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COMPARING 2-CROSSED MODULES WITH GRAY 3-GROUPS

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Abstract. In this paper, we have constructed the close relationship between 2-crossed modules and Gray 3-groupoids with a single object (Gray 3-groups). Using both the equivalence between 2-crossed modules and Gray 3-groups, and the Gray category structure over the category of chain complexes of vector spaces; we describe linear representations as certain 3-functors.

1. Introduction

Whitehead in [29] introduced the concept of crossed modules of groups as an algebraic model for homotopy 2-types. As an algebraic model for homotopy 3-types, Conduché, [14], defined the notion of 2-crossed modules and showed how to obtain a 2-crossed module from a 2-truncated simplicial group. This model extends canonically to a 2-truncated simplicial group (cf. [13]) and is also equivalent to the notion of crossed square introduced by Loday and Guin-Walery in [27]. For this connection, see [15]. As an alternative algebraic model for homotopy 3-types, in [10], Brown and Gilbert gave a lead, from the automorphism structure for crossed modules, to the notion of braided regular crossed modules. This structure is equivalent to Conduché's 2-crossed module. There is also an equivalence between the category of braided regular crossed modules and that of 2-truncated simplicial groups. For this equivalence see [3] in terms of Carrasco-Cegarra pairings operators given in [13] and examined in [26].

Gray, in [19], has developed tensor products for 2-categories. As an algebraic aspect of this structures, the construction of the tensor product has been restricted to the notion of 2-groupoids and this gives naturally another basic example for 3-types. Then, Joyal and Tierney in [21], proved that Gray groupoids model all homotopy 3-types. Since 2-crossed modules are algebraic models of homotopy 3-types and the 2-crossed module underlying a Gray 3-group has a natural almost geometric description (cf. [6]), in this work, we give an explicit comparison between 2-crossed modules and Gray 3-groups. In order to better understand the verification of each axiom in this comparison, we have intensively given diagrams representing these axioms visually. Furthermore, the concept of a 3-crossed

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module, which is equivalent to a 3-truncated simplicial group, has been introduced in [2] as an algebraic model for homotopy 4-type. This structure can be regarded as a suitable model for extending the comparison between 2-crossed modules and Gray 3-groups to the next higher dimension and provided that the corresponding notion of Gray 4-group can be defined.

In the literature, it is relatively to find some references to the construction of a Gray 3-groupoid or a 2-groupoid enrichment for the category **Ch** of positive chain complexes over vector spaces, for instance in the papers [7] and [8]. For further results about the Gray category structure for positive chain complexes see also Kamps and Porter's work [22]. They have mainly proved that the category of chain complexes of length-2, $\mathbf{Ch}^2_K$, over vector spaces has a Gray 3-groupoid structure. In this context, Barker in [5], using the fact that the category of chain complexes of length 1, $\mathbf{Ch}^1_K$, has a 2-groupoid structure, has defined the linear representation of crossed modules or equivalently cat^1-groups (cf. [24]), as a 2-functor $\Phi : \mathfrak{C} \to \mathbf{Ch}^1_K$, where $\mathfrak{C}$ is a cat^1-group obtained from a crossed module. The functorial image of $\mathfrak{C}$ under Φ lies within a sub 2-groupoid with a single object; $\mathbf{Aut}(\delta)$ of $\mathbf{Ch}^1_K$, called automorphism cat^1-group. Elgueta in [18] has constructed an alternative representation of 2-groups or equivalently cat^1-groups in the 2-category of finite dimensional 2-vector spaces as defined by Kapranov and Voevodsky [23]. As a 2-dimensional version of these results, Al-asady, in [1], has considered a linear representation of a cat^2-group $\mathfrak{C}^2$, as a lax 3-functor $\mathfrak{C}^2 \to \mathbf{Aut}(\delta) \leqslant \mathbf{Ch}^2_K$, where δ is the chain complex of length 2 of vector spaces.

In the last section, using the detailed comparison between 2-crossed modules and Gray 3-groups given in sections (3),(4) of this work and evaluating the results of how linear representations of the above-mentioned algebraic models are constructed, we define an indirect linear representation for 2-crossed modules.

Contents

2. Preliminaries

2.1. 2-Crossed modules. Crossed modules were introduced by Whitehead in [29]. A crossed module $\mathfrak{X} := (M, N, \partial)$ consists of groups M, N together with a homomorphism $\partial : M \to N$ and a left action $N \times M \to M$ of N on M given by $(n, m) \mapsto {}^n m$, satisfying the conditions: (i) $\partial({}^n m) = n\partial(m)n^{-1}$ and (ii) ${}^{\partial(m)}m' = mm'm^{-1}$ for all $n \in N,\ m, m' \in M$.

Condition (ii) is called the *Peiffer identity*. A structure with the same data as a crossed module and satisfying the first condition but not the Peiffer identity is called a *pre-crossed module*.

Recall from [14] that *a 2-crossed module* of groups consists of a complex of groups

$$\mathcal{L} := L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

together with (a) actions of N on M and L so that ∂_2, ∂_1 are morphisms of N-groups, and (b) an N-equivariant function

$$\{-,-\} : M \times M \longrightarrow L$$

called a Peiffer lifting. This data must satisfy the following axioms:

$$\begin{array}{llrcl}
\mathbf{PL1}: & & \partial_2\{m,m'\} & = & mm'(m^{-1})^{\partial_1 m}(m')^{-1} \\
\mathbf{PL2}: & & \{\partial_2 l, m\} & = & l^m(l)^{-1} \\
\mathbf{PL3}: & & \{m, \partial_2 l\} & = & {}^m(l)^{\partial_1 m}(l)^{-1} \\
\mathbf{PL4}: & (i) & \{m, m'm''\} & = & \{m,m'\}^{\partial_1 m(m')}\{m,m''\} \\
 & (ii) & \{mm', m''\} & = & {}^m\{m',m''\}\{m, {}^{\partial_1 m'}m''\} \\
\mathbf{PL5}: & & \{\partial_2 l, \partial_2 l'\} & = & [l,l'] \\
\mathbf{PL6}: & & {}^n\{m,m'\} & = & \{{}^n m, {}^n m'\}
\end{array}$$

for all $l, l' \in L$, $m, m', m'' \in M$ and $n \in N$.

2.2. Gray 3-group(oid)s. Recall that a *small category* $\mathcal{A}$ consists of an object set A_0, a set of morphisms A_1, source and target maps from A_1 to A_0, a map $e : A_0 \to A_1$ which gives the identity morphisms at an object and a partially defined function $A_1 \times A_1 \to A_1$ which gives the composition of two morphisms. We will show a small category (A_1, A_0) and diagramatically as

$$A_1 \underset{e}{\overset{s,t}{\rightrightarrows}} A_0.$$

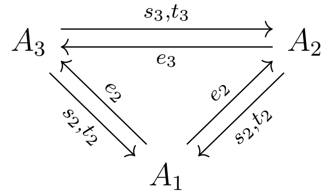
For the set of morphisms A_1, and $x, y \in A_0$ the set of morphisms from x to y is written $A_1(x,y)$ and termed a hom-set. Then for $a \in A_1(x,y)$, we have $s(a) = x$ and $t(a) = y$. We will usually write e_x for $e(x)$ and $b \circ a$ for the composite of the morphisms $a : x \to y$ and $b : y \to z$. The elements of A_0 are also called 0-cells and the elements of A_1 are called 1-cells between 0-cells.

A *groupoid* $\mathcal{A}$ is a small category in which every morphism (or every 1-cell) is an isomorphism (or invertible), that is, for any 1-cell $(a : x \to y) \in A_1(x,y)$, there is a 1-cell $(a^{-1} : y \to x) \in A_1(y,x)$, such that $a^{-1} \circ a = e_x$ and $a \circ a^{-1} = e_y$. If $A_1(x,y)$ is empty whenever x and y are distinct (that is $s = t$), then $\mathcal{A}$ is called totally disconnected. Note that a groupoid with a single 0-cell can be regarded as a group. For a survey of application of groupoids and introduction to their literature, see [9, 10].

We now recall the definition of a *Gray 3-groupoid* from Martins and Picken's work [25]. For this definiton see also Wang [28]. Their definition is slightly different from the ones of Kamps-Porter [22] and Crans [16].

A *Gray 3-groupoid* $\mathcal{A}$ is given by a set A_0 of 0-cells, a set A_1 of 1-cells, a set A_2 of 2-cells and a set A_3 of 3-cells, and maps $s_i, t_i : A_k \to A_{i-1}$ where $i = 1, ..., k$ such that:

1. $s_2 \circ s_3 = s_2$ and $t_2 \circ t_3 = t_2$ as maps $A_3 \to A_1$.
2. $s_1 = s_1 \circ s_2 = s_1 \circ s_3$ and $t_1 = t_1 \circ t_2 = t_1 \circ t_3$ as maps $A_3 \to A_0$.
3. $s_1 = s_1 \circ s_2$ and $t_1 = t_1 \circ t_2$ as maps $A_2 \to A_0$.
4. There exists a 2-vertical composition $J\#_3 J'$ of 3-cells if $t_3(J') = s_3(J)$. Then, $A_3 \underset{e_3}{\overset{s_3,t_3}{\rightleftarrows}} A_2$ is a groupoid with this composition.
5. There exists a vertical composition
$$\Gamma'\#_2\Gamma = \begin{bmatrix}\Gamma \\ \Gamma'\end{bmatrix}$$
of 2-cells if $t_2(\Gamma) = s_2(\Gamma')$. Then, $A_2 \underset{e_2}{\overset{s_2,t_2}{\rightleftarrows}} A_1$ is a groupoid with the composition $\#_2$.
6. There exists a 1-vertical composition $J'\#_1 J$ of 3-cells if $s_2(J') = t_2(J)$. Then, $A_3 \underset{e_2}{\overset{s_2,t_2}{\rightleftarrows}} A_1$ is a groupoid with this composition. In this case, we have two different groupoids over A_1; (A_3, A_1) and (A_2, A_1). Then, $s_3, t_3 : A_3 \to A_2$ are functors between two categories and these are considered as groupoid morphisms.
7. The 1-vertical and 2-vertical compositions of 3-cells satisfy the *interchange law*;
$$(J_1'\#_3 J_1)\#_1(J'\#_3 J) = (J_1'\#_1 J')\#_3(J_1\#_1 J).$$
According to these conditions, we can say that 2-vertical and 1-vertical compositions of 3-cells and vertical compositions of 2-cells give a structure of 2-groupoid (cf. [20]) shown pictorially as;



where A_1 is the set of 0-cells, A_2 is the set of 1-cells and A_3 is the set of 2-cells for this structure.

8. (**Whiskering by 1-cells**) For each $x, y \in A_0$, it can be defined a 2-groupoid $\mathcal{A}(x,y)$ of all 1-, 2- and 3-cells b such that $s_1(b) = x$ and $t_1(b) = y$. Given a 1-cell $\eta : y \to z$, there is a 2-groupoid map $\natural_1\eta : \mathcal{A}(x,y) \to \mathcal{A}(y,z)$. Similarly if $\eta' : w \to x$, there is a 2-groupoid map $\eta'\natural_1 : \mathcal{A}(x,y) \to \mathcal{A}(w,y)$.

9. There exists a horizontal composition $\eta\natural_1\eta'$ of 1-cells if $s_1(\eta) = t_1(\eta')$, which is to be associative and to define a groupoid with set of objects A_0 and set of 1-cells A_1.

10. Given $\eta, \eta' \in A_1$;

$$\natural_1\eta \circ \natural_1\eta' = \natural_1(\eta'\eta), \quad \eta\natural_1 \circ \eta'\natural_1 = (\eta\eta')\natural_1 \quad \text{and} \quad \eta\natural_1 \circ \natural_1\eta' = \natural_1\eta' \circ \eta\natural_1,$$

whenever these compositions make sense.

11. There are two horizontal compositions of 2-cells

$$\begin{bmatrix} & \Gamma' \\ \Gamma & \end{bmatrix} = (\Gamma\natural_1 t_2(\Gamma'))\#_2(s_2(\Gamma)\natural_1\Gamma') \quad \text{and} \quad \begin{bmatrix} \Gamma & \\ & \Gamma' \end{bmatrix} = (t_2(\Gamma)\natural_1\Gamma')\#_2(\Gamma\natural_1 s_2(\Gamma'))$$

and of 3-cells:

$$\begin{bmatrix} & J' \\ J & \end{bmatrix} = (J\natural_1 t_2(J'))\#_1(s_2(J)\natural_1 J') \quad \text{and} \quad \begin{bmatrix} J & \\ & J' \end{bmatrix} = (t_2(J)\natural_1 J')\#_1(J\natural_1 s_2(J')).$$

It follows from the previous axioms that they are associative.

12. (**Interchange 3-cells**) For any 2-cells Γ and Γ', there is a 3-cell (called an interchange 3-cell)

$$\begin{bmatrix} & \Gamma' \\ \Gamma & \end{bmatrix} = s_3(\Gamma\#\Gamma') \xrightarrow{(\Gamma\#\Gamma')} t_3(\Gamma\#\Gamma') = \begin{bmatrix} \Gamma & \\ & \Gamma' \end{bmatrix}$$

13. (**2-functoriality**) For any 3-cells

$$\Gamma_1 = s_3(J) \xrightarrow{J} t_3(J) = \Gamma_2 \ \text{ and } \ \Gamma'_1 = s_3(J') \xrightarrow{J'} t_3(J') = \Gamma'_2\,,$$

with $s_1(J') = t_1(J)$ the following upwards compositions (1-vertical compositions) of 3-cells coincide:

$$\begin{bmatrix} & \Gamma'_1 \\ \Gamma_1 & \end{bmatrix} \xrightarrow{(\Gamma_1\#\Gamma'_1)} \begin{bmatrix} \Gamma_1 & \\ & \Gamma'_1 \end{bmatrix} \xrightarrow{\left[\begin{smallmatrix} J & \\ & J' \end{smallmatrix}\right]} \begin{bmatrix} \Gamma_2 & \\ & \Gamma'_2 \end{bmatrix}$$

and

$$\begin{bmatrix} & \Gamma'_1 \\ \Gamma_1 & \end{bmatrix} \xrightarrow{\left[\begin{smallmatrix} & J' \\ J & \end{smallmatrix}\right]} \begin{bmatrix} & \Gamma'_2 \\ \Gamma_2 & \end{bmatrix} \xrightarrow{(\Gamma_2\#\Gamma'_2)} \begin{bmatrix} \Gamma_2 & \\ & \Gamma'_2 \end{bmatrix}$$

This of course means that the collection $\Gamma\#\Gamma'$, for arbitrary 2-cells Γ and Γ' with $s_1(\Gamma') = t_1(\Gamma)$ defines a natural transformation between the 2-functors of 11. Note that by using the interchange condition for the vertical and upwards compositions, we only need to verify this condition for the case when either J or J' is an identity.(This is the way this axiom appears written in [22, 16, 6])

14. (**1-functoriality**) For any three 2-cells $\gamma \xrightarrow{\Gamma} \phi \xrightarrow{\Gamma'} \psi$ and $\gamma'' \xrightarrow{\Gamma''} \phi''$ with $s_2(\Gamma') = t_2(\Gamma)$ and $t_1(\Gamma) = t_1(\Gamma') = s_1(\Gamma'')$ the following 2-vertical compositions of 3-cells coincide:

    (**a**)

$$\begin{bmatrix}\gamma\natural_1\Gamma'' \\ \Gamma\natural_1\phi'' \\ \Gamma'\natural_1\phi''\end{bmatrix} \xrightarrow{\begin{bmatrix}\Gamma\#\Gamma'' \\ \Gamma'\natural_1\phi''\end{bmatrix}} \begin{bmatrix}\Gamma\natural_1\gamma'' \\ \phi\natural_1\Gamma'' \\ \Gamma'\natural_1\phi''\end{bmatrix} \xrightarrow{\begin{bmatrix}\Gamma\natural_1\gamma'' \\ \Gamma'\#\Gamma''\end{bmatrix}} \begin{bmatrix}\Gamma\natural_1\gamma'' \\ \Gamma'\natural_1\gamma'' \\ \psi\natural_1\Gamma''\end{bmatrix}$$

and

$$\begin{bmatrix}\gamma\natural_1\Gamma'' \\ \Gamma\natural_1\phi'' \\ \Gamma'\natural_1\phi''\end{bmatrix} \xrightarrow{\begin{bmatrix}\Gamma \\ \Gamma'\end{bmatrix}\#\Gamma''} \begin{bmatrix}\Gamma\natural_1\gamma'' \\ \Gamma'\natural_1\gamma'' \\ \psi\natural_1\Gamma''\end{bmatrix}$$

and so, we can write

$$\begin{bmatrix}\Gamma\natural_1\gamma'' \\ \Gamma'\#\Gamma''\end{bmatrix}\#_3\begin{bmatrix}\Gamma\#\Gamma'' \\ \Gamma'\natural_1\phi''\end{bmatrix} = \begin{bmatrix}\Gamma \\ \Gamma'\end{bmatrix}\#\Gamma''$$

Similarly,

    (**b**)

$$\begin{bmatrix}\gamma''\natural_1\Gamma \\ \gamma''\natural_1\Gamma' \\ \Gamma''\natural_1\psi\end{bmatrix} \xrightarrow{\begin{bmatrix}\gamma''\natural_1\Gamma \\ \Gamma''\#\Gamma'\end{bmatrix}} \begin{bmatrix}\gamma''\natural_1\Gamma \\ \Gamma''\natural_1\phi \\ \phi''\natural_1\Gamma'\end{bmatrix} \xrightarrow{\begin{bmatrix}\Gamma''\#\Gamma \\ \phi''\natural_1\Gamma'\end{bmatrix}} \begin{bmatrix}\Gamma''\natural_1\gamma \\ \phi''\natural_1\Gamma \\ \phi''\natural_1\Gamma'\end{bmatrix}$$

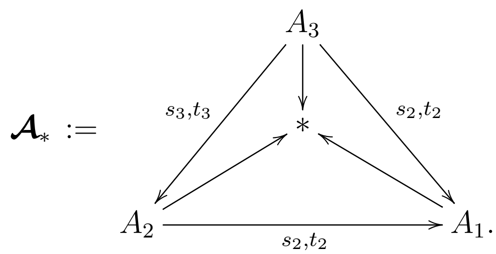
and

$$\begin{bmatrix}\gamma''\natural_1\Gamma \\ \gamma''\natural_1\Gamma' \\ \Gamma''\natural_1\psi\end{bmatrix} \xrightarrow{\Gamma''\#\begin{bmatrix}\Gamma \\ \Gamma'\end{bmatrix}} \begin{bmatrix}\Gamma''\natural_1\gamma \\ \phi''\natural_1\Gamma \\ \phi''\natural_1\Gamma'\end{bmatrix}$$

and so, we can write

$$\begin{bmatrix}\Gamma''\#\Gamma \\ \phi''\natural_1\Gamma'\end{bmatrix}\#_3\begin{bmatrix}\gamma''\natural_1\Gamma \\ \Gamma''\#\Gamma'\end{bmatrix} = \Gamma''\#\begin{bmatrix}\Gamma \\ \Gamma'\end{bmatrix}.$$

A Gray 3-group, [4], is a Gray 3-groupoid with a single 0-cell $*$. We can show it pictorially as;



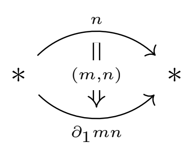
We will denote the category of Gray 3-groups by **Gray**.

