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KLEISLI CATEGORIES, T-CATEGORIES AND INTERNAL CATEGORIES

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ABSTRACT. We investigate the properties of the Kleisli category $\mathsf{Kl}\,T$ of a monad (T, λ, μ) on a category $\mathbb{E}$ and in particular the existence of (some kind of) pullbacks. This culminates when the monad is cartesian. In this case, we show that any T-category in $\mathbb{E}$ in the sense of A. Burroni coincides with a special kind of internal category in $\mathsf{Kl}\,T$. Thus, it is the case in particular for T-operads and T-multicategories. More unexpectedly, this, in turn, sheds new conceptual lights on internal categories and n-categories.

1. Introduction

It is well known from [Eilenberg and Moore, 1965] that any monad (T, λ, μ) on a category $\mathbb{E}$ determines an adjoint pair $(U^T, F^T) : \mathsf{Alg}\,T \rightleftarrows \mathbb{E}$. The Kleisli category $\mathsf{Kl}\,T$ of this monad [Kleisli, 1965] is then produced by the canonical decomposition of the functor F^T:

$$\mathbb{E} \xrightarrow{\bar{F}^T} \mathsf{Kl}\,T \xrightarrow{K_T} \mathsf{Alg}\,T$$

into a bijective on objects functor $\bar{F}^T$ followed by a fully faithful one K_T. If the properties of the category $\mathsf{Alg}\,T$ of T-algebras are well known, those of $\mathsf{Kl}\,T$ have been neglected. After some recalls, a first part of this work (Section 3) will give us the opportunity to investigate them. For instance, the functor $\bar{F}^T$ becomes an inclusion as soon as the natural transformation λ is a monomorphism. More generally we shall show how, step by step, the several assumptions of a cartesian monad (T, λ, μ) surprisingly organize the properties of $\mathsf{Kl}\,T$. When the monad (T, λ, μ) is fully cartesian, we show that:

1. the bijective on object natural functor $\bar{F}^T : \mathbb{E} \longrightarrow \mathsf{Kl}\,T$ is actually an inclusion;
2. the subcategory $\mathbb{E}$ then appears to be *left cancelable* in $\mathsf{Kl}\,T$, i.e. such that $h \in \mathbb{E}$ and $g \in \mathbb{E}$ imply $f \in \mathbb{E}$ when $h = g.f$ in $\mathsf{Kl}\,T$;
3. when $\mathbb{E}$ is finitely complete, not only the category $\mathsf{Alg}\,T$ of algebras on the monad is finitely complete as well, but the Kleisli category $\mathsf{Kl}\,T$, which is not finitely complete, is however such that any map $f : X \longrightarrow Y$ in (the subcategory) $\mathbb{E}$ has a pullback along any map in $\mathsf{Kl}\,T$ which still belongs to $\mathbb{E}$; in other words, $\mathbb{E}$ becomes a pullback stable subcategory of the Kleisli category $\mathsf{Kl}\,T$.

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The notion of T-category in a category $\mathbb{E}$ endowed with a monad (T, λ, μ) was introduced in [Burroni, 1971]. It is a kind of a mix of a relational algebra on T in the sense of [Barr, 1969] and of "something" which looks like an internal category, but shifted by this monad, see precise definition in Section 6. Ultimately, the category T-$Cat\mathbb{E}$ of T-categories appears to be a fully faithful extension of $\mathsf{Alg}\,T \hookrightarrow T\text{-}Cat\mathbb{E}$. A second part of this work (Sections 6 and 7) will investigate what is exactly the "something" in question. First, it easily appears that a T-category is a special kind of 3-truncated simplicial object in the Kleisli category $\mathsf{Kl}\,T$ of this monad.

Then, from the observation 3) above, we shall show that, when the monad is cartesian, the previous 3-truncated simplical object in $\mathsf{Kl}\,T$ is actually underlying a regular internal category in $\mathsf{Kl}\,T$, and that any T-category coincides with this kind of internal category. So, according to [Leinster, 2004], T-operads and T-multicategories appear to be internal categories in $\mathsf{Kl}\,T$. On the other hand, when the monad is cartesian, we get a natural notion of T-groupoid, see Section 7.10.

We are also able to localize these results with respect to a pullback stable class Σ in $\mathbb{E}$ when the monad is only Σ-cartesian, see Section 7.2.

Conversely, and more unexpectedly, any internal category in $\mathbb{E}$ will appear to be a special kind of $\mathbb{G}$-category where $(\mathbb{G}, \sigma, \pi)$ is the monad on the category $\mathrm{Pt}\mathbb{E}$ of split epimorphisms in $\mathbb{E}$ whose category of algebras $\mathsf{Alg}\mathbb{G}$ is known to be nothing but the category $Grd\mathbb{E}$ of internal groupoids in $\mathbb{E}$, see [Bourn, 1987]. In this way, the inclusion $Grd\mathbb{E} \hookrightarrow Cat\mathbb{E}$ is thoroughly produced by a standard construction on this only monad.

From that we shall show how internal n-categories and n-groupoids are related to this monad as well. So that the following whole tower of fibrations:

$$\ldots\; n{-}Cat\mathbb{E} \overset{(\,)_{n-1}}{\to} (n-1){-}Cat\mathbb{E} \;\ldots\ldots\; 2{-}Cat\mathbb{E} \overset{(\,)_1}{\to} Cat\mathbb{E} \overset{(\,)_0}{\to} \mathbb{E}$$

is entirely ruled by the split epimorphisms in $\mathbb{E}$ and the monad $(\mathbb{G}, \sigma, \pi)$. Beyond the heuristic interest of this result, it could appear very useful when combinatorial diagrammatic calculations will have been developed on computers. The last part of this article (Sections 9 to 11) is devoted to examples.

For some fresh results on Burroni's T-categories in another direction see [Tholen and Yeganeh, 2021]. I thank Cl. Berger and N. Arkor for their bibliographic suggestions.

The ideas of this work came to my mind during a talk of M. Batanin in Nice for the *Homotopical days* (7-9 Dec. 2022), see [Batanin and De Leger, 2019] and also [Batanin, 2008]. The article is organized along the following lines.

Section 2: recalls about monads (T, λ, μ), cartesian monads, cartesian and autonomous adjunctions. Section 3: step by step properties of the Kleisli category $\mathsf{Kl}\,T$. Section 4: brief recalls about internal categories. Section 5: recalls about internal groupoids and the monad $(\mathbb{G}, \sigma, \pi)$ on $\mathrm{Pt}\mathbb{E}$. Section 6: recalls about T-categories; their simplicial description in $\mathsf{Kl}\,T$. Section 7: when T-categories in $\mathbb{E}$ coincide with a special kind of internal categories in $\mathsf{Kl}\,T$. Section 8: when internal categories in $\mathbb{E}$ coincide with a special kind of $\mathbb{G}$-categories. Section 9: extensions of the results of the previous section to internal n-categories and n-groupoids. It is well known that any internal category

$X_\bullet$ in a finitely complete category $\mathbb{C}$ produces a cartesian monad $(T_{X_\bullet}, \lambda_{X_\bullet}, \mu_{X_\bullet})$ on the slice category $\mathbb{C}/X_0$; Section 10 is devoted to make explicit all the results of Section 7 concerning this cartesian monad. If $\mathsf{Alg}\, T_{X_\bullet}$ is well known to be nothing but the category $DisF/X_\bullet$ of the discrete fibrations above $X_\bullet$, the category of $T_{X_\bullet}$-categories surprisingly coincides with the whole slice category $Cat\mathbb{C}/X_\bullet$. Section 11 is devoted to translate the results of this same section 7 to the T-operads and the T-multicategories themselves and to their algebras.

2. Monads

2.1. Basics. Let us briefly recall the basics about monads. A monad on a category $\mathbb{E}$ is a triple (T, λ, μ) of an endofunctor T and two natural transformations:

$$Id_{\mathbb{E}} \xrightarrow{\lambda} T \xleftarrow{\mu} T^2$$

satisfying $\mu.\mu T = \mu.T\mu$ and $\mu.\lambda T = 1_T = \mu.T\lambda$. An adjoint pair $(U, F, \lambda, \epsilon) : \bar{\mathbb{E}} \rightleftarrows \mathbb{E}$ determines the monad $(T, \lambda, \mu) = (U.F, \lambda, U\epsilon F)$ on $\mathbb{E}$.

A *T-algebra* [Eilenberg and Moore, 1965] on an object X is given by a map $\xi : T(X) \longrightarrow X$ satisfying $\xi.\lambda_X = 1_X$ and $\xi.\mu_X = \xi.T(\xi)$. Accordingly the pair $(T(X), \mu_X)$ produces a T-algebra on $T(X)$. A morphism $f : (X, \xi) \longrightarrow (Y, \gamma)$ of T-algebras is given by a map $f : X \longrightarrow Y$ such that $f.\xi = \gamma.T(f)$.

This construction determines the category $\mathsf{Alg}\, T$ of T-algebras and the forgetful functor $U^T : \mathsf{Alg}\, T \longrightarrow \mathbb{E} : (X, \xi) \mapsto X$ which is obviously conservative. It has the functor $F^T : \mathbb{E} \longrightarrow \mathsf{Alg}\, T$ defined by $F^T(X) = (T(X), \mu_X)$ as left adjoint which makes U^T a left exact functor. The monad associated with the adjoint pair (U^T, F^T) recovers the initial monad (T, λ, μ). From an adjoint pair $(U, F, \lambda, \epsilon) : \bar{\mathbb{E}} \rightleftarrows \mathbb{E}$ and its associated monad $(T, \lambda, \mu) = (U.F, \lambda, U\epsilon F)$ we get a comparison functor $A_{(U,F)} : \bar{\mathbb{E}} \longrightarrow \mathsf{Alg}\, T$, defined by $A_{(U,F)}(Z) = (U(Z), U(\epsilon_Z))$, making the following adjoint pairs commute:

$$\begin{tikzcd}
\bar{\mathbb{E}} \arrow[rr, "A_{(U,F)}"] \arrow[dr, shift left, "U"] & & \mathsf{Alg}\, T \arrow[dl, shift left, "U^T"] \\
& \mathbb{E} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^T"] &
\end{tikzcd}$$

The functor U is said to be *monadic* when the comparison functor $A_{(U,F)}$ is an equivalence of categories.

A comonad (C, ϵ, ν) is the dual of a monad; it determines the category ColgC of coalgebras and a coadjoint pair $(F_C, U_C) : \mathsf{ColgC} \rightleftarrows \mathbb{E}$. Any adjoint pair $(U, F, \lambda, \epsilon) : \bar{\mathbb{E}} \rightleftarrows \mathbb{E}$ determines the comonad $(C, \epsilon, \nu) = (F.U, \epsilon, F\lambda U)$ on $\bar{\mathbb{E}}$ and a comparison functor $C_{(U,F)} : \mathbb{E} \longrightarrow \mathsf{ColgC}$, defined by $C_{(U,F)}(X) = (F(X), F(\lambda_X))$, making the following coadjoint pairs commute:

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "C_{(U,F)}"] \arrow[dr, shift left, "F"] & & \mathsf{ColgC} \arrow[dl, shift left, "F_C"] \\
& \bar{\mathbb{E}} \arrow[ul, shift left, "U"] \arrow[ur, shift left, "U_C"] &
\end{tikzcd}$$

The functor F is said to be *comonadic* when the comparison functor $C_{(U,F)}$ is an equivalence of categories.

2.2. Cartesian monads. A functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ is said to be *cartesian* when $\mathbb{C}$ has pullbacks and F preserves them. A natural transformation $\nu : F \Rightarrow G$ between any pair of functors is said to be *cartesian* when, given any map $f : X \longrightarrow Y \in \mathbb{C}$, the following square is a pullback in $\mathbb{D}$:

$$\begin{tikzcd}
F(X) \arrow[r, "\nu_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\nu_Y"'] & G(Y)
\end{tikzcd}$$

A monad (T, λ, μ) is *cartesian* when the three ingredients are cartesian. A. Burroni [Burroni, 1971] was deeply involved in the cartesian monad induced by the free adjunction $Cat \rightleftarrows Gph$ between categories and directed graphs. More specifically, from T. Leinster [Leinster, 1998], the free monoid monad (M, λ, μ) on Set is a cartesian one: it is the restriction of Burroni's monad to directed graphs with only one object. Sections 4.6 will be devoted to a cartesian monad associated with any internal category $X_\bullet$.

By the following lemma, when λ is cartesian, λ is the equalizer of λT and $T\lambda$.

2.3. Lemma. *Given any cosplit parallel pair in a category* $\mathbb{E}$*:*

$$\begin{tikzcd}
X \arrow[r, shift left=2, "m"] \arrow[r, shift right=2, "m'"'] & \check{X} \arrow[l, "g" description]
\end{tikzcd}$$

any equality $m.k = m'.h$ *implies* $k = h$*. Accordingly the pullback of the maps* m *and* m' *produces their equalizer. So:*
1) given any monad (T, λ, μ)*, if* λ *is cartesian, then* λ *is the equalizer of the pair* $(\lambda_T, T(\lambda))$*;*
2) any cartesian functor preserves the equalizers of cosplit parallel pairs.

2.4. Proposition. *Let* (T, λ, μ) *be a monad on* $\mathbb{E}$ *where* μ *is cartesian; the two following conditions are equivalent:*
1) λ *is the equalizer of* λT *and* $T\lambda$*, and 2)* λ *is cartesian.*

Proof. When μ is cartesian, the natural transformations λT and $T\lambda$ are necessarily cartesian as well, being splittings of the cartesian μ. Now, for any map $f : X \longrightarrow Y$ in Σ, consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, tail, "\lambda_X"] \arrow[d, "f"'] & T(X) \arrow[r, shift left=2, "\lambda_{T(X)}"] \arrow[r, shift right=2, "T(\lambda_X)"'] \arrow[d, "T(f)"] & T^2(X) \arrow[l, dashed, "\mu_X" description] \arrow[d, "T^2(f)"] \\
Y \arrow[r, tail, "\lambda_Y"'] & T(Y) \arrow[r, shift left=2, "\lambda_{T(Y)}"] \arrow[r, shift right=2, "T(\lambda_Y)"'] & T^2(Y) \arrow[l, dashed, "\mu_Y" description]
\end{tikzcd}$$

Any of the right hand side commutative squares is a pullback. Moreover, under assumption 1), the two horizontal ones are pullbacks. Accordingly, the "box lemma" for pullbacks makes the left hand square a pullback as well. ■

Later on, we shall need the following:

2.5. Definition. *A monad (T, λ, μ) is said to be half-cartesian when the endofunctor T is cartesian and λ is the equalizer of the pair $(\lambda_T, T(\lambda))$.*

Accordingly, a monad is cartesian if and only if it is half-cartesian and μ is cartesian.

2.6. Cartesian adjoint pairs. It is then natural to call *cartesian adjoint pair*, any adjoint pair $(U, F) : \mathbb{C} \rightleftarrows \mathbb{D}$ such that $\mathbb{C}$ and $\mathbb{D}$ has pullbacks, the functor F is cartesian, the natural transformations $\lambda : Id_{\mathbb{D}} \Rightarrow U.F$ and $\epsilon : F.U \Rightarrow Id_{\mathbb{C}}$ are cartesian. Then the induced monad (T, λ, μ) on $\mathbb{D}$ is clearly a cartesian monad, since $\mu = U(\epsilon_F)$. The induced comonad (C, ϵ, ν) on $\mathbb{C}$ is cartesian as well: the functor $C = F.U$ is cartesian and the natural transformation ν, being a section of the natural transformation ϵ_C, is cartesian as soon as so is ϵ.

2.7. Proposition. *Given any cartesian adjoint pair $(U, F) : \mathbb{C} \rightleftarrows \mathbb{D}$, the natural transformation $\mu : T^2 \longrightarrow T$ of the induced monad on $\mathbb{D}$ is such that the following diagram is a kernel equivalence relation:*

$$\begin{tikzcd}
T^3(X) \arrow[r, shift left=2, "T(\mu_X)"] \arrow[r, shift right=2, "\mu_{T(X)}"'] & T^2(X) \arrow[l, "T(\lambda_{T(X)})" description] \arrow[r, "\mu_X"'] & T(X)
\end{tikzcd}$$

Conversely, suppose that (T, λ, μ) is a cartesian monad. The adjoint pair (U^T, F^T) : $\mathsf{Alg}T \rightleftarrows \mathbb{E}$ is a cartesian one if and only if the natural transformation μ satisfies the above property. In this case, given any T-algebra $\xi : T(X) \longrightarrow X$, the following diagram produces a kernel equivalence relation:

$$\begin{tikzcd}
T^2(X) \arrow[r, shift left=2, "T(\xi)"] \arrow[r, shift right=2, "\mu_X"'] & T(X) \arrow[l, "T(\lambda_X)" description] \arrow[r, "\xi"'] & X
\end{tikzcd}$$

Proof. The first assertion is the consequence of the fact that the commutative square underlying the diagram in question is the image by the cartesian functor U of the following pullback:

$$\begin{tikzcd}
(F.U)^2(F(X)) \arrow[r, "\epsilon_{F.U.F(X)}"] \arrow[d, "F.U(\epsilon_{F(X)})"'] & F.U.F(X) \arrow[d, "\epsilon_{F(X)}"] \\
F.U.F(X) \arrow[r, "\epsilon_{F(X)}"'] & F(X)
\end{tikzcd}$$

Now suppose the monad is cartesian. Given any T-algebra $x : T(X) \longrightarrow X$ on the object X ans applying μ-cartesianness to the map x, we get a pullback, in such a way that the map $T(\mu_X)$ delineates the composition map of an internal category in $\mathbb{E}$ (see Section 4 below):

$$\begin{tikzcd}
T^3(X) \arrow[r, shift left=3, "\mu_{T(X)}"] \arrow[r, "T(\mu_X)" description] \arrow[r, shift right=3, "T^2(x)"'] & T^2(X) \arrow[r, shift left=3, "\mu_X"] \arrow[r, shift right=3, "T(x)"'] & T(X) \arrow[l, "T(\lambda_X)" description]
\end{tikzcd}$$

When, in addition, μ satisfies the condition in question, the pair $(\mu_{T(X)}, T(\mu_X))$ is the kernel pair of μ_X and this category is actually a groupoid (see Theorem 5.7 below). Any morphism of T-algebras: $f : (X, x) \longrightarrow (Y, y)$ produces the following vertical discrete fibration between groupoids:

$$\begin{tikzcd}
T^3(X) \arrow[r, shift left=2, "\mu_{T(X)}"] \arrow[r, "T(\mu_X)" description] \arrow[r, shift right=2, "T^2(x)"'] \arrow[d, "T^3(f)"'] & T^2(X) \arrow[r, shift left=2, "\mu_X"] \arrow[r, shift right=2, "T(x)"'] \arrow[d, "T^2(f)"] & T(X) \arrow[l, "T(\lambda_X)" description] \arrow[d, "T(f)"] \\
T^3(Y) \arrow[r, shift left=2, "\mu_{T(Y)}"] \arrow[r, "T(\mu_Y)" description] \arrow[r, shift right=2, "T^2(y)"'] & T^2(Y) \arrow[r, shift left=2, "\mu_Y"] \arrow[r, shift right=2, "T(y)"'] & T(Y) \arrow[l, "T(\lambda_Y)" description]
\end{tikzcd}$$

So, it is a discrete cofibration as well, and the following rightward left hand side commutative square is a pullback:

$$\begin{tikzcd}
T^2(X) \arrow[r, shift right=1, "T(x)"'] \arrow[d, "T^2(f)"'] & T(X) \arrow[l, shift right=1, "\lambda_{T(X)}"'] \arrow[r, shift right=1, "x"'] \arrow[d, "T(f)"] & X \arrow[l, shift right=1, "\lambda_X"'] \arrow[d, "f"] \\
T^2(Y) \arrow[r, shift right=1, "T(y)"'] & T(Y) \arrow[l, shift right=1, "\lambda_{T(Y)}"'] \arrow[r, shift right=1, "y"'] & Y \arrow[l, shift right=1, "\lambda_Y"']
\end{tikzcd}$$

So, the rightward right hand side commutative square is a pullback as well, since, composed with the leftward right hand side pullback (λ is cartesian), it gives rise to the pullback obtained by composition of the two left hand side pullbacks. This exactly means that the co-unit $\epsilon : F^T.U^T \to Id_{\mathsf{Alg}\,T}$ of the comonad on $\mathsf{Alg}\,T$ is cartesian. The last assertion is obtained by applying ϵ-cartesianness to ϵ itself. ■

So, let us introduce the following:

2.8. Definition. *A monad (T, λ, μ) is said to be hypercartesian when it is cartesian and the natural transformation μ is such that:*

$$\begin{tikzcd}
T^3(X) \arrow[r, shift left=2, "T(\mu_X)"] \arrow[r, shift right=2, "\mu_{T(X)}"'] & T^2(X) \arrow[l, "T(\lambda_{T(X)})" description] \arrow[r, "\mu_X"'] & T(X)
\end{tikzcd}$$

is a kernel equivalence relation for any object X.

2.9. Autonomous adjoint pairs. Let us introduce the following:

2.10. Definition. *An adjoint pair $(U, F) : \bar{\mathbb{E}} \rightleftarrows \mathbb{E}$ is said to be autonomous when U is monadic and F comonadic.*

In other words, an adjunction is autonomous when it does not expand in new adjunctions via the algebra or co-algebra constructions. The aim of this section is to prove that any half-cartesian monad makes the adjoint pair $(U^T, F^T) : \mathsf{Alg}\,T \rightleftarrows \mathbb{E}$ an autonomous one.

2.11. PROPOSITION. *Let* $(U, F) : \bar{\mathbb{E}} \rightleftarrows \mathbb{E}$ *be an adjoint pair and* (T, λ, μ) *its associated monad on* $\mathbb{E}$. *The following conditions are equivalent:*
1) the natural transformation λ *is the equalizer of the pair* $(\lambda_T, T(\lambda))$*;*
2) the comparison functor $C_{(U,F)} : \mathbb{E} \longrightarrow \mathsf{ColgC}$ *is fully faithful.*
Any of these conditions implies that the functor $F : \mathbb{E} \longrightarrow \bar{\mathbb{E}}$ *is conservative.*

PROOF. Suppose 1). Let $h : F(X) \longrightarrow F(Y)$ be a map in $\bar{\mathbb{E}}$ making the following left hand side square commute:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\lambda_X)"] \arrow[d, "h"'] & FT(X) \arrow[d, "FU(h)"] & X \arrow[r, tail, "\lambda_X"] \arrow[d, dotted, "k"'] & T(X) \arrow[r, shift left, "\lambda_{T(X)}"] \arrow[r, shift right, "T(\lambda_X)"'] \arrow[d, "U(h)"'] & T^2(X) \arrow[d, "TU(h)"] \\
F(Y) \arrow[r, "F(\lambda_Y)"'] & FT(Y) & Y \arrow[r, tail, "\lambda_Y"'] & T(Y) \arrow[r, shift left, "\lambda_{T(Y)}"] \arrow[r, shift right, "T(\lambda_Y)"'] & T^2(Y)
\end{tikzcd}$$

then the two right hand side squares of the right hand side diagram commute and, by 1), we get the dotted factorization k in $\mathbb{E}$ such that $U(h).\lambda_X = \lambda_Y.k$. It remains to show that $F(k) = h$, which is a consequence of $UF(k).\lambda_X = \lambda_Y.k = U(h).\lambda_X$. The unicity of such a k is a consequence of the fact that λ_Y is a monomorphism.

