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FROM RAMSEY DEGREES TO RAMSEY EXPANSIONS VIA WEAK AMALGAMATION

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Abstract. In this paper we provide another argument to support the recently reinvigorated interest in treating Ramsey-type phenomena categorically. Using purely categorical strategies we construct a Ramsey expansion for every category of finite objects with finite small Ramsey degrees. Our construction is based on the relationship between small Ramsey degrees, weak amalgamation, and recent results about weak Fraïssé categories. Starting from the fact that weak Fraïssé categories allow for certain model-theoretic properties to be reflected in the free ω-cocompletion of the category, we show that classes with finite Ramsey degrees have weak amalgamation and then invoke the machinery of weak Fraïssé categories to perform the construction. This improves previous similar results where an analogous construction was carried out under the assumption that everything sits comfortably in a bigger class with enough infrastructure, and that in this wider context there is an ultrahomogeneous structure under whose umbrella the construction takes place.

1. Introduction

The intimate relationship between category theory and structural Ramsey theory has been evident from the very beginnings of structural Ramsey theory in early 1970's. It was Leeb who pointed out already in 1970 (see [16]) that the use of category theory can be quite helpful when trying to understand combinatorial phenomena that deal not with a single combinatorial object, but require analyzing entire classes of structures. The categorical approach to Ramsey theory was explicitly applied already in 1972 by Graham, Leeb and Rothschild in [7] where a proof of Rota's conjecture that finite vector spaces are Ramsey is given. In 1973 Leeb published a booklet [17] about categorical treatment of finite Ramsey phenomena, and in 1977 Nešetřil and Rödl published their celebrated result, the Nešetřil-Rödl theorem, using the language of category theory [23].

This all changed in the second half of 1980's when one can see a sudden shift away from category theory and towards the language of model theory. Perhaps the most striking

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example of the shift is Nešetřil and Rödl's paper [25] where they reprove the Nešetřil-Rödl theorem, but this time in the context of first-order structures.

Some quarter of a century later the interest in categorical treatment of Ramsey-related phenomena was reinvigorated when several new Ramsey-type results were published where the authors explicitly relied in their proofs not only on the choice of objects, but also on the choice of morphisms between them (see for example [20, 29, 30]). This feeling was formalized in a 2017 paper [19] where the authors showed that the Ramsey property is a genuine categorical property (in the sense that it is invariant under categorical equivalence).

Treating Ramsey-type phenomena categorically requires a significant change in the point of view, but also brings many benefits such as systematic treatment of dual Ramsey phenomena, results that are not limited to relational languages, and the use of abstract constructions available in category theory which are often not easy to mimic in the context of first-order structures.

Ramsey property is closely related to another fundamental structural property – amalgamation. On the one hand, amalgamation lies at the heart of the *partite method*, one of the most powerful tools for proving the Ramsey property for a class of finite structures (see, e.g. the proof of the famous Nešetřil-Rödl Theorem [23, 25]). On the other hand, it is an easy but fundamental result of Nešetřil and Rödl from 1977 that every class of structures with the Ramsey property has the amalgamation property [23].

Many natural classes of finite structures with the amalgamation property (such as finite graphs and finite partially ordered sets) do not enjoy the Ramsey property. It is quite common, though, that the structures can be expanded by a few carefully chosen relations so that the resulting class of expanded structures has the Ramsey property. We then say that the class of structures has a *Ramsey expansion*.

In the late 1990s it was observed that many concrete classes of finite structures where a Ramsey expansion had been identified also enjoyed a weaker property of having *finite Ramsey degrees* [4, 5, 6]. For Fraïssé classes, that this is not a mere coincidence was proven in one direction in [9], who prove that classes with a Ramsey expansion have finite Ramsey degrees, and in the other by the second author in [30], showing that small Ramsey degrees suffice for the existence of a Ramsey expansion. A more combinatorial proof of the same fact can be found in [27], and a reinterpretation of the latter proof in the language of category theory in [18]. All these proofs make key use of the fact that the classes are Fraïssé, in the sense that everything sits comfortably in a bigger class with enough infrastructure, and that in this wider context there is an ultrahomogeneous structure under whose umbrella the construction takes place.

In this paper we show that the assumption about the ambient class in which an ultrahomogeneous object oversees the construction is unnecessary. Imposing no additional assumptions, for each category of finite objects with finite small Ramsey degrees (the definitions are given below) we construct a Ramsey expansion. The expansion is not constructed from scratch, though. Section 3 starts with our first insight, where we show (Theorem 3.2) that classes with finite Ramsey degrees have the weak amalgamation property. This is an analogue of Nešetřil and Rödl's result [23] that the Ramsey property implies amalgamation. In the remainder of the section we recall from [15, 14] how an ambient category and a weakly homogeneous object in it can be constructed from a category with weak amalgamation by taking the free ω-cocompletion of the original category.

In Section 4 we then upgrade the results from [18] to show that if everything sits in a bigger category in which there is a weakly homogeneous and locally finite object universal for the category we are trying to expand, then there is a convenient expansion which can be trimmed down to a Ramsey expansion. Finally, in Section 5 we put all the ingredients together to prove the main result of the paper. As a corollary, we specialize the main result of the paper to arbitrary first-order structures, and then give a dual result about the relationship of small dual Ramsey degrees and dual Ramsey expansions.

2. Preliminaries

Let us quickly fix some notation and conventions. All the categories in this paper are locally small. Let $\mathbf{C}$ be a category. By $\mathrm{Ob}(\mathbf{C})$ we denote the class of all the objects in $\mathbf{C}$. Hom-sets in $\mathbf{C}$ will be denoted by $\hom_{\mathbf{C}}(A, B)$, or simply $\hom(A, B)$ when $\mathbf{C}$ is clear from the context. The identity morphism will be denoted by id_A and the composition of morphisms by $\cdot$ (dot). If $\hom_{\mathbf{C}}(A, B) \neq \varnothing$ we write $A \xrightarrow{\mathbf{C}} B$. Recall that a morphism is *mono* if it is left cancellable, and *epi* if it is right cancellable. Let $\mathrm{iso}_{\mathbf{C}}(A, B)$ denote the set of all the invertible morphisms from $\hom_{\mathbf{C}}(A, B)$, and let $\mathrm{Aut}_{\mathbf{C}}(A) = \mathrm{iso}_{\mathbf{C}}(A, A)$ denote the set of all the *automorphisms of* A. We write $A \cong B$ to denote that $\mathrm{iso}_{\mathbf{C}}(A, B) \neq \varnothing$. A *skeleton* of a category $\mathbf{C}$ is a full subcategory $\mathbf{S}$ of $\mathbf{C}$ such that no two objects in $\mathbf{S}$ are isomorphic and for every $C \in \mathrm{Ob}(\mathbf{C})$ there is an $S \in \mathrm{Ob}(\mathbf{S})$ such that $C \cong S$. As usual, $\mathbf{C}^{\mathrm{op}}$ denotes the opposite category. Whenever the category $\mathbf{C}$ is fixed we shall simply write $\hom(A, B)$, $\mathrm{Aut}(C)$, $t(A)$, etc.

A category $\mathbf{C}$ is *directed* if for all $A, B \in \mathrm{Ob}(\mathbf{C})$ there is a $C \in \mathrm{Ob}(\mathbf{C})$ such that $A \xrightarrow{\mathbf{C}} C$ and $B \xrightarrow{\mathbf{C}} C$. A subcategory $\mathbf{D}$ of $\mathbf{C}$ is *cofinal in* $\mathbf{C}$ if for every $C \in \mathrm{Ob}(\mathbf{C})$ there is a $D \in \mathrm{Ob}(\mathbf{D})$ such that $C \xrightarrow{\mathbf{C}} D$. Directedness and cofinality as we have introduced them here refer to the underlying preorder $\xrightarrow{\mathbf{C}}$ of the category $\mathbf{C}$, which differs from other uses of these two notions that can be found in the literature (see for example [1]).

An *ω-chain* in $\mathbf{C}$ is a functor $F : \omega \to \mathbf{C}$ where ω denotes the poset category $0 < 1 < 2 < \dots$. A category $\mathbf{D}$ is *ω-cocomplete* if every ω-chain in $\mathbf{D}$ has a colimit in $\mathbf{D}$. An *ω-cocompletion* of a category $\mathbf{C}$ is an ω-cocomplete category $\mathbf{D}$ together with an embedding $E : \mathbf{C} \to \mathbf{D}$. An ω-cocompletion $E : \mathbf{C} \to \mathbf{D}$ of a category $\mathbf{C}$ is *free* if for every other ω-cocompletion $E' : \mathbf{C} \to \mathbf{D}'$ of $\mathbf{C}$ there is a unique (up to natural isomorphism) ω-colimit preserving functor $H : \mathbf{D} \to \mathbf{D}'$ such that $H \circ E$ is naturally isomorphic to E'.

For $k \in \mathbb{N}$, a k-coloring of a set S is any mapping $\chi : S \to k$, where, as usual, we identify k with $\{0, 1, \dots, k-1\}$. For positive integers $k, t \in \mathbb{N}$ and objects $A, B, C \in \mathrm{Ob}(\mathbf{C})$ such that $A \xrightarrow{\mathbf{C}} B$ we write $C \longrightarrow (B)^A_{k,t}$ to denote that for every k-coloring $\chi : \hom(A, C) \to k$ there is a morphism $w \in \hom(B, C)$ such that $|\chi(w \cdot \hom(A, B))| \leqslant t$.

(For a set of morphisms F we let $w \cdot F = \{w \cdot f : f \in F\}$.) In case $t = 1$ we write $C \longrightarrow (B)^A_k$.

A category $\mathbf{C}$ has the *Ramsey property* if for every integer $k \in \mathbb{N}$ and all $A, B \in \mathrm{Ob}(\mathbf{C})$ there is a $C \in \mathrm{Ob}(\mathbf{C})$ such that $C \longrightarrow (B)^A_k$.