3. From 2-crossed modules to Gray 3-groups

In this section, we will construct a Gray 3-group $\boldsymbol{\mathcal{A}}_*$ from a 2-crossed module $\boldsymbol{\mathcal{L}}$. Thus, we will define a functor $\boldsymbol{\Theta} : \mathbf{X_2Mod} \longrightarrow \mathbf{Gray}$.

Let $\boldsymbol{\mathcal{L}} := L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$ be a 2-crossed module together with the Peiffer lifting map $\{-,-\} : M \times M \to L$. Suppose $A_0 = \{*\}$ and $A_1 = N$. Then, any element n in N can be regarded as a 1-cell in $\mathcal{A}_*$. That is, $n : * \to *$ where $s_1(n) = t_1(n) = *$. The horizontal composition of 1-cells is given by the group operation in N.

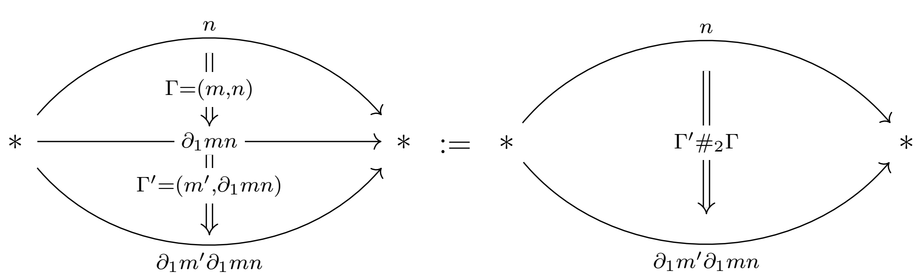
Using the group action of N on M, we can create the semi-direct product group $A_2 = M \rtimes N$ together with the operation $(m,n)(m',n') = (m^n m', nn')$ for $m, m' \in M$ and $n, n' \in N$. An element $\Gamma = (m,n)$ of A_2 can be considered as a 2-cell from n to $\partial_1 mn$, so we can define source, target maps between A_2 and A_1 as follows: for $\Gamma = (m,n) \in (M \rtimes N) = A_2$, the 1-source of this 2-cell is n and so $s_2(m,n) = n$ and 1-target of this 2-cell is $t_2(m,n) = \partial_1 mn$. The 0-source and 0-target of (m,n) is $*$. We can represent a 2-cell (m,n) in $\boldsymbol{\mathcal{A}}_*$ pictorially as:



The vertical composition of $\Gamma = (m,n)$ and $\Gamma' = (m', \partial_1 mn)$ in A_2 is given by

$$\Gamma' \#_2 \Gamma = \begin{bmatrix} \Gamma \\ \Gamma' \end{bmatrix} = (m', \partial_1 mn) \#_2 (m,n) = (m'm, n)$$

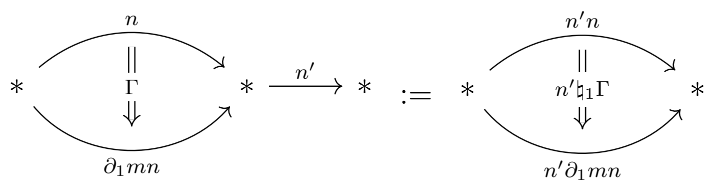
with $t_2(\Gamma) = \partial_1 mn = s_2(\Gamma')$. The vertical composition $\#_2$ of 2-cells can be pictured as follows:



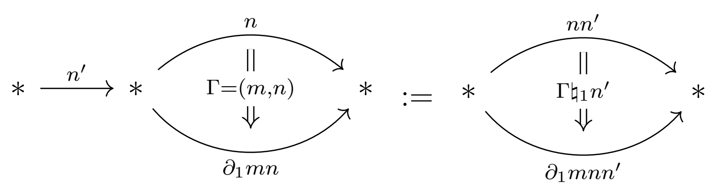
For this composition, we have clearly $s_2(\Gamma'\#_2\Gamma) = n = s_2(\Gamma)$ and $t_2(\Gamma'\#_2\Gamma) = \partial_1 m'\partial_1 mn = t_2(\Gamma')$. For a 2-cell; $\Gamma = (m, n)$ in A_2, the inverse of Γ with $\#_2$ is defined by $(\Gamma^{-1})^{\#_2} = (m^{-1}, \partial_1 mn)$. The identitiy map $e_2 : A_1 \to A_2$ is defined by $e_2(n) = (1_M, n)$. Thus, we have $s_2e_2 = t_2e_2 = id_{A_1}$. Obviously, $(\Gamma^{-1})^{\#_2}\#_2\Gamma = (1, n) = e_2(n) = e_2(s_2(\Gamma))$ and $\Gamma\#_2(\Gamma^{-1})^{\#_2} = (m, n)\#_2(m^{-1}, \partial_1 mn) = (1_M, \partial_1 mn) = e_2(t_2(\Gamma))$. Thus, we get the following result:

3.1. Proposition. $A_2 \underset{e_2}{\overset{s_2,t_2}{\rightrightarrows}} A_1$ *is a groupoid with the vertical composition* $\#_2$ *of 2-cells.*

3.2. The whiskerings of a 1-cell on a 2-cell. The whiskering of a 1-cell $n' \in A_1$ on $\Gamma = (m, n) \in A_2$ on the left side is $n'\natural_1\Gamma = ({}^{n'}m, n'n)$. We can show it diagramatically by



The left whiskering of n' on Γ appears on the left in the notation $n'\natural_1\Gamma$, but on the right in the picture. For this definition, we can see that $s_2(n'\natural_1\Gamma) = n'\natural_1 s_2(\Gamma)$ and $t_2(n'\natural_1\Gamma) = n'\natural_1 t_2(\Gamma)$. Similarly, the right whiskering of n' on $\Gamma = (m, n)$ is given by $\Gamma\natural_1 n' = (m, nn')$ shown pictorially by

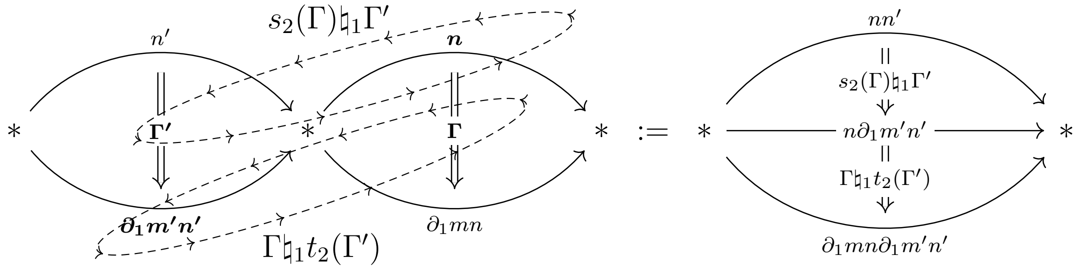


For this definition, clearly we have $s_2(\Gamma\natural_1 n') = s_2(\Gamma)\natural_1 n'$ and $t_2(\Gamma\natural_1 n') = t_2(\Gamma)\natural_1 n'$.

3.3. The horizontal compositions of 2-cells. Let $\Gamma = (m, n) : n \Rightarrow \partial_1 mn$ and $\Gamma' = (m', n') : n' \Rightarrow \partial_1 m'n'$ be 2-cells. Using the left and right whiskerings of 1-cells on 2-cells, we can define the horizontal composition $\begin{bmatrix} & \Gamma' \\ \Gamma & \end{bmatrix}$ of Γ and Γ' by

$$\begin{aligned} \begin{bmatrix} & \Gamma' \\ \Gamma & \end{bmatrix} &= (\Gamma\natural_1 t_2(\Gamma'))\#_2(s_2(\Gamma)\natural_1\Gamma') \\ &= ((m, n)\natural_1\partial_1 m'n')\#_2(n\natural_1(m', n')) \\ &= ((m, n\partial_1 m'n')\#_2({}^{n}(m'), nn') \\ &= (m^{n}(m'), nn'). \end{aligned}$$

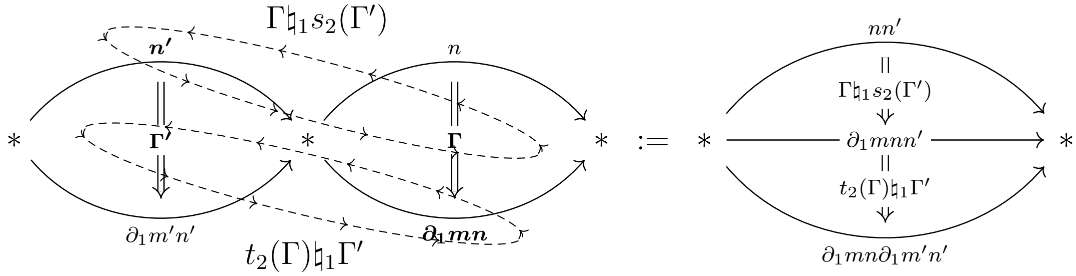
This can be represented by the diagram below:



On the other hand, the horizontal composition $\left[\begin{smallmatrix}\Gamma & \\ & \Gamma'\end{smallmatrix}\right]$ is defined by

$$\begin{aligned}\begin{bmatrix}\Gamma & \\ & \Gamma'\end{bmatrix} =&(t_2(\Gamma)\natural_1\Gamma')\#_2(\Gamma\natural_1 s_2(\Gamma'))\\ =&(\partial_1 mn\natural_1(m',n'))\#_2((m,n)\natural_1 n')\\ =&({}^{\partial_1 mn}(m'),\partial_1 mnn')\#_2(m,nn')\\ =&({}^{\partial_1 mn}(m')m,nn')\end{aligned}$$

and similarly, we can show this by a diagram



Note that $\left[\begin{smallmatrix} & \Gamma' \\ \Gamma & \end{smallmatrix}\right] \neq \left[\begin{smallmatrix}\Gamma & \\ & \Gamma'\end{smallmatrix}\right]$ since ∂_1 is not a crossed module. We have clearly,

$$s_2\left(\begin{bmatrix} & \Gamma' \\ \Gamma & \end{bmatrix}\right) = s_2(m^n(m'),nn') = nn' = s_2(\Gamma)s_2(\Gamma')$$

and

$$t_2\left(\begin{bmatrix} & \Gamma' \\ \Gamma & \end{bmatrix}\right) = \partial_1(m^n(m'))nn' = \partial_1 mn\partial_1 m'n' = t_2(\Gamma)t_2(\Gamma')$$

and similarly,

$$s_2\left(\begin{bmatrix}\Gamma & \\ & \Gamma'\end{bmatrix}\right) = s_2({}^{\partial_1 mn}(m')m,nn') = nn' = s_2(\Gamma)s_2(\Gamma')$$

and

$$t_2\left(\begin{bmatrix}\Gamma & \\ & \Gamma'\end{bmatrix}\right) = \partial_1({}^{\partial_1 mn}(m')m)nn' = \partial_1 mn\partial_1 m'n' = t_2(\Gamma)t_2(\Gamma').$$

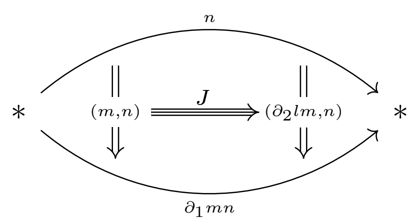
3.4. The set of 3-cells. Now, we can define the group of 3-cells in $\boldsymbol{\mathcal{A}}_*$. Using the group actions of M and N on L, we can create the semi-direct product group $A_3 = L \rtimes M \rtimes N$ with the multiplication

$$(l, m, n)(l', m', n') = (l^n(l')\{\partial_2({}^n(l'))^{-1}, m\}, m^n(m'), nn')$$

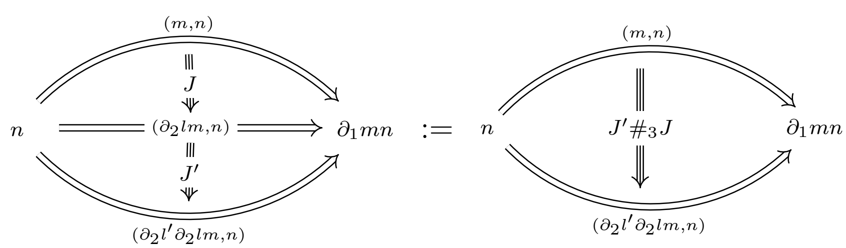
where $\{-,-\} : M \times M \to L$ is the Peiffer lifting of the 2-crossed module $\mathcal{L}$. Using the equality $l\{\partial_2 l^{-1}, m\} = {}^m l$, we can rewrite it as

$$(l, m, n)(l', m', n') = (l^m({}^n l'), m^n m', nn').$$

Any 3-cell in A_3 can be given by an element $J = (l, m, n)$ in $L \rtimes M \rtimes N$ for $l \in L,\ m \in M,\ n \in N$. The 2-source of a 3-cell J is given by $s_3(J) = (m, n)$ and 2-target is given by $t_3(J) = (\partial_2 lm, n)$. Clearly, $s_2(J) = n$ and $t_2(J) = \partial_1 mn$. We can show a 3-cell in A_3 by a diagram;



3.5. The 2-vertical composition of 3-cells. Let $J = (l, m, n) : (m, n) \Rrightarrow (\partial_2 lm, n)$ and $J' = (l', \partial_2 lm, n) : (\partial_2 lm, n) \Rrightarrow (\partial_2 l' \partial_2 lm, n)$ be 3-cells with $s_3(J') = t_3(J)$. The 2-vertical composition $J'\#_3 J$ of J and J' represented by the diagram below



can be given by

$$J'\#_3 J = \begin{bmatrix} J \\ J' \end{bmatrix} = (l', \partial_2 lm, n)\#_3(l, m, n) = (l'l, m, n).$$

For this definition, we obtain clearly

$$s_3(J'\#_3 J) = s_3(l'l, m, n) = (m, n) = s_3(J) \text{ and}$$

$$t_3(J'\#_3 J) = t_3(l'l, m, n) = (\partial_2 l' \partial_2 lm, n) = t_3(J').$$

The identity map $e_3 : A_2 \to A_3$ is defined by $e_3(m, n) = (1_L, m, n)$. We clearly have $s_3 e_3 = t_3 e_3 = id_{A_2}$. The inverse $(J^{-1})^{\#_3}$ of a 3-cell $J = (l, m, n)$ is given by $(J^{-1})^{\#_3} =$

$(l^{-1}, \partial_2 lm, n)$. We have $s_3((J^{-1})^{\#_3}) = (\partial_2 lm, n) = t_3(J)$ and $t_3((J^{-1})^{\#_3}) = (m, n) = s_3(J)$ and

$$(J^{-1})^{\#_3} \#_3 J = (l^{-1}, \partial_2 lm, n) \#_3 (l, m, n) = (1_L, m, n) = e_3(s_3(J))$$

and

$$J \#_3 (J^{-1})^{\#_3} = (l, m, n) \#_3 (l^{-1}, \partial_2 lm, n) = (1_L, \partial_2 lm, n) = e_3(t_3(J))$$