Conversely suppose 2). Let $l : Z \longrightarrow T(X)$ be a map such that $\lambda_{T(X)}.l = T(\lambda_X).l$. We get the following commutative square in $\bar{\mathbb{E}}$:

$$\begin{tikzcd}
F(Z) \arrow[r, "F(\lambda_Z)"] \arrow[d, "\epsilon_{F(X)}.F(l)"'] & FT(Z) \arrow[d, "FU(\epsilon_{F(X)}.F(l))"] \\
F(X) \arrow[r, "F(\lambda_X)"'] & FT(X)
\end{tikzcd}$$

Since:

$$FU(\epsilon_{F(X)}.F(l)).F(\lambda_Z) = FU(\epsilon_{F(X)}).FT(l).F(\lambda_Z) = FU(\epsilon_{F(X)}).F(\lambda_{T(X)}).F(l) = F(l)$$

While:

$$F(\lambda_X).\epsilon_{F(X)}.F(l) = \epsilon_{FT(X)}.FT(\lambda_X).F(l) = \epsilon_{FT(X)}.F(\lambda_{T(X)}).F(l) = F(l).$$

According to 2), there is a map $k : Z \longrightarrow X$ such that $F(k) = \epsilon_{F(X)}.F(l)$. Whence $\lambda_X.k = l$, by $\epsilon_{F(X)}.F(\lambda_X.k) = F(k) = \epsilon_{F(X)}.F(l)$. It remains to show the unicity of the factorization. Let k' be be such that $\lambda_X.k' = l$. Since $C_{(U,F)}$ is faithful, checking $k = k'$ is equivalent to checking $F(k) = F(k')$. We get: $F(k') = \epsilon_{F(X)}.F(\lambda_X).F(k') = \epsilon_{F(X)}.F(\lambda_Z.k') = \epsilon_{F(X)}.F(l) = F(k)$. The last assertion is straightforward since $F = F_C.C_{(U,F)}$ and F_C is conservative. ■

2.12. Proposition. *Let* $(U,F) : \bar{\mathbb{E}} \rightleftarrows \mathbb{E}$ *be an adjoint pair and* (T,λ,μ) *its associated monad on* $\mathbb{E}$*. Suppose the functor* F *cartesian. Then the following conditions are equivalent:*
1) the natural transformation λ *is the equalizer of the pair* $(\lambda_T, T(\lambda))$*;*
2) the functor F *is conservative.*
Under any of these conditions, the functor F *is comonadic.*

Proof. Suppose 1). We shall show that the comparison functor $C_{(U,F)}$ is an equivalence of categories. Accordingly the functor F will be comonadic and thus conservative. So, let us show that $C_{(U,F)}$ is essentially surjective. Let $a : W \longrightarrow F.U(W) = C(W)$ be a coalgebra structure on W in $\bar{\mathbb{E}}$. By the identities $\epsilon_W.a = 1_W$ and $F.U(a).a = F(\lambda_{U(W)}).a$, it produces a 2-truncated split simplicial object in $\bar{\mathbb{E}}$:

$$\begin{tikzcd}
W \arrow[r, shift right=1.5, "a"'] & F.U(W) \arrow[l, shift right=1.5, "\epsilon_W"'] \arrow[r, shift left=1, "F(\lambda_{U(W)})"] \arrow[r, shift right=5, "F.U(a)"'] & (F.U)^2(W) \arrow[l, shift right=5, "\epsilon_{F.U(W)}"'] \arrow[l, shift left=1, "F.U(\epsilon_W)"]
\end{tikzcd}$$

Accordingly, a is the equalizer of the pair $(F.U(a), F(\lambda_{U(W)}))$ in $\bar{\mathbb{E}}$. We have to find an object J in $\mathbb{E}$ such that $C_{(U,F)}(J) = (F(J), F(\lambda_J) \simeq (W,a)$ in ColgC. For that take the equalizer $j : J \longrightarrow U(W)$ in $\mathbb{E}$ of the cosplit parallel pair $(U(a), \lambda_{U(W)})$. Since F is cartesian, this equalizer is preserved by F. So the natural comparison $\gamma : F(J) \longrightarrow W$ such that $a.\gamma = F(j)$ in $\bar{\mathbb{E}}$ is an isomorphism. It remains to check that the following square commutes:

$$\begin{tikzcd}
W \arrow[r, "a"] & F.U(W) \\
F(J) \arrow[u, "\gamma"] \arrow[r, "F(\lambda_J)"'] & F.T(J) \arrow[u, "F.U(\gamma)"']
\end{tikzcd}$$

namely that $F.U(\gamma).F(\lambda_J) = F(j)$. We shall check it by composition with the monomorphism $F.U(a)$: $F.U(a).F.U(\gamma).F(\lambda_J) = F.U.F(j).F(\lambda_J) = F(\lambda_{U(W)}).F(j)$ while $F.U(a).F(j) = F(\lambda_{U(W)}).F(j)$ by definition of j.

Conversely, suppose 2). We have to show that the natural transformation λ is the equalizer of the pair $(\lambda_T, T(\lambda))$. For that, take the equalizer $j : J \rightarrowtail X$ in $\mathbb{E}$ of the cosplit pair $(\lambda_{T(X)}, T(\lambda_X))$ and denote $\gamma : X \longrightarrow J$ the natural comparison such that $\lambda_X = j.\gamma$. Since F is cartesian, this equalizer j is preserved by F, and, $F\lambda_X$ being necessarily the equalizer of the pair $(F(\lambda_{T(X)}), F.T(\lambda_X))$ thanks to the retraction $\epsilon_{F(X)}$, the map $F(\gamma) : F(X) \longrightarrow F(J)$ is an isomorphism is $\bar{\mathbb{E}}$. Now, since F is conservative, γ is an isomorphism, and λ_X is the equalizer of the pair $(\lambda_{T(X)}, T(\lambda_X))$. ■

2.13. Corollary. *Let* (T,λ,μ) *be a monad with a cartesian endofunctor* T*, then the following conditions are equivalent:*
1) the monad is half-cartesian;
2) the endofunctor T *is conservative.*

Under any of these conditions, the adjoint pair $(U^T, F^T) : \mathsf{Alg}\,T \rightleftarrows \mathbb{E}$ *is an autonomous one.*

Moreover, when the monad is hypercartesian, the adjoint pair $(U^T, F^T) : \mathsf{Alg}\,T \rightleftarrows \mathbb{E}$ *is a cartesian one.*

Proof. The functor U^T being monadic, it reflects pullbacks; so, T is cartesian if and only if F^T is cartesian. Applying the previous proposition, the natural transformation λ is the equalizer of the pair $(\lambda_T, T(\lambda))$ if and only if F^T is conservative. This is the case, U^T being conservative, if and only if T is conservative. Then F^T is comonadic; since U^T is monadic, the adjoint pair $(U^T, F^T) : \mathsf{Alg}\,T \rightleftarrows \mathbb{E}$ an autonomous one. The last assertion is a consequence of Proposition 2.7. ■

3. Kleisli category of a monad

The canonical decomposition of the functor $F^T : \mathbb{E} \xrightarrow{\bar{F}^T} \mathsf{Kl}\,T \xrightarrow{K_T} \mathsf{Alg}\,T$ into a functor F^T which is bijective on objects followed by a fully faithful functor K_T produces the *Kleisli category* $\mathsf{Kl}\,T$ of the monad [Kleisli, 1965]. Accordingly, the functor $\bar{F}^T$ remains a left adjoint to : $\bar{U}^T = U^T.K_T : \mathsf{Kl}\,T \longrightarrow \mathbb{E}$, and obviously this adjoint pair recovers the monad (T, λ, μ) as well. Moreover the functor $\bar{U}^T$ is conservative as a composition of two conservative functors.

By the adjoint bijection $\mathrm{Hom}_{\mathsf{Kl}\,T}(X, Y) \simeq \mathrm{Hom}_{\mathbb{E}}(X, TY)$, any map $X \dashrightarrow Y$ in $\mathsf{Kl}\,T$ is given by a map $\alpha : X \longrightarrow T(Y)$ in $\mathbb{E}$; we call the map $\alpha \in \mathbb{E}$, the *support* of this map in $\mathsf{Kl}\,T$ which we shall then denote by "α" : $X \dashrightarrow Y$. Given any other map "β" : $Y \dashrightarrow Z$ in $\mathsf{Kl}\,T$, we get "β"."α" = "$\mu_Z.T(\beta).\alpha$" in $\mathsf{Kl}\,T$. In this way, the natural transformation $\epsilon_X : \bar{F}^T.\bar{U}^T(X) = T(X) \dashrightarrow X$ is given by "1"$_{T(X)}$ and, for any map "α" : $X \dashrightarrow Y$ in $\mathsf{Kl}\,T$, its support α is the unique map in $\mathbb{E}$ such that "α" = "1"$_{T(X)}.\bar{F}^T(\alpha)$ in $\mathsf{Kl}\,T$. So, given any map $f : X \longrightarrow Y$ in $\mathbb{E}$, we get $\bar{F}^T(f) =$ "$\lambda_Y.f$", and $\bar{U}^T$("α") $= \mu_Y.T(\alpha) : T(X) \longrightarrow T(Y)$ in $\mathbb{E}$.

We shall now investigate, step by step, how the assumptions of a cartesian monad surprisingly organizes the properties of $\mathsf{Kl}\,T$.

3.1. Consequences of constraints on λ. Let (T, λ, μ) be a monad on $\mathbb{E}$.

3.2. Proposition. *The endofunctor T of the monad is faithful if and only if the natural transformation λ is monomorphic. The functor $\bar{F}^T : \mathbb{E} \longrightarrow \mathsf{Kl}\,T$ is then an inclusion.*

Proof. Suppose λ is monomorphic. Given a parallel pair (f, g) of maps between X and Y such that $T(f) = T(g)$. Then $T(f).\lambda_X = T(g).\lambda_X$. So, $\lambda_Y.f = \lambda_Y.g$ and $f = g$.

Conversely suppose the endofunctor T faithful. Given a parallel pair (f, g) of maps between X and Y such that $\lambda_Y.f = \lambda_Y.g$, we get $T(\lambda_Y).T(f) = T(\lambda_Y).T(g)$. Since $T(\lambda_Y)$ is a monomorphism as a retract of μ_Y, we get $T(f) = T(g)$; and $f = g$. The last assertion is then straightforward since $\bar{F}T$ is bijective on objects. ■

So, when λ is monomorphic, we are in the rather weird situation of a bijective on objects inclusion $\bar{F}^T : \mathbb{E} \rightarrowtail \mathsf{Kl}\,T$ which admits a right adjoint $\bar{U}^T$. The endofunctor of the induced comonad $(C = \bar{F}^T.\bar{U}^T, \epsilon, \nu)$ on $\mathsf{Kl}\,T$ coincides with T on objects and maps in $\mathbb{E}$ and we get $C(\text{“}\alpha\text{”}) = \mu_Y.T(\alpha)$. Whence the following diagram in $\mathsf{Kl}\,T$ where μ_X coincides with $C(\text{“}1\text{”}_{T(X)})$:

$$\begin{tikzcd}
X \arrow[d, dotted, "\text{“}\alpha\text{”}"'] \arrow[r, shift right, "\lambda_X"'] & T(X) \arrow[l, dotted, shift right, "\text{“}1\text{”}_{T(X)}"'] \arrow[d, "\mu_Y.\alpha"'] \arrow[r, shift left, "\lambda_{T(X)}"] \arrow[r, shift right=3, "T(\lambda_X)"'] & T^2(X) \arrow[l, dotted, shift right=3, "\text{“}1\text{”}_{T^2(X)}"'] \arrow[l, shift left=-1, "\mu_X"] \arrow[d, "T(\mu_Y.\alpha)"] \\
Y \arrow[r, shift right, "\lambda_Y"'] & T(Y) \arrow[l, dotted, shift right, "\text{“}1\text{”}_{T(Y)}"'] \arrow[r, shift left, "\lambda_{T(Y)}"] \arrow[r, shift right=3, "T(\lambda_Y)"'] & T^2(Y) \arrow[l, dotted, shift right=3, "\text{“}1\text{”}_{T^2(Y)}"'] \arrow[l, shift left=-1, "\mu_Y"]
\end{tikzcd}$$

Thanks to the horizontal 2-truncated split simplicial objects, the map λ_X appears to be the equalizer in $\mathsf{Kl}\,T$ of the pair $(\lambda_{T(X)}, T(\lambda_X))$.

3.3. Proposition. *Let $j : \mathbb{E} \rightarrowtail \bar{\mathbb{E}}$ be a bijective on objects inclusion. When j admits a right adjoint $T : \bar{\mathbb{E}} \longrightarrow \mathbb{E}$, the induced monad (T, λ, μ) on $\mathbb{E}$ has its λ monomorphic. Moreover we get $\mathsf{Kl}\,T = \bar{\mathbb{E}}$.*

Proof. Let us denote by $\epsilon_X : X \dashleftarrow T(X)$ the co-unit in $\bar{\mathbb{E}}$ of this adjunction. The map $1_X : X \longrightarrow X$ produces a unique map $\lambda_X : X \longrightarrow T(X)$ in $\mathbb{E}$ such that $\epsilon_X.\lambda_X = 1_X$ and the following diagram in $\bar{\mathbb{E}}$:

$$\begin{tikzcd}
X \arrow[r, shift right, "\lambda_X"'] & T(X) \arrow[l, dotted, shift right, "\epsilon_X"'] \arrow[r, shift left, "\lambda_{T(X)}"] \arrow[r, shift right=3, "T(\lambda_X)"'] & T^2(X) \arrow[l, dotted, shift right=3, "\epsilon_{T(X)}"'] \arrow[l, shift left=-1, "\mu_X"]
\end{tikzcd}$$

which makes ϵ_X the coequalizer of the pair $(\epsilon_{T(X)}, T(\epsilon_X))$ in $\bar{\mathbb{E}}$, and λ_X the equalizer of the pair $(\lambda_{T(X)}, T(\lambda_X))$ in $\bar{\mathbb{E}}$. So, $\lambda_X \in \mathbb{E}$ is a monomorphism in $\bar{\mathbb{E}}$, therefore in $\mathbb{E}$. Since the co-unit ϵ_X in $\bar{\mathbb{E}}$ is the coequalizer of the pair $(\epsilon_{T(X)}, T(\epsilon_X))$, the comparison functor $A_{(T,j)} : \bar{\mathbb{E}} \longrightarrow \mathsf{Alg}\,T$ is fully faithful. Now consider the following commutative diagram:

$$\begin{tikzcd}
\bar{\mathbb{E}} \arrow[rr, "A_{(T,j)}"] \arrow[dr, shift left, "T"] & & \mathsf{Alg}\,T \arrow[dl, shift left, "U^T"] \\
& \mathbb{E} \arrow[ul, shift left, "j"] \arrow[ur, shift left, "F^T"] &
\end{tikzcd}$$

Since $\mathbb{E} \rightarrowtail \bar{\mathbb{E}}$ is bijective on objects, we get $\bar{\mathbb{E}} = \mathsf{Kl}\,T$. ∎

From Proposition 2.11, we get the following:

3.4. Corollary. *Let (T, λ, μ) be a monad on $\mathbb{E}$. The following conditions are equivalent:*
1) the natural transformation λ is the equalizer of the pair $(\lambda_T, T(\lambda))$;
2) a map “α” $: X \dashrightarrow Y$ in $\mathsf{Kl}\,T$ lies in $\mathbb{E}$ if and only if $\lambda_{T(Y)}.\alpha = T(\lambda_Y).\alpha$ in $\mathbb{E}$.
The inclusion $\bar{F}^T : \mathbb{E} \rightarrowtail \mathsf{Kl}\,T$ is then conservative; this last point means that the inverse in $\mathsf{Kl}\,T$ of a map f of (the subcategory) $\mathbb{E}$ belongs to $\mathbb{E}$.

Proof. The condition 2) of Proposition 2.11 means that, when the following square commutes in $\mathsf{Kl}\,T$, the map "α" belongs to $\mathbb{E}$:

$$\begin{tikzcd}
X \arrow[r, "\lambda_X"] \arrow[d, dotted, "``\alpha\text{''}"'] & T(X) \arrow[d, "\mu_Y.T(\alpha)"] \\
Y \arrow[r, "\lambda_Y"'] & T(Y)
\end{tikzcd}$$

The commutation in $\mathsf{Kl}\,T$ is: $T(\lambda_Y).\alpha = \lambda_{T(Y)}.\mu_Y.T(\alpha).\lambda_X$ in $\mathbb{E}$; now this last term is clearly: $\lambda_{T(Y)}.\mu_Y.\lambda_{T(Y)}.\alpha = \lambda_{T(Y)}.\alpha$. When 1) is true, the endofunctor T is conservative; since $T = \bar{U}^T.\bar{F}^T$, the functor $\bar{F}^T$ is conservative as well. ■

Let us introduce the following:

3.5. Definition. *A class Σ of maps in a category $\mathbb{E}$ is said to be left cancellable, when it is such that $g.f \in \Sigma$ and $g \in \Sigma$ imply $f \in \Sigma$.*

3.6. Proposition. *Let (T, λ, μ) be a monad with λ monomorphic. Then the following conditions are equivalent:*
1) the natural transformation λ is cartesian;
2) the inclusion $\bar{F}^T : \mathbb{E} \rightarrowtail \mathsf{Kl}\,T$ makes $\mathbb{E}$ a left cancellable subcategory of $\mathsf{Kl}\,T$.
Any of these conditions implies that λ is the equalizer of the pair $(\lambda_T, T(\lambda))$.

Proof. Suppose 1). Let $h : X \longrightarrow Z \in \mathbb{E}$, $g : Y \longrightarrow Z \in \mathbb{E}$, and "$\phi$" $: X \dashrightarrow Y \in \mathsf{Kl}\,T$ be such that $g.$"ϕ" $= h$ in $\mathsf{Kl}\,T$. This means that $T(g).\phi = \lambda_Z.h$ in $\mathbb{E}$. According to the previous corollary, we must show that $\lambda_{T(Y)}.\phi = T(\lambda_Y).\phi$ in $\mathbb{E}$. Now consider the following diagram in $\mathbb{E}$:

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "h"'] & T(Y) \arrow[r, shift left=2, "\lambda_{T(Y)}"] \arrow[r, shift right=2, "T(\lambda_Y)"'] \arrow[d, "T(g)"'] & T^2(Y) \arrow[l, dashed, "\mu_Y" description] \arrow[d, "T^2(g)"] \\
Z \arrow[r, "\lambda_Z"'] & T(Z) \arrow[r, shift left=2, "\lambda_{T(Z)}"] \arrow[r, shift right=2, "T(\lambda_Z)"'] & T^2(Z) \arrow[l, dashed, "\mu_Z" description]
\end{tikzcd}$$

Since the leftward right hand side commutative square is a pullback, it is enough to check our equality via composition with μ_Y (trivial) and by $T^2(g)$ which is straightforward since the left hand side square commutes.

Conversely suppose 2). We have to show that the following left hand side square is a pullback in $\mathbb{E}$. So let (h, ϕ) be a pair of maps in $\mathbb{E}$ such that $T(g).\phi = \lambda_Z.h$. This means that the following right hand side triangle commutes in $\mathsf{Kl}\,T$:

$$\begin{tikzcd}
Y \arrow[r, "\lambda_Y"] \arrow[d, "g"'] & T(Y) \arrow[d, "T(g)"] \\
Z \arrow[r, "\lambda_Z"'] & T(Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, dotted, "``\phi\text{''}"] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

According to our assumption, the map "ϕ" is in $\mathbb{E}$, which means that there is a map $f : X \longrightarrow Y$ in $\mathbb{E}$ such that $\phi = \lambda_Y.f$ and $g.f = h$ in $\mathbb{E}$. This map f is unique since λ_Y is monomorphic. So, the square in question is a pullback. ■

Finally we get:

3.7. Proposition. *Let $j : \mathbb{E} \rightarrowtail \bar{\mathbb{E}}$ be a bijective on objects left cancellable inclusion. When j admits a right adjoint $T : \bar{\mathbb{E}} \longrightarrow \mathbb{E}$, the induced monad (T, λ, μ) on $\mathbb{E}$ is such that λ is cartesian.*

Proof. According to Proposition 3.3, λ_X is monomorphic and $\bar{\mathbb{E}} = \mathsf{Kl}\,T$. Since $\mathbb{E}$ is left cancellable in $\bar{\mathbb{E}}$, the previous proposition asserts that λ is cartesian. ■

3.8. Consequences of constraints on T. Rephrasing Corollary 2.13 we get:

3.9. Corollary. *Let (T, λ, μ) be a monad with a cartesian endofunctor T. When λ is the equalizer of the pair $(\lambda_T, T\lambda)$, the injective functor $\bar{F}^T : \mathbb{E} \rightarrowtail \mathsf{Kl}\,T$ is conservative, cartesian and then it reflects the pullbacks of $\mathbb{E}$.*

3.10. Consequences of constraints on μ.

3.11. Definition. *A pullback stable class Σ of maps in a category $\mathbb{E}$ is a class of maps which admit pullbacks along any map in $\mathbb{E}$ and whose pullbacks stay in Σ.*

3.12. Proposition. *Let (T, λ, μ) be a monad with a cartesian endofunctor T and a cartesian natural transformation μ. Then, the class $\bar{F}^T(\Sigma)$ is pullback stable in $\mathsf{Kl}\,T$.*

Proof. Starting with a map f, consider the following left hand side pullback in $\mathbb{E}$:

$$\begin{tikzcd}
V \arrow[r, "h"] \arrow[d, "\phi"'] & U \arrow[d, "\psi"] \\
T(X) \arrow[r, "T(f)"'] & T(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, dotted, "\bar{F}^T(h)"] \arrow[d, dotted, "``\phi"'"'] & U \arrow[d, dotted, "``\psi"'"] \\
X \arrow[r, dotted, "\bar{F}^T(f)"'] & Y
\end{tikzcd}$$

This pullback in $\mathbb{E}$ determines a right hand side commutative square in $\mathsf{Kl}\,T$. Let us show that it is a pullback in this category. Since $K_T : \mathsf{Kl}\,T \longrightarrow \mathsf{Alg}\,T$ is fully faithful it is sufficient to check it in $\mathsf{Alg}\,T$; now, since U^T reflects pullbacks, it is sufficient to check that its image by U^T, which is given the following vertical rectangle, is a pullback in $\mathbb{E}$:

$$\begin{tikzcd}
T(V) \arrow[r, "T(h)"] \arrow[d, "T(\phi)"'] & T(U) \arrow[d, "T(\psi)"] \\
T^2(X) \arrow[r, "T^2(f)"] \arrow[d, "\mu_X"'] & T^2(Y) \arrow[d, "\mu_Y"] \\
T(X) \arrow[r, "T(f)"'] & T(Y)
\end{tikzcd}$$

This is the case since the lower square is a pullback, μ being cartesian, and the upper one as well since, T being cartesian, the image by T of our above left hand side pullback is preserved by T. ■

From Propositions 3.6, 3.12 and 2.12, we get:

3.13. Corollary. *Let (T, λ, μ) be a cartesian monad on $\mathbb{E}$. We then get a bijective on objects inclusion $\bar{F}^T : \mathbb{E} \rightarrowtail \mathsf{Kl}\,T$, which makes $\mathbb{E}$ a left cancellable subcategory of $\mathsf{Kl}\,T$ which is pullback stable in $\mathsf{Kl}\,T$. This inclusion functor is cartesian and conservative. It admits a right adjoint $\bar{U}^T$ which necessarily makes this inclusion comonadic.*

And from Proposition 3.7 the following:

3.14. Corollary. *Let $j : \mathbb{E} \rightarrowtail \bar{\mathbb{E}}$ be a bijective on objects left cancellable and pullback stable inclusion. When j admits a right adjoint $T : \bar{\mathbb{E}} \longrightarrow \mathbb{E}$, the induced monad (T, λ, μ) on $\mathbb{E}$ is cartesian and $\bar{\mathbb{E}}$ is the Kleisli category of this monad.*

Proof. By Proposition 3.3, we know that λ is cartesian and $\bar{\mathbb{E}} = \mathsf{Kl}\,T$. The endofunctor $T.j : \mathbb{E} \longrightarrow \mathbb{E}$ is cartesian since so are T (being a right adjoint) and j ($\mathbb{E}$ being pullback stable in $\mathsf{Kl}\,T$, the inclusion j preserves the pullbacks). It remains to check that μ is cartesian. For that, given any map $f \in \mathbb{E}$, consider the following leftward pullback square in $\mathsf{Kl}\,T$:

$$\begin{tikzcd}
 & & T(X) \arrow[dll, dotted, "“1_{T(X)}”"'] \arrow[dl, "\check{f}"] \arrow[dl, phantom] \arrow[dll, phantom] \arrow[dl, phantom] \arrow[dl, phantom] \arrow[ddl, "T(f)"] \\
X \arrow[d, "f"] & P \arrow[l, dotted, "“\phi_f”"] \arrow[d, "\bar{f}"] & \\
Y & T(Y) \arrow[l, dotted, "“1_{T(Y)}”"] &
\end{tikzcd}$$

The commutation of this square in $\mathsf{Kl}\,T$ means $T(f).\phi_f = \bar{f}$ in $\mathbb{E}$. The commutation of the quadrangle in $\mathsf{Kl}\,T$ produces a factorization $\check{f}$ in $\mathsf{Kl}\,T$; since $\mathbb{E}$ is left cancellable in $\mathsf{Kl}\,T$, the commutation of the vertical right hand side triangle makes $\check{f}$ in $\mathbb{E}$. So we get $\bar{f}.\check{f} = T(f)$ in $\mathbb{E}$ and “ϕ_f”.$\check{f}$ = “$1_{T(X)}$” in $\mathsf{Kl}\,T$ which means $\phi_f.\check{f} = 1_{T(X)}$ in $\mathbb{E}$. Checking $\phi_f.\check{f} = 1_{T(X)}$ in $\mathbb{E}$ will prove that the quadrangle is a pullback in $\mathsf{Kl}\,T$, which, being preserved by T, will prove, in turn, that μ is cartesian. We shall check $\phi_f.\check{f} = 1_{T(X)}$, by composition with $\bar{f}$ and “ϕ_f” in $\mathsf{Kl}\,T$. 1): $\bar{f}.\phi_f.\check{f} = T(f).\check{f} = \bar{f} = \bar{f}.1_{T(X)}$; 2)“$\phi_f$”$.\phi_f.\check{f}$ = “ϕ_f”$.1_{T(X)}$ in $\mathsf{Kl}\,T$ is equivalent to $\phi_f.\check{f}.\phi_f = \phi_f$ in $\mathbb{E}$, which is true since $\phi_f.\check{f} = 1_{T(X)}$. ∎