For $A \in \mathrm{Ob}(\mathbf{C})$ let $t_{\mathbf{C}}(A)$, the *Ramsey degree of A in* $\mathbf{C}$ (sometimes called the *small* Ramsey degree) denotes the least positive integer n such that for all $k \in \mathbb{N}$ and all $B \in \mathrm{Ob}(\mathbf{C})$ there exists a $C \in \mathrm{Ob}(\mathbf{C})$ such that $C \longrightarrow (B)^A_{k,n}$, if such an integer exists. Otherwise put $t_{\mathbf{C}}(A) = \infty$. A category $\mathbf{C}$ has *finite small Ramsey degrees* if $t_{\mathbf{C}}(A) < \infty$ for all $A \in \mathrm{Ob}(\mathbf{C})$.

2.1. Example. The Ramsey property imposes severe restrictions on the class of finite structures under consideration. A class $\mathbf{K}$ of finite structures with the Ramsey property for embeddings has amalgamation [23] and consists of rigid objects [21]. Therefore, the class of all finite graphs does not have the Ramsey property (finite graphs are in general not rigid), but the class of all finite ordered graphs does [23]. Analogously, the class of all finite partial orders does not have the Ramsey property, but the class of finite structures consisting of a finite partial order and a linear extension of that partial order does. The fact that such a natural class of finite objects such as graphs does not enjoy such a relevant property such as the Ramsey property is somewhat annoying, so the introduction of the concept of small Ramsey degrees [4, 5, 6] rectified this injustice: the class of all finite graphs has finite small Ramsey degrees. In fact, it is an open question [3] if any Fraïssé class in a finite relational language has finite small Ramsey degrees.

As in [18] we shall be working in the following setup. Let $\mathbf{C}$ be a locally small category and let $\mathbf{C}_{\mathit{fin}}$ be a full subcategory of $\mathbf{C}$ such that the following holds:

(C1) all the morphisms in $\mathbf{C}$ are mono;

(C2) $\mathrm{Ob}(\mathbf{C}_{\mathit{fin}})$ is a set;

(C3) for all $A, B \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}})$ the set $\hom(A, B)$ is finite;

(C4) for every $F \in \mathrm{Ob}(\mathbf{C})$ there is an $A \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}})$ such that $A \to F$; and

(C5) for every $B \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}})$ the set $\{A \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}}) : A \to B\}$ is finite.

We think of objects in $\mathbf{C}_{\mathit{fin}}$ as templates of finite objects in $\mathbf{C}$. For the remainder of the section let us fix a locally small category $\mathbf{C}$ and its full subcategory $\mathbf{C}_{\mathit{fin}}$ which satisfies (C1)–(C5).

Let $\mathbf{A}$ be a full subcategory of $\mathbf{C}$. An object $F \in \mathrm{Ob}(\mathbf{C})$ is *ultrahomogeneous for* $\mathbf{A}$ if for every $A \in \mathrm{Ob}(\mathbf{A})$ and every pair of morphisms $e_1, e_2 \in \hom(A, F)$ there is a $g \in \mathrm{Aut}(F)$ such that $g \cdot e_1 = e_2$:

$$\begin{tikzcd}
F \arrow[rr, "g"] & & F \\
& A \arrow[ul, "e_1"'] \arrow[ur, "e_2"] &
\end{tikzcd}$$

An object $F \in \mathrm{Ob}(\mathbf{C})$ is *ultrahomogeneous in* $\mathbf{C}$ *(with respect to* $\mathbf{C}_{\mathit{fin}}$*)* if it is ultrahomogeneous for $\mathbf{C}_{\mathit{fin}}$.

Recall that an object F is universal for $\mathbf{A}$ if $A \to F$ for all $A \in \mathrm{Ob}(\mathbf{A})$. We shall say that F is *universal in* $\mathbf{C}$ *(with respect to* $\mathbf{C}_{\mathit{fin}}$*)* if it is universal for $\mathbf{C}_{\mathit{fin}}$. Let us define the *age of* F *in* $\mathbf{C}$ *with respect to* $\mathbf{C}_{\mathit{fin}}$ as

$$\mathrm{Age}_{(\mathbf{C},\mathbf{C}_{\mathit{fin}})}(F) = \{A \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}}) : A \xrightarrow{\mathbf{C}} F\}.$$

Clearly, every F is universal for its age. Whenever $\mathbf{C}$ and $\mathbf{C}_{\mathit{fin}}$ are fixed we shall simply write $\mathrm{Age}(F)$.

Let $\mathbf{D}$ be a full subcategory of a locally small category $\mathbf{C}$. An $F \in \mathrm{Ob}(\mathbf{C})$ is *locally finite for* $\mathbf{D}$ if:

- for every $A, B \in \mathrm{Ob}(\mathbf{D})$ and every $e \in \hom(A, F)$, $f \in \hom(B, F)$ there exist $D \in \mathrm{Ob}(\mathbf{D})$, $r \in \hom(D, F)$, $p \in \hom(A, D)$ and $q \in \hom(B, D)$ such that $r \cdot p = e$ and $r \cdot q = f$:

$$\begin{tikzcd}
D \arrow[rr, "r"] & & F & \\
& A \arrow[ul, "p"'] \arrow[ur, "e"'] & & B \arrow[ulll, "q"] \arrow[ul, "f"']
\end{tikzcd}$$

- and for every $H \in \mathrm{Ob}(\mathbf{C})$, $r' \in \hom(H, F)$, $p' \in \hom(A, H)$ and $q' \in \hom(B, H)$ such that $r' \cdot p' = e$ and $r' \cdot q' = f$ there is an $s \in \hom(D, H)$ such that the diagram below commutes

$$\begin{tikzcd}
D \arrow[rr, "r"] \arrow[rrrr, bend left, dotted, "s"] & & F & & H \arrow[ll, "r'"'] \\
& A \arrow[ul, "p"'] \arrow[ur, "e"] \arrow[urrr, "p'"'] & & B \arrow[ulll, "q"] \arrow[ul, "f"] \arrow[ur, "q'"'] &
\end{tikzcd}$$

An $F \in \mathrm{Ob}(\mathbf{C})$ is *locally finite in* $\mathbf{C}$ *(with respect to* $\mathbf{C}_{\mathit{fin}}$*)* if it is locally finite for $\mathbf{C}_{\mathit{fin}}$.

2.2. Example. The notion of local finiteness as presented here captures the usual model-theoretic notion: a first-order structure is locally finite if every finitely generated substructure is finite. All relational structures are trivially locally finite. This can change in the presence of functional symbols in the language. For example, the additive group of the integers $(\mathbb{Z}, +)$ is not locally finite since every 1-generated subgroup is infinite.

An *expansion* of a category $\mathbf{C}$ is a category $\mathbf{C}^*$ together with a faithful functor $U : \mathbf{C}^* \to \mathbf{C}$ which is surjective on objects. We shall generally follow the convention that $A, B, C, \ldots$ denote objects from $\mathbf{C}$ while $A^*, B^*, C^*, \ldots$ denote objects from $\mathbf{C}^*$. Since U is injective on hom-sets we may safely assume that $\hom_{\mathbf{C}^*}(A^*, B^*) \subseteq \hom_{\mathbf{C}}(A, B)$ where $A = U(A^*)$, $B = U(B^*)$. In particular, $\mathrm{id}^*_A = \mathrm{id}_A$ for $A = U(A^*)$. Moreover, it is safe to drop subscripts $\mathbf{C}$ and $\mathbf{C}^*$ in $\hom_{\mathbf{C}}(A, B)$ and $\hom_{\mathbf{C}^*}(A^*, B^*)$, so we shall simply write $\hom(A, B)$ and $\hom(A^*, B^*)$, respectively. Let $U^{-1}(A) = \{A^* \in \mathrm{Ob}(\mathbf{C}^*) : U(A^*) = A\}$. Note that this is not necessarily a set. Therefore, we say that an expansion $U : \mathbf{C}^* \to \mathbf{C}$ is *precompact* (cf. [26]) if $U^{-1}(A)$ is a set for all $A \in \mathrm{Ob}(\mathbf{C})$, and it is a finite set for all $A \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}})$.

An expansion $U : \mathbf{C}^* \to \mathbf{C}$ is *reasonable* (cf. [9]) if for every $e \in \hom(A, B)$ and every $A^* \in U^{-1}(A)$ there is a $B^* \in U^{-1}(B)$ such that $e \in \hom(A^*, B^*)$:

$$\begin{tikzcd}
A^* \arrow[r, "e"] \arrow[d, dashed, mapsto, "U"'] & B^* \arrow[d, dashed, mapsto, "U"] \\
A \arrow[r, "e"'] & B
\end{tikzcd}$$

An expansion $U : \mathbf{C}^* \to \mathbf{C}$ has *unique restrictions* if for every $B^* \in \mathrm{Ob}(\mathbf{C}^*)$ and every $e \in \hom(A, U(B^*))$ there is a *unique* $A^* \in U^{-1}(A)$ such that $e \in \hom(A^*, B^*)$:

$$\begin{tikzcd}
B^*\restriction_e = A^* \arrow[r, "e"] \arrow[d, dashed, mapsto, "U"'] & B^* \arrow[d, dashed, mapsto, "U"] \\
A \arrow[r, "e"'] & B
\end{tikzcd}$$

We denote this unique A^* by $B^*\restriction_e$ and refer to it as the *restriction of B^* along e.*

2.3. Example. Let $\mathbf{C}$ be the category of finite and countably infinite graphs with embeddings as morphisms, let $\mathbf{C}^*$ be the category of finite and countably infinite linearly ordered graphs with embeddings as morphisms, and let $U : \mathbf{C}^* \to \mathbf{C}$ be the functor that forgets the order. Then this is a precompact reasonable expansion with unique restrictions.

Let $\mathbf{A}$ and $\mathbf{A}^*$ be categories and $U : \mathbf{A}^* \to \mathbf{A}$ and expansion. Following [26] we say that $U : \mathbf{A}^* \to \mathbf{A}$ *has the expansion property* if for every $A \in \mathrm{Ob}(\mathbf{A})$ there exists a $B \in \mathrm{Ob}(\mathbf{A})$ such that $A^* \to B^*$ for all $A^*, B^* \in \mathrm{Ob}(\mathbf{A}^*)$ with $U(A^*) = A$ and $U(B^*) = B$. If $\mathbf{A}^*$ is directed and all the morphisms in $\mathbf{A}$ are mono, and if $U : \mathbf{A}^* \to \mathbf{A}$ is a reasonable expansion with unique restrictions such that $U^{-1}(A)$ is finite for all $A \in \mathrm{Ob}(\mathbf{A})$ then $U : \mathbf{A}^* \to \mathbf{A}$ has the expansion property if and only if for every $D^* \in \mathrm{Ob}(\mathbf{A}^*)$ there is a $B \in \mathrm{Ob}(\mathbf{A})$ such that for all $B^* \in \mathrm{Ob}(\mathbf{A}^*)$ with $U(B^*) = B$ we have $D^* \to B^*$ [18].

