CONNECTING GENERALIZED PRIESTLEY DUALITY TO HOFMANN-MISLOVE-STRALKA DUALITY

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Abstract.   We connect Priestley duality for distributive lattices and its generalization to distributive meet-semilattices to Hofmann-Mislove-Stralka duality for semilattices. Among other things, this involves consideration of various morphisms between algebraic frames. We also show how Stone duality for boolean algebras and generalized boolean algebras fits as a particular case of the general picture we develop.

Contents

1. Introduction

The celebrated Stone duality [Sto36] establishes a dual equivalence between the categories of boolean algebras and what we now call Stone spaces (zero-dimensional compact Hausdorff spaces). Since Stone's groundbreaking work, numerous dualities have been developed for various categories of algebras. Stone himself generalized his duality for boolean algebras to distributive lattices [Sto37]. The resulting dual spaces are now known as spectral spaces and play an important role in algebraic geometry as Zariski spectra of commutative rings.

In [Pri70, Pri72] Priestley developed another duality for distributive lattices by means of certain ordered Stone spaces, which became known as Priestley spaces. They form a subcategory of the category of ordered topological spaces studied by Nachbin [Nac65] and have numerous applications in diverse areas such as natural dualities [CD98], for-

mal concept analysis [GW99, DP02], computer science [Pan13, Geh16], and modal logic [Gol89, CJ99].

There are various generalizations of Priestley duality. It was extended to all lattices by Urquhart [Urq78] (see also [Har92, Har97, HD97, Plo08, GvG14, MJ14a]). Duality theory for distributive lattices with operators and the theory of their canonical extensions was developed in [Gol89, GJ94, GJ04], and it was further generalized to lattices with operators in [GH01] (see also [MJ14b]). Our main interest is in generalized Priestley duality for distributive semilattices developed in [BJ08, BJ11] and [HP08]. Our aim is to connect this duality, as well as Priestley and Stone dualities, to the Pontryagin-style duality for semilattices developed by Hofmann, Mislove, and Stralka [HMS74], which we will refer to as *HMS duality*.

The classic Pontryagin duality (see, e.g., [HR94, Sec. 24]) states that the category of locally compact abelian groups is self-dual. As a corollary, the categories of abelian groups and compact Hausdorff abelian groups are dual to each other, as are the categories of torsion abelian groups and Stone abelian groups (see [HR94, Sec. 24] or [Joh82, Ch. VI(3,4)]). A version of Pontryagin duality for semilattices by Hofmann, Mislove, and Stralka [HMS74] states that the categories of meet-semilattices and Stone meet-semilattices are dual to each other. Since Stone meet-semilattices are exactly the algebraic lattices, at the object level this result is a reformulation of an earlier result of Nachbin [Nac49] (see also Birkhoff and Frink [BF48]) that there is a 1-1 correspondence between semilattices and algebraic lattices. The restriction of this duality to the distributive case yields that the categories of distributive semilattices and distributive algebraic lattices are equivalent. This provides an important link to pointfree topology [Joh82, PP12] since distributive algebraic lattices are exactly the algebraic frames.

There are various morphisms to consider between distributive meet-semilattices, which give rise to various morphisms between algebraic frames. The study of the resulting categories is one of our aims. In addition, we show that prime and pseudoprime elements of algebraic lattices, which have been extensively studied in domain theory in connection with continuous lattices [GHK+03], can be used to analyze the spectra of prime and optimal filters of distributive meet-semilattices that play a crucial role in generalized Priestley duality. It is this analysis that connects generalized Priestley duality to HMS duality.

By a meet-semilattice we mean a poset in which all finite meets exist, including the empty meet. Thus, a meet-semilattice M has a top element, but M may not have a bottom element. By HMS duality, the category MS of meet-semilattices is dual to the category StoneMS of Stone meet-semilattices (that is, topological meet-semilattices whose topology is a Stone topology). Since Stone meet-semilattices are exactly the algebraic lattices, working with left adjoints of StoneMS-morphisms yields the category AlgLat$_{Sup}$ of algebraic lattices and maps between them that preserve arbitrary joins and compact elements (see [GHK+03, p. 272]). Thus, HMS duality yields that MS is equivalent to AlgLat$_{Sup}$ (see [GHK+03, p. 274]). This equivalence is obtained by the functors

$$\mathscr{F}: \mathsf{MS} \to \mathsf{AlgLat_{Sup}} \quad \text{and} \quad \mathscr{K}: \mathsf{AlgLat_{Sup}} \to \mathsf{MS}.$$

The functor $\mathscr{F}$ sends each $M \in \mathsf{MS}$ to the algebraic lattice of its filters ordered by inclusion. The functor $\mathscr{K}$ sends each $L \in \mathsf{AlgLat_{Sup}}$ to the meet-semilattice of its compact elements ordered by the dual of the restriction of the order on L.

Let DMS be the full subcategory of MS consisting of distributive meet-semilattices. As we pointed out above, distributive algebraic lattices are exactly the algebraic frames, and we denote by $\mathsf{AlgFrm_{Sup}}$ the full subcategory of $\mathsf{AlgLat_{Sup}}$ consisting of algebraic frames. We point out that morphisms in $\mathsf{AlgFrm_{Sup}}$ preserve arbitrary suprema, but may not be frame homomorphisms. Restricting the equivalence of MS and $\mathsf{AlgLat_{Sup}}$ to the distributive case yields that DMS is equivalent to $\mathsf{AlgFrm_{Sup}}$ (see Corollary 2.6).

A generalization of Priestley duality to distributive meet-semilattices was developed in [BJ08, BJ11]. The key ingredient of this duality is the notion of an optimal filter of $M \in \mathsf{DMS}$, which is best described by means of the distributive envelope of M. We present two versions of generalized Priestley duality, for distributive meet-semilattices with and without a bottom element. When the bottom is present, a DMS-morphism may or may not preserve it. This results in two dualities for bounded distributive meet-semilattices and generalized Priestley spaces (see Theorem 3.7). Each generalizes to a duality between distributive meet-semilattices and pointed generalized Priestley spaces with appropriate morphisms (see Theorem 3.11 and Corollary 3.13).

Our main observation in connecting generalized Priestley duality to HMS duality is that the categories $\mathsf{AlgFrm_{Sup}}$ and PGPS are dual to each other. This we do by constructing the contravariant functors

$$\mathscr{V}^a \colon \mathsf{PGPS} \to \mathsf{AlgFrm_{Sup}} \quad \text{and} \quad \mathscr{Y} \colon \mathsf{AlgFrm_{Sup}} \to \mathsf{PGPS}.$$

The functor $\mathscr{V}^a$ is a version of the upper Vietoris functor, which is constructed by working with admissible closed upsets of pointed generalized Priestley spaces (see Definition 4.1). The functor $\mathscr{Y}$ is constructed by working with pseudoprime and prime elements of algebraic frames. In Theorem 4.29 we prove that the functors $\mathscr{V}^a$ and $\mathscr{Y}$ establish the duality of PGPS and $\mathsf{AlgFrm_{Sup}}$. This together with the equivalence of $\mathsf{AlgFrm_{Sup}}$ and DMS yields the duality of DMS and PGPS. We also show how these results restrict to the bounded case.

It is natural to consider several stronger notions of morphisms between algebraic frames. Recalling that our morphisms preserve arbitrary suprema and compact elements, obvious choices are to consider those morphisms that preserve all finite infima (resp. nonempty finite infima) of compact elements or only those finite infima of compact elements that are compact. The latter correspond to those DMS-morphisms that preserve existing finite suprema (resp. existing nonempty finite suprema), which is equivalent to the inverse image of an optimal filter being optimal (see Remark 5.34). The former are simply frame homomorphisms that preserve compact elements, and correspond to those DMS-morphisms that pull prime filters back to prime filters (see Lemma 5.33). We characterize the PGPS-morphisms that correspond to these classes of morphisms between algebraic frames, thus yielding a series of duality results, which in particular imply the results of [BJ08, BJ11] and [HP08].

We also show how Stone and Priestley dualities fit in the general picture developed in this paper. We consider Priestley duality for distributive lattices with and without bottom. The latter involves working with pointed Priestley spaces. On the frame side, the bounded case requires working with coherent frames; that is, algebraic frames in which all finite meets of compact elements are compact (see Theorem 6.5). The non-bounded case requires working with arithmetic frames; that is, algebraic frames where nonempty finite meets of compact elements are compact, but the frame itself may not be compact (see Theorem 6.14). Similarly, we consider two versions of Stone duality, for boolean algebras and for generalized boolean algebras. For the latter we work with pointed Stone spaces. On the frame side, boolean algebras give rise to Stone frames (see Theorem 7.3), while generalized boolean algebras to locally Stone frames (see Theorem 7.13). As a special case we derive the dualities of Halmos [Hal56] and Cignoli et. al. [CLP91].

The following diagram summarizes the connection between HMS duality and generalized Priestley duality. The horizontal arrows below the labeled arrows are their restrictions. The arrows $\rightarrowtail$ represent being a subcategory while the arrows $\hookrightarrow$ being equivalent to a subcategory. The blue color indicates the results obtained in this paper. The corresponding diagrams summarizing a similar picture for various categories of distributive lattices and boolean algebras are given at the ends of Sections 6 and 7, respectively.
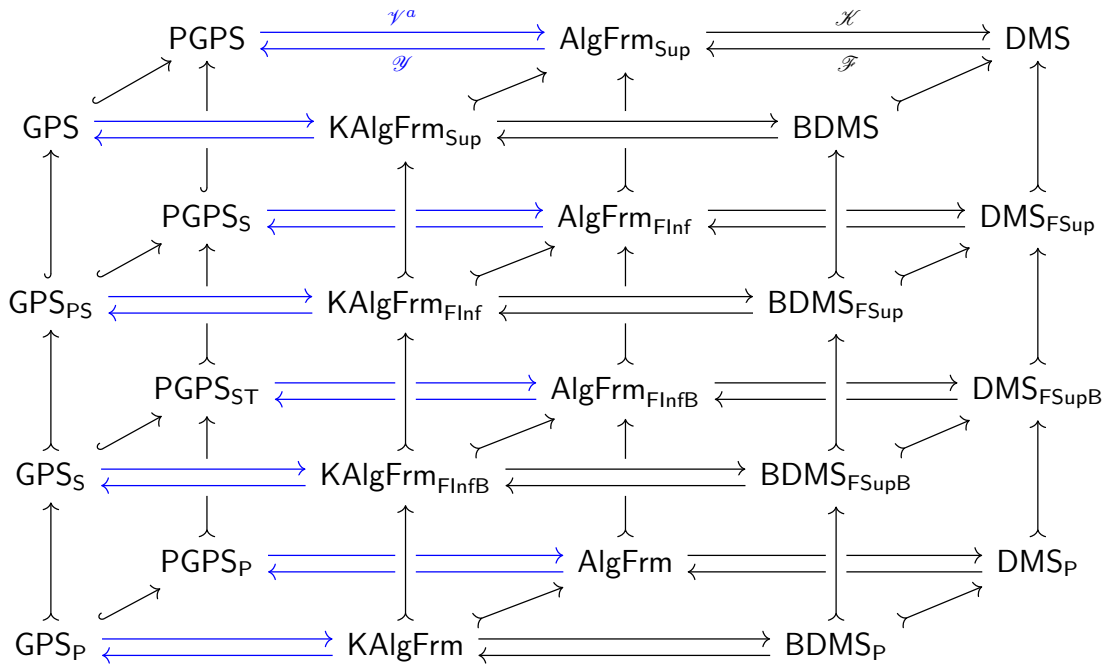

Figure 1: Connecting generalized Priestley duality and HMS duality

The tables below describe the categories listed in Figure 1. The order of the categories in each table corresponds to the diagram from top to bottom.

Categories of pointed generalized Priestley spaces		
Category	Morphisms	Location
PGPS	generalized Priestley morphisms	Def. 3.8
PGPS$_S$	strong Priestley morphisms	Def. 5.8(2)
PGPS$_{ST}$	PGPS$_S$-morphisms s.t. $f[X^-] \subseteq Y^-$ when m, n are isolated	"
PGPS$_P$	PGPS$_S$-morphisms s.t. $f[X_0] \subseteq Y_0$	Def. 5.22

Categories of generalized Priestley spaces		
Category	Morphisms	Location
GPS	generalized Priestley morphisms	Def. 3.6
GPS$_{PS}$	partial strong Priestley morphisms	Def. 5.16(2)
GPS$_S$	strong Priestley morphisms	Def. 5.5(2)
GPS$_P$	GPS$_S$-morphisms s.t. $f[X_0] \subseteq Y_0$	Def. 5.22

Categories of algebraic frames		
Category	Morphisms	Location
AlgFrm$_{Sup}$	maps preserving suprema and compact elements	Def. 2.5(2)
AlgFrm$_{FInf}$	AlgFrm$_{Sup}$-morphisms satisfying (FInf)	Def. 5.3(1)
AlgFrm$_{FInfB}$	bounded AlgFrm$_{FInf}$-morphisms	"
AlgFrm	frame homomorphisms preserving compact elements	Def. 5.20

Categories of compact algebraic frames		
Category	Morphisms	Location
KAlgFrm$_{Sup}$	AlgFrm$_{Sup}$-morphisms	Def. 2.9(2)
KAlgFrm$_{FInf}$	AlgFrm$_{FInf}$-morphisms	Def. 5.3(2)
KAlgFrm$_{FInfB}$	AlgFrm$_{FInfB}$-morphisms	"
KAlgFrm	AlgFrm-morphisms	Def. 5.30(1)

Categories of distributive meet-semilattices		
Category	Morphisms	Location
DMS	meet-semilattice homomorphisms	Def. 2.5(1)
DMS$_{FSup}$	DMS-morphisms preserving existing nonempty finite sups	Def. 5.1(1)
DMS$_{FSupB}$	bounded DMS$_{FSup}$-morphisms	Def. 5.1(2)
DMS$_P$	DMS-morphisms satisfying (P)	Def. 5.26

Categories of bounded distributive meet-semilattices		
Category	Morphisms	Location
BDMS	DMS-morphisms	Def. 2.9(1)
BDMS$_{FSup}$	DMS$_{FSup}$-morphisms	Def. 5.1(1)
BDMS$_{FSupB}$	DMS$_{FSupB}$-morphisms	Def. 5.1(2)
BDMS$_P$	DMS$_P$-morphisms	Def. 5.30(2)

2. Hofmann-Mislove-Stralka duality

We recall that a *meet-semilattice* is a poset M in which meets of finite subsets exist. In particular, M has a top element, which we denote by 1_M or 1 if the context is clear. A *meet-semilattice homomorphism* is a map $\alpha\colon M_1 \to M_2$ preserving all finite meets. In particular, $\alpha(1_{M_1}) = 1_{M_2}$.

A *topological meet-semilattice* is a meet-semilattice M such that M is also a topological space in which the meet operation is continuous. If the topology on M is a Stone topology, then we call M a *Stone meet-semilattice*.

2.1. DEFINITION.

(1) *Let* MS *be the category of meet-semilattices and meet-semilattice homomorphisms.*

(2) *Let* StoneMS *be the category of Stone meet-semilattices and continuous meet-semilattice homomorphisms.*

Hofmann, Mislove, and Stralka [HMS74] developed a duality between MS and StoneMS that is reminiscent of Pontryagin duality. We will refer to it as *HMS duality*. There are two contravariant functors establishing this duality. The functor MS $\to$ StoneMS sends $M \in$ MS to the Stone meet-semilattice $\hom_{\mathsf{MS}}(M, 2)$, and the functor StoneMS $\to$ MS sends $L \in$ StoneMS to $\hom_{\mathsf{StoneMS}}(L, 2)$. Since meet-semilattice homomorphisms $M \to 2$ correspond to filters of M, we can alternatively work with filters of M, which is more convenient for our purposes. We thus define the functors establishing HMS duality as follows.

For $M \in$ MS, let $\mathsf{Filt}(M)$ be the poset of filters of M ordered by inclusion. For $a \in M$ let $\sigma(a) = \{F \in \mathsf{Filt}(M) \mid a \in F\}$, and topologize $\mathsf{Filt}(M)$ by the subbasis

$$\{\sigma(a) \mid a \in M\} \cup \{\sigma(b)^c \mid b \in M\}.$$

Then $\mathsf{Filt}(M)$ is a Stone meet-semilattice. Moreover, if $\alpha\colon M_1 \to M_2$ is an MS-morphism, then $\alpha^{-1}\colon \mathsf{Filt}(M_2) \to \mathsf{Filt}(M_1)$ is a StoneMS-morphism. This defines a contravariant functor $\mathsf{Filt}\colon$ MS $\to$ StoneMS.

To define a contravariant functor in the other direction, for $L \in$ StoneMS let $\mathsf{ClopFilt}(L)$ be the poset of clopen filters of L ordered by inclusion. Then $\mathsf{ClopFilt}(L)$ is a meet-semilattice in which finite meets are finite intersections (and L is the top element). Moreover, if $\alpha\colon L_1 \to L_2$ is a StoneMS-morphism, then $\alpha^{-1}\colon \mathsf{ClopFilt}(L_2) \to \mathsf{ClopFilt}(L_1)$ is an MS-morphism. This defines a contravariant functor $\mathsf{ClopFilt}\colon$ StoneMS $\to$ MS, which together with Filt yields HMS duality:

2.2. THEOREM. [HMS74, Thm. 3.9] MS *is dually equivalent to* StoneMS.

As we pointed out in the introduction, Stone meet-semilattices are exactly the algebraic lattices. To see this, we recall that an element k of a complete lattice L is *compact* if $k \leq \bigvee S$ implies $k \leq \bigvee T$ for some finite $T \subseteq S$, and that L is *algebraic* if the poset $\mathsf{K}(L)$ of compact elements of L is join-dense in L (meaning that each element of L is a

join of compact elements). By HMS duality, each Stone meet-semilattice is isomorphic to $\mathsf{Filt}(M)$ for some $M \in \mathsf{MS}$. Clearly $\mathsf{Filt}(M)$ is a complete lattice, where meet is set-theoretic intersection and join is the filter generated by the union. Moreover, compact elements of $\mathsf{Filt}(M)$ are precisely the principal filters $\uparrow a$ (see [HMS74, Prop. 3.8]), which clearly join-generate $\mathsf{Filt}(M)$. Thus, each Stone meet-semilattice is an algebraic lattice.

Conversely, if L is an algebraic lattice, we consider the topology λ on L generated by the subbasis

$$\{\uparrow k \mid k \in \mathsf{K}(L)\} \cup \{(\uparrow l)^c \mid l \in \mathsf{K}(L)\}.$$

Since $\{\uparrow k \mid k \in \mathsf{K}(L)\}$ is a basis for the Scott topology [GHK+03, Cor. II-1.15], λ is the Lawson topology [GHK+03, Def. III-1.1]. Therefore, λ is a Stone topology [GHK+03, Thm. III-1.10], and with this topology, $L \in \mathsf{StoneMS}$ [GHK+03, Thm. III-2.8].

Let $\alpha \colon L_1 \to L_2$ be a map between Stone meet-semilattices. By [HMS74, Thm. II.3.25], α is a $\mathsf{StoneMS}$-morphism iff α preserves arbitrary infima and directed suprema. But, α preserves arbitrary infima iff it has a left adjoint $\beta \colon L_2 \to L_1$, which then preserves arbitrary suprema. Moreover, α preserves directed suprema iff β preserves compact elements [GHK+03, Cor. IV-1.12]. We thus obtain that $\mathsf{StoneMS}$ is dually isomorphic to the following category:

2.3. DEFINITION. *Let $\mathsf{AlgLat_{Sup}}$ be the category of algebraic lattices and maps between them preserving arbitrary suprema and compact elements.*

The above observation together with Theorem 2.2 yields the following version of HMS duality for meet-semilattices:

2.4. COROLLARY. [GHK+03, p. 274] MS *is equivalent to* $\mathsf{AlgLat_{Sup}}$.

It is this version that we will mainly be working with in this paper. As we pointed out in the introduction, the object level of this equivalence goes back to Nachbin [Nac49, Thm. 1] (see also [BF48, Thm. 2]). However, Nachbin worked with join-semilattices and the corresponding algebraic lattices of ideals. Since we are working with meet-semilattices, our functor from MS to $\mathsf{AlgLat_{Sup}}$ is the filter functor. We next describe explicitly how it acts on morphisms.