4. Internal categories

From now on, we shall suppose any ground category $\mathbb{E}$ has pullbacks and terminal object 1. Given any map f, we shall denote the kernel equivalence $R[f]$ of this map (which is underlying an internal groupoid $R[f]_\bullet$ in $\mathbb{E}$) in the following way:

$$\begin{tikzcd}
R_2[f] \arrow[r, shift left=2.5, "p_0^f"] \arrow[r, "p_1^f" description] \arrow[r, shift right=2.5, "p_2^f"'] & R[f] \arrow[r, shift left=2.5, "p_0^f"] \arrow[r, shift right=2.5, "p_1^f"'] & X \arrow[l, "s_0^f" description] \arrow[r, "f"] & Y
\end{tikzcd}$$

and given any commutative square, as on the right hand side, we denote by $R(\phi)$ the induced map between the kernel equivalences:

$$\begin{tikzcd}
R[f] \arrow[r, shift left=3, "p_0^f"] \arrow[r, shift right=3, "p_1^f"'] \arrow[d, "R(\phi)"'] & X \arrow[l, "s_0^f" description] \arrow[r, "f"] \arrow[d, "\phi"] & Y \arrow[d, "\psi"] \\
R[f'] \arrow[r, shift left=3] \arrow[r, shift right=3] & X' \arrow[l] \arrow[r, "f'"'] & Y'.
\end{tikzcd}$$

which is underlying an internal functor $R(\phi)_\bullet : R[f]_\bullet \longrightarrow R[f']_\bullet$. Let us recall the following useful Barr-Kock Theorem [Bourn and Gran, 2004]:

4.1. Lemma. *When the above right hand side square is a pullback, the left hand part of the diagram is a discrete fibration between groupoids, which implies that any vertical commutative square is a pullback. Conversely, if the left hand side part of the diagram is a discrete fibration, and if f is a pullback stable strong epimorphism (it is the case when f is split), then the right hand side square is a pullback.*

4.2. Basics. Internal categories have been introduced by Ch. Ehrhesmann in [Ehresmann, 1963]. We deliberately choose the simplicial notations. For the basics on simplicial objects, see, for instance, Chapter VII.5 in [Mac Lane, 1971]. An internal category in $\mathbb{E}$ is is a 3-truncated simplicial object $X_\bullet$ in $\mathbb{E}$, namely a diagram:

$$\begin{tikzcd}
X_\bullet : & X_3 \arrow[r, shift left=9, "d_3^{X_3}"] \arrow[r, shift left=3, "d_2^{X_3}" description] \arrow[r, shift right=3, "d_1^{X_3}" description] \arrow[r, shift right=9, "d_0^{X_3}"'] & X_2 \arrow[l, shift left=6, "s_2^{X_3}" description] \arrow[l, "s_1^{X_3}" description] \arrow[l, shift right=6, "s_0^{X_3}" description] \arrow[r, shift left=6, "d_2^{X_2}"] \arrow[r, "d_1^{X_2}" description] \arrow[r, shift right=6, "d_0^{X_2}"'] & X_1 \arrow[l, shift left=3, "s_1^{X_2}" description] \arrow[l, shift right=3, "s_0^{X_2}" description] \arrow[r, shift left=4, "d_1^{X_1}"] \arrow[r, shift right=4, "d_0^{X_1}"'] & X_0 \arrow[l, "s_0^{X_1}" description]
\end{tikzcd}$$

(where we shall drop the upper indexes when there is no ambiguities) subject to the following identities:

$$\begin{array}{llll}
d_i.d_{j+1} = d_j.d_i, & i \leq j & d_i.s_j = s_{j-1}.d_i, & i < j \\
s_{j+1}.s_i = s_i.s_j, & i \leq j & d_i.s_j = 1, & i = j, j+1 \\
 & & d_i.s_j = s_j.d_{i-1}, & i > j+1
\end{array}$$

where the object X_2 (resp. X_3) is obtained by the pullback of $d_0^{X_1}$ along $d_1^{X_1}$ (resp. $d_0^{X_2}$ along $d_2^{X_2}$). An internal functor is a simplicial morphism between this kind of 3-truncated simplicial objects. Let us recall some classical classes of internal functors:

4.3. Definition. *An internal functor $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ is a discrete cofibration (resp. fibration) when the following right hand side square horizontally indexed by* 0 *(resp. by* 1*) is* *a pullback:*

$$\begin{tikzcd}
X_2 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] \arrow[d, "f_2"'] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "f_1"] & X_0 \arrow[l, "s_0" description] \arrow[d, "f_0"] \\
Y_2 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & Y_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

We denote by $Cat\mathbb{E}$ the category of internal categories in $\mathbb{E}$, and by $(\ )_0 : Cat\mathbb{E} \longrightarrow \mathbb{E}$ the forgetful functor associating with any internal category $X_\bullet$ its "object of objects" X_0. Since $\mathbb{E}$ has pullbacks, so has the category $Cat\mathbb{E}$, since, by commutation of limits, it is easy to see that the limits in $Cat\mathbb{E}$ are built levelwise in $\mathbb{E}$. So, the forgetful functor $(\ )_0$ is cartesian.

The functor $(\ )_0$ is actually a fibration whose cartesian maps are the internal *fully faithful functors* (obtained by a joint pullback) and whose maps in the fibers are the internal functors which are "identities on objects" (*ido-functors* or *idomorphisms* for short).

It is clear that the fiber $Cat_1\mathbb{E}$ above the terminal object 1 is nothing but the pointed category $Mon\mathbb{E}$ of internal monoids in $\mathbb{E}$. Any fiber $Cat_Y\mathbb{E}$ above an object Y, with $Y \neq 1$, has an initial object with the discrete equivalence relation $\Delta_Y = R[1_Y]$ and a terminal one with the indiscrete one $\nabla_Y = R[\tau_Y]$, where $\tau_Y : Y \longrightarrow 1$ is the terminal map. So, the left exact fully faithful functor $\nabla : \mathbb{E} \longrightarrow Cat\mathbb{E}$ admits the fibration $(\ )_0$ as left adjoint and makes the pair $((\ )_0, \nabla)$ a *fibered reflection* in the sense of [Bourn, 1987] (see also section 9.1 below). A functor $f_\bullet$ is then cartesian with respect to $(\ )_0$ (namely, internally fully faithful) if and only if the following left hand side square is a pullback in $Cat\mathbb{E}$, or, equivalently the right hand side one is a pullback in $\mathbb{E}$:

$$\begin{tikzcd}
X_\bullet \arrow[r, "f_\bullet"] \arrow[d] & Y_\bullet \arrow[d] \\
\nabla_{X_\bullet} \arrow[r, "\nabla_{f_\bullet}"'] & \nabla_{Y_\bullet}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "{(d_0,d_1)}"'] & Y_1 \arrow[d, "{(d_0,d_1)}"] \\
X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0
\end{tikzcd}$$

As for any left exact fibration, we get:

4.4. Proposition. *1) The cartesian maps (= internally fully faithful) functors are stable under composition and pullback.*

2) Given any commutative square in $Cat\mathbb{E}$ where both $x_\bullet$ and $y_\bullet$ are cartesian maps:

$$\begin{tikzcd}
X_\bullet \arrow[r, "x_\bullet"] \arrow[d, "f_\bullet"'] & X'_\bullet \arrow[d, "f'_\bullet"] \\
Y_\bullet \arrow[r, "y_\bullet"'] & Y'_\bullet
\end{tikzcd}$$

then it is a pullback:
1) if and only if its image by $(\ )_0$ is a pullback
2) in particular when $f_\bullet$ and $f'_\bullet$ are ido-functors.

The dual category $X_\bullet^{op}$ of $X_\bullet$ is the internal category where the role of $d_0^{X_1}$ and $d_1^{X_1}$ are interchanged. In the context of monads, we have the following:

4.5. Proposition. *Let (T, λ, μ) be any monad on $\mathbb{E}$ with μ cartesian. Then, for any algebra (X, ξ), the following diagram produces an internal category $\overline{T}(X, \xi)$ in $\mathsf{Alg}\,T$:*

$$\begin{tikzcd}
(T^3(X), \mu_{T^2(X)}) \arrow[r, shift left=6, "\mu_{T(X)}"] \arrow[r, "T(\mu_X)" description] \arrow[r, shift right=6, "T^2(\xi)"'] & (T^2(X), \mu_{T(X)}) \arrow[l, shift right=3, "T(\lambda_{T(X)})"'] \arrow[l, shift left=3, "T^2(\lambda_X)"] \arrow[r, shift left=3, "\mu_X"] \arrow[r, shift right=3, "T(\xi)"'] & (T(X), \mu_X) \arrow[l, "T(\lambda_X)" description]
\end{tikzcd}$$

determining the following commutative square:

$$\begin{tikzcd}
\mathsf{Alg}T \arrow[r, "\bar{T}"] \arrow[d, "U^T"'] & Cat\mathsf{Alg}T \arrow[d, "(\,)_0"] \\
\mathbb{E} \arrow[r, "F^T"'] & \mathsf{Alg}T
\end{tikzcd}$$

Moreover any morphism $f : (X, \xi) \longrightarrow (Y, \gamma)$ of T-algebras produces a discrete fibration $\bar{T}(f) : \bar{T}(X, \xi) \longrightarrow \bar{T}(Y, \gamma)$ in $Cat\mathsf{Alg}T$.

Proof. First, we have the following pullbacks in $\mathbb{E}$ since μ is cartesian:

$$\begin{tikzcd}
T^4(X) \arrow[r, "T^3(\xi)"] \arrow[d, "\mu_{T^2(X)}"'] & T^3(X) \arrow[r, "T^2(\xi)"] \arrow[d, "\mu_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \\
T^3(X) \arrow[r, "T^2(\xi)"'] & T^2(X) \arrow[r, "T(\xi)"'] & T(X)
\end{tikzcd}$$

Then $T(\mu_X)$, thanks to the axioms of T-algebra on X, furnishes a composition map. The following pullback in $\mathbb{E}$:

$$\begin{tikzcd}
T^2(X) \arrow[r, "\mu_X"] \arrow[d, "T^2(f)"'] & T(X) \arrow[d, "T(f)"] \\
T^2(Y) \arrow[r, "\mu_Y"'] & T(Y)
\end{tikzcd}$$

determines the last assertion. ∎

There is a comonad (Dec, ϵ, ν) on the simplicial objects which is stable on $Cat\mathbb{E}$ as soon as $\mathbb{E}$ has pullbacks. We shall briefly describe this endofunctor Dec and the co-unit $\epsilon : Dec \longrightarrow 1_{Cat\mathbb{E}}$ because they will be useful in Section 8. Let us start with the lower internal category $X_\bullet$ and consider the following diagram, where X_4 is determined by the pullback of $d_3 : X_3 \longrightarrow X_2$ along $d_0 : X_3 \longrightarrow X_2$:

$$\begin{tikzcd}
DecX_\bullet : \arrow[d, "\epsilon_{X_\bullet}"'] & X_4 \arrow[r, shift left=4.5, "d_0"] \arrow[r, shift left=1.5, "d_1"] \arrow[r, shift right=1.5, "d_2"] \arrow[r, shift right=4.5, "d_3"'] \arrow[d, "d_4"'] & X_3 \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=3, "d_2"'] \arrow[d, "d_3"'] & X_2 \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] \arrow[d, "d_2"] & X_1 \arrow[l, "s_0"'] \arrow[d, "d_1"] \\
X_\bullet : & X_3 \arrow[r, shift left=4.5, "d_0"] \arrow[r, shift left=1.5, "d_1"] \arrow[r, shift right=1.5, "d_2"] \arrow[r, shift right=4.5, "d_3"'] & X_2 \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=3, "d_2"'] & X_1 \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] & X_0 \arrow[l, "s_0"']
\end{tikzcd}$$

The category $DecX_\bullet$ is given by the upper row (in Set, it is the sum of all the coslice categories $Y/\mathbb{E}$), while the co-unit of the comonad is given by the vertical internal functor which is a discrete cofibration.

4.6. The cartesian monad $(T_{X_\bullet}, \lambda_{X_\bullet}, \mu_{X_\bullet})$ on $\mathbb{E}/X_0$. Given any object Y, the slice category $\mathbb{E}/Y$ has the maps with codomain Y as objects, and the commutative triangles above Y as morphisms. Given any map $g : Z \longrightarrow Y$, the composition with g determines a functor $\Sigma_g : \mathbb{E}/Z \longrightarrow \mathbb{E}/Y$ which admits as right adjoint the pullback functor $g^* : \mathbb{E}/Y \longrightarrow \mathbb{E}/Z$ along g.

Let $X_\bullet$ be an internal category in $\mathbb{E}$. It produces a cartesian monad on the slice category $\mathbb{E}/X_0$ whose endofunctor is $T_{X_\bullet} = \Sigma_{d_0}.d_1^*$. The following diagram where any leftward plain square is a pullback describes vertically the behaviour of the endofunctor $T_{X_\bullet}$ from left to right. The associated natural transformations $\lambda_{X_\bullet}$ and $\mu_{X_\bullet}$ are precisely described by the upper horizontal dotted arrows σ_0^h and δ_1^h which are induced by the middle horizontal ones:

$$\begin{tikzcd}
Z \arrow[r, tail, dotted, shift left, "\sigma_0^h"] \arrow[d, "h"'] & d_1^*(Z) \arrow[l, shift left, "\delta_1^h"] \arrow[d, "d_1^*(h)"] & (d_1.d_2)^*(Z) \arrow[l, dotted, shift right, "\delta_1^h"'] \arrow[l, shift left, "\delta_2^h"] \arrow[d, "(d_1.d_2)^*(h)"] \\
X_0 \arrow[r, tail, dotted, shift left, "s_0"] & X_1 \arrow[l, shift left, "d_1"] \arrow[d, "d_0"] & X_2 \arrow[l, dotted, shift right, "d_1"'] \arrow[l, shift left, "d_2"] \arrow[d, "d_0"] \\
 & X_0 & X_1 \arrow[l, dotted, shift right, "d_0"'] \arrow[l, shift left, "d_1"] \arrow[d, "d_0"] \\
 & & X_0
\end{tikzcd}$$

$$\begin{tikzcd}
h \arrow[r, tail, "\sigma_0^h"'] & T_{X_\bullet}(h) & T^2_{X_\bullet}(h) \arrow[l, "\delta_1^h"']
\end{tikzcd}$$

It is a cartesian monad on $\mathbb{E}/X_0$ since the upper plain part of the diagram is made of pullbacks. It is well known (see for instance [Johnstone, 1977]) that the algebras of this monad coincides with the internal discrete fibrations above $X_\bullet$; namely, we have $\mathsf{Alg}T_{X_\bullet} = DisF/X_\bullet$.

5. Internal groupoids

5.1. Basics. A category $X_\bullet$ is a groupoid if and only if any map is invertible. It is a property, which, internally speaking, is equivalent to saying that the following square in the 3-truncated simplicial object defining $X_\bullet$ is a pullback:

$$\begin{tikzcd}
X_2 \arrow[r, "d_1"] \arrow[d, "d_0"'] & X_1 \arrow[d, "d_0"] \\
X_1 \arrow[r, "d_0"'] & X_0
\end{tikzcd}$$

It is easy to check (via the Yoneda embedding) that a category $X_\bullet$ is a groupoid if and only if any commutative square in the 3-truncated simplicial object of its definition is a pullback, see [Bourn, 1987]. The category $Grd\mathbb{E}$ of internal groupoids is the full subcategory of $Cat\mathbb{E}$ whose objects are the groupoids, and it determines a sub-fibration:

$$\begin{tikzcd}
Grd\mathbb{E} \arrow[r, tail] \arrow[d, "(\ )_0"'] & Cat\mathbb{E} \arrow[d, "(\ )_0"] \\
\mathbb{E} \arrow[r, equal] & \mathbb{E}
\end{tikzcd}$$

The fibre $Grd_Y\mathbb{E}$ has the same initial and terminal objects as the fibre $Cat_Y\mathbb{E}$.

5.2. The fibration of points. We denote by $\mathrm{Pt}\mathbb{E}$ the category whose objects are the split epimorphisms $(g, t) : X \rightleftarrows Y$ in $\mathbb{E}$ and whose morphisms are the commutative squares between them:

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, shift right, "g"'] & X' \arrow[d, shift right, "g'"'] \\
Y \arrow[r, "y"'] \arrow[u, shift right, "t"'] & Y' \arrow[u, shift right, "t'"']
\end{tikzcd}$$

We denote by $¶_\mathbb{E} : \mathrm{Pt}\mathbb{E} \longrightarrow \mathbb{E}$ the functor which associates with any split epimorphism (g, t) its codomain Y, and associates with any morphism (y, x) the map y. It is a left exact fibration whose cartesian maps are those squares which are pullback of split epimorphisms in $\mathbb{E}$; it is called the *fibration of points* [Bourn, 1991]. The class ¶ of cartesian maps is stable under composition and pullbacks in $\mathrm{Pt}\mathbb{E}$, it is left cancellable and contains the isomorphisms. Accordingly, it determines a bijective on objects inclusion $j_¶ : \Sigma_¶ \rightarrowtail \mathrm{Pt}\mathbb{E}$, where $\Sigma_¶$ denotes the subcategory of $\mathrm{Pt}\mathbb{E}$ whose morphisms belong to the class ¶, which is left cancellable and pullback stable in $\mathrm{Pt}\mathbb{E}$.

The fibre above Y is denoted by $\mathrm{Pt}_Y\mathbb{E}$ and an object of this fibre is called a (generalized) *point* of Y, while any morphism in a fiber is, for short, called an *idomorphism* (=having an identity as lower map y). The left exact change of base functor produced by the map $\psi : Y \to Y' \in \mathbb{E}$ is the pullback along it and denoted by: $\psi^* : \mathrm{Pt}_{Y'}\mathbb{E} \to \mathrm{Pt}_Y\mathbb{E}$.

5.3. THE MONAD $(\mathbb{G}, \sigma, \pi)$ ON $\mathrm{Pt}\mathbb{E}$. The endofunctor $\mathbb{G}$ on $\mathrm{Pt}\mathbb{E}$ defined by $\mathbb{G}(g,t) = (p_0^g, s_0^g)$ is underlying a monad described by the following diagram in $\mathbb{E}$, where $t_1 = (t.g, 1_X)$:

$$\begin{tikzcd}
X \arrow[r, dotted, "t_1"] \arrow[d, shift right, "g"'] & R[g] \arrow[d, shift right, "p_0^g"'] & R_2[g] \arrow[l, dotted, "p_2^g"'] \arrow[d, shift right, "p_0^g"'] \\
Y \arrow[r, dotted, "t"'] \arrow[u, shift right, "t"'] & X \arrow[u, shift right, "s_0^g"'] & R[g] \arrow[l, dotted, "p_1^g"] \arrow[u, shift right, "s_0^g"'] \\
(g,t) \arrow[r, dotted, "\sigma_{(g,t)}"'] & \mathbb{G}(g,t) & \mathbb{G}^2(g,t) \arrow[l, dotted, "\pi_{(g,t)}"]
\end{tikzcd}$$

It is clear that the maps $\sigma_{(g,t)}$ and $\pi_{(g,t)}$ belong to the class ¶ of cartesian maps. Although being not strictly cartesian, this monad shares many properties with this notion.

5.4. Σ-CARTESIAN MONADS. Given any *pullback stable class* Σ of maps in a category $\mathbb{E}$, we shall call Σ-*cartesian* any functor or natural transformation which only satisfies the cartesian condition on the maps in Σ:

5.5. DEFINITION. *Given any pullback stable class Σ of maps in a category $\mathbb{E}$ and any monad (T, λ, μ) on $\mathbb{E}$, we shall say that:*
1) this monad is Σ-cartesian, when:
i) the endofunctor T preserves the maps in Σ and is Σ-cartesian ;
ii) the natural transformations λ is the equalizer of the pair $(\lambda_T, T(\lambda))$;
iii) the natural transformations μ is Σ-cartesian;
2) this monad is strongly Σ-cartesian when:
i) it is Σ-cartesian;
ii) any λ_X and any μ_X belong to Σ.

Remark. *1) A Σ-cartesian monad is such that λ is Σ-cartesian;*
2) a monad is a strongly Σ-cartesian one if and only if:
i) the endofunctor T preserves the maps in Σ and is Σ-cartesian ;
ii) the natural transformations λ and μ are Σ-cartesian;
iii) any λ_X and any μ_X belongs to Σ..

PROOF. The point 1) is obtained by the proof of Proposition 2.4 restricted to maps in Σ. As soon as λ_X belongs to Σ and λ is Σ-cartesian, then the natural transformations λ is the equalizer of the pair $(\lambda_T, T(\lambda))$, whence 2). ■

5.6. PROPOSITION. *[Bourn, 1987] The endofunctor $\mathbb{G}$ is cartesian. It preserves and reflects the maps of the class ¶. The monad $(\mathbb{G}, \sigma, \pi)$ is strongly ¶-cartesian. Furthermore, given any object (g,t) in $\mathrm{Pt}\mathbb{E}$, the following diagram is a kernel equivalence relation in $\mathrm{Pt}\mathbb{E}$ with its (levelwise) quotient:*

$$\begin{tikzcd}
\mathbb{G}^3(g,t) \arrow[r, shift left=3, "\pi_{\mathbb{G}(g,t)}"] \arrow[r, shift right=3, "\mathbb{G}\pi_{(g,t)}"'] & \mathbb{G}^2(g,t) \arrow[l, "\mathbb{G}\sigma_{\mathbb{G}(g,t)}"'] \arrow[r, "\pi_{(g,t)}"'] & \mathbb{G}(g,t)
\end{tikzcd}$$

Proof. The endofunctor $\mathbb{G}$ is cartesian because it is the result of a cartesian construction. The second assertion comes from Lemma 4.1. We already noticed that the maps $\sigma_{(g,t)}$ and $\pi_{(g,t)}$ belong to the class ¶. It remains to show that the natural transformations σ and π are ¶-cartesian. The second point is a consequence of the fact that the pullbacks in Pt$\mathbb{E}$ are levelwise and of Lemma 4.1, while the first one is a consequence of the further fact that, in the pullback of a split epimorphism, the splitting is pullbacked as well. The last assertion comes from the fact that in the diagram of a kernel equivalence relation:

$$\begin{tikzcd}
R_2[f] \arrow[r, dotted, shift left=2.5ex, "p_0^f"] \arrow[r, "p_1^f" description] \arrow[r, shift right=2.5ex, "p_2^f"'] & R[f] \arrow[r, dotted, shift left=2ex, "p_0^f"] \arrow[r, shift right=2ex, "p_1^f"'] & X \arrow[l, dotted, "s_0^f" description] \arrow[r, dotted, "f"] & Y
\end{tikzcd}$$

the diagram in plain arrow is again a kernel equivalence relation. ∎

Again from [Bourn, 1987], recall the following:

5.7. Theorem. *Any algebra $\alpha : \mathbb{G}(g,t) \to (g,t)$ of this monad is necessarily a ¶-cartesian map. The category of algebras of the monad $(\mathbb{G}, \sigma, \pi)$ on* Pt$\mathbb{E}$ *is the category Grd$\mathbb{E}$ of internal groupoids in $\mathbb{E}$. The functor $U^{\mathbb{G}} : Grd\mathbb{E} \longrightarrow \mathbb{E}$ associates $(d_0, s_0) : X_1 \rightleftarrows X_0$ with any groupoid $X_\bullet$, while the functor $F^{\mathbb{G}} : \mathbb{E} \longrightarrow Grd\mathbb{E}$ associates the internal groupoid $R_\bullet[g]$ with split epimorphism (g,t).*

Proof. We refer to [Bourn, 1987] for the proof. The main conceptual aspects of the proof are detailed in the proof of Proposition 7.12. ∎

Accordingly an internal groupoid is given by a reflexive graph in $\mathbb{E}$ as on the right hand side, completed by a map $d_2 : R[d_0] \longrightarrow X_1$ which makes the following diagram :

$$\begin{tikzcd}
X_\bullet : & R_2[d_0] \arrow[r, shift left=3ex, "p_0"] \arrow[r, shift left=1ex, "p_1" description] \arrow[r, shift right=1ex, "p_2" description] \arrow[r, dotted, shift right=3ex, "d_3=R[d_2]"'] & R[d_0] \arrow[r, shift left=3ex, "p_0"] \arrow[r, "p_1" description] \arrow[r, dotted, shift right=3ex, "d_2"'] & X_1 \arrow[l, shift right=1.5ex, "s_0" description] \arrow[l, shift left=1.5ex, "s_1" description] \arrow[r, shift left=2ex, "d_0"] \arrow[r, shift right=2ex, "d_1"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

a 3-truncated simplicial object; namely the map d_2 must: 1) complete a 2-simplicial object, and 2) satisfy $d_2.R(d_2) = d_2.p_2$. In the set theoretical context, the map d_2 is defined by $d_2(\alpha, \beta) = \beta.\alpha^{-1}$ for any pair (α, β) of arrows with same domain. ∎

5.8. Corollary. *The adjoint pair $(U^{\mathbb{G}}, F^{\mathbb{G}}) : Grd\mathbb{E} \rightleftarrows \mathbb{E}$ is autonomous and the comonad induced on Grd$\mathbb{E}$ coincides with the restriction to Grd$\mathbb{E}$ of the comonad (Dec, ϵ, ν) on Cat$\mathbb{E}$. The endofunctor $Dec : Cat\mathbb{E} \longrightarrow Cat\mathbb{E}$ reflects the groupoids.*

Proof. By Proposition 5.6, the endofunctor $\mathbb{G}$ is cartesian, and the monad $(\mathbb{G}, \sigma, \pi)$ is strongly ¶-cartesian with a ¶-cartesian σ; so, σ is the equalizer of the pair $(\sigma_{\mathbb{G}}, \mathbb{G}(\sigma))$, and the monad is half-cartesian. Then apply Corollary 2.13: accordingly the adjoint pair

$(U^{\mathbb{G}}, F^{\mathbb{G}}) : Grd\mathbb{E} \rightleftarrows \mathbb{E}$ is autonomous. As for the second point, starting with an internal groupoid $X_\bullet$, observe that the endofunctor Dec on $Cat\mathbb{E}$ is stable on $Grd\mathbb{E}$:

$$\begin{tikzcd}
DecX_\bullet : \arrow[d, "\epsilon_{X_\bullet}"'] & R_3[d_0] \arrow[r, shift left=3, "d_0"] \arrow[r, shift left=1, "d_1" description] \arrow[r, shift right=1, "d_2" description] \arrow[r, shift right=3, "d_3"'] \arrow[d, "d_4"'] & R_2[d_0] \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] \arrow[d, "d_3"'] & R[d_0] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "d_2"] & X_1 \arrow[l, "s_0" description] \arrow[d, "d_1"] \\
X_\bullet : & R_2[d_0] \arrow[r, shift left=3, "d_0"] \arrow[r, shift left=1, "d_1" description] \arrow[r, shift right=1, "d_2" description] \arrow[r, shift right=3, "d_3"'] & R[d_0] \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

since the upper horizontal diagram is the groupoid $R_\bullet[d_0]$ and that this $DecX_\bullet$ is nothing but $F^{\mathbb{G}}.U^{\mathbb{G}}(X_\bullet)$.