2.4. Example. The expansion property is a generalization of the *ordering property* introduced in [22, 24]. In many cases expanding the structures from a class with no Ramsey property with appropriately chosen linear orders results in a class of expanded structures with the Ramsey property. But there are notable exceptions: in [11] the authors prove that no expansion of the class of finite distributive lattices by linear orders satisfies the Ramsey property. However, an expansion of the class of all finite distributive lattices by particular ternary relations will result in a class with the Ramsey property.

Let F be a locally finite object. For $A \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}})$ such that $A \to F$ and $e_1, e_2 \in \hom(A, F)$ let $N_F(e_1, e_2) = \{f \in \mathrm{Aut}(F) : f \cdot e_1 = e_2\}$. Then

$$M_F = \{N_F(e_1, e_2) : A \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}}),\ A \xrightarrow{\mathbf{C}} F \text{ and } e_1, e_2 \in \hom(A, F)\}.$$

is a base of a topology τ_F on $\mathrm{Aut}(F)$ [18].

Let $U : \mathbf{C}^* \to \mathbf{C}$ be a reasonable precompact expansion with unique restrictions and let $F \in \mathrm{Ob}(\mathbf{C})$. For $A \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}})$, $e \in \hom(A, F)$ and $A^* \in U^{-1}(A)$ let

$$N(e, A^*) = \{F^* \in U^{-1}(F) : e \in \hom(A^*, F^*)\}.$$

Then $\mathcal{S}_F = \{N(e, A^*) : U(A^*) \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}}), e \in \hom(U(A^*), F)\}$ is a base of clopen sets of a topology σ_F on $U^{-1}(F)$ [18].

Each reasonable expansion with unique restrictions yields an action of $\mathrm{Aut}(F)$ on $U^{-1}(F)$ for every $F \in \mathrm{Ob}(\mathbf{C})$: for $F \in \mathrm{Ob}(\mathbf{C})$, $g \in \mathrm{Aut}(F)$ and $F^* \in U^{-1}(F)$ let $F^* \cdot g$ denote the unique element of $U^{-1}(F)$ satisfying $g \in \hom(F^* \cdot g, F^*)$. (See [18] for details.) This action is continuous with respect to topologies τ_F on $\mathrm{Aut}(F)$ and σ_F on $U^{-1}(F)$ and will be referred to as *logical*.

That every Fraïssé class with finite Ramsey degrees has a Ramsey expansion was first established in 2016 in [30]. A combinatorial proof of the same fact was then given in [27], and this was put into the context of category theory in [18]. The categorical version takes the following form:

2.5. Theorem. *[30, 27, 18] Let $\mathbf{C}$ be a locally small category and let $\mathbf{C}_{\mathit{fin}}$ be a full subcategory of $\mathbf{C}$ such that (C1) – (C5) hold. Let $F \in \mathrm{Ob}(\mathbf{C})$ be an ultrahomogeneous locally finite object, and let $\mathbf{A}$ be the full subcategory of $\mathbf{C}_{\mathit{fin}}$ spanned by* $\mathrm{Age}(F)$. *Then the following are equivalent:*

(1) $\mathbf{A}$ has finite Ramsey degrees.

(2) There is a reasonable precompact expansion with unique restrictions $U : \mathbf{C}^* \to \mathbf{C}$ and a full subcategory $\mathbf{A}^*$ of $\mathbf{C}^*$ which is directed, has the Ramsey property and $U\restriction_{\mathbf{A}^*} : \mathbf{A}^* \to \mathbf{A}$ has the expansion property.

3. Weak amalgamation and weak Fraïssé categories

The weak amalgamation property was introduced independently by Ivanov [8] and Kechris and Rosendal [10] to describe the behaviour of generic automorphisms of first-order structures. In this setting we let $\mathbf{C}$ be a locally small category whose morphisms are mono. We say that $\mathbf{C}$ has the *weak amalgamation property* [8, 10, 15] if for any $A \in \mathrm{Ob}(\mathbf{C})$ there is $A' \in \mathrm{Ob}(\mathbf{C})$ and a morphism $f \in \hom(A, A')$ so that whenever we are given $B, C \in \mathrm{Ob}(\mathbf{C})$ and morphisms $g \in \hom(A', B)$ and $h \in \hom(A', C)$ there are $D \in \mathrm{Ob}(\mathbf{C})$ and morphisms $r \in \hom(B, D)$ and $s \in \hom(C, D)$ so that $r \cdot g \cdot f = s \cdot h \cdot f$:

$$\begin{tikzcd}
& A' \arrow[r, "g"] & B \arrow[dr, "r"] & \\
A \arrow[ur, "f"] \arrow[dr, "f"'] & & & D \\
& A' \arrow[r, "h"'] & C \arrow[ur, "s"'] &
\end{tikzcd}$$

When f is as above we call such an f an *amalgamation arrow for A*.

3.1. Example. Clearly, every class with the amalgamation property has the weak amalgamation property, but the converse is not true [12]. Here is an example from [12]. Let $\mathbf{G}$ be the class of all finite acyclic undirected graphs in which no two vertices of degree greater than 2 are adjacent. Then $\mathbf{G}$ fails the amalgamation property but has the weak amalgamation property.

3.2. Theorem. *Let* $\mathbf{C}$ *be a locally small directed category with finite Ramsey degrees. Then* $\mathbf{C}$ *has the weak amalgamation property.*

Proof. The following weakening of the amalgamation property will be useful during the course of the proof. Let $\mathbf{C}$ be a locally small category whose morphisms are mono and fix $k < \omega$. We say that $A \in \mathrm{Ob}(\mathbf{C})$ has *2-out-of-k-amalgamation* if for any $B_0, \dots, B_{k-1} \in \mathrm{Ob}(\mathbf{C})$ and morphisms $g_i \in \hom(A, B_i)$ there are $C \in \mathrm{Ob}(\mathbf{C})$, $i \neq j < k$ and morphisms $r \in \hom(B_i, C)$ and $s \in \hom(B_j, C)$ with $r \cdot g_i = s \cdot g_j$:

$$\begin{tikzcd}
& & & C & & & \\
B_0 & \cdots & B_i \arrow[ur, "r"] & \cdots & B_j \arrow[ul, "s"'] & \cdots & B_{k-1} \\
& & & A \arrow[ulll, "g_0"] \arrow[ul, "g_i"] \arrow[ur, "g_j"'] \arrow[urrr, "g_{k-1}"'] & & &
\end{tikzcd}$$

The proof now proceeds in two steps.

Claim 1. Assume that $t_{\mathbf{C}}(A) = k - 1$ for some $A \in \mathrm{Ob}(\mathbf{C})$. Then A has 2-out-of-k-amalgamation.

Proof. Fix morphisms $f_i \in \hom(A, B_i)$ with $B_i \in \mathrm{Ob}(\mathbf{C})$ for each $i < k$. Then use the fact that $\mathbf{C}$ is directed to find some $C \in \mathrm{Ob}(\mathbf{C})$ and morphisms $g_i \in \hom(B_i, C)$ for each $i < k$. Next find $D \in \mathrm{Ob}(\mathbf{C})$ satisfying

$$D \longrightarrow (C)^A_{k,k-1}.$$

Consider the coloring $\chi\colon \hom(A, D) \to k$ where $\chi(h) = i < k-1$ if i is least so that there is $g \in \hom(B_i, D)$ with $h = g \cdot f_i$, or set $\chi(h) = k-1$ if there is no such $i < k-1$. Then there is an $x \in \hom(C, D)$ such that $|\chi(x \cdot \hom(A, C))| \leqslant k - 1$. In particular, there is some color $j < k$ which is avoided, that is, $j \notin \chi(x \cdot \hom(A, C))$. Then consider the value of $\chi(x \cdot g_j \cdot f_j) = i \neq j$. So there is $g \in \hom(B_i, D)$ with $g \cdot f_i = x \cdot g_j \cdot f_j$, showing that f_i and f_j can be amalgamated. This concludes the proof of Claim 1.

Claim 2. Suppose for every $A \in \mathrm{Ob}(\mathbf{C})$ there is some $k \in \mathbb{N}$ so that A has 2-out-of-k-amalgamation. Then $\mathbf{C}$ has the weak amalgamation property.

Proof. Suppose $\mathbf{C}$ failed to have the weak amalgamation property as witnessed by $A \in \mathrm{Ob}(\mathbf{C})$. Then the identity map id_A is not an amalgamation arrow. Set $C_0 = A$ and $\mathrm{id}_A = f_0 \in \hom(A, C_0)$. Now suppose $f_i \in \hom(A, C_i)$ has been defined. Then f_i is not an amalgamation arrow, so we may find $B_{i+1}, C_{i+1} \in \mathrm{Ob}(\mathbf{C})$ and morphisms

$g_i \in \hom(C_i, B_{i+1})$ and $h_i \in \hom(C_i, C_{i+1})$ so that $g_i \cdot f_i$ and $h_i \cdot f_i$ cannot be amalgamated. Then set $f_{i+1} = h_i \cdot f_i$. We can continue this as long as we would like, obtaining arrows $\{g_i \cdot f_i : i < \omega\}$ no pair of which can be amalgamated. Therefore, A does not have 2-out-of-k-amalgamation. This concludes the proof of Claim 2 and proof of the theorem. ■

3.3. Example. Let $\mathbf{K}$ be the class of finite graphs G with the property that there is no injective homomorphism $C_4 \to G$, where, as usual, C_4 denotes the cycle on four vertices. In [28] the authors show that the class K does not have the weak amalgamation property (with respect to embeddings). Therefore, by the above theorem, the class $\mathbf{K}$ does not have finite small Ramsey degrees.