Let $\alpha \colon M_1 \to M_2$ be an MS-morphism. The left adjoint of $\alpha^{-1} \colon \mathsf{Filt}(M_2) \to \mathsf{Filt}(M_1)$ is the map $\ell \colon \mathsf{Filt}(M_1) \to \mathsf{Filt}(M_2)$ which sends $F \in \mathsf{Filt}(M_1)$ to

$$\ell(F) = \bigwedge \{G \in \mathsf{Filt}(M_2) \mid F \subseteq \alpha^{-1}(G)\} = \bigwedge \{G \in \mathsf{Filt}(M_2) \mid \alpha[F] \subseteq G\}$$
$$= \bigwedge \{G \in \mathsf{Filt}(M_2) \mid \uparrow\alpha[F] \subseteq G\} = \uparrow\alpha[F].$$

Let $\mathscr{F} \colon \mathsf{MS} \to \mathsf{AlgLat_{Sup}}$ be the functor that sends each $M \in \mathsf{MS}$ to $\mathsf{Filt}(M) \in \mathsf{AlgLat_{Sup}}$ and each MS-morphism $\alpha \colon M_1 \to M_2$ to the $\mathsf{AlgLat_{Sup}}$-morphism $\ell \colon \mathsf{Filt}(M_1) \to \mathsf{Filt}(M_2)$.

The functor in the other direction is the compact element functor $\mathscr{K} \colon \mathsf{AlgLat_{Sup}} \to \mathsf{MS}$ which sends each algebraic lattice L to the poset $\mathsf{K}(L)$ of compact elements of L ordered by the dual $\geq$ of the restriction of $\leq$ to $\mathsf{K}(L)$. Since $(\mathsf{K}(L), \leq)$ is a sub-join-semilattice of L, $(\mathsf{K}(L), \geq)$ is a meet-semilattice, where the top element is 0 (the bottom element

of L). Again, our approach is dual to that of Nachbin [Nac49] who worked with the join-semilattice $(\mathsf{K}(L), \leq)$.

The functor $\mathscr{K} \colon \mathsf{AlgLat}_{\mathsf{Sup}} \to \mathsf{MS}$ sends an $\mathsf{AlgLat}_{\mathsf{Sup}}$-morphism $\alpha \colon L_1 \to L_2$ to its restriction $\mathsf{K}(L_1) \to \mathsf{K}(L_2)$. This is well defined since α sends compact elements to compact elements, and it is an MS-morphism because a finite join of compact elements is compact, α preserves suprema, and we work with $\geq$ on $\mathsf{K}(L_1)$ and $\mathsf{K}(L_2)$.

We conclude this section by observing that the equivalence of Corollary 2.4 restricts to the distributive case. Recall that a meet-semilattice M is *distributive* if whenever $a, b, c \in M$ with $a \wedge b \leq c$, there are $a', b' \in M$ with $a \leq a'$, $b \leq b'$, and $a' \wedge b' = c$.

$$
\begin{array}{ccc}
a' & & b' \\
& c & \\
a & & b \\
& a \wedge b &
\end{array}
$$
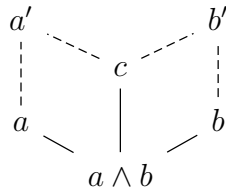
It is well known that M is distributive iff $\mathsf{Filt}(M)$ is a distributive lattice (see, e.g., [Grä11, p. 167] for the dual result that a join-semilattice is distributive iff the lattice of its ideals is distributive).

We recall [PP12, p. 10] that a complete lattice L is a *frame* if it satisfies the join infinite distributive law $a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$ for each $a \in L$ and $S \subseteq L$. An *algebraic frame* is a frame that is an algebraic lattice. It is well known (see, e.g., [Joh82, p. 309]) that a distributive algebraic lattice is a frame.

2.5. DEFINITION.

(1) *Let* DMS *be the full subcategory of* MS *consisting of distributive meet-semilattices.*

(2) *Let* $\mathsf{AlgFrm}_{\mathsf{Sup}}$ *be the full subcategory of* $\mathsf{AlgLat}_{\mathsf{Sup}}$ *consisting of algebraic frames.*

As a consequence of Corollary 2.4 we obtain:

2.6. COROLLARY. DMS *is equivalent to* $\mathsf{AlgFrm}_{\mathsf{Sup}}$.

As we pointed out in the introduction, Sup in the subscript indicates that morphisms in $\mathsf{AlgFrm}_{\mathsf{Sup}}$ preserve arbitrary suprema, but they may not be frame homomorphisms. Note that $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphisms preserve 0 but may not preserve 1.

2.7. DEFINITION.

(1) *A* DMS-*morphism* $\alpha \colon M_1 \to M_2$ *is* bounded *if whenever* M_1 *and* M_2 *are bounded, then* $\alpha(0) = 0$. *Let* $\mathsf{DMS}_{\mathsf{B}}$ *be the wide subcategory of* DMS *whose morphisms are bounded.*

(2) *An* $\mathsf{AlgFrm}_{\mathsf{Sup}}$-*morphism* $\alpha \colon L_1 \to L_2$ *is* bounded *if whenever* L_1 *and* L_2 *are compact, then* $\alpha(1) = 1$. *Let* $\mathsf{AlgFrm}_{\mathsf{SupB}}$ *be the wide subcategory of* $\mathsf{AlgFrm}_{\mathsf{Sup}}$ *whose morphisms are bounded.*

For an algebraic frame L, since $\mathsf{K}(L)$ is ordered by $\geq$, $\mathsf{K}(L)$ has a bottom iff L is compact. Therefore, for algebraic frames L_1 and L_2, an $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphism $\alpha\colon L_1 \to L_2$ is an $\mathsf{AlgFrm}_{\mathsf{SupB}}$-morphism iff its restriction $\alpha\colon \mathsf{K}(L_1) \to \mathsf{K}(L_2)$ is a $\mathsf{DMS}_{\mathsf{B}}$-morphism. Thus, the following is an immediate consequence of Corollary 2.6.

2.8. COROLLARY. $\mathsf{DMS}_{\mathsf{B}}$ *is equivalent to* $\mathsf{AlgFrm}_{\mathsf{SupB}}$.

We recall that a meet-semilattice M is *bounded* if it has a bottom, and that a frame L is *compact* if the top element of L is compact.

2.9. DEFINITION.

   (1) *Let* BDMS *be the full subcategory of* DMS *and* $\mathsf{BDMS}_{\mathsf{B}}$ *the full subcategory of* $\mathsf{DMS}_{\mathsf{B}}$ *whose objects are bounded meet-semilattices.*

   (2) *Let* $\mathsf{KAlgFrm}_{\mathsf{Sup}}$ *be the full subcategory of* $\mathsf{AlgFrm}_{\mathsf{Sup}}$ *and* $\mathsf{KAlgFrm}_{\mathsf{SupB}}$ *the full subcategory of* $\mathsf{AlgFrm}_{\mathsf{SupB}}$ *whose objects are compact algebraic frames.*

As an immediate consequence of Corollaries 2.6 and 2.8, we obtain:

2.10. COROLLARY.

   (1) BDMS *is equivalent to* $\mathsf{KAlgFrm}_{\mathsf{Sup}}$.

   (2) $\mathsf{BDMS}_{\mathsf{B}}$ *is equivalent to* $\mathsf{KAlgFrm}_{\mathsf{SupB}}$.

3. Priestley duality and its generalizations

In this section we recall Priestley duality and its generalizations. We start by briefly describing Priestley duality. For a bounded distributive lattice M, let X_M be the set of prime filters of M ordered by inclusion, and let $\varphi\colon M \to \wp(X_M)$ be the map given by $\varphi(a) = \{x \in X_M \mid a \in x\}$. We topologize X_M by letting

$$\{\varphi(a) \mid a \in M\} \cup \{\varphi(b)^c \mid b \in M\}$$

be a subbasis for the topology. Then X_M is a compact space such that if $x, y \in X_M$ with $x \not\leq y$, then there is a clopen upset U of X_M containing x and missing y. Such spaces are called *Priestley spaces*.

3.1. DEFINITION.

   (1) *Let* DL *be the category of bounded distributive lattices and bounded lattice homomorphisms.*

   (2) *Let* PS *be the category of Priestley spaces and continuous order-preserving maps.*

We then have a contravariant functor $\mathsf{DL} \to \mathsf{PS}$ which sends M to X_M and a DL-morphism $\alpha\colon M_1 \to M_2$ to $\alpha^{-1}\colon X_{M_2} \to X_{M_1}$. We also have a contravariant functor $\mathsf{PS} \to \mathsf{DL}$ which sends $X \in \mathsf{PS}$ to the lattice $\mathsf{ClopUp}(X)$ of clopen upsets of X and a PS-morphism $f\colon X \to Y$ to the DL-morphism $f^{-1}\colon \mathsf{ClopUp}(Y) \to \mathsf{ClopUp}(X)$. These functors establish Priestley duality:

3.2. THEOREM. [Pri70, Pri72] DL *is dually equivalent to* PS.

We will be interested in the generalization of Priestley duality to distributive meet-semilattices established in [BJ08, BJ11] (see also [HP08] for a similar duality for distributive join-semilattices, but with more restrictive morphisms). There are two versions of this duality for BDMS and DMS. We first describe the duality for BDMS from which we derive the duality for DMS.

Let X be a Priestley space. For a closed set $C \subseteq X$, let $\max C$ be the set of maximal points and $\min C$ the set of minimal points of C. It is well known that for every $x \in C$ there are $y \in \max C$ and $z \in \min C$ such that $z \leq x \leq y$.

Let X_0 be a fixed dense subset of X. We call a clopen upset U of X *admissible* if $\max(X \setminus U) \subseteq X_0$. Let $\mathscr{A}(X)$ be the set of admissible clopen upsets of X. For $x \in X$ set $\mathscr{I}_x = \{U \in \mathscr{A}(X) \mid x \notin U\}$.

3.3. DEFINITION. *A* generalized Priestley space *is a tuple* $X = \langle X, \tau, \leq, X_0 \rangle$ *satisfying*

(1) $\langle X, \tau, \leq \rangle$ *is a Priestley space;*

(2) X_0 *is a dense subset of* X*;*

(3) X_0 *is order-dense in* X *(meaning that for each $x \in X$ there is $y \in X_0$ with $x \leq y$);*

(4) $x \in X_0$ *iff* $\mathscr{I}_x$ *is directed;*

(5) *for all $x, y \in X$, we have $x \leq y$ iff $\forall U \in \mathscr{A}(X)$, $x \in U \Longrightarrow y \in U$.*

3.4. REMARK.

(1) By Definition 3.3(3), $\max X \subseteq X_0$. Thus, $\varnothing \in \mathscr{A}(X)$, so $\varnothing \in \mathscr{I}_x$, and hence $\mathscr{I}_x$ is nonempty for each $x \in X$.

(2) If $X_0 = X$, then $\mathscr{A}(X) = \mathsf{ClopUp}(X)$, so Conditions (2)–(5) of Definition 3.3 become redundant, and hence X becomes a Priestley space.

Let $R \subseteq X \times Y$ be a relation between sets X and Y. For $U \subseteq Y$ we follow the standard notation in modal logic and write $\square_R U = \{x \in X \mid R[x] \subseteq U\}$.

3.5. DEFINITION. *A* generalized Priestley morphism *between generalized Priestley spaces X, Y is a relation $R \subseteq X \times Y$ satisfying*

(1) *If $x \not{R} y$, then there is $U \in \mathscr{A}(Y)$ with $R[x] \subseteq U$ and $y \notin U$.*

(2) *If $U \in \mathscr{A}(Y)$, then $\square_R U \in \mathscr{A}(X)$.*

We call R total *if $R^{-1}[Y] = X$.*

This gives rise to two categories.

3.6. DEFINITION. *Let* GPS *be the category of generalized Priestley spaces and generalized Priestley morphisms, and let* $\mathsf{GPS_T}$ *be the wide subcategory of* GPS *whose morphisms are total.*

The identity morphism on $X \in$ GPS is $\leq$ and the composition $S * R$ of two morphisms $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is defined by

$$x \, (S * R) \, z \Longleftrightarrow \forall U \in \mathscr{A}(Z), \, x \in \Box_R \Box_S U \Longrightarrow z \in U.$$

We then have the following generalization of Priestley duality:

3.7. THEOREM. [BJ11, Sec. 6] BDMS *is dually equivalent to* GPS *and* $\mathsf{BDMS_B}$ *is dually equivalent to* $\mathsf{GPS_T}$.

The contravariant functors establishing these dual equivalences are constructed as follows. The functor $\mathscr{A}\colon$ GPS $\to$ BDMS sends $X \in$ GPS to $\mathscr{A}(X)$ and a GPS-morphism $R \subseteq X \times Y$ to $\Box_R\colon \mathscr{A}(Y) \to \mathscr{A}(X)$. The relation R is total iff $\Box_R$ preserves the bottom, so we obtain the restriction $\mathscr{A}\colon \mathsf{GPS_T} \to \mathsf{BDMS_B}$.

To define the contravariant functor $\mathscr{X}\colon$ BDMS $\to$ GPS we recall the definitions of prime and optimal filters of $M \in$ BDMS. A filter P of M is *prime* if $F_1 \cap F_2 \subseteq P$ implies $F_1 \subseteq P$ or $F_2 \subseteq P$ for any $F_1, F_2 \in \mathsf{Filt}(M)$. To define optimal filters, we require the definition of the distributive envelope $\mathsf{D}(M)$ of M. Let $\mathsf{BDMS_{FSupB}}$ be the wide subcategory of BDMS whose morphisms preserve existing finite suprema. Then the forgetful functor $\mathsf{U}\colon \mathsf{DL} \to \mathsf{BDMS_{FSupB}}$ has a left adjoint $\mathsf{D}\colon \mathsf{BDMS_{FSupB}} \to \mathsf{DL}$, and we call $\mathsf{D}(M)$ the *distributive envelope* of M. There are various constructions of $\mathsf{D}(M)$ (see [CH78], [BJ11, Sec. 3], or [HP08, Thm. 1.3]). What matters to us is that M embeds into $\mathsf{D}(M)$ and we identify M with its image in $\mathsf{D}(M)$. Then a filter F of M is *optimal* if $F = P \cap M$ for some prime filter P of $\mathsf{D}(M)$. Thus, optimal filters of M are the restrictions of prime filters of $\mathsf{D}(M)$ to M. For other characterizations of optimal filters we refer to [BJ11, Sec. 4].

Let $\mathsf{Opt}(M)$ be the set of optimal filters of M. Then the set $\mathsf{Pr}(M)$ of prime filters of M is contained in $\mathsf{Opt}(M)$. We order $\mathsf{Opt}(M)$ by inclusion, and topologize it by letting

$$\{\varphi(a) \mid a \in M\} \cup \{\varphi(b)^c \mid b \in M\}$$

be a subbasis for the topology τ, where $\varphi(a) = \{x \in \mathsf{Opt}(M) \mid a \in x\}$. This yields the generalized Priestley space $\mathscr{X}(M) := \langle \mathsf{Opt}(M), \tau, \subseteq, \mathsf{Pr}(M) \rangle$.

If $\alpha\colon M_1 \to M_2$ is a BDMS-morphism, we define $R_\alpha \subseteq \mathsf{Opt}(M_2) \times \mathsf{Opt}(M_1)$ by $x \, R_\alpha \, y$ if $\alpha^{-1}(x) \subseteq y$ and let $\mathscr{X}(\alpha) = R_\alpha$. We have that R_α is total iff $\alpha(0) = 0$. This defines the functor $\mathscr{X}\colon$ BDMS $\to$ GPS and its restriction $\mathscr{X}\colon \mathsf{BDMS_B} \to \mathsf{GPS_T}$, and the functors $\mathscr{A}, \mathscr{X}$ establish the dual equivalences of Theorem 3.7.

To generalize this to a duality for DMS, we will work with pointed generalized Priestley spaces. This approach is similar to the one undertaken in [BMR17, Sec. 3] where Esakia duality for Heyting algebras was generalized to a duality for brouwerian algebras.

3.8. DEFINITION. *We call a tuple $(X, m) = \langle X, \tau, \leq, X_0, m \rangle$ a pointed generalized Priestley space if*

(1) $\langle X, \tau, \leq \rangle$ *is a Priestley space;*

(2) *m is the unique maximum of X;*

(3) *X_0 is a dense subset of $X \setminus \{m\}$;*

(4) *X_0 is order-dense in $X \setminus \{m\}$;*

(5) *$x \in X_0$ iff $\mathscr{I}_x$ is nonempty and directed;*

(6) *$x \leq y$ iff $\forall U \in \mathscr{A}(X)$, $x \in U \implies y \in U$.*

Let PGPS be the category of pointed generalized Priestley spaces and generalized Priestley morphisms.

3.9. REMARK. We recall that a pointed Stone space is a pair (X, p) where X is a Stone space and $p \in X$. Thus, every Stone space can be made into a pointed Stone space. On the other hand, for a generalized Priestley space X to be made into a pointed generalized Priestley space, X must have a unique maximum. Nevertheless, GPS is equivalent to a full subcategory of PGPS, as we detail in the next remark.

3.10. REMARK. By Definition 3.8, $m \notin X_0$, so $\varnothing \notin \mathscr{A}(X)$ and $\mathscr{I}_m = \varnothing$. In fact, $\mathscr{I}_x = \varnothing$ iff $x = m$. Moreover, m is an isolated point iff $\{m\}$ is the bottom of $\mathscr{A}(X)$. The full subcategory of PGPS consisting of those (X, m) where m is an isolated point is equivalent to GPS. The equivalence is obtained as follows. If $(X, m) \in$ PGPS and m is isolated, then $X^- := X \setminus \{m\}$ is a generalized Priestley space. Also, if $(X, m), (Y, n) \in$ PGPS with m, n isolated and $R \subseteq X \times Y$ is a PGPS-morphism, then $R^- := R \cap (X^- \times Y^-)$ is a GPS-morphism. Conversely, let $X \in$ GPS. If X^+ is obtained from X by adding a new isolated top and $(X^+)_0 := X_0$, then $X^+ \in$ PGPS. Also, if $R \subseteq X \times Y$ is a GPS-morphism, then $R^+ := R \cup (X^+ \times \{n\})$ is a PGPS-morphism. Thus, we obtain two functors which yield an equivalence of GPS and the full subcategory of PGPS consisting of those $(X, m) \in$ PGPS in which m is an isolated point.

The contravariant functor from PGPS to DMS is defined the same way as the contravariant functor $\mathscr{A}$ above and we use the same letter to denote it. The only difference is that if (X, m) is a pointed generalized Priestley space, then $\varnothing \notin \mathscr{A}(X)$. Moreover, $\mathscr{A}(X)$ is bounded iff $\{m\} \in \mathscr{A}(X)$, which happens iff m is an isolated point of X. If $R \subseteq X \times Y$ is a generalized Priestley morphism with $(X, m), (Y, n) \in$ PGPS, then $\mathscr{A}(R) = \Box_R$. The only difference is that $m \in \Box_R U$ for each $U \in \mathscr{A}(Y)$.

The contravariant functor from DMS to PGPS is defined by a slight modification of the contravariant functor $\mathscr{X}$. Again, we use the same letter to denote it. The main difference is that if $M \in$ DMS, then we work with $X_M = \mathsf{Opt}(M) \cup \{M\}$, so M becomes the unique maximum of X_M. Then M has a bottom iff $\{M\}$ is an isolated point of X_M. If

$\alpha \colon M_1 \to M_2$ is a DMS-morphism, then $\mathscr{X}(\alpha) = R_\alpha \subseteq X_{M_2} \times X_{M_1}$. In this case, $y \, R_\alpha \, M_1$ for each $y \in X_{M_2}$.

Consequently, $\mathscr{A} \colon$ PGPS $\to$ DMS and $\mathscr{X} \colon$ DMS $\to$ PGPS yield the following generalization of Theorem 3.7, a version of which was established in [BJ11, Thm. 9.2]:

3.11. THEOREM. DMS *is dually equivalent to* PGPS.

3.12. DEFINITION. *Let* PGPS$_\mathsf{T}$ *be the wide subcategory of* PGPS *consisting of those* PGPS-*morphisms* $R \subseteq X \times Y$ *such that if* $X^-, Y^- \in$ GPS, *then* $R^- \subseteq X^- \times Y^-$ *is total.*

We have the following corollary of Theorem 3.11, which generalizes the duality of Theorem 3.7 between BDMS$_\mathsf{B}$ and GPS$_\mathsf{T}$.

3.13. COROLLARY. DMS$_\mathsf{B}$ *is dually equivalent to* PGPS$_\mathsf{T}$.

PROOF. Let $\alpha \colon M_1 \to M_2$ be a DMS-morphism with M_1, M_2 bounded. Then $X_{M_1}^-, X_{M_2}^- \in$ GPS (see Remark 3.10). It is sufficient to show that α is a DMS$_\mathsf{B}$-morphism iff R_α^- is total. First suppose that α is a DMS$_\mathsf{B}$-morphism, so $\alpha(0) = 0$. If $x \in \mathsf{Opt}(M_2)$, then $0 \notin \alpha^{-1}(x)$ since $0 \notin x$. Therefore, by [BJ11, Lem. 4.7], there is $y \in \mathsf{Opt}(M_1)$ with $\alpha^{-1}(x) \subseteq y$. This implies that $R_\alpha^-[x] \neq \varnothing$. Conversely, let R_α^- be total. If $\alpha(0) \neq 0$, then $\uparrow\alpha(0)$ is a proper filter. Therefore, there is $x \in \mathsf{Opt}(M_2)$ with $\uparrow\alpha(0) \in x$. This implies that $0 \in \alpha^{-1}(x)$, so $\alpha^{-1}(x) = M_1$. Thus, $R_\alpha^-[x] = \varnothing$, and hence R_α^- is not total. The obtained contradiction proves that $\alpha(0) = 0$. ∎

4. Connecting generalized Priestley duality to HMS duality

In this section we connect generalized Priestley duality to HMS duality. To do so, we define contravariant functors $\mathscr{V}^a \colon$ PGPS $\to$ AlgFrm$_\mathsf{Sup}$ and $\mathscr{Y} \colon$ AlgFrm$_\mathsf{Sup} \to$ PGPS and prove that they establish a dual equivalence, yielding our main result. This together with Corollary 2.6 gives Theorem 3.11. As a consequence, we derive Theorem 3.7 from Corollary 2.10 and Corollary 3.13 from Corollary 2.8, thus obtaining the top layer of Figure 1.