As for the last point, consider the following vertical discrete fibration in $Cat\mathbb{E}$:

$$\begin{tikzcd}
DecX_\bullet : \arrow[d, "\epsilon_{X_\bullet}"'] & X_4 \arrow[r, shift left=3, "d_0"] \arrow[r, shift left=1, "d_1" description] \arrow[r, shift right=1, "d_2" description] \arrow[r, shift right=3, "d_3"'] \arrow[d, "d_4"'] & X_3 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] \arrow[d, "d_3"'] & X_2 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "d_2"] & X_1 \arrow[l, "s_0" description] \arrow[d, "d_1"] \\
X_\bullet : & X_3 \arrow[r, shift left=3, "d_0"] \arrow[r, shift left=1, "d_1" description] \arrow[r, shift right=1, "d_2" description] \arrow[r, shift right=3, "d_3"'] & X_2 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

and suppose the upper horizontal part is groupoid. Since the vertical discrete fibration $h_\bullet$ has its $h_0 = d_1$ split by s_0, then the lower horizontal row is a groupoid: denote by $\gamma : X_2 \longrightarrow X_2$ the involutive mapping producing the inversion in the upper groupoid. Then $d_2.\gamma.s_1 : X_1 \longrightarrow X_1$ determines an involutive map which produces an inversion for the lower row and makes it a groupoid as well. ■

5.9. Corollary. *The category* PtE *is a subcategory of* $Kl\mathbb{G}$*; the morphisms of this last category are the commutative squares in* $\mathbb{E}$*:*

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, shift right=1, "g"'] & X' \arrow[d, shift right=1, "g'"'] \\
Y \arrow[r, "y"'] \arrow[u, shift right=1, dotted, "t"'] & Y' \arrow[u, shift right=1, dotted, "t'"']
\end{tikzcd}$$

not necessarily respecting the sections.

Proof. The first assertion is a consequence of Proposition 3.2. The morphisms in $Kl\mathbb{G}$ between the two vertical objects of the previous diagram are given by the internal functors between $R[g]$ and $R[g']$ which produce (and are produced by) the commutative diagrams in question since g and g', being split, are the quotients of these kernel equivalence relations. ■

5.10. THE MONAD $(T_{X_\bullet}, \lambda_{X_\bullet}, \mu_{X_\bullet})$ WHEN $X_\bullet$ IS A GROUPOID.

5.11. PROPOSITION. *Given any internal category $X_\bullet$, the monad $(T_{X_\bullet}, \lambda_{X_\bullet}, \mu_{X_\bullet})$ is hypercartesian if and only if $X_\bullet$ is a groupoid.*

PROOF. Suppose $X_\bullet$ is a groupoid. We have to show that the monad is hypercartesian. For that, let us reproduce below a part of the upper part of the diagram of Section 4.6 and let us complete it on the left hand side:

$$\begin{tikzcd}
d_1^*(Z) \arrow[d, "d_1^*(h)"] & (d_1.d_2)^*(Z) \arrow[l, dotted, shift right, "\delta_1^h"'] \arrow[l, shift left, "\delta_2^h"'] \arrow[d, "(d_1.d_2)^*(h)"] & (d_1.d_2.d_3)^*(Z) \arrow[l, dotted, shift right=2, "\delta_1^h"'] \arrow[l, "\delta_2^h" description] \arrow[l, shift left=2, "\delta_3^h"] \arrow[d, "(d_1.d_2.d_3)^*(h)"] \\
X_1 & R[d_1] \arrow[l, dotted, shift right, "d_1"'] \arrow[l, shift left, "d_2"] & R_2[d_1] \arrow[l, dotted, shift right=2, "d_1"'] \arrow[l, "d_2" description] \arrow[l, shift left=2, "d_3"]
\end{tikzcd}$$

In this diagram, any commutative vertical square is a pullback. Since $X_\bullet$ is a groupoid, we get the following kernel equivalence relations on the lower row:

$$\begin{tikzcd}
X_1 & R[d_1] \arrow[l, "d_1"'] & R_2[d_1] \arrow[l, shift right, "d_1"'] \arrow[l, shift left, "d_2"]
\end{tikzcd}$$

which is lifted by pullback on the upper row as a kernel equivalence relation:

$$\begin{tikzcd}
d_1^*(Z) & (d_1.d_2)^*(Z) \arrow[l, "\delta_1^h"'] & (d_1.d_2.d_3)^*(Z) \arrow[l, shift right, "\delta_1^h"'] \arrow[l, shift left, "\delta_2^h"]
\end{tikzcd}$$

which is the hypercartesian condition for the cartesian monad $(T_{X_\bullet}, \lambda_{X_\bullet}, \mu_{X_\bullet})$.

Conversely, suppose this monad is hypercartesian. The hypercartesian condition applied to the terminal object 1_{X_0} of $\mathbb{E}/X_0$ says that in the following diagram which is nothing but the $DecX_\bullet^{op}$:

$$\begin{tikzcd}
X_1 \arrow[r, "s_1" description] & X_2 \arrow[l, dotted, shift right=2, "d_1"'] \arrow[l, shift left=2, "d_2"] & X_3 \arrow[l, dotted, shift right=2, "d_1"'] \arrow[l, dotted, "d_2" description] \arrow[l, shift left=2, "d_3"]
\end{tikzcd}$$

the dotted part of the diagram is a kernel equivalence relation, and consequently that $DecX_\bullet^{op}$ is a groupoid. According to Corollary 5.8, $X_\bullet^{op}$, and thus $X_\bullet$, is a groupoid. ■

This gives a conceptual way to a straightforward result in *Set*:

5.12. COROLLARY. *The domain of any discrete fibration above a groupoid in $Cat\mathbb{E}$ is a groupoid as well.*

Proof. We know that any $T_{X_\bullet}$-algebra produces a discrete fibration above the groupoid $X_\bullet$:

$$\begin{tikzcd}
Z_2 \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=3, "d_2"'] \arrow[d, "h_2"'] & Z_1 \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] \arrow[d, "h_1"] & Z_0 \arrow[l, "s_0" description] \arrow[d, "h_0"] \\
R[d_0] \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=3, "d_2"'] & X_1 \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

By Proposition 2.7, any $T_{X_\bullet}$-algebra $d_0 : T_{X_\bullet}(h) \longrightarrow h$ produces a kernel equivalence relation:

$$\begin{tikzcd}
T^2_{X_\bullet}(h) \arrow[r, shift left=3, "T_{X_\bullet}(d_0)"] \arrow[r, shift right=3, "\mu_{X_\bullet h}"'] & T_{X_\bullet}(h) \arrow[l] \arrow[r, two heads, "d_0"] & h
\end{tikzcd}$$

which, here, is nothing but:

$$\begin{tikzcd}
Z_2 \arrow[r, shift left=3, "d_1"] \arrow[r, shift right=3, "d_0"'] & Z_1 \arrow[l] \arrow[r, two heads, "d_0"] & Z_0
\end{tikzcd}$$

and shows that the category $Z_\bullet$ is a groupoid. ■

6. T-categories

Now, let (T, λ, μ) be any monad on $\mathbb{E}$. The notion of T-category has been introduced by A. Burroni, see [Burroni, 1971], as a mix of a relational algebra in the sense of Barr [Barr, 1969] and of something which looks like a kind of internal category, but shifted by this monad. For that, he first introduced the notion of *pointed T-graph* in $\mathbb{E}$ as a triple (d_0, δ_1, s_0) of maps:

$$\begin{tikzcd}
 & X_1 \arrow[dl, shift right, "d_0"'] \arrow[dr, "\delta_1"] & \\
X_0 \arrow[ur, shift right, "s_0"'] \arrow[rr, "\lambda_{X_0}"'] & & T(X_0)
\end{tikzcd}$$

such that $d_0.s_0 = 1_{X_0}$ and $\delta_1.s_0 = \lambda_{X_0}$ (Axioms 1). According to our notations, it is nothing but a reflexive graph in the Kleisli category $\mathsf{Kl}T$:

$$\begin{tikzcd}
X_1 \arrow[r, shift left=3, dotted, "\text{“}\delta_1\text{”}"] \arrow[r, shift right=3, "\bar{F}^T(d_0)"'] & X_0 \arrow[l, "\bar{F}^T(s_0)" description]
\end{tikzcd}$$

Then building the pullback of the map δ_1 along $T(d_0)$ in $\mathbb{E}$ (in plain arrows in the following diagram):

$$\begin{tikzcd}
 & & X_2 \arrow[dl, shift right, "d_0^1"'] \arrow[dl, dashed, shift left=2, "d_1^1"] \arrow[dr, "\delta_2"] & & \\
 & X_1 \arrow[dl, shift right, "d_0"'] \arrow[dr, "\delta_1"] \arrow[ur, dashed, "s_0^1" description] & & T(X_1) \arrow[dl, shift right, "T(d_0)"'] \arrow[dr, "T(\delta_1)"] & \\
X_0 \arrow[ur, shift right, "s_0"'] \arrow[rr, dotted, "\lambda_{X_0}"'] & & T(X_0) \arrow[ur, shift right, "T(s_0)"'] & & T^2(X_0) \arrow[ll, "\mu_{X_0}"]
\end{tikzcd}$$

Burroni first observed that the section s_0 of d_0 produces, through the existence of $T(s_0)$ a section s_0^1 of d_0^1 such that $\delta_2.s_0^1 = T(s_0).\delta_1$ (Observation 2) and that the identity $\delta_1.s_0.d_0 = \lambda X_0.d_0 = T(d_0).\lambda_{X_1}$ produces a map $s_1^1 : X_1 \longrightarrow X_2$ such that $d_0^1.s_1^1 = s_0.d_0$ and $\delta_2.s_1^1 = \lambda X_1$ (Observation 3).

Then *he demanded a "composition" map* $d_1^1 : X_2 \longrightarrow X_1$ in $\mathbb{E}$ satisfying: $d_0.d_1^1 = d_0.d_0^1$ and $\delta_1.d_1^1 = \mu_{X_0}.T(\delta_1).\delta_2$ (Axioms 4) which, with our notation, delineates the beginning of a 2-truncated simplicial object in $\mathsf{Kl}\,T$:

$$\begin{tikzcd}[column sep=large]
X_2 \arrow[r, dotted, shift left=8, "``\delta_2\text{''}"] \arrow[r, "\bar{F}^T(d_1^1)" description] \arrow[r, shift right=8, "\bar{F}^T(d_0^1)"'] & X_1 \arrow[l, shift right=4, "\bar{F}^T(s_1^1)" description] \arrow[l, shift left=4, "\bar{F}^T(s_0^1)" description] \arrow[r, dotted, shift left=5, "``\delta_1\text{''}"] \arrow[r, shift right=5, "\bar{F}^T(d_0)"'] & X_0 \arrow[l, "\bar{F}^T(s_0)" description]
\end{tikzcd}$$

Then Buronni demands Axioms 7 (*neutrality*): $d_1^1.s_0^1 = 1_{X_1}$ and $d_1^1.s_1^1 = 1_{X_1}$ which completes the previous diagram into a plain 2-truncated simplicial object in $\mathsf{Kl}\,T$. Finally, constructing the pullback of the map δ_2 along $T(d_0^1)$ in $\mathbb{E}$:

$$\begin{tikzcd}
 & & X_3 \arrow[dl, shift right=2, "d_0^2"'] \arrow[dl, dashed, "d_1^2" description] \arrow[dl, dashed, shift left=3, "d_2^2"] \arrow[dr, "\delta_3"] & & \\
 & X_2 \arrow[dl, shift right=2, "d_0^1"'] \arrow[dl, shift left=2, "d_1^1"] \arrow[dr, "\delta_2"] \arrow[ur, dashed, shift right, "s_0^2" description] & & T(X_2) \arrow[dl, shift right=2, "T(d_0^1)"'] \arrow[dl, shift left=2, "T(d_1^1)"] \arrow[dr, "T(\delta_2)"] & \\
X_1 \arrow[ur, "s_0^1" description] & & T(X_1) \arrow[ur, dashed, "T(s_0^1)" description] & & T^2(X_1) \arrow[ll, "\mu_{X_1}"]
\end{tikzcd}$$

Burroni observed that:
1) from: $\delta_1.d_0^1.d_0^2 = T(d_0).\delta_2.d_0^2 = T(d_0).T(d_0^1).\delta_3 = T(d_0).T(d_1^1).\delta_3$
we get a morphism $d_1^2 : X_3 \longrightarrow X_2$ such that $d_0^1.d_1^2 = d_0^1.d_0^2$ and $\delta_2.d_1^2 = T(d_1^1).\delta_3$ (Observations 5);
2) and from: $\delta_1.d_1^1.d_0^2 = \mu_{X_0}.T(\delta_1).\delta_2.d_0^2 = \mu_{X_0}.T(\delta_1).T(d_0^1).\delta_3$
$= \mu_{X_0}.T^2(d_0).T(\delta_2).\delta_3 = T(d_0).\mu_{X_1}.T(\delta_2).\delta_3$, we get a morphism $d_2^2 : X_3 \longrightarrow X_2$ such that: $d_0^1.d_2^2 = d_1^1.d_0^2$ and $\delta_2.d_2^2 = \mu_{X_1}.T(\delta_2).\delta_3$ (Observations 6).
Then he added Axiom 8 (*associativity*) $d_1^1.d_1^2 = d_1^1.d_2^2$ to complete the definition of a T-category.

Further observations: the splitting s_0^1 of d_0^1 produces, via the map $T(s_0^1)$, a splitting $s_0^2 : X_1 \longrightarrow X_2$ of d_0^2 such that $\delta_3.s_0^2 = T(s_0^1).\delta_2$. And from: $d_1^1.s_0^1.d_0 = \lambda_{X_0}.d_0 = T(d_0).\lambda_{X_1}$

we get a map $s_1^1 : X_1 \longrightarrow X_2$ such that $d_0^1.s_1^1 = s_0.d_0$ and $\delta_2.s_1^1 = \lambda_{X_1}$. Finally from $\delta_2.s_1^1.d_0^1 = \lambda_{X_1}.d_0^1 = T(d_0^1).\lambda_{X_1}$, we get a map $s_2^2 : X_2 \longrightarrow X_3$ such that $d_0^2.s_2^2 = s_1^1.d_0^1$ and $\delta_3.s_2^2 = \lambda_{X_2}$. Again, with our notation, this delineates a 3-truncated object in the category $\mathsf{Kl}\,T$ (where the higher degeneracies are omitted):

$$\begin{tikzcd}
X_3 \arrow[r, dotted, shift left=12, "“\delta_3”"] \arrow[r, leftarrow, shift left=8, "\bar{F}^T(s_2^2)" description] \arrow[r, shift left=4, "\bar{F}^T(d_2^2)" description] \arrow[r, leftarrow, "\bar{F}^T(s_1^2)" description] \arrow[r, shift right=4, "\bar{F}^T(d_1^2)" description] \arrow[r, leftarrow, shift right=8, "\bar{F}^T(s_0^2)" description] \arrow[r, shift right=12, "\bar{F}^T(d_0^2)"'] & X_2 \arrow[r, dotted, shift left=9, "“\delta_2”"] \arrow[r, leftarrow, shift left=5, "\bar{F}^T(s_1^1)" description] \arrow[r, "\bar{F}^T(d_1^1)" description] \arrow[r, leftarrow, shift right=5, "\bar{F}^T(s_0^1)" description] \arrow[r, shift right=9, "\bar{F}^T(d_0^1)"'] & X_1 \arrow[r, dotted, shift left=5, "“\delta_1”"] \arrow[r, leftarrow, "\bar{F}^T(s_0)" description] \arrow[r, shift right=5, "\bar{F}^T(d_0)"'] & X_0
\end{tikzcd}$$

A morphism of T-categories, namely a T*-functor*, is a morphism (f_0, f_1) of pointed T-graph:

$$\begin{tikzcd}
& T(X_0) \arrow[rr, "T(f_0)"] & & T(Y_1) \\
X_1 \arrow[ur, "\delta_1"] \arrow[rr, "f_1"] \arrow[dr, "d_0"'] & & Y_1 \arrow[ur, "\delta_1"'] \arrow[dr, "d_0"] & \\
& X_0 \arrow[rr, "f_0"'] & & Y_0
\end{tikzcd}$$

which preserves the "composition maps" d_1^1. Any T-functor naturally induces a morphism of 3-truncated simplicial objects in $\mathsf{Kl}\,T$. Whence the category T-Cat$\mathbb{E}$ of T-categories whose objects will be denoted $X_\bullet^T$ and morphisms $h_\bullet^T : X_\bullet^T \longrightarrow Y_\bullet^T$. We get a forgetful functor $(\ )_T$: T-Cat$\mathbb{E} \longrightarrow \mathbb{E}$ associating the object X_0 with the T-category $X_0 \overset{d_0}{\leftarrow} X_1 \overset{\delta_1}{\rightarrow} T(X_0)$.

We get an *injective fully faithful injective functor* $\mathsf{T}\sharp\mathsf{C} : \mathsf{Alg}\,T \rightarrowtail$ T-Cat$\mathbb{E}$ where $\mathsf{T}\sharp\mathsf{C}_{\mathbb{E}}(X, \xi)$ has the following underlying pointed T-graph:

$$\begin{tikzcd}
& T(X) \arrow[dl, shift right, "\xi"'] \arrow[dr, "1_{T(X)}"] & \\
X \arrow[ur, shift right, "\lambda_X"'] \arrow[rr, "\lambda_X"'] & & T(X)
\end{tikzcd}$$

the structure of T-category being produced by the following diagram:

$$\begin{tikzcd}
& & T^2(X) \arrow[dl, shift right, "T(\xi)"'] \arrow[dl, dashed, shift left, "\mu_X"] \arrow[dr, "1_{T^2(X)}"] & & \\
& T(X) \arrow[dl, shift right, "\xi"'] \arrow[dr, "1_{T(X)}"] & & T^2(X) \arrow[dl, "T(\xi)"'] \arrow[dr, "T(1_{T(X)})"] & \\
X \arrow[ur, shift right, "\lambda_X"'] \arrow[rr, dotted, "\lambda_X"'] & & T(X) & & T^2(X) \arrow[ll, dotted, "\mu_X"]
\end{tikzcd}$$

The functor $\mathsf{T}\sharp\mathsf{C}_{\mathbb{E}}$ makes the following diagram commute:

$$\begin{tikzcd}
\mathsf{Alg}\,T \arrow[r, tail, "\mathsf{T}\sharp\mathsf{C}_{\mathbb{E}}"] \arrow[d, "U^T"'] & \mathrm{T}{-}Cat\mathbb{E} \arrow[d, "(\,)_T"] \\
\mathbb{E} \arrow[r, equal] & \mathbb{E}
\end{tikzcd}$$

In this way, *the category* $\mathrm{T}\text{-}Cat\mathbb{E}$ *appears as a natural extension of the category* $\mathsf{Alg}\,T$.

Warning: However an internal category $X_\bullet : X_1 \underset{d_0}{\overset{d_1}{\substack{\longrightarrow\\ \xleftarrow{s_0}\\ \longrightarrow}}} X_0$ in $\mathbb{E}$ does not induce in general a structure of T-category on the following pointed T-graph:

$$\begin{tikzcd}
& X_1 \arrow[dl, shift right, "d_0"'] \arrow[dr, "\lambda_{X_0}.d_1"] & \\
X_0 \arrow[ur, shift right, "s_0"'] \arrow[rr, "\lambda_{X_0}"'] & & T(X_0)
\end{tikzcd}$$

since the pullback of $T(d_0)$ along $\lambda_{X_0}.d_1$ does not coincide with X_2 in general. When the endofunctor T is cartesian, so is $\bar{F}^T : \mathbb{E} \longrightarrow \mathsf{Kl}\,T$. However, for the same reason, the internal category $\bar{F}^T(X_\bullet)$ in $\mathsf{Kl}\,T$ does not coincide with a T-category.

6.1. PROPOSITION. *Suppose T and λ cartesian. Then the image by the inclusion $\bar{F}^T : \mathbb{E} \rightarrowtail \mathsf{Kl}\,T$ of an internal category is a T-category. So, we really get an inclusion functor $Cat(\bar{F}^T) : Cat\mathbb{E} \rightarrowtail T\text{-}Cat\mathbb{E}$.*

PROOF. Start with an internal category $X_\bullet$ in $\mathbb{E}$. We have $\lambda_{X_0}.d_1 = T(d_1).\lambda_{X_1}$. Now, since the following whole rectangle is a pullback in $\mathbb{E}$

$$\begin{tikzcd}
X_2 \arrow[r, "\lambda_{X_2}"] \arrow[d, "d_0"'] & T(X_2) \arrow[r, "T(d_2)"] \arrow[d, "T(d_0)"'] & T(X_1) \arrow[d, "T(d_0)"] \\
X_1 \arrow[r, "\lambda_{X_1}"'] & T(X_1) \arrow[r, "T(d_1)"'] & T(X_0)
\end{tikzcd}$$

the composition map $d_1^1 : X_2 \longrightarrow X_1$ of $X_\bullet$ in $\mathbb{E}$ produces the map d_1^1 demanded by the definition of a T-category. The satisfaction of the other axioms immedialely follows. ∎

7. When T-categories in $\mathbb{E}$ coincide with (a special kind of) internal categories in $\mathsf{Kl}\,T$

In this section, we are going to investigate the T-categories in the setting of the Σ-cartesian monads and to show that some specific class of T-categories coincides with some specific class of internal categories in the Kleisli category $\mathsf{Kl}\,T$ of the monad. This will be the case thanks to the following:

7.1. Proposition. *Given any pullback stable class Σ in $\mathbb{E}$ and any Σ-cartesian monad (T, λ, μ), then:*
1) the class Σ is pullback stable in $\mathsf{Kl}\,T$;
2) given a pair (g, h) of maps in $\Sigma \times \mathbb{E}$ with same codomain Z, if there is a map $\phi : X \dashrightarrow Y$ in $\mathsf{Kl}\,T$ such that $g.\phi = h$, then ϕ belongs to $\mathbb{E}$.

