=\!=} c_{(i,\ell_i)} \overset{\text{(C(i,ℓ_i))}}{=\!=\!=} \tau_{(i,\ell_i-1)}[\lambda] \cdots\! \tau_{(i,\ell_i-1)}[\mu] \overset{\text{(D(i,ℓ_i-1))}}{=\!=\!=} c_{(i,\ell_i-1)}
\end{cases}
$$

$$
\text{(c)}\begin{cases}
t_n'[2,1] \overset{\text{(S(n))}}{=\!=\!=} \tau_{(n,0)}[\mu] \cdots\!\cdots \tau_{(n,0)}[\lambda] \overset{\text{(C(n,1))}}{=\!=\!=} c_{(n,1)} \overset{\text{(D(n,1))}}{=\!=\!=} \tau_{(n,1)}[\mu] \cdots\!\cdots \tau_{(n,1)}[\lambda] \overset{\text{(C(n,2))}}{=\!=\!=} c_{(n,2)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots \\
(y,x)^2 \overset{\text{(B2)}}{=\!=\!=} \tau_{(n,\ell_n)}[\lambda] \cdots\! \tau_{(n,\ell_n)}[\mu] \overset{\text{(D(n,ℓ_n))}}{=\!=\!=} c_{(n,\ell_n)} \overset{\text{(C(n,ℓ_n))}}{=\!=\!=} \tau_{(n,\ell_n-1)}[\lambda] \cdots\! \tau_{(n,\ell_n-1)}[\mu] \overset{\text{(D(n,ℓ_n-1))}}{=\!=\!=} c_{(n,\ell_n-1)},
\end{cases}
$$

where

$$
\begin{aligned}
t_i'[1,2] := t_i'\Big(& (y,x)^1, (f_1,g_1)^1, \ldots, (f_k,g_k)^1, \\
& (p_1(x,x,y),p_1(x,y,y))^1, \ldots, (p_m(x,x,y),p_m(x,y,y))^1, \\
& (p_1(x,x,y),p_1(x,y,y))^1, \ldots, (p_m(x,x,y),p_m(x,y,y))^1, \\
& (y,x)^2, (f_1,g_1)^2, \ldots, (f_k,g_k)^2, \\
& (p_1(x,x,y),p_1(x,y,y))^2, \ldots, (p_m(x,x,y),p_m(x,y,y))^2, \\
& (p_1(x,x,y),p_1(x,y,y))^2, \ldots, (p_m(x,x,y),p_m(x,y,y))^2 \Big), \\
t_i'[2,1] := t_i'\Big(& (y,x)^1, (f_1,g_1)^1, \ldots, (f_k,g_k)^1, \\
& (p_1(x,x,y),p_1(x,y,y))^1, \ldots, (p_m(x,x,y),p_m(x,y,y))^1, \\
& (p_1(x,x,y),p_1(x,y,y))^2, \ldots, (p_m(x,x,y),p_m(x,y,y))^2, \\
& (y,x)^2, (f_1,g_1)^2, \ldots, (f_k,g_k)^2, \\
& (p_1(x,x,y),p_1(x,y,y))^2, \ldots, (p_m(x,x,y),p_m(x,y,y))^2, \\
& (p_1(x,x,y),p_1(x,y,y))^1, \ldots, (p_m(x,x,y),p_m(x,y,y))^1 \Big), \\
\tau_{(i,j)}[\mu] := \tau_{(i,j)}\Big(& (D_1^{(i,j)})^1, \ldots, (D_{a_{(i,j)}}^{(i,j)})^1, (E_1^{(i,j)})^2, \ldots, (E_{b_{(i,j)}}^{(i,j)})^2, \\
& \mu_1^{(i,j)}\big((D_1^{(i,j)})^1, \ldots, (D_{a_{(i,j)}}^{(i,j)})^1\big), \mu_2^{(i,j)}\big((E_1^{(i,j)})^2, \ldots, (E_{b_{(i,j)}}^{(i,j)})^2\big)\Big), \\
\tau_{(i,j)}[\lambda] := \tau_{(i,j)}\Big(& (D_1^{(i,j)})^1, \ldots, (D_{a_{(i,j)}}^{(i,j)})^1, (E_1^{(i,j)})^2, \ldots, (E_{b_{(i,j)}}^{(i,j)})^2, \\
& \lambda_1^{(i,j)}\big((D_1^{(i,j)})^1, \ldots, (D_{a_{(i,j)}}^{(i,j)})^1\big), \lambda_2^{(i,j)}\big((E_1^{(i,j)})^2, \ldots, (E_{b_{(i,j)}}^{(i,j)})^2\big)\Big).
\end{aligned}
$$

Since the relation $\sim$ as in Proposition 2.2 is reflexive, we can assume, without loss of generality, that $\ell_0 = \cdots = \ell_n =: \ell$. As before, by possibly artificially increasing the number of variables the terms of the form $\tau_{(i,j)}$, $\mu_1^{(i,j)}$, $\mu_2^{(i,j)}$, $\lambda_1^{(i,j)}$ or $\lambda_2^{(i,j)}$ depend on, we can also assume, without loss of generality, that we have integers $a, b \geqslant 0$ such that $a_{(i,j)} = a$ and $b_{(i,j)} = b$ for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, \ell\}$, and omit the upper index from the elements $D_\alpha^{(i,j)}, E_\alpha^{(i,j)}$. Moreover, we can, without loss of generality, assume that $a = b$ and $D_\alpha = E_\alpha$ for all $\alpha \in \{1, \ldots, a\}$. Furthermore, up to adding some D_α's to the list of (f_j, g_j)'s, and up to adding (y, x), some (f_j, g_j)'s and some $(p_j(x, x, y), p_j(x, y, y))$'s to the list of D_α's, we can assume that $a = k + m + 1$ and $D_1 = (y, x)$, $D_2 = (f_1, g_1), \ldots, D_{k+1} = (f_k, g_k)$ and $D_{k+2} = (p_1(x, x, y), p_1(x, y, y)), \ldots,$ $D_{k+m+1} = (p_m(x, x, y), p_m(x, y, y))$, keeping the elements

$$(y, x), (f_1, g_1) \ldots, (f_k, g_k), (p_1(x, x, y), p_1(x, y, y)) \ldots, (p_m(x, x, y), p_m(x, y, y))$$

of P pairwise distinct. Thus, Equations (a), (b(i)) for all $i \in \{1, \ldots, n-1\}$ and (c) imply the existence of an integer $\ell \geqslant 0$, terms $c_{(i,1)}, \ldots, c_{(i,\ell)}$, $(k+m+1)$-ary terms $\mu_1^{(i,j)}, \mu_2^{(i,j)}, \lambda_1^{(i,j)}, \lambda_2^{(i,j)}$ and $(2(k+m+2))$-ary terms $\tau_{(i,j)}$ for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, \ell\}$ such that the following equalities hold. Using variables $u, v_1, \ldots, v_k, w_1, \ldots, w_m$ and $u', v_1', \ldots, v_k', w_1', \ldots, w_m'$, Equations (B1) and (B2) read

$$u = \tau_{(0,0)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \mu_1^{(0,0)}(u, \vec{v}, \vec{w}), \mu_2^{(0,0)}(u', \vec{v}', \vec{w}')), \tag{B1'}$$

$$u' = \tau_{(n,\ell)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \lambda_1^{(n,\ell)}(u, \vec{v}, \vec{w}), \lambda_2^{(n,\ell)}(u', \vec{v}', \vec{w}')), \tag{B2'}$$