3.4. Example. Another example can be found in [13]. This time let $\mathbf{K}$ be the class of finite graphs in which different cycles of the same length are disjoint. Then $\mathbf{K}$ fails the weak amalgamation property and hence does not have finite small Ramsey degrees.

The remainder of this section is devoted to the exposition of key notions of weak Fraïssé theory [15] which is one of the building blocks of our construction. Let us stress that in [15] the author considers the free ω-cocompletion of a category and demonstrates how certain model-theoretic properties of the original category reflect on some special objects in the cocompletion.

A subcategory $\mathbf{D}$ of $\mathbf{C}$ is *weakly dominating in* $\mathbf{C}$ [15] if it is cofinal in $\mathbf{C}$ and for every $D \in \mathrm{Ob}(\mathbf{D})$ there exist a $D' \in \mathrm{Ob}(\mathbf{D})$ and a $j \in \hom_{\mathbf{D}}(D, D')$ such that for every $C \in \mathrm{Ob}(\mathbf{C})$ and every $f \in \hom_{\mathbf{C}}(D', C)$ there is an $E \in \mathrm{Ob}(\mathbf{D})$ and a morphism $g \in \hom_{\mathbf{C}}(C, E)$ such that $g \cdot f \cdot j$ is a morphism in $\mathbf{D}$.

$$\begin{tikzcd}
 & C \arrow[dr, "g"] & & \mathbf{C} \\
D \arrow[r, "j"] \arrow[rr, bend right, "g \cdot f \cdot j"'] & D' \arrow[u, "f"] & E & \mathbf{D}
\end{tikzcd}$$

A category $\mathbf{C}$ is a *weak Fraïssé category* [15] if it is directed, has the weak amalgamation property and is weakly dominated by a countable subcategory.

A sequence $W = (W_n, w_n^m)_{n \leqslant m \in \omega}$ is a *weak Fraïssé sequence* [15] if the following is satisfied:

- for every $C \in \mathrm{Ob}(\mathbf{C})$ there is an $n \in \omega$ such that $C \xrightarrow{\mathbf{C}} W_n$; and
- for every $n \in \omega$ there exists an $m \geqslant n$ such that for every $f \in \hom_{\mathbf{C}}(W_m, C)$ there are $k \geqslant m$ and $g \in \hom_{\mathbf{C}}(C, W_k)$ satisfying $g \cdot f \cdot w_n^m = w_n^k$.

$$\begin{tikzcd}
\cdots \arrow[r] & W_n \arrow[r, "w_n^m"] \arrow[dr, "w_n^m"'] & W_m \arrow[rr, "w_m^k"] & & W_k \arrow[r] & \cdots \\
& & W_m \arrow[r, "f"'] & C \arrow[ur, "g"'] & &
\end{tikzcd}$$

If follows immediately that every category with a weak Fraïssé sequence is directed and has the weak amalgamation property [15].

A particularly important class of weak Fraïssé categories is the category of chains formed from a category with a weak Fraïssé limit [15, 14, 2]. Let $\omega = \{0, 1, 2, \ldots\}$ denote the chain of nonnegative integers treated here as a poset category (that is, for all $n, m \in \omega$ such that $n \leqslant m$ there is a single morphism $n \to m$). A *sequence in a category* $\mathbf{C}$ is any functor $X : \omega \to \mathbf{C}$. We shall find it more convenient to describe functors $X : \omega \to \mathbf{C}$ as $(X_n, x_n^m)_{n \leqslant m \in \omega}$ where $X_n = X(n) \in \mathrm{Ob}(\mathbf{C})$ and $x_n^m \in \hom_{\mathbf{C}}(X_n, X_m)$ is the image under X of the only morphism $n \to m$. Then, clearly, $x_n^n = \mathrm{id}_{X_n}$ and $x_m^k \cdot x_n^m = x_n^k$ whenever $n \leqslant m \leqslant k$.

The next step is to define morphisms between sequences in such a way that a morphism from a sequence X to a sequence Y induces a morphism from the colimit of X into the colimit of Y whenever the category of sequences is embedded into a category in which sequences X and Y have colimits. This is a two-phase process: the idea is to start with the category of sequences and transformations $\sigma_1\mathbf{C}$ and then factor this category by a congruence to arrive at the category $\sigma_0\mathbf{C}$ of sequences and morphisms between them. Hence, a morphism of sequences is an equivalence class of transformations between them.

Let $(X_n, x_n^m)_{n \leqslant m \in \omega}$ and $(Y_n, y_n^m)_{n \leqslant m \in \omega}$ be sequences in $\mathbf{C}$. A *transformation* from $(X_n, x_n^m)_{n \leqslant m \in \omega}$ to $(Y_n, y_n^m)_{n \leqslant m \in \omega}$ is a pair (F, φ) where $\varphi : \omega \to \omega$ is a functor such that $\varphi(\omega)$ is cofinal in ω and $F : X \to Y \circ \varphi$ is a natural transformation. In other words, φ is a nondecreasing cofinal map $\omega \to \omega$ (that is, $i \leqslant j \Rightarrow \varphi(i) \leqslant \varphi(j)$ and for every n there is an m such that $n \leqslant \varphi(m)$) and there is a family of arrows $F_n : X_n \to Y_{\varphi(n)}$, $n \in \omega$, such that

$$\begin{tikzcd}
X_n \arrow[r, "x_n^m"] \arrow[d, "F_n"'] & X_m \arrow[d, "F_m"] \\
Y_{\varphi(n)} \arrow[r, "y_{\varphi(n)}^{\varphi(m)}"'] & Y_{\varphi(m)}
\end{tikzcd} \quad \text{for all } n \leqslant m \in \omega.$$

All sequences in $\mathbf{C}$ and transformations between them form a category that we denote by $\sigma_1\mathbf{C}$.

Two transformations $(F, \varphi), (G, \psi) : (X_n, x_n^m)_{n \leqslant m \in \omega} \to (Y_n, y_n^m)_{n \leqslant m \in \omega}$ are *equivalent*, in symbols $(F, \varphi) \approx (G, \psi)$, if for every $n \in \omega$ there exists an $m \geqslant \max\{\varphi(n), \psi(n)\}$ such that

$$\begin{tikzcd}
X_n \arrow[d, "F_n"'] \arrow[dr, "G_n"] & & \\
Y_{\varphi(n)} \arrow[rr, bend right, "y^m_{\varphi(n)}"'] & Y_{\psi(n)} \arrow[r, "y^m_{\psi(n)}"] & Y_m
\end{tikzcd}$$

It is easy to check that $\approx$ is a congruence of $\sigma_1\mathbf{C}$, so let $\sigma_0\mathbf{C} = \sigma_1\mathbf{C}/{\approx}$ be the factor category. Clearly, each morphism is an equivalence class of transformations. Just as a quick demonstration of the interaction of all these notions let us show the following

3.5. Lemma. *Let* $\mathbf{C}$ *be a locally small category whose morphisms are mono. Then all the morphisms in* $\sigma_0\mathbf{C}$ *are mono.*

Proof. Let $X = (X_n, x_n^m)_{n \leqslant m \in \omega}$, $Y = (Y_n, y_n^m)_{n \leqslant m \in \omega}$ and $Z = (Z_n, z_n^m)_{n \leqslant m \in \omega}$ be sequences in $\mathbf{C}$, and let $(F, \varphi) : Y \to Z$ and $(G, \psi), (H, \theta) : X \to Y$ be transformations such that $(F, \varphi) \cdot (G, \psi) \approx (F, \varphi) \cdot (H, \theta)$. We are going to show that $(G, \psi) \approx (H, \theta)$.

Note that $(F, \varphi) \cdot (G, \psi) \approx (F, \varphi) \cdot (H, \theta)$ means that for every $n \in \omega$ there is an $m \geqslant \max\{\varphi(\psi(n)), \varphi(\theta(n))\}$ such that

$$z^m_{\varphi(\psi(n))} \cdot F_{\psi(n)} \cdot G_n = z^m_{\varphi(\theta(n))} \cdot F_{\theta(n)} \cdot H_n \tag{1}$$

$$\begin{tikzcd}
X_n \arrow[d, "G_n"'] \arrow[dr, "H_n"] & & \\
Y_{\psi(n)} \arrow[d, "F_{\psi(n)}"'] & Y_{\theta(n)} \arrow[d, "F_{\theta(n)}"] & \\
Z_{\varphi(\psi(n))} \arrow[rr, bend right] & Z_{\varphi(\theta(n))} \arrow[r] & Z_m
\end{tikzcd}$$

Since φ, ψ and θ are nondecreasing and cofinal, $\varphi \circ \theta$ and $\varphi \circ \psi$ are also nondecreasing and cofinal, so there is a $k \in \omega$ such that $\theta(k) > \theta(n)$ and $\varphi(\theta(k)) > m$, and an $s \in \omega$ such that $\psi(s) > \theta(k)$ and $\varphi(\psi(s)) > \varphi(\theta(k))$:

$$\begin{tikzcd}
X_n \arrow[d, "G_n"'] \arrow[dr, "H_n"] & & & & \\
Y_{\psi(n)} \arrow[d, "F_{\psi(n)}"'] \arrow[rrrr, bend left] & Y_{\theta(n)} \arrow[d, "F_{\theta(n)}"] \arrow[rr] & & Y_{\theta(k)} \arrow[d, "F_{\theta(k)}"] \arrow[r] & Y_{\psi(s)} \arrow[d, "F_{\psi(s)}"] \\
Z_{\varphi(\psi(n))} \arrow[rr, bend right] \arrow[rrrr, bend right] & Z_{\varphi(\theta(n))} \arrow[r] & Z_m \arrow[r] & Z_{\varphi(\theta(k))} \arrow[r] & Z_{\varphi(\psi(s))}
\end{tikzcd}$$