Proof. First, since λ is the equalizer of the pair $(\lambda_T, T(\lambda))$, $\mathbb{E}$ is a subcategory of $\mathsf{Kl}\,T$.
1) By the same proof as the one of Proposition 3.12, restricted to the maps in Σ, any map in Σ is pullback stable in $\mathsf{Kl}\,T$.
2) Again, by a careful inspection, the same proof as the one of Proposition 3.6 works, when it is restricted to the maps in Σ. ■

7.2. T-categories and Σ-cartesian monads. Now, let be given any pointed T-graph in $\mathbb{E}$:

$$\begin{tikzcd}
& X_1 \arrow[dl, shift right, "d_0"'] \arrow[dr, "\delta_1"] & \\
X_0 \arrow[ur, shift right, "s_0"'] \arrow[rr, "\lambda_{X_0}"'] & & T(X_0)
\end{tikzcd}$$

7.3. Proposition. *Let Σ be a pullback stable class of morphisms in $\mathbb{E}$ and (T, λ, μ) be a Σ-cartesian monad. Then there is a bijection between the T-categories $X_\bullet^T$ having its underlying pointed T-graph with leg $d_0 \in \Sigma$ and the internal categories in $\mathsf{Kl}\,T$:*

$$\begin{tikzcd}[column sep=large]
X_\bullet : & X_3
\arrow[r, dotted, shift left=12, "\text{``}\delta_3\text{''}"]
\arrow[r, leftarrow, shift left=8, "s_2^2" description]
\arrow[r, shift left=4, "d_2^2" description]
\arrow[r, leftarrow, "s_1^2" description]
\arrow[r, shift right=4, "d_1^2" description]
\arrow[r, leftarrow, shift right=8, "s_0^2" description]
\arrow[r, shift right=12, "d_0^2"']
& X_2
\arrow[r, dotted, shift left=8, "\text{``}\delta_2\text{''}"]
\arrow[r, leftarrow, shift left=4, "s_1^1" description]
\arrow[r, "d_1^1" description]
\arrow[r, leftarrow, shift right=4, "s_0^1" description]
\arrow[r, shift right=8, "d_0^1"']
& X_1
\arrow[r, dotted, shift left=4, "\text{``}\delta_1\text{''}"]
\arrow[r, leftarrow, "s_0" description]
\arrow[r, shift right=4, "d_0"']
& X_0
\end{tikzcd}$$

$\mathsf{Kl}\,T$ having leg $d_0 : X_1 \longrightarrow X_0 \in \Sigma$ and section $s_0 \in \mathbb{E}$.

Proof. Starting with such a T-category $X_\bullet^T$, according to the previous proposition, the following diagrams are pullbacks in $\mathsf{Kl}\,T$ since d_0 is in Σ:

$$\begin{tikzcd}
X_3 \arrow[r, "d_0^2"] \arrow[d, dotted, "\text{``}\delta_3\text{''}"'] & X_2 \arrow[r, "d_0^1"] \arrow[d, dotted, "\text{``}\delta_2\text{''}"'] & X_1 \arrow[d, dotted, "\text{``}\delta_1\text{''}"] \\
X_2 \arrow[r, "d_0^1"'] & X_1 \arrow[r, "d_0"'] & X_0
\end{tikzcd}$$

and the above 3-truncated simplicial object in $\mathsf{Kl}\,T$ is underlying an internal category in $\mathsf{Kl}\,T$.

Conversely, starting with any internal category in $\mathsf{Kl}\,T$ with $d_0 : X_1 \longrightarrow X_0 \in \Sigma$ and $s_0 \in \mathbb{E}$, the pullbacks involved in the definition of an internal category are obtained as above, and then they coincide with Burroni's construction. By the second part of the same proposition and the identity $d_0.d_1^1 = d_0.d_0^1$, the composition map $d_1^1 : X_2 \longrightarrow X_1$ necessarily belongs to $\mathbb{E}$, since $d_0 \in \Sigma$; in the same way, since $d_0^1 \in \Sigma$ as well, by $d_0^1.d_1^2 = d_0^1.d_0^2$, the map d_1^2 is in $\mathbb{E}$ too; finally the identity $d_0^1.d_2^2 = d_1^1.d_0^2$ assures us that d_2^2 is in $\mathbb{E}$; accordingly, the internal category $X_\bullet$ in $\mathsf{Kl}\,T$ is underlying a T-category in $\mathbb{E}$. ■

Similarly, by the same previous proposition, we know that any internal functor (h_0, h_1) between internal categories in $\mathsf{Kl}\,T$ with $d_0 \in \Sigma$ and having $h_0 \in \mathbb{E}$ is such that $h_1 \in \mathbb{E}$; accordingly the full subcategory T_Σ-$Cat\mathbb{E}$ of T-$Cat\mathbb{E}$ whose objects are the T-categories with $d_0 \in \Sigma$ coincides with the subcategory $Cat_\Sigma\mathsf{Kl}\,T$ of internal categories in $\mathsf{Kl}\,T$ having $d_0 \in \Sigma$ and $s_0 \in \mathbb{E}$ and internal functors (h_0, h_1) in $\mathsf{Kl}\,T$ with $h_0 \in \mathbb{E}$. It is obtained by the following pullback:

$$\begin{tikzcd}
\mathrm{T}_\Sigma{-}Cat\mathbb{E} \arrow[rr] \arrow[d] & & Cat\mathsf{Alg}\,T \arrow[d, "D^0_{\mathsf{Alg}\,T}"] \\
Pt_\Sigma\mathbb{E} \arrow[r, tail] & Pt\mathbb{E} \arrow[r, "Pt(F^T)"'] & Pt\mathsf{Alg}\,T
\end{tikzcd}$$

where $Pt_\Sigma\mathbb{E}$ denotes the full subcategory of $Pt\mathbb{E}$ whose objects are the split epimorphism (f, s) in $\mathbb{E}$ with $f \in \Sigma$.

When, in addition, *the class* Σ *contains the identity maps, is stable under composition and left cancellable*, the situation becomes even clearer since we are now assured that the map s_0 and the composition map $d_1 : X_2 \longrightarrow X_1$ belong to Σ. So, we get a fully faithful inclusion $j : Cat_\Sigma\mathbb{E} \rightarrowtail T_\Sigma\text{-}Cat\mathbb{E}$ where $Cat_\Sigma\mathbb{E}$ denotes the full subcategory of $Cat\mathbb{E}$ whose objects are the internal categories in Σ.

7.4. T-CATEGORIES AND CARTESIAN MONADS. Now, we get our more meaningful result:

7.5. THEOREM. *When the monad (T, λ, μ) is cartesian, a T-category coincides with an internal category $X_\bullet^T$ in $\mathsf{Kl}\,T$ whose leg d_0 belongs to the subcategory $\mathbb{E}$. A T-functor, coincides with an internal functor in $\mathsf{Kl}\,T$ whose image by the functor $(\ )_0 : Cat(\mathsf{Kl}\,T) \longrightarrow \mathsf{Kl}\,T$ belongs to the subcategory $\mathbb{E}$. The image by the cartesian inclusion functor $\bar{F}^T : \mathbb{E} \rightarrowtail \mathsf{Kl}\,T$ of any internal category is a T-category. The image of any T-category by the fully faithful functor $K_T : \mathsf{Kl}\,T \longrightarrow \mathsf{Alg}\,T$ produces an internal category in $\mathsf{Alg}\,T$.*

PROOF. It is a corollary of the previous proposition where $\Sigma = \mathbb{E}$. ■

By Proposition III.2.21 in [Burroni, 1971], the author observed that, when the monad (T, λ, μ) is cartesian, the image by the functor $\bar{U}^T = U^T.K_T : \mathsf{Kl}\,T \longrightarrow \mathbb{E}$ of the 3-truncated simplicial object in $\mathsf{Kl}\,T$ induced by a T-category in $\mathbb{E}$ is an internal category in $\mathbb{E}$, but he did not produce the previous characterization; for that the Proposition 3.6 concerning the behaviour of the maps of $\mathbb{E}$ inside $\mathsf{Kl}\,T$ and the Proposition 3.12 concerning the existence of a certain class of pullbacks in $\mathsf{Kl}\,T$ are needed. According to the previous proposition, the category T-$Cat\mathbb{E}$ is defined by any of the following pullbacks:

$$\begin{tikzcd}
\mathrm{T}{-}Cat\mathbb{E} \arrow[r, tail] \arrow[d] & Cat\mathsf{Kl}\,T \arrow[d, "D^0_{\mathsf{Kl}\,T}"] \\
\mathrm{Pt}\mathbb{E} \arrow[r, tail, "\mathrm{Pt}(\bar{F}^T)"'] & \mathrm{Pt}\mathsf{Kl}\,T
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{T}{-}Cat\mathbb{E} \arrow[r] \arrow[d] & Cat\mathsf{Alg}\,T \arrow[d, "D^0_{\mathsf{Alg}\,T}"] \\
\mathrm{Pt}\mathbb{E} \arrow[r, "\mathrm{Pt}(F^T)"'] & \mathrm{Pt}\mathsf{Alg}\,T
\end{tikzcd}$$

Let us denote by $(\ )_T : T\text{-}Cat\mathbb{E} \longrightarrow \mathbb{E}$ the forgetful functor associating with any T-category $X_\bullet^T$ its "object of objects" X_0.

7.6. Proposition. *When the monad (T, λ, μ) is cartesian, the category $T\text{-}Cat\mathbb{E}$ has pullbacks and the forgetful functor $(\ )_T$ is cartesian; it is a fibration such that, in the following commutative diagram, the inclusion $Cat(\bar{F}^T) : Cat\mathbb{E} \rightarrowtail T\text{-}Cat\mathbb{E}$ is fully faithful, cartesian and preserves the cartesian maps:*

$$\begin{tikzcd}
Cat\mathbb{E} \arrow[r, tail, "Cat(\bar{F}^T)"] \arrow[d, "(\ )_0"'] & \mathrm{T}-Cat\mathbb{E} \arrow[d, "(\ )_T"] \\
\mathbb{E} \arrow[r, equal] & \mathbb{E}
\end{tikzcd}$$

Proof. Let $f_\bullet : X_\bullet^T \longrightarrow Z_\bullet^T$ and $g_\bullet : Y_\bullet^T \longrightarrow Z_\bullet^T$ be a pair of T-functors. Consider the following levelwise pullbacks in $\mathbb{E}$:

$$\begin{tikzcd}
P_0 \arrow[r, "p_0^X"] \arrow[d, "p_0^Y"'] & X_0 \arrow[d, "f_0"] & P_1 \arrow[r, "p_1^X"] \arrow[d, "p_1^Y"'] & X_1 \arrow[d, "f_1"] \\
Y_0 \arrow[r, "g_0"'] & Z_0 & Y_1 \arrow[r, "g_1"'] & Z_1
\end{tikzcd}$$

The split epimorphisms (d_0, s_0) in $\mathbb{E}$ produce a split epimorphism $(d_0^P, s_0^P) : P_1 \rightleftarrows P_0$ in $\mathbb{E}$. And since the injection $\mathbb{E} \rightarrowtail \mathsf{Kl}\,T$ is cartesian (Proposition 3.9), the maps "δ_1" $: X_1 \dashrightarrow X_0$ in $\mathsf{Kl}\,T$ produces a map "δ_1" $: P_1 \dashrightarrow P_0$ in $\mathsf{Kl}\,T$; from that the structure $P_\bullet^T$ of T-category on this induced pointed graph follows. By this construction, the functors $(\ )_T$ and $Cat(\bar{F}^T)$ are cartesian.

To show that this functor is a fibration, we have first to check that the classical construction of the cartesian maps above a map $f : X \longrightarrow Y_0$ is valid in $\mathsf{Kl}\,T$, namely to build some joint pullbacks in $\mathsf{Kl}\,T$. So let $Y_\bullet^T$ be a T-category and $f : X \longrightarrow Y_0$ any map in $\mathbb{E}$. The following diagram where any square is a pullback in $\mathsf{Kl}\,T$ makes it explicit:

$$\begin{tikzcd}
X_1 \arrow[r, "\check{\bar{f}}"] \arrow[d, "\bar{\phi}"'] & \check{X} \arrow[r, dotted, "``\check{\delta}_1\text{''}"] \arrow[d, "\phi"'] & X \arrow[d, "f"] \\
\bar{X} \arrow[r, "\bar{f}"'] \arrow[d, "\bar{d}_0"'] & Y_1 \arrow[r, dotted, "``\delta_1\text{''}"'] \arrow[d, "d_0"] & Y_0 \\
X \arrow[r, "f"'] & Y_0 &
\end{tikzcd}$$

Whence a morphism of pointed graphs in $\mathsf{Kl}\,T$:

$$\begin{tikzcd}
X_1 \arrow[rr, "\bar{f}.\bar{\phi}"] \arrow[d, shift right=2, "\bar{d}_0.\bar{\phi}"'] \arrow[d, dotted, shift left=2, "``\check{\delta}_1\text{''}.\check{\bar{f}}"] & & Y_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, dotted, shift left=2, "``\delta_1\text{''}"] \\
X \arrow[u, tail] \arrow[rr, "f"'] & & Y_0 \arrow[u, tail]
\end{tikzcd}$$

which, by general arguments, endows the left hand side reflexive graph with an internal category structure $X_\bullet^T$ in $\mathsf{Kl}T$. By Theorem 7.5, this internal category in $\mathsf{Kl}T$ is a T-category.

We have now to check the universal property: so, let $Z_\bullet^T$ be another T-category and $g_\bullet : Z_\bullet^T \longrightarrow Y_\bullet^T$ a T-functor, such that $g_0 = f.h$ for some $h : Z_0 \longrightarrow X$ in $\mathbb{E}$. Again, by general arguments, we certainly get an internal functor $h_\bullet : Z_\bullet^T \longrightarrow X_\bullet^T$ in $\mathsf{Kl}T$ such that $h_0 = h \in \mathbb{E}$. Since both $Z_\bullet$ and $X_\bullet$ are T-categories and h_0 belongs to $\mathbb{E}$, then $h_\bullet$ is a T-functor, see Theorem 7.5. Since $\bar{F}^T$ preserves pullbacks, $Cat(\bar{F}^T)$ preserves the fully faithful internal functors, namely the cartesian maps with respect to $(\ )_0$. ■

In our context, the construction of the endofunctor Dec can be extended to T-categories:

7.7. Proposition. *Given any cartesian monad (T, λ, μ), there is an endofunctor Dec on T-$Cat\mathbb{E}$ which mimicks the endofunctor Dec on $Cat\mathbb{E}$. However the co-unit ϵ does not belong to T-$Cat\mathbb{E}$.*

Proof. Consider the upper part of the following vertical diagram in $\mathsf{Kl}T$:

$$\begin{tikzcd}
DecX_\bullet : \arrow[d, "\epsilon_{X_\bullet}"'] & X_4 \arrow[r, shift left=4.5, "d_0"] \arrow[r, shift left=1.5, "d_1" description] \arrow[r, shift right=1.5, "d_2" description] \arrow[r, shift right=4.5, "d_3"'] \arrow[d, dotted, "\delta_4"'] & X_3 \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=3, "d_2"'] \arrow[d, dotted, "``\delta_3\text{''}"'] & X_2 \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] \arrow[d, dotted, "``\delta_2\text{''}"] & X_1 \arrow[l, "s_0" description] \arrow[d, dotted, "``\delta_1\text{''}"] \\
X_\bullet : & X_3 \arrow[r, shift left=4.5, "d_0"] \arrow[r, shift left=1.5, "d_1" description] \arrow[r, shift right=1.5, "d_2" description] \arrow[r, shift right=4.5, dotted, "``\delta_3\text{''}"'] & X_2 \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=3, dotted, "``\delta_2\text{''}"'] & X_1 \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, dotted, "``\delta_1\text{''}"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

where X_4 is defined by the pullback in $\mathsf{Kl}T$ of the map $d_0 : X_3 \longrightarrow X_2 \in \mathbb{E}$ along the map $\delta_3 : X_3 \dashrightarrow X_2 \in \mathsf{Kl}T$. ■

7.8. Proposition. *Given any cartesian monad (T, λ, μ), the fully faithful inclusion $Cat(\bar{F}^T) : Cat\mathbb{E} \rightarrowtail T$-$Cat\mathbb{E}$ admit a right adjoint $\mathbb{R}$ which preserves the cartesian maps (=fully faithful functors).*

Proof. Let us start with a T-category $X_\bullet^T$ and define $\mathbb{R}(X_\bullet^T)$ by the following fully faithful internal functor in $Cat\mathbb{E}$:

$$\begin{tikzcd}
\bar{X}_1 \arrow[r, "\check{\lambda}_1"] \arrow[d, shift right=3, "\bar{d}_0"'] \arrow[d, shift left=3, "\bar{d}_1"] & T(X_1) \arrow[rd, "T(\delta_1)"] \arrow[d, shift right=2, "T(d_0)"'] & \\
X_0 \arrow[r, "\lambda_{X_0}"'] \arrow[u, "\bar{s}_0" description] & T(X_0) \arrow[u, shift right=2, "T(\bar{s}_0)"'] & T^2(X_0) \arrow[l, "\mu_{X_0}"]
\end{tikzcd}$$

It determines the left hand side T-functor in T-$Cat\mathbb{E}$ where $\bar{\lambda}_1$ is the factorization of $\check{\lambda}_1$ through λ_{X_1}:

$$\begin{tikzcd}
\bar{X}_1 \arrow[r, "\bar{\lambda}_1"'] \arrow[rr, bend left, "\check{\lambda}_1"] \arrow[d, shift right=2, "\bar{d}_0"'] \arrow[d, shift left=2, "\bar{d}_1"] & X_1 \arrow[r, "\lambda_{X_1}"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, dotted, "``\delta_1\text{''}"] & T(X_1) \arrow[dr, "T(\delta_1)"] \arrow[d, shift right=1, "T(d_0)"'] & \\
 & & & T^2(X_0) \arrow[dl, "\mu_{X_0}"] \\
X_0 \arrow[r, equal] \arrow[u, "\bar{s}_0" description] & X_0 \arrow[r, "\lambda_{X_0}"'] \arrow[u, "s_0" description] & T(X_0) \arrow[u, shift right=1, "T(\bar{s}_0)"'] &
\end{tikzcd}$$

since "δ_1".$\bar{\lambda}_1 = \bar{d}_1$ in $\mathsf{Kl}\,T$ means $\delta_1.\bar{\lambda}_1 = \lambda_{X_0}.\bar{d}_1$ in $\mathbb{E}$ which is true since $\lambda_{X_0}.\bar{d}_1 = \mu_{X_0}.T(\delta_1).\check{\lambda}_1 = \mu_{X_0}.T(\delta_1).\lambda_{X_1}.\bar{\lambda}_1 = \mu_{X_0}.\lambda_{T(X_0)}.\delta_1.\bar{\lambda}_1 = \delta_1.\bar{\lambda}_1$.

Now, let $h_\bullet : Z_\bullet \longrightarrow X_\bullet^T$ be any T-functor with $Z_\bullet \in Cat\mathbb{E}$. This means that we get $\delta_1.h_1 = \lambda_{X_0}.h_0.d_1$ in $\mathbb{E}$. We have to check that the map $h_1 : Z_1 \longrightarrow X_1$ factors through $\bar{\lambda}_1$ or equivalently that $\lambda_{X_1}.h_1$ factors through $\check{\lambda}_1$. For that it is enough to check that we have an internal functor in $Cat\mathbb{E}$:

$$\begin{tikzcd}
Z_1 \arrow[r, "\lambda_{X_1}.h_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & T(X_1) \arrow[dr, "T(\delta_1)"] \arrow[d, shift right=1, "T(d_0)"'] & \\
 & & T^2(X_0) \arrow[dl, "\mu_{X_0}"] \\
Z_0 \arrow[r, "\lambda_{X_0}.h_0"'] \arrow[u, "s_0" description] & T(X_0) \arrow[u, shift right=1, "T(\bar{s}_0)"'] &
\end{tikzcd}$$

which is straightforward. The functor $\mathbb{R}$ preserves the fully faithful functors since the functor $K_T : \mathsf{Kl}\,T \longrightarrow \mathsf{Alg}T$ is cartesian as soon as the monad is cartesian. ∎

Now, when the monad is cartesian, the injective functor:
$\mathsf{T}\sharp\mathsf{C}_{\mathbb{E}} : \mathsf{Alg}\,T \rightarrowtail$ T-Cat$\mathbb{E} \rightarrowtail Cat(\mathsf{Kl}\,T)$ is defined by the following diagram:

$$\begin{tikzcd}
T^2(X) \arrow[r, shift left=6, dotted, "``1\text{''}_{T^2(X)}"] \arrow[r, "\mu_X" description] \arrow[r, shift right=6, "T(\xi)"'] & T(X) \arrow[l, shift right=3, "\lambda_{T(X)}"'] \arrow[l, shift left=3, "T(\lambda_X)"] \arrow[r, shift left=4, dotted, "``1\text{''}_{T(X)}"] \arrow[r, shift right=4, "\xi"'] & X \arrow[l, "\lambda_X" description]
\end{tikzcd}$$

which makes commute the following diagram:

$$\begin{tikzcd}
\mathsf{Alg}\,T \arrow[r, tail] \arrow[d, "U^T"'] & Cat(\mathsf{Kl}\,T) \arrow[d, "(\,)_0"] \\
\mathbb{E} \arrow[r, equal] & \mathbb{E}
\end{tikzcd}$$

7.9. Proposition. *The upper injective functor is cartesian. We get* $\mathbb{R}(\mathsf{T}\sharp\mathsf{C}_{\mathbb{E}}(X,\xi)) = \Delta.U^T(X,\xi) = \Delta_X$.

Proof. The first point is straightforward since the pullbacks of internal categories are levelwise and that it is also the case for the T-algebras when T is cartesian. For the second point, given any T-algebra (X, ξ), consider the following joint pullback diagram:

$$\begin{tikzcd}
X \arrow[rr, "1_X"] \arrow[d, "\lambda_X"'] & & X \arrow[d, "\lambda_X"] \\
T(X) \arrow[r, "\lambda_{T(X)}"'] \arrow[d, "\xi"'] & T^2(X) \arrow[r, "\mu_X"'] \arrow[d, "T(\xi)"] & T(X) \\
X \arrow[r, "\lambda_X"'] & T(X) &
\end{tikzcd}$$

■

The composition functor $\mathsf{Alg}\,T \rightarrowtail Cat(\mathsf{Kl}\,T) \overset{Cat(\bar{F}^T)}{\rightarrowtail} Cat(\mathsf{Alg}T)$ gives rise, for any $X \in \mathbb{E}$ to the following internal category in $\mathsf{Alg}\,T$:

$$\begin{tikzcd}
(T^3(X), \mu_{T^2(X)}) \arrow[r, shift left=4, "\mu_{T(X)}"] \arrow[r, shift left=2, leftarrow, "T(\lambda_{T(X)})" description] \arrow[r, "T(\mu_X)" description] \arrow[r, shift right=2, leftarrow, "T^2(\lambda_X)" description] \arrow[r, shift right=4, "T^2(\xi)"'] & (T^2(X), \mu_{T(X)}) \arrow[r, shift left=2, "\mu_X"] \arrow[r, leftarrow, "T(\lambda_X)" description] \arrow[r, shift right=2, "T(\xi)"'] & (T(X), \mu_X)
\end{tikzcd}$$

For further developments on these internal categories, see [Batanin and Berger, 2017]. On the other hand, the "intersection" of the inclusions $Cat(\bar{F}^T) : Cat\mathbb{E} \rightarrowtail Cat(\mathsf{Kl}\,T)$ and $\mathsf{Alg}T \rightarrowtail Cat(\mathsf{Kl}\,T)$ is clearly the empty set.

7.10. T-groupoids. Is the notion of T-groupoid meaningful? Probably not in general, but it is clear that when a T-category coincides with an internal category in $\mathsf{Kl}\,T$, it is legitimate to say that a T-category:

$$\begin{tikzcd}
X_\bullet : & X_3 \arrow[r, shift left=6, dotted, "``\delta_3"'"] \arrow[r, shift left=4, leftarrow, "s_2^2" description] \arrow[r, shift left=2, "d_2^2" description] \arrow[r, leftarrow, "s_1^2" description] \arrow[r, shift right=2, "d_1^2" description] \arrow[r, shift right=4, leftarrow, "s_0^2" description] \arrow[r, shift right=6, "d_0^2"'] & X_2 \arrow[r, shift left=4, dotted, "``\delta_2"'"] \arrow[r, shift left=2, leftarrow, "s_1^1" description] \arrow[r, "d_1^1" description] \arrow[r, shift right=2, leftarrow, "s_0^1" description] \arrow[r, shift right=4, "d_0^1"'] & X_1 \arrow[r, shift left=2, dotted, "``\delta_1"'"] \arrow[r, leftarrow, "s_0" description] \arrow[r, shift right=2, "d_0"'] & X_0
\end{tikzcd}$$

is a T-groupoid when:

$$\begin{tikzcd}
X_2 \arrow[r, shift left=2, "d_1^1"] \arrow[r, leftarrow, "s_0^1" description] \arrow[r, shift right=2, "d_0^1"'] & X_1 \arrow[r, "d_0"'] & X_0
\end{tikzcd}$$

is a kernel equivalence relation in $\mathbb{E}$.

7.11. When T-algebras produce T-groupoids. We shall try now to answer the question: when is the image $\mathsf{T}\sharp\mathsf{C}_{\mathbb{E}}(X, \xi)$ an internal groupoid in $\mathsf{Kl}\,T$?

7.12. Proposition. *Let Σ be a pullback stable class of morphisms and (T, λ, μ) be a strongly Σ-cartesian monad on $\mathbb{E}$. Suppose that the object X is such that the following diagram is a kernel equivalence relation:*

$$\begin{tikzcd}
T^3(X) \arrow[r, shift left=2, "T(\mu_X)"] \arrow[r, shift right=2, "\mu_{T(X)}"'] & T^2(X) \arrow[l, "T(\lambda_{T(X)})" description] \arrow[r, "\mu_X"'] & T(X)
\end{tikzcd}$$

Then the object $T(X)$ satisfies the same property. Suppose moreover that Σ is a bijective on objects left cancellable subcategory of $\mathbb{E}$ containing all the isomorphisms. Then any algebra $\xi : T(X) \longrightarrow X$ on X belongs to Σ and the following diagram is a kernel equivalence relation:

$$\begin{tikzcd}
T^2(X) \arrow[r, shift left=2, "T(\xi)"] \arrow[r, shift right=2, "\mu_X"'] & T(X) \arrow[l, "T(\lambda_X)" description] \arrow[r, "\xi"'] & X
\end{tikzcd}$$

So, the T-category $\mathsf{T}\sharp\mathsf{C}_{\mathbb{E}}(X, \xi)$ is actually a T-groupoid, i.e. an internal groupoid in the Kleisli category $\mathsf{Kl}\, T$.