where we write $\vec{v}$ for $v_1 \ldots, v_k$ and $\vec{w}$ for $w_1, \ldots, w_m$, and analogously for $\vec{v}'$ and $\vec{w}'$. Equations (T(1)) and (T($i+1$)), and Equations (S(i)) and (S(n)) read

$$t_i'(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}') = \tau_{(i-1,\ell)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \lambda_1^{(i-1,\ell)}(u, \vec{v}, \vec{w}), \lambda_2^{(i-1,\ell)}(u', \vec{v}', \vec{w}')), \tag{T(i)'}$$

$$t_i'(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w}) = \tau_{(i,0)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \mu_1^{(i,0)}(u, \vec{v}, \vec{w}), \mu_2^{(i,0)}(u', \vec{v}', \vec{w}')), \tag{S(i)'}$$

for all $i \in \{1, \ldots, n\}$. Equations (C(0, 1)), (C(0, $j+1$)), (C(i, 1)), (C(i, $j+1$)), (C(n, 1)), (C(n, $j+1$)), and Equations (D(0, j)), (D(0, ℓ_0)), (D(i, j)), (D(i, ℓ_i)), (D(n, j)), (D(n, ℓ_n)) read

$$c_{(i,j)} = \tau_{(i,j-1)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \lambda_1^{(i,j-1)}(u, \vec{v}, \vec{w}), \lambda_2^{(i,j-1)}(u', \vec{v}', \vec{w}')), \tag{C(i, j)'}$$

$$c_{(i,j)} = \tau_{(i,j)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \mu_1^{(i,j)}(u, \vec{v}, \vec{w}), \mu_2^{(i,j)}(u', \vec{v}', \vec{w}')), \tag{D(i, j)'}$$

for all $i \in \{0, \ldots, n\}$ and $j \in \{1, \ldots, \ell\}$ where the variables on which the $c_{(i,j)}$'s depend are left inexplicit. Equations (M(0, 0)), (M(0, j)), (M(0, ℓ_0)), (M(i, 0)), (M(i, j)), (M(i, ℓ_i)), (M(n, 0)), (M(n, j)), (M(n, ℓ_n)), and Equations (N(0, 0)), (N(0, j)), (N(0, ℓ_0)), (N(i, 0)), (N(i, j)), (N(i, ℓ_i)), (N(n, 0)), (N(n, j)), (N(n, ℓ_n)) read

$$\mu_\alpha^{(i,j)}\Big((y, x), (f_1, g_1), \ldots, (f_k, g_k), (p_1(x, x, y), p_1(x, y, y)), \ldots, (p_m(x, x, y), p_m(x, y, y))\Big)$$

$$= \lambda_\alpha^{(i,j)}\Big((y, x), (f_1, g_1), \ldots, (f_k, g_k), (p_1(x, x, y), p_1(x, y, y)), \ldots, (p_m(x, x, y), p_m(x, y, y))\Big) \tag{Z(α, i, j)}$$

for all $i \in \{0, \ldots, n\}$, $j \in \{0, \ldots, \ell\}$ and $\alpha \in \{1, 2\}$. Equations $(\mathrm{C}(i,j)')$ and $(\mathrm{D}(i,j)')$ imply the equation

$$
\begin{aligned}
&\tau_{(i,j)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \lambda_1^{(i,j)}(u, \vec{v}, \vec{w}), \lambda_2^{(i,j)}(u', \vec{v}', \vec{w}')) \\
&= \tau_{(i,j+1)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \mu_1^{(i,j+1)}(u, \vec{v}, \vec{w}), \mu_2^{(i,j+1)}(u', \vec{v}', \vec{w}'))
\end{aligned}
\qquad (\mathrm{W}(i,j))
$$

for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, \ell - 1\}$. Equation $(\mathrm{Z}(\alpha, i, j))$ means more explicitly that

$$
\begin{aligned}
&\mu_\alpha^{(i,j)}(y, f_1, \ldots, f_k, p_1(x, x, y), \ldots, p_m(x, x, y)) \\
&= \lambda_\alpha^{(i,j)}(y, f_1, \ldots, f_k, p_1(x, x, y), \ldots, p_m(x, x, y))
\end{aligned}
\qquad (\mathrm{Z}(\alpha, i, j)\mathrm{a})
$$

and

$$
\begin{aligned}
&\mu_\alpha^{(i,j)}(x, g_1, \ldots, g_k, p_1(x, y, y), \ldots, p_m(x, y, y)) \\
&= \lambda_\alpha^{(i,j)}(x, g_1, \ldots, g_k, p_1(x, y, y), \ldots, p_m(x, y, y))
\end{aligned}
\qquad (\mathrm{Z}(\alpha, i, j)\mathrm{b})
$$

for all $i \in \{0, \ldots, n\}$, $j \in \{0, \ldots, \ell\}$ and $\alpha \in \{1, 2\}$. We display the Equations $(\mathrm{B1}')$, $(\mathrm{B2}')$, $(\mathrm{T}(i)')$, $(\mathrm{S}(i)')$ and $(\mathrm{W}(i,j))$ in the diagram

$$(14)$$

where

$$
\begin{aligned}
\tau_{(i,j)}[[\mu]] &:= \tau_{(i,j)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \mu_1^{(i,j)}(u, \vec{v}, \vec{w}), \mu_2^{(i,j)}(u', \vec{v}', \vec{w}')), \\
\tau_{(i,j)}[[\lambda]] &:= \tau_{(i,j)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \lambda_1^{(i,j)}(u, \vec{v}, \vec{w}), \lambda_2^{(i,j)}(u', \vec{v}', \vec{w}')),
\end{aligned}
$$

$$t'_i[[1,2]] := t'_i(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}'),$$
$$t'_i[[2,1]] := t'_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w}).$$

In comparison to $t'_i[[1,2]]$, the second occurrences of $\vec{w}$ and $\vec{w}'$, respectively, are swapped in $t'_i[[2,1]]$. Furthermore, for each $\tau_{(i,j)}$, the terms $\mu_1^{(i,j)}, \mu_2^{(i,j)}, \lambda_1^{(i,j)}, \lambda_2^{(i,j)}$ satisfy the Equations $(Z(\alpha,i,j)a)$ and $(Z(\alpha,i,j)b)$. We call an equation of form (W) if it is of the form

$$\omega(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \gamma_1(u, \vec{v}, \vec{w}), \gamma_2(u', \vec{v}', \vec{w}'))$$
$$=\chi(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \delta_1(u, \vec{v}, \vec{w}), \delta_2(u', \vec{v}', \vec{w}')),$$

of form (T) if it is of the form

$$\xi(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}')$$
$$=\omega(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \gamma_1(u, \vec{v}, \vec{w}), \gamma_2(u', \vec{v}', \vec{w}')),$$

and of form (S) if it is of the form

$$\xi(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w})$$
$$=\chi(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \delta_1(u, \vec{v}, \vec{w}), \delta_2(u', \vec{v}', \vec{w}'))$$

where ω, χ, ξ and $\gamma_1, \gamma_2, \delta_1, \delta_2$ are arbitrary terms with appropriate arity. In comparison to an equation of form (T), the second occurrences of the variables $\vec{w}$ and $\vec{w}'$ in the first line, respectively, are swapped in an equation of form (S). Then all Equations $(W(i,j))$, $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, \ell-1\}$, are of form (W), all Equations $(T(i)')$, $i \in \{1, \ldots, n\}$, are of form (T), and all Equations $(S(i)')$, $i \in \{1, \ldots, n\}$, are of form (S), and we transform Diagram (14) for illustrative reasons into the following diagram:

$$\underline{(B1')} \quad \underline{(W)} \quad \cdots \quad \underline{(W)} \quad \underline{(T)} \quad \underline{(S)} \quad \underline{(W)} \quad \cdots \quad \underline{(W)} \quad \underline{(T)} \quad \underline{(S)} \quad \underline{(W)} \quad \cdots \quad \underline{(W)} \quad \underline{(B2')}$$