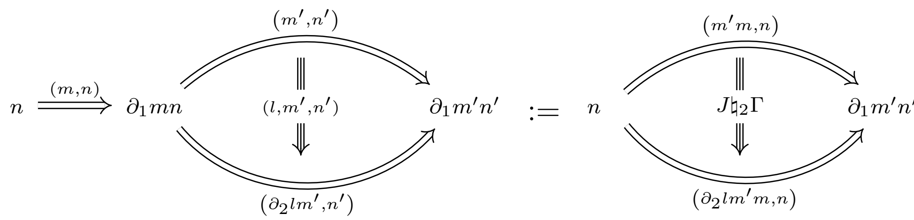
So, we obtain the following result:

3.6. Proposition. $A_3 \underset{e_3}{\overset{s_3,t_3}{\rightrightarrows}} A_2$ *is a groupoid with the 2-vertical composition $\#_3$ of 3-cells.*

3.7. The whiskerings of a 2-cell on a 3-cell. Let $\Gamma = (m, n)$ be a 2-cell and $J = (l, m', \partial_1 mn)$ be a 3-cell with $t_2(\Gamma) = s_2(J)$. The right whiskering of Γ on J is given by

$$J \natural_2 \Gamma = (l, m', \partial_1 mn) \natural_2 (m, n) = (l, m'm, n).$$

This can be represented pictorially as



where $n' = \partial_1 mn$. For this definition, we have clearly

$$s_3(J \natural_2 \Gamma) = (m'm, n) = (m', \partial_1 mn) \#_2 (m, n) = s_3(J) \#_2 \Gamma$$

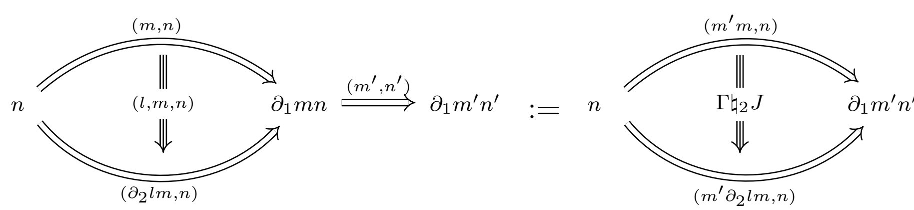
and

$$t_3(J \natural_2 \Gamma) = (\partial_2 lm'm, n) = (\partial_2 lm', \partial_1 mn) \#_2 (m, n) = t_3(J) \#_2 \Gamma.$$

The left whiskering of a 2-cell $\Gamma = (m', \partial_1 mn)$ on a 3-cell $J = (l, m, n)$ with $t_2(J) = s_2(\Gamma)$ is given by

$$\Gamma \natural_2 J = (m', \partial_1 mn) \natural_2 (l, m, n) = ({}^{m'}l, m'm, n) = (l\{\partial_2 l^{-1}, m'\}, m'm, n).$$

This can be represented pictorially as



where $n' = \partial_1 mn$. For this definition, we have clearly

$$s_3(\Gamma \natural_2 J) = (m'm, n) = (m', \partial_1 mn)\#_2(m, n) = \Gamma \#_2 s_3(J)$$

and

$$t_3(\Gamma \natural_2 J) = t_3(l\{\partial_2 l^{-1}, m'\}, m'm, n) = (\partial_2 l \partial_2\{\partial_2 l^{-1}, m'\}m'm, n)$$

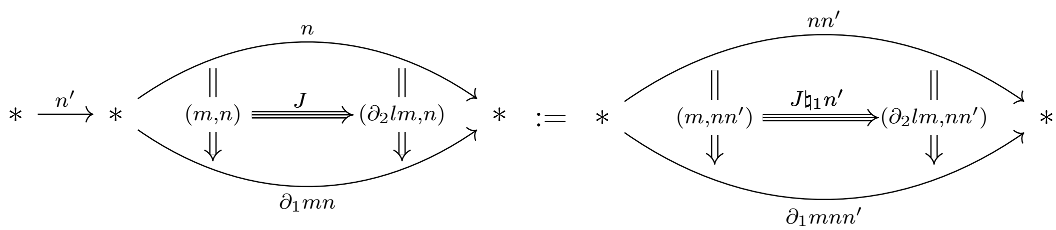
and from the Peiffer lifting axiom (**PL1**)

$$\partial_2\{\partial_2 l^{-1}, m'\} = \partial_2 l^{-1} m' \partial_2 l^{\partial_1 \partial_2 l^{-1}} (m')^{-1}$$

and so; since $\partial_1 \partial_2 l^{-1} = 1$, we have;

$$\begin{aligned} t_3(\Gamma \natural_2 J) =& (\partial_2 l \partial_2 l^{-1} m' \partial_2 l^{\partial_1 \partial_2 l^{-1}} (m')^{-1} m'm, n) \\ =& (m' \partial_2 lm, n) \\ =& (m', \partial_1 mn)\#_2(\partial_2 lm, n) \\ =& \Gamma \#_2 t_3(J). \end{aligned}$$

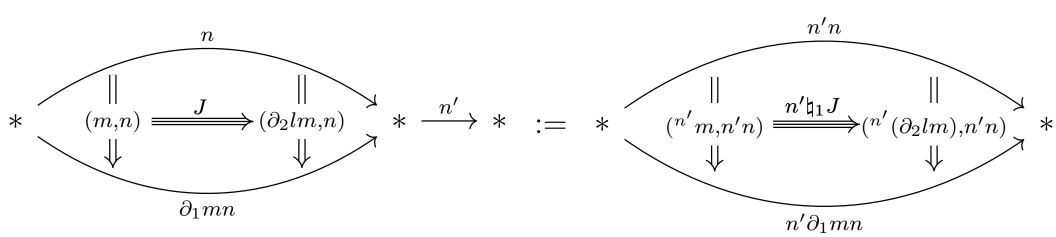
3.8. The whiskerings of a 1-cell on a 3-cell. Let $n' : * \to *$ be a 1-cell and $J = (l, m, n)$ be a 3-cell. The right whiskering of n' on J as shown in the following diagram:



is defined by $J \natural_1 n' = (l, m, nn')$. For this definition clearly;

$$s_3(J \natural_1 n') = (m, nn') = s_3(J)\natural_1 n' \quad \text{and} \quad t_3(J \natural_1 n') = (\partial_2 lm, nn') = t_3(J)\natural_1 n'.$$

The left whiskering of a 1-cell $n' : * \to *$ on a 3-cell $J = (l, m, n)$ represented by the diagram



is defined by

$$n' \natural_1 J = n' \natural_1 (l, m, n) = ({}^{n'}l, {}^{n'}m, n'n).$$

For this definition, we have clearly,

$$s_3(n'\natural_1 J) = ({}^{n'}m, n'n) = n'\natural_1(m,n) = n'\natural_1 s_3(J)$$

and

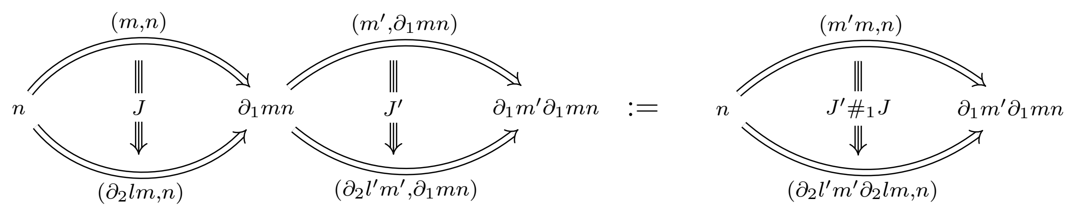
$$t_3(n'\natural_1 J) = (\partial_2({}^{n'}l)^{n'}m, n'n) = ({}^{n'}(\partial_2 lm), n'n) = n'\natural_1(\partial_2 lm, n) = n'\natural_1 s_3(J).$$

On the other hand; we have $s_2(n'\natural_1 J) = n'n = n's_2(J)$ and $t_2(n'\natural_1 J) = n'\partial_1 mn = n't_2(J)$.

3.9. The 1-vertical composition of 3-cells. Let $J = (l,m,n)$ and $J' = (l', m', \partial_1 mn)$ be 3-cells with $s_2(J') = t_2(J)$. The 1-vertical composition $J\#_1 J'$ of J and J' is given by

$$J'\#_1 J = (l'({}^{m'}l), m'm, n) = (l'l\{\partial_2 l^{-1}, m'\}, m'm, n).$$

The 1-vertical composition of these 3-cells can be represented pictorially by



For this definition, we have

$$s_3(J'\#_1 J) = (m'm, n) = (m', \partial_1 mn)\#_2(m,n) = s_3(J')\#_2 s_3(J)$$

and

$$t_3(J'\#_1 J) = (\partial_2 l'm'\partial_2 lm, n) = (\partial_2 l'm', \partial_1 mn)\#_2(\partial_2 lm, n) = t_3(J')\#_2 t_3(J).$$

Similarly, we have $s_2(J'\#_1 J) = n = s_2(J)$ and $t_2(J'\#_1 J) = \partial_1 m'\partial_1 mn = t_2(J')$. The identity map $e_2 : A_1 \to A_3$ is defined by $e_2(n) = (1_L, 1_M, n)$. Clearly, $s_2 e_2 = t_2 e_2 = id_{A_1}$.

Using the 2-vertical composition of 3-cells and whiskerings of 2-cells on 3-cells, we can also give the 1-vertical composition of 3-cells as follows:

$$\begin{aligned}
J'\#_1 J =& (l'l\{\partial_2 l^{-1}, m'\}, m'm, n)\\
=& (l', m'\partial_2 lm, n)\#_3(l\{\partial_2 l^{-1}, m'\}, m'm, n)\\
=& \left((l', m', \partial_1 mn)\natural_2(\partial_2 lm, n)\right)\#_3\left((m', \partial_1 mn)\natural_2(l,m,n)\right)\\
=& (J'\natural_2 t_3(J))\#_3(s_3(J')\natural_2 J)\\
=& \begin{bmatrix} s_3(J')\natural_2 J \\ J'\natural_2 t_3(J)\end{bmatrix}
\end{aligned}$$

and similarly

$$\begin{aligned} J'\#_1 J =&(l'l\{\partial_2 l^{-1}, m'\}, m'm, n)\\ =&({}^{\partial_2 l'}(l\{\partial_2 l^{-1}, m'\})l', m'm, n)\\ =&({}^{\partial_2 l'}({}^{m'}l)l', m'm, n)\\ =&({}^{\partial_2 l'}({}^{m'}l), \partial_2 l'm'm, n)\#_3(l', m'm, n)\\ =&((\partial_2 l'm', \partial_1 mn)\natural_2(l, m, n))\#_3((l', m', \partial_1 mn)\natural_2(m, n))\\ =&(t_3(J')\natural_2 J)\#_3(J'\natural_2 s_3(J))\\ =&\begin{bmatrix} J'\natural_2 s_3(J)\\ t_3(J')\natural_2 J\end{bmatrix}. \end{aligned}$$

For the 3-cell $J = (l, m, n)$ the 1-vertical inverse $(J^{-1})^{\#_1}$ is given by

$$(J^{-1})^{\#_1} = (l^{-1}\{\partial_2 l, m^{-1}\}, m^{-1}, \partial_1 mn).$$

Clearly, we have;

$$(J^{-1})^{\#_1}\#_1 J = (l^{-1}\{\partial_2 l, m^{-1}\}, m^{-1}, \partial_1 mn)\#_1(l, m, n) = (l^{-1}\{\partial_2 l, m^{-1}\}l\{\partial_2 l^{-1}, m^{-1}\}, 1_M, n).$$

From Peiffer lifting axioms; $\{\partial_2 l^{-1}, m^{-1}\} = l^{-1}({}^{m^{-1}}l)$ and $\{\partial_2 l, m^{-1}\} = l({}^{m^{-1}}l^{-1})$, we have $(J^{-1})^{\#_1}\#_1 J = (1_L, 1_M, n) = e_2(n) = e_2(s_2(J))$. Similarly, we obtain

$$\begin{aligned} J\#_1(J^{-1})^{\#_1} =&(l, m, n)\#_1(l^{-1}\{\partial_2 l, m^{-1}\}, m^{-1}, \partial_1 mn)\\ =&(ll^{-1}\{\partial_2 l, m^{-1}\}\{\partial_2(l^{-1}\{\partial_2 l, m^{-1}\})^{-1}, m\}, 1_M, \partial_1 mn). \end{aligned}$$

From Peiffer lifting axioms, we have,

$$\{\partial_2(l^{-1}\{\partial_2 l, m^{-1}\})^{-1}, m\} = \{\partial_2({}^{m^{-1}}l), m\} = ({}^{m^{-1}}l)l^{-1} \text{ and } l^{-1}\{\partial_2 l, m^{-1}\} = {}^{m^{-1}}l^{-1}$$

and then, $J\#_1(J^{-1})^{\#_1} = (1_L, 1_M, \partial_1 mn) = e_2(\partial_1 mn) = e_2 t_2(J)$. Thus, we get the following result:

3.10. Proposition. $A_3 \underset{e_2}{\overset{s_2,t_2}{\rightrightarrows}} A_1$ *is a groupoid with the 1-vertical composition* $\#_1$ *of 3-cells.*

3.11. The interchange law for $\#_1$ and $\#_3$ of 3-cells. Let J and J' be 3-cells in A_3 with $s_3(J') = t_3(J)$. Define $J = (l, m, n)$ and $J' = (l', \partial_2 lm, n)$. The 2-vertical composition of J and J' is given by $(J'\#_3 J) = (l'l, m, n)$.

On the other hand, J_1 and J_1' be 3-cells in A_3 with $s_3(J_1') = t_3(J_1)$. Define

$$J_1 = (l_1, m_1, \partial_1 mn) \quad\text{and}\quad J_1' = (l_1', \partial_2 l_1 m_1, \partial_1 mn).$$

The 2-vertical composition of J_1 and J_1' is given by

$$(J_1'\#_3J_1)=(l_1'l_1,m_1,\partial_1mn).$$

Since $s_2(J_1'\#_3J_1)=t_2(J'\#_3J)$, the 1-vertical composition of 3-cells $(J_1'\#_3J_1)$ and $(J'\#_3J)$ can be given by

$$\begin{aligned}(J_1'\#_3J_1)\#_1(J'\#_3J)=&(l_1'l_1,m_1,\partial_1mn)\#_1(l'l,m,n)\\=&(\underbrace{l_1'l_1l'l\{\partial_2(l'l)^{-1},m_1\}}_{\textbf{(A)}},m_1m,n).\end{aligned}$$

Since $t_2(J')=s_2(J_1')$, the 1-vertical composition of J',J_1' in A_3 can be given by

$$\begin{aligned}J_1'\#_1J'=&(l_1',\partial_2l_1m_1,\partial_1mn)\#_1(l',\partial_2lm,n)\\=&(l_1'l'\{\partial_2(l')^{-1},\partial_2l_1m_1\},\partial_2l_1m_1\partial_2lm,n)\end{aligned}$$

and since $s_2(J_1)=t_2(J)$, the 1-vertical composition of J,J_1 in A_3 can be given by

$$\begin{aligned}(J_1\#_1J)=&(l_1,m_1,\partial_1mn)\#_1(l,m,n)\\=&(l_1l\{\partial_2l^{-1},m_1\},m_1m,n)\end{aligned}$$

Since $s_3(J_1'\#_1J')=t_3(J_1\#_1J)$, the 2-vertical composition of 3-cells $(J_1'\#_1J')$ and $(J_1\#_1J)$ is given by

$$(J_1'\#_1J')\#_3(J_1\#_1J)=(\underbrace{l_1'l'\{\partial_2(l')^{-1},\partial_2l_1m_1\}l_1l\{\partial_2(l)^{-1},m_1\}}_{\textbf{(B)}},m_1m,n).$$

It must be that $(\mathbf{A})=(\mathbf{B})$. For these equalities, we have;

$$\begin{aligned}\textbf{(A)}=&l_1'l_1l'l\{\partial_2(l'l)^{-1},m_1\}\\=&l_1'l_1l'l\{\partial_2(l)^{-1}\partial_2(l')^{-1},m_1\}\\=&l_1'l_1l'l^{\partial_2(l)^{-1}}\{\partial_2(l')^{-1},m_1\}\{\partial_2(l)^{-1},{}^{\partial_1\partial_2(l')^{-1}}(m_1)\}&&(\because\ \mathbf{PL4}(ii))\\=&l_1'l_1l'll^{-1}\{\partial_2(l')^{-1},m_1\}l\{\partial_2(l)^{-1},m_1\}\\=&l_1'l_1l'll^{-1}(l')^{-1}({}^{m_1}l')l(l)^{-1}({}^{m_1}l)&&(\because\ \mathbf{PL2})\\=&l_1'l_1({}^{m_1}l')({}^{m_1}l)\end{aligned}$$

and

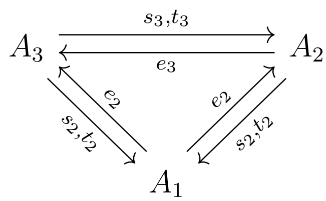
$$\begin{aligned}\textbf{(B)}=&l_1'l'\{\partial_2(l')^{-1},\partial_2l_1m_1\}l_1l\{\partial_2(l)^{-1},m_1\}\\=&l_1'l'\{\partial_2(l')^{-1},\partial_2l_1\}^{\partial_1\partial_2(l')^{-1}(\partial_2l_1)}\{\partial_2(l')^{-1},m_1\}l_1l\{\partial_2(l)^{-1},m_1\}&&(\because\ \mathbf{PL4}(i))\\=&l_1'l'[(l')^{-1},l_1]l_1\{\partial_2(l')^{-1},m_1\}(l_1)^{-1}l_1l(l)^{-1}({}^{m_1}l)&&(\because\ \mathbf{PL5})\\=&l_1'l'(l')^{-1}l_1l'(l_1)^{-1}l_1(l')^{-1}({}^{m_1}l')(l_1)^{-1}l_1ll^{-1}({}^{m_1}l)\\=&l_1'l_1({}^{m_1}l')({}^{m_1}l).\end{aligned}$$

Thus, we have

$$(J'_1\#_3J_1)\#_1(J'\#_3J) = (J'_1\#_1J')\#_3(J_1\#_1J).$$

Consequently, the interchange law for $\#_1$ and $\#_3$ is satisfied. We can give the following result:

3.12. Proposition. *The 2-vertical and 1-vertical compositions of 3-cells and vertical compositions of 2-cells give a structure of 2-groupoid shown pictorially as;*



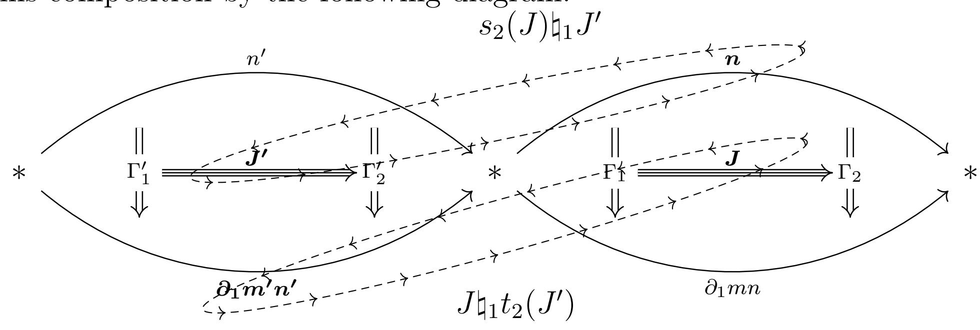
where A_1 is the set of 0-cells, A_2 is the set of 1-cells and A_3 is the set of 2-cells for this structure.