Proof. Since μ_X is in Σ and μ is Σ-cartesian, the map $T(\mu_{T(X)})$ delineates the composition map of an internal category in $\mathbb{E}$:

$$\begin{tikzcd}
T^4(X) \arrow[r, shift left=3, "\mu_{T^2(X)}"] \arrow[r, "T(\mu_{T(X)})" description] \arrow[r, shift right=3, "T^2(\mu_X)"'] & T^3(X) \arrow[r, shift left=2, "\mu_{T(X)}"] \arrow[r, shift right=2, "T(\mu_X)"'] & T^2(X) \arrow[l, "T(\lambda_{T(X)})" description]
\end{tikzcd}$$

When the diagram in question is a kernel equivalence relation, its image by T is a kernel equivalence relation (i.e. the pair $(T(\mu_{T(X)}), T^2(\mu_X))$ is the kernel pair of $T(\mu_X)$), and this category is actually a groupoid. By duality, the pair $(\mu_{T^2(X)}, T(\mu_{T(X)}))$ is the kernel equivalence relation of $\mu_{T(X)}$ and we get the first assertion.

Any T-algebra $\xi : T(X) \longrightarrow X$ produces the following diagram:

$$\begin{tikzcd}
T^4(X) \arrow[r, shift left=3, "\mu_{T^2(X)}"] \arrow[r, "T(\mu_{T(X)})" description] \arrow[r, shift right=3, "T^2(\mu_X)"'] \arrow[d, "T^3(\xi)"'] & T^3(X) \arrow[r, shift left=2, "\mu_{T(X)}"] \arrow[r, shift right=2, "T(\mu_X)"'] \arrow[d, "T^2(\xi)"] & T^2(X) \arrow[l, "T(\lambda_{T(X)})" description] \arrow[d, "T(\xi)"] \\
T^3(X) \arrow[r, shift left=3, "\mu_{T(X)}"] \arrow[r, "T(\mu_X)" description] \arrow[r, shift right=3, "T^2(\xi)"'] & T^2(X) \arrow[r, shift left=2, "\mu_X"] \arrow[r, shift right=2, "T(\xi)"'] & T(X) \arrow[l, "T(\lambda_X)" description]
\end{tikzcd}$$

Now, since the square in question is a pullback, the maps $T(\xi)$ and $T^2(\xi)$ makes the lower row an internal groupoid. So, the involutive "inversion" mapping $\gamma_\xi : T^2(X) \longrightarrow T^2(X)$ of this groupoid exchanges the maps μ_X and $T(\xi)$. Since, by assumption, the isomorphism γ_X is in Σ which is a subcategory of $\mathbb{E}$, then $T(\xi) = \mu_X.\gamma_\xi$ belongs to Σ. Now, the identity $\lambda_X.\xi = T(\xi).\lambda_{T(X)}$ shows that $\xi \in \Sigma$, since Σ is left cancellable.

It remains to check the last assertion. Since the lower row of the diagram above is a groupoid, by duality, the following diagram is a kernel equivalence relation:

$$\begin{tikzcd}
T^3(X) \arrow[r, shift left, "T(\mu_X)"] \arrow[r, shift right, "T^2(\xi)"'] & T^2(X) \arrow[r, "T(\xi)"'] & T(X)
\end{tikzcd}$$

The endofunctor T is conservative since λ is the equalizer of the pair $(\lambda_T, T(\lambda))$ (since λ is in Σ and λ Σ-cartesian); accordingly it reflects the pullbacks of maps in Σ, and the following diagram is thus a kernel equivalence relation:

$$\begin{tikzcd}
T^2(X) \arrow[r, shift left, "\mu_X"] \arrow[r, shift right, "T(\xi)"'] & T(X) \arrow[r, "\xi"'] & X
\end{tikzcd}$$

■

7.13. Corollary. *Let (T, λ, μ) be a cartesian monad on $\mathbb{E}$. Suppose the object X is such that the following diagram is a kernel equivalence relation:*

$$\begin{tikzcd}
T^3(X) \arrow[r, shift left=2, "T(\mu_X)"] \arrow[r, shift right=2, "\mu_{T(X)}"'] & T^2(X) \arrow[l, "T(\lambda_X)" description] \arrow[r, "\mu_X"'] & T(X)
\end{tikzcd}$$

Then any T-algebra $\xi : T(X) \longrightarrow X$ on X is such that the T-category $\mathsf{T}\sharp\mathsf{C}_{\mathbb{E}}(X, \xi)$ is actually a T-groupoid, i.e. an internal groupoid in $\mathsf{Kl}\,T$.

In Section 10.5 we shall produce a cartesian monad where this condition is satisfied for any object X.

8. When internal categories in $\mathbb{E}$ coincide with $\mathbb{G}$-categories in Pt$\mathbb{E}$

In this section we shall show that the category $Cat\mathbb{E}$ of internal categories in $\mathbb{E}$ coincides with a specific subcategory of the category of $\mathbb{G}$-categories in Pt$\mathbb{E}$.

The monad $(\mathbb{G}, \sigma, \pi)$ being strongly $\P$-cartesian on Pt$\mathbb{E}$ (Section 5), we get:

8.1. Proposition. *The full subcategory $\mathbb{G}_{\P}$-CatPt$\mathbb{E}$ of $\mathbb{G}$-CatPt$\mathbb{E}$ whose objects are the $\mathbb{G}$-categories with a $\P$-cartesian 0-leg coincides with the category whose objects are the discrete fibrations $h_\bullet : X_\bullet \longrightarrow Y_\bullet$ in $\mathbb{E}$, where $h_0 : X_0 \longrightarrow Y_0$ is endowed with a given splitting t_0 and whose morphisms are the commutative squares between discrete fibrations in $Cat\mathbb{E}$ as on the left hand side:*

$$\begin{tikzcd}
X_\bullet \arrow[r, "\psi_\bullet"] \arrow[d, "h_\bullet"'] & \bar{X}_\bullet \arrow[d, "\bar{h}_\bullet"] \\
Y_\bullet \arrow[r, "\phi_\bullet"'] & \bar{Y}_\bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0 \arrow[r, "\psi_0"] \arrow[d, shift right, "h_0"'] & \bar{X}_0 \arrow[d, shift right, "\bar{h}_0"'] \\
Y_0 \arrow[r, "\phi_0"'] \arrow[u, shift right, "t_0"'] & \bar{Y}_0 \arrow[u, shift right, "\bar{t}_0"']
\end{tikzcd}$$

such that the above right hand side square is a morphism in Pt$\mathbb{E}$*.*

Proof. Apply Propositions 7.3 and 5.9. ■

We shall be interested now in the full subcategory of $\mathbb{G}_\P$-$Cat\mathrm{Pt}\mathbb{E}$ whose objects are the $\mathbb{G}$-categories having, in addition, an idomorphic 1-leg.

8.2. Theorem. *The full subcategory of $\mathbb{G}$-$Cat\mathrm{Pt}\mathbb{E}$ whose objects are the $\mathbb{G}$-categories with a ¶-cartesian 0-leg and an idomorphic 1-leg is isomorphic to the category $Cat\mathbb{E}$ of internal categories in $\mathbb{E}$.*

Proof. Let $h_\bullet : X_\bullet \longrightarrow Y_\bullet$ be a $\mathbb{G}$-category with a ¶-cartesian 0-leg, namely a vertical discrete fibration with a section t_0 of h_0:

$$\begin{tikzcd}
X_0 \arrow[r, "s_0^Y"] \arrow[d, shift right=1, "h_0"'] & X_1 \arrow[l, shift right=3, "d_{0}X"'] \arrow[l, shift left=3, "d_{1}X"] \arrow[d, "h_1"] & X_2 \arrow[l, "d_1"] \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_2"] \arrow[d, "h_2"] \\
Y_0 \arrow[r, "s_0"] \arrow[u, shift right=1, "t_0"'] & Y_1 \arrow[l, shift right=3, "d_{0}Y"'] \arrow[l, shift left=3, "d_{1}Y"] & Y_2 \arrow[l, "d_1"] \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_2"]
\end{tikzcd}$$

we shall denote by t_i the induced section of h_i. Its underlying $\mathbb{G}$-graph:

$$(h_0, t_0) \overset{(d_0^Y, d_0^X)}{\longleftarrow} (h_1, t_1) \overset{(d_1^X, R(d_1^X)).\sigma(h_1, t_1)}{\longrightarrow} \mathbb{G}(h_0, t_0)$$

is the following one:

$$\begin{tikzcd}
 & X_1 \arrow[dl, "d_{0}X"'] \arrow[dr, "(d_1^X.t_1.h_1, d_1^X)"] \arrow[dd, shift right=1, "h_1"'] & \\
X_0 \arrow[dd, shift right=1, "h_0"'] & & R[h_0] \arrow[dd, shift right=1, "p_0^{h_0}"'] \\
 & Y_1 \arrow[uu, shift right=1, "t_1"'] \arrow[dl, "d_{0}Y"] \arrow[dr, "d_1^X.t_1"'] & \\
Y_0 \arrow[uu, shift right=1, "t_0"'] & & X_0 \arrow[uu, shift right=1, "s_0^{h_0}"']
\end{tikzcd}$$

Saying that its 1-leg is idomorphic is saying that $d_1^X.t_1 = 1_{Y_1}$. Whence:
1) $h_0 = h_0.d_1^X.t_1 = d_1^Y.h_1.t_1 = d_1^Y$, and 2) $t_0 = t_0.d_1^Y.s_0^Y = d_1^X.t_1.s_0^Y = s_0^Y$, and consequently $(h_0, t_0) = (d_1, s_0)$. Similarly we have $d_2^X.t_2 = 1_{Y_2}$. Whence:
1) $h_1 = h_1.d_2^X.t_2 = d_2^Y.h_2.t_2 = d_2^Y$, and 2) $t_1 = t_1.d_2^Y.s_1^Y = d_2^X.t_2.s_1^Y = s_1^Y$, and consequently $(h_1, t_1) = (d_2, s_1)$. Accordingly, we get the following diagram:

$$\begin{tikzcd}
Y_1 \arrow[r, "s_0"] \arrow[d, shift right=1, "d_1"'] & Y_2 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] \arrow[d, "d_2"] & Y_3 \arrow[l, "d_1"] \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_2"] \arrow[d, "d_3"] \\
Y_0 \arrow[r, "s_0"] \arrow[u, shift right=1, "s_0"'] & Y_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] & Y_2 \arrow[l, "d_1"] \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_2"]
\end{tikzcd}$$

which is nothing but the discrete fibration $\epsilon_{Y_\bullet} : DecY_\bullet \longrightarrow Y_\bullet$ with the section s_0, and nothing more. So that a $\mathbb{G}_\P$-category with an idomorphic 1-leg is just an internal category in $\mathbb{E}$. Conversely any internal category $Y_\bullet$ in $\mathbb{E}$ produce the above $\mathbb{G}_\P$-category with an idomorphic 1-leg. ■

Now, consider the injective functor $\mathbb{G}\sharp\mathsf{C} : \mathsf{Alg}\mathbb{G} = Grd\mathbb{E} \rightarrowtail \mathbb{G}\text{-}Cat(\mathrm{Pt}\mathbb{E})$; according to Proposition 7.12, any $\mathbb{G}$-algebra is $\P$-cartesian, so that the pointed $\mathbb{G}$-graph underlying $\mathbb{G}\sharp\mathsf{C}((d_0, s_0), (d_1, d_2))$ (following the notations of Section 5):

$$\begin{tikzcd}
 & \mathbb{G}(d_0,s_0) \arrow[dl, "{(d_1,d_2)}"'] \arrow[dr, "1_{\mathbb{G}(d_0,s_0)}"] & \\
(d_0,s_0) \arrow[rr, "{\sigma(d_0,s_0)}"'] & & \mathbb{G}(d_0,s_0)
\end{tikzcd}$$

has a $\P$-cartesian 0-leg and an idomorphic 1-leg. Accordingly, the injective functor $\mathbb{G}\sharp\mathsf{C}$ factors through $Cat\mathbb{E}$, producing the natural inclusion $Grd\mathbb{E} \rightarrowtail Cat\mathbb{E}$. So, not only the monad $(\mathbb{G}, \sigma, \pi)$ on $\mathrm{Pt}\mathbb{E}$ produces the category $Grd\mathbb{E} = \mathsf{Alg}\mathbb{G}$ of internal groupoids, but also it entirely rules the construction of $Cat\mathbb{E}$ and the previous inclusion.

9. Internal n-groupoids and n-categories

In this section, we shall show that the constructions and results of the previous section about the monad $(\mathbb{G}, \sigma, \pi)$ have a natural extension to the internal (strict) n-groupoids and n-categories.

9.1. The monad $(\mathbb{G}_F, \sigma_F, \pi_F)$. We shall first introduce a locating process for the monad $(\mathbb{G}, \sigma, \pi)$ up to a fibration. So let $F : \bar{\mathbb{E}} \longrightarrow \mathbb{E}$ be any fibration whose underlying functor is cartesian; we denote by F_W the fiber above $W \in \mathbb{E}$.

9.2. Lemma. *Given any fibration $F : \bar{\mathbb{E}} \longrightarrow \mathbb{E}$ whose underlying functor is cartesian, then any fiber F_W has pullbacks.*

Proof. Given pair (f, g) of maps with same codomain in the fiber $F_{F(Y)}$ and their pullback in $\bar{\mathbb{E}}$:

$$\begin{tikzcd}
P \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d, "f"] \\
Z \arrow[r, "g"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F(P) \arrow[r, "F(p_X)"] \arrow[d, "F(p_Y)"'] & F(X) \arrow[d, equal, "F(f)"] \\
F(Z) \arrow[r, equal, "F(g)"'] & F(Y)
\end{tikzcd}$$

we get $F(p_X) = F(p_Y) = \gamma$, where γ is an isomorphism in $\mathbb{E}$. Taking $\zeta : \bar{P} \longrightarrow P$ the cartesian map above γ^{-1} with domain P furnishes the pullback of the pair (f, g) inside the fiber $F_{F(Y)}$. ∎

Let us denote by $\mathrm{Pt}_F\bar{\mathbb{E}}$ the full subcategory of $\mathrm{Pt}\bar{\mathbb{E}}$ whose objects are the split epimorphisms in a fiber of F, it is obtained by the following left hand side pullback where $1_{\mathbb{E}}(W) = (1_W, 1_W)$:

$$\begin{tikzcd}
\mathrm{Pt}_F\bar{\mathbb{E}} \arrow[r, tail, "\iota_{\bar{\mathbb{E}}}"'] \arrow[d] \arrow[rr, bend left, "\P_F"] & \mathrm{Pt}\bar{\mathbb{E}} \arrow[r, "\P_{\bar{\mathbb{E}}}"'] \arrow[d, "\mathrm{Pt}F"] & \bar{\mathbb{E}} \\
\mathbb{E} \arrow[r, tail, "1_{\mathbb{E}}"'] & \mathrm{Pt}\mathbb{E} &
\end{tikzcd}$$

This makes $\mathrm{Pt}_F\bar{\mathbb{E}}$ a cartesian category and any functor in this diagram is a cartesian one. This produces, in addition, the upper horizontal cartesian functor $\P_F$ which becomes a subfibration of $\P_{\bar{\mathbb{E}}}$ by the following *specification of the base-change functor* ψ^*: starting with any split epimorphism $(f, s) : X \rightleftarrows Y$ in the fiber $F_{F(Y)}$, take the pullback in $\bar{\mathbb{E}}$ as on the left hand side which produces a cartesian map above ψ in $\mathrm{Pt}\bar{\mathbb{E}}$:

$$\begin{tikzcd}
\bar{X} \arrow[r, "\bar{\psi}"] \arrow[d, shift right, "\bar{f}"'] & X \arrow[d, shift right, "f"'] & & F(\bar{X}) \arrow[r, "F(\bar{\psi})"] \arrow[d, "F(\bar{f})"'] & F(X) \arrow[d, equal, "F(f)"] \\
Z \arrow[r, "\psi"'] \arrow[u, shift right, "\bar{s}"'] & Y \arrow[u, shift right, "s"'] & & F(Z) \arrow[r, "F(\psi)"'] & F(Y)
\end{tikzcd}$$

So, in its right hand side image by F in $\mathbb{E}$ which is a pullback, the map $F(\bar{f}) = \gamma$ is an isomorphism whose inverse is $F(\bar{s})$. Taking the cartesian isomorphism $\zeta : \check{X} \longrightarrow \bar{X}$ above γ^{-1} with codomain $\bar{X}$ produces the desired split epimorphism $(\bar{f}.\zeta, \zeta^{-1}.\bar{s}) : \check{X} \rightleftarrows Z$ in the fiber above $F(Z)$. *From now on we shall use the previous specification in the construction of the base-change functors* ψ^*.

We shall denote by $\P_F$ the class of the cartesian maps with respect to the fibration $\P_F$ (namely pullbacks between split epimorphisms belonging to a fiber) and, again, we shall call idomorphims the morphisms (y, x) in $\mathrm{Pt}_F\bar{\mathbb{E}}$ whose lower map y in $\bar{\mathbb{E}}$ is an identity map. Modulo the above precisions, *the monad* $(\mathbb{G}, \sigma, \pi)$ *on* $\mathrm{Pt}\bar{\mathbb{E}}$ *is stable on* $\mathrm{Pt}_F\bar{\mathbb{E}}$; for sake of clarity, we shall denote it by $(\mathbb{G}_F, \sigma_F, \pi_F)$.

9.3. Proposition. *The endofunctor* $\mathbb{G}_F$ *on* $\mathrm{Pt}_F\bar{\mathbb{E}}$ *is cartesian. It preserves and reflects the class* $\P_F$. *The monad* $(\mathbb{G}_F, \sigma_F, \pi_F)$ *is strongly* $\P_F$*-cartesian. Furthermore, given any object* (g, t) *in* $\mathrm{Pt}_F\bar{\mathbb{E}}$, *the following diagram is a kernel equivalence relation in* $\mathrm{Pt}\bar{\mathbb{E}}$ *with its (levelwise) quotient:*

$$\begin{tikzcd}
\mathbb{G}^3_F(g,t) \arrow[r, shift left=2, "\pi_F\mathbb{G}_F(g,t)"] \arrow[r, shift right=2, "\mathbb{G}_{F}\pi_{F}(g,t)"'] & \mathbb{G}^2_F(g,t) \arrow[l, "\mathbb{G}_F(\sigma_{F\mathbb{G}_F(g,t)})"'] \arrow[r, "\pi_{F(g,t)}"'] & \mathbb{G}_F(g,t)
\end{tikzcd}$$

Proof. It is just Proposition 5.6 restricted to the full subcategory $\mathrm{Pt}_F\bar{\mathbb{E}}$ of $\mathrm{Pt}\bar{\mathbb{E}}$ since the inclusion $\iota_{\bar{\mathbb{E}}}$ preserves the cartesian maps and the monad $(\mathbb{G}_F, \sigma_F, \pi_F)$ is just the restriction to $\mathrm{Pt}_F\bar{\mathbb{E}}$ of the monad $(\mathbb{G}, \sigma, \pi)$ on $\mathrm{Pt}\bar{\mathbb{E}}$. ■

9.4. Proposition. *Any algebra* $\alpha : \mathbb{G}_F(g, t) \to (g, t)$ *of this monad necessarily belongs to the class* $\P_F$. *The category of algebras of the monad* $(\mathbb{G}_F, \sigma_F, \pi_F)$ *on* $\mathrm{Pt}_F\bar{\mathbb{E}}$ *is the full subcategory* $Grd^F\bar{\mathbb{E}}$ *of* $Grd\bar{\mathbb{E}}$ *whose objects are the internal groupoids in the fibers of* F.

Proof. This time, it is just a restriction of Theorem 5.7 to the full subcategory $\mathrm{Pt}_F\bar{\mathbb{E}}$ of $\mathrm{Pt}\bar{\mathbb{E}}$. ■

We shall denote by $(\ )_0^F$ the diagonal functor of the following commutative square:

$$\begin{tikzcd}
\mathsf{Alg}\mathbb{G}_F = Grd^F\bar{\mathbb{E}} \arrow[r, tail] \arrow[d, "U^{\mathbb{G}_F}"'] \arrow[dr, dashed, "(\ )_0^F"] & Grd\bar{\mathbb{E}} \arrow[d, "(\ )_0"] \\
\mathrm{Pt}_F\bar{\mathbb{E}} \arrow[loop left, "\mathbb{G}_F"] \arrow[r, "\P_F"'] & \bar{\mathbb{E}}
\end{tikzcd}$$

9.5. Definition. *A functor* $F : \bar{\mathbb{E}} \longrightarrow \mathbb{E}$ *is called a fibered reflection when it is cartesian and is a fibration such that any fiber* F_W *of* F *has a terminal object* $T(W)$ *which is stable under any base-change functor.*

The easiest examples of fibered reflection are the fibrations $(\ )_0 : Cat\mathbb{E} \longrightarrow \mathbb{E}$ and $(\ )_0 : Grd\mathbb{E} \longrightarrow \mathbb{E}$, when $\mathbb{E}$ is a cartesian category with a terminal object; in both cases, the terminal object in the fiber above the object X in $\mathbb{E}$ being the undiscrete equivalence relation $\nabla_X = R[\tau_X]$, where $\tau_X : X \longrightarrow 1$ is the terminal map. We are going to show now that when F is a fibered reflection, so are the forgetful functors $(\ )_0^F : Grd^F\mathbb{E} \longrightarrow \mathbb{E}$ and $(\ )_0^F : Cat^F\mathbb{E} \longrightarrow \mathbb{E}$. First, the previous terminology comes from the following:

9.6. Proposition. *[Bourn, 1988] A functor* $F : \bar{\mathbb{E}} \longrightarrow \mathbb{E}$ *is a fibered reflection if and only if the following conditions hold:*
1) the functor F *is cartesian and has a right adjoint right inverse* $T : \mathbb{E} \longrightarrow \bar{\mathbb{E}}$ *such that the unit* $\eta_X : X \longrightarrow TF(X)$ *of this co-adjoint pair is such that* $F(\eta_X) = 1_{F(X)}$*;*
2) for any map $h : Z \longrightarrow F(X)$ *in* $\bar{\mathbb{E}}$*, there is a map* $\bar{h} : \bar{Z} \longrightarrow X$ *in* $\mathbb{E}$ *such that* $F(\bar{h}) = h$ *and the following square is a pullback:*

$$\begin{tikzcd}
\bar{Z} \arrow[r, "\bar{h}"] \arrow[d, "\eta_{\bar{Z}}"'] & X \arrow[d, "\eta_X"] \\
TF(\bar{Z}) = T(Z) \arrow[r, "TF(\bar{h})=T(h)"'] & TF(X)
\end{tikzcd}$$

Proof. Suppose F is a fibered reflection. Choose a terminal object $T(W)$ in the fiber F_W; this determines a right adjoint right inverse T of F with $\eta_X : X \longrightarrow TF(X)$ the terminal map in the fiber $F_{F(X)}$. Then a map $f : X \longrightarrow Y$ in $\bar{\mathbb{E}}$ is cartesian with respect to F if and only if the following left hand side square is a pullback in $\bar{\mathbb{E}}$, see Section 1 in [Bourn, 1988] for instance:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] & & \check{Z} \arrow[r, "\check{h}"] \arrow[d, "\check{\eta}"'] & X \arrow[d, "\eta_X"] \\
TF(X) \arrow[r, "TF(f)"'] & TF(Y) & & T(Z) \arrow[r, "T(h)"'] & TF(X)
\end{tikzcd}$$

Now starting with a map $h : Z \longrightarrow F(X)$ in $\bar{\mathbb{E}}$, take the above right hand side pullback in $\mathbb{E}$. Since F is cartesian, its image by F is a pullback in $\mathbb{E}$, and $F(\check{\eta})$ is an isomorphism.

Taking $\gamma : \bar{Z} \longrightarrow \check{Z}$ the invertible cartesian map above $F(\check{\eta})^{-1}$ with codomain $\check{Z}$, produces the desired map $\bar{h} = \check{h}.\gamma : \bar{Z} \longrightarrow X$ of condition 2).