THE FUNCTOR $\mathscr{V}^a$. Let $X := \langle X, \tau, \leq, X_0, m \rangle$ be a pointed generalized Priestley space. We generalize the notion of an admissible clopen upset to an admissible closed upset of X. Recall that $U \in \mathsf{ClopUp}(X)$ is admissible if $\max(X \setminus U) \subseteq X_0$. This is equivalent to $X \setminus U = \downarrow(X_0 \setminus U)$, which motivates our definition of admissible closed upsets.

4.1. DEFINITION. *A closed upset C of a pointed generalized Priestley space X is admissible if $X \setminus C = \downarrow(X_0 \setminus C)$. Let $\mathscr{V}^a(X)$ be the set of admissible closed upsets of X.*

4.2. REMARK. The notation $\mathscr{V}^a(X)$ is motivated by the fact that $\mathscr{V}^a(X)$ is a version of the upper Vietoris functor applied to X.

It is well known that in Priestley spaces, each closed upset is an intersection of clopen upsets. This result generalizes to admissible closed upsets of pointed generalized Priestley spaces.

4.3. LEMMA. *A closed upset C of a pointed generalized Priestley space X is admissible iff it is an intersection of admissible clopen upsets.*

PROOF. First suppose that C is an intersection of a family $\mathscr{S}$ of admissible clopen upsets. Let $x \notin C$. Then there is $U \in \mathscr{S}$ with $x \notin U$. Since U is admissible, there is $y \in X_0 \setminus U$ with $x \leq y$. From $y \notin U$ and $C \subseteq U$ it follows that $y \notin C$. Therefore, $X \setminus C \subseteq {\downarrow}(X_0 \setminus C)$, hence the equality. Thus, C is admissible.

Conversely, suppose that C is admissible and $x \notin C$. Then there is $y \in X_0 \setminus C$ with $x \leq y$. For each $z \in C$ we have $z \not\leq y$. Therefore, there is an admissible clopen U_z with $z \in U_z$ and $y \notin U_z$. Since C is closed, it is compact, and the U_z cover C. Thus, there are $z_1, \ldots, z_n \in C$ such that $C \subseteq U_{z_1} \cup \cdots \cup U_{z_n}$. Clearly each U_{z_i} is in $\mathscr{I}_y$. Because $y \in X_0$, $\mathscr{I}_y$ is directed, so there is $V \in \mathscr{I}_y$ with each U_{z_i} contained in V. Therefore, $C \subseteq V$ and $y \notin V$. Since $x \leq y$, $y \notin V$, and V is an upset, we have $x \notin V$. Thus, for each $x \notin C$ there is an admissible clopen upset containing C and missing x. Consequently, C is an intersection of admissible clopen upsets. ∎

4.4. LEMMA. *If $X \in \mathsf{PGPS}$, then $(\mathscr{V}^a(X), \supseteq) \in \mathsf{AlgFrm}_{\mathsf{Sup}}$ and $\mathsf{K}(\mathscr{V}^a(X)) = \mathscr{A}(X)$.*

PROOF. We first show that $\mathscr{V}^a(X)$ is a complete lattice. We have $X \in \mathscr{V}^a(X)$, so $\mathscr{V}^a(X)$ has a bottom. Let $\mathscr{S} \subseteq \mathscr{V}^a(X)$ and set $C = \bigcap \mathscr{S}$. Then C is a closed upset. Lemma 4.3 yields that C is admissible, so $C \in \mathscr{V}^a(X)$. But then C is the join of $\mathscr{S}$ in $\mathscr{V}^a(X)$, and hence $\mathscr{V}^a(X)$ is a complete lattice.

We next show that $\mathscr{V}^a(X)$ is distributive. Since the order on $\mathscr{V}^a(X)$ is $\supseteq$, it is enough to show that $(A \vee B) \wedge C \supseteq (A \wedge C) \vee (B \wedge C)$ for each $A, B, C \in \mathscr{V}^a(X)$. We first show that if $x \in X_0 \setminus ((A \vee B) \wedge C)$, then $x \notin (A \wedge C) \vee (B \wedge C)$. Since join in $\mathscr{V}^a(X)$ is intersection and meet is the least admissible closed upset containing the union, from $x \notin (A \vee B) \wedge C$ it follows that $x \notin (A \cap B) \cup C$. Therefore, $x \notin (A \cup C) \cap (B \cup C)$, and so $x \notin A \cup C$ or $x \notin B \cup C$. If $x \notin A \cup C$, Lemma 4.3 yields that there exist $U_{A,x}, U_{C,x} \in \mathscr{A}(X)$ such that $A \subseteq U_{A,x}$, $C \subseteq U_{C,x}$, and $x \notin U_{A,x}, U_{C,x}$. Since $x \in X_0$ and $U_{A,x}, U_{C,x} \in \mathscr{I}_x$, there is $U_x \in \mathscr{I}_x$ with $U_{A,x}, U_{C,x} \subseteq U_x$, so $A \cup C \subseteq U_x$. Similarly, if $x \notin B \cup C$, there is $V_x \in \mathscr{I}_x$ such that $B \cup C \subseteq V_x$. Thus, if $x \in X_0 \setminus ((A \vee B) \wedge C)$, then $x \notin A \wedge C$ or $x \notin B \wedge C$, and hence $x \notin (A \wedge C) \vee (B \wedge C)$. Since $(A \vee B) \wedge C$ is admissible, if $y \notin (A \vee B) \wedge C$, then there is $x \in X_0 \setminus ((A \vee B) \wedge C)$ such that $y \leq x$, and so $y \notin (A \wedge C) \vee (B \wedge C)$. This proves that $(A \vee B) \wedge C \supseteq (A \wedge C) \vee (B \wedge C)$.

It is left to prove that $(\mathscr{V}^a(X), \supseteq)$ is algebraic. By Lemma 4.3, $C \in \mathscr{V}^a(X)$ is the intersection of $U \in \mathscr{A}(X)$ containing it. Thus, it suffices to show that $\mathsf{K}(\mathscr{V}^a(X)) = \mathscr{A}(X)$. First, let $U \in \mathscr{A}(X)$. If $U \leq \bigvee \mathscr{S}$ for some $\mathscr{S} \subseteq \mathscr{V}^a(X)$, then $\bigcap \mathscr{S} \subseteq U$. Since X is compact, there is a finite subset $\mathscr{T}$ of $\mathscr{S}$ with $\bigcap \mathscr{T} \subseteq U$, so $U \leq \bigvee \mathscr{T}$. Therefore, $U \in \mathsf{K}(\mathscr{V}^a(X))$. Conversely, let $C \in \mathsf{K}(\mathscr{V}^a(X))$. Since $C = \bigcap \{U \in \mathscr{A}(X) \mid C \subseteq U\}$ and C is compact, there exist $U_1, \ldots, U_n \in \mathscr{A}(X)$ such that $C = U_1 \cap \cdots \cap U_n$. Thus, $C \in \mathscr{A}(X)$, so $\mathsf{K}(\mathscr{V}^a(X)) = \mathscr{A}(X)$. Consequently, $(\mathscr{V}^a(X), \supseteq)$ is an algebraic frame. ∎

The above lemma defines $\mathscr{V}^a$ on objects. We next define $\mathscr{V}^a$ on morphisms.

4.5. LEMMA. *Let $X, Y \in \mathsf{PGPS}$. If $R \subseteq X \times Y$ is a generalized Priestley morphism, then $\square_R \colon \mathscr{V}^a(Y) \to \mathscr{V}^a(X)$ is a well-defined $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphism.*

PROOF. To see that $\square_R$ is well defined, let $C \in \mathscr{V}^a(Y)$. Then $C = \bigcap\{U \in \mathscr{A}(X) \mid C \subseteq U\}$ by Lemma 4.3. Since $\square_R$ preserves arbitrary intersections,

$$\square_R C = \square_R \bigcap\{U \in \mathscr{A}(X) \mid C \subseteq U\} = \bigcap\{\square_R U \mid U \in \mathscr{A}(X) \text{ and } C \subseteq U\},$$

which is an intersection of admissible clopen upsets. Thus, $\square_R C \in \mathscr{V}^a(X)$ by Lemma 4.3. Since joins in $\mathscr{V}^a(X)$ are intersections, it is clear that $\square_R$ preserves arbitrary joins. Finally, by Lemma 4.4, compact elements of $\mathscr{V}^a(X)$ are admissible clopen upsets. Therefore, $\square_R$ preserves compact elements by the definition of a generalized Priestley morphism. Thus, $\square_R$ is an $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphism. ∎

4.6. PROPOSITION. *There is a contravariant functor $\mathscr{V}^a \colon \mathsf{PGPS} \to \mathsf{AlgFrm}_{\mathsf{Sup}}$ which sends $X \in \mathsf{PGPS}$ to $\mathscr{V}^a(X)$ and a PGPS-morphism R to $\square_R$.*

PROOF. By Lemmas 4.4 and 4.5, $\mathscr{V}^a$ is well defined. If R is the identity morphism on $X \in \mathsf{PGPS}$, then R is $\leq$. Therefore, for $U \in \mathscr{V}^a(X)$, we have

$$\square_R U = \{x \in X \mid R[x] \subseteq U\} = \{x \in X \mid {\uparrow}x \subseteq U\} = U.$$

Thus, $\square_R$ is the identity morphism on $\mathscr{V}^a(X)$. Next, let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be PGPS-morphisms. We show that $\square_R \circ \square_S = \square_{S*R}$. Let $C \in \mathscr{V}^a(Z)$. By Lemma 4.3, $C = \bigcap\{U \in \mathscr{A}(Z) \mid C \subseteq U\}$. Since for each $U \in \mathscr{A}(Z)$ we have $\square_R \square_S U = \square_{S*R} U$ (see [BJ11, p. 106]) and $\square_R, \square_S$ commute with arbitrary intersections, we obtain

$$\square_R \square_S C = \bigcap\{\square_R \square_S U \mid U \in \mathscr{A}(Z) \text{ and } C \subseteq U\}$$
$$= \bigcap\{\square_{S*R} U \mid U \in \mathscr{A}(Z) \text{ and } C \subseteq U\} = \square_{S*R} C.$$

Thus, $\square_R \circ \square_S = \square_{S*R}$, and hence $\mathscr{V}^a$ is a contravariant functor. ∎

THE FUNCTOR $\mathscr{Y}$. Let L be a complete lattice. We recall (see, e.g., [GHK+03, p. 50]) that the *way below* relation on L is defined by $a \ll b$ if, for each $S \subseteq L$, from $b \leq \bigvee S$ it follows that $a \leq \bigvee T$ for some finite $T \subseteq S$. We also recall (see, e.g., [GHK+03, p. 54]) that a complete lattice L is *continuous* if $b = \bigvee\{a \in L \mid a \ll b\}$ for each $b \in L$. Since $a \in L$ is compact iff $a \ll a$, for $a \in \mathsf{K}(L)$, we have $a \leq b$ iff $a \ll b$. Therefore, every algebraic lattice is a continuous lattice.

Let $p \in L \setminus \{1\}$. We recall that p is *(meet-)prime* if $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. In distributive lattices prime elements are exactly (meet-)irreducible elements, where p is *irreducible* if $a \wedge b = p$ implies $a = p$ or $b = p$. Next, recall that p is *pseudoprime* if for each $n \geq 1$, from $a_1 \wedge \cdots \wedge a_n \ll p$ it follows that $a_i \leq p$ for some i (see [GHK+03, Prop. I-3.25]).

4.7. DEFINITION. *For a complete lattice L, let $\mathsf{P}(L)$ be the set of prime elements and $\mathsf{PP}(L)$ the set of pseudoprime elements of L.*

4.8. DEFINITION. *For an algebraic frame L, let $Y_L = \mathsf{PP}(L) \cup \{1\}$.*

As we pointed out in Section 2, the topology λ generated by the subbasis

$$\{\uparrow k \mid k \in \mathsf{K}(L)\} \cup \{(\uparrow l)^c \mid l \in \mathsf{K}(L)\}$$

turns L into a Stone meet-semilattice. This, in particular, implies that $\langle L, \lambda, \leq \rangle$ is a Priestley space. We restrict the topology and order on L to Y_L. We then have:

4.9. LEMMA. *If L is an algebraic frame, then $\langle Y_L, \lambda, \leq \rangle$ is a Priestley space.*

PROOF. Since L is a Priestley space, it suffices to show that Y_L is a closed subset of L. Let $a \in L \setminus Y_L$. Then $a \neq 1$ and $a \notin \mathsf{PP}(L)$. Therefore, there are $b_1, \ldots, b_n \in L$ with $b_1 \wedge \cdots \wedge b_n \ll a$ and $b_i \not\leq a$ for each i. Because L is an algebraic lattice, there are $k_i \in \mathsf{K}(L)$ with $k_i \leq b_i$ but $k_i \not\leq a$. Clearly $k_1 \wedge \cdots \wedge k_n \ll a$. Since $a = \bigvee (\downarrow a \cap \mathsf{K}(L))$, there are $t_1, \ldots, t_p \in \downarrow a \cap \mathsf{K}(L)$ with $k_1 \wedge \cdots \wedge k_n \leq t_1 \vee \cdots \vee t_p$. Let $t = t_1 \vee \cdots \vee t_p$. Then $t \in \downarrow a \cap \mathsf{K}(L)$ and $k_1 \wedge \cdots \wedge k_n \leq t$. Therefore, $\uparrow t \cap (\uparrow k_1)^c \cap \cdots \cap (\uparrow k_n)^c$ is an open neighborhood of a. Moreover, if $x \in \uparrow t \cap (\uparrow k_1)^c \cap \cdots \cap (\uparrow k_n)^c$, then $k_1 \wedge \cdots \wedge k_n \leq t \leq x$ and $k_1, \ldots, k_n \not\leq x$. Since t is compact, $k_1 \wedge \cdots \wedge k_n \leq t \leq x$ implies $k_1 \wedge \cdots \wedge k_n \ll x$. Thus, $k_1 \wedge \cdots \wedge k_n \ll x$ but $k_i \not\leq x$ for each i. Since $x \neq 1$, we conclude that $x \notin Y_L$. This implies that $\uparrow t \cap (\uparrow k_1)^c \cap \cdots \cap (\uparrow k_n)^c$ is an open neighborhood of a that misses Y_L. Thus, Y_L is a closed subset of L. ∎

4.10. REMARK. The proof of Lemma 4.9 does not require that L is a frame. It only requires that L is an algebraic lattice.

We next show that the tuple $\langle Y_L, \lambda, \leq, \mathsf{P}(L), 1 \rangle$ is a pointed generalized Priestley space. For this we need the following lemma. The proof of (1) can for example be found in [GHK+03, Cor. I-3.10]. We include the proof of (2) because we were unable to find a reference for it.

4.11. LEMMA. *Let L be an algebraic frame.*

(1) *If $a, b \in L$ with $a \not\leq b$, then there is $p \in \mathsf{P}(L)$ with $a \not\leq p$ and $b \leq p$.*

(2) *Let U be a clopen upset of L. Then $U = \uparrow k_1 \cup \cdots \cup \uparrow k_n$ for some $k_i \in \mathsf{K}(L)$.*

PROOF. (1). In a continuous lattice each element is a meet of irreducible elements (see, e.g., [GHK+03, Cor. I-3.10]). Since every algebraic lattice is continuous and irreducible elements are prime in distributive lattices, the result follows.

(2). We first show that $U = \bigcup\{\uparrow k \mid k \in \mathsf{K}(L) \cap U\}$. Let $a \in U$. Since U is an upset, $\uparrow a \subseteq U$, so $\uparrow a \cap U^c = \varnothing$. Because L is an algebraic lattice, $\uparrow a = \bigcap\{\uparrow k \mid k \in \mathsf{K}(L), k \leq a\}$. Therefore, $U^c \cap \bigcap\{\uparrow k \mid k \in \mathsf{K}(L), k \leq a\} = \varnothing$. Since U^c is closed, compactness of L shows there are $k_1, \ldots, k_n \in \mathsf{K}(L)$ with $k_i \leq a$ and $U^c \cap \uparrow k_1 \cap \cdots \cap \uparrow k_n = \varnothing$. If $k = k_1 \vee \cdots \vee k_n$, then $k \in \mathsf{K}(L)$, $k \leq a$, and $\uparrow k = \uparrow k_1 \cap \cdots \cap \uparrow k_n$, so $\uparrow k \subseteq U$. Thus, $U = \bigcup\{\uparrow k \mid k \in \mathsf{K}(L) \cap U\}$, as desired. Since U is closed, hence compact, this union is a finite union, completing the proof. ∎

4.12. LEMMA. $\mathsf{P}(L)$ *is dense in* $\mathsf{PP}(L) = Y_L \setminus \{1\}$.

PROOF. Since $\{\uparrow k \mid k \in \mathsf{K}(L)\}$ is closed under finite intersections,

$$\{\uparrow k \cap (\uparrow l_1)^c \cap \cdots \cap (\uparrow l_n)^c \mid k, l_1, \ldots, l_n \in \mathsf{K}(L)\}$$

is a basis for the topology on Y_L. Therefore, it suffices to show that if $k, l_1, \ldots, l_n \in \mathsf{K}(L)$ and $U = \uparrow k \cap (\uparrow l_1)^c \cap \cdots \cap (\uparrow l_n)^c$ with $U \cap \mathsf{PP}(L) \neq \varnothing$, then $U \cap \mathsf{P}(L) \neq \varnothing$. Let $q \in \mathsf{PP}(L)$ with $q \in U$. Then $k \leq q$ and each $l_i \not\leq q$. If $l_1 \wedge \cdots \wedge l_n \leq k$, then $l_1 \wedge \cdots \wedge l_n \ll q$ since k is compact. This contradicts $q \in \mathsf{PP}(L)$. Therefore, $l_1 \wedge \cdots \wedge l_n \not\leq k$, so by Lemma 4.11(1) there is $p \in \mathsf{P}(L)$ with $k \leq p$ and $l_1 \wedge \cdots \wedge l_n \not\leq p$, so each $l_i \not\leq p$. Thus, $p \in U$, and hence $U \cap \mathsf{P}(L) \neq \varnothing$. ∎

4.13. LEMMA. *Let U be a clopen upset of Y_L. Then U is admissible iff $U = \uparrow k \cap Y_L$ for some $k \in \mathsf{K}(L)$.*

PROOF. First suppose that $U = \uparrow k \cap Y_L$ for some $k \in \mathsf{K}(L)$. If $a \in Y_L \setminus U$, then $k \not\leq a$. By Lemma 4.11(1), there is $p \in \mathsf{P}(L)$ with $k \not\leq p$ and $a \leq p$. Thus, $p \in \mathsf{P}(L) \setminus U$. This shows that U is admissible.