We note that here the first line (B1') encodes the equality of u and $\tau_{(0,0)}[[\mu]]$, the first of the lines (W) encodes the equality of $\tau_{(0,0)}[[\lambda]]$ and $\tau_{(0,1)}[[\mu]]$, and so on. We show that we can replace each equation of form (W) by an equation of form (T) followed by an equation of form (S) in the following sense. Then we get the diagram

where, following the dashed lines, equations of form (T) and of form (S) alternate. More precisely, we show that, for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, \ell-1\}$, Equation $(W(i,j))$ implies the existence of a $(2(k+2m+1))$-ary term $\pi_{(i,j)}$ such that

$$\tau_{(i,j)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \lambda_1^{(i,j)}(u, \vec{v}, \vec{w}), \lambda_2^{(i,j)}(u', \vec{v}', \vec{w}'))$$
$$= \pi_{(i,j)}(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}'), \tag{T(i,j)'}$$

$$\pi_{(i,j)}(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w})$$
$$= \tau_{(i,j+1)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \mu_1^{(i,j+1)}(u, \vec{v}, \vec{w}), \mu_2^{(i,j+1)}(u', \vec{v}', \vec{w}')). \tag{S(i,j)'}$$

Equation $(\mathrm{T}(i,j)')$ is of form (T) and Equation $(\mathrm{S}(i,j)')$ is of form (S). One sees immediately that the choice

$$\pi_{(i,j)}(u,\vec{v},\vec{w},\vec{\tilde{w}},u',\vec{v}',\vec{w}',\vec{\tilde{w}}) := \tau_{(i,j)}(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\lambda_1^{(i,j)}(u,\vec{v},\vec{w}),\lambda_2^{(i,j)}(u',\vec{v}',\vec{w}'))$$

fulfills both equations. By setting $N := n(\ell+1)+\ell$,

$$\sigma_{i(\ell+1)+j+1} := \tau_{(i,j)} \text{ for all } i \in \{0,\dots,n\},\ j \in \{0,\dots,\ell\},$$
$$s_{i(\ell+1)+j+1} := \pi_{(i,j)} \text{ for all } i \in \{0,\dots,n\},\ j \in \{0,\dots,\ell-1\},$$
$$s_{i(\ell+1)} := t'_i \text{ for all } i \in \{1,\dots,n\},$$
$$\eta_\alpha^{(i(\ell+1)+j+1)} := \mu_\alpha^{(i,j)} \text{ for all } i \in \{0,\dots,n\},\ j \in \{0,\dots,\ell\},\ \alpha \in \{1,2\},$$
$$\epsilon_\alpha^{(i(\ell+1)+j+1)} := \lambda_\alpha^{(i,j)} \text{ for all } i \in \{0,\dots,n\},\ j \in \{0,\dots,\ell\},\ \alpha \in \{1,2\},$$

we get an integer $N \geqslant 1$, $(2(k+m+2))$-ary terms $\sigma_1,\dots,\sigma_{N+1}$, $(2(k+2m+1))$-ary terms $s_1,\dots,s_N$ and $(k+m+1)$-ary terms $\eta_1^{(i)},\eta_2^{(i)},\epsilon_1^{(i)},\epsilon_2^{(i)}$, for all $i \in \{1,\dots,N+1\}$, satisfying, by construction, all the desired equations.

Now we show that the existence of the terms as in the statement implies that $\mathbb{V}$ is a weakly Mal'tsev category. Equation (7) means that $(f_i(x,y),g_i(x,y)) \in P$ for all $i \in \{1,\dots,k\}$. Hence Equation (10a) implies that

$$q_1(y,x)$$
$$= \sigma_1\Big(q_1(y,x),q_1(f_1,g_1),\dots,q_1(f_k,g_k),q_1(p_1(x,x,y),p_1(x,y,y)),\dots,q_1(p_m(x,x,y),p_m(x,y,y)),$$
$$q_2(y,x),q_2(f_1,g_1),\dots,q_2(f_k,g_k),q_2(p_1(x,x,y),p_1(x,y,y)),\dots,q_2(p_m(x,x,y),p_m(x,y,y)),$$
$$\eta_1^{(1)}\Big(q_1(y,x),q_1(f_1,g_1),\dots,q_1(f_k,g_k),q_1(p_1(x,x,y),p_1(x,y,y)),\dots,q_1(p_m(x,x,y),p_m(x,y,y))\Big),$$
$$\eta_2^{(1)}\Big(q_2(y,x),q_2(f_1,g_1),\dots,q_2(f_k,g_k),q_2(p_1(x,x,y),p_1(x,y,y)),\dots,q_2(p_m(x,x,y),p_m(x,y,y))\Big)\Big),$$

where we write (f_i,g_i) as shorter notation for $(f_i(x,y),g_i(x,y))$. Furthermore, we have, for all $i \in \{1,\dots,N+1\}$ and $\alpha \in \{1,2\}$, that

$$\eta_\alpha^{(i)}\Big(q_\alpha(y,x),q_\alpha(f_1,g_1),\dots,q_\alpha(f_k,g_k),q_\alpha(p_1(x,x,y),p_1(x,y,y)),\dots,q_\alpha(p_m(x,x,y),p_m(x,y,y))\Big)$$
$$=q_\alpha\Big(\eta_\alpha^{(i)}\big((y,x),(f_1,g_1),\dots,(f_k,g_k),(p_1(x,x,y),p_1(x,y,y)),\dots,(p_m(x,x,y),p_m(x,y,y))\big)\Big)$$
$$=q_\alpha\Big(\epsilon_\alpha^{(i)}\big((y,x),(f_1,g_1),\dots,(f_k,g_k),(p_1(x,x,y),p_1(x,y,y)),\dots,(p_m(x,x,y),p_m(x,y,y))\big)\Big)$$
$$=\epsilon_\alpha^{(i)}\Big(q_\alpha(y,x),q_\alpha(f_1,g_1),\dots,q_\alpha(f_k,g_k),q_\alpha(p_1(x,x,y),p_1(x,y,y)),\dots,q_\alpha(p_m(x,x,y),p_m(x,y,y))\Big),$$

where we used Equations (8) and the definition of the algebra structure on $P \subseteq \mathsf{F}(x,y) \times$

$\mathsf{F}(x, y)$. Thus, we have that

$$
\sigma_i\Big(q_1\big(y, x\big), q_1\big(f_1, g_1\big), \ldots, q_1\big(f_k, g_k\big), q_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_1\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
q_2\big(y, x\big), q_2\big(f_1, g_1\big), \ldots, q_2\big(f_k, g_k\big), q_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_2\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
\eta_1^{(i)}\Big(q_1\big(y, x\big), q_1\big(f_1, g_1\big), \ldots, q_1\big(f_k, g_k\big), q_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_1\big(p_m(x, x, y), p_m(x, y, y)\big)\Big),
$$
$$
\eta_2^{(i)}\Big(q_2\big(y, x\big), q_2\big(f_1, g_1\big), \ldots, q_2\big(f_k, g_k\big), q_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_2\big(p_m(x, x, y), p_m(x, y, y)\big)\Big)\Big)
$$
$$
= \sigma_i\Big(q_1\big(y, x\big), q_1\big(f_1, g_1\big), \ldots, q_1\big(f_k, g_k\big), q_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_1\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
q_2\big(y, x\big), q_2\big(f_1, g_1\big), \ldots, q_2\big(f_k, g_k\big), q_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_2\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
\epsilon_1^{(i)}\Big(q_1\big(y, x\big), q_1\big(f_1, g_1\big), \ldots, q_1\big(f_k, g_k\big), q_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_1\big(p_m(x, x, y), p_m(x, y, y)\big)\Big),
$$
$$
\epsilon_2^{(i)}\Big(q_2\big(y, x\big), q_2\big(f_1, g_1\big), \ldots, q_2\big(f_k, g_k\big), q_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_2\big(p_m(x, x, y), p_m(x, y, y)\big)\Big)\Big)
$$