3.13. The horizontal compositions of 3-cells. The horizontal composition $\begin{bmatrix} & J' \\ J & \end{bmatrix}$ of 3-cells $J=(l,m,n):\Gamma_1 \Rrightarrow \Gamma_2$ and $J'=(l',m',n'):\Gamma'_1 \Rrightarrow \Gamma'_2$ in A_3, where $\Gamma_1=(m,n)$, $\Gamma_2=(\partial_2 lm,n)$ and $\Gamma'_1=(m',n')$, $\Gamma'_2=(\partial_2 l'm',n')$ is given by

$$\begin{aligned}\begin{bmatrix} & J' \\ J & \end{bmatrix} &=(J\natural_1 t_2(J'))\#_1(s_2(J)\natural_1 J')\\ &=((l,m,n)\natural_1\partial_1 m'n')\#_1(n\natural_1(l',m',n'))\\ &=(l,m,n\partial_1 m'n')\#_1({}^n(l'),{}^n(m'),nn')\\ &=(l^m({}^n(l')),m^n(m'),nn')\\ &=(l^n(l')\{\partial_2({}^n(l'))^{-1},m\},m^n(m'),nn')\end{aligned}$$

We can show this composition by the following diagram:

$\begin{bmatrix} & J' \\ J & \end{bmatrix} :=$



For this definition, we have

$$s_3\left(\begin{bmatrix} & J' \\ J & \end{bmatrix}\right) = (m^n m', nn') = \begin{bmatrix} & (m',n') \\ (m,n) & \end{bmatrix} = \begin{bmatrix} & s_3(J') \\ s_3(J) & \end{bmatrix}$$

and

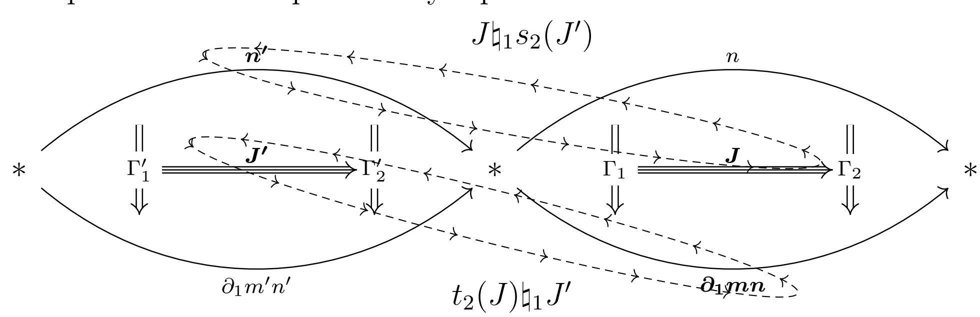
$$
\begin{aligned}
t_3\left(\begin{bmatrix} & J' \\ J & \end{bmatrix}\right) =& (\partial_2(l^n(l')\{\partial_2({}^n(l'))^{-1}, m\})m^n(m'), nn') \\
=& (\partial_2(l^n(l'))\partial_2\{\partial_2({}^n(l'))^{-1}, m\}m^n(m'), nn') \\
=& (\partial_2(l)\partial_2({}^n(l'))\partial_2({}^n(l')^{-1})m\partial_2({}^n(l'))^{\partial_1\partial_2(({}^n(l'))^{-1})}m^{-1}m^n(m'), nn') && (\because \mathbf{PL1}) \\
=& (\partial_2(l)m\partial_2({}^n(l'))^n(m'), nn') && (\because \partial_1\partial_2 = 1) \\
=& (\partial_2(l)m^n(\partial_2(l')m'), nn') \\
=& (\partial_2(l)m, m\partial_1 m'n')\#_2(\partial_2({}^n l')^n(m'), nn') \\
=& ((\partial_2(l)m, n)\natural_1\partial_1 m'n')\#_2(n\natural_1(\partial_2 l'm', n')) \\
=& \begin{bmatrix} & (\partial_2 l'm', n') \\ (\partial_2 lm, n) & \end{bmatrix} = \begin{bmatrix} & t_3(J') \\ t_3(J) & \end{bmatrix}.
\end{aligned}
$$

On the other hand, we can define the horizontal composition $\left[\begin{smallmatrix} J & \\ & J' \end{smallmatrix}\right]$ by

$$
\begin{aligned}
\begin{bmatrix} J & \\ & J' \end{bmatrix} =& (t_2(J)\natural_1 J')\#_1(J\natural_1 s_2(J')) \\
=& (\partial_1 mn\natural_1(l', m', n'))\#_1((l, m, n)\natural_1 n') \\
=& ({}^{\partial_1 mn}(l')^{\partial_1 mn}(m'), \partial_1 mnn')\#_1((l, m, nn')) \\
=& ({}^{\partial_1 mn}(l')^{\partial_1 mn(m')}(l), {}^{\partial_1 mn}(m')m, nn') \\
=& ({}^{\partial_1 mn}(l')^{\partial_1 mn}(l\{\partial_2 l^{-1}, m'\}), {}^{\partial_1 mn}(m')m, nn') \\
=& ({}^{\partial_1 mn}(l'l\{\partial_2 l^{-1}, m'\}), {}^{\partial_1 mn}(m')m, nn').
\end{aligned}
$$

Similarly, we can represent this composition by a picture

$\begin{bmatrix} J & \\ & J' \end{bmatrix} :=$



For this definiton, we obtain

$$
s_3\left(\begin{bmatrix} J & \\ & J' \end{bmatrix}\right) = \left({}^{\partial_1 m}({}^n m')m, nn'\right) = \begin{bmatrix} s_3(J) & \\ & s_3(J') \end{bmatrix}
$$

and

$$
\begin{aligned}
t_3\left(\begin{bmatrix} J & \\ & J' \end{bmatrix}\right) =& (\partial_2({}^{\partial_1 mn}(l'l\{\partial_2 l^{-1}, m'\}))^{\partial_1 mn}(m')m, nn') \\
=& (\partial_2({}^{\partial_1 mn}(l'l l^{-1\,m'} l))^{\partial_1 mn}(m')m, nn') \qquad (\because \mathbf{PL2}) \\
=& (\partial_2({}^{\partial_1 mn}(l'^{m'} l))^{\partial_1 mn}(m')m, nn') \\
=& \partial_2({}^{\partial_1 mn}(l'))\partial_2({}^{\partial_1 mn}({}^{m'} l))^{\partial_1 mn}(m')m, nn') \\
=& ({}^{\partial_1 mn}(\partial_2(l'))^{\partial_1 mn}(m')\partial_2 l(({}^{\partial_1 mn}(m'))^{-1})^{\partial_1 mn}(m')m, nn') \\
=& ({}^{\partial_1 mn}(\partial_2(l')m')\partial_2(l)m, nn') \\
=& (\partial_1 mn\natural_1(\partial_2(l')m', n'))\#_2((\partial_2(l)m, n)\natural_1 n') \\
=& \begin{bmatrix} \Gamma_2 & \\ & \Gamma_2' \end{bmatrix} = \begin{bmatrix} t_3(J) & \\ & t_3(J') \end{bmatrix}.
\end{aligned}
$$

3.14. The interchange 3-cell. For any 2-cells $\Gamma = (m, n)$ and $\Gamma' = (m', n')$, the interchange 3-cell is defined by

$$\Gamma\#\Gamma' = (\{m, {}^n m'\}^{-1}, m^n m', nn').$$

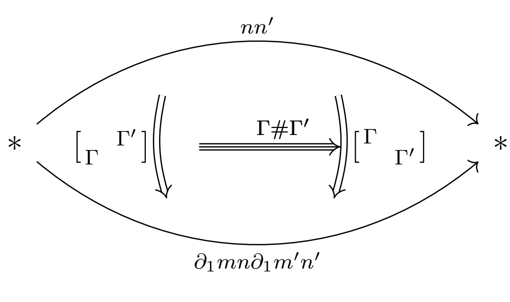
For this interchange 3-cell, we have

$$s_3(\Gamma\#\Gamma') = (m^n m', nn') = \begin{bmatrix} & \Gamma' \\ \Gamma & \end{bmatrix}$$

and

$$
\begin{aligned}
t_3(\Gamma\#\Gamma') =& (\partial_2\{m, {}^n m'\}^{-1} m^n m', nn') \\
=& ((m^n m'(m)^{-1}({}^{\partial_1 m}({}^n m')^{-1}))^{-1} m^n m', nn') \qquad (\because \mathbf{PL1}) \\
=& ({}^{\partial_1 m}({}^n m')m^n m'^{-1} m^{-1} m^n m', nn') \\
=& ({}^{\partial_1 m}({}^n m')m, nn') \\
=& \begin{bmatrix} \Gamma & \\ & \Gamma' \end{bmatrix}
\end{aligned}
$$

Thus, we can say that the interchange 3-cell $\Gamma\#\Gamma'$ is a 3-cell from $\left[{}_{\Gamma}\ {}^{\Gamma'}\right]$ to $\left[{}^{\Gamma}\ {}_{\Gamma'}\right]$ in A_3. We can represent the interchange 3-cell by the following diagram,



where

$$s_2(\Gamma\#\Gamma') = nn' = s_2(\Gamma)s_2(\Gamma') \quad \text{and} \quad t_2(\Gamma\#\Gamma') = \partial_1 mn\partial_1 m'n' = t_2(\Gamma)t_2(\Gamma').$$

3.15. 2-FUNCTORIALITY. Consider the 2-cells $\Gamma_1 = (m, n)$, $\Gamma_2 = (\partial_2 lm, n)$, $\Gamma'_1 = (m', n')$ and $\Gamma'_2 = (\partial_2 l'm', n')$ and 3-cells $J = (l, m, n)$ and $J' = (l', m', n')$ with

$$\Gamma_1 = s_3(J) \xrightarrow{J} t_3(J) = \Gamma_2 \quad \text{and} \quad \Gamma'_1 = s_3(J') \xrightarrow{J'} t_3(J') = \Gamma'_2 \,.$$

We know that

$$\begin{bmatrix} & \Gamma'_1 \\ \Gamma_1 & \end{bmatrix} = m^n m', nn' \qquad \text{and} \qquad \begin{bmatrix} \Gamma_1 & \\ & \Gamma'_1 \end{bmatrix} = ({}^{\partial_1 m}({}^n m')m, nn')$$

and

$$\begin{bmatrix} & \Gamma'_2 \\ \Gamma_2 & \end{bmatrix} = (\partial_2 lm^n(\partial_2 l'm'), nn') \quad \text{and} \quad \begin{bmatrix} \Gamma_2 & \\ & \Gamma'_2 \end{bmatrix} = ({}^{\partial_1 mn}(\partial_2 l'm')\partial_2 lm, nn').$$

Our aim is to show the following equality:

$$\begin{bmatrix} J & \\ & J' \end{bmatrix} \#_3(\Gamma_1 \# \Gamma'_1) = (\Gamma_2 \# \Gamma'_2) \#_3 \begin{bmatrix} & J' \\ J & \end{bmatrix}.$$

On the left side of the equality, we have already

$$\begin{bmatrix} J & \\ & J' \end{bmatrix} = ({}^{\partial_1 mn}(l'l\{\partial_2 l^{-1}, m'\}), {}^{\partial_1 mn}(m')m, nn').$$

and

$$\Gamma_1 \# \Gamma'_1 = (m, n) \# (m', n') = (\{m, {}^n m'\}^{-1}, m^n m', nn').$$

Since $t_3(\Gamma_1 \# \Gamma'_1) = \begin{bmatrix} \Gamma_1 & \\ & \Gamma'_1 \end{bmatrix} = s_3\left(\begin{bmatrix} J & \\ & J' \end{bmatrix}\right)$, we obtain

$$\begin{aligned} \begin{bmatrix} J & \\ & J' \end{bmatrix} \#_3(\Gamma_1 \# \Gamma'_1) &= ({}^{\partial_1 mn}(l'l\{\partial_2 l^{-1}, m'\}), {}^{\partial_1 mn}(m')m, nn') \#_3 (\{m, {}^n m'\}^{-1}, m^n m', nn') \\ &= (\underbrace{{}^{\partial_1 mn}(l'l\{\partial_2 l^{-1}, m'\})\{m, {}^n m'\}^{-1}}_{\mathbf{A}}, m^n m', nn') \end{aligned}$$

where

$$s_3\left(\begin{bmatrix} J & \\ & J' \end{bmatrix} \#_3(\Gamma_1 \# \Gamma'_1)\right) = \begin{bmatrix} & \Gamma'_1 \\ \Gamma_1 & \end{bmatrix} \text{ and } t_3\left(\begin{bmatrix} J & \\ & J' \end{bmatrix} \#_3(\Gamma_1 \# \Gamma'_1)\right) = \begin{bmatrix} \Gamma_2 & \\ & \Gamma'_2 \end{bmatrix}.$$

On the right side, we have already

$$\begin{bmatrix} & J' \\ J & \end{bmatrix} = (l^n(l')\{\partial_2({}^n(l'))^{-1}, m\}, m^n(m'), nn')$$

and

$$\Gamma_2 \# \Gamma'_2 = (\partial_2 lm, n) \# (\partial_2 l'm', n') = (\{\partial_2 lm, {}^n(\partial_2 l'm')\}^{-1}, \partial_2 lm^n(\partial_2 l'm'), nn').$$

Since, $s_3(\Gamma_2\#\Gamma_2') = \left[{}_{\Gamma_2}\ {}^{\Gamma_2'}\right] = t_3\left(\left[{}_J\ {}^{J'}\right]\right)$, we obtain

$$\begin{aligned}
&(\Gamma_2\#\Gamma_2')\#_3\begin{bmatrix} & J' \\ J & \end{bmatrix} \\
= \ &(\{\partial_2 lm,^n(\partial_2 l'm')\}^{-1}, \partial_2 lm^n(\partial_2 l'm'), nn')\#_3(l^n(l')\{\partial_2(^n(l'))^{-1}, m\}, m^n(m'), nn') \\
= \ &(\underbrace{\{\partial_2 lm,^n(\partial_2 l'm')\}^{-1} l^n(l')\{\partial_2(^n(l'))^{-1}, m\}}_{\mathbf{B}}, m^n(m'), nn')
\end{aligned}$$

where

$$s_3\left((\Gamma_2\#\Gamma_2')\#_3\begin{bmatrix} & J' \\ J & \end{bmatrix}\right) = \begin{bmatrix} & \Gamma_1' \\ \Gamma_1 & \end{bmatrix} \text{ and } t_3\left((\Gamma_2\#\Gamma_2')\#_3\begin{bmatrix} & J' \\ J & \end{bmatrix}\right) = \begin{bmatrix} \Gamma_2 & \\ & \Gamma_2' \end{bmatrix}.$$