Conversely, let U be admissible. Since U is a clopen upset of Y_L and Y_L is a closed subspace of L, there is a clopen upset V of L with $U = V \cap Y_L$ (see [BH21, Lem. 4.4]). By Lemma 4.11(2), there are $k_i \in \mathsf{K}(L)$ with $V = \uparrow k_1 \cup \cdots \cup \uparrow k_n$. Consequently, $U = (\uparrow k_1 \cup \cdots \cup \uparrow k_n) \cap Y_L$. If $k_1 \wedge \cdots \wedge k_n \notin U$, then since U is admissible, there is $p \in \mathsf{P}(L) \setminus U$ with $k_1 \wedge \cdots \wedge k_n \leq p$. But then $k_i \leq p$ for some i, yielding $p \in U$, which is false. Therefore, $k_1 \wedge \cdots \wedge k_n \in U$, and so $k_i \leq k_1 \wedge \cdots \wedge k_n$ for some i. Thus, $k_i \leq k_j$ for each j, and hence $U = \uparrow k_i \cap Y_L$. ∎

4.14. LEMMA. *For $a, b \in Y_L$ we have $a \leq b$ iff $\forall U \in \mathscr{A}(Y_L)$, $a \in U \Longrightarrow b \in U$.*

PROOF. Suppose that $a \leq b$. Let $U \in \mathscr{A}(Y_L)$. Since U is an upset of Y_L, if $a \in U$, then $b \in U$. Conversely, suppose that $a \not\leq b$. Since $\mathsf{K}(L)$ is join-dense in L, there is $k \in \mathsf{K}(L)$ with $k \leq a$ and $k \not\leq b$. Let $U = \uparrow k \cap Y_L$. Then U is admissible by Lemma 4.13, $a \in U$, and $b \notin U$. ∎

4.15. LEMMA. *For $a \in Y_L$ we have that $\mathscr{I}_a$ is nonempty and directed iff $a \in \mathsf{P}(L)$.*

PROOF. Let $a \in \mathsf{P}(L)$. Then $a \neq 1$, so there is $k \in \mathsf{K}(L)$ with $k \not\leq a$. By Lemma 4.13, $\uparrow k \cap Y_L$ is admissible and does not contain a. Therefore, $\mathscr{I}_a$ is nonempty. To see it is directed, let $U, V \in \mathscr{I}_a$. Then there are $k, l \in \mathsf{K}(L)$ with $U = \uparrow k \cap Y_L$ and $V = \uparrow l \cap Y_L$. Therefore, $k, l \not\leq a$, so $k \wedge l \not\leq a$ since $a \in \mathsf{P}(L)$. Thus, there is $t \in \mathsf{K}(L)$ with $t \leq k \wedge l$ and $t \not\leq a$. Then $\uparrow t \cap Y_L \in \mathscr{I}_a$ and contains both U, V. This proves that $\mathscr{I}_a$ is directed.

Conversely, suppose that $\mathscr{I}_a$ is nonempty and directed. Since $\mathscr{I}_a$ is nonempty, there is $k \in \mathsf{K}(L)$ with $\uparrow k \cap Y_L \in \mathscr{I}_a$ by Lemma 4.13. Thus, $k \not\leq a$, and so $a \neq 1$. If $a \notin \mathsf{P}(L)$, then there are $x, y \in L$ with $x \wedge y \leq a$ and $x, y \not\leq a$. The latter implies that there are $k, l \in \mathsf{K}(L)$ with $k \leq x$, $l \leq y$, and $k, l \not\leq a$. Set $U = \uparrow k \cap Y_L$ and $V = \uparrow l \cap Y_L$. Then $U, V \in \mathscr{I}_a$. Therefore, there is $W \in \mathscr{I}_a$ with $U, V \subseteq W$. By Lemma 4.13, there is $t \in \mathsf{K}(L)$ with $W = \uparrow t \cap Y_L$. But then $t \leq k, l$, so $t \leq k \wedge l \leq a$, and hence $W \notin \mathscr{I}_a$. The obtained contradiction proves that $a \in \mathsf{P}(L)$. ∎

4.16. PROPOSITION. *If L is an algebraic frame, then $\mathscr{Y}(L) := \langle Y_L, \lambda, \leq, \mathsf{P}(L), 1 \rangle$ is a pointed generalized Priestley space.*

PROOF. By Lemma 4.9, $\langle Y_L, \lambda, \leq \rangle$ is a Priestley space. It is clear that 1 is the unique maximum of Y_L. By Lemma 4.12, $\mathsf{P}(L)$ is dense in $Y_L \setminus \{1\}$. By Lemma 4.11(1), $\mathsf{P}(L)$ is order-dense in $Y_L \setminus \{1\}$. By Lemma 4.15, $a \in \mathsf{P}(L)$ iff $\mathscr{I}_a$ is nonempty and directed. Finally, by Lemma 4.14, $a \leq b$ iff $\forall U \in \mathscr{A}(Y_L)\,(a \in U \Rightarrow b \in U)$. Thus, $\mathscr{Y}(L)$ is a pointed generalized Priestley space. ∎

 We next turn to morphisms.

4.17. DEFINITION. *Let L_1, L_2 be algebraic frames and $\alpha \colon L_1 \to L_2$ an $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphism. If r is the right adjoint of α, we define $R_\alpha \subseteq Y_{L_2} \times Y_{L_1}$ by $p\, R_\alpha\, q$ if $r(p) \leq q$ in L_1.*

4.18. REMARK. It is not necessarily the case that if $p \in Y_{L_2}$, then $r(p) \in Y_{L_1}$. In Lemma 5.11 we will show exactly when $r[Y_{L_2}] \subseteq Y_{L_1}$.

4.19. LEMMA. *Let $\alpha \colon L_1 \to L_2$ be an $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphism. Then $R_\alpha \subseteq Y_{L_2} \times Y_{L_1}$ is a generalized Priestley morphism.*

PROOF. Let $p \in Y_{L_2}$ and $q \in Y_{L_1}$ with $p\, \not\mathrel{R}_\alpha\, q$. Then $r(p) \not\leq q$, so there is $k \in \mathsf{K}(L_1)$ with $k \leq r(p)$ and $k \not\leq q$. If $U = {\uparrow}k \cap Y_{L_1}$, then U is admissible by Lemma 4.13 and $q \notin U$. Let $q' \in R_\alpha[p]$. Then $r(p) \leq q'$. Therefore, $k \leq q'$, so $q' \in U$. Thus, $R_\alpha[p] \subseteq U$. This shows that R_α satisfies Definition 3.5(1). To see that R_α also satisfies Definition 3.5(2), suppose that $U \in \mathscr{A}(Y_{L_1})$. Then $U = {\uparrow}k \cap Y_{L_1}$ for some $k \in \mathsf{K}(L_1)$ by Lemma 4.13.

4.20. CLAIM. $\Box_{R_\alpha} U = {\uparrow}\alpha(k) \cap Y_{L_2}$.

PROOF. Suppose that $p \in {\uparrow}\alpha(k) \cap Y_{L_2}$. Then $\alpha(k) \leq p$, so $k \leq r(p)$. Let $q \in Y_{L_1}$ with $p\, R_\alpha\, q$. Then $r(p) \leq q$, so $k \leq q$, and hence $q \in U$. Therefore, $R_\alpha[p] \subseteq U$, yielding that $p \in \Box_{R_\alpha} U$. This proves that ${\uparrow}\alpha(k) \cap Y_{L_2} \subseteq \Box_{R_\alpha} U$. If $p \notin {\uparrow}\alpha(k) \cap Y_{L_2}$, then $\alpha(k) \not\leq p$, so $k \not\leq r(p)$. By Lemma 4.11(1), there is $q \in \mathsf{P}(L_1)$ with $k \not\leq q$ and $r(p) \leq q$. Therefore, $p\, R_\alpha\, q$ and $q \notin U$. Thus, $p \notin \Box_{R_\alpha} U$, and so $\Box_{R_\alpha} U = {\uparrow}\alpha(k) \cap Y_{L_2}$. ∎

 The claim together with Lemma 4.13 shows that $\Box_{R_\alpha} U \in \mathscr{A}(Y_{L_2})$. Thus, R_α is a generalized Priestley morphism. ∎

4.21. PROPOSITION. *There is a contravariant functor $\mathscr{Y} \colon \mathsf{AlgFrm}_{\mathsf{Sup}} \to \mathsf{PGPS}$ which sends $L \in \mathsf{AlgFrm}_{\mathsf{Sup}}$ to $\mathscr{Y}(L)$ and an $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphism α to R_α.*

PROOF. Proposition 4.16 and Lemma 4.19 show that $\mathscr{Y}$ is well defined. If α is the identity on L, then $p\, R_\alpha\, q$ iff $r(p) \leq q$ iff $p \leq q$, so R_α is equal to $\leq$. Therefore, $\mathscr{Y}$ sends identity morphisms to identity morphisms. To show that $\mathscr{Y}$ preserves composition, let $\alpha \colon L_1 \to L_2$ and $\beta \colon L_2 \to L_3$ be $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphisms. Write $R = R_\beta$, $S = R_\alpha$, and $T = R_{\beta\alpha}$. We show that $T = S * R$.

First suppose that $x\,T\,z$, so $r_{\beta\alpha}(x) \leq z$, and hence $r_\alpha r_\beta(x) \leq z$ because $r_{\beta\alpha} = r_\alpha r_\beta$. Let U be an admissible clopen upset of Y_{L_3}. By Lemma 4.13, $U = {\uparrow}k \cap Y_{L_3}$ for some $k \in \mathsf{K}(L_3)$. By Claim 4.20, $\square_R\square_S U = \square_R({\uparrow}\alpha(k) \cap Y_{L_2}) = {\uparrow}\beta\alpha(k) \cap Y_{L_1}$. Therefore,

$$x \in \square_R\square_S U \implies \beta\alpha(k) \leq x \implies k \leq r_\alpha r_\beta(x) \leq z,$$

so $z \in U$. Thus, $x\,(S * R)\,z$.

Conversely, if $x\,\slashed{T}\,z$, then there is $U \in \mathscr{A}(Y_{L_3})$ with $T[x] \subseteq U$ and $z \notin U$. We show that $x \in \square_R\square_S U$. Let $x\,R\,y$ and $y\,S\,t$. Then $r_\beta(x) \leq y$ and $r_\alpha(y) \leq t$, so $r_\alpha r_\beta(x) \leq t$. Therefore, $r_{\beta\alpha}(x) \leq t$, so $x\,T\,t$, and hence $t \in T[x]$. This yields $t \in U$. Thus, $x \in \square_R\square_S U$ as desired. Since $z \notin U$, it is false that $x\,(S * R)\,z$. This shows that $R_{\beta\alpha} = R_\alpha * R_\beta$. Consequently, $\mathscr{Y}$ is a contravariant functor. ∎

DUAL EQUIVALENCE OF $\mathsf{AlgFrm_{Sup}}$ **AND** PGPS. We now show that the functors $\mathscr{V}^a$ and $\mathscr{Y}$ yield a dual equivalence of $\mathsf{AlgFrm_{Sup}}$ and PGPS. To do so, we produce natural isomorphisms $\Upsilon\colon 1_{\mathsf{PGPS}} \to \mathscr{Y} \circ \mathscr{V}^a$ and $\eta\colon 1_{\mathsf{AlgFrm_{Sup}}} \to \mathscr{V}^a \circ \mathscr{Y}$.

Let $X \in \mathsf{PGPS}$. Define $\varepsilon_X\colon X \to Y_{\mathscr{V}^a(X)}$ by $\varepsilon_X(x) = {\uparrow}x$ for each $x \in X$.

4.22. LEMMA. *For $X \in \mathsf{PGPS}$, we have:*

(1) *ε_X is well defined.*

(2) *ε_X is an order-isomorphism.*

(3) *ε_X is a homeomorphism.*

(4) *$\varepsilon_X[X_0] = (Y_{\mathscr{V}^a(X)})_0$.*

PROOF. (1). Let $x \in X$. For each $y \notin {\uparrow}x$ we have $x \not\leq y$, so there is $U \in \mathscr{A}(X)$ with $x \in U$ and $y \notin U$ by Definition 3.8(6). Thus, ${\uparrow}x = \bigcap\{U \in \mathscr{A}(X) \mid x \in U\}$, and hence ${\uparrow}x \in \mathscr{V}^a(X)$.

We next show that ${\uparrow}x \in Y_{\mathscr{V}^a(X)}$. If ${\uparrow}x = X$, then this is trivial. Otherwise, we show that ${\uparrow}x$ is pseudoprime. Suppose that $C_i \in \mathscr{V}^a(X)$ with $C_1 \wedge \cdots \wedge C_n \ll {\uparrow}x$. Since ${\uparrow}x = \bigcap\{U \in \mathscr{A}(X) \mid x \in U\}$, recalling that $\mathscr{V}^a(X)$ is ordered by $\supseteq$, there is $U \in \mathscr{A}(X)$ with $C_1 \wedge \cdots \wedge C_n \leq U \leq {\uparrow}x$. Therefore, $x \in U \subseteq C_1 \wedge \cdots \wedge C_n$. We claim that $x \in C_1 \cup \cdots \cup C_n$. If not, then there are $V_i \in \mathscr{A}(X)$ with $C_i \subseteq V_i$ and $x \notin V_1 \cup \cdots \cup V_n$. Thus, $U \subseteq C_1 \wedge \cdots \wedge C_n \subseteq V_1 \wedge \cdots \wedge V_n$. We show this implies that $U \subseteq V := V_1 \cup \cdots \cup V_n$.

We first show $U \cap X_0 \subseteq V$. Let $y \in U \cap X_0$. If $y \notin V$, then $V_i \in \mathscr{I}_y$ for each i. Since $y \in X_0$, there is $W \in \mathscr{I}_y$ with $V \subseteq W$. Therefore, $V_1, \ldots, V_n \subseteq W$, and so $W \leq V_1, \ldots, V_n$, which implies $W \leq V_1 \wedge \cdots \wedge V_n$. Thus, $V_1 \wedge \cdots \wedge V_n \subseteq W$, and hence $U \subseteq W$. This is a contradiction since $y \in U$. Consequently, $U \cap X_0 \subseteq V$.

Now, if $U \not\subseteq V$, then $U \cap V^c$ is a nonempty open subset of X, and $U \cap V^c \subseteq X \setminus \{m\}$ since $m \in V$. Because X_0 is dense in $X \setminus \{m\}$, we have $X_0 \cap U \cap V^c \neq \varnothing$. This contradicts the inclusion $U \cap X_0 \subseteq V$. Therefore, $U \subseteq V = V_1 \cup \cdots \cup V_n$. But this is false since $x \in U$. Thus, $x \in C_1 \cup \cdots \cup C_n$, proving the claim. This implies that $C_i \leq {\uparrow}x$ for some i, so ${\uparrow}x \in Y_{\mathscr{V}^a(X)}$, and hence ε_X is well defined.

(2). It is clear that ε_X is order preserving and order reflecting. Thus, ε_X is 1-1. To see that it is onto, let $C \in Y_{\mathscr{V}^a(X)}$. If C is the top of $Y_{\mathscr{V}^a(X)}$, then $C = \uparrow m$, and hence $C = \varepsilon_X(m)$. Suppose that C is a pseudoprime. We show that $\min C$ is a singleton. Otherwise for each distinct pair $x, y \in \min C$ there is $U_{xy} \in \mathscr{A}(X)$ with $x \in U_{xy}$ and $y \notin U_{xy}$. Therefore, $C \nsubseteq U_{xy}$. The various U_{xy} cover C. By Lemma 4.3, C is an intersection from $\mathscr{A}(X)$, so by compactness, there are $V \in \mathscr{A}(X)$ and distinct pairs $x_1, y_1, \ldots, x_n, y_n \in \min C$ with $C \subseteq V \subseteq U_1 \cup \cdots \cup U_n$, where we write U_i for $U_{x_i y_i}$. In $\mathscr{V}^a(X)$ this says that $U_1 \wedge \cdots \wedge U_n \leq V \leq C$. By Lemma 4.4, $V \in \mathsf{K}(\mathscr{V}^a(X))$, implying that $U_1 \wedge \cdots \wedge U_n \ll C$. Because C is a pseudoprime, this forces $U_i \leq C$ for some i, so $C \subseteq U_i$. This is false by construction of the U_i. Therefore, $\min C = \{x\}$, and so $C = \uparrow x$ for some $x \in X$. Thus, ε_X is onto, hence an order-isomorphism.

(3). Since X and $Y_{\mathscr{V}^a(X)}$ are compact Hausdorff, by (2) it is sufficient to show that ε_X is continuous. For this, since the topology on $Y_{\mathscr{V}^a(X)}$ is generated by clopen upsets and their complements, by Lemmas 4.11(2) and 4.13 it suffices to show that if V is an admissible clopen upset of $Y_{\mathscr{V}^a(X)}$, then $\varepsilon_X^{-1}(V)$ is clopen. By Lemmas 4.4 and 4.13, there is $U \in \mathscr{A}(X)$ with $V = \uparrow U \cap Y_{\mathscr{V}^a(X)} = \{W \in Y_{\mathscr{V}^a(X)} \mid U \leq W\}$. We have

$$\varepsilon_X(x)^{-1}(V) = \varepsilon_X(x)^{-1}(\uparrow U \cap Y_{\mathscr{V}^a(X)}) = \{x \in X \mid \uparrow x \in \uparrow U \cap Y_{\mathscr{V}^a(X)}\}$$
$$= \{x \in X \mid U \leq \uparrow x\} = \{x \in X \mid \uparrow x \subseteq U\} = U.$$

Thus, ε_X is continuous, hence a homeomorphism.

(4). First let $x \in X_0$. Then $x \neq m$, so $\uparrow x$ is not the top of $\mathscr{V}^a(X)$. Suppose that $C, D \in \mathscr{V}^a(X)$ with $C \wedge D \leq \uparrow x$, so $x \in C \wedge D$. If $x \notin C \cup D$, then there are $U, V \in \mathscr{A}(X)$ with $C \subseteq U$, $D \subseteq V$, and $x \notin U \cup V$. Therefore, $U, V \in \mathscr{I}_x$, so from $x \in X_0$ it follows that there is $W \in \mathscr{I}_x$ with $U \cup V \subseteq W$. But $C \wedge D$ is the intersection of all $U' \in \mathscr{A}(X)$ with $C \cup D \subseteq U'$. This contradicts $x \in C \wedge D$. Thus, $x \in C$ or $x \in D$, so $C \leq \uparrow x$ or $D \leq \uparrow x$. Consequently, $\uparrow x \in \mathsf{P}(\mathscr{V}^a(X)) = (Y_{\mathscr{V}^a(X)})_0$.

Conversely, let $C \in (Y_{\mathscr{V}^a(X)})_0$. Since ε_X is onto, $C = \uparrow x$ for some $x \in X$. If $x \notin X_0$, then either $\mathscr{I}_x = \varnothing$ or $\mathscr{I}_x$ is not directed. If $\mathscr{I}_x = \varnothing$, then $x = m$, so $\uparrow x$ is the top of $\mathscr{V}^a(X)$, and hence not in $(Y_{\mathscr{V}^a(X)})_0$, a contradiction. If $\mathscr{I}_x$ is not directed, then there are $U, V \in \mathscr{I}_x$ but no larger admissible clopen upset misses x. Since the meet $U \wedge V$ in $\mathscr{V}^a(X)$ is the intersection of all admissible clopen upsets containing $U \cup V$, all such contain x, and so $x \in U \wedge V$. Therefore, $U \wedge V \leq \uparrow x$. This contradicts $\uparrow x \in (Y_{\mathscr{V}^a(X)})_0$. Thus, $x \in X_0$. ∎

4.23. DEFINITION. *Let $X \in \mathsf{PGPS}$. Define $\Upsilon_X \subseteq X \times Y_{\mathscr{V}^a(X)}$ by $x \, \Upsilon_X \, C$ iff $\varepsilon_X(x) \leq C$ in $\mathscr{V}^a(X)$.*

To prove that $\Upsilon \colon 1_{\mathsf{PGPS}} \to \mathscr{Y} \circ \mathscr{V}^a$ is a natural isomorphism, we require the following two lemmas.

4.24. LEMMA. *Let $X, Y \in \mathsf{PGPS}$ and suppose that $f \colon X \to Y$ is an order-isomorphism and homeomorphism with $f[X_0] = Y_0$. Define $R_f \subseteq X \times Y$ by $x \, R_f \, y$ if $f(x) \leq y$.*

(1) *R_f is a generalized Priestley morphism.*

(2) *If $g\colon Y \to Z$ is another map satisfying the same hypotheses as f, then $R_{g\circ f} = R_g * R_f$.*

(3) *R_f is a PGPS-isomorphism.*

PROOF. (1) and (2) follow from the same proof as [BJ08, Lem. 9.3] since f is a strong Priestley morphism (see Definition 5.5(2)). To see (3), for the identity morphism $1_X\colon X \to X$ we have $x\, R_{1_X}\, y$ iff $x \le y$, so R_{1_X} is equal to $\le$, an identity morphism in PGPS. Since f is an order-isomorphism and homeomorphism, $R_{f^{-1}}$ is a generalized Priestley morphism by (1), and it is the inverse of R_f by (2). Therefore, R_f is a PGPS-isomorphism. ∎

4.25. LEMMA. *Let $X \in$ PGPS. For $U \in \mathscr{A}(X)$ set $V = {\uparrow}U \cap Y_{\mathscr{V}^a(X)}$. Then $V \in \mathscr{A}(Y_{\mathscr{V}^a(X)})$ and $\Box_{\Upsilon_X} V = U$.*

PROOF. That $V \in \mathscr{A}(Y_{\mathscr{V}^a(X)})$ follows from Lemmas 4.4 and 4.13. Moreover,

$$\Box_{\Upsilon_X} V = \{x \in X \mid \Upsilon_X[x] \subseteq V\} = \{x \in X \mid x\, \Upsilon_X\, C \Rightarrow C \in V\}$$
$$= \{x \in X \mid x\, \Upsilon_X\, C \Rightarrow C \subseteq U\} = \{x \in X \mid C \subseteq {\uparrow}x \Rightarrow C \subseteq U\}$$
$$= \{x \in X \mid {\uparrow}x \subseteq U\} = U,$$

where the first equality on the last line holds since ${\uparrow}x$ is admissible by Lemma 4.22(1). ∎

4.26. PROPOSITION. *Let $X \in$ PGPS. Then Υ_X is a generalized Priestley morphism and $\Upsilon\colon 1_{\mathsf{PGPS}} \to \mathscr{Y} \circ \mathscr{V}^a$ is a natural isomorphism.*

PROOF. Let $X \in$ PGPS. By Lemma 4.22, $\varepsilon_X\colon X \to Y_{\mathscr{V}^a(X)}$ is an order-isomorphism and homeomorphism such that $\varepsilon_X[X_0] = (Y_{\mathscr{V}^a(X)})_0$. Therefore, Υ_X is a PGPS-isomorphism by Lemma 4.24.