for all $i \in \{1, \ldots, N + 1\}$. Next we have that

$$
\sigma_i\Big(q_1\big(y, x\big), q_1\big(f_1, g_1\big), \ldots, q_1\big(f_k, g_k\big), q_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_1\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
q_2\big(y, x\big), q_2\big(f_1, g_1\big), \ldots, q_2\big(f_k, g_k\big), q_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_2\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
\epsilon_1^{(i)}\Big(q_1\big(y, x\big), q_1\big(f_1, g_1\big), \ldots, q_1\big(f_k, g_k\big), q_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_1\big(p_m(x, x, y), p_m(x, y, y)\big)\Big),
$$
$$
\epsilon_2^{(i)}\Big(q_2\big(y, x\big), q_2\big(f_1, g_1\big), \ldots, q_2\big(f_k, g_k\big), q_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_2\big(p_m(x, x, y), p_m(x, y, y)\big)\Big)\Big)
$$
$$
= s_i\Big(q_1\big(y, x\big), q_1\big(f_1, g_1\big), \ldots, q_1\big(f_k, g_k\big), q_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_1\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
q_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_1\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
q_2\big(y, x\big), q_2\big(f_1, g_1\big), \ldots, q_2\big(f_k, g_k\big), q_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_2\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
q_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_2\big(p_m(x, x, y), p_m(x, y, y)\big)\Big)
$$
$$
= s_i\Big(q_1\big(y, x\big), q_1\big(f_1, g_1\big), \ldots, q_1\big(f_k, g_k\big), q_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_1\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
q_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_2\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
q_2\big(y, x\big), q_2\big(f_1, g_1\big), \ldots, q_2\big(f_k, g_k\big), q_2\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_2\big(p_m(x, x, y), p_m(x, y, y)\big),
$$
$$
q_1\big(p_1(x, x, y), p_1(x, y, y)\big), \ldots, q_1\big(p_m(x, x, y), p_m(x, y, y)\big)\Big)
$$

$$= \sigma_{i+1}\Big(q_1(y,x), q_1(f_1,g_1), \ldots, q_1(f_k,g_k), q_1(p_1(x,x,y), p_1(x,y,y)), \ldots, q_1(p_m(x,x,y), p_m(x,y,y)),$$

$$q_2(y,x), q_2(f_1,g_1), \ldots, q_2(f_k,g_k), q_2(p_1(x,x,y), p_1(x,y,y)), \ldots, q_2(p_m(x,x,y), p_m(x,y,y)),$$

$$\eta_1^{(i+1)}\Big(q_1(y,x), q_1(f_1,g_1), \ldots, q_1(f_k,g_k), q_1(p_1(x,x,y), p_1(x,y,y)), \ldots, q_1(p_m(x,x,y), p_m(x,y,y))\Big),$$

$$\eta_2^{(i+1)}\Big(q_2(y,x), q_2(f_1,g_1), \ldots, q_2(f_k,g_k), q_2(p_1(x,x,y), p_1(x,y,y)), \ldots, q_2(p_m(x,x,y), p_m(x,y,y))\Big)\Big)$$

for all $i \in \{1,\ldots,N\}$, where we applied Equation (9a) in the first step. In the second step we used that

$$q_1(p_i(x,x,y), p_i(x,y,y)) = q_2(p_i(x,x,y), p_i(x,y,y)) \tag{15}$$

for all $i \in \{1,\ldots,m\}$, which is true by Equation (6) and the definition of (q_1,q_2) as the cokernel pair of $[e_1,e_2]$. In the third step we utilized Equation (9b). Equation (10b) implies that

$$\sigma_{N+1}\Big(q_1(y,x), q_1(f_1,g_1), \ldots, q_1(f_k,g_k), q_1(p_1(x,x,y), p_1(x,y,y)), \ldots, q_1(p_m(x,x,y), p_m(x,y,y)),$$

$$q_2(y,x), q_2(f_1,g_1), \ldots, q_2(f_k,g_k), q_2(p_1(x,x,y), p_1(x,y,y)), \ldots, q_2(p_m(x,x,y), p_m(x,y,y)),$$

$$\epsilon_1^{(N+1)}\Big(q_1(y,x), q_1(f_1,g_1), \ldots, q_1(f_k,g_k), q_1(p_1(x,x,y), p_1(x,y,y)), \ldots, q_1(p_m(x,x,y), p_m(x,y,y))\Big),$$

$$\epsilon_2^{(N+1)}\Big(q_2(y,x), q_2(f_1,g_1), \ldots, q_2(f_k,g_k), q_2(p_1(x,x,y), p_1(x,y,y)), \ldots, q_2(p_m(x,x,y), p_m(x,y,y))\Big)\Big)$$

$$= q_2(y,x).$$

Combining the above identities, we get that $q_1(y,x) = q_2(y,x)$ which means that $\mathbb{V}$ is a weakly Mal'tsev category by Lemma 3.4. ∎

3.6. EXAMPLE. We consider the special case $N = 0$ in Theorem 3.5, i.e., we suppose that the variety $\mathbb{V}$ admits binary terms $f_1, g_1, \ldots, f_k, g_k$, ternary terms $p_1, \ldots, p_m$, where $k, m \geqslant 0$ are integers, a $(2(k+m+2))$-ary term σ_1 and $(k+m+1)$-ary terms $\eta_1^{(1)}, \eta_2^{(1)}, \epsilon_1^{(1)}, \epsilon_2^{(1)}$ that fulfill all the required conditions. Using Equations (10), Equation (8a) for $\eta_1^{(1)}$ and $\epsilon_1^{(1)}$, and Equation (8b) for $\eta_2^{(1)}$ and $\epsilon_2^{(1)}$, we get that the identity

$$y = \sigma_1(y, f_1, \ldots, f_k, p_1(x,x,y), \ldots, p_m(x,x,y), x, g_1, \ldots, g_k, p_1(x,y,y), \ldots, p_m(x,y,y),$$

$$\eta_1^{(1)}(y, f_1, \ldots, f_k, p_1(x,x,y), \ldots, p_m(x,x,y)),$$

$$\eta_2^{(1)}(x, g_1, \ldots, g_k, p_1(x,y,y), \ldots, p_m(x,y,y)))$$

$$= \sigma_1(y, f_1, \ldots, f_k, p_1(x,x,y), \ldots, p_m(x,x,y), x, g_1, \ldots, g_k, p_1(x,y,y), \ldots, p_m(x,y,y),$$

$$\epsilon_1^{(1)}(y, f_1, \ldots, f_k, p_1(x,x,y), \ldots, p_m(x,x,y)),$$

$$\epsilon_2^{(1)}(x, g_1, \ldots, g_k, p_1(x,y,y), \ldots, p_m(x,y,y)))$$

$$= x$$

holds in $\mathbb{V}$. In particular, $\mathbb{V}$ is a Mal'tsev variety with Mal'tsev term $p(x,y,z) := x$.