The rest of the proof reduces to straightforward calculation. Multiplying (1) by $z_m^{\varphi(\psi(s))}$ from the left we get

$$z_m^{\varphi(\psi(s))} \cdot z^m_{\varphi(\psi(n))} \cdot F_{\psi(n)} \cdot G_n = z_m^{\varphi(\psi(s))} \cdot z^m_{\varphi(\theta(n))} \cdot F_{\theta(n)} \cdot H_n$$

that is

$$z^{\varphi(\psi(s))}_{\varphi(\psi(n))} \cdot F_{\psi(n)} \cdot G_n = z^{\varphi(\psi(s))}_{\varphi(\theta(n))} \cdot F_{\theta(n)} \cdot H_n.$$

Using the fact that $F : Y \to Z$ is a transformation as the next step we have

$$F_{\psi(s)} \cdot y^{\psi(s)}_{\psi(n)} \cdot G_n = F_{\psi(s)} \cdot y^{\psi(s)}_{\theta(n)} \cdot H_n.$$

Finally, $F_{\psi(s)}$ is mono as a morphism in $\mathbf{C}$, whence

$$y^{\psi(s)}_{\psi(n)} \cdot G_n = y^{\psi(s)}_{\theta(n)} \cdot H_n.$$

This shows that $(G, \psi) \approx (H, \theta)$. ■

The category $\mathbf{C}$ embeds fully into $\sigma_0\mathbf{C}$ as follows. For $A \in \mathrm{Ob}(\mathbf{C})$ let $\overline{A} = (A, \mathrm{id}_A)_{n \leqslant m \in \omega}$ denote the constant sequence such that $A_n = A$ for all $n \in \omega$ and $a_n^m = \mathrm{id}_A$ for all $n \leqslant m \in \omega$. Every morphism $f \in \hom_{\mathbf{C}}(A, B)$ gives rise to a unique transformation $\overline{f} = (\mathrm{Const}_f, \mathrm{id}_\omega)$ where $(\mathrm{Const}_f)_n = f$, $n \in \omega$. It is easy to check that $J : \mathbf{C} \to \sigma_0\mathbf{C} : A \mapsto \overline{A} : f \mapsto \overline{f}/{\approx}$ is indeed a full functor injective on objects, and hence an embedding [14].

It is a bit technical but easy to show (see [14]) that $\sigma_0\mathbf{C}$ is a cocompletion of $\mathbf{C}$: for every sequence $X = (X_n, x_n^m)_{n \leqslant m \in \omega}$ in $\mathbf{C}$ we have simply added a formal colimit to $\sigma_0\mathbf{C}$ and adjusted the morphisms so that X is the colimit of $J(X)$ in $\sigma_0\mathbf{C}$. More precisely, the following holds in $\sigma_0\mathbf{C}$ (see [14] for details):

$$X = \mathrm{colim}(\overline{X_0} \xrightarrow{\overline{x}_0^1} \overline{X_1} \xrightarrow{\overline{x}_1^2} \cdots)$$

Clearly, every element in $\mathrm{Ob}(\sigma_0\mathbf{C})$ is a colimit of a sequence in $\mathbf{C}$.

It comes as no surprise that weak Fraïssé sequences should demonstrate a certain level of homogeneity in $\sigma_0\mathbf{C}$ with respect to objects from $J(\mathbf{C})$. In the setting of weak Fraïssé theory the corresponding notion is referred to as weak homogeneity.

Let $\mathbf{C}$ be a locally small category whose morphisms are mono, let $\mathbf{D}$ be a full subcategory of $\mathbf{C}$ and let $S \in \mathrm{Ob}(\mathbf{C})$. We say that S is *weakly homogeneous for* $\mathbf{D}$ [15] if for every $A \in \mathrm{Ob}(\mathbf{D})$ and every $f \in \hom(A, S)$ there is a $B \in \mathrm{Ob}(\mathbf{D})$, an $e \in \hom(A, B)$ and an $i \in \hom(B, S)$ such that

(W1) $f = i \cdot e$, and

(W2) for every $j \in \hom(B, S)$ there is an $h \in \mathrm{Aut}(S)$ such that $i \cdot e = h \cdot j \cdot e$:

$$\begin{tikzcd}
 & B \arrow[r, "i"] & S \\
A \arrow[ur, "e"] \arrow[dr, "e"'] \arrow[urr, bend left, "f"] & & \\
 & B \arrow[r, "j"'] & S \arrow[uu, "h"']
\end{tikzcd}$$

3.6. Theorem. *[15] Let* $\mathbf{C}$ *be a category.*

(a) $\mathbf{C}$ *is a weak Fraïssé category if and only if there is a weak Fraïssé sequence in* $\mathbf{C}$.

(b) *A category may have, up to isomorphism, at most one weak Fraïssé sequence.*

(c) *If* $\mathbf{C}$ *is a weak Fraïssé category and* W *a weak Fraïssé sequence in* $\mathbf{C}$ *then* W *as an object of* $\sigma_0\mathbf{C}$ *is weakly homogeneous for* $\mathbf{C}$.

In case $\mathbf{C}$ is a class of finite first-order structures of the same first-order signature Θ, we can think of $\mathbf{C}$ as a category where embeddings serve as morphisms, and in this particular case we can take $\sigma_0\mathbf{C}$ to be the class of all structures isomorphic to the unions of countable chains in $\mathbf{C}$. Recall that the morphisms between sequences in $\sigma_0\mathbf{C}$ are defined in a rather convoluted manner to ensure that in the context of first-order structures a morphism from a sequence X to a sequence Y correspond uniquely to an embedding of the colimit of X into the colimit of Y.

4. Weak homogeneity and precompact expansions

The purpose of this section is to prove a generalization of Theorem 2.5 where ultrahomogeneity is replaced by weak homogeneity. Interestingly, the proof as we have presented it in [18] remains largely the same, so we cover only the differences here. We strongly suggest the reader to have a copy of the proof given in [18] at hand while reading this section.

4.1. Lemma. *Let* $U : \mathbf{C}^* \to \mathbf{C}$ *be a reasonable expansion with unique restrictions. Let* F *be a locally finite object of* $\mathbf{C}$ *which is weakly homogeneous for its age and let* $G = \mathrm{Aut}(F)$. *Then for* $F^*, F_1^* \in U^{-1}(F)$ *we have* $F_1^* \in \overline{F^* \cdot G}$ *(where the closure is computed in* σ_F*) if and only if* $\mathrm{Age}(F_1^*) \subseteq \mathrm{Age}(F^*)$.

Proof. $(\Rightarrow)$ The same as the proof of direction $(\Rightarrow)$ in [18, Lemma 5.11].

$(\Leftarrow)$ Assume that $\mathrm{Age}(F_1^*) \subseteq \mathrm{Age}(F^*)$ and let us show that every neighborhood of F_1^* intersects $F^* \cdot G$. Let $N(e, A^*)$ be a neighborhood of F_1^*. Then $e \in \hom_{\mathbf{C}^*}(A^*, F_1^*)$ and $e \in \hom_{\mathbf{C}}(A, F)$. Our intention now is to show that $e \in \hom_{\mathbf{C}^*}(A^*, F^* \cdot g)$ for some $g \in G$.

Since F is weakly homogeneous there exists a $B \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}})$ and morphisms $h \in \hom_{\mathbf{C}}(A, B)$ and $i \in \hom_{\mathbf{C}}(B, F)$ such that $e = i \cdot h$. Let $B_1^* = F_1^*\restriction_i$. Then $B_1^* \in \mathrm{Age}(F_1^*) \subseteq \mathrm{Age}(F^*)$, so there exists a morphism $f \in \hom(B_1^*, F^*)$. Since F is weakly homogeneous, there is a $g \in \mathrm{Aut}(F)$ such that $g \cdot f \cdot h = i \cdot h = e$:

$$\begin{tikzcd}
 & B \arrow[r, "i"] & F \\
A \arrow[ur, "h"] \arrow[dr, "h"'] \arrow[urr, bend left, "e"] & & \\
 & B \arrow[r, "f"'] & F \arrow[uu, "g"']
\end{tikzcd}$$

But then (see [18, Lemma 5.1 (c)]):

$$\begin{tikzcd}
A^* \arrow[r, "h"'] \arrow[rrr, bend left, "e"] \arrow[d, dashed, maps to, "U"'] & B_1^* \arrow[r, "f"'] \arrow[d, dashed, maps to, "U"'] & F^* \arrow[r, "g"'] \arrow[d, dashed, maps to, "U"'] & F^* \cdot g^{-1} \arrow[d, dashed, maps to, "U"] \\
A \arrow[r, "h"] \arrow[rrr, bend right, "e"'] & B \arrow[r, "f"] & F \arrow[r, "g"] & F
\end{tikzcd}$$

whence follows that $F^* \cdot g^{-1} \in N(e, A^*)$. ∎

4.2. Proposition. *Let* $U : \mathbf{C}^* \to \mathbf{C}$ *be a reasonable precompact expansion with unique restrictions. Let* F *be a locally finite weakly homogeneous object in* $\mathbf{C}$ *and assume that* $U^{-1}(F)$ *is compact with respect to the topology* σ_F*. Let* $G = \mathrm{Aut}(F)$ *and let* $F^* \in U^{-1}(F)$ *be arbitrary. Then* $U{\restriction}_{\mathrm{Age}(F^*)} : \mathrm{Age}(F^*) \to \mathrm{Age}(F)$ *has the expansion property if and only if* $\mathrm{Age}(F^*) = \mathrm{Age}(F_1^*)$ *for all* $F_1^* \in \overline{F^* \cdot G}$*.*

Proof. (⇒) The same as the proof of direction (⇒) in [18, Lemma 5.12].

(⇐) Assume that $\mathrm{Age}(F^*) \subseteq \mathrm{Age}(F_1^*)$ for all $F_1^* \in \overline{F^* \cdot G}$. Let $A^* \in \mathrm{Age}(F^*)$ be arbitrary and let $A = U(A^*)$. For $e \in \hom(A, F)$ let

$$X_e = \overline{F^* \cdot G} \cap N(e, A^*).$$

Let us show that

$$\overline{F^* \cdot G} = \bigcup \{X_e : e \in \hom(A, F)\}.$$

The inclusion $\supseteq$ is trivial, while the inclusion $\subseteq$ follows from the assumption. Namely, if $F_1^* \in \overline{F^* \cdot G}$ then $\mathrm{Age}(F^*) \subseteq \mathrm{Age}(F_1^*)$; so $A^* \in \mathrm{Age}(F_1^*)$, or, equivalently, there is a morphism $f \in \hom(A^*, F_1^*)$, whence $F_1^* \in X_f$.