It is left to show that Υ is natural. Let $R \subseteq X \times Y$ be a generalized Priestley morphism. We must show that $\Upsilon_Y * R = \mathscr{Y}\mathscr{V}^a(R) * \Upsilon_X$. Since $\mathscr{V}^a(R) = \Box_R$ and $\mathscr{Y}\mathscr{V}^a(R) = R_{\Box_R}$, we must show that $\Upsilon_Y * R = R_{\Box_R} * \Upsilon_X$.

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ R\ \ } & Y \\
{\scriptstyle \Upsilon_X}\big\downarrow & & \big\downarrow{\scriptstyle \Upsilon_Y} \\
Y_{\mathscr{V}^a(X)} & \xrightarrow[\ R_{\Box_R}\ ]{} & Y_{\mathscr{V}^a(Y)}
\end{array}
$$

Let $x \in X$ and $C \in Y_{\mathscr{V}^a(Y)}$. By Lemma 4.25, quantifying $V = {\uparrow}U \cap Y_{\mathscr{V}^a(Y)}$, we have

$$x\, (\Upsilon_Y * R)\, C \Leftrightarrow (\forall V)(x \in \Box_R \Box_{\Upsilon_Y} V \Rightarrow C \in V)$$
$$\Leftrightarrow (\forall U)(x \in \Box_R U \Rightarrow C \subseteq U).$$

On the other hand, $\Box_R$ is an $\mathsf{AlgFrm_{Sup}}$-morphism by Lemma 4.5. By Claim 4.20, $\Box_{R_{\Box_R}} V = ({\uparrow}\Box_R U) \cap Y_{\mathscr{V}^a(X)}$. Therefore, applying Lemma 4.25, we obtain

$$x\, (R_{\Box_R} * \Upsilon_X)\, C \Leftrightarrow (\forall V)(x \in \Box_{\Upsilon_X} \Box_{R_{\Box_R}} V \Rightarrow C \in V)$$
$$\Leftrightarrow (\forall U)(x \in \Box_{\Upsilon_X}(({\uparrow}\Box_R U) \cap Y_{\mathscr{V}^a(X)}) \Rightarrow C \subseteq U)$$
$$\Leftrightarrow (\forall U)(x \in \Box_R U \Rightarrow C \subseteq U).$$

This shows that $\Upsilon_Y * R = R_{\Box_R} * \Upsilon_X$, and hence Υ is a natural isomorphism. ∎

We now turn to the natural isomorphism $\eta \colon 1_{\mathsf{AlgFrm}_{\mathsf{Sup}}} \to \mathscr{V}^a \circ \mathscr{Y}$. Let L be an algebraic frame. Define $\eta_L \colon L \to \mathscr{V}^a(Y_L)$ by $\eta_L(a) = \uparrow a \cap Y_L$. To prove that η_L is well defined, we need the following generalization of Lemma 4.13.

4.27. LEMMA. *Let L be an algebraic frame and C a closed upset of Y_L. Then C is admissible iff $C = \uparrow a \cap Y_L$ for some $a \in L$.*

PROOF. Suppose that $C = \uparrow a \cap Y_L$ for some $a \in L$. If $q \in Y_L \setminus C$, then $a \not\leq q$. By Lemma 4.11(1), there is $p \in \mathsf{P}(L)$ with $a \not\leq p$ and $q \leq p$. Thus, C is admissible. Conversely, suppose that C is admissible. By Lemmas 4.3 and 4.13,

$$C = \bigcap \{\uparrow k \cap Y_L \mid k \in \mathsf{K}(L), C \subseteq \uparrow k \cap Y_L\}.$$

Set $S = \{k \in \mathsf{K}(L) \mid C \subseteq \uparrow k \cap Y_L\}$ and $a = \bigvee S$. If $p \in C$, then $p \in \uparrow k \cap Y_L$ and so $k \leq p$ for each $k \in S$. Therefore, $a \leq p$, which gives $p \in \uparrow a \cap Y_L$. For the reverse inclusion, let $p \in \uparrow a \cap Y_L$. Then $a \leq p$, so $k \leq p$ for each $k \in S$. This yields $p \in \bigcap \{\uparrow k \cap Y_L \mid k \in S\} = C$. Thus, $C = \uparrow a \cap Y_L$. ∎

4.28. PROPOSITION. *Let L be an algebraic frame. Then η_L is an $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphism and $\eta \colon 1_{\mathsf{AlgFrm}_{\mathsf{Sup}}} \to \mathscr{V}^a \circ \mathscr{Y}$ is a natural isomorphism.*

PROOF. That η_L is well defined follows from Lemma 4.27. To see that η_L is an $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphism, let $S \subseteq L$ and $a = \bigvee S$. Then $\uparrow a = \bigcap \{\uparrow s \mid s \in S\}$, so

$$\eta_L(a) = \uparrow a \cap Y_L = \bigcap \{\uparrow s \cap Y_L \mid s \in S\} = \bigvee \{\uparrow s \cap Y_L \mid s \in S\} = \bigvee \{\eta_L(s) \mid s \in S\}.$$

Therefore, η_L preserves arbitrary joins. To see it preserves compact elements, let $k \in \mathsf{K}(L)$. Then $\eta_L(k) = \uparrow k \cap Y_L$, which is compact by Lemmas 4.4 and 4.13. Therefore, η_L is an $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphism. It is clearly 1-1, and is onto by Lemma 4.27. Thus, η_L is an isomorphism.

To show naturality, let $\alpha \colon L_1 \to L_2$ be an $\mathsf{AlgFrm}_{\mathsf{Sup}}$-morphism with right adjoint r. Then $R_\alpha \subseteq Y_{L_2} \times Y_{L_1}$ is given by $p \, R_\alpha \, q$ if $r(p) \leq q$.

$$\begin{array}{ccc}
L_1 & \xrightarrow{\quad \alpha \quad} & L_2 \\
{\scriptstyle \eta_{L_1}}\downarrow & & \downarrow{\scriptstyle \eta_{L_2}} \\
\mathscr{V}^a(Y_{L_1}) & \xrightarrow[\quad \Box_{R_\alpha} \quad]{} & \mathscr{V}^a(Y_{L_2})
\end{array}$$

Let $a \in L_1$. Then $\eta_{L_2}(\alpha(a)) = \uparrow\alpha(a) \cap Y_{L_2}$. Also,

$$\Box_{R_\alpha}\eta_{L_1}(a) = \Box_{R_\alpha}(\uparrow a \cap Y_{L_1}) = \{x \in Y_{L_2} \mid R_\alpha[x] \subseteq \uparrow a \cap Y_{L_1}\}$$
$$= \{x \in Y_{L_2} \mid (\forall q \in Y_{L_1})(r(x) \leq q \Rightarrow a \leq q)\}.$$

By Lemma 4.11(1), $r(x) = \bigwedge(\uparrow r(x) \cap \mathsf{P}(L))$. Therefore,

$$\Box_{R_\alpha}\eta_{L_1}(a) = \{x \in Y_{L_2} \mid a \leq r(x)\} = \{x \in Y_{L_2} \mid \alpha(a) \leq x\} = \uparrow\alpha(a) \cap Y_{L_2}.$$

Thus, $\Box_{R_\alpha}\eta_{L_1}(a) = \uparrow\alpha(a) \cap Y_{L_2}$. This proves naturality, and hence η is a natural isomorphism. ∎

Propositions 4.6, 4.21, 4.26, and 4.28 yield our main result.

4.29. THEOREM. *The contravariant functors $\mathscr{V}^a$ and $\mathscr{Y}$ establish a dual equivalence between* $\mathsf{AlgFrm_{Sup}}$ *and* PGPS.

Putting Theorem 4.29 and Corollary 2.6 together yields Theorem 3.11. But we can say more:

4.30. THEOREM. *The functors establishing the duality of Theorem* 3.11 *are the compositions of the functors of Theorem* 4.29 *and Corollary* 2.6.

PROOF. Let $M \in \mathsf{DMS}$ and $L = \mathscr{F}(M)$. By [BJ13, Rem. 3.2] and [BJ11, Prop. 4.8], pseudoprime elements of L are precisely the optimal filters of M. Therefore,

$$\mathscr{Y}\mathscr{F}(M) = Y_L = \mathsf{PP}(L) \cup \{M\} = \mathsf{Opt}(M) \cup \{M\} = \mathscr{X}(M).$$

If $\alpha \colon M_1 \to M_2$ is a DMS-morphism, then as we saw in Section 2, $\mathscr{F}(\alpha)$ is the left adjoint of α^{-1}, and hence α^{-1} is the right adjoint of $\mathscr{F}(\alpha)$. Therefore, it follows from the definitions of $\mathscr{X}(\alpha)$ and $\mathscr{Y}\mathscr{F}(\alpha)$ that they coincide. In the opposite direction, if $X \in \mathsf{PGPS}$, then $\mathscr{K}\mathscr{V}^a(X) = \mathscr{A}(X)$ by Lemma 4.4. If $R \subseteq X \times Y$ is a generalized Priestley morphism, then $\mathscr{V}^a(R) = \square_R$ and $\mathscr{K}\mathscr{V}^a(R)$ is the restriction of $\square_R$ to $\mathscr{K}\mathscr{V}^a(Y) = \mathscr{A}(Y)$, which is exactly $\mathscr{A}(R)$. This shows that $\mathscr{X} = \mathscr{Y} \circ \mathscr{F}$ and $\mathscr{A} = \mathscr{K} \circ \mathscr{V}^a$.

■

We conclude this section by discussing what happens when we restrict our attention to bounded distributive meet-semilattices and compact algebraic frames. Let $X \in \mathsf{PGPS}$. As we pointed out in Remark 3.10, $\mathscr{A}(X)$ is bounded iff m is an isolated point of X. By Corollary 2.10(1), this is equivalent to $\mathscr{V}^a(X)$ being compact. By Remark 3.10, the full subcategory of PGPS consisting of those $X \in \mathsf{PGPS}$ in which m is isolated is equivalent to GPS. Thus, the diagram above restricts to the categories BDMS, $\mathsf{KAlgFrm_{Sup}}$, and GPS, which further restricts to $\mathsf{BDMS_B}$, $\mathsf{KAlgFrm_{SupB}}$, and $\mathsf{GPS_T}$ by Corollary 2.10(2) and Theorem 3.7.

The restrictions of the functors $\mathscr{F}, \mathscr{K}$ do not require any modification. The functors $\mathscr{A}, \mathscr{X}$ are modified as in Theorem 3.7. The functor $\mathscr{V}^a$ is the same, but this time defined on GPS. Because of this, $\varnothing \in \mathscr{V}^a(X)$ for each $X \in \mathsf{GPS}$. In fact, the maps $C \mapsto C \cap X$ and $D \mapsto D \cup \{m\}$ are inverse isomorphisms between $\mathscr{V}^a(X)$ and $\mathscr{V}^a(X^+)$, under which $\varnothing$ corresponds to $\{m\}$ (see Remark 3.10).

Finally, we need to slightly modify $\mathscr{Y}$. Indeed, if L is a compact algebraic frame, then 1 is an isolated point of Y_L. Therefore, $\mathsf{PP}(L) \in \mathsf{GPS}$. Also, if α is an $\mathsf{KAlgFrm_{Sup}}$-morphism, then R_α^- is a GPS-morphism (see Remark 3.10). Thus, we can modify $\mathscr{Y}$ by sending L to $\mathsf{PP}(L)$ and α to R_α^-. Using the same letter $\mathscr{Y}$ for this modified functor, we arrive at the following:

4.31. COROLLARY. *The functors of Theorem 4.30 restrict to yield an equivalence and dual equivalence of:*

(1) $\mathsf{KAlgFrm_{Sup}}$, BDMS, *and* GPS.

(2) $\mathsf{KAlgFrm_{SupB}}$, $\mathsf{BDMS_B}$, *and* $\mathsf{GPS_T}$.

Putting Theorem 4.29 and Corollaries 2.6 and 4.31(1) together yields the top layer of Figure 1.

5. Various morphisms between algebraic frames

So far we worked with the maps between algebraic frames that preserve arbitrary suprema and compact elements. This resulted in the category $\mathsf{AlgFrm_{Sup}}$, which is equivalent to DMS (Corollary 2.6) and dually equivalent to PGPS (Theorem 4.29). The equivalence and dual equivalence of their bounded versions was established in Corollary 4.31. As we pointed out in the introduction, there are several stronger notions of morphism between algebraic frames that are natural to consider. In this section we turn our attention to those and the corresponding morphisms between generalized Priestley spaces, thus obtaining the bottom three layers of Figure 1.

STRONG PRIESTLEY MORPHISMS.

5.1. DEFINITION.

(1) *Let* $\mathsf{DMS_{FSup}}$ *be the wide subcategory of* DMS *whose morphisms preserve all existing nonempty finite suprema, and let* $\mathsf{BDMS_{FSup}}$ *be the full subcategory of* $\mathsf{DMS_{FSup}}$ *whose objects are bounded.*

(2) *Let* $\mathsf{DMS_{FSupB}}$ *be the wide subcategory of* $\mathsf{DMS_{FSup}}$ *whose morphisms are bounded, and let* $\mathsf{BDMS_{FSupB}}$ *be the full subcategory of* $\mathsf{DMS_{FSupB}}$ *whose objects are bounded.*

5.2. REMARK. In [BJ08, BJ11] morphisms of $\mathsf{BDMS_{FSupB}}$ are called *sup-homomorphisms*.

Let L_1, L_2 be algebraic frames and $\alpha \colon L_1 \to L_2$ an $\mathsf{AlgFrm_{Sup}}$-morphism. Since we work with the dual orders on $\mathsf{K}(L_1)$ and $\mathsf{K}(L_2)$, the restriction $\alpha|_{\mathsf{K}(L_1)}$ is a $\mathsf{DMS_{FSup}}$-morphism iff the following condition is satisfied:

$$\text{If } \varnothing \neq S \subseteq \mathsf{K}(L_1) \text{ is finite and } \bigwedge S \in \mathsf{K}(L_1), \text{ then } \alpha\left(\bigwedge S\right) = \bigwedge \alpha[S]. \qquad \text{(FInf)}$$

5.3. DEFINITION.

(1) *Let* $\mathsf{AlgFrm_{FInf}}$ *be the wide subcategory of* $\mathsf{AlgFrm_{Sup}}$ *whose morphisms satisfy* (FInf), *and let* $\mathsf{AlgFrm_{FInfB}}$ *be the wide subcategory of* $\mathsf{AlgFrm_{FInf}}$ *whose morphisms are bounded.*

(2) *Let* $\mathsf{KAlgFrm_{FInf}}$ *be the full subcategory of* $\mathsf{AlgFrm_{FInf}}$ *and* $\mathsf{KAlgFrm_{FInfB}}$ *the full subcategory of* $\mathsf{AlgFrm_{FInfB}}$ *consisting of compact algebraic frames.*

As an immediate consequence of Corollaries 2.6, 2.8, 2.10, and the above observation, we obtain:

5.4. THEOREM.

(1) $\mathsf{AlgFrm}_{\mathsf{FInf}}$ *is equivalent to* $\mathsf{DMS}_{\mathsf{FSup}}$.

(2) $\mathsf{AlgFrm}_{\mathsf{FInfB}}$ *is equivalent to* $\mathsf{DMS}_{\mathsf{FSupB}}$.

(3) $\mathsf{KAlgFrm}_{\mathsf{FInf}}$ *is equivalent to* $\mathsf{BDMS}_{\mathsf{FSup}}$.

(4) $\mathsf{KAlgFrm}_{\mathsf{FInfB}}$ *is equivalent to* $\mathsf{BDMS}_{\mathsf{FSupB}}$.

We next describe the corresponding categories of generalized Priestley spaces utilizing the notion of strong Priestley morphisms from [BJ08, BJ11].

5.5. DEFINITION. *Let* X, Y *be generalized Priestley spaces.*

(1) *A generalized Priestley morphism* $R \subseteq X \times Y$ is *functional if* $R[x]$ *has a least element for each* $x \in X$. *Let* GPS_F *be the wide subcategory of* GPS *consisting of functional generalized Priestley morphisms.*

(2) *An order-preserving map* $f: X \to Y$ *is a* strong Priestley morphism *if* $U \in \mathscr{A}(Y)$ *implies* $f^{-1}(U) \in \mathscr{A}(X)$. *Let* GPS_S *be the category of generalized Priestley spaces and strong Priestley morphisms.*

The categories GPS_F and GPS_S consist of the same objects. If $R \subseteq X \times Y$ is a functional generalized Priestley morphism, then sending x to the least element of $R[x]$ defines a strong Priestley morphism $f_R: X \to Y$. Conversely, if $f: X \to Y$ is a strong Priestley morphism, then $R_f \subseteq X \times Y$ defined by $x\, R_f\, y$ iff $f(x) \leq y$ is a functional generalized Priestley morphism. Moreover,

$$f_{S*R} = f_S \circ f_R, \ R_{g \circ f} = R_g * R_f, \ R = R_{f_R}, \text{ and } f = f_{R_f}.$$

We thus obtain:

5.6. THEOREM. [BJ08, Prop. 9.5] *The categories* GPS_F *and* GPS_S *are isomorphic.*

In [BJ08, Thm. 9.6] it was shown that GPS_S is dually equivalent to $\mathsf{BDMS}_{\mathsf{FSupB}}$. This together with Theorem 5.4(4) yields:

5.7. COROLLARY. *The category* $\mathsf{KAlgFrm}_{\mathsf{FInfB}}$ *is equivalent to* $\mathsf{BDMS}_{\mathsf{FSupB}}$ *and dually equivalent to* GPS_S.

The notions of functional generalized Priestley morphism and strong Priestley morphism directly generalize to the pointed case. Let (X, m) and (Y, n) be pointed generalized Priestley spaces with m, n isolated. If $R \subseteq X \times Y$ is a functional Priestley morphism, then R^- may not be functional since $R^-[x]$ could be empty for some $x \in X^-$. Similarly, if $f: X \to Y$ is a strong Priestley morphism, then the restriction $f^-: X^- \to Y^-$ may only be a partial function. For R^- to be functional, and hence for f^- to be a total function, an additional condition is required. We thus arrive at the following wide subcategories of PGPS consisting of two kinds of functional generalized Priestley morphisms, and their corresponding categories of pointed generalized Priestley spaces and strong Priestley morphisms.

5.8. DEFINITION.

(1) *Let* $\mathsf{PGPS_F}$ *be the wide subcategory of* PGPS *consisting of functional morphisms, and let* $\mathsf{PGPS_{FT}}$ *be the wide subcategory of* $\mathsf{PGPS_F}$ *whose morphisms* $R \subseteq X \times Y$ *additionally satisfy* $x \neq m$ *implies* $R[x] \neq \{n\}$ *provided* m, n *are isolated.*

(2) *Let* $\mathsf{PGPS_S}$ *be the category of pointed generalized Priestley spaces and strong Priestley morphisms, and let* $\mathsf{PGPS_{ST}}$ *be the wide subcategory of* $\mathsf{PGPS_S}$ *whose morphisms* $f \colon X \to Y$ *additionally satisfy* $f[X^-] \subseteq Y^-$ *provided* m, n *are isolated.*

5.9. REMARK. The subscript F abbreviates functional, S abbreviates strong, and T abbreviates total because a $\mathsf{PGPS_S}$-morphism f is a $\mathsf{PGPS_{ST}}$-morphism iff $f^- \colon X^- \to Y^-$ is a total function provided m, n are isolated.

We point out that identity morphisms in $\mathsf{PGPS_S}$ and $\mathsf{PGPS_{ST}}$ are identity functions and composition is usual function composition. As a direct generalization of [BJ08, Prop. 9.5], we obtain:

5.10. PROPOSITION.

(1) $\mathsf{PGPS_F}$ *is isomorphic to* $\mathsf{PGPS_S}$.

(2) $\mathsf{PGPS_{FT}}$ *is isomorphic to* $\mathsf{PGPS_{ST}}$.

To describe the corresponding categories of distributive meet-semilattices and algebraic frames, we require the following two lemmas.

5.11. LEMMA. *Let* $\alpha \colon L_1 \to L_2$ *be an* $\mathsf{AlgFrm_{Sup}}$*-morphism and* $r \colon L_2 \to L_1$ *its right adjoint. The following are equivalent.*

(1) α *is an* $\mathsf{AlgFrm_{FInf}}$*-morphism.*

(2) $r[Y_{L_2}] \subseteq Y_{L_1}$.