To prove the necessary equality for this axiom, we must show that $\mathbf{A} = \mathbf{B}$. Using the Peiffer lifting axioms, we have

$$
\begin{aligned}
\mathbf{B} &= (\{\partial_2 lm, {}^{n}(\partial_2 l' m')\}^{-1} l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}) \\
&= (\{\partial_2 lm, \partial_2({}^{n}(l'))^{n} m'\}^{-1}) l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}) \\
&= \Big(\Big(\{\partial_2 lm, \partial_2({}^{n}(l'))\}^{\partial_1\partial_2 lm(\partial_2({}^{n}(l')))}(\{\partial_2 lm, {}^{n} m'\})\Big)^{-1} l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}\Big) \\
&= \Big(\Big(\{\partial_2 lm, \partial_2({}^{n}(l'))\}^{\partial_2({}^{\partial_1 m}(({}^{n}(l'))))}(\{\partial_2 lm, {}^{n} m'\})\Big)^{-1} l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}\Big) \\
&= \Big((\{\partial_2 lm, \partial_2({}^{n}(l'))\}^{\partial_1 mn}(l')\{\partial_2 lm, {}^{n} m'\}({}^{\partial_1 mn}(l'))^{-1})^{-1} l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}\Big) \\
&= \Big(({}^{\partial_2 l}(\{m, \partial_2({}^{n}(l'))\})\{\partial_2 l, {}^{\partial_1 m}(\partial_2({}^{n}(l')))\}^{\partial_1 mn}(l')\{\partial_2 lm, {}^{n} m'\}({}^{\partial_1 mn}(l'))^{-1})^{-1} l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}\Big) \\
&= \Big(({}^{\partial_2 l}\left({}^{mn}l'({}^{\partial_1 mn}l')^{-1}\right)\{\partial_2 l, \partial_2({}^{\partial_1 mn}(l'))\}^{\partial_1 mn}(l')\{\partial_2 lm, {}^{n} m'\}({}^{\partial_1 mn}(l'))^{-1})^{-1} l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}\Big) \\
&= \Big(({}^{\partial_2 l}\left({}^{mn}l'({}^{\partial_1 mn}l')^{-1}\right)[l, {}^{\partial_1 mn}(l')]^{\partial_1 mn}(l')\{\partial_2 lm, {}^{n} m'\}({}^{\partial_1 mn}(l'))^{-1})^{-1} l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}\Big) \\
&= \Big(l({}^{mn}l'({}^{\partial_1 mn}l')^{-1})l^{-1}l^{\partial_1 mn}(l')l^{-1}({}^{\partial_1 mn}(l'))^{-1\,\partial_1 mn}(l')\{\partial_2 lm, {}^{n} m'\}({}^{\partial_1 mn}(l'))^{-1}\Big)^{-1} \\
&\qquad\qquad l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}) \\
&= \Big((l^{mn}l' l^{-1}\{\partial_2 lm, {}^{n} m'\}({}^{\partial_1 mn}(l'))^{-1})^{-1} l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}\Big) \\
&= \Big((l^{mn}l' l^{-1\,\partial_2 l}(\{m, {}^{n} m'\})\{\partial_2 l, {}^{\partial_1 mn}(m')\}({}^{\partial_1 mn}(l'))^{-1})^{-1} l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}\Big) \\
&= \Big(\Big(l^{mn}l' l^{-1} l(\{m, {}^{n} m'\})l^{-1}l({}^{\partial_1 mn(m')}l^{-1})({}^{\partial_1 mn}(l'))^{-1}\Big)^{-1} l^{n}(l')\{\partial_2({}^{n}(l'))^{-1}, m\}\Big) \\
&= \Big(\Big(l^{mn}l'(\{m, {}^{n} m'\})({}^{\partial_1 mn(m')}l^{-1})({}^{\partial_1 mn}(l'))^{-1}\Big)^{-1} l^{n}(l')({}^{n}(l'))^{-1}({}^{mn}(l'))\Big) \\
&= \Big({}^{\partial_1 mn}(l'))({}^{\partial_1 mn(m')}l)(\{m, {}^{n} m'\})^{-1}({}^{mn}l')^{-1}l^{-1}l^{n}(l')({}^{n}(l'))^{-1}({}^{mn}(l'))\Big) \\
&= \Big({}^{\partial_1 mn}(l'))({}^{\partial_1 mn(m')}l)(\{m, {}^{n} m'\})^{-1}\Big) \\
&= \Big({}^{\partial_1 mn}(l'^{m'}l)(\{m, {}^{n} m'\})^{-1}\Big) \\
&= ({}^{\partial_1 mn}(l' l\{\partial_2 l^{-1}, m'\})(\{m, {}^{n} m'\})^{-1}) \\
&= \mathbf{A}
\end{aligned}
$$

and thus, we obtain

$$
\begin{bmatrix} J & \\ & J' \end{bmatrix} \#_3(\Gamma_1 \# \Gamma_1') = (\Gamma_2 \# \Gamma_2') \#_3 \begin{bmatrix} & J' \\ J & \end{bmatrix}.
$$

3.16. 1-FUNCTORIALITY. For the 2-cells $\Gamma = (m, n)$, $\Gamma' = (m', \partial_1 mn)$ and $\Gamma'' = (m'', n'')$ given by the following diagrams;

$$
n \xRightarrow{(m,n)} \partial_1 mn \xRightarrow{(m',\partial_1 mn)} \partial_1 m' \partial_1 mn \quad \text{and} \quad n'' \xRightarrow{(m'',n'')} \partial_1 m'' n''
$$

by taking $\gamma = n$, $\gamma'' = n''$, $\phi = \partial_1 mn$, $\phi'' = \partial_1 m''n''$, $\psi = \partial_1 m'\partial_1 mn$, first we must show that

$$\begin{bmatrix}\Gamma\natural_1\gamma'' \\ \Gamma'\#\Gamma''\end{bmatrix}\#_3\begin{bmatrix}\Gamma\#\Gamma'' \\ \Gamma'\natural_1\phi''\end{bmatrix} = \begin{bmatrix}\Gamma \\ \Gamma'\end{bmatrix}\#\Gamma''.$$

On the left side, we have already

$$\Gamma\#\Gamma'' = (\{m,{}^n m''\}^{-1}, m^n m'', nn'')$$

and

$$\Gamma'\natural_1\phi'' = \Gamma'\natural_1\partial_1 m''n'' = (m', \partial_1 mn)\natural_1(\partial_1 m''n'') = (m', \partial_1 mn\partial_1 m''n'')$$

and so, we have

$$\begin{aligned}\begin{bmatrix}\Gamma\#\Gamma'' \\ \Gamma'\natural_1\phi''\end{bmatrix} =&(\Gamma'\natural_1\phi'')\natural_2(\Gamma\#\Gamma'') \\ =&(m', \partial_1 mn\partial_1 m''n'')\natural_2(\{m,{}^n m''\}^{-1}, m^n m'', nn'') \\ =&({}^{m'}(\{m,{}^n m''\}^{-1}), m'm^n m'', nn'').\end{aligned}$$

Similarly, we have

$$\Gamma'\#\Gamma'' = (m', \partial_1 mn)\#(m'', n'') = (\{m',{}^{\partial_1 mn} m''\}^{-1}, m'({}^{\partial_1 mn}m''), \partial_1 mnn'')$$

and $\Gamma\natural_1\gamma'' = (m, n)\natural_1 n'' = (m, nn'')$ and so, we have

$$\begin{aligned}\begin{bmatrix}\Gamma\natural_1\gamma'' \\ \Gamma'\#\Gamma''\end{bmatrix} =&(\Gamma'\#\Gamma'')\natural_2(\Gamma\natural_1\gamma'') \\ =&(\{m',{}^{\partial_1 mn} m''\}^{-1}, m'({}^{\partial_1 mn}m''), \partial_1 mnn'')\natural_2(m, nn'') \\ =&(\{m',{}^{\partial_1 mn} m''\}^{-1}, m'({}^{\partial_1 mn}m'')m, nn').\end{aligned}$$

Since

$$s_3\left(\begin{bmatrix}\Gamma\natural_1\gamma'' \\ \Gamma'\#\Gamma''\end{bmatrix}\right) = t_3\left(\begin{bmatrix}\Gamma\#\Gamma'' \\ \Gamma'\natural_1\phi''\end{bmatrix}\right),$$

we obtain,

$$\begin{bmatrix}\Gamma\natural_1\gamma'' \\ \Gamma'\#\Gamma''\end{bmatrix}\#_3\begin{bmatrix}\Gamma\#\Gamma'' \\ \Gamma'\natural_1\phi''\end{bmatrix} = (\{m',{}^{\partial_1 mn}(m'')\}^{-1}({}^{m'}(\{m,{}^n m''\}^{-1})), m'm^n m'', nn'').$$

On the right side, we have already

$$\begin{bmatrix}\Gamma \\ \Gamma'\end{bmatrix} = (m', \partial_1 mn)\#_2(m, n) = (m'm, n)$$

and

$$\begin{bmatrix}\Gamma \\ \Gamma'\end{bmatrix}\#\Gamma'' = (m'm, n)\#(m'', n'') = (\{m'm,{}^n m''\}^{-1}, m'm^n m'', nn'')$$

where

$$\begin{aligned}\{m'm,{}^{n}m''\}^{-1} =& \left({}^{m'}(\{m,{}^{n}m''\})\{m',{}^{\partial_1 mn}(m'')\}\right)^{-1} \qquad (\because \textbf{PL4}(ii))\\ =&\{m',{}^{\partial_1 mn}(m'')\}^{-1}({}^{m'}(\{m,{}^{n}m''\})^{-1}).\end{aligned}$$

Consequently,

$$\begin{aligned}\begin{bmatrix}\Gamma\\ \Gamma'\end{bmatrix}\#\Gamma'' =&(\{m',{}^{\partial_1 mn}(m'')\}^{-1}({}^{m'}(\{m,{}^{n}m''\}^{-1})), m'm^{n}m'', nn'')\\ =&\begin{bmatrix}\Gamma\natural_1\gamma''\\ \Gamma'\#\Gamma''\end{bmatrix}\#_3\begin{bmatrix}\Gamma\#\Gamma''\\ \Gamma'\natural_1\phi''\end{bmatrix}\end{aligned}$$

Now, for the same 2-cells, we must show that

$$\begin{bmatrix}\Gamma''\#\Gamma\\ \phi''\natural_1\Gamma'\end{bmatrix}\#_3\begin{bmatrix}\gamma''\natural_1\Gamma\\ \Gamma''\#\Gamma'\end{bmatrix} = \Gamma''\#\begin{bmatrix}\Gamma\\ \Gamma'\end{bmatrix}.$$

On the left side, we have already

$$\gamma''\natural_1\Gamma = n''\natural_1(m,n) = ({}^{n''}m, n''n)$$

and

$$\Gamma''\#\Gamma' = (m'',n'')\#(m',\partial_1 mn) = (\{m'',{}^{n''}m'\}^{-1}, m''({}^{n''}m'), n''\partial_1 mn)$$

and so,

$$\begin{aligned}\begin{bmatrix}\gamma''\natural_1\Gamma\\ \Gamma''\#\Gamma'\end{bmatrix} =&(\Gamma''\#\Gamma')\natural_2(\gamma''\natural_1\Gamma)\\ =&(\{m'',{}^{n''}m'\}^{-1}, m''({}^{n''}m')({}^{n''}m), n''n).\end{aligned}$$

Similarly,

$$\Gamma''\#\Gamma = (m'',n'')\#(m,n) = (\{m'',{}^{n''}m\}^{-1}, m''({}^{n''}m), n''n)$$

and

$$\phi''\natural_1\Gamma' = \partial_1 m''n''\natural_1(m',\partial_1 mn) = ({}^{\partial_1 m''n''}(m'), \partial_1 m''n''\partial_1 mn)$$

so, we obtain

$$\begin{aligned}\begin{bmatrix}\Gamma''\#\Gamma\\ \phi''\natural_1\Gamma'\end{bmatrix} =&(\phi''\natural_1\Gamma')\natural_2(\Gamma''\#\Gamma)\\ =&({}^{\partial_1 m''n''}(m'), \partial_1 m''n''\partial_1 mn)\natural_2(\{m'',{}^{n''}m\}^{-1}, m''({}^{n''}m), n''n)\\ =&({}^{{}^{\partial_1 m''n''}(m')}(\{m'',{}^{n''}m\}^{-1}), {}^{\partial_1 m''n''}(m')m''({}^{n''}m), n''n).\end{aligned}$$

Therefore, we obtain

$$\begin{bmatrix}\Gamma''\#\Gamma\\ \phi''\natural_1\Gamma'\end{bmatrix}\#_3\begin{bmatrix}\gamma''\natural_1\Gamma\\ \Gamma''\#\Gamma'\end{bmatrix} = (\underbrace{{}^{{}^{\partial_1 m''n''}(m')}(\{m'',{}^{n''}m\}^{-1})\{m'',{}^{n''}m'\}^{-1}}_{\mathbf{A}}, m''({}^{n''}(m'm)), n''n).$$

On the right side, we have already

$$\begin{bmatrix}\Gamma\\ \Gamma'\end{bmatrix} = (m', \partial_1 mn)\#_2(m,n) = (m'm, n)$$

and

$$\Gamma''\#\begin{bmatrix}\Gamma\\ \Gamma'\end{bmatrix} = (m'', n'')\#(m'm, n) = (\{m'',^{n''}(m'm)\}^{-1}, m''(^{n''}(m'm)), n''n)$$

where

$$\begin{aligned}\Big(\{m'',^{n''}(m'm)\}\Big)^{-1} &= \Big(\{m'',^{n''}(m')^{n''}(m))\}\Big)^{-1}\\ &= \Big(\{m'',^{n''}m'\}^{\partial_1 m''n''(m')}\{m'',^{n''}m\}\Big)^{-1} \qquad (\because \mathbf{PL4}(i))\\ &= \Big(^{\partial_1 m''n''(m')}(\{m'',^{n''}m\}^{-1})\{m'',^{n''}m'\}^{-1}\Big)\\ &= \mathbf{A}.\end{aligned}$$

Consequently, we obtain

$$\begin{bmatrix}\Gamma''\#\Gamma\\ \phi''\natural_1\Gamma'\end{bmatrix}\#_3\begin{bmatrix}\gamma''\natural_1\Gamma\\ \Gamma''\#\Gamma'\end{bmatrix} = \Gamma''\#\begin{bmatrix}\Gamma\\ \Gamma'\end{bmatrix}.$$

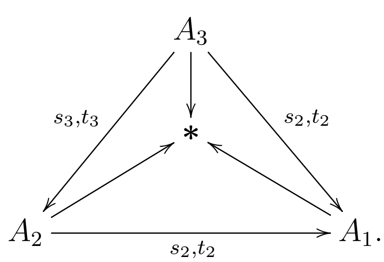
Therefore, we have verified all Gray 3-group axioms, so this is functorial and hence defines a functor from the category of 2-crossed modules of groups to the category of Gray 3-groups:

$$\mathbf{\Theta} : \mathbf{X_2Mod} \longrightarrow \mathbf{Gray}.$$

4. From Gray 3-groups to 2-crossed modules

Let $\boldsymbol{\mathcal{A}}_*$ be a Gray 3-group shown as

$\boldsymbol{\mathcal{A}}_* :=$



We will construct a 2-crossed module $L^* \xrightarrow{\partial_2} M^* \xrightarrow{\partial_1} N$ with the Peiffer lifting map $\{-,-\}^* : M^* \times M^* \longrightarrow L^*$. Since $A_1 \underset{e_1}{\overset{s_1,t_1}{\rightrightarrows}} *$ is a totally disconnected groupoid, it can be regarded as a group and so we can take $A_1 = N$. We know that $A_2 \underset{e_2}{\overset{s_2,t_2}{\rightrightarrows}} A_1$ is a groupoid together with the operation $\#_2$ of 2-cells. Define a set in A_2 by $M^* = \{\Gamma \in A_2 : s_2(\Gamma) = 1_N\}$. In this case, any element of M^* is given by the form $\Gamma : 1_N \Rightarrow n$ as a 2-cell in $\boldsymbol{\mathcal{A}}_*$. The set M^* is a group with the operation given by

$$\Gamma\Gamma' = \begin{bmatrix} & \Gamma' \\ \Gamma & \end{bmatrix} = (\Gamma\natural_1 t_2(\Gamma'))\#_2\Gamma' = (\Gamma\natural_1 n')\#_2\Gamma'$$

for $\Gamma : 1_N \Rightarrow n$ and $\Gamma' : 1_N \Rightarrow n'$ in M^* with $s_2(\Gamma) = s_2(\Gamma') = 1_N$. Firstly, we show that M^* is a group together with this operation. For any elements $\Gamma : 1_N \Rightarrow n$, $\Gamma' : 1_N \Rightarrow n'$ and $\Gamma'' : 1_N \Rightarrow n''$ in M^*, we have;

$$\begin{aligned}
(\Gamma\Gamma')\Gamma'' =& ((\Gamma\Gamma')\natural_1 t_2(\Gamma''))\#_2\Gamma'' \\
=& (((\Gamma\natural_1 t_2(\Gamma'))\#_2\Gamma')\natural_1 t_2(\Gamma''))\#_2\Gamma'' \\
=& (\Gamma\natural_1 t_2(\Gamma')\natural_1 t_2(\Gamma''))\#_2((\Gamma'\natural_1 t_2(\Gamma''))\#_2\Gamma'') \\
=& (\Gamma\natural_1 t_2(\Gamma'\Gamma''))\#_2((\Gamma'\natural_1 t_2(\Gamma''))\#_2\Gamma'') \\
=& (\Gamma\natural_1 t_2(\Gamma'\Gamma''))\#_2(\Gamma'\Gamma'') \\
=& \Gamma(\Gamma'\Gamma'')
\end{aligned}$$

and also, $\Gamma^{-1} : 1_N \Rightarrow n^{-1}$ and $e_2(1_{A_1})$ is an identity element in M^*. So we have

$$\Gamma\Gamma^{-1} = (\Gamma\natural_1 t_2(\Gamma^{-1}))\#_2\Gamma^{-1} = e_2(1_{A_1}) \quad \text{and} \quad \Gamma^{-1}\Gamma = (\Gamma^{-1}\natural_1 t_2(\Gamma))\#_2\Gamma = e_2(1_{A_1}).$$

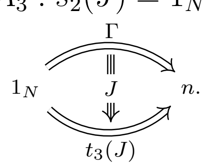
Therefore, M^* is a group with the operation given above. Moreover,

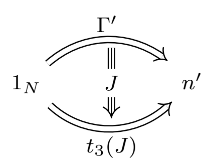
$$(\Gamma^{-1})\natural_1 n = (\Gamma)^{-1\#_2} \quad \text{and} \quad ((\Gamma)^{-1\#_2})\natural_1 n^{-1} = \Gamma^{-1}.$$

Since $t_2|_{M^*}(\Gamma\Gamma') = nn' = t_2|_{M^*}(\Gamma)t_2|_{M^*}(\Gamma')$ for $\Gamma, \Gamma' \in M^*$, the map $\partial_1 = t_2|_{M^*}$ is a homomorphism of groups. The action of element $p \in N$ on $\Gamma : 1_N \Rightarrow n \in M^*$ is given by ${}^p\Gamma = p\natural_1\Gamma\natural_1 p^{-1}$. For this action, we have

$$\partial_1({}^p\Gamma) = t_2|_M({}^p\Gamma) = pnp^{-1} = pt_2|_{M^*}(\Gamma)p^{-1} = p\partial_1(\Gamma)p^{-1}$$

and so ∂_1 is a pre-crossed module.