3.7. EXAMPLE. [Distributive lattices] In [22], it is shown that a variety $\mathbb{L}$ of lattices is a weakly Mal'tsev category if and only if it is a variety of distributive lattices. For such a variety $\mathbb{L}$, we want to find integers $k, m, N \geqslant 0$ and terms $f_1, g_1, \ldots, f_k, g_k, p_1, \ldots, p_m,$ $s_1, \ldots, s_N,$ $\sigma_1, \ldots, \sigma_{N+1}$ and $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ for all $i \in \{1, \ldots, N + 1\}$ that fulfill the conditions of Theorem 3.5. We observe that we can perform the following calculation in $\mathbb{L}$ to show that $q_1(y, x) = q_2(y, x)$:

$$
\begin{aligned}
&q_1(y, x) \\
=\,&q_1(y, x) \wedge (q_1(y, x) \vee q_1(x, y)) && \text{(absorption law)} \\
=\,&q_1(y, x) \wedge (q_1(y, y) \vee q_1(x, x)) && (q_1 \text{ morphism and commutativity of } \vee) \\
=\,&q_1(y, x) \wedge (q_2(y, y) \vee q_2(x, x)) && (q_1(y, y) = q_2(y, y) \text{ and } q_1(x, x) = q_2(x, x)) \\
=\,&q_1(y, x) \wedge (q_2(y, x) \vee q_2(x, y)) && (q_2 \text{ morphism and commutativity of } \vee) \\
=\,&q_1(y, x) \wedge (q_2(y, x) \vee q_1(x, y)) && (q_2(x, y) = q_1(x, y)) \\
=\,&(q_1(y, x) \wedge q_2(y, x)) \vee (q_1(y, x) \wedge q_1(x, y)) && \text{(distributivity of } \wedge \text{ over } \vee) \\
=\,&(q_1(y, x) \wedge q_2(y, x)) \vee (q_1(y, y) \wedge q_1(x, x)) && (q_1 \text{ morphism and commutativity of } \wedge) \\
=\,&(q_2(y, x) \wedge q_1(y, x)) \vee (q_1(y, y) \wedge q_1(x, x)) && \text{(commutativity of } \wedge) \\
=\,&(q_2(y, x) \wedge q_1(y, x)) \vee (q_2(y, y) \wedge q_2(x, x)) && (q_1(y, y) = q_2(y, y) \text{ and } q_1(x, x) = q_2(x, x)) \\
=\,&(q_2(y, x) \wedge q_1(y, x)) \vee (q_2(y, x) \wedge q_2(x, y)) && (q_2 \text{ morphism and commutativity of } \wedge) \\
=\,&q_2(y, x) \wedge (q_1(y, x) \vee q_2(x, y)) && \text{(distributivity of } \wedge \text{ over } \vee) \\
=\,&q_2(y, x) \wedge (q_1(y, x) \vee q_1(x, y)) && (q_2(x, y) = q_1(x, y)) \\
=\,&q_2(y, x) \wedge (q_1(y, y) \vee q_1(x, x)) && (q_1 \text{ morphism and commutativity of } \vee) \\
=\,&q_2(y, x) \wedge (q_2(y, y) \vee q_2(x, x)) && (q_1(y, y) = q_2(y, y) \text{ and } q_1(x, x) = q_2(x, x)) \\
=\,&q_2(y, x) \wedge (q_2(y, x) \vee q_2(x, y)) && (q_2 \text{ morphism and commutativity of } \vee) \\
=\,&q_2(y, x). && \text{(absorption law)}
\end{aligned}
$$

In the proof of Theorem 3.5, we showed, given terms satisfying the conditions as in the statement, that $q_1(y, x) = q_2(y, x)$ in the following way:

$$
\begin{aligned}
&q_1(y, x) \\
=\,&\sigma_1(\eta) && \text{(Equation (10a))} \\
=\,&\sigma_1(\epsilon) && ((\text{M}) \text{ and Equations (8) for } i = 1) \\
=\,&s_1(1, 2) && (\text{Equation (9a) for } i = 1) \\
=\,&s_1(2, 1) && (\text{Equation (15))} \\
=\,&\sigma_2(\eta) && (\text{Equation (9b) for } i = 1) \\
=\,&\sigma_2(\epsilon) && ((\text{M}) \text{ and Equations (8) for } i = 2) \\
=\,&s_2(1, 2) && (\text{Equation (9a) for } i = 2) \\
=\,&s_2(2, 1) && (\text{Equation (15))}
\end{aligned}
$$

$$=\sigma_3(\eta) \hspace{4cm} \text{(Equation (9b) for } i=2)$$

$$\vdots$$

$$=\sigma_{N+1}(\epsilon) \hspace{3cm} \text{((M) and Equations (8) for } i=N+1)$$

$$=q_2(y,x), \hspace{4cm} \text{(Equation (10b))}$$

where we set

$$\sigma_i(\eta) := \sigma_i\Big(q_1(y,x), q_1(f_1,g_1), \ldots, q_1(p_1(x,x,y), p_1(x,y,y)), \ldots,$$
$$q_2(y,x), q_2(f_1,g_1), \ldots, q_2(p_1(x,x,y), p_1(x,y,y)), \ldots,$$
$$\eta_1^{(i)}\big(q_1(y,x), q_1(f_1,g_1), \ldots, q_1(p_1(x,x,y), p_1(x,y,y)), \ldots\big),$$
$$\eta_2^{(i)}\big(q_2(y,x), q_2(f_1,g_1), \ldots, q_2(p_1(x,x,y), p_1(x,y,y)), \ldots\big)\Big),$$

$$\sigma_i(\epsilon) := \sigma_i\Big(q_1(y,x), q_1(f_1,g_1), \ldots, q_1(p_1(x,x,y), p_1(x,y,y)), \ldots,$$
$$q_2(y,x), q_2(f_1,g_1), \ldots, q_2(p_1(x,x,y), p_1(x,y,y)), \ldots,$$
$$\epsilon_1^{(i)}\big(q_1(y,x), q_1(f_1,g_1), \ldots, q_1(p_1(x,x,y), p_1(x,y,y)), \ldots\big),$$
$$\epsilon_2^{(i)}\big(q_2(y,x), q_2(f_1,g_1), \ldots, q_2(p_1(x,x,y), p_1(x,y,y)), \ldots\big)\Big),$$

$$s_j(1,2) := s_i\Big(q_1(y,x), q_1(f_1,g_1), \ldots,$$
$$q_1(p_1(x,x,y), p_1(x,y,y)), \ldots,$$
$$q_1(p_1(x,x,y), p_1(x,y,y)), \ldots,$$
$$q_2(y,x), q_2(f_1,g_1), \ldots,$$
$$q_2(p_1(x,x,y), p_1(x,y,y)), \ldots,$$
$$q_2(p_1(x,x,y), p_1(x,y,y)), \ldots\Big),$$

$$s_j(2,1) := s_i\Big(q_1(y,x), q_1(f_1,g_1), \ldots,$$
$$q_1(p_1(x,x,y), p_1(x,y,y)), \ldots,$$
$$q_2(p_1(x,x,y), p_1(x,y,y)), \ldots,$$
$$q_2(y,x), q_2(f_1,g_1), \ldots,$$
$$q_2(p_1(x,x,y), p_1(x,y,y)), \ldots,$$
$$q_1(p_1(x,x,y), p_1(x,y,y)), \ldots\Big)$$

for all $i \in \{1, \ldots, N+1\}$ and all $j \in \{1, \ldots, N\}$ and (M) means that we used that q_1 and q_2 are morphisms. Hence our ansatz is that the first step, where we use the absorption law, should correspond to the step where we use Equation (10a). And the last step, where we use again the absorption law, should correspond to the step where we use Equation (10b). A step in our concrete calculation for distributive lattices where we use an equation should either correspond to a step from $\sigma_i(\epsilon)$ to $s_i(1,2)$ or from $s_i(2,1)$ to $\sigma_{i+1}(\eta)$ for some i. A