(3) $r \colon Y_{L_2} \to Y_{L_1}$ *is a strong Priestley morphism whose corresponding functional generalized Priestley morphism is* R_α.

PROOF. (1)$\Rightarrow$(2). Let $p \in Y_{L_2}$. Since r preserves arbitrary meets, $r(1) = 1$, so we may assume that $p \in \mathsf{PP}(L_2)$. If $r(p) = 1$, there is nothing to prove. Suppose that $r(p) \neq 1$. We prove that $r(p) \in \mathsf{PP}(L_1)$. Let $a_1, \ldots, a_n \in L_1$ with $a_1 \wedge \cdots \wedge a_n \ll r(p)$. Since L_1 is algebraic, there is $k \in \mathsf{K}(L_1)$ with $a_1 \wedge \cdots \wedge a_n \leq k \leq r(p)$. We show that $a_i \leq r(p)$ for some i. If not, then there are compact l_i with $l_i \leq a_i$ and $l_i \not\leq r(p)$. Since $l_1 \wedge \cdots \wedge l_n \leq k$ and L_1 is distributive, $(l_1 \vee k) \wedge \cdots \wedge (l_n \vee k) = k$. Thus, the meet of $l_1 \vee k, \ldots, l_n \vee k$ exists in $\mathsf{K}(L_1)$, so $\alpha(k) = \alpha(l_1 \vee k) \wedge \cdots \wedge \alpha(l_n \vee k)$ by (1). Since α preserves joins and L_2 is distributive,

$$\alpha(k) = \bigwedge_{i=1}^{n} (\alpha(l_i) \vee \alpha(k)) = \left(\bigwedge_{i=1}^{n} \alpha(l_i) \right) \vee \alpha(k).$$

Therefore, $\alpha(l_1) \wedge \cdots \wedge \alpha(l_n) \leq \alpha(k)$. Since $k \leq r(p)$ implies $\alpha(k) \leq p$ and $\alpha(k) \in \mathsf{K}(L_2)$, we have $\alpha(l_1) \wedge \cdots \wedge \alpha(l_n) \ll p$. Because $p \in \mathsf{PP}(L_2)$, it follows that $\alpha(l_i) \leq p$ for some i. Consequently, $l_i \leq r(p)$ for some i. The obtained contradiction proves that $r(p) \in \mathsf{PP}(L_1)$.

$(2)\Rightarrow(3)$. Let $p \in Y_{L_2}$. By (2), $r(p) \in Y_{L_1}$. Since $q \in R_\alpha[p]$ iff $r(p) \leq q$, we see that $r(p)$ is the least element of $R_\alpha[p]$. Thus, R_α is functional, and r is its corresponding strong Priestley morphism.

$(3)\Rightarrow(1)$. Let $S \subseteq \mathsf{K}(L_1)$ be nonempty finite and $\bigwedge S \in \mathsf{K}(L_1)$. Since α is order preserving, $\alpha(\bigwedge S) \leq \bigwedge \alpha[S]$. Suppose that $\bigwedge \alpha[S] \not\leq \alpha(\bigwedge S)$. By Lemma 4.11(1), there is $p \in \mathsf{P}(L_2)$ with $\bigwedge \alpha[S] \not\leq p$ and $\alpha(\bigwedge S) \leq p$. This yields $\bigwedge S \leq r(p)$. Since $\bigwedge S \in \mathsf{K}(L_1)$, we see that $\bigwedge S \ll r(p)$, so $s \leq r(p)$ for some $s \in S$ since $r(p) = 1$ or $r(p) \in \mathsf{PP}(L_1)$. Therefore, $\alpha(s) \leq p$, and so $\bigwedge \alpha[S] \leq p$. The obtained contradiction proves that $\bigwedge \alpha[S] \leq \alpha(\bigwedge S)$, hence the equality. Thus, α is an $\mathsf{AlgFrm_{FInf}}$-morphism. ∎

5.12. LEMMA. *Let L_1 and L_2 be compact algebraic frames, $\alpha\colon L_1 \to L_2$ an $\mathsf{AlgFrm_{Sup}}$-morphism, and $r\colon L_2 \to L_1$ its right adjoint. The following are equivalent.*

(1) *α is an $\mathsf{AlgFrm_{FInfB}}$-morphism.*

(2) *$r[Y_{L_2}] \subseteq Y_{L_1}$ and $r(p) \neq 1$ for each $p \in \mathsf{PP}(L_2)$.*

(3) *$r\colon Y_{L_2} \to Y_{L_1}$ is a strong Priestley morphism whose corresponding functional generalized Priestley morphism is R_α and $R_\alpha[p] \neq \{1\}$ for each $p \in \mathsf{PP}(L_2)$.*

PROOF. $(1)\Rightarrow(2)$. By Lemma 5.11, $r[Y_{L_2}] \subseteq Y_{L_1}$. Since L_1, L_2 are compact and α is a $\mathsf{AlgFrm_{FInfB}}$-morphism, we have $\alpha(1) = 1$. If $p \in \mathsf{PP}(L_2)$ with $r(p) = 1$, then $\alpha(1) \leq p$, so $p = 1$, which is false. Therefore, $r(p) \neq 1$.

$(2)\Rightarrow(3)$. By Lemma 5.11, r is a strong Priestley morphism whose corresponding functional generalized Priestley morphism is R_α. If $p \in \mathsf{PP}(L_2)$, then $r(p) \neq 1$ by (2). Therefore, by Lemma 4.11(1), there is $q \in \mathsf{P}(L_1)$ with $r(p) \leq q$. Consequently, $q \in R_\alpha[p]$, and so $R_\alpha[p] \neq \{1\}$.

$(3)\Rightarrow(1)$. By Lemma 5.11, α is an $\mathsf{AlgFrm_{FInf}}$-morphism. It then suffices to show that $\alpha(1) = 1$. If $\alpha(1) \neq 1$, then there is $p \in \mathsf{P}(L_2)$ with $\alpha(1) \leq p$. This yields $1 \leq r(p)$, so $R_\alpha[p] = \{1\}$, contradicting (3). Therefore, $\alpha(1) = 1$, and thus α is an $\mathsf{AlgFrm_{FInfB}}$-morphism. ∎

Lemmas 5.11 and 5.12 give the following:

5.13. THEOREM.

(1) *The duality of Theorem 4.29 between PGPS and $\mathsf{AlgFrm_{Sup}}$ restricts to a duality between $\mathsf{PGPS_F}$ and $\mathsf{AlgFrm_{FInf}}$ and yields a duality between $\mathsf{PGPS_S}$ and $\mathsf{AlgFrm_{FInf}}$.*

(2) *The duality also restricts to a duality between $\mathsf{PGPS_{FT}}$ and $\mathsf{AlgFrm_{FInfB}}$ and yields a duality between $\mathsf{PGPS_{ST}}$ and $\mathsf{AlgFrm_{FInfB}}$.*

PROOF. (1). Let X be a pointed generalized Priestley space. Then Υ_X is a functional morphism since the least element of $\Upsilon_X[x]$ is $\varepsilon_X(x)$ by the definition of Υ_X and Lemma 4.22(1). Therefore, Υ_X is a $\mathsf{PGPS_F}$-isomorphism. In addition, if L is an algebraic frame, then it follows from the proof of Proposition 4.28 that η_L is a poset isomorphism. Therefore, it is an $\mathsf{AlgFrm_{FInfB}}$-isomorphism. From these observations, Proposition 5.10(1), and Lemma 5.11 it follows that the duality of Theorem 4.29 restricts to a duality between $\mathsf{PGPS_F}$ and $\mathsf{AlgFrm_{FInf}}$.

(2). The proof is similar to that of (1) except that Lemma 5.12 is used instead of Lemma 5.11. ∎

As a consequence of Theorems 5.4 and 5.13, we obtain:

5.14. COROLLARY.

(1) $\mathsf{AlgFrm_{FInf}}$ is equivalent to $\mathsf{DMS_{FSup}}$ and dually equivalent to $\mathsf{PGPS_S}$.

(2) $\mathsf{AlgFrm_{FInfB}}$ is equivalent to $\mathsf{DMS_{FSupB}}$ and dually equivalent to $\mathsf{PGPS_{ST}}$.

5.15. REMARK. If $\alpha\colon L_1 \to L_2$ is an $\mathsf{AlgFrm_{FInf}}$-morphism, then $\mathscr{Y}(\alpha) = R_\alpha$. Applying Lemma 5.11, the corresponding strong Priestley morphism is the right adjoint r to α restricted to Y_{L_2}. Therefore, we may view that $\mathscr{Y}\colon \mathsf{AlgFrm_{FInf}} \to \mathsf{PGPS_S}$ acts on morphisms by sending α to $r\colon Y_{L_2} \to Y_{L_1}$.

If $R \subseteq X \times Y$ is a $\mathsf{PGPS_S}$-morphism, then $\mathscr{V}^a(R) = \square_R$. Let f be the strong Priestley morphism corresponding to R. Then $\square_R C = f^{-1}(C)$ for each $C \in \mathscr{V}^a(Y)$ (see [BJ08, Lem. 9.2]). Thus, we may view that $\mathscr{V}^a\colon \mathsf{PGPS_S} \to \mathsf{AlgFrm_{FInf}}$ acts on morphisms by sending a strong Priestley morphism f to f^{-1}. Similar observations apply to $\mathscr{Y}\colon \mathsf{AlgFrm_{FInfB}} \to \mathsf{PGPS_{ST}}$ and $\mathscr{V}^a\colon \mathsf{PGPS_{ST}} \to \mathsf{AlgFrm_{FInfB}}$.

We next restrict the dualities of Theorem 5.13 to compact algebraic frames. As is customary, by a *partial function* from X to Y we mean a function from a subset of X to Y. We denote such partial function by $f\colon X \dashrightarrow Y$.

5.16. DEFINITION.

(1) *Let* $X, Y \in \mathsf{GPS}$. *A* partial strong Priestley morphism *between X and Y is a partial function $f\colon X \dashrightarrow Y$ whose domain is a clopen downset of X such that $U \in \mathscr{A}(Y)$ implies $X \setminus f^{-1}(Y \setminus U) \in \mathscr{A}(X)$.*

(2) *Let* $\mathsf{GPS_{PS}}$ *be the category of generalized Priestley spaces and partial strong Priestley morphisms.*

5.17. PROPOSITION. $\mathsf{GPS_{PS}}$ *is equivalent to the full subcategory of* $\mathsf{PGPS_S}$ *consisting of generalized Priestley spaces with isolated maxima.*

PROOF. By Remark 3.10, GPS is equivalent to the full subcategory of PGPS whose objects have isolated maxima. Let (X, m) and (Y, n) be generalized Priestley spaces with m, n isolated. As in the remark, set $X^- = X \setminus \{m\}$ and $Y^- = Y \setminus \{n\}$. By Proposition 5.10(1), functional morphisms between X and Y correspond to strong Priestley morphisms. Therefore, it is enough to show that the latter correspond to partial strong Priestley morphisms between X^- and Y^-.

Let $f \colon X \to Y$ be a strong Priestley morphism. Set $C = f^{-1}(Y^-)$. Then $f \colon C \to Y^-$ is a well-defined function, and we show that $f \colon X^- \dashrightarrow Y^-$ is a partial strong Priestley morphism. First, C is a clopen downset of X since Y^- is a clopen downset of Y and f is a continuous order-preserving function. It is then a clopen downset of X^- since $m \notin C$. Let $U \in \mathscr{A}(Y^-)$. Then $V := U \cup \{n\} \in \mathscr{A}(Y)$ and we have

$$
\begin{aligned}
X^- \setminus f^{-1}(Y^- \setminus U) &= \{x \in X^- \mid x \notin f^{-1}(Y^- \setminus U)\} \\
&= \{x \in X^- \mid x \notin C \text{ or } x \in C \ \& \ f(x) \in U\} \\
&= \{x \in X^- \mid f(x) = n \text{ or } f(x) \in U\} \\
&= \{x \in X^- \mid f(x) \in V\} \\
&= X^- \cap f^{-1}(V) \in \mathscr{A}(X^-)
\end{aligned}
$$

because $f^{-1}(V) \in \mathscr{A}(X)$. Therefore, $f \colon X^- \dashrightarrow Y^-$ is a partial strong Priestley morphism.

Conversely, let $f \colon X^- \dashrightarrow Y^-$ be a partial strong Priestley morphism, and let C be the domain of f. Extend f to a function $g \colon X \to Y$ by setting $g(x) = n$ for each $x \in X \setminus C$. Since f is order preserving and n is the top of Y, we see that g is order preserving. To show that g is a strong Priestley morphism, let $V \in \mathscr{A}(Y)$. Set $U = V \setminus \{n\} \in \mathscr{A}(Y^-)$. We have $X^- \setminus f^{-1}(Y^- \setminus U) = (X^- \setminus C) \cup f^{-1}(U)$. Therefore,

$$
\begin{aligned}
g^{-1}(V) = \{x \in X \mid g(x) \in V\} &= g^{-1}(n) \cup g^{-1}(U) \\
&= \{m\} \cup (X^- \setminus C) \cup f^{-1}(U) \\
&= \{m\} \cup (X^- \setminus f^{-1}(Y^- \setminus U)) \in \mathscr{A}(X)
\end{aligned}
$$

since $X^- \setminus f^{-1}(Y^- \setminus U) \in \mathscr{A}(X^-)$. Thus, g is a strong Priestley morphism. Consequently, $\mathsf{GPS_{PS}}$ is equivalent to the full subcategory of $\mathsf{PGPS_S}$ consisting of generalized Priestley spaces with isolated maxima. ∎

As an immediate consequence of Proposition 5.17 we obtain:

5.18. THEOREM. *The duality of Theorem* 5.13 *between* $\mathsf{PGPS_S}$ *and* $\mathsf{AlgFrm_{FInf}}$ *restricts to a duality between* $\mathsf{GPS_{PS}}$ *and* $\mathsf{KAlgFrm_{FInf}}$.

This together with Theorem 5.4(3) gives:

5.19. COROLLARY. *The category* $\mathsf{KAlgFrm_{FInf}}$ *is equivalent to* $\mathsf{BDMS_{FSup}}$ *and dually equivalent to* $\mathsf{GPS_{PS}}$.

Putting Corollaries 5.7, 5.14, and 5.19 together yields the middle two layers of Figure 1.

STRONG PRIESTLEY MORPHISMS PRESERVING PRIME ELEMENTS. To obtain the bottom layer of Figure 1, we turn to the most natural category of algebraic frames, in which morphisms are frame homomorphisms preserving compact elements (that is, $\mathsf{AlgFrm_{Sup}}$-morphisms preserving finite infima).

5.20. DEFINITION. *Let* AlgFrm *be the wide subcategory of* $\mathsf{AlgFrm_{Sup}}$ *whose morphisms preserve finite infima.*

5.21. REMARK. Clearly AlgFrm is also a wide subcategory of $\mathsf{AlgFrm_{FInfB}}$.

We next describe the wide subcategory of $\mathsf{PGPS_{ST}}$ that is dually equivalent to AlgFrm.

5.22. DEFINITION. *Let* $\mathsf{PGPS_P}$ *denote the wide subcategory of* $\mathsf{PGPS_S}$ *whose morphisms* $f \colon X \to Y$ *satisfy* $f[X_0] \subseteq Y_0$, *and define* $\mathsf{GPS_P}$ *similarly.*

5.23. REMARK. Let $f \colon X \to Y$ be a $\mathsf{PGPS_P}$-morphism and suppose that m, n are isolated. Then $\{m\} \in \mathscr{A}(X)$, so $\max(X \setminus \{m\}) \subseteq X_0$, which implies that $\max X^- \subseteq X_0$. Let $x \in X^-$. By Definition 3.8(4), there is $z \in X_0$ with $x \le z$. Therefore, $f(x) \le f(z)$, and $f(z) \in Y_0$ by hypothesis. Consequently, $f(x) \ne n$. Thus, $f[X^-] \subseteq Y^-$, and hence $\mathsf{PGPS_P}$ is a wide subcategory of $\mathsf{PGPS_{ST}}$.

5.24. LEMMA. *Let* $\alpha \colon L_1 \to L_2$ *be an* $\mathsf{AlgFrm_{Sup}}$-*morphism and* $r \colon L_2 \to L_1$ *its right adjoint. The following are equivalent.*

(1) α *is a frame homomorphism.*

(2) $r \colon Y_{L_2} \to Y_{L_1}$ *is a strong Priestley morphism and* $r[\mathsf{P}(L_2)] \subseteq \mathsf{P}(L_1)$.

(3) *Let* S *be a finite subset of* $\mathsf{K}(L_1)$ *and* $k \in \mathsf{K}(L_2)$ *with* $k \le \bigwedge \alpha[S]$. *Then there is* $c \in \mathsf{K}(L_1)$ *with* $c \le \bigwedge S$ *and* $k \le \alpha(c)$.

PROOF. (1)$\Rightarrow$(3). Let S be a finite subset of $\mathsf{K}(L_1)$ and $k \in \mathsf{K}(L_2)$ with $k \le \bigwedge \alpha[S]$. By (1), $k \le \alpha(\bigwedge S)$. Since k is compact, $\alpha(\bigwedge S) = \bigvee \{\alpha(c) \mid c \in \mathsf{K}(L_1), c \le \bigwedge S\}$, and the join is directed, there is $c \in \mathsf{K}(L_1)$ with $c \le \bigwedge S$ and $k \le \alpha(c)$.

(3)$\Rightarrow$(2). We first show that if $S \subseteq \mathsf{K}(L_1)$ is finite with $\bigwedge S \in \mathsf{K}(L_1)$, then $\alpha(\bigwedge S) = \bigwedge \alpha[S]$. The inequality $\alpha(\bigwedge S) \le \bigwedge \alpha[S]$ holds since α is order preserving. Let $k \in \mathsf{K}(L_2)$ with $k \le \bigwedge \alpha[S]$. By (3), there is $c \in \mathsf{K}(L_1)$ with $c \le \bigwedge S$ and $k \le \alpha(c)$. Therefore, $k \le \alpha(c) \le \alpha(\bigwedge S)$. Since $\bigwedge \alpha[S]$ is the join of the compact elements below it, we see that $\bigwedge \alpha[S] \le \alpha(\bigwedge S)$, hence the equality. This by Lemma 5.12 implies that r is a strong Priestley morphism.

We next show that if $p \in \mathsf{P}(L_2)$, then $r(p) \in \mathsf{P}(L_1)$. By the previous paragraph, $\alpha(1) = \alpha(\bigwedge \varnothing) = \bigwedge \alpha[\varnothing] = 1$. Therefore, $p \ne 1$ implies that $r(p) \ne 1$. Let $a, b \in L_1$ with $a \wedge b \le r(p)$. Then $\alpha(a \wedge b) \le p$. Suppose that $\alpha(a) \wedge \alpha(b) \not\le p$. Then there is $k \in \mathsf{K}(L_2)$ with $k \le \alpha(a) \wedge \alpha(b)$ and $k \not\le p$. By (3), there is $c \in \mathsf{K}(L_1)$ with $c \le a \wedge b$ and $k \le \alpha(c)$. Therefore, $\alpha(c) \not\le p$, so $c \not\le r(p)$. This contradicts $c \le a \wedge b \le r(p)$. Thus, $\alpha(a) \wedge \alpha(b) \le p$, so $\alpha(a) \le p$ or $\alpha(b) \le p$ because $p \in \mathsf{P}(L_2)$. Consequently, $a \le r(p)$ or $b \le r(p)$, and hence $r(p) \in \mathsf{P}(L_1)$.

$(2) \Rightarrow (1)$. It is sufficient to show that α preserves binary meets and $\alpha(1) = 1$. Let $a, b \in L_1$. Then $\alpha(a \wedge b) \leq \alpha(a) \wedge \alpha(b)$ since α is order preserving. Suppose that $\alpha(a) \wedge \alpha(b) \not\leq \alpha(a \wedge b)$. By Lemma 4.11(1), there is $p \in \mathsf{P}(L_2)$ with $\alpha(a) \wedge \alpha(b) \not\leq p$ and $\alpha(a \wedge b) \leq p$. This implies that $a \wedge b \leq r(p)$. By (2), $a \leq r(p)$ or $b \leq r(p)$. Thus, $\alpha(a) \leq p$ or $\alpha(b) \leq p$, and hence $\alpha(a) \wedge \alpha(b) \leq p$. The obtained contradiction shows that $\alpha(a) \wedge \alpha(b) \leq \alpha(a \wedge b)$, hence the equality. If $\alpha(1) \neq 1$, then there is $p \in \mathsf{P}(L_2)$ with $\alpha(1) \leq p$. By (2), $1 \leq r(p) \in \mathsf{P}(L_1)$, a contradiction. Thus, $\alpha(1) = 1$. $\blacksquare$

5.25. THEOREM. *The duality of Theorem 5.13(2) between $\mathsf{PGPS_{ST}}$ and $\mathsf{AlgFrm_{FInfB}}$ restricts to a duality between $\mathsf{PGPS_P}$ and AlgFrm.*

PROOF. Let X be a pointed generalized Priestley space. By Lemma 4.22, ε_X is a $\mathsf{PGPS_P}$-isomorphism. In addition, if $L \in \mathsf{AlgFrm}$, then η_L is a poset isomorphism, so a frame isomorphism. Thus, Lemma 5.24 yields that the duality of Theorem 5.13(2) between $\mathsf{PGPS_{ST}}$ and $\mathsf{AlgFrm_{FInfB}}$ restricts to a duality between $\mathsf{PGPS_P}$ and AlgFrm. $\blacksquare$

Lemma 5.24(3) suggests the following definition. To simplify notation, we denote the set of upper bounds of a subset S of a poset by S^u.