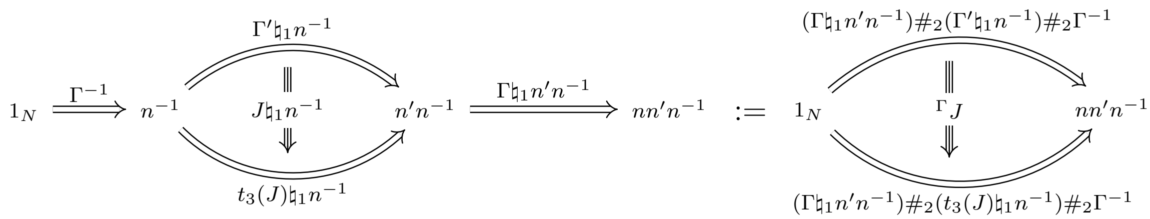
We know that $A_3 \underset{e_2}{\overset{s_2,t_2}{\rightleftarrows}} A_1$ is a groupoid with the 1-vertical composition $\#_1$ of 3-cells. Define a set in A_3 by $A_3^* = \{J \in A_3 : s_2(J) = 1_N\}$. For this description, any element in A_3^* can be illustrated by the picture 

An action of $(\Gamma : 1_N \Rightarrow n) \in M^*$ on $J :=$ 

in A_3^* is given by

$${}^\Gamma J = (\Gamma\natural_1 n'n^{-1})\natural_2(J\natural_1 n^{-1})\natural_2\Gamma^{-1}.$$

This action can be represented pictorially as



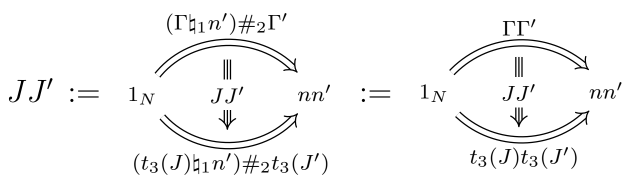
where

$$\begin{aligned} t_3(^{\Gamma}J) =&(\Gamma\natural_1 n'n^{-1})\#_2(t_3(J)\natural_1 n^{-1})\#_2\Gamma^{-1}\\ =&((\Gamma\natural_1 n')\#_2 t_3(J))\natural_1 n^{-1}\#_2\Gamma^{-1}\\ =&\Gamma t_3(J)\Gamma^{-1}\end{aligned}$$

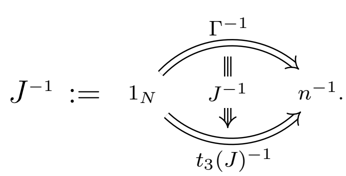
and $t_2(^{\Gamma}J) = nn'n^{-1} = t_2(\Gamma)t_2(J)t_2(\Gamma)^{-1}$. For this definition A_3^* is a group with the operation by

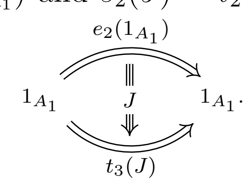
$$JJ' = \begin{bmatrix} & J' \\ J & \end{bmatrix} = (J\natural_1 t_2(J'))\#_1 J' = (J\natural_1 n')\#_1 J'$$

for any $J, J' \in A_3^*$. This operation can be represented by the following diagram



For this operation, the inverse J^{-1} of J is given by

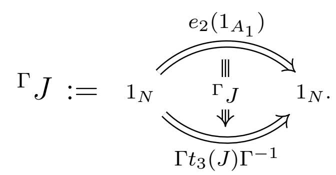


Define a set in A_3^* by $L^* = {A_3}^*(1_{A_1}) = \{J \in {A_3}^* : s_3(J) = e_2(1_{A_1})$ and $s_2(J) = t_2(J) = 1_{A_1}\}$. For this description, any element in L^* is given by the form  The

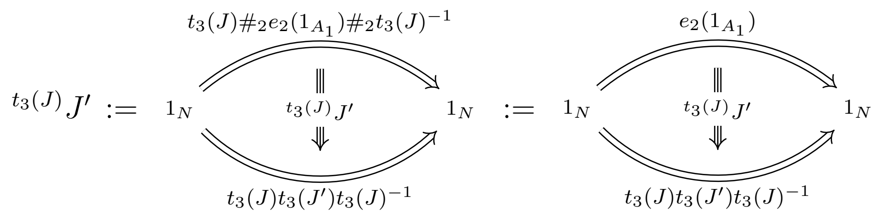
group operation in L^* is given by

$$JJ' = \begin{bmatrix} & J' \\ J & \end{bmatrix} = (J\natural_1 t_2(J'))\#_1 J' = (J\natural_1 1_{A_1})\#_1 J' = J\#_1 J'.$$

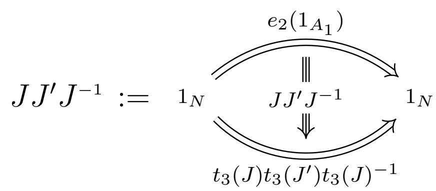
The map $\partial_2 : L^* \to M^*$ is given by the restriction of t_3 to L^*. Since $t_3|_{L^*}(JJ') = t_3|_{L^*}(J)t_3|_{L^*}(J')$ for $J, J' \in L^*$, ∂_2 is a homomorphism of groups. The action of $\Gamma : 1_N \Rightarrow n$ on $J \in L^*$ is given by: ${}^{\Gamma}J = (\Gamma\natural_1 n^{-1})\natural_2(J\natural_1 n^{-1})\natural_2\Gamma^{-1}$ and we can show it pictorially by



For this action, we have



On the other hand, we have



Therefore, we have $\partial_2({}^{\Gamma}J) = \Gamma\partial_2(J)\Gamma^{-1}$ and ${}^{\partial_2(J)}J' = JJ'J^{-1}$ and so, ∂_2 is a crossed module. Since $\partial_1\partial_2(J) = t_2(t_3(J)) = 1_N$ for all $J \in L^*$, the diagram $L^* \xrightarrow{\partial_2} M^* \xrightarrow{\partial_1} N$ is a complex of groups.

We can define the Peiffer Lifting $\{-,-\}^* : M^* \times M^* \to L^*$ by

$$\{\Gamma, \Gamma'\}^* = \left[{}_{(\Gamma\#\Gamma')^{-1\#_3}} \quad {}^{e_3\left(s_3((\Gamma\#\Gamma')^{-1\#_3})\right)^{-1}}\right].$$

For $\Gamma : 1_N \Rightarrow n$ and $\Gamma' : 1_N \Rightarrow n'$ in M^*, we have

$$\partial_2\{\Gamma, \Gamma'\}^* = \left[{}_{t_3(\Gamma\#\Gamma')^{-1\#_3}} \quad {}^{\left(s_3((\Gamma\#\Gamma')^{-1\#_3})\right)^{-1}}\right]$$

where

$$\left(s_3((\Gamma\#\Gamma')^{-1\#_3})\right)^{-1} = ((n\natural_1\Gamma')\#_2\Gamma)^{-1} \quad \text{and} \quad t_3(\Gamma\#\Gamma')^{-1\#_3} = (\Gamma\natural_1 n')\#_2\Gamma'$$

and

$$((n\natural_1\Gamma')\#_2\Gamma)^{-1} = (((n\natural_1\Gamma')\#_2\Gamma)^{-1\#_2})\natural_1(n')^{-1}n^{-1}.$$

So, we have

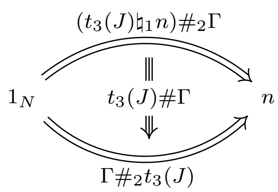
$$\begin{aligned}
\partial_2\{\Gamma,\Gamma'\}^* =&((\Gamma\natural_1 n')\#_2\Gamma')((n\natural_1\Gamma')\#_2\Gamma)^{-1}\\
=&(((\Gamma\natural_1 n')\#_2\Gamma')\natural_1(n')^{-1}n^{-1})\#_2((((n\natural_1\Gamma')\#_2\Gamma)^{-1\#_2})\natural_1(n')^{-1}n^{-1})\\
=&((\Gamma\Gamma')\natural_1(n')^{-1}n^{-1})\#_2((\Gamma)^{-1\#_2}\natural_1(n')^{-1}n^{-1})\#_2(n\natural_1(\Gamma')^{-1\#_2}\natural_1(n')^{-1}n^{-1})\\
=&((\Gamma\Gamma')\natural_1(n')^{-1}n^{-1})\#_2(((\Gamma)^{-1}\natural_1 n)\natural_1(n')^{-1}n^{-1})\#_2(n\natural_1(\Gamma')^{-1}\natural_1 n^{-1})\\
=&((\Gamma\Gamma')\natural_1(n')^{-1}n^{-1})\#_2((\Gamma)^{-1}\natural_1 n(n')^{-1}n^{-1})\#_2({}^{t_2(\Gamma)}(\Gamma')^{-1})\\
=&((\Gamma\Gamma')\natural_1(n')^{-1}n^{-1})\#_2((\Gamma)^{-1}({}^{t_2(\Gamma)}(\Gamma')^{-1}))\\
=&\Gamma\Gamma'(\Gamma)^{-1}({}^{\partial_1(\Gamma)}(\Gamma')^{-1})
\end{aligned}$$

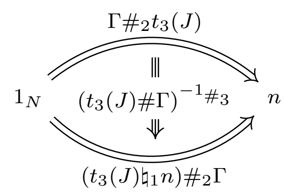
and clearly this is the first axiom of Peiffer Lifting.

Now, we show that $\{t_3(J),\Gamma\}^* = J^{\Gamma}(J)^{-1}$. We know that

$$\{t_3(J),\Gamma\}^* = \begin{bmatrix} & e_3\left(s_3((t_3(J)\#\Gamma)^{-1\#_3})\right)^{-1} \\ (t_3(J)\#\Gamma)^{-1\#_3} & \end{bmatrix}$$

where

$t_3(J)\#\Gamma$:= 

and $(t_3(J)\#\Gamma)^{-1\#_3}$:= 

On the other hand, we have

$$(s_3((t_3(J)\#\Gamma)^{-1\#_3})) = \Gamma\#_2 t_3(J) \quad\text{and}\quad (s_3((t_3(J)\#\Gamma)^{-1\#_3}))^{-1} = (\Gamma\#_2 t_3(J))^{-1}$$

where

$$(\Gamma\#_2 t_3(J))^{-1} = ((t_3(J)^{-1\#_2})\natural_1 n^{-1})\#_2(((\Gamma)^{-1\#_2})\natural_1 n^{-1}) : 1_N \Rightarrow n^{-1}.$$

We have also

$$((t_3(J)\#\Gamma)^{-1\#_3})\natural_1 n^{-1} = J\#_1 e_3(\Gamma\natural_1 n^{-1})\#_1(J\natural_1 n^{-1})\#_1(e_3(t_3(J))\natural_1 n^{-1}).$$

Thus, we have

$$\begin{aligned}
\{t_3(J),\Gamma\}^* =&\begin{bmatrix} & e_3\left(((t_3(J)^{-1\#_2})\natural_1 n^{-1})\#_2(((\Gamma)^{-1\#_2})\natural_1 n^{-1})\right) \\ (t_3(J)\#\Gamma)^{-1\#_3} & \end{bmatrix}\\
=&(((t_3(J)\#\Gamma)^{-1\#_3})\natural_1 n^{-1})\#_1 e_3\left(((t_3(J)^{-1\#_2})\natural_1 n^{-1})\#_2(((\Gamma)^{-1\#_2})\natural_1 n^{-1})\right)\\
=&(((t_3(J)\#\Gamma)^{-1\#_3})\natural_1 n^{-1})\#_1 e_3((t_3(J)^{-1\#_2})\natural_1 n^{-1})\#_1 e_3(((\Gamma)^{-1\#_2})\natural_1 n^{-1})\\
=&J\#_1 e_3(\Gamma\natural_1 n^{-1})\#_1(J^{-1}\natural_1 n^{-1})\#_1 e_3(t_3(J)\natural_1 n^{-1})\#_1 e_3((t_3(J)^{-1\#_2})\natural_1 n^{-1})\#_1\\
&\qquad e_3(((\Gamma)^{-1\#_2})\natural_1 n^{-1})\\
=&J\#_1 e_3(\Gamma\natural_1 n^{-1})\#_1(J^{-1}\natural_1 n^{-1})\#_1 e_3(\Gamma^{-1})\\
=&J\#_1\left((\Gamma\natural_1 n^{-1})\natural_2(J^{-1}\natural_1 n^{-1})\natural_2\Gamma^{-1}\right)\\
=&J^{\Gamma}(J)^{-1}
\end{aligned}$$

and thus the second axiom of the Peiffer Lifting is satisfied. Using the 1-and 2-functorialities, the other Peiffer lifting axioms can be shown similarly.

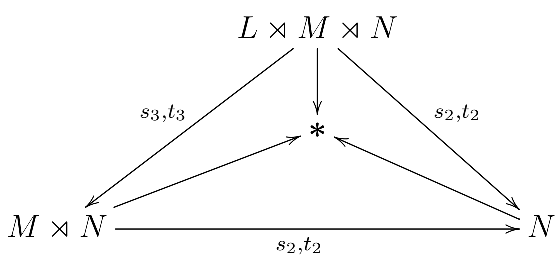
Therefore, we have defined a functor from the category of Gray 3-groups to that of 2-crossed modules denoted by $\Delta : \mathbf{Gray} \longrightarrow \mathbf{X_2Mod}$.

5. The equivalence between $\mathbf{X_2Mod}$ and $\mathbf{Gray}$

In the previous sections, we obtained functors between the categories of 2-crossed modules and Gray 3-groups: $\mathbf{\Theta} : \mathbf{X_2Mod} \longrightarrow \mathbf{Gray}$ and $\Delta : \mathbf{Gray} \longrightarrow \mathbf{X_2Mod}$. We will prove that $\mathbf{X_2Mod}$ is equivalent to $\mathbf{Gray}$.