By construction each X_e is open in $\overline{F^* \cdot G}$. Since $\overline{F^* \cdot G}$ is compact (as a closed subspace of the compact space $U^{-1}(F)$), there is a finite sequence $e_0, \ldots, e_{k-1} \in \hom(A, F)$ such that

$$\overline{F^* \cdot G} = \bigcup \{X_{e_j} : j < k\}.$$

Since F is locally finite, there exist $B \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}})$ and morphisms $r \in \hom(B, F)$ and $p_i \in \hom(A, B)$ such that $r \cdot p_i = e_i$, $i < k$. Moreover, F is weakly homogeneous so for $r \in \hom(B, F)$ there exist $C \in \mathrm{Ob}(\mathbf{C}_{\mathit{fin}})$ and morphisms $h \in \hom(B, C)$ and $i \in \hom(C, F)$ such that (W1), that is $i \cdot h = r$, and (W2) are satisfied. Let us show that for every $C^* \in U^{-1}(C)$ we have that $A^* \xrightarrow{\mathbf{C}^*} C^*$.

Take any $C^* \in \mathrm{Age}(F^*)$ such that $U(C^*) = C$ and let $s \in \hom(C^*, F^*)$ be any morphism. Then, $s \in \hom(C, F)$, so by (W2) there is a $g \in G$ such that $g \cdot s \cdot h = i \cdot h = r$. Furthermore, let $B^* = C^*{\restriction}_h$. Since $g \in \hom(F^*, F^* \cdot g^{-1})$ we have that

$$\begin{tikzcd}
F^* \cdot g^{-1} \arrow[d, mapsto, dashed, "U"'] & B^* \arrow[l, "g \cdot s \cdot h"] \arrow[r, "h"] \arrow[d, mapsto, dashed, "U"'] & C^* \arrow[r, "s"'] \arrow[d, mapsto, dashed, "U"'] & F^* \arrow[lll, bend right, "g"'] \arrow[d, mapsto, dashed, "U"] \\
F & B \arrow[l, "r"'] \arrow[r, "h"] & C \arrow[r, "s"] & F \arrow[lll, bend left, "g"]
\end{tikzcd}$$

In particular, $r = g \cdot s \cdot h \in \hom(B^*, F^* \cdot g^{-1})$, so $B^* \in \mathrm{Age}(F^* \cdot g^{-1})$. Now, $F^* \cdot g^{-1} \in \overline{F^* \cdot G} = \bigcup\{X_{e_j} : j < k\}$, so $F^* \cdot g^{-1} \in X_{e_i}$ for some i. Moreover, $r \cdot p_i = e_i$ by the construction of B. Therefore:

$$\begin{tikzcd}
A^* \arrow[r, "e_i"] \arrow[d, mapsto, dashed, "U"'] & F^* \cdot g^{-1} \arrow[d, mapsto, dashed, "U"'] & B^* \arrow[l, "r"'] \arrow[d, mapsto, dashed, "U"] \\
A \arrow[r, "e_i"] \arrow[rr, bend right, "p_i"'] & F & B \arrow[l, "r"']
\end{tikzcd}$$

Let $A_1^* = B^*\restriction_{p_i}$. Since $B^* = F^* \cdot g^{-1}\restriction_r$ we have $A_1^* = (F^* \cdot g^{-1}\restriction_r)\restriction_{p_i} = F^* \cdot g^{-1}\restriction_{r \cdot p_i} = F^* \cdot g^{-1}\restriction_{e_i} = A^*$. Consequently, $p_i \in \hom(A^*, B^*)$ which, together with $h \in \hom(B^*, C^*)$ concludes the proof that $A^* \overset{\mathbf{C}^*}{\longrightarrow} C^*$. ∎

Putting it all together and having in mind parts of the proof from [18] that do not depend on F being homogeneous we finally get the following:

4.3. Theorem. *Let* $\mathbf{C}$ *be a locally small category and let* $\mathbf{C}_{\mathit{fin}}$ *be a full subcategory of* $\mathbf{C}$ *such that (C1) – (C5) hold. Let* $F \in \mathrm{Ob}(\mathbf{C})$ *be a weakly homogeneous locally finite object, and let* $\mathbf{A}$ *be the full subcategory of* $\mathbf{C}_{\mathit{fin}}$ *spanned by* $\mathrm{Age}(F)$. *Then the following are equivalent:*

(1) $\mathbf{A}$ *has finite Ramsey degrees.*

(2) There is a reasonable precompact expansion with unique restrictions $U : \mathbf{C}^* \to \mathbf{C}$ *and a full subcategory* $\mathbf{A}^*$ *of* $\mathbf{C}^*$ *which is directed, has the Ramsey property and* $U\restriction_{\mathbf{A}^*} : \mathbf{A}^* \to \mathbf{A}$ *has the expansion property.*

For reader's convenience we conclude the section with the quick recapitulation of the construction of $\mathbf{C}^*$ and $\mathbf{A}^*$. The category $\mathbf{C}^*$ is constructed from $\mathbf{C}$ by adding structure to objects of $\mathbf{C}$. The language convenient for the efficient description of the additional structure is that of essential colorings which capture the small Ramsey degree of an object by identifying the "unavoidable coloring" (see [30] and also [27]). Given a locally finite $F \in \mathrm{Ob}(\mathbf{C})$, a coloring $\lambda : \hom(A, B) \to t$, $t \geqslant 2$, is *essential at* B if for every coloring $\chi : \hom(A, F) \to k$ there is a $w \in \hom(B, F)$ such that $\ker \lambda \subseteq \ker \chi^{(w)}$, where $\chi^{(w)}(f) = \chi(w \cdot f)$. A coloring $\gamma : \hom(A, F) \to t$, $t \geqslant 2$, is *essential* if for every $B \in \mathrm{Ob}(\mathbf{A})$ such that $A \overset{\mathbf{C}}{\longrightarrow} B$ and every $w \in \hom(B, F)$ the coloring $\gamma^{(w)} : \hom(A, B) \to t$ is essential at B. The key observation now is that for every $A \in \mathrm{Ob}(\mathbf{A})$ there is an essential coloring

$\gamma_A : \hom(A, F) \to t_{\mathbf{A}}(A)$ (see [30, 27] for the original statement and [18] for the proof of the result in the categorical setting).

Let $\mathbf{C}^*$ be the category whose objects are pairs $C_\theta = (C, \theta)$ where $C \in \mathrm{Ob}(\mathbf{C})$ and $\theta = (\theta_A)_{A \in \mathrm{Ob}(\mathbf{A})}$ is a family of colorings

$$\theta_A : \hom(A, C) \to t_{\mathbf{A}}(A)$$

indexed by the objects of $\mathbf{A}$. Morphisms in $\mathbf{C}^*$ are morphisms from $\mathbf{C}$ that preserve colorings. More precisely, f is a morphism from $C_\theta = (C, \theta)$ to $D_\delta = (D, \delta)$ in $\mathbf{C}^*$ if $f \in \hom(C, D)$ and

$$\delta(f \cdot e) = \theta(e), \text{ for all } e \in \bigcup_{A \in \mathrm{Ob}(\mathbf{A})} \hom(A, C).$$

Let $U : \mathbf{C}^* \to \mathbf{C}$ be the obvious forgetful functor $(C, \theta) \mapsto C$ and $f \mapsto f$. This is a reasonable precompact expansion with unique restrictions [18].

Let $\gamma = (\gamma_A)_{A \in \mathrm{Ob}(\mathbf{A})}$, and let $F_\gamma = (F, \gamma) \in \mathrm{Ob}(\mathbf{C}^*)$ be an arbitrary expansion of F (the weakly homogeneous locally finite object from the proof). As we have seen, expansions with unique restrictions induce group actions. Moreover, the action of $G = \mathrm{Aut}(F)$ on $U^{-1}(F)$ is continuous [18]. Since $U^{-1}(F)$ is compact [18] there is an $F^* = (F, \varphi^*) \in \overline{F_\gamma \cdot G}$ such that $\overline{F^* \cdot G}$ is minimal with respect to inclusion. We then let $\mathbf{A}^* = \mathrm{Age}(F^*)$.

5. The Main Result

We have now set up all the infrastructure necessary for the main result of the paper. We shall say that $\mathbf{C}$ is a *category of finite objects* if $\mathbf{C}$ has a skeleton $\mathbf{S}$ with the following properties:

- $\mathbf{S}$ is a countable category,
- $\hom_{\mathbf{S}}(A, B)$ is finite for all $A, B \in \mathbf{S}$,
- for every $B \in \mathrm{Ob}(\mathbf{S})$ the set $\{A \in \mathrm{Ob}(\mathbf{S}) : A \xrightarrow{\mathbf{S}} B\}$ is finite, and
- every $A \in \mathrm{Ob}(\mathbf{S})$ is locally finite for $\mathbf{S}$.

Whenever $\mathbf{C}$ is a category of finite structures, we take $\mathbf{C}_{\mathit{fin}}$ to be whole of $\mathbf{C}$. Consequently, a precompact expansion $U : \mathbf{C}^* \to \mathbf{C}$ has the property that $U^{-1}(A)$ is finite for all $A \in \mathrm{Ob}(\mathbf{C})$.

5.1. Theorem. *Let* $\mathbf{C}$ *be a directed category of finite objects whose morphisms are mono. Then* $\mathbf{C}$ *has finite Ramsey degrees if and only if there exists a category* $\mathbf{C}^*$ *with the Ramsey property and a reasonable precompact expansion with unique restrictions* $U : \mathbf{C}^* \to \mathbf{C}$ *with the expansion property. (Such an expansion is usually referred to as a* Ramsey expansion.*)*

Proof. ($\Leftarrow$) See [18].

($\Rightarrow$) Since $\mathbf{C}$ is a category of finite objects it has a skeleton $\mathbf{S}$ with the properties listed above.

Step 1. Let us first show that there exists a category $\mathbf{S}^*$ with the Ramsey property and a reasonable expansion with unique restrictions $V : \mathbf{S}^* \to \mathbf{S}$ with the expansion property.