step where we use that q_1 or q_2 is a morphism and an equation should correspond to a step from $\sigma_i(\eta)$ to $\sigma_i(\epsilon)$. A step where we use that $q_1(y,y) = q_2(y,y)$, $q_1(x,x) = q_2(x,x)$ or $q_1(x,y) = q_2(x,y)$ should correspond to a step from $s_i(1,2)$ to $s_i(2,1)$. For $N = 5$ we found integers $k, m \geq 0$ and terms $f_1, g_1, \ldots, f_k, g_k$, $p_1, \ldots, p_m$, $s_1, \ldots, s_5$, $\sigma_1, \ldots, \sigma_6$ and $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ for all $i \in \{1, \ldots, 6\}$ such that

$$q_1(y,x) \wedge (q_1(y,x) \vee q_1(x,y)) = \sigma_1(\eta),$$
$$q_1(y,x) \wedge (q_1(y,y) \vee q_1(x,x)) = \sigma_1(\epsilon) = s_1(1,2),$$
$$q_1(y,x) \wedge (q_2(y,y) \vee q_2(x,x)) = s_1(2,1) = \sigma_2(\eta),$$
$$q_1(y,x) \wedge (q_2(y,x) \vee q_2(x,y)) = \sigma_2(\epsilon) = s_2(1,2),$$
$$q_1(y,x) \wedge (q_2(y,x) \vee q_1(x,y)) = s_2(2,1),$$
$$(q_1(y,x) \wedge q_2(y,x)) \vee (q_1(y,x) \wedge q_1(x,y)) = \sigma_3(\eta),$$
$$(q_1(y,x) \wedge q_2(y,x)) \vee (q_1(y,y) \wedge q_1(x,x)) = \sigma_3(\epsilon),$$
$$(q_2(y,x) \wedge q_1(y,x)) \vee (q_1(y,y) \wedge q_1(x,x)) = s_3(1,2),$$
$$(q_2(y,x) \wedge q_1(y,x)) \vee (q_2(y,y) \wedge q_2(x,x)) = s_3(2,1) = \sigma_4(\eta),$$
$$(q_2(y,x) \wedge q_1(y,x)) \vee (q_2(y,x) \wedge q_2(x,y)) = \sigma_4(\epsilon),$$
$$q_2(y,x) \wedge (q_1(y,x) \vee q_2(x,y)) = s_4(1,2),$$
$$q_2(y,x) \wedge (q_1(y,x) \vee q_1(x,y)) = s_4(2,1) = \sigma_5(\eta),$$
$$q_2(y,x) \wedge (q_1(y,y) \vee q_1(x,x)) = \sigma_5(\epsilon) = s_5(1,2),$$
$$q_2(y,x) \wedge (q_2(y,y) \vee q_2(x,x)) = s_5(2,1) = \sigma_6(\eta),$$
$$q_2(y,x) \wedge (q_2(y,x) \vee q_2(x,y)) = \sigma_6(\epsilon),$$

and the conditions of Theorem 3.5 are satisfied. One can check that $k := 0$, $m := 3$ and the following terms fulfill the requirements:

$$p_1(x,y,z) := x, \qquad p_2(x,y,z) := y, \qquad p_3(x,y,z) := z,$$

$$s_1(u, w_1, w_2, w_3, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, u', w_1', w_2', w_3', \tilde{w}_1', \tilde{w}_2', \tilde{w}_3') := u \wedge (\tilde{w}_3 \vee \tilde{w}_1),$$
$$s_2(u, w_1, w_2, w_3, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, u', w_1', w_2', w_3', \tilde{w}_1', \tilde{w}_2', \tilde{w}_3') := u \wedge (u' \vee \tilde{w}_2'),$$
$$s_3(u, w_1, w_2, w_3, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, u', w_1', w_2', w_3', \tilde{w}_1', \tilde{w}_2', \tilde{w}_3') := (u' \wedge u) \vee (\tilde{w}_3 \wedge \tilde{w}_1),$$
$$s_4(u, w_1, w_2, w_3, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, u', w_1', w_2', w_3', \tilde{w}_1', \tilde{w}_2', \tilde{w}_3') := u' \wedge (u \vee \tilde{w}_2'),$$
$$s_5(u, w_1, w_2, w_3, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, u', w_1', w_2', w_3', \tilde{w}_1', \tilde{w}_2', \tilde{w}_3') := u' \wedge (\tilde{w}_3 \vee \tilde{w}_1),$$

$$\sigma_1(u, w_1, w_2, w_3, u', w_1', w_2', w_3', a, b) := u \wedge a,$$
$$\sigma_2(u, w_1, w_2, w_3, u', w_1', w_2', w_3', a, b) := u \wedge b,$$
$$\sigma_3(u, w_1, w_2, w_3, u', w_1', w_2', w_3', a, b) := (u \wedge u') \vee a,$$
$$\sigma_4(u, w_1, w_2, w_3, u', w_1', w_2', w_3', a, b) := (u' \wedge u) \vee b,$$

$$\sigma_5(u, w_1, w_2, w_3, u', w_1', w_2', w_3', a, b) := u' \wedge a,$$
$$\sigma_6(u, w_1, w_2, w_3, u', w_1', w_2', w_3', a, b) := u' \wedge b,$$

$$\eta_1^{(1)}(u, w_1, w_2, w_3) := u \vee w_2, \qquad \epsilon_1^{(1)}(u, w_1, w_2, w_3) := w_3 \vee w_1,$$
$$\eta_2^{(1)}(u, w_1, w_2, w_3) := u, \qquad \epsilon_2^{(1)}(u, w_1, w_2, w_3) := u,$$
$$\eta_1^{(2)}(u, w_1, w_2, w_3) := u, \qquad \epsilon_1^{(2)}(u, w_1, w_2, w_3) := u,$$
$$\eta_2^{(2)}(u, w_1, w_2, w_3) := w_3 \vee w_1, \qquad \epsilon_2^{(2)}(u, w_1, w_2, w_3) := u \vee w_2,$$
$$\eta_1^{(3)}(u, w_1, w_2, w_3) := u \wedge w_2, \qquad \epsilon_1^{(3)}(u, w_1, w_2, w_3) := w_3 \wedge w_1,$$
$$\eta_2^{(3)}(u, w_1, w_2, w_3) := u, \qquad \epsilon_2^{(3)}(u, w_1, w_2, w_3) := u,$$
$$\eta_1^{(4)}(u, w_1, w_2, w_3) := u, \qquad \epsilon_1^{(4)}(u, w_1, w_2, w_3) := u,$$
$$\eta_2^{(4)}(u, w_1, w_2, w_3) := w_3 \wedge w_1, \qquad \epsilon_2^{(4)}(u, w_1, w_2, w_3) := u \wedge w_2,$$
$$\eta_1^{(5)}(u, w_1, w_2, w_3) := u \vee w_2, \qquad \epsilon_1^{(5)}(u, w_1, w_2, w_3) := w_3 \vee w_1,$$
$$\eta_2^{(5)}(u, w_1, w_2, w_3) := u, \qquad \epsilon_2^{(5)}(u, w_1, w_2, w_3) := u,$$
$$\eta_1^{(6)}(u, w_1, w_2, w_3) := u, \qquad \epsilon_1^{(6)}(u, w_1, w_2, w_3) := u,$$
$$\eta_2^{(6)}(u, w_1, w_2, w_3) := w_3 \vee w_1, \qquad \epsilon_2^{(6)}(u, w_1, w_2, w_3) := u \vee w_2.$$

4. Another Mal'tsev-like property

In this section, we discuss the observation that the condition which is obtained from the syntactic characterization of weakly Mal'tsev varieties from Theorem 3.5 by dropping Equation (7) characterizes the varieties in which any reflexive regular relation is an equivalence relation. A *regular relation*, in the context of finitely complete categories, is a monomorphism $r \colon R \rightarrowtail X \times Y$ which is *regular*, i.e., an equalizer of a parallel pair of morphisms $f, g \colon X \times Y \to Z$.