Conversely, let F be a functor satisfying the two above conditions. Condition 1) implies that any map $f : X \longrightarrow Y$ making the above left hand side square a pullback is cartesian with respect to F, while Condition 2) guarantees the existence of a cartesian map above any map h. Then $\eta_X : X \longrightarrow TF(X)$ is necessarily the terminal map in the fiber $F_{F(X)}$. ■

9.7. Proposition. *When the fibration $F : \bar{\mathbb{E}} \longrightarrow \mathbb{E}$ is a fibered reflection, so is the forgetful functor $(\ )_0^F : Grd^F\bar{\mathbb{E}} \longrightarrow \bar{\mathbb{E}}$.*

Proof. The kernel equivalence relation $R[\eta_X]$ produces a groupoid $R_\bullet[\eta_X]$ in the fiber $F_{F(X)}$ (since $F(\eta_X) = 1_{F(X)}$) such that $R_0[\eta_X] = X$, and it is clearly a terminal object among the groupoids $Z_\bullet$ in the fiber $F_{F(X)}$ such that $Z_0 = X$.

Let us check that $(\ )_0^F$ is a fibration whose base-change functors preserve these terminal objects. So, let $X_\bullet$ be any internal groupoid in a fiber of F and $h : Z \longrightarrow X_0$ be any map in $\bar{\mathbb{E}}$; then consider the following left hand side pullback in the following left hand side diagram in $\bar{\mathbb{E}}$, where $\bar{\eta}_1 X_\bullet$ is the factorization of the pair $(d_0^{X_\bullet}, d_1^{X_\bullet})$:

$$\begin{tikzcd}
\bar{Z}_1 \arrow[r, "\bar{\eta}"] \arrow[d, "\bar{h}_1"'] & R[\eta_Z] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] \arrow[d, "R(h)"'] & Z \arrow[l] \arrow[r, "\eta_Z"] \arrow[d, "h"] & TF(Z) \arrow[d, "TF(h)"] & & \bar{Z}_\bullet \arrow[r, "\bar{\eta}_\bullet"] \arrow[d, "\bar{h}_\bullet"'] & R_\bullet[\eta_Z] \arrow[d, "R_\bullet(h)"] \\
X_1 \arrow[r, "\bar{\eta}_1 X_\bullet"'] & R[\eta_{X_0}] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & X_0 \arrow[l] \arrow[r, "\eta_{X_0}"'] & TF(X_0) & & X_\bullet \arrow[r, "\bar{\eta}_\bullet X_\bullet"'] & R_\bullet[\eta_{X_0}]
\end{tikzcd}$$

This produces an internal groupoid in $\bar{\mathbb{E}}$, since this pullback in $\bar{\mathbb{E}}$ is underlying the right hand side pullback in $Grd\bar{\mathbb{E}}$. The map $\bar{\eta}$ is not necessarily inside a fiber, but certainly $F(\bar{\eta})$ is an isomorphism since $F(\bar{\eta}_1 X_\bullet) = 1_{F(X_0)}$. Take the invertible cartesian map $\zeta_1 : Z_1 \longrightarrow \bar{Z}_1$ above $F(\bar{\eta})^{-1}$ with codomain $\bar{Z}_1$, then the associated internal groupoid $Z_\bullet$ (which is isomorphic to $\bar{Z}_\bullet$) belongs to $Grd_F\bar{\mathbb{E}}$. It is then straighforward to check that the internal functor $h_\bullet.\zeta_\bullet : Z_\bullet \longrightarrow X_\bullet$ is the desired cartesian map above h with respect to the functor: $(\ )_0^F : Grd^F\bar{\mathbb{E}} \longrightarrow \bar{\mathbb{E}}$. This construction makes $R_\bullet(h)$ a cartesian map above h, which means that the terminal object $R_\bullet[\eta_{X_0}]$ in the fiber of $(\ )_0^F$ is preserved by the base-change functor along h; in other words this means that the fibration $(\ )_0^F$ is a fibered reflection. ■

9.8. Theorem. *The full subcategory of $\mathbb{G}_F$-$Cat(\mathrm{Pt}_F\bar{\mathbb{E}})$ whose objects are the $\mathbb{G}_F$-categories with a $\P_F$-cartesian 0-leg and an idomorphic 1-leg is isomorphic to the full subcategory $Cat^F\bar{\mathbb{E}}$ of $Cat\bar{\mathbb{E}}$ whose objects are the internal categories in the fibers of F. The inclusion $\mathbb{G}\sharp\mathsf{C}_F : \mathsf{Alg}\mathbb{G}_F \rightarrowtail Cat^F\bar{\mathbb{E}}$ coincides with the following upper one:*

$$\begin{tikzcd}
Grd^F\bar{\mathbb{E}} \arrow[r, tail] \arrow[d, "(\ )_0^F"'] & Cat^F\bar{\mathbb{E}} \arrow[d, "(\ )_0^F"] \\
\bar{\mathbb{E}} \arrow[r, equal] & \bar{\mathbb{E}}
\end{tikzcd}$$

When F is a fibered reflection, so is $(\ )_0^F : Cat^F\bar{\mathbb{E}} \longrightarrow \bar{\mathbb{E}}$.

Proof. Once again, the first point is only a straighforward restriction of Theorem 7.5 to the full subcategory $\mathrm{Pt}_F\bar{\mathbb{E}}$ of $\mathrm{Pt}\bar{\mathbb{E}}$. And the second one holds since the previous proof for $Grd^F\bar{\mathbb{E}}$ is still valid for $Cat^F\bar{\mathbb{E}}$, which means, as expected, that the terminal objects in the fibers $Grd^F_X\bar{\mathbb{E}}$ and $Cat^F_X\bar{\mathbb{E}}$ are the same one. ■

9.9. When the fibration F has protomodular fibers. Recall from [Bourn, 1991], that *a category* $\mathbb{C}$ *is protomodular* when any base-change functor of the fibration $¶_{\mathbb{C}}$: $\mathrm{Pt}\mathbb{C} \longrightarrow \mathbb{C}$ is conservative, that any protomodular category is a Mal'tsev one in the sense of [Cardoni, Lambek and Pedicchio, 1991] (namely any reflexive relation in $\mathbb{C}$ is an equivalence relation), and that any internal category in a Mal'tsev category is an internal groupoid [Carboni, Pedicchio and Pirovano, 1992].

The easiest examples of protomodular category are the category Gp of groups and $\mathsf{Gp}\mathbb{E}$ of internal groups in $\mathbb{E}$ when $\mathbb{E}$ is cartesian. More generally any fiber of the fibration $(\ )_0 : Grd\mathbb{E} \longrightarrow \mathbb{E}$ is protomodular, again see [Bourn, 1991], and when $\mathbb{C}$ is protomodular, so is the category $Grd\mathbb{C}$. When the fibration F has protomodular fibers, then the extension determined by Theorem 9.8 does not produce anything new:

9.10. Proposition. *When the fibration F has protomodular fibers, the inclusion* $\mathbb{G}\sharp\mathsf{C}_F$: $\mathsf{Alg}\mathbb{G}_F = Grd^F\bar{\mathbb{E}} \rightarrowtail Cat^F\bar{\mathbb{E}}$ *is an isomorphism of categories, and the fibered reflection* $(\ )_0^F : Grd^F\bar{\mathbb{E}} \longrightarrow \bar{\mathbb{E}}$ *has protomodular fibers as well.*

Proof. Following what we just recalled, internal categories and internal groupoids do coincide inside the protomodular fibers of $F : \bar{\mathbb{E}} \longrightarrow \mathbb{E}$; whence the first point. Now, the fiber $(\ )_0^{F,X}$ of $(\ )_0^F$ above the object $X \in \bar{\mathbb{E}}$ is a cartesian subcategory of the category $Grd(F_{F(X)})$ which, as we just recalled above, is protomodular since so is $F_{F(X)}$. Accordingly, so is this fiber $(\ )_0^{F,X}$. ■

9.11. 2-categories and 2-groupoids. (Strict) 2-categories have been introduced by Benabou [Benabou, 1965] and Maranda [Maranda, 1965] as Cat-enriched categories. They are examples of double categories in the sense of Ehresmann [Ehresmann, 1965] as well, namely as special cases of internal categories. Internally speaking, the category 2-$Cat\mathbb{E}$ of internal 2-categories in $\mathbb{E}$ is nothing but the full subcategory of the category $Cat(Cat\mathbb{E})$ of double categories whose objects are the internal categories in the fibers of the fibration $(\ )_0 : Cat\mathbb{E} \longrightarrow \mathbb{E}$, see Section VI.2 in [Bourn, 1988]. So we are in the situation investigated in Section 9.1 with the fibration $F = (\ )_0$. This section will be devoted to the translation of the results of Section 9.1, and this will show how, again, the monad $(\mathbb{G}, \sigma, \pi)$ entirely rules the construction of the category 2-$Cat\mathbb{E}$. Similarly, the category 2-$Grd\mathbb{E}$ of internal 2-groupoids in $\mathbb{E}$ is nothing but the full subcategory of the category $Grd(Grd\mathbb{E})$ of double groupoids whose objects are the internal groupoids in the fibers of the fibration $(\ )_0$: $Grd\mathbb{E} \longrightarrow \mathbb{E}$.

Let us begin by the category 2-$Grd\mathbb{E}$. In this way, 2-$Grd\mathbb{E} = Grd^{(\ )_0}Grd\mathbb{E}$. When there is no ambiguity, a 2-groupoid will be denoted by the central part $X^2_\bullet$ of the internal groupoid defining it in a fiber of $(\ )_0 : Grd_E E \longrightarrow \mathbb{E}$:

$$\begin{tikzcd}
X^2_\bullet \times_0 X^2_\bullet \arrow[r, shift left=3, "d^{2,2}_\bullet"] \arrow[r, "d^{2,1}_\bullet" description] \arrow[r, shift right=3, "d^{2,0}_\bullet"'] & X^2_\bullet \arrow[r, shift left=3, "d^{2,1}_\bullet"] \arrow[r, shift right=3, "d^{2,0}_\bullet"'] & X^1_\bullet \arrow[l, "s^{2,0}_\bullet" description]
\end{tikzcd}$$

where the left hand side object is a pullback in this fiber. The internal groupoid $X^1_\bullet$ is called the groupoid of 1-morphisms, while the internal groupoid $X^2_\bullet$ is called the groupoid of 2-morphisms or 2-cells. Let us translate now the results of Section 9.1 with $F = (\ )_0 : Grd\mathbb{E} \longrightarrow \mathbb{E}$. For that and for sake of simplicity we shall denote:
1) by $\mathrm{Pt}_0 Grd\mathbb{E}$ the category $\mathrm{Pt}_{(\ )_0} Grd\mathbb{E}$ whose objects are the split epimorphisms between internal groupoids lying in a fiber of $(\ )_0$,
2) by $\P_0\mathbb{E}$ the fibration $\P_{(\ )_0} : \mathrm{Pt}_0 Grd\mathbb{E} = \mathrm{Pt}_{(\ )_0} Grd\mathbb{E} \longrightarrow Grd\mathbb{E}$ associating with any split epimorphism of this kind its codomain,
3) and by $(\mathbb{G}_1, \sigma_1, \pi_1)$ the monad $(\mathbb{G}_{(\ )_0}, \sigma_{(\ )_0}, \pi_{(\ )_0})$ on the category $\mathrm{Pt}_0 Grd\mathbb{E}$.
So we get:

9.12. PROPOSITION. *1) The category* $2\text{-}Grd\mathbb{E}$ *is isomorphic to* $\mathsf{Alg}\mathbb{G}_1$. *The forgetful functor* $(\ )_1 : 2\text{-}Grd\mathbb{E} \longrightarrow Grd\mathbb{E}$ *associating the groupoid* $X^1_\bullet$ *with the* 2*-groupoid* $X^2_\bullet$ *is a fibered reflection.*
2) The inclusion $\mathbb{G}\sharp\mathsf{C}_1 : \mathsf{Alg}\mathbb{G}_1 = 2\text{-}Grd\mathbb{E} \rightarrowtail Cat^{(\ )_0} Grd\mathbb{E}$ *is an isomorphism of categories and the fibered reflection* $(\ )_1 : 2\text{-}Grd\mathbb{E} \longrightarrow Grd\mathbb{E}$ *has protomodular fibers.*

PROOF. For the first point, just apply Propositions 9.3, 9.4 and 9.7 to the fibration $(\ )_0 : Grd\mathbb{E} \longrightarrow \mathbb{E}$. For the second one, apply Proposition 9.10. ■

Let us translate now the results of Section 9.1 related with $F = (\ )_0 : Cat\mathbb{E} \longrightarrow \mathbb{E}$. In this way, $2\text{-}Cat\mathbb{E} = Cat^{(\ )_0} Cat\mathbb{E}$. Again, when there is no ambiguity, a 2-category will be denoted by the central part $X^2_\bullet$ of the internal category producing it in a fiber of of $(\ )_0 : Cat\mathbb{E} \longrightarrow \mathbb{E}$:

$$\begin{tikzcd}
X^2_\bullet \times_0 X^2_\bullet \arrow[r, shift left=3, "d^{2,2}_\bullet"] \arrow[r, "d^{2,1}_\bullet" description] \arrow[r, shift right=3, "d^{2,0}_\bullet"'] & X^2_\bullet \arrow[r, shift left=3, "d^{2,1}_\bullet"] \arrow[r, shift right=3, "d^{2,0}_\bullet"'] & X^1_\bullet \arrow[l, "s^{2,0}_\bullet" description]
\end{tikzcd}$$

where the left hand side object is a pullback in this fiber of $(\ )_0 : Cat\mathbb{E} \longrightarrow \mathbb{E}$. The internal category $X^1_\bullet$ is called the category of 1-morphisms, while the internal category $X^2_\bullet$ is called the category of 2-morphisms or 2-cells. Again for sake of simplicity, we shall denote:
1) by $\mathrm{Pt}_0 Cat\mathbb{E}$ the category $\mathrm{Pt}_{(\ )_0} Cat\mathbb{E}$ whose objects are the split epimorphisms between internal categories lying in a fiber of $(\ )_0$,
2) by $\P^C_0\mathbb{E}$ the fibration $\P_{(\ )_0} : \mathrm{Pt}_0 Cat\mathbb{E} = \mathrm{Pt}_{(\ )_0} Cat\mathbb{E} \longrightarrow Cat\mathbb{E}$ associating with any split epimorphism of this kind its codomain,
3) and by $(\mathbb{G}^C_1, \sigma^C_1, \pi^C_1)$ the monad $(\mathbb{G}_{(\ )_0}, \sigma_{(\ )_0}, \pi_{(\ )_0})$ on the category $\mathrm{Pt}_0 Cat\mathbb{E}$.
Now translating the results of Section 9.1 we get:

9.13. THEOREM. *1) The category* $\mathsf{Alg}\mathbb{G}_1^C$ *is the full subcategory* 2_G-$Cat\mathbb{E}$ *of* $Grd(Cat\mathbb{E})$ *whose objects are the internal groupoids in the fibers of* $(\ )_0 : Cat\mathbb{E} \longrightarrow \mathbb{E}$*, namely the* 2*-categories with invertible* 2*-cells. The forgetful functor* $(\ )_1 : 2_G$-$Cat\mathbb{E} \longrightarrow Cat\mathbb{E}$ *associating the category* $X_\bullet^1$ *with the* 2*-category* $X_\bullet^2$ *is a fibered reflection.*
2) The full subcategory of $\mathbb{G}_1^C$-$Cat(\mathrm{Pt}_0 Cat\mathbb{E})$ *whose objects are the* $\mathbb{G}_1^C$*-categories with a* $\P_0^C\mathbb{E}$*-cartesian* 0*-leg and an idomorphic* 1*-leg is isomorphic to the category of internal categories in the fibers of* $(\ )_0$*, namely to the category* 2-$Cat\mathbb{E}$ *of internal* 2*-categories.*
3) The inclusion $\mathbb{G}\sharp\mathsf{C}_1 : \mathsf{Alg}\mathbb{G}_1^C \rightarrowtail 2$-$Cat\mathbb{E}$ *coincides with the following upper horizontal one in the following commutative diagram:*

$$\begin{tikzcd}
2_G{-}Cat\mathbb{E} \arrow[r, tail] \arrow[d, "(\ )_1"'] & 2{-}Cat\mathbb{E} \arrow[d, "(\ )_1"] \\
Cat\mathbb{E} \arrow[r, equal] & Cat\mathbb{E}
\end{tikzcd}$$

where the vertical functors are fibered reflections.

PROOF. Apply Proposition 9.4 and Theorem 9.8. ■

9.14. n-CATEGORIES AND n-GROUPOIDS. Internally speaking, the category $(n+1)$-$Cat\mathbb{E}$ of internal $(n+1)$-categories in $\mathbb{E}$ is defined by induction from the construction of the fibered reflection $(\ )_1 : 2$-$Cat\mathbb{E} \longrightarrow Cat\mathbb{E}$, see for instance [Bourn, 1990]. Suppose we have defined the fibered reflection $(\ )_{n-1} : n$-$Cat\mathbb{E} \longrightarrow (n$-$1)$-$Cat\mathbb{E}$. Then the category $(n+1)$-$Cat\mathbb{E}$ of $(n+1)$-categories is the full subcategory of $Cat(n$-$Cat\mathbb{E})$ whose objects are the internal categories in the fibers of $(\ )_{n-1}$. We are now in the situation investigated in Section 9.1 with $F = (\ )_{n-1}$. This section will be devoted to the translation of the results of Section 9.1, and this will show how, this time, the monad $(\mathbb{G}, \sigma, \pi)$ entirely rules the construction of the category $(n+1)$-$Cat\mathbb{E}$. Similarly, the category $(n+1)$-$Grd\mathbb{E}$ of internal $(n+1)$-groupoids in $\mathbb{E}$ is inductively defined as the full subcategory of the category $Grd(n$-$Grd\mathbb{E})$ whose objects are the internal groupoids in the fibers of the fibration $(\ )_{n-1} : n$-$Grd\mathbb{E} \longrightarrow (n-1)$-$Grd\mathbb{E}$.

Let us begin by the category $(n+1)$-$Grd\mathbb{E}$. When there is no ambiguity, a $(n+1)$-groupoid will be denoted by the central part $X_\bullet^{n+1}$ of the internal groupoid defining it in a fiber of $(\ )_{n-1} : n$-$Grd\mathbb{E} \longrightarrow (n-1)$-$Grd\mathbb{E}$:

$$\begin{tikzcd}
X_\bullet^{n+1} \times_{n_1} X_\bullet^{n+1} \arrow[r, shift left=2, "d_\bullet^{n+1,2}"] \arrow[r, "d_\bullet^{n+1,1}" description] \arrow[r, shift right=2, "d_\bullet^{n+1,0}"'] & X_\bullet^{n+1} \arrow[r, shift left=2, "d_\bullet^{n+1,1}"] \arrow[r, shift right=2, "d_\bullet^{n+1,0}"'] & X_\bullet^{n} \arrow[l, "s_\bullet^{n+1,0}" description]
\end{tikzcd}$$

where the left hand side object is a pullback in this fiber. The n-groupoid $X_\bullet^n$ is called the n-groupoid of n-morphisms, while the n-groupoid $X_\bullet^{n+1}$ is called the n-groupoid of $(n+1)$-morphisms or $(n+1)$-cells. Let us translate now the results of Section 9.1 with $F = (\ )_{n-1}$. For that and for sake of simplicity we shall denote:
1) by $\mathrm{Pt}_{n-1} n$-$Grd\mathbb{E}$ the category $\mathrm{Pt}_{(\ )_{n-1}} n$-$Grd\mathbb{E}$ whose objects are the split epimorphisms between internal n-groupoids lying in a fiber of $(\ )_{n-1}$,

2) by $\P_{n-1}\mathbb{E}$ the fibration $\P_{(\,)_{n-1}} : \mathrm{Pt}_{n-1}n\text{-}Grd\mathbb{E} \longrightarrow n\text{-}Grd\mathbb{E}$ associating with any split epimorphism of this kind its codomain,
3) and by $(\mathbb{G}_n, \sigma_n, \pi_n)$ the monad $(\mathbb{G}_{(\,)_{n-1}}, \sigma_{(\,)_{n-1}}, \pi_{(\,)_{n-1}})$ on the category $\mathrm{Pt}_{n-1}n\text{-}Grd\mathbb{E}$. Now translating the results of the Sections 9.1 and 9.9, we get:

9.15. Proposition. *1) The category $(n+1)\text{-}Grd\mathbb{E}$ of internal $(n+1)$-groupoids is isomorphic to $\mathsf{Alg}\mathbb{G}_n$. The forgetful functor $(\,)_n : (n+1)\text{-}Grd\mathbb{E} \longrightarrow n\text{-}Grd\mathbb{E}$ associating the n-groupoid $X^n_\bullet$ with the $(n+1)$-groupoid $X^{n+1}_\bullet$ is a fibered reflection.*
2) The inclusion: $\mathbb{G}\sharp\mathsf{C}_n : \mathsf{Alg}\mathbb{G}_n = (n+1){-}Grd\mathbb{E} \rightarrowtail Cat^{(\,)_{n-1}}n{-}Grd\mathbb{E}$ is an isomorphism of categories and the fibered reflection: $(\,)_n : (n+1)\text{-}Grd\mathbb{E} \longrightarrow n\text{-}Grd\mathbb{E}$ has protomodular fibers.

Proof. For the first point, apply Propositions 9.3, 9.4 and 9.7 to the fibered reflection $(\,)_{n-1} : n\text{-}Grd\mathbb{E} \longrightarrow (n\text{-}1)\text{-}Grd\mathbb{E}$. For the second one, apply Proposition 9.10. ■

Let us translate now the results of Section 9.1 related to the fibration $(\,)_{n-1} : n\text{-}Cat\mathbb{E} \longrightarrow (n-1)\text{-}Cat\mathbb{E}$. In this way, $(n+1)\text{-}Cat\mathbb{E} = Cat^{(\,)_{n-1}}n\text{-}Cat\mathbb{E}$. Again, when there is no ambiguity, a $(n+1)$-category will be denoted by the central part $X^{n+1}_\bullet$ of the internal category producing it in a fiber of $(\,)_{n-1} : n\text{-}Cat\mathbb{E} \longrightarrow (n-1)\text{-}Cat\mathbb{E}$:

$$\begin{tikzcd}
X^{n+1}_\bullet \times_{n-1} X^{n+1}_\bullet \arrow[r, shift left=2, "d^{n+1,2}_\bullet"] \arrow[r, "d^{n+1,1}_\bullet" description] \arrow[r, shift right=2, "d^{n+1,0}_\bullet"'] & X^{n+1}_\bullet \arrow[r, shift left=2, "d^{n+1,1}_\bullet"] \arrow[r, shift right=2, "d^{n+1,0}_\bullet"'] & X^n_\bullet \arrow[l, "s^{n+1,0}_\bullet" description]
\end{tikzcd}$$

where the left hand side object is a pullback in this fiber. The internal category $X^n_\bullet$ is called the category of n-morphisms, while the internal category $X^{n+1}_\bullet$ is called the category of $(n+1)$-morphisms or $(n+1)$-cells. Again for sake of simplicity, we shall denote:
1) by $\mathrm{Pt}_{n-1}Cat\mathbb{E}$ the category $\mathrm{Pt}_{(\,)_{n-1}}n\text{-}Cat\mathbb{E}$ whose objects are the split epimorphisms between internal n-categories lying in a fiber of $(\,)_{n-1}$,
2) by $\P^C_{n-1}\mathbb{E}$ the fibration $\P_{(\,)_{n-1}} : \mathrm{Pt}_{n-1}n\text{-}Cat\mathbb{E} \longrightarrow n\text{-}Cat\mathbb{E}$ associating with any split epimorphism of this kind its codomain,
3) and by $(\mathbb{G}^C_n, \sigma^C_n, \pi^C_n)$ the monad $(\mathbb{G}_{(\,)_{n-1}}, \sigma_{(\,)_{n-1}}, \pi_{(\,)_{n-1}})$ on the category $\mathrm{Pt}_{n-1}n\text{-}Cat\mathbb{E}$. Now translating the results of Section 9.1 we get:

9.16. Theorem. *1) The category $\mathsf{Alg}\mathbb{G}^C_n$ is the full subcategory $(n+1)_G\text{-}Cat\mathbb{E}$ of $(n+1)$-$Cat\mathbb{E}$ whose objects are the internal groupoids in the fibers of $(\,)_{n-1} : n\text{-}Cat\mathbb{E} \longrightarrow (n-1)$-$Cat\mathbb{E}$, namely the $(n+1)$-categories with invertible $(n+1)$-cells. The forgetful functor $(\,)_{n-1} : n\text{-}Cat\mathbb{E} \longrightarrow (n-1)\text{-}Cat\mathbb{E}$ associating the n-category $X^n_\bullet$ with the $(n+1)$-category $X^{n+1}_\bullet$ is a fibered reflection.*
2) The full subcategory of $\mathbb{G}^C_n\text{-}Cat(\mathrm{Pt}_{n-1}n\text{-}Cat\mathbb{E})$ whose objects are the $\mathbb{G}^C_n$-categories with a $\P^C_{n-1}\mathbb{E}$-cartesian 0-leg and an idomorphic 1-leg is isomorphic to the category of internal categories in the fibers of $(\,)_{n-1}$, namely to $(n+1)\text{-}Cat\mathbb{E}$.
3) The inclusion $\mathbb{G}\sharp\mathsf{C}^C_n : \mathsf{Alg}\mathbb{G}^C_n \rightarrowtail (n+1)\text{-}Cat\mathbb{E}$ coincides with the following upper

horizontal one in the following commutative diagram:

$$\begin{tikzcd}
(n+1)_G{-}Cat\mathbb{E} \arrow[r, tail] \arrow[d, "(\,)_n"'] & (n+1){-}Cat\mathbb{E} \arrow[d, "(\,)_n"] \\
n-Cat\mathbb{E} \arrow[r, equal] & n-Cat\mathbb{E}
\end{tikzcd}$$

where the vertical functors are fibered reflections.