The category $\mathbf{S}$ is clearly directed, so the fact that it has finite Ramsey degrees ensures by Theorem 3.2 that it also has the weak amalgamation property. Since $\mathbf{S}$ is trivially weakly dominated by itself, it follows that $\mathbf{S}$ is a weak amalgamation category and it has a weak Fraïssé sequence $W = (W_n, w_n^m)_{n \leqslant m \in \omega}$ (Theorem 3.6 (a)). Let $\mathbf{D} = \sigma_0 \mathbf{S}$ and let $\mathbf{D}_{\mathit{fin}} = J(\mathbf{S})$. Let us show that (C1) – (C5) are satisfied.

(C1) Morphisms in $\mathbf{D}$ are mono by Lemma 3.5

(C2) $\mathrm{Ob}(\mathbf{D}_{\mathit{fin}}) = \mathrm{Ob}(J(\mathbf{S}))$ is a set because $\mathrm{Ob}(\mathbf{S})$ is a set by the assumption (actually, a countable one).

(C3) and (C5) follow from the fact that $\mathbf{S}$ is a skeleton of a category of finite objects.

(C4) take any $X = (X_n, x_n^m)_{n \leqslant m \in \omega} \in \mathrm{Ob}(\mathbf{D})$ and note that there is a morphism $\overline{X_0} \to X$ given by

$$\begin{tikzcd}
X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "x_0^1"'] & X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "x_0^2"'] & \cdots \\
X_0 \arrow[r, "x_0^1"'] & X_1 \arrow[r, "x_1^2"'] & X_2 \arrow[r, "x_2^3"'] & \cdots
\end{tikzcd}$$

It is easy to see that W as an object of $\mathbf{D}$ is universal for $J(\mathbf{S})$. Moreover, W is weakly homogeneous for $J(\mathbf{S})$ by Theorem 3.6 (c). Let us show that W is locally finite for $J(\mathbf{S})$.

Take any $A, B \in \mathrm{Ob}(\mathbf{S})$ and arbitrary morphisms $(E, \varepsilon)/{\approx} \in \hom_{\mathbf{D}}(\overline{A}, W)$, $(F, \varphi)/{\approx} \in \hom_{\mathbf{D}}(\overline{B}, W)$. Without loss of generality we may assume that the transformations (E, ε) and (F, φ) are chosen so that $\varepsilon(0) = \varphi(0)$. Let $k = \varepsilon(0) = \varphi(0)$. Then the fact that every object in $\mathbf{S}$ is locally finite for $\mathbf{S}$ implies that there is a $D \in \mathrm{Ob}(\mathbf{S})$ and morphisms $p \in \hom_{\mathbf{S}}(A, D)$, $q \in \hom_{\mathbf{S}}(B, D)$ and $r \in \hom_{\mathbf{S}}(D, W_k)$ such that

$$\begin{tikzcd}
D \arrow[rr, "r"] & & W_k & \\
& A \arrow[ul, "p"] \arrow[ur, "E_0"'] & & B \arrow[lllu, "q"] \arrow[ul, "F_0"']
\end{tikzcd}$$

and D is the "smallest" such object in the sense that for every other contender there is a morphism from D into the contender such that everything commutes.

Let (G, γ) be the transformation $\overline{D} \to W$ there $\gamma : \omega \to \omega : n \mapsto n + k$ and $G_n \in \hom_{\mathbf{S}}(D, W_{n+k})$ is given by $G_n = w_k^{k+n} \cdot r$:

$$\begin{tikzcd}
 & & D \arrow[r, "\mathrm{id}"] \arrow[d, "G_0=r"'] & D \arrow[r, "\mathrm{id}"] \arrow[d, "G_1"'] & D \arrow[r, "\mathrm{id}"] \arrow[d, "G_2"'] & \cdots \\
W_0 \arrow[r] & \cdots \arrow[r] & W_k \arrow[r, "w_k^{k+1}"'] & W_{k+1} \arrow[r, "w_{k+1}^{k+2}"'] & W_{k+2} \arrow[r, "w_{k+2}^{k+3}"'] & \cdots
\end{tikzcd}$$

Let us show that $(G,\gamma)\cdot\overline{p}\approx(E,\varepsilon)$. Take any $n\in\omega$ and let $m=1+\max\{k+n,\varepsilon(n)\}$. Note that $E_n=w_k^{\varepsilon(n)}\cdot E_0$ because (E,ε) is a transformation $\overline{A}\to W$ (recall that $k=\varepsilon(0)$):

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "E_0"'] & A \arrow[d, "E_n"] \\
W_k \arrow[r, "w_k^{\varepsilon(n)}"'] & W_{\varepsilon(n)}
\end{tikzcd}$$

Therefore, $w^m_{\varepsilon(n)}\cdot E_n=w^m_{\varepsilon(n)}\cdot w_k^{\varepsilon(n)}\cdot E_0=w^m_k\cdot E_0=w^m_{k+n}\cdot w_k^{k+n}\cdot r\cdot p=w^m_{k+n}\cdot G_n\cdot p.$

$$\begin{tikzcd}
& A \arrow[ddl, bend right, "E_0"'] \arrow[d, "p"'] \arrow[ddr, "E_n"] & & \\
& D \arrow[dl, "r"'] \arrow[d, "G_n"] & & \\
W_k \arrow[r, "w_k^{k+n}"'] & W_{k+n} \arrow[rr, bend right, "w^m_{k+n}"'] & W_{\varepsilon(n)} \arrow[r, "w^m_{\varepsilon(n)}"] & W_m
\end{tikzcd}$$

The proof of $(G,\gamma)\cdot\overline{q}\approx(F,\varphi)$ is analogous.

To complete the proof that W is locally finite for $J(\mathbf{S})$ assume now that there is a contender $\overline{C}\in J(\mathbf{S})$ with morphisms $\overline{u}/{\approx}:\overline{A}\to\overline{C}$, $\overline{v}/{\approx}:\overline{B}\to\overline{C}$ and $(H,\theta)/{\approx}:\overline{C}\to W$, where $u\in\hom_{\mathbf{S}}(A,C)$ and $v\in\hom_{\mathbf{S}}(B,C)$. Without loss of generality we may assume that the transformation (H,θ) was chosen so that $\theta(0)=k$. Then we have the following configuration in $\mathbf{S}$:

$$\begin{tikzcd}
D \arrow[rr, "r"] & & W_k & & C \arrow[ll, "H_0"'] \\
& A \arrow[ul, "p"] \arrow[ur, "E_0"] \arrow[urrr, "u"] & & B \arrow[ulll, "q"] \arrow[ul, "F_0"'] \arrow[ur, "v"'] &
\end{tikzcd}$$

By the choice of D there is a morphism $s\in\hom_{\mathbf{S}}(D,C)$ such that

$$\begin{tikzcd}
D \arrow[rr, "r"] \arrow[rrrr, bend left, "s"] & & W_k & & C \arrow[ll, "H_0"'] \\
& A \arrow[ul, "p"] \arrow[ur, "E_0"] \arrow[urrr, "u"] & & B \arrow[ulll, "q"] \arrow[ul, "F_0"'] \arrow[ur, "v"'] &
\end{tikzcd}$$

Note that everything still commutes when this diagram is transposed to $\mathbf{D}$. The proof is straightforward. For example, $(H,\theta)\cdot\overline{s}\approx(G,\gamma)$ follows from

$$\begin{tikzcd}
D \arrow[r, "\mathrm{id}"] \arrow[d, "s"] \arrow[dd, bend right, "G_0=r"'] & D \arrow[d, "s"'] \arrow[dd, bend left, "G_n"] \\
C \arrow[r, "\mathrm{id}"] \arrow[d, "H_0"] & C \arrow[d, "H_n"'] \\
W_k \arrow[r, "w_k^{k+n}"'] & W_{k+n}
\end{tikzcd}$$

Therefore, $\mathbf{D}$ is a locally small category and, $\mathbf{D}_{\mathit{fin}} = J(\mathbf{S}) \cong \mathbf{S}$ is a full subcategory of $\mathbf{D}$ and (C1) – (C5) hold. Moreover, W is a weakly homogeneous locally finite object and $\mathrm{Age}(F)$ is the whole of $\mathbf{D}_{\mathit{fin}} \cong \mathbf{S}$. Since $\mathbf{S}$ has finite Ramsey degrees, Theorem 4.3 ensures that there is a reasonable precompact expansion with unique restrictions $V : \mathbf{D}^* \to \mathbf{D}$ and a full subcategory $\mathbf{S}^*$ of $\mathbf{D}^*$ which is directed, has the Ramsey property and $V\restriction_{\mathbf{S}^*} : \mathbf{S}^* \to \mathbf{S}$ has the expansion property.