We call a class $\mathcal{M}$ of binary relations in a finitely complete category $\mathfrak{C}$ *pullback-stable* if, for any relation $r \colon R \rightarrowtail X \times Y$ in $\mathcal{M}$ and any morphisms $f \colon A \to X$ and $g \colon B \to Y$, the relation $\rho \colon P \rightarrowtail A \times B$ given by any pullback

$$\begin{array}{ccc} P & \overset{\rho}{\rightarrowtail} & A \times B \\ {\scriptstyle p_R}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f \times g} \\ R & \underset{r}{\rightarrowtail} & X \times Y, \end{array}$$

lies also in $\mathcal{M}$. The classes $\mathcal{M}_{\mathrm{all}}$ of all binary relations in $\mathfrak{C}$, $\mathcal{M}_{\mathrm{strong}}$ of strong binary relations and $\mathcal{M}_{\mathrm{reg}}$ of regular binary relations are pullback-stable.

4.1. DEFINITION. [$\mathcal{M}$-Mal'tsev category] *Let $\mathfrak{C}$ be a finitely complete category and $\mathcal{M}$ be a pullback-stable class of binary relations in $\mathfrak{C}$. We call $\mathfrak{C}$ an $\mathcal{M}$-Mal'tsev category if any reflexive relation $r\colon R \rightarrowtail X \times X$ in $\mathcal{M}$ is an equivalence relation.*

Let $\mathfrak{C}$ be a finitely complete category. By Definition 2.4, it is $\mathcal{M}_{\text{all}}$-Mal'tsev if and only if it is a Mal'tsev category. Furthermore, by Proposition 2.8, it is $\mathcal{M}_{\text{strong}}$-Mal'tsev if and only if it is a weakly Mal'tsev category. In this section, we consider $\mathcal{M}_{\text{reg}}$-Mal'tsev categories.

The following proposition recovers Proposition 2.5 and parts of Proposition 2.8. In [17], the authors show similarly the following equivalences for $\mathcal{M}$ being a proper class of binary relations in a category with pullbacks and equalizers (see also [24], Theorem 1).

4.2. PROPOSITION. *Let $\mathfrak{C}$ be a finitely complete category and $\mathcal{M}$ be a pullback-stable class of binary relations in $\mathfrak{C}$. Then the following conditions are equivalent:*

1. *The category $\mathfrak{C}$ is an $\mathcal{M}$-Mal'tsev category.*

2. *Any reflexive relation in $\mathcal{M}$ is symmetric.*

3. *Any reflexive relation in $\mathcal{M}$ is transitive.*

4. *Any relation in $\mathcal{M}$ is difunctional.*

PROOF. The implications "1. $\Rightarrow$ 2." and "1. $\Rightarrow$ 3." are trivial. The implication "4. $\Rightarrow$ 1." holds since, as it is well-known, any reflexive relation is an equivalence relation if and only if it is difunctional. For the implication "2. $\Rightarrow$ 4.", we consider a relation $r = (r_1, r_2)\colon R \rightarrowtail X \times Y$ in $\mathcal{M}$. Since $\mathcal{M}$ is assumed to be pullback-stable, the relation $\rho\colon P \rightarrowtail R \times R$ in the pullback

$$
\begin{array}{ccc}
P & \xrightarrow{\ \rho\ } & R \times R \\
{\scriptstyle p_R}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle r_1 \times r_2} \\
R & \xrightarrow[\ r\ ]{} & X \times Y,
\end{array}
$$

lies also in $\mathcal{M}$. Furthermore, it is clear that it is reflexive. Hence, by assumption, ρ is symmetric. It is a classical result that ρ being symmetric is equivalent to r being difunctional. For the implication "3. $\Rightarrow$ 4.", we note that r is difunctional if and only if ρ is transitive. ∎

As in the preceding section, we will adapt a proof of the classical theorem of Mal'tsev to the case of $\mathcal{M}_{\text{reg}}$-Mal'tsev varieties. This time, we use the characterization of Mal'tsev categories as those finitely complete categories in which every reflexive relation is symmetric as recalled in Proposition 2.5. Let

$$
R \xrightarrow[\quad\quad]{\ r\ } \mathsf{F}(x,y) \times \mathsf{F}(x,y)
$$

be the smallest relation on $\mathsf{F}(x,y)$ which contains (x,x), (x,y) and (y,y). This means that R consists precisely of the elements

$$
p((x,x),(x,y),(y,y)) \in \mathsf{F}(x,y) \times \mathsf{F}(x,y)
$$

where $p \in \mathsf{F}(x, y, z)$. Hence

$$R = \{(p(x, x, y), p(x, y, y)) \mid p \in \mathsf{F}(x, y, z)\} \subseteq \mathsf{F}(x, y) \times \mathsf{F}(x, y). \tag{16}$$

4.3. LEMMA. *The variety $\mathbb{V}$ is Mal'tsev if and only if the relation R contains (y, x).*

PROOF. "$\Rightarrow$": Let $\mathbb{V}$ be a Mal'tsev variety. We see from Equation (16) that R is reflexive since we can write

$$(t(x, y), t(x, y)) = (p(x, x, y), p(x, y, y))$$

for any binary term $t(x, y) \in \mathsf{F}(x, y)$ if we set $p(x, y, z) := t(x, z)$. Hence R is symmetric and $(y, x) \in R$.

"$\Leftarrow$": Let $s \colon S \rightarrowtail A \times A$ be a reflexive relation on an algebra A. We show that if $(a, b) \in S$ then also $(b, a) \in S$. For this let $f \colon \mathsf{F}(x, y) \to A$ be the unique morphism such that $f(x) = a$ and $f(y) = b$. It is easy to see that there exists a morphism $\varphi \colon R \to S$ such that the diagram

$$\begin{array}{ccc}
R & \xrightarrow{\ \ r\ \ } & \mathsf{F}(x, y) \times \mathsf{F}(x, y) \\
\varphi \downarrow & & \downarrow f \times f \\
S & \xrightarrow{\ \ s\ \ } & A \times A,
\end{array} \tag{17}$$

commutes since R is the smallest relation which contains (x, x), (x, y) and (y, y) and $(f \times f)(x, x) = (a, a)$, $(f \times f)(x, y) = (a, b)$ and $(f \times f)(y, y) = (b, b)$ lie in S. The assumption $(y, x) \in R$ implies that $(b, a) = (f \times f)(y, x)$ is contained in S. ∎

4.4. LEMMA. *The relation R contains (y, x) if and only if there exists a ternary term $p \in \mathsf{F}(x, y, z)$ such that $p(x, x, y) = y$ and $p(x, y, y) = x$.*

PROOF. This is clear by Equation (16). ∎

To characterize the varieties which are $\mathcal{M}_{\mathrm{reg}}$-Mal'tsev, we consider the smallest *regular* relation on $\mathsf{F}(x, y)$ which contains (x, x), (x, y) and (y, y). This relation can be obtained by first constructing the cokernel pair (Q', q_1', q_2') of the relation R. The equalizer (E, e) of q_1', q_2' yields the desired regular relation and we get a unique morphism $j \colon R \to E$ such that $ej = r$.

$$\begin{array}{ccc}
 & E & \\
j \nearrow & & \searrow e \\
R \xrightarrow{\ \ r\ \ } & \mathsf{F}(x, y) \times \mathsf{F}(x, y) & \underset{q_2'}{\overset{q_1'}{\rightrightarrows}} Q'
\end{array}$$

4.5. LEMMA. *A variety $\mathbb{V}$ is $\mathcal{M}_{reg}$-Mal'tsev if and only if E contains (y, x).*

PROOF. "⇒": By definition of an $\mathcal{M}_{\mathrm{reg}}$-Mal'tsev category, the reflexive regular relation E is symmetric. Hence $(y, x) \in E$.