Let $\mathcal{L} : L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$ be a 2-crossed module with the Peiffer lifting $\{-,-\} : M \times M \to L$ in $\mathbf{X_2Mod}$. If we apply the functor $\mathbf{\Theta}$ to this 2-crossed module, we obtained the following Gray 3-group:

$\mathbf{\Theta}(\mathcal{L}) :=$



Now we apply the functor Δ to this Gray 3-group $\mathbf{\Theta}(\mathcal{L})$. We will obtain a 2-crossed module which is isomorphic to $\mathcal{L}$ in each step. We know that in $\mathbf{\Theta}(\mathcal{L})$, the 1-cells are the elements of N and 2-cells are given by the form $(m,n) : n \Rightarrow \partial_1 mn$. Then;

$$M^* = {A_2}^* = \{(m,n) : s_2(m,n) = 1_N\} = \{(m,1) : m \in M\} \cong M.$$

Similarly,

$${A_3}^* = \{(l,m,n) : s_2(l,m,n) = n = 1_N\} = \{(l,m,1) : l \in L, m \in M\}$$

and so we have,

$$L^* = \{(l,m,1) : s_3(l,m,1) = (m,1) = e_2(1_N) = (1_M,1_N)\} = \{(l,1,1) : l \in L\} \cong L.$$

We know that for any 2-cells $\Gamma = (m,1) : 1_N \Rightarrow \partial_1 m = n$, $\Gamma' = (m',1) : 1 \Rightarrow \partial_1 m' = n'$ in M^*, the group operation in M^* is given by,

$$\Gamma\Gamma' = (m,1)(m',1) = \begin{bmatrix} & {}^{(m',1)} \\ {}_{(m,1)} & \end{bmatrix} = (m, \partial_1 m')\#_2(m',1) = (mm',1)$$

and the group operation in L^* is given by $JJ' = (l,1,1)(l',1,1) = (ll',1,1)$. For these elements, the Peiffer Lifting is

$$\begin{aligned}
\{\Gamma,\Gamma'\}^* &= \begin{bmatrix} & e_3\left(s_3((\Gamma\#\Gamma')^{-1\#_3})\right)^{-1} \\ (\Gamma\#\Gamma')^{-1\#_3} & \end{bmatrix} \\
&= \begin{bmatrix} & e_3(s_3(\{m,m'\}^{-1},mm',1)^{-1\#_3})^{-1} \\ (\{m,m'\}^{-1},mm',1)^{-1\#_3} & \end{bmatrix} \\
&= \begin{bmatrix} & e_3(\partial_2\{m,m'\}^{-1}mm',1)^{-1}) \\ (\{m,m'\},\partial_2\{m,m'\}^{-1}mm',1) & \end{bmatrix} \\
&= \begin{bmatrix} & (1,(m^{-1})^{\partial_1 m}(m'),1) \\ (\{m,m'\},{}^{\partial_1 m}(m')m,1) & \end{bmatrix} \\
&= (\{m,m'\},1,1)
\end{aligned}$$

where $\{-,-\}$ is the Peiffer lifting of the 2-crossed module $\mathcal{L}$. Thus, we have $\Delta\mathbf{\Theta}(\mathcal{L}) \cong \mathcal{L}$.

Let $\mathcal{A}_*$ be any Gray 3-group. If we apply the functor Δ to $\mathcal{A}_*$, we obtained a 2-crossed module as $L^* \xrightarrow{\partial_2} M^* \xrightarrow{\partial_1} N$ with the Peiffer lifting map $\{-,-\}^* : M^* \times M^* \to L^*$ given above. If we apply the functor Θ to this 2-crossed module $\Delta(\mathcal{A}_*)$, we have $\Theta\Delta(A_1) = N$ and since $\Delta(\ A_2 \underset{e_2}{\overset{s_2,t_2}{\rightleftarrows}} A_1\ ) =\ M^* \xrightarrow{\partial_1} N$, by applying the functor Θ, we have

$$\Theta(\ M^* \xrightarrow{\partial_1} N\ ) :=\ M^* \rtimes N \underset{\overline{e_2}}{\overset{\overline{s_2},\overline{t_2}}{\rightleftarrows}} N$$

where $\overline{s_2}(\Gamma,n) = n$ and $\overline{t_2}(\Gamma,n) = t_2(\Gamma)\natural_1 n$ with $\Gamma : 1 \Rightarrow n'$ in M^*. We must show that $(M^* \rtimes N \rightrightarrows N) \cong (A_2 \rightrightarrows A_1)$. Define a groupoid morphism

$$\eta\ :\ \begin{array}{ccc} A_2 & \xrightarrow{\eta_1} & M^* \rtimes N \\ {\scriptstyle s_2}\downarrow\downarrow{\scriptstyle t_2} & & {\scriptstyle \overline{s_2}}\downarrow\downarrow{\scriptstyle \overline{t_2}} \\ A_1 & \underset{\eta_0 = id}{=\!=\!=} & N \end{array}$$

by $\eta_1(\Gamma) = (\Gamma\#_2 e_2 s_2 \Gamma^{-1}, s_2(\Gamma))$ and $\eta_0 = id$. In this case, we obtain $\overline{s_2}(\Gamma\#_2 e_2 s_2\Gamma^{-1}, s_2(\Gamma)) = s_2(\Gamma)$ and $\overline{t_2}(\Gamma\#_2 e_2 s_2 \Gamma^{-1}, s_2(\Gamma)) = t_2(\Gamma)$. Conversely, define a groupoid morphism

$$\psi\ :\ \begin{array}{ccc} M^* \rtimes N & \xrightarrow{\psi_1} & A_2 \\ {\scriptstyle \overline{s_2}}\downarrow\downarrow{\scriptstyle \overline{t_2}} & & {\scriptstyle s_2}\downarrow\downarrow{\scriptstyle t_2} \\ N & \underset{\Psi_0 = id}{=\!=\!=} & A_1 \end{array}$$

by $\psi_1(\Gamma,n) = \Gamma\#_2 e_2(n)$ where $\Gamma : 1 \Rightarrow n'$ in M^*. Therefore, for all $\Gamma \in A_2$, we have

$$\psi_1 \circ \eta_1(\Gamma) = \psi_1(\Gamma\#_2 e_2 s_2 \Gamma^{-1}, s_2(\Gamma)) = \Gamma\#_2 e_2 s_2 \Gamma^{-1}\#_2 e_2 s_2 \Gamma = \Gamma$$

and for all $(\Gamma, n) \in M^* \rtimes N$ with $\Gamma : 1 \Rightarrow n'$, we have

$$\begin{aligned}\eta_1 \circ \psi_1(\Gamma, n) =& \eta_1(\Gamma \#_2 e_2(n)) \\ =& (\Gamma \#_2 e_2(n) \#_2 e_2 s_2(\Gamma \#_2 e_2 n)^{-1}, s_2(\Gamma \#_2 e_2(n))) \\ =& (\Gamma \#_2 e_2(n) \#_2 e_2(n)^{-1}, n) \qquad\qquad (\because\ s_2(\Gamma) = 1) \\ =& (\Gamma, n).\end{aligned}$$

Thus, we have $(A_2 \rightrightarrows A_1) \cong (M^* \rtimes N \rightrightarrows N)$. Now, we must show that

$$(A_3 \rightrightarrows A_2) \cong (L^* \rtimes M^* \rtimes N \rightrightarrows M^* \rtimes N).$$

Define a groupoid morphism

$$\beta\ :\ \begin{array}{ccc} A_3 & \xrightarrow{\beta_1} & L^* \rtimes M^* \rtimes N \\ {\scriptstyle s_3}\downarrow\downarrow{\scriptstyle t_3} & & {\scriptstyle \overline{s_3}}\downarrow\downarrow{\scriptstyle \overline{t_3}} \\ A_2 & \xrightarrow[\beta_0]{} & M^* \rtimes N \end{array}$$

by $\beta_1(J) = (J \#_1 e_3 s_3 J^{-1}, s_3(J) \natural_1 t_2(J)^{-1}, t_2(J))$ and $\beta_0(\Gamma) = \eta_1(\Gamma)$. Then, by taking $J = (l, m, n)$, we can check that by

$$\begin{aligned}\beta_1(J) =& (J \#_1 e_3 s_3 J^{-1}, s_3(J) \natural_1 t_2(J)^{-1}, t_2(J)) \\ =& ((l, m, n) \#_1 (1, {}^{n^{-1}}m^{-1}, n^{-1}), (m, n) \natural_1 n^{-1}, n) \\ =& ((l, 1, 1), (m, 1), n) \in L^* \rtimes M^* \rtimes N.\end{aligned}$$

Conversely, define a groupoid morphism

$$\alpha\ :\ \begin{array}{ccc} L^* \rtimes M^* \rtimes N & \xrightarrow{\alpha_1} & A_3 \\ {\scriptstyle \overline{s_3}}\downarrow\downarrow{\scriptstyle \overline{t_3}} & & {\scriptstyle s_3}\downarrow\downarrow{\scriptstyle t_3} \\ M^* \rtimes N & \xrightarrow[\alpha_0]{} & A_2 \end{array}$$

by $\alpha_1(J, \Gamma, n) = J \#_1 e_3(\Gamma) \#_1 e_3(n)$ and $\alpha_0(\Gamma, n) = \psi_1(\Gamma, n)$ where $s_3(J) = e_2(1_{A_1})$, $\Gamma : 1 \Rightarrow n'$. In this case, by taking $J = (l, 1, 1) \in L^*, \Gamma = (m, 1) \in M^*$ and $n \in N$ we can check it by

$$\alpha_1(J, \Gamma, n) = (l, 1, 1) \#_1 e_3(m, 1) \#_1 (1, 1, m) = (l, 1, 1) \#_1 (1, m, 1) \#_1 (1, 1, n) = (l, m, n).$$

On the other hand, for all $J \in A_3$, we have

$$\begin{aligned}\alpha_1 \circ \beta_1(J) =& \alpha_1(J \#_1 (e_3 s_3 J)^{-1}, s_3(J) \natural_1 t_2(J)^{-1}, t_2(J)) \\ =& J \#_1 e_3 s_3 J^{-1} \#_1 e_3(s_3 J \natural_1 t_2(J)^{-1}) \natural_1 e_3 t_2(J) \\ =& J \# e_3 s_3 J^{-1} \#_1 e_3 s_3 J \#_1 e_3 t_2(J)^{-1} \#_1 e_3 t_2(J) \\ =& J\end{aligned}$$

and similarly for all $(J,\Gamma,n)\in L^*\rtimes M^*\rtimes N$, we have

$$\begin{aligned}\beta_1\circ\alpha_1(J,\Gamma,n)=&\beta_1(J\#_1e_3\Gamma\#_1e_3n)\\=&(J\#_1e_3(\Gamma)\#_1e_3(n)\#_1e_3s_3(J\#_1e_3\Gamma\#_1e_3(n))^{-1},\\&\quad s_3(J\#_1e_3\Gamma\#e_3n)\natural_1t_2(J\#_1e_3\Gamma\#_1e_3n)^{-1},t_2(J\#_1e_3\Gamma\#e_3n))\\=&(J\#_1e_3(\Gamma)\#_1e_3(n)\#_1e_3n^{-1}\#_1e_3\Gamma^{-1},(\Gamma\natural_1n)\natural_1n^{-1},n)\quad(\because\ s_3(J)=e_2(1_{A_1}))\\=&(J,\Gamma,n)\end{aligned}$$

By taking $J=(l,1,1)\in L^*$, $\Gamma=(m,1)\in M^*$ and $n\in N$, we can check it by

$$\begin{aligned}\beta_1\circ\alpha_1((l,1,1),(m,1),n)=&\beta_1((l,1,1)\#_1e_3(m,1)\#_1e_3(n))\\=&\beta_1((l,1,1)\#_1(1,m,1)\#_1(1,1,n))\\=&\beta_1(l,m,n)\\=&((l,1,1),(m,1),n).\end{aligned}$$

Therefore, we have; $\left(A_3\underset{e_3}{\overset{s_3,t_3}{\rightleftarrows}}A_2\right)\cong(L^*\rtimes M^*\rtimes N\rightrightarrows M^*\rtimes N)$. Consequently, we obtain that $\Theta\Delta(\mathcal{A}_*)\cong A_*$ and $\Delta\boldsymbol{\Theta}(\mathcal{L})\cong\mathcal{L}$. Thus, we get the following result.

5.1. Theorem. $\mathbf{X_2Mod}$ *is equivalent to* **Gray**.

6. A linear representation of 2-crossed modules

A common approach to representations of groups is via modules over a group or an algebra [12], [17]. Linear representations of a group G are in one-to-one correspondence with modules over its group algebra, $K(G)$, see [5], where K is the group algebra functor from the category of groups to that of algebras. A linear representation of a cat^1-group or (indirectly) a crossed module has been obtained by Barker [5]. Barker's result, of course, was a 2-dimensional generalisation of a linear representation of groups. In [5], Barker has proven that the category $\mathbf{Ch}^1_K$ of chain complexes over vector spaces on a fixed field K is a 2-category. Using this result, a linear representation of a crossed module or equivalently of a cat^1-group $\mathfrak{C}$ is a 2-functor $\mathfrak{C}\longrightarrow\mathbf{Aut}(\delta)\leqslant\mathbf{Ch}^1_K$, where $\mathbf{Aut}(\delta)$ is a cat^1-group obtained from $\mathbf{Ch}^1_K$. The subcategory $\mathbf{Aut}(\delta)$ is considered automorphism cat^1-group. In $\mathbf{Ch}^1_K$, by considering only the invertible chain maps over a fixed linear transformation $\delta:V_1\longrightarrow V_0$ of vector spaces, $\mathbf{Aut}(\delta)$ has a 2-groupoid structure with a single object δ. In this section, we will explain 2-dimensional version of these results for 2-crossed modules.

6.1. A Gray 3-group from chain complexes of length-2. Let K be a field and $\mathcal{V}_i(i\in\mathbb{Z})$ be vector spaces over K. Consider the chain complexes of linear transformations

$$\mathcal{V}:=\cdots\longrightarrow V_n\xrightarrow{d_n}V_{n-1}\longrightarrow\cdots V_1\xrightarrow{d_1}V_0\xrightarrow{d_0}V_{-1}\longrightarrow\cdots$$

A chain map between chain complexes $\mathcal{V}$ and $\mathcal{V}'$; $F : \mathcal{V} \longrightarrow \mathcal{V}'$ consists of components $F_i : V_i \longrightarrow V_i{}'$ such that $F_{i-1}d_i = d_i{}'F_i$ for all $i \in \mathbb{Z}$ where each F_i is a linear transformation. We can say that the following diagram is commutative.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & V_{n+1} & \xrightarrow{d_{n+1}} & V_n & \xrightarrow{d_n} & V_{n-1} & \xrightarrow{d_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle F_{n+1}} & & \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_{n-1}} & & \\
\cdots & \longrightarrow & V'_{n+1} & \xrightarrow[d'_{n+1}]{} & V'_n & \xrightarrow[d'_n]{} & V'_{n-1} & \xrightarrow[d'_{n-1}]{} & \cdots
\end{array}$$

Let $F : \mathcal{V} \longrightarrow \mathcal{V}'$ and $G : \mathcal{V}' \longrightarrow \mathcal{V}''$ be chain maps. The composition $GF : \mathcal{V} \longrightarrow \mathcal{V}''$ is defined $(GF)_i = G_iF_i$ for all i, where G_iF_i is the usual composition of linear transformations.

Let F and G be chain maps from the chain complex $\mathcal{V}$ to the chain complex $\mathcal{V}'$. A chain homotopy from F to G ; $H : F \simeq G$ consists of a linear map $H'_n : \mathcal{V}_n \longrightarrow \mathcal{V}'{}_{n+1}$ satisfying the condition

$$G_n - F_n = d'_{n+1}H'_n + H'_{n-1}d_n$$

for each $n \in \mathbb{Z}$.

The category of chain complexes will be shown by **Ch**. Kamps and Porter in [22] showed that **Ch** has a 2-groupoid enriched Gray category. We will consider in this section non-negative chain complexes in which the subscripts are non-negative integers. Now, recall from [1] and [22],the construction of a Gray category structure from the chain complexes of length-2 of vector spaces. Suppose that

$$\mathcal{V} := V_2 \xrightarrow{\delta_2} V_1 \xrightarrow{\delta_1} V_0$$

is a chain complex of vector spaces of length-2. By considering all chain complexes of length-2 as objects, we can create the category $\mathbf{Ch}^2_K$ whose morphisms are chain maps between chain complexes of length-2.

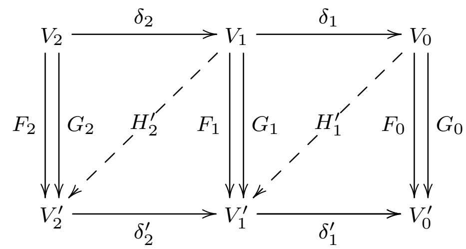
A chain map $F = (F_2, F_1, F_0)$ from $\mathcal{V}$ to $\mathcal{V}'$ is given by following commutative diagram:

$$\begin{array}{ccccc}
V_2 & \xrightarrow{\delta_2} & V_1 & \xrightarrow{\delta_1} & V_0 \\
{\scriptstyle F_2}\downarrow & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\
V'_2 & \xrightarrow[\delta'_2]{} & V'_1 & \xrightarrow[\delta'_1]{} & V'_0
\end{array}$$

where F_i is a linear transformation for $i = 0, 1, 2$.

Thus, we can consider the chain maps $F := (F_2, F_1, F_0)$ as 1-cells for $\mathbf{Ch}^2_K$. Now suppose that F and G are chain maps between the chain complexes of length-2 $\mathcal{V}$ and $\mathcal{V}'$. A 1-homotopy $(H, F) := ((H'_1, H'_2), F)$ from F to G with the chain homotopy components

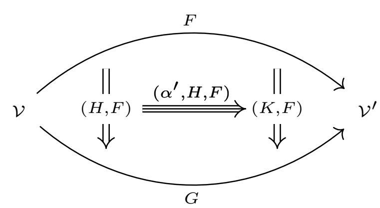
H_1', H_2' can be represented pictorially as



For the homotopy components H_1' and H_2' the following conditions are satisfied.

1. $\delta_1' H_1' = G_0 - F_0$,
2. $H_1'\delta_1 + \delta_2' H_2' = G_1 - F_1$,
3. $H_2'\delta_2 = G_2 - F_2$.