Step 2. Since $\mathbf{S}$ is a skeleton of $\mathbf{C}$ we can now easily expand $\mathbf{S}^*$ to $\mathbf{C}^*$ and $V : \mathbf{S}^* \to \mathbf{S}$ to $U : \mathbf{C}^* \to \mathbf{C}$ as follows. Let $F : \mathbf{C} \to \mathbf{S}$ be a functor that takes $C \in \mathrm{Ob}(\mathbf{C})$ to the unique $S \in \mathrm{Ob}(\mathbf{S})$ such that $C \cong S$. Next, for each $C \in \mathrm{Ob}(\mathbf{C})$ fix an isomorphism $\eta_C : C \to F(C)$ and define F on morphisms so that F takes $f : C \to D$ to $F(f) : F(C) \to F(D)$ where $F(f) = \eta_D \cdot f \cdot \eta_C^{-1}$. This turns η into a natural transformation $\mathrm{ID} \to F$. Now take $\mathbf{C}^*$ to be the pullback of $\mathbf{C} \xrightarrow{F} \mathbf{S} \xleftarrow{V} \mathbf{S}^*$ in the quasicategory of all locally small categories and functors between them, and let $U : \mathbf{C}^* \to \mathbf{C}$ be the corresponding functor in the pullback diagram:

$$\begin{tikzcd}
\mathbf{C}^* \arrow[r, "F^*"] \arrow[d, "U"'] & \mathbf{S}^* \arrow[d, "V"] \\
\mathbf{C} \arrow[r, "F"'] & \mathbf{S}
\end{tikzcd}$$

It is easy to see that $\mathbf{S}^*$ is a skeleton of $\mathbf{C}^*$ and that $C^* \cong F^*(C^*)$ for every $C^* \in \mathrm{Ob}(\mathbf{C}^*)$. Therefore, $\mathbf{C}^*$ has the Ramsey property and $U : \mathbf{C}^* \to \mathbf{C}$ is a reasonable expansion with unique restrictions and with the expansion property. ∎

The above result easily specializes to classes of first-order structures. Let Θ be a first-order signature (that is, a set of functional, relational and constant symbols) and let $\mathbf{K}$ be a class of finite Θ-structures. Then $\mathbf{K}$ can be treated as a category with embeddings as morphisms. So, when we stipulate that a class of first-order structures has certain properties we have introduced in the context of categories (Ramsey property, finite Ramsey degrees, …) we have this interpretation in mind. Historically, structural Ramsey phenomena we consider in this paper were first identified in classes of first-order structures and were later transposed to the context of abstract categories. For historical reasons we shall, therefore, say that a class of first-order structures $\mathbf{K}$ has the *joint embedding property* if for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ there is a $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{A} \hookrightarrow \mathcal{C} \hookleftarrow \mathcal{B}$. Clearly, $\mathbf{K}$ has the joint embedding property if and only if $\mathbf{K}$ is directed as a category.

Let $\Theta^* \supseteq \Theta$ be a first-order signature that contains Θ and let $\mathbf{K}^*$ be a class of Θ^*-structures. Let $\mathbf{K}$ be a class of Θ-structures and let $U : \mathbf{K}^* \to \mathbf{K}$ be the forgetful functor that forgets the extra structure from $\Theta^* \setminus \Theta$ so that U takes a structure from $\mathbf{K}^*$ to its Θ-reduct (and takes homomorphisms to themselves). We then say that $\mathbf{K}^*$ is an *expansion of* $\mathbf{K}$, and that it is a *reasonable expansion with unique restrictions and with the expansion property* if $U : \mathbf{K}^* \to \mathbf{K}$ is.

5.2. Corollary. *Let Θ be a first-order signature and let $\mathbf{K}$ be a class of finite Θ-structures such that there are at most countably many pairwise nonisomorphic structures in $\mathbf{K}$ and $\mathbf{K}$ has the joint-embedding property. Then: $\mathbf{K}$ has finite Ramsey degrees if and only if there exists a first-order signature $\Theta^* \supseteq \Theta$ and a class $\mathbf{K}^*$ of Θ^*-structures such that $\mathbf{K}^*$ has the Ramsey property and $\mathbf{K}^*$ is a reasonable precompact expansion of $\mathbf{K}$ with unique restrictions and with the expansion property.*

Proof. ($\Leftarrow$): Immediate from Theorem 5.1.

($\Rightarrow$): To show that this direction also follows from Theorem 5.1 we have to show that the expansion constructed in Theorem 5.1 can be performed in the realm of first-order structures.

Let $\mathcal{A}_m$, $m \in \mathbb{N}$, be an enumeration of all representatives of isomorphism types in $\mathbf{K}$ where $\mathcal{A}_m$ is a Θ-structure whose underlying set is $A_m = \{a_{m,1}, \dots, a_{m,n_m}\}$, $n_m = |A_m|$. Clearly (with a slight abuse of terminology), $\mathbf{A} = \{\mathcal{A}_m : m \in \mathbb{N}\}$ is the skeleton of $\mathbf{K}$.

The first step in the proof of Theorem 5.1 begins with the construction of the ambient category $\mathbf{D} = \sigma_0\mathbf{S}$ in which weakly homogeneous locally finite universal objects dwell. Since $\mathbf{A}$ is a set of finite Θ-structures in the context of first-order structures we can take $\mathbf{D} = \sigma_0\mathbf{A}$ to be the class of all structures isomorphic to the unions of countable chains in $\mathbf{A}$. Recall that the morphisms between sequences are defined so that a morphism from a sequence X to a sequence Y corresponds uniquely to an embedding of the colimit of X into the colimit of Y. Therefore, the ambient category $\mathbf{D}$ is a category whose objects are finite or countably infinite Θ-structures which can be constructed as limits of countable chains in $\mathbf{A}$. Morphisms in $\mathbf{D}$ are embeddings.

Next, Theorem 4.3 is invoked to produce a Ramsey expansion by adding additional structure to objects from $\mathbf{D}$ so that for each $\mathcal{D} \in \mathrm{Ob}(\mathbf{D})$ we add to $\mathrm{Ob}(\mathbf{D}^*)$ all possible pairs $(\mathcal{D}, (\delta_\mathcal{A})_{\mathcal{A} \in \mathrm{Ob}(\mathbf{A})})$ where $\delta_\mathcal{A} : \hom(\mathcal{A}, \mathcal{D}) \to t_\mathbf{A}(\mathcal{A})$ is an arbitrary coloring. This particular construction specializes to first-order structures as follows (cf. [27]). Let $t_\mathbf{K}(\mathcal{A}_m) = t_\mathbf{A}(\mathcal{A}_m) = t_m \in \mathbb{N}$ and let us expand Θ with countably many relational symbols $R_{m,j}$, $m \in \mathbb{N}$, $1 \leqslant j \leqslant t_m$, where the arity of $R_{m,j}$ is $n_m = |A_m|$:

$$\Theta' = \Theta \cup \{R_{m,j} : m \in \mathbb{N}, 1 \leqslant j \leqslant t_m\}.$$

Now, for each Θ-structure $\mathcal{D} = (D, \Theta^\mathcal{D}) \in \mathrm{Ob}(\mathbf{D})$ we add to $\mathrm{Ob}(\mathbf{D}^*)$ all possible Θ'-structures

$$\mathcal{D}^* = (D, \Theta^\mathcal{D} \cup \{R_{m,j}^{\mathcal{D}^*} : m \in \mathbb{N}, 1 \leqslant j \leqslant t_m\})$$

where, for each $m \in \mathbb{N}$:

- $R^{\mathcal{D}^*}_{m,1}, \ldots, R^{\mathcal{D}^*}_{m,t_m}$ are pairwise disjoint, some of them possibly empty;
- if $(d_1, \ldots, d_{n_m}) \in R^{\mathcal{D}^*}_{m,j}$ then the map $\left(\begin{smallmatrix} a_{m,1} & \ldots & a_{m,n_m} \\ d_1 & \ldots & d_{n_m} \end{smallmatrix}\right)$ is an embedding $\mathcal{A}_m \hookrightarrow \mathcal{D}$, $1 \leqslant j \leqslant t_m$; and
- if $\left(\begin{smallmatrix} a_{m,1} & \ldots & a_{m,n_m} \\ d_1 & \ldots & d_{n_m} \end{smallmatrix}\right)$ is an embedding $\mathcal{A}_m \hookrightarrow \mathcal{D}$ then $(d_1, \ldots, d_{n_m}) \in R^{\mathcal{D}^*}_{m,j}$ for some $1 \leqslant j \leqslant t_m$.

Morphisms in $\mathbf{D}^*$ are embeddings. By Theorem 4.3 the obvious forgetful functor $V : \mathbf{D}^* \to \mathbf{D}$ which takes a Θ' structure to its Θ-reduct is a reasonable precompact expansion with unique restrictions and there is a full subcategory $\mathbf{A}^*$ of $\mathbf{D}^*$ which is directed, has the Ramsey property and $V\restriction_{\mathbf{A}^*} : \mathbf{A}^* \to \mathbf{A}$ has the expansion property.

The final step in the proof of Theorem 5.1 consists of spreading the construction that we performed on $\mathbf{A}$ to the whole of $\mathbf{K}$, and in the context of first-order structures this reduces to constructing isomorphic copies of elements of $\mathbf{A}^*$ simply by renaming elements.

■

Moreover, the dual of Theorem 5.1 holds as well. We shall say that $\mathbf{C}$ is a *category of finite quotients* if $\mathbf{C}$ has a skeleton $\mathbf{S}$ with the following properties:

- $\mathbf{S}$ is a countable category,
- $\hom_{\mathbf{S}}(A, B)$ is finite for all $A, B \in \mathbf{S}$,
- for every $B \in \mathrm{Ob}(\mathbf{S})$ the set $\{A \in \mathrm{Ob}(\mathbf{S}) : B \xrightarrow{\mathbf{S}} A\}$ is finite, and
- every $A \in \mathrm{Ob}(\mathbf{S})$ is dually locally finite for $\mathbf{S}$.

Here, $A \in \mathrm{Ob}(\mathbf{S})$ is *dually locally finite for* $\mathbf{S}$ if $A \in \mathrm{Ob}(\mathbf{S})$ is locally finite for $\mathbf{S}^{\mathrm{op}}$. Continuing in the same fashion, we say that $\mathbf{C}$ is *dually directed* if $\mathbf{C}^{\mathrm{op}}$ is directed; that $\mathbf{C}$ has *small dual Ramsey degrees* if $\mathbf{C}^{\mathrm{op}}$ has small Ramsey degrees; that $\mathbf{C}$ has *dual Ramsey property* if $\mathbf{C}^{\mathrm{op}}$ has the Ramsey property; that an expansion $U : \mathbf{C} \to \mathbf{D}$ is *dually reasonable* if $U : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}^{\mathrm{op}}$ is reasonable; that an expansion $U : \mathbf{C} \to \mathbf{D}$ has *unique quotients* if $U : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}^{\mathrm{op}}$ has unique restrictions; and that an expansion $U : \mathbf{C} \to \mathbf{D}$ has *the dual expansion property* if $U : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}^{\mathrm{op}}$ has the expansion property.

5.3. Corollary. *Let $\mathbf{C}$ be a dually directed category of finite quotients whose morphisms are epi. Then $\mathbf{C}$ has finite dual Ramsey degrees if and only if there exists a category $\mathbf{C}^*$ with the dual Ramsey property and a dually reasonable precompact expansion with unique quotients $U : \mathbf{C}^* \to \mathbf{C}$ with the dual expansion property.*

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