"⇐": Let A be an algebra in $\mathbb{V}$ and $s\colon S \rightarrowtail A \times A$ be a reflexive regular relation on A. We show that if $(a, b) \in S$ then also $(b, a) \in S$. We consider the extended version

of Diagram (17), where (Q'', q_1'', q_2'') is the cokernel pair of s and, by assumption, s is the equalizer of q_1'', q_2''. Furthermore, ψ is the unique map such that the upper and lower right-hand side squares of the diagram commute. Then

$$q_1''(f \times f)e = \psi q_1' e = \psi q_2' e = q_2''(f \times f)e,$$

and there exists a unique map $\omega\colon E \to S$ such that $s\omega = (f \times f)e$. Since $(y, x) \in E$ by assumption and $(f \times f)(y, x) = (b, a)$, it follows that $(b, a) \in S$. ∎

4.6. THEOREM. *A finitary one-sorted variety $\mathbb{V}$ of universal algebras is an $\mathcal{M}_{reg}$-Mal'tsev category if and only if there exist integers $k, m, N \geqslant 0$, binary terms $f_1, g_1, \ldots, f_k, g_k \in \mathsf{F}(x, y)$, ternary terms $p_1, \ldots, p_m \in \mathsf{F}(x, y, z)$, $(2(k + 2m + 1))$-ary terms $s_1, \ldots, s_N$, $(2(k + m + 2))$-ary terms $\sigma_1, \ldots, \sigma_{N+1}$ and, for all $i \in \{1, \ldots, N + 1\}$, $(k + m + 1)$-ary terms $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ such that the following identities (on variables x, y, u, u', $v_1, \ldots, v_k$, $v_1', \ldots, v_k'$, $w_1, \ldots, w_m$ and $w_1', \ldots, w_m'$) hold, where we write $\vec{v}$ for $v_1, \ldots, v_k$ and $\vec{w}$ for $w_1, \ldots, w_m$, and analogously for $\vec{v}'$ and $\vec{w}'$:*

$$\eta_\alpha^{(i)}(y, f_1(x, y), \ldots, f_k(x, y), p_1(x, x, y), \ldots, p_m(x, x, y))$$
$$= \epsilon_\alpha^{(i)}(y, f_1(x, y), \ldots, f_k(x, y), p_1(x, x, y), \ldots, p_m(x, x, y)), \tag{18a}$$

$$\eta_\alpha^{(i)}(x, g_1(x, y), \ldots, g_k(x, y), p_1(x, y, y), \ldots, p_m(x, y, y))$$
$$= \epsilon_\alpha^{(i)}(x, g_1(x, y), \ldots, g_k(x, y), p_1(x, y, y), \ldots, p_m(x, y, y)), \tag{18b}$$

for all $i \in \{1, \ldots, N + 1\}$ and $\alpha \in \{1, 2\}$;

$$\sigma_i(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(i)}(u, \vec{v}, \vec{w}), \epsilon_2^{(i)}(u', \vec{v}', \vec{w}'))$$
$$= s_i(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}'), \tag{19a}$$

$$s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w})$$
$$= \sigma_{i+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(i+1)}(u, \vec{v}, \vec{w}), \eta_2^{(i+1)}(u', \vec{v}', \vec{w}')), \tag{19b}$$

for all $i \in \{1, \ldots, N\}$ and

$$u = \sigma_1(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(1)}(u, \vec{v}, \vec{w}), \eta_2^{(1)}(u', \vec{v}', \vec{w}')), \tag{20a}$$
$$u' = \sigma_{N+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(N+1)}(u, \vec{v}, \vec{w}), \epsilon_2^{(N+1)}(u', \vec{v}', \vec{w}')). \tag{20b}$$

PROOF. By Lemma 4.5, we know that $\mathbb{V}$ is $\mathcal{M}_{\mathrm{reg}}$-Mal'tsev if and only if $(y, x) \in E$. To construct E, we first build the cokernel pair (Q', q_1', q_2') by means of the coequalizer q' of the maps $\iota_1' r, \iota_2' r$, where ι_1', ι_2' are the coproduct inclusions into $\mathsf{F}(x, y)^2 + \mathsf{F}(x, y)^2$, as in the following diagram:
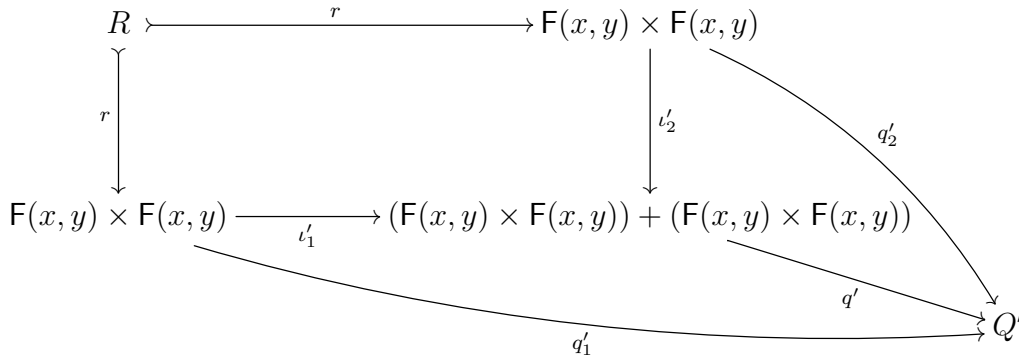
The algebra Q' is given by the quotient of $\mathsf{F}(x, y)^2 + \mathsf{F}(x, y)^2$, with respect to, due to Equation (16), the smallest congruence $C' \subseteq (\mathsf{F}(x, y)^2 + \mathsf{F}(x, y)^2)^2$ containing all pairs of the form

$$\Big(\iota_1'\big(p(x, x, y), p(x, y, y)\big), \iota_2'\big(p(x, x, y), p(x, y, y)\big)\Big),$$

where $p \in \mathsf{F}(x, y, z)$. The algebra E is then given by all pairs $(s, t) \in \mathsf{F}(x, y) \times \mathsf{F}(x, y)$ such that $q'(\iota_1'(s, t)) = q'(\iota_2'(s, t))$. Thus, the condition $(y, x) \in E$ is equivalent to $(\iota_1'(y, x), \iota_2'(y, x)) \in C'$. Proceeding from here analogously as in the proof of Theorem 3.5 yields the claim. We note that, in comparison to the proof of Theorem 3.5, here $P = \{(s(x, y), t(x, y)) \in \mathsf{F}(x, y) \times \mathsf{F}(x, y) \mid s(x, x) = t(x, x)\}$ is replaced by $\mathsf{F}(x, y) \times \mathsf{F}(x, y)$, which explains why the condition $f_i(x, x) = g_i(x, x)$ for all $i \in \{1, \ldots, k\}$ does not appear here. ∎

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