Thus, we can consider the 1-homotopies (H, F) from F to G as 2-cells for $\mathbf{Ch}_K^2$. Now, we briefly describe a 3-cell for $\mathbf{Ch}_K^2$, using the definition of a 2-homotopy between 1-homotopies given in [1]. Suppose that $(H, F) := (H_1', H_2', F)$ and $(K, F) := (K_1', K_2', F)$ are 1-homotopies from F to G. A 2-homotopy from (H, F) to (K, F) is given by a triple $\alpha := (\alpha', H, F)$ where $\alpha' : V_0 \to V_2'$ is the homotopy component linear map satisfying the conditions; $\delta_2'\alpha' = K_1' - H_1'$ and $\alpha'\delta_1 = K_2' - H_2'$. Therefore, we can represent the cells in $\mathbf{Ch}_K^2$ pictorially as



Now, we give the source and target maps. For any 3-cell (α', H, F) these maps are given by

$$s_3(\alpha', H, F) = (H, F)\ ,\ s_2(\alpha', H, F) = F \ \text{ and } \ s_1(\alpha', H, F) = \mathcal{V}.$$

and similarly

$$t_3(\alpha', H, F) = (K, F)\ ,\ t_2(\alpha', H, F) = G \ \text{ and } \ t_1(\alpha', H, F) = \mathcal{V}'.$$

We will give the definitions of vertical and horizontal compositions of 2-cells and 3-cells. The 2-vertical composition of $\alpha := (\alpha', H, F)$ and $\beta := (\beta', K, F)$ is defined by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \beta \#_3 \alpha := (\beta' + \alpha', H, F)$$

where $t_3(\alpha) = s_3(\beta)$, that is $K_1' = H_1' + \delta_2'\alpha'$ and $K_2' = H_2 + \alpha'\delta_1$.

For any 2-cells, $(H,F) : F \Longrightarrow G$ and $(K,G) : G \Longrightarrow T$, the vertical composition $\#_2$ is given by $K\#_2H : F \Longrightarrow T$ where the chain homotopy component is $(K\#_2H)' = K' + H'$ with $K' = (K_1', K_2')$ and $H' = (H_1', H_2')$. For any 1-cell $F' : \mathcal{V} \to \mathcal{V}'$ and a 2-cell (K,G), the right whiskering of F' on (K,G) is given by $(K,G)\natural_1 F' = (K_1'F_0', K_2'F_1', GF')$ where $(K,G) : G \Longrightarrow G'$ is a 1-homotopy. Similarly, the left whiskering of a 1-cell $G : \mathcal{V}' \to \mathcal{V}''$ on a 2-cell $(H,F) : F \Longrightarrow F' : \mathcal{V} \to \mathcal{V}'$ is given by $G\natural_1(H,F) = (G_1H_1', G_2H_2', GF)$.

The horizontal compositions of 2-cells

$$\Gamma = (K,G) = ((K_1', K_2'), (G_2, G_1, G_0)) : G \Rightarrow G'$$

and

$$\Gamma' = (H,F) = ((H_1', H_2'), (F_2, F_1, F_0)) : F \Rightarrow F'$$

are given by

$$\begin{bmatrix} & \Gamma' \\ \Gamma & \end{bmatrix} = (K_1'F_0' + G_1H_1', K_2'F_1' + G_2H_2', GF)$$

and

$$\begin{bmatrix} \Gamma & \\ & \Gamma' \end{bmatrix} = (K_1'F_0 + G_1'H_1', K_2'F_1 + G_2'H_2', GF).$$

For any 3-cells $\beta := (\beta', K, G) : (K,G) \Rrightarrow (K', G)$ and $\alpha := (\alpha', H, F) : (H,F) \Rrightarrow (H', F)$, the horizontal composition of α and β is given by

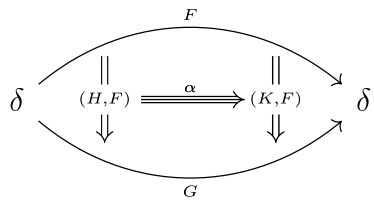
$$\begin{bmatrix} & \beta \\ \alpha & \end{bmatrix} = (G_2\alpha' + \beta' F_0', (K_1'F_0' + G_1H_1', K_2F_1' + G_2H_2'), GF).$$

Similarly, $[\,{}^{\alpha}\;{}_{\beta}\,]$ can be defined. The verification of Gray 3-group axioms for these structures, can be found in [1] and [22]. Therefore, we can say that $\mathbf{Ch}_K^2$ has a Gray category structure.

Suppose now that $\delta := V_2 \xrightarrow{\delta_2} V_1 \xrightarrow{\delta_1} V_0$ is a fixed chain complex of vector spaces of length-2. The automorphism cat^2-group (cf. [24]) as a Gray 3-groupoid with a single object δ; $\mathbf{Aut}(\delta)$ was defined by Al-Asady in [1]. This structure is a Gray 3-group and consists of

1. $\mathbf{Aut}(\delta)_0 = \{\delta\}$ as a set of 0-cells,
2. $\mathbf{Aut}(\delta)_1$ is the chain automorphisms $F : (F_2, F_1, F_0) : \delta \Longrightarrow \delta$ where each F_i is a linear isomorphism from V_i to V_i,
3. $\mathbf{Aut}(\delta)_2$ is the group of all 1-homotopies (H,F) from F to G,
4. $\mathbf{Aut}(\delta)_3$ is the group of all 2-homotopies (α', H, F) from (H,F) to (K,F).

Thus, $\mathbf{Aut}(\delta)$ can be considered as a Gray 3-group. Any 3-cell in $\mathbf{Aut}(\delta)$ can be represented pictorially as



6.2. The linear representation defined. In section 5, we have established the equivalence between the categories of Gray 3-groups and 2-crossed modules. We have, from a 2-crossed module

$$\mathcal{L} := L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

a Gray 3-group;

$$\Theta(\mathcal{L}) := \mathcal{A}_* = \begin{cases} A_0 = \{*\} \ \text{ and } \ A_1 = N, \\ (A_2 \overset{s_2,t_2}{\underset{e_2}{\rightleftarrows}} A_1) \ = \ (M \rtimes N \overset{s,t}{\underset{e}{\rightleftarrows}} N)\,, \\ (A_3 \overset{s_3,t_3}{\underset{e_3}{\rightleftarrows}} A_2) \ = \ (L \rtimes M \rtimes N \overset{s,t}{\underset{e}{\rightleftarrows}} M \rtimes N) \end{cases}$$

and this may be thought of as a graded set with 4 non-empty levels, the lowest of which is a singleton and various graded maps. Thus, we may look for a linear representation of a 2-crossed module or its associated Gray 3-group as a 3-functor Φ into a *suitable 3-category* taking elements of N to 1-cells, the elements of $M \rtimes N$ to 2-cells and the elements of $L \rtimes M \rtimes N$ to 3-cells, so as to preserve the structures. This suitable 3-category is $\mathbf{Ch}^2_K$.

For the 0-cell $A_0 = \{*\}$, we can define as

$$\big(\ \Phi(*) = \delta\ \big) := \Big(\ V_2 \xrightarrow{\delta_2} V_1 \xrightarrow{\delta_1} V_0\ \Big)$$

where δ is a chain complex of length-2 over vector spaces.

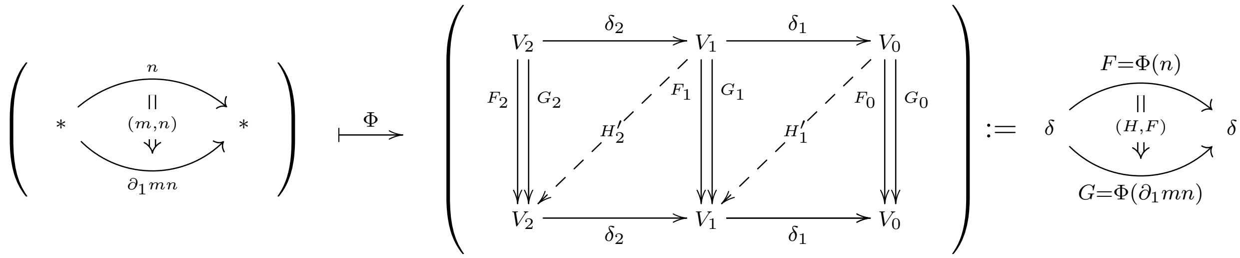
For any $n \in N$, as a 1-cell, we can define $\Phi(n) = F_i = (F_2, F_1, F_0)$ as a chain map from δ to δ. That is

$$\Big(\ * \xrightarrow{\quad n \quad} *\ \Big) \ \overset{\Phi}{\longmapsto} \ \left(\begin{array}{ccccc} V_2 & \xrightarrow{\delta_2} & V_1 & \xrightarrow{\delta_1} & V_0 \\ {\scriptstyle F_2}\downarrow & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\ V_2 & \xrightarrow[\delta_2]{} & V_1 & \xrightarrow[\delta_1]{} & V_0 \end{array} \right) := \ \delta \xrightarrow{\Phi(n)=F} \delta$$

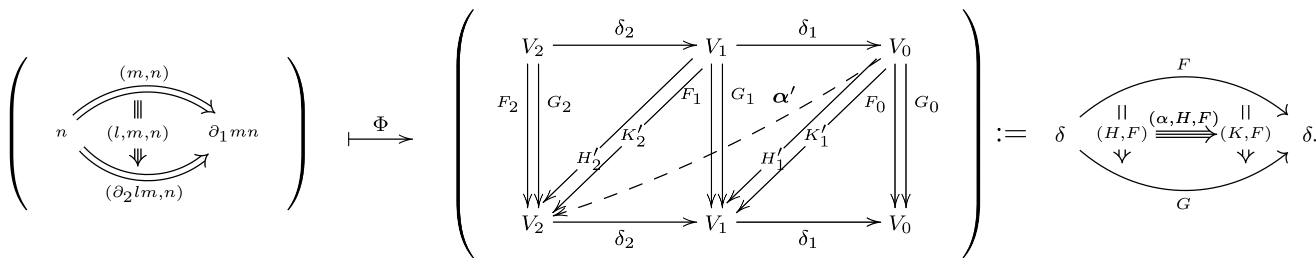
where F_i is a linear isomorphism of vector spaces for each i.

For any 2-cell $(m,n) : n \Longrightarrow \partial_1 mn$ in $M \rtimes N$, we can define $(\Phi(m,n) : \Phi(n) \Longrightarrow \Phi(\partial_1 mn)) := (F \Longrightarrow G)$ as a 1-homotopy in $\mathbf{Aut}(\delta)$.

We can represent it pictorially as



For any 3-cell $(l,m,n) : ((m,n) \Rrightarrow (\partial_2 lm, n) : n \Longrightarrow \partial_1 mn)$ in $L \rtimes M \rtimes N$, we can define $\Phi(l,m,n)$ as a 2-homotopy from $\Phi(m,n)$ to $\Phi(\partial_2 lm, n)$. We can picture it by



Since a 2-crossed module $\mathcal{L}$ itself is not a category, we should not expect to construct a direct definition of 2-crossed module representation functorially. But it was shown that a 2-crossed module can be thought as a Gray 3-group. Thus, an important criterion for a definition of a 2-crossed module representation is that it should be equivalent to a representation of the corresponding Gray 3-group $\mathcal{A}_*$ as defined above. Then a definition of a linear representation of the 2-crossed module $\mathcal{L}$ would be to first pass to the associated Gray 3-group $\Theta(\mathcal{L}) := \mathcal{A}_*$ as suggested above and find a representation, which will give as a mapping into the Gray 3-group $\mathbf{Aut}(\delta)$ for our choice of δ, and then we could then pass back to the associated 2-crossed module of $\mathbf{Aut}(\delta)$. Therefore, we can give the following result.

6.3. Proposition. *A linear representation of the 2-crossed module $\mathcal{L}$ or associated Gray 3-group $\mathcal{A}_*$ is a 3-functor*

$$\Phi : \mathcal{A}_* \longrightarrow \mathbf{Ch}^2_K$$

as defined above.

Therefore, the image of $\mathcal{A}_*$ lies in $\mathbf{Aut}(\delta)$, where δ is the chain complex of length-2.

References

[1] J. Al-asady. *Representations of Crossed Squares and Cat²-Groups.* University of Leicester, Dep. of Math. (Ph. D. Thesis), 2018.

[2] Z. Arvasi, T.S. Kuzpınarı and E.Ö. Uslu. Three-crossed modules. *Homology, Homotopy and Applications.* 11, 2, 161-187, 2009.

[3] Z. Arvasi and E. Ulualan. 3-types of simplicial groups and braided regular crossed modules. *Homology, Homotopy and Applications*. 9, 1, 139-161, 2007.

[4] J.C. Baez and M. Neuchl. Higher dimensional algebra I: Braided monodial 2-categories . *Adv. Math.* 121, 2, 196-244, 1996.

[5] M.F. Barker. *Representations of crossed Modules and cat*^{1}*-groups*. University of Wales, Bangor (Ph. D. Thesis) 2003.

[6] C. Berger. Double loop spaces, braided monoidal categories and algebraic 3-type of space. Higher homotopy structures in topology and mathematical Physics (Poughkeepsie, NY, 1996), 49-66, *Contemp. Math.* 227, Amer. Math. Soc., Providence, RI, 1999.

[7] D. Bourn. M´ethode n-cat´egorique d'interpr´etation des complexes et des extensions abeliennes de longueur n, Preprint, *Inst. Math. Louvain-la-Neuve*, Rapport no 45, 1982.

[8] D. Bourn. Produits tensoriels coh´erents de complexes de chaˆıne. *Bull. Soc. Math. Belg.* 41, 219–247, 1989.

[9] R. Brown. Possible connections between whiskered categories and groupoids, Leibniz algebras, automorphism structures and local-to-global questions. *Journal of Homotopy and Related Structures* 5, 1, 305-318, 2010.

[10] R. Brown and N.D. Gilbert. Algebraic models of 3-types and automorphism structures for crossed modules. *Proc. London Math. Soc.*, 3, 59, 51-73, 1989.

[11] R. Brown and P.J. Higgins. Tensor products and homotopies for ω-groupoids and crossed complexes. *Journal of Pure and Applied Algebra* 47, 1-33, 1987.

[12] M. Burrow. Representation theory of finite groups. *Academic Paperbacks*, 1965.

[13] P. Carrasco and A.M. Cegarra. Group-theoretic algebraic models for homotopy types. *Journal of Pure and Applied Algebra*, 75, 195-235, 1991.

[14] D. Conduché. Modules croisés généralisés de longueur 2. *Journal of Pure and Applied Algebra*, 34, 155-178, 1984.

[15] D. Conduché. Simplicial crossed modules and mapping cones. *Georgian Mathematical Journal*, 10, 4, 623-636, 2003.

[16] S.E. Crans. A tensor product for Gray categories. *Theory and Applications of Categories*, 5, 12-69, 1999.

[17] C.W. Curtis and I. Reiner. Representation theory of finite groups and associative algebras. Interscience, 1962.

[18] J. Elgueta. Representation theory of 2-groups on Kapranov and Voevodsky's 2-vector spaces. *Advances in Mathematics*, 213, 53-92, 2007.

[19] J.W. Gray. Formal category theory: Adjointness for 2-categories. *Lecture Notes in Math.* 391, Springer Berlin, 1974.

[20] K.A. Hardie, K.H. Kamps and R.W. Kieboom. A homotopy 2-groupoid of a Haussdorff Space. Papers in honour of Bernhard Banaschewski (Cape Town,1996). *Appl. Categ. Structures* 8 (2000), no. 1-2, 209-234. Representation Theory of 2-groups on Kapranov and Voevodsky's 2-vector spaces, *Advances in Mathematics*, 213, 53-92, 2007.

[21] A. Joyal and M. Tierney. Algebraic homotopy types, Handwritten lecture notes, 1984.

[22] K.H. Kamps and T. Porter. 2-groupoid enrichments in homotopy theory and algebra. *K-Theory*, 25, 4, 373-409, 2002.

[23] M. Kapranov and V. Voevodsky. 2-categories and Zamelodchikov tetrahedra equations. *Proc. Symp. Pure math.* 56, 177-260, 1994.

[24] J.-L. Loday. Spaces with finitely many non-trivial homotopy groups. *Journal of Pure and Applied Algebra*, 24, 2, 179-202, 1982.

[25] J.F. Martins and R. Picken. The fundamental Gray 3-groupoid of a smooth manifold and local 3-dimensional holonomy based on a 2-crossed module. *Differential Geometry and its Applications*, 29, 2, 179-206, 2011.

[26] A. Mutlu and T. Porter. Applications of Peiffer pairings in the Moore complexes of a simplicial group. *Theory and Applications of Categories*, 4, 7, 148-173, 1998.

[27] D.G. Walery and J.-L. Loday. Obstructions a l'excision en K-theorie algebrique. *Sipringer Lecture Notes in Math.*, 854, 179-216, 1981.

[28] W. Wang. On 3-gauge transformations, 3-curvatures, and Gray-categories. *J. Math. Phys.* 55, 4, 043506, 2014.

[29] J.H.C. Whitehead. Combinatorial homotopy II. *Bull. Amer. Math. Soc.*, 55, 453-496, 1949.

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