5.26. DEFINITION. *We denote by $\mathsf{DMS_P}$ the wide subcategory of DMS whose morphisms $\alpha \colon M_1 \to M_2$ satisfy the following condition:*

$$\text{If } S \subseteq M_1 \text{ is finite and } x \in \alpha[S]^u, \text{ then } \exists c \in S^u : \alpha(c) \leq x. \tag{P}$$

5.27. REMARK. The subscript P in the above definition is motivated by Lemma 5.33(2) where we show that α is a $\mathsf{DMS_P}$-morphism iff α pulls prime filters back to prime filters.

5.28. LEMMA. *$\mathsf{DMS_P}$ is a wide subcategory of $\mathsf{DMS_{FSupB}}$.*

PROOF. Let $\alpha \colon M_1 \to M_2$ be a $\mathsf{DMS_P}$-morphism. Let $\varnothing \neq S$ be a finite subset of M_1 such that $\bigvee S$ exists in M_1. Since α is order preserving, $\bigvee \alpha[S] \leq \alpha(\bigvee S)$. Let $x \in \alpha[S]^u$. Since α is a $\mathsf{DMS_P}$-morphism, there is $c \in S^u$ such that $\alpha(c) \leq x$. Therefore, $\bigvee S \leq c$, so $\alpha(\bigvee S) \leq \alpha(c)$, and hence $\alpha(\bigvee S) \leq x$. Thus, $\alpha(\bigvee S) = \bigvee \alpha[S]$, and so α is a $\mathsf{DMS_{FSup}}$-morphism. To see that it is a $\mathsf{DMS_{FSupB}}$-morphism, suppose that M_1, M_2 are bounded. Setting $S = \varnothing$, (P) implies that for $x = 0$ there is $c \in M_2$ with $\alpha(c) \leq 0$. This forces $\alpha(c) = 0$, and therefore, $\alpha(0) = 0$. Consequently, $\mathsf{DMS_P}$ is a wide subcategory of $\mathsf{DMS_{FSupB}}$. $\blacksquare$

Putting Theorems 5.4(1), 5.25 and Lemmas 5.24, 5.28 together yields:

5.29. THEOREM. *AlgFrm is equivalent to $\mathsf{DMS_P}$ and dually equivalent to $\mathsf{PGPS_P}$.*

5.30. DEFINITION.

(1) *Let $\mathsf{KAlgFrm}$ be the full subcategory of AlgFrm consisting of compact algebraic frames.*

(2) *Let $\mathsf{BDMS_P}$ be the full subcategory of $\mathsf{DMS_P}$ consisting of bounded distributive meet-semilattices.*

As an immediate consequence of Theorem 5.29 we obtain:

5.31. THEOREM. KAlgFrm *is equivalent to* BDMS$_\mathsf{P}$ *and dually equivalent to* GPS$_\mathsf{P}$.

Putting Theorems 5.29 and 5.31 together yields the bottom layer of Figure 1. We point out that BDMS$_\mathsf{P}$ is a proper subcategory of BDMS$_{\mathsf{FSupB}}$ (see Example 5.32). This contrasts with the bounded distributive lattice case, where the full subcategories of BDMS$_{\mathsf{FSupB}}$ and BDMS$_\mathsf{P}$ consisting of lattices are equal (see Remark 6.18).

5.32. EXAMPLE. Let M be the distributive meet-semilattice shown below.
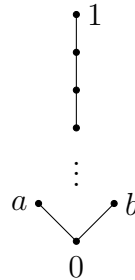
Set $F = M \setminus \{a, b, 0\}$ and define $\alpha \colon M \to M$ by

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in F, \\ x & \text{if } x \in \{a, b, 0\}. \end{cases}$$

Then α is a DMS$_{\mathsf{FSupB}}$-morphism. We show that α is not a DMS$_\mathsf{P}$-morphism. For, let $x \in F$ with $x < 1$. If $S = \{a, b\}$, then $x \in \alpha[S]^u$, but since $S^u = F$, there is no $c \in S^u$ with $\alpha(c) \leq x$. Therefore, Condition (P) does not hold, and hence α is not a DMS$_\mathsf{P}$-morphism. This shows that BDMS$_\mathsf{P}$ is a proper wide subcategory of BDMS$_{\mathsf{FSupB}}$. Corollary 5.19 and Theorem 5.31 then show that KAlgFrm is a proper wide subcategory of KAlgFrm$_{\mathsf{FInfB}}$ and GPS$_\mathsf{P}$ is a proper wide subcategory of GPS$_\mathsf{S}$.

We conclude this section by pointing out that Theorem 5.31 yields the duality result of Hansoul and Poussart [HP08]. To see this, we recall that a nonempty downset I of a meet-semilattice M is an *ideal* if $a, b \in I$ implies $\uparrow a \cap \uparrow b \cap I \neq \varnothing$. It is easy to see that I is an ideal iff for each finite subset S of I, we have $\bigcap_{s \in S} \uparrow s \cap I \neq \varnothing$. As usual, an ideal I is *proper* if $I \neq M$ and a proper ideal I is *prime* if $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

5.33. LEMMA. *Let* $M_1, M_2 \in$ DMS *and* $\alpha \colon M_1 \to M_2$ *be a* DMS-*morphism. The following are equivalent.*

(1) α *is a* DMS$_\mathsf{P}$-*morphism.*

(2) *If P is a prime filter of M_2, then $\alpha^{-1}(P)$ is a prime filter of M_1.*

(3) *If I is an ideal of M_2, then $\alpha^{-1}(I)$ is an ideal of M_1.*

PROOF. (1)$\Rightarrow$(2). Let P be a prime filter of M_2. Then $\alpha^{-1}(P)$ is a filter of M_1. By (1) and Lemma 5.28, α is a $\mathsf{DMS_{FSupB}}$-morphism. Because P is proper, there is $x \in M_2 \setminus P$. Set $S = \alpha^{-1}(\downarrow x)$. Then $x \in \alpha[S]^u$. By (P), there is $c \in S^u$ with $\alpha(c) \leq x$. Consequently, $\alpha(c) \notin P$, so $c \notin \alpha^{-1}(P)$, and hence $\alpha^{-1}(P)$ is a proper filter. To see it is prime, suppose that F, G are filters of M_1 with $F \cap G \subseteq \alpha^{-1}(P)$. Then $\uparrow\alpha[F], \uparrow\alpha[G]$ are filters of M_2. We show that $\uparrow\alpha[F] \cap \uparrow\alpha[G] \subseteq P$. Let $x \in \uparrow\alpha[F] \cap \uparrow\alpha[G]$. Then $\alpha(a), \alpha(b) \leq x$ for some $a \in F$ and $b \in G$. By (1), there is $c \in M_1$ with $a, b \leq c$ and $\alpha(c) \leq x$. Therefore, $c \in F \cap G$, so $\alpha(c) \in P$. This yields $x \in P$, as desired. Thus, since P is prime, $\uparrow\alpha[F] \subseteq P$ or $\uparrow\alpha[G] \subseteq P$, so $F \subseteq \alpha^{-1}(P)$ or $G \subseteq \alpha^{-1}(P)$. Consequently, $\alpha^{-1}(P)$ is a prime filter of M_1.

(2)$\Rightarrow$(3). We first show that the pullback of a prime ideal is a prime ideal. If I is a prime ideal of M_2, then $M_2 \setminus I$ is a prime filter (see, e.g., [BJ11, Prop. 2.3]). By (2), $\alpha^{-1}(M_2 \setminus I)$ is a prime filter of M_1. Since $\alpha^{-1}(M_2 \setminus I) = M_1 \setminus \alpha^{-1}(I)$, we conclude that $\alpha^{-1}(I)$ is a prime ideal of M_1. Finally, by the prime filter theorem for distributive meet-semilattices (see, e.g., [Grä11, p. 168] for the dual statement for distributive join-semilattices), each ideal is an intersection of prime ideals, and hence the pullback of an ideal is an ideal.

(3)$\Rightarrow$(1). Suppose that S is a finite subset of M_1 and $x \in M_2$ with $\alpha(s) \leq x$ for each $s \in S$. Since $\downarrow x$ is an ideal of M_2, (3) implies that $\alpha^{-1}(\downarrow x)$ is an ideal of M_1. Therefore, because $S \subseteq \alpha^{-1}(\downarrow x)$, there is $c \in \alpha^{-1}(\downarrow x)$ with $s \leq c$ for each $s \in S$. Thus, α is a $\mathsf{DMS_P}$-morphism. ∎

5.34. REMARK. Let $\alpha \colon M_1 \to M_2$ be a $\mathsf{DMS_{FSupB}}$-morphism. An argument similar to [BJ08, Lem. 9.7] shows that α is a $\mathsf{DMS_{FSupB}}$-morphism iff the α-preimage of an optimal filter is an optimal filter. On the other hand, Lemma 5.33 shows that α is a $\mathsf{DMS_P}$-morphism iff the α-preimage of a prime filter is a prime filter. This shows how morphisms in $\mathsf{DMS_{FSupB}}$ and $\mathsf{DMS_P}$ compare to each other in the language of prime and optimal filters.

5.35. REMARK. Since distributive meet- and join-semilattices are order-duals of each other, it follows from Lemma 5.33 that $\mathsf{BDMS_P}$ is isomorphic to the category of distributive join-semilattices considered in [HP08]. Thus, [HP08, Thm. 1.12] is a consequence of Theorem 5.31 and [BJ08, Prop. 13.6].

6. Priestley duality from the perspective of HMS duality

In this section we show how Priestley duality fits in the general picture we developed in this paper. We recall [Joh82, Sec. II.3] that an algebraic frame L is *coherent* if finite meets of compact elements are compact. Therefore, L is coherent iff $\mathsf{K}(L)$ is a bounded sublattice of L. In particular, each coherent frame is compact.

6.1. DEFINITION. *Let* CohFrm *be the full subcategory of* $\mathsf{KAlgFrm}$ *consisting of coherent frames.*

The restriction of the functor $\mathscr{K}$ to CohFrm lands in DL. Conversely, if a distributive meet-semilattice is a bounded lattice, then $\mathscr{F}(M)$ is a coherent frame because $\mathscr{K}\mathscr{F}(M)$ consists of principal upsets and $\uparrow a \vee \uparrow b = \uparrow a \cap \uparrow b = \uparrow(a \vee b)$. Thus, $\mathscr{K}$ and $\mathscr{F}$ restrict to yield an equivalence of CohFrm and DL, and we arrive at the following well-known result, which is the pointfree version of Stone duality for distributive lattices:

6.2. THEOREM. [Joh82, p. 65] CohFrm *is equivalent to* DL.

6.3. REMARK. Johnstone [Joh82], like Nachbin [Nac49], worked with the ideal functor rather than the filter functor. Also, Johnstone worked with the category CohLoc of coherent locales, the objects of which are the same as those of CohFrm, but the morphisms of CohLoc are the right adjoints of morphisms in CohFrm.

6.4. LEMMA.

(1) *Let* $L \in$ AlgFrm. *If* $L \in$ CohFrm, *then* $\mathsf{PP}(L) = \mathsf{P}(L)$.

(2) *Let* $X \in$ GPS. *If* $X = X_0$, *then* $\mathscr{V}^a(X)$ *is the set of closed upsets of* X, *and hence* $\mathscr{V}^a(X) \in$ CohFrm.

PROOF. (1). Since $\mathsf{P}(L) \subseteq \mathsf{PP}(L)$, we only need to prove the other inclusion. Let $p \in \mathsf{PP}(L)$ and $a, b \in L$ with $a \wedge b \leq p$. If $a, b \not\leq p$, then there are $k, l \in \mathsf{K}(L)$ with $k \leq a$, $l \leq b$, and $k, l \not\leq p$. We have $k \wedge l \leq a \wedge b \leq p$. Since L is coherent, $k \wedge l \in \mathsf{K}(L)$, so $k \wedge l \leq p$ implies $k \wedge l \ll p$. Because $p \in \mathsf{PP}(L)$, either $k \leq p$ or $l \leq p$. The obtained contradiction proves that $p \in \mathsf{P}(L)$.

(2). Let $X_0 = X$. Then all clopen upsets and closed upsets are admissible. Therefore, $\mathscr{A}(X)$ is all clopen upsets and $\mathscr{V}^a(X)$ is all closed upsets, and hence $\mathscr{A}(X)$ is a bounded sublattice of $\mathscr{V}^a(X)$. Since $\mathscr{K}\mathscr{V}^a(X) = \mathscr{A}(X)$ by Lemma 4.4, we conclude that $\mathscr{V}^a(X)$ is a coherent frame. ∎

Let $\langle X, \tau, \leq \rangle$ be a Priestley space. Then $\langle X, \tau, \leq, X \rangle$ is a generalized Priestley space. Moreover, a map between Priestley spaces is a Priestley morphism iff it is a GPS$_\mathsf{P}$-morphism between the corresponding generalized Priestley spaces. Thus, we may view PS as a full subcategory of GPS$_\mathsf{P}$.

6.5. THEOREM. CohFrm *is dually equivalent to* PS.

PROOF. We have that CohFrm is a full subcategory of KAlgFrm and we may view PS as a full subcategory of GPS$_\mathsf{P}$. Therefore, by Lemma 6.4 the dual equivalence between KAlgFrm and GPS$_\mathsf{P}$ of Theorem 5.31 restricts to a dual equivalence between CohFrm and PS. ∎

Putting Theorems 6.2 and 6.5 together yields Priestley duality:

6.6. THEOREM. CohFrm *is equivalent to* DL *and dually equivalent to* PS.

6.7. REMARK. The functors establishing the duality of DL and PS are the compositions of the functors establishing the equivalence of DL and CohFrm and the duality of CohFrm and PS. Indeed, if $M \in$ DL, then the Priestley space X_M of M is equal to $\mathsf{P}(\mathscr{F}(M))$. If α is a DL-morphism, it follows from the proof of Theorem 4.30 that $\mathscr{X}(\alpha) = \mathscr{Y}\mathscr{F}(\alpha)$. Therefore, $\mathscr{X} = \mathscr{Y} \circ \mathscr{F}$. In the opposite direction, if $X \in$ PS, then $\mathsf{ClopUp}(X) = \mathscr{A}(X)$, which is $\mathsf{K}(\mathscr{V}^a(X))$ by Lemma 4.4. Moreover, if f is a PS-morphism, then $\mathscr{V}^a(f) = f^{-1}$ by Remark 5.15 and $\mathscr{K}\mathscr{V}^a(f)$ is the restriction of f^{-1} to $\mathscr{A}(X)$ which is $\mathscr{A}(f)$. Thus, $\mathscr{A} = \mathscr{K} \circ \mathscr{V}^a$.

We next show how to view the duality for distributive lattices with top but possibly without bottom from this perspective. This can be done by working with pointed Priestley spaces.

VARIOUS MORPHISMS BETWEEN POINTED PRIESTLEY SPACES.

6.8. DEFINITION. *Let $X = \langle X, \tau, \leq, X_0, m \rangle \in$ PGPS. If $X_0 = X \setminus \{m\}$, then we call X a pointed Priestley space. Let PPS be the full subcategory of PGPS consisting of pointed Priestley spaces, and define $\mathsf{PPS_S}$, $\mathsf{PPS_{ST}}$, and $\mathsf{PPS_P}$ similarly.*

We also introduce arithmetic frames in analogy with arithmetic lattices [GHK+03, p. 117]. Arithmetic frames are also known as M-frames (see, e.g., [IM09, p. 2]).

6.9. DEFINITION. *We call an algebraic frame L arithmetic if*

$$a, b \in \mathsf{K}(L) \implies a \wedge b \in \mathsf{K}(L).$$

Note that coherent frames are simply compact arithmetic frames.

6.10. DEFINITION. *Let ArFrm be the full subcategory of AlgFrm consisting of arithmetic frames, and define $\mathsf{ArFrm_{Sup}}$, $\mathsf{ArFrm_{FInf}}$, and $\mathsf{ArFrm_{FInfB}}$ similarly.*

6.11. THEOREM. *$\mathsf{ArFrm_{Sup}}$ is dual to PPS, $\mathsf{ArFrm_{FInf}}$ is dual to $\mathsf{PPS_S}$, $\mathsf{ArFrm_{FInfB}}$ is dual to $\mathsf{PPS_{ST}}$, and ArFrm is dual to $\mathsf{PPS_P}$.*

PROOF. Let L be an arithmetic frame. Observe that the proof of Lemma 6.4(1) only uses that L is arithmetic, so it yields that $\mathscr{Y}(L)$ is a pointed Priestley space. Next, let X be a pointed Priestley space. Then $\mathscr{A}(X)$ is all nonempty clopen upsets and $\mathscr{V}^a(X)$ is all nonempty closed upsets of X. Therefore, the same argument as in Lemma 6.4(2) yields that $\mathscr{V}^a(X)$ is an arithmetic frame. It is left to apply Theorems 4.29, 5.13, and 5.25. ∎

6.12. DEFINITION. *Let $\mathsf{DL_M^-}$, $\mathsf{DL^-}$, $\mathsf{DL_B^-}$, and $\mathsf{DL_P^-}$ be the full subcategories of DMS, $\mathsf{DMS_{FSup}}$, $\mathsf{DMS_{FSupB}}$, and $\mathsf{DMS_P}$, respectively, whose objects are lattices.*

6.13. REMARK. Objects in each of $\mathsf{DL_M^-}$, $\mathsf{DL^-}$, $\mathsf{DL_B^-}$, and $\mathsf{DL_P^-}$ are distributive lattices with top, but possibly without bottom. Morphisms of $\mathsf{DL_M^-}$ are meet-semilattice homomorphisms, morphisms of $\mathsf{DL^-}$ are lattice homomorphisms, morphisms of $\mathsf{DL_B^-}$ are lattice homomorphisms which preserve bottom when it exists, and morphisms of $\mathsf{DL_P^-}$ are lattice homomorphisms which pull prime filters back to prime filters.

We point out that not every DL_B^--morphism is a DL_P^--morphism. For example, let M be a decreasing chain with top but no bottom, and let $\alpha\colon M \to M$ be defined by $\alpha(a) = 1$ for each $a \in M$. Then α is a DL_B^--morphism but not a DL_P^--morphism.

For an algebraic frame L, we have that L is arithmetic iff $\mathscr{K}(L)$ is a distributive lattice (possibly without bottom), which happens iff $\mathscr{Y}(L)$ is a pointed Priestley space. Therefore, putting Theorems 5.4, 5.29, and 6.11 together yields:

6.14. THEOREM.

(1) $\mathsf{ArFrm}_\mathsf{Sup}$ *is equivalent to* DL_M^- *and dually equivalent to* PPS.

(2) $\mathsf{ArFrm}_\mathsf{FInf}$ *is equivalent to* DL^- *and dually equivalent to* PPS_S.

(3) $\mathsf{ArFrm}_\mathsf{FInfB}$ *is equivalent to* DL_B^- *and dually equivalent to* PPS_ST.

(4) ArFrm *is equivalent to* DL_P^- *and dually equivalent to* PPS_P.

VARIOUS MORPHISMS BETWEEN PRIESTLEY SPACES.

6.15. DEFINITION.

(1) *Let* PS_R *be the category of Priestley spaces with generalized Priestley morphisms.*

(2) *Let* PS_PS *be the category of Priestley spaces with partial strong Priestley morphisms.*

6.16. REMARK.

(1) By restricting the equivalence of Proposition 5.17, we obtain that PS_R is equivalent to the full subcategory of PPS and PS_PS to the full subcategory of PPS_S consisting of Priestley spaces with isolated maxima.

(2) The full subcategory of PPS_ST consisting of pointed Priestley spaces with isolated maxima is equivalent to PS. To see this, if $f\colon X \to Y$ is a PPS_ST-morphism, then $f^{-1}(n) = \{m\}$. Therefore, its restriction $f^-\colon X^- \to Y^-$ is a PS-morphism. Such morphisms automatically satisfy $f[X_0] \subseteq Y_0$ since $X_0 = X \setminus \{m\}$ and $Y_0 = Y \setminus \{n\}$, so are PPS_P-morphisms. Consequently, PS is also equivalent to the full subcategory of PPS_P consisting of pointed Priestley spaces with isolated maxima.

6.17. DEFINITION.

(1) *Let* $\mathsf{CohFrm}_\mathsf{Sup}$ *and* $\mathsf{CohFrm}_\mathsf{FInf}$ *be the full subcategories of* $\mathsf{ArFrm}_\mathsf{Sup}$ *and* $\mathsf{ArFrm}_\mathsf{FInf}$, *respectively, consisting of coherent frames.*

(2) *Let* DL_M *and* DL_L *be the full subcategories of* BDMS *and* $\mathsf{BDMS}_\mathsf{FSup}$, *respectively, whose objects are lattices.*

6.18. REMARK. Objects of $\mathsf{DL_M}$ and $\mathsf{DL_L}$ are bounded distributive lattices. Morphisms of $\mathsf{DL_M}$ are meet-semilattice homomorphisms and morphisms of $\mathsf{DL_L}$ are lattice homomorphisms (not necessarily preserving bottom). Observe that the full subcategories of $\mathsf{BDMS_{FSupB}}$ and $\mathsf{BDMS_P}$ consisting of lattices are both equal to DL. To see this, it is clear that morphisms of $\mathsf{BDMS_{FSupB}}$ preserve finite joins and 0, so are DL-morphisms. Since $\mathsf{BDMS_P}$ is a wide subcategory of $\mathsf{BDMS_{FSupB}}$, we also obtain that $\mathsf{BDMS_P}$-morphisms are DL-morphisms. Similarly, the full subcategory of $\mathsf{ArFrm_{FInfB}}$ consisting of coherent frames is equal to CohFrm.

An arithmetic frame L is coherent iff $\mathscr{K}(L)$ is a bounded distributive lattice, which happens iff $\mathscr{Y}(L)$ is a pointed Priestley space with isolated top. Therefore, Theorem 6.14 and Remark 6.16 yield:

6.19. THEOREM.

(1) $\mathsf{CohFrm_{Sup}}$ is equivalent to $\mathsf{DL_M}$ and dually equivalent to $\mathsf{PS_R}$.

(2) $\mathsf{CohFrm_{FInf}}$ is equivalent to $\mathsf{DL_L}$ and dually equivalent to $\mathsf{PS_{PS}}$.

6.20. REMARK. The dual equivalence between $\mathsf{DL_M}$ and $\mathsf{PS_R}$ was first established in [CLP91], but the authors worked with join-preserving rather than meet-preserving maps between bounded distributive lattices.

Putting Theorems 6.6, 6.14, and 6.19 together yields Figure 2. The tables after the figure describe the listed categories.
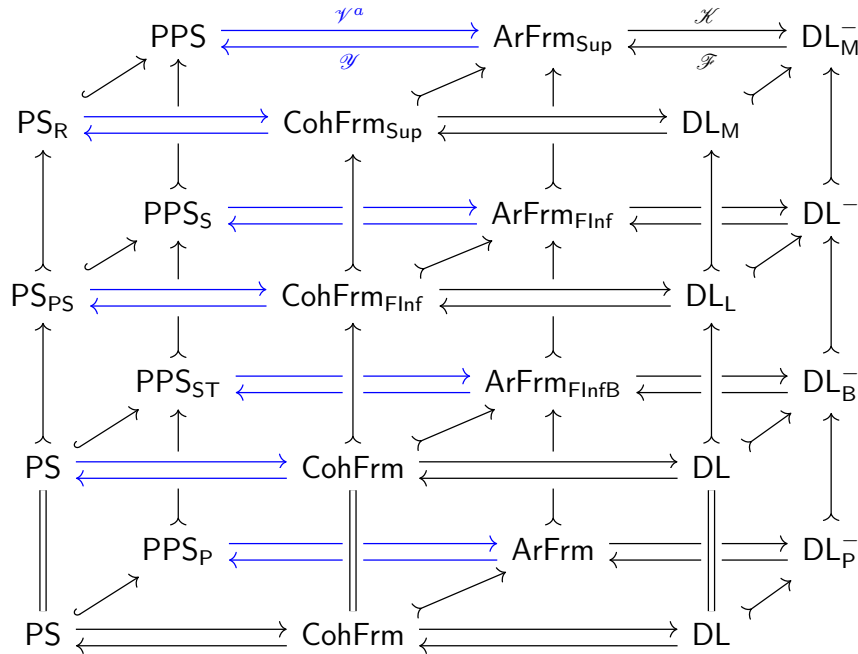
Figure 2: Connecting Priestley duality and HMS duality

Categories of pointed Priestley spaces		
Category	**Morphisms**	**Location**
PPS	PGPS-morphisms	Def. 6.8
PPS$_S$	PGPS$_S$-morphisms	"
PPS$_{ST}$	PGPS$_{ST}$-morphisms	"
PPS$_P$	PGPS$_P$-morphisms	"

Categories of Priestley spaces		
Category	**Morphisms**	**Location**
PS$_R$	GPS-morphisms	Def. 6.15
PS$_{PS}$	GPS$_S$-morphisms	"
PS	PS-morphisms	Def. 3.1(2)

Categories of arithmetic frames		
Category	**Morphisms**	**Location**
ArFrm$_{Sup}$	AlgFrm$_{Sup}$-morphisms	Def. 6.10
ArFrm$_{FInf}$	AlgFrm$_{FInf}$-morphisms	"
ArFrm$_{FInfB}$	AlgFrm$_{FInfB}$-morphisms	"
ArFrm	AlgFrm-morphisms	"

Categories of coherent frames		
Category	**Morphisms**	**Location**
CohFrm$_{Sup}$	AlgFrm$_{Sup}$-morphisms	Def. 6.17(1)
CohFrm$_{FInf}$	AlgFrm$_{FInf}$-morphisms	"
CohFrm	AlgFrm-morphisms	Def. 6.1

Categories of distributive lattices		
Category	**Morphisms**	**Location**
DL$_M^-$	DMS-morphisms	Def. 6.12
DL$^-$	DMS$_{FSup}$-morphisms	"
DL$_B^-$	DMS$_{FSupB}$-morphisms	"
DL$_P^-$	DMS$_P$-morphisms	"

Categories of bounded distributive lattices		
Category	**Morphisms**	**Location**
DL$_M$	DMS-morphisms	Def. 6.17(2)
DL$_L$	DMS$_{FSup}$-morphisms	"
DL	DL-morphisms	Def. 3.1(1)

7. Stone duality for generalized boolean algebras via HMS duality

We conclude the paper by discussing how to view Stone duality for boolean algebras and generalized boolean algebras from this perspective.

7.1. DEFINITION.

(1) *Let* BA *be the category of boolean algebras and boolean homomorphisms.*

(2) *Let* Stone *be the category of Stone spaces and continuous maps.*

We may view BA as a full subcategory of DL and Stone as a full subcategory of PS. For a frame L, we recall that the *pseudocomplement* of $a \in L$ is

$$a^* = \bigvee \{x \in L \mid a \wedge x = 0\}$$

and that a is *complemented* if $a \vee a^* = 1$. Then L is *zero-dimensional* if the complemented elements are join-dense in L.

7.2. DEFINITION. [Ban89] *A* Stone frame *is a compact zero-dimensional frame. Let* StoneFrm *be the category of Stone frames and frame homomorphisms.*

Since in Stone frames compact elements are exactly complemented elements, every Stone frame is coherent, and every frame homomorphism between Stone frames preserves compact elements. Thus, StoneFrm is a full subcategory of CohFrm, and we obtain Stone duality for boolean algebras as a consequence of Theorem 6.6:

7.3. THEOREM. [Sto36, Ban89] BA *is equivalent to* StoneFrm *and dually equivalent to* Stone.

PROOF. It is well known that $B \in$ BA implies $\mathscr{F}(B) \in$ StoneFrm and $L \in$ StoneFrm implies $\mathscr{K}(L) \in$ BA. Thus, the equivalence between DL and CohFrm (see Theorem 6.2) restricts to an equivalence between BA and StoneFrm.

We next show that the dual equivalence between CohFrm and PS (see Theorem 6.5) restricts to a dual equivalence between StoneFrm and Stone. For this it is enough to observe that from $L \in$ StoneFrm it follows that the order on $\mathsf{P}(L)$ is equality, and that X a Stone space implies $\mathscr{V}^a(X)$ is a Stone frame. For the latter, since $\mathscr{K}\mathscr{V}^a(X) = \mathscr{A}(X) = \mathsf{Clop}(X)$, we see that $\mathscr{V}^a(X)$ is zero-dimensional, hence a Stone frame. For the former, let L be a Stone frame and $p, q \in \mathsf{P}(L)$ with $p < q$. Then $q \not\leq p$, so there is $k \in \mathsf{K}(L)$ with $k \leq q$ and $k \not\leq p$. The latter together with $k \wedge k^* \leq p$ implies that $k^* \leq p \leq q$. Therefore, $k \vee k^* \leq q$. Since L is a Stone frame, k is complemented, so $k \vee k^* = 1$. Thus, $q = 1$, a contradiction. Consequently, the order on $\mathsf{P}(L)$ is equality. ∎

VARIOUS MORPHISMS BETWEEN GENERALIZED BOOLEAN ALGEBRAS. We recall that a generalized boolean algebra is a distributive lattice M with bottom such that $[0, a]$ is a boolean algebra for each $a \in M$. Since we work with distributive lattices with top, we consider the order-dual of M. Therefore, by a *generalized boolean algebra M* we mean a distributive lattice M with top such that $[a, 1]$ is a boolean algebra for each $a \in M$.

7.4. DEFINITION. *Let* $\mathsf{GBA_M}$, GBA, $\mathsf{GBA_B}$, *and* $\mathsf{GBA_P}$ *be the full subcategories of* $\mathsf{DL_M^-}$, $\mathsf{DL^-}$, $\mathsf{DL_B^-}$, *and* $\mathsf{DL_P^-}$, *respectively, whose objects are generalized boolean algebras.*

7.5. REMARK. Objects in each of GBA$_M$, GBA, GBA$_B$, and GBA$_P$ are generalized boolean algebras. Morphisms of GBA$_M$ are meet-semilattice homomorphisms and morphisms of GBA are lattice homomorphisms (not necessarily preserving bottom). Morphisms of GBA$_B$ are lattice homomorphisms that preserve bottom when it exists and morphisms of GBA$_P$ are lattice homomorphisms which pull prime filters back to prime filters.

We next generalize Stone frames as follows (see, e.g., [BK23]).

7.6. DEFINITION. *A locally Stone frame is an algebraic zero-dimensional frame.*

7.7. REMARK. It is easy to see that a frame L is a locally Stone frame iff compact complemented elements are join-dense in L.

Clearly Stone frames are compact locally Stone frames. It is also straightforward to see that each locally Stone frame is an arithmetic frame.

7.8. DEFINITION. *Let* LStoneFrm$_{Sup}$, LStoneFrm$_{FInf}$, LStoneFrm$_{FInfB}$, *and* LStoneFrm *be the full subcategories of* ArFrm$_{Sup}$, ArFrm$_{FInf}$, ArFrm$_{FInfB}$, *and* ArFrm, *respectively, whose objects are locally Stone frames.*

We next define pointed Stone spaces as special pointed Priestley spaces.

7.9. DEFINITION. *A pointed Stone space is a pointed Priestley space (X, m) such that $\leq$ restricts to the identity on X^-.*

7.10. REMARK. A standard definition of a pointed Stone space is that it is a Stone space X with a designated point $m \in X$ (see Remark 3.9). The definition above is different in that we make m the maximum of X. This definition fits nicer in the more general picture of pointed Priestley spaces developed in this paper. It also makes sense from the perspective of Stone duality for generalized boolean algebras. Indeed, if we order the dual space X_M of a generalized boolean algebra M by inclusion, then M is the maximum of X_M.

7.11. DEFINITION. *Let* PStone *be the full subcategory of* PPS *whose objects are pointed Stone spaces, and define* PStone$_S$, PStone$_{ST}$, *and* PStone$_P$ *similarly.*

7.12. REMARK. Objects of PStone, PStone$_S$, PStone$_{ST}$, and PStone$_P$ are pointed Stone spaces. Morphisms of PStone are PPS-morphisms and hence are relations. Morphisms of PStone$_S$ are strong Priestley morphisms. If (X, m) is a pointed Stone space, then $U \in \mathscr{A}(X)$ iff U is a clopen subset of X containing m. Consequently, a PStone$_S$-morphism $f : (X, m) \to (Y, n)$ is a continuous function with $f(m) = n$. A PStone$_S$-morphism f is a PStone$_{ST}$-morphism provided $f^{-1}(n) = \{m\}$ when m, n are isolated. In this case f restricts to a continuous function from X^- to Y^-. Thus, Stone is equivalent to the full subcategory of PStone$_{ST}$ consisting of pointed Stone spaces with isolated top, as well as to the corresponding full subcategory of PStone$_P$.

An arithmetic frame L is locally Stone iff $\mathscr{K}(L)$ is a generalized Boolean algebra, which happens iff $\mathscr{Y}(L)$ is a pointed Stone space with an isolated maximum. Therefore, Theorem 6.14 and Remark 7.12 yield:

7.13. THEOREM.

(1) LStoneFrm$_{\mathsf{Sup}}$ *is equivalent to* GBA$_{\mathsf{M}}$ *and dually equivalent to* PStone.

(2) LStoneFrm$_{\mathsf{FInf}}$ *is equivalent to* GBA *and dually equivalent to* PStone$_{\mathsf{S}}$.

(3) LStoneFrm$_{\mathsf{FInfB}}$ *is equivalent to* GBA$_{\mathsf{B}}$ *and dually equivalent to* PStone$_{\mathsf{ST}}$.

(4) LStoneFrm *is equivalent to* GBA$_{\mathsf{P}}$ *and dually equivalent to* PStone$_{\mathsf{P}}$.

VARIOUS MORPHISMS BETWEEN BOOLEAN ALGEBRAS.

7.14. DEFINITION.

(1) *Let* BA$_{\mathsf{M}}$ *and* BA$_{\mathsf{L}}$ *be the full subcategories of* DL$_{\mathsf{M}}$ *and* DL$_{\mathsf{L}}$, *respectively, whose objects are boolean algebras.*

(2) *Let* StoneFrm$_{\mathsf{Sup}}$ *and* StoneFrm$_{\mathsf{FInf}}$ *be the full subcategories of* ArFrm$_{\mathsf{Sup}}$ *and* ArFrm$_{\mathsf{FInf}}$, *respectively, whose objects are Stone frames.*

(3) *Let* Stone$_{\mathsf{R}}$ *and* Stone$_{\mathsf{PS}}$ *be the full subcategories of* PS$_{\mathsf{R}}$ *and* PS$_{\mathsf{PS}}$, *respectively, whose objects are Stone spaces.*

7.15. REMARK.

(1) Objects in both BA$_{\mathsf{M}}$ and BA$_{\mathsf{L}}$ are boolean algebras. Morphisms of BA$_{\mathsf{M}}$ are meet-semilattice homomorphisms and those of BA$_{\mathsf{L}}$ are lattice homomorphisms (which may not preserve bottom).

(2) By restricting the equivalence of Remark 6.16(1), we obtain that Stone$_{\mathsf{R}}$ is equivalent to the full subcategory of PStone and Stone$_{\mathsf{PS}}$ to the full subcategory of PStone$_{\mathsf{S}}$ consisting of pointed Stone spaces with isolated maxima.

(3) By Remark 6.16(2), the full subcategories of PStone$_{\mathsf{ST}}$ and PStone$_{\mathsf{P}}$ consisting of pointed Stone spaces with isolated maxima are equivalent to Stone.

Let $L \in$ LStoneFrm$_{\mathsf{Sup}}$. Then L is a Stone frame iff $\mathscr{K}(L)$ is a boolean algebra, which happens iff $\mathscr{Y}(L)$ is a Stone space. Therefore, Theorem 7.13 yields:

7.16. THEOREM.

(1) StoneFrm$_{\mathsf{Sup}}$ *is equivalent to* BA$_{\mathsf{M}}$ *and dually equivalent to* Stone$_{\mathsf{R}}$.

(2) StoneFrm$_{\mathsf{FInf}}$ *is equivalent to* BA$_{\mathsf{L}}$ *and dually equivalent to* Stone$_{\mathsf{PS}}$.

7.17. REMARK. The dual equivalence between $\mathsf{BA_M}$ and $\mathsf{Stone_R}$ was first established in [Hal56], but Halmos worked with join-preserving rather than meet-preserving maps between boolean algebras.

Putting together Theorems 7.13 and 7.16 yields Figure 3, which is similar to Figure 2. The tables after the figure describe the listed categories.
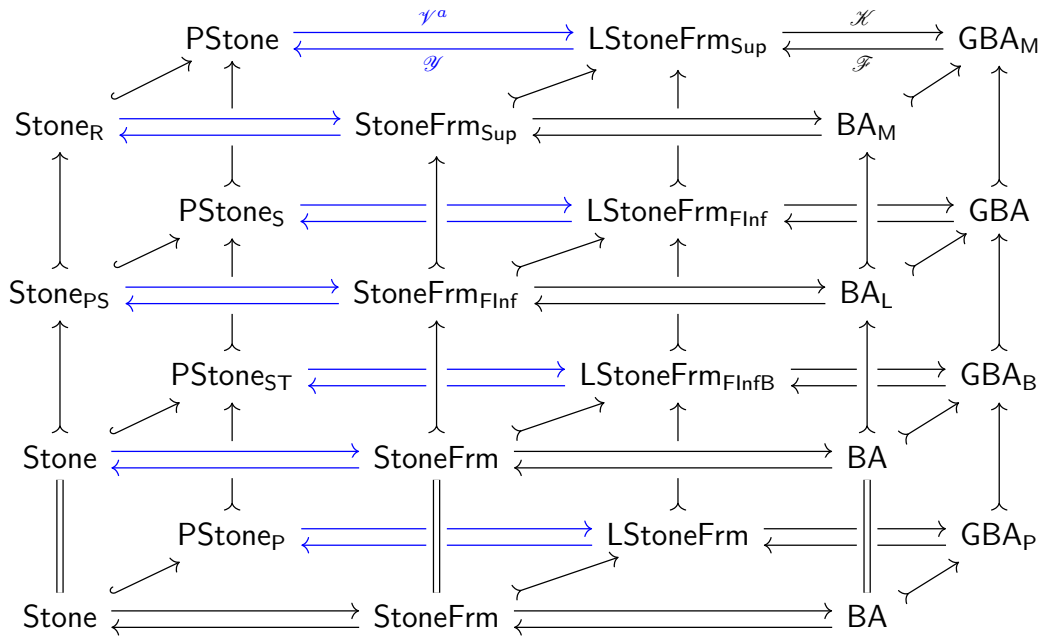
Figure 3: Connecting Stone duality and HMS duality

Categories of pointed Stone spaces		
Category	Morphisms	Location
PStone	PGPS-morphisms	Def. 7.11
$\mathsf{PStone_S}$	$\mathsf{PGPS_S}$-morphisms	"
$\mathsf{PStone_{ST}}$	$\mathsf{PGPS_{ST}}$-morphisms	"
$\mathsf{PStone_P}$	$\mathsf{PGPS_P}$-morphisms	"

Categories of Stone spaces		
Category	Morphisms	Location
$\mathsf{Stone_R}$	PGPS-morphisms	Def. 7.14(3)
$\mathsf{Stone_S}$	$\mathsf{PGPS_{ST}}$-morphisms	"
Stone	$\mathsf{PGPS_P}$-morphisms	Def. 7.1(2)

Categories of locally Stone frames		
Category	Morphisms	Location
LStoneFrm$_{\text{Sup}}$	AlgFrm$_{\text{Sup}}$-morphisms	Def. 7.8
LStoneFrm$_{\text{FInf}}$	AlgFrm$_{\text{FInf}}$-morphisms	"
LStoneFrm$_{\text{FInfB}}$	AlgFrm$_{\text{FInfB}}$-morphisms	"
LStoneFrm	AlgFrm-morphisms	"

Categories of Stone frames		
Category	Morphisms	Location
StoneFrm$_{\text{Sup}}$	AlgFrm$_{\text{Sup}}$-morphisms	Def. 7.14(2)
StoneFrm$_{\text{FInf}}$	AlgFrm$_{\text{FInf}}$-morphisms	"
StoneFrm	frame homomorphisms	Def. 7.2

Categories of boolean algebras		
Category	Morphisms	Location
GBA$_{\text{M}}$	DMS-morphisms	Def. 7.4
GBA	DMS$_{\text{FSup}}$-morphisms	"
GBA$_{\text{B}}$	DMS$_{\text{FSupB}}$-morphisms	"
GBA$_{\text{P}}$	DMS$_{\text{P}}$-morphisms	"

Categories of generalized boolean algebras		
Category	Morphisms	Location
BA$_{\text{M}}$	DMS-morphisms	Def. 7.14(1)
BA$_{\text{L}}$	DMS$_{\text{FSupB}}$-morphisms	"
BA	boolean homomorphisms	Def. 7.1(1)

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