Proof. Apply Proposition 9.4 and Theorem 9.8. ■

Accordingly, the construction of the following tower of fibered reflections is entirely ruled by the monad $(\mathbb{G}, \sigma, \pi)$ on Pt$\mathbb{E}$:

$$... \ n{-}Cat\mathbb{E} \overset{(\,)_{n-1}}{\to} (n-1){-}Cat\mathbb{E} \ ...... \ 2{-}Cat\mathbb{E} \overset{(\,)_1}{\to} Cat\mathbb{E} \overset{(\,)_0}{\to} \mathbb{E}$$

10. The $T_{X_\bullet}$-categories

10.1. The general case. In section 4.6 we observed that, when $X_\bullet$ is an internal category, the monad $(T_{X_\bullet}, \lambda_{X_\bullet}, \mu_{X_\bullet})$ on $\mathbb{E}/X_0$ is cartesian, and that the algebras of this monad coincide with the discrete fibrations above $X_\bullet$, so that $\mathsf{Alg}\, T_{X_\bullet} = DisF/X_\bullet$. We are now going to investigate what are that the $T_{X_\bullet}$-categories.

10.2. Proposition. *Given any internal category $X_\bullet$ in the category $\mathbb{E}$, then the category $T_{X_\bullet}$-$Cat(\mathbb{E}/X_0)$ is isomorphic to $Cat\mathbb{E}/X_\bullet$.*

Proof. A pointed $T_{X_\bullet}$-graph on an object $g_0 : Y_0 \longrightarrow X_0$ of $\mathbb{E}/X_0$ is given by a diagram of the following kind in $\mathbb{E}$, where $g_0.d_0^Y = \gamma = d_0.d_1^*(g_0).\bar{d}_1 = d_0.g_1$:

$$\begin{tikzcd}
 & Y_1 \arrow[dl, shift right, "d_0^Y"'] \arrow[dr, "\bar{d}_1=(d_1^Y\,,g_1)"] \arrow[ddl, "\gamma"] & \\
Y_0 \arrow[ur, shift right, "s_0^Y"'] \arrow[rr, "\sigma_0^{g_0}"] \arrow[d, "g_0"'] & & d_1^*(Y_0) \arrow[ll, dotted, bend right=15] \arrow[d, "d_1^*(g_0)"] \\
X_0 \arrow[rr, "s_0"] & & X_1 \arrow[ll, dotted, shift right=2, "d_1"'] \arrow[ll, shift left=2, "d_0"]
\end{tikzcd}$$

satisfying $(d_1^Y, g_1).s_0^Y = \sigma_0^{g_0} = (1_{Y_0}, s_0.g_0)$, namely $d_1^Y.s_0^Y = 1_{Y_0}$ and $g_1.s_0^Y = s_0.g_0$. Accordingly it is equivalent to a morphism of internal reflexif graphs in $\mathbb{E}$:

$$\begin{tikzcd}
Y_0 \arrow[r, "s_0^Y"] \arrow[d, "g_0"'] & Y_1 \arrow[l, shift right=3, "d_0^Y"'] \arrow[l, shift left=3, "d_1^Y"] \arrow[d, "g_1"] \\
X_0 \arrow[r, "s_0"] & X_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"]
\end{tikzcd}$$

We have to build now the pullback of $\bar{d}_1 = (d_1^Y, g_1)$ along $T_{X_\bullet}(d_0^Y)$, namely the pullback of d_1^Y along d_0^Y whose domain is denoted Y_2:

$$\begin{tikzcd}
Y_1 \arrow[r, shift left] \arrow[d, "d_0^Y"'] & d_1^*(Y_1) \arrow[l, shift left, "\delta_1^{g_0.d_0}"] \arrow[d, "T_{X_\bullet}(g_0)"] & Y_2 \arrow[l, "{(d_2^Y, g_1.d_0^Y)}"'] \arrow[d, "d_0^Y"] \\
Y_0 \arrow[r, tail, shift left, "\sigma_0^{g_0}"] \arrow[d, "g_0"'] & d_1^*(Y_0) \arrow[l, shift left, "\delta_1^{g_0}"] \arrow[d, "d_1^*(g_0)"] & Y_1 \arrow[l, "{(d_1^Y, g_1)}"] \\
X_0 \arrow[r, tail, shift left, "s_0"] & X_1 \arrow[l, shift left, "d_1"] &
\end{tikzcd}$$

This induces a map $g_2 : Y_2 \longrightarrow X_2$ such that $g_1.d_0^Y = d_0.g_2$ and $g_1.d_2^Y = d_2.g_2$. Accordingly, we get the following diagram where the two central "vertical" triangles commute and where $g_2 = d_2^*(g_1).\bar{d}_2$, with $\bar{d}_2 = (d_2^Y, g_1.d_0^Y)$:

$$\begin{tikzcd}
& & & & & Y_2 \arrow[dlll, dashed, bend right, "d_0^Y"'] \arrow[dlll, dotted, "d_1^Y"] \arrow[dl, dashed, "\bar{d}_2"] \\
& & Y_1 \arrow[dll, dashed, "d_0^Y"'] \arrow[dl, dashed, "\bar{d}_1"'] \arrow[ddl, "g_1"] & & d_2^*(Y_1) \arrow[ll, "\delta_1^{d_0.g_1}"] \arrow[dl, dashed, "T_{X_\bullet}(\bar{d}_1)"'] \arrow[ddl, "d_2^*(g_1)"] & \\
Y_0 \arrow[r, tail, shift left, "\sigma_0^{g_0}"] \arrow[d, "g_0"'] & d_1^*(Y_0) \arrow[l, shift left, "\delta_1^{g_0}"] \arrow[d, "d_1^*(g_0)"'] & & (d_1.d_2)^*(Y_0) \arrow[ll, shift right, "\delta_1^{g_0}"'] \arrow[ll, shift left, "\delta_2^{g_0}"] \arrow[d, "(d_1.d_2)^*(g_0)"'] & & \\
X_0 \arrow[r, tail, shift left, "s_0"] & X_1 \arrow[l, shift left, "d_1"] \arrow[d, "d_0"] & & X_2 \arrow[ll, shift right, "d_1"'] \arrow[ll, shift left, "d_2"] \arrow[d, "d_0"] & & \\
& X_0 & & X_1 \arrow[ll, shift right, "d_0"'] \arrow[ll, shift left, "d_1"] & &
\end{tikzcd}$$

The structure of $T_{X_\bullet}$-category on g_0 is then completed by the data of a map $d_1^Y : Y_2 \longrightarrow Y_1$ in $\mathbb{E}/X_0$ such that Burroni's Axioms 4, 7, 8 hold. The first part of Axioms 4 is $d_0^Y.d_1^Y = d_0^Y.d_0^Y$ (which implies that d_1^Y is a map in the slice category $\mathbb{E}/X_0$), while the second part is $\bar{d}_1.d_1^Y = \mu_{X_\bullet}(g_0).T_{X_\bullet}(\bar{d}_1).\bar{d}_2 = \delta_1^{g_0}.T_{X_\bullet}(\bar{d}_1).\bar{d}_2$. This second part is equivalent to $d_1^Y.d_1^Y = d_1^Y.d_2^Y$ and $g_1.d_1^Y = d_1.g_2$, which would complete the structure of an internal functor:

$$\begin{tikzcd}
Y_0 \arrow[r, tail, "s_0^Y"] \arrow[d, "g_0"'] & Y_1 \arrow[l, shift right=2, "d_0^Y"'] \arrow[l, shift left=2, "d_1^Y"] \arrow[d, "g_1"] & Y_2 \arrow[l, shift right=2, "d_0^Y"'] \arrow[l, "d_1^Y"] \arrow[l, shift left=2, "d_2^Y"] \arrow[d, "g_2"] \\
X_0 \arrow[r, tail, "s_0"] & X_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] & X_2 \arrow[l, shift right=2, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=2, "d_2"]
\end{tikzcd}$$

provided that neutrality and associativity of the composition map $d_1^Y : Y_2 \longrightarrow Y_1$ hold, which is straightforward with Axioms 7 and 8. ■

The inclusion $Cat(\mathbb{E}/X_0) = Cat\mathbb{E}/\Delta_{X_0} \rightarrowtail T_{X_\bullet}\text{-}Cat(\mathbb{E}/X_0) = Cat\mathbb{E}/X_\bullet$ is given by composition, in $Cat\mathbb{E}$, with the inclusion functor $\Delta_{X_0} \rightarrowtail X_\bullet$; while its coadjoint $\mathbb{R}$ (see Proposition 7.8) is obtained by the pullback in $Cat\mathbb{E}$ along this inclusion functor.

10.3. The $T_{X_\bullet}$-groupoids.

10.4. Proposition. *Given any internal category $X_\bullet$ in the category $\mathbb{E}$, a $T_{X_\bullet}$-groupoid is functor above $X_\bullet$ whose domain is a groupoid. Accordingly, the category $T_{X_\bullet}$-$Grd(\mathbb{E}/X_0)$ is given by the following pullback:*

$$\begin{tikzcd}
T_{X_\bullet}\text{--}Grd(\mathbb{E}/X_0) \arrow[r, tail] \arrow[d] & Cat\mathbb{E}/X_\bullet \arrow[d, "dom"] \\
Grd\mathbb{E} \arrow[r, tail] & Cat\mathbb{E}
\end{tikzcd}$$

Proof. According to Section 7.10, a $T_{X_\bullet}$-category gives rise to a $T_{X_\bullet}$-groupoid if and only if the map $d_1^Y : Y_2 \longrightarrow Y_1$ produces the following kernel equivalence relation:

$$\begin{tikzcd}
Y_0 & Y_1 \arrow[l, "d_0^Y"'] & Y_2 \arrow[l, shift right, "d_0^Y"'] \arrow[l, shift left, "d_1^Y"]
\end{tikzcd}$$

which is equivalent to the fact that $Y_\bullet$ is a groupoid. ■

10.5. The $T_{X_\bullet}$-categories when $X_\bullet$ is a groupoid. By Section 5.10, we know that:

$$\begin{tikzcd}
T_{X_\bullet} & T_{X_\bullet}^2 \arrow[l, two heads, "\mu_{X_\bullet}"'] & T_{X_\bullet}^3 \arrow[l, shift right, "\mu_{X_\bullet}T_{X_\bullet}"'] \arrow[l, shift left, "T_{X_\bullet}(\mu_{X_\bullet})"]
\end{tikzcd}$$

is a kernel equivalence relation if and only if $X_\bullet$ is a groupoid. In this case, by Corollary 7.13, any $T_{X_\bullet}$-algebra produces a $T_{X_\bullet}$-groupoid and we get the following string of inclusions:

$$\begin{tikzcd}
\mathsf{Alg}\, T_{X_\bullet} \arrow[r, tail, "\mathsf{T_{X_\bullet}\sharp C}"] \arrow[d, equal] & T_{X_\bullet}\text{--}Grd(\mathbb{E}/X_0) \arrow[r, tail] \arrow[d, equal] & T_{X_\bullet}\text{--}Cat(\mathbb{E}/X_0) \arrow[d, equal] \\
DFib/X_\bullet \arrow[r, tail] & Grd\mathbb{E}/X_\bullet \arrow[r, tail] & Cat\mathbb{E}/X_\bullet
\end{tikzcd}$$

11. T-operads and T-multicategories

About thirty years after Burroni's work [Burroni, 1971] (which was published in french), his ideas have been independantly rediscovered by Leinster [Leinster, 1998] and Hermida [Hermida, 2000]. According to the historical note, p. 63, of Leinster's encyclopedia about operads [Leinster, 2004], the notions of operad and multicategory gradually emerged from multiple horizons until they found a name, the first one in May [May, 1972] and the second one in Lambek [Lambek, 1969], before being completely stabilized. Finally, starting with $\mathbb{E} = Set$ and (M, λ, μ) the free monoid monad which is cartesian as we recalled above, *operads appeared to coincide with M-categories with only one object*, while *muticategories appeared to coincide with M-categories* [Leinster, 1998]. Then Leinster introduced the terminolgy T-operads and T-multicategory for the same notions related to any cartesian monad (T, λ, μ). So, in the cartesian context, T-multicategory in the sense of Leinster coincides with T-category in the sense of Burroni. See also [Hermida, 2000] and [Crutwell and Shulman, 2010].

So, given any cartesian monad (T, λ, μ) and following our results, and with respect to our notations related to the inclusion $\bar{F}^T : \mathbb{E} \rightarrowtail \mathsf{Kl}\,T$, a T-multicategory in $\mathbb{E}$ is nothing but an internal category in $\mathsf{Kl}\,T$:

$$\begin{tikzcd}
X_3 \arrow[r, dotted, shift left=8, "``\delta_3\text{''}"] \arrow[r, shift left=4, "d_2"] \arrow[r, shift right=4, "d_1"] \arrow[r, shift right=8, "d_0"'] & X_2 \arrow[l, "s_1"'] \arrow[r, dotted, shift left=6, "``\delta_2\text{''}"] \arrow[r, "d_1"] \arrow[r, shift right=6, "d_0"'] & X_1 \arrow[l, shift right=3, "s_1"'] \arrow[l, shift left=3, "s_0"] \arrow[r, dotted, shift left=4, "``\delta_1\text{''}"] \arrow[r, shift right=4, "d_0"'] & X_0 \arrow[l, "s_0"']
\end{tikzcd}$$

Warning: Leinster's designation of a T-operad in terms of "generalized monoid" could be a bit confusing, because, beyond the undisputable existence of a unit e and of an internal "operation" m, a T-operad is an actual internal category in $\mathsf{Kl}\,T$:

$$\begin{tikzcd}
X \times_{\delta_1} X \arrow[r, dotted, shift left=6, "``\delta_2\text{''}"] \arrow[r, "m"] \arrow[r, shift right=6, "\tau_0^X"'] & X \arrow[l, shift right=3, "s_1"'] \arrow[l, shift left=3, "s_0"] \arrow[r, dotted, shift left=4, "``\delta_1\text{''}"] \arrow[r, shift right=4, "\tau_X"'] & 1 \arrow[l, "e"']
\end{tikzcd}$$

since the object 1 does not stay a terminal object in $\mathsf{Kl}\,T$, unless $T(1) \simeq 1$, and consequently the map m is far from being a classical binary operation.

11.1. The cartesian monad $(T_{X_\bullet^T}, \lambda_{X_\bullet^T}, \mu_{X_\bullet^T})$. Given any cartesian monad (T, λ, μ) on $\mathbb{E}$ and any T-category $X_\bullet^T$, Leinster introduced in [Leinster, 1998] a notion of algebras associated with them. Indeed, on the model of Section 4.6, we get a cartesian monad $(T_{X_\bullet^T}, \lambda_{X_\bullet^T}, \mu_{X_\bullet^T})$ on the slice category $\mathbb{E}/X_0$:
1) we get a cartesian functor on $\mathbb{E}/X_0$ since, in the cartesian context, $\mathbb{E}$ becomes a pullback

stable subcategory of $\mathsf{Kl}\,T$:

$$\begin{tikzcd}
Z \arrow[r, tail, "\sigma_0^h"] \arrow[d, "h"'] & d_1^*(Z) \arrow[l, dotted, shift left=2, "{\text{“}\delta_1^h\text{”}}"] \arrow[d, "d_1^*(h)"] & (d_1.d_2)^*(Z) \arrow[l, "\delta_1^h"'] \arrow[l, dotted, shift left=2, "{\text{“}\delta_2^h\text{”}}"] \arrow[d, "(d_1.d_2)^*(h)"] \\
X_0 \arrow[r, tail, "s_0"] & X_1 \arrow[l, dotted, shift left=2, "{\text{“}\delta_1\text{”}}"] \arrow[d, "d_0"] & X_2 \arrow[l, "d_1"'] \arrow[l, dotted, shift left=2, "{\text{“}\delta_2\text{”}}"] \arrow[d, "d_0"] \\
 & X_0 & X_1 \arrow[l, "d_0"'] \arrow[l, dotted, shift left=2, "{\text{“}\delta_1\text{”}}"] \arrow[d, "d_0"] \\
 & & X_0
\end{tikzcd}$$

$$\begin{tikzcd}
h \arrow[r, tail, "\sigma_0^h"'] & T_{X_\bullet^T}(h) & T^2_{X_\bullet^T}(h) \arrow[l, "\delta_1^h"]
\end{tikzcd}$$

2) and, in addition, since the diagram $X_0 \overset{s_0}{\rightarrowtail} X_1 \overset{d_1}{\leftarrow} X_2$ lies in $\mathbb{E}$, so does: $Z \overset{\sigma_0^h}{\rightarrowtail} d_1^*(Z) \overset{\delta_1^h}{\leftarrow} (d_1.d_2)^*(Z)$; accordingly, this monad is entirely defined inside the slice category $\mathbb{E}/X_0$. So, in our terms, we get:

11.2. Proposition. *When (T, λ, μ) is a cartesian monad, the algebras of the monad $(T_{X_\bullet^T}, \lambda_{X_\bullet^T}, \mu_{X_\bullet^T})$ are the* discrete fibrations *above $X_\bullet^T$ in the category $Cat\mathsf{Kl}\,T$:*

$$\begin{tikzcd}
Z_0 \arrow[r, "s_0"] \arrow[d, "h_0"'] & Z_1 \arrow[l, bend right, "d_0"'] \arrow[l, dotted, bend left, "{\text{“}\delta_1\text{”}}"] \arrow[d, "h_1"] \\
X_0 \arrow[r, "s_0"] & X_1 \arrow[l, bend right, "d_0"'] \arrow[l, dotted, bend left, "{\text{“}\delta_1\text{”}}"]
\end{tikzcd}$$

in other words, we get $\mathsf{Alg}\,T_{X_\bullet^T} = DisF(T\text{-}Cat\mathbb{E}/X_\bullet^T)$.

Proof. Straightforward from the classical result on the monad $(T_{X_\bullet}, \lambda_{X_\bullet}, \mu_{X_\bullet})$ ■

11.3. The $T_{X_\bullet^T}$-categories. It remains to characterize the $T_{X_\bullet^T}$-categories. On the model of what happens for an ordinary internal category $X_\bullet$, we get:

11.4. Proposition. *The $T_{X_\bullet^T}$-categories are the T-functors above $X_\bullet^T$:*

$$\begin{tikzcd}
Y_0 \arrow[r, "s_0"] \arrow[d, "g_0"'] & Y_1 \arrow[l, bend right, "d_0"'] \arrow[l, dotted, bend left, "{\text{“}\delta_1\text{”}}"] \arrow[d, "g_1"] \\
X_0 \arrow[r, "s_0"] & X_1 \arrow[l, bend right, "d_0"'] \arrow[l, dotted, bend left, "{\text{“}\delta_1\text{”}}"]
\end{tikzcd}$$

in other words, we get $T_{X_\bullet^T}\text{-}Cat(\mathbb{E}/X_0) = T\text{-}Cat\mathbb{E}/X_\bullet^T$.

Proof. Let us follow step by step the proof of Proposition 10.2. A pointed $T_{X^T_\bullet}$-graph on an object $g_0 : Y_0 \longrightarrow X_0$ of $\mathbb{E}/X_0$ is given by a diagram of the following kind in $\mathsf{Kl}\,T$, where $g_0.d_0^Y = \gamma = d_0.d_1^*(g_0).\bar{d}_1 = d_0.g_1$ which implies $g_1 \in \mathbb{E}$:

$$\begin{tikzcd}
 & Y_1 \arrow[dl, shift right, "d_0^Y"'] \arrow[ddl, "\gamma"] \arrow[dr, "\bar{d}_1=(“\delta_1^Y”{,}g_1)"] & \\
Y_0 \arrow[ur, shift right, "s_0^Y"'] \arrow[rr, "\sigma_0^{g_0}"] \arrow[d, "g_0"'] & & d_1^*(Y_0) \arrow[ll, dotted, bend right] \arrow[d, "d_1^*(g_0)"] \\
X_0 \arrow[rr, "s_0"] & & X_1 \arrow[ll, dotted, bend right, "“\delta_1”"'] \arrow[ll, bend left, "d_0"]
\end{tikzcd}$$

satisfying $(“\delta_1^Y”, g_1).s_0^Y = \sigma_0^{g_0} = (1_{Y_0}, s_0.g_0)$, namely $“\delta_1^Y”.s_0^Y = 1_{Y_0}$ and $g_1.s_0^Y = s_0.g_0$. Accordingly it is equivalent to a morphism of pointed T-graphs:

$$\begin{tikzcd}
Y_0 \arrow[r, "s_0^Y"] \arrow[d, "g_0"'] & Y_1 \arrow[l, bend right, "d_0^Y"'] \arrow[l, dotted, bend left, "“\delta_1^Y”"] \arrow[d, "g_1"] \\
X_0 \arrow[r, "s_0"] & X_1 \arrow[l, bend right, "d_0"'] \arrow[l, dotted, bend left, "“\delta_1”"]
\end{tikzcd}$$

We have to build now the pullback of $T^T_{X_\bullet}(d_0^Y)$ along $\bar{d}_1 = (“\delta_1^Y”, g_1)$ in $\mathsf{Kl}\,T$, which is nothing but the pullback of d_0^Y along $“\delta_1^Y”$ in $\mathsf{Kl}\,T$, whose domain is denoted Y_2:

$$\begin{tikzcd}
Y_1 \arrow[r, shift left] \arrow[d, "d_0^Y"'] & d_1^*(Y_1) \arrow[l, dotted, shift left, "“\delta_1^{g_0.d_0}”"] \arrow[d, "T_{X^T_\bullet}(g_0)"] & Y_2 \arrow[l, "(“\delta_2^Y”{,}g_1.d_0^Y)"'] \arrow[d, "d_0^Y"] \\
Y_0 \arrow[r, shift left, "\sigma_0^{g_0}"] \arrow[d, "g_0"'] & d_1^*(Y_0) \arrow[l, dotted, shift left, "“\delta_1^{g_0}”"] \arrow[d, "d_1^*(g_0)"] & Y_1 \arrow[l, "(“\delta_1^Y”{,}g_1)"] \\
X_0 \arrow[r, shift left, "s_0"] & X_1 \arrow[l, dotted, shift left, "“\delta_1”"]
\end{tikzcd}$$

This induces a map $g_2 : Y_2 \longrightarrow X_2$ in $\mathsf{Kl}\,T$ such that $g_1.d_0^Y = d_0.g_2$ (which implies that g_2 belongs to $\mathbb{E}$) and $g_1.“\delta_2^Y” = “\delta_2”.g_2$. Accordingly, we get the following diagram in $\mathsf{Kl}\,T$ where the two central "vertical" triangles commute in $\mathbb{E}$ and where $g_2 = d_2^*(g_1).\bar{d}_2$, with $\bar{d}_2 = (“\delta_2^Y”, g_1.d_0^Y)$:

$$\begin{tikzcd}
 & & & & & Y_2 \arrow[lllld, shift left, "d_0^Y"'] \arrow[lllld, shift right, dashed, "d_1^Y"] \arrow[ld, "\bar{d}_2"] \\
 & & Y_1 \arrow[lld, "d_0^Y"'] \arrow[ld, "\bar{d}_1"'] \arrow[lldd, "g_1"] & & d_1^*(Y_1) \arrow[ll, "\delta_1^{d_0\cdot g_1}"] \arrow[ld, "T_{X_\bullet^T}(\bar{d}_1)"'] \arrow[ldd, "d_2^*(g_1)"] & \\
Y_0 \arrow[r, tail, shift left, "\sigma_0^{g_0}"] \arrow[d, "g_0"'] & d_1^*(Y_0) \arrow[l, dotted, shift left, "\text{“}\delta_1^{g_0}\text{”}"] \arrow[d, "d_1^*(g_0)"'] & & (d_1.d_2)^*(Y_0) \arrow[ll, shift right, "\delta_1^{g_0}"'] \arrow[ll, dotted, shift left, "\text{“}\delta_2^{g_0}\text{”}"] \arrow[d, "(d_1.d_2)^*(g_0)"'] & & \\
X_0 \arrow[r, tail, shift left, "s_0"] & X_1 \arrow[l, dotted, shift left, "\text{“}\delta_1\text{”}"] \arrow[d, "d_0"] & & X_2 \arrow[ll, shift right, "d_1"'] \arrow[ll, dotted, shift left, "\text{“}\delta_2\text{”}"] \arrow[d, "d_0"] & & \\
 & X_0 & & X_1 \arrow[ll, shift right, "d_0"'] \arrow[ll, dotted, shift left, "\text{“}\delta_1\text{”}"] & &
\end{tikzcd}$$

The structure of $T_{X_\bullet^T}$-category on g_0 is then completed by the data of a map $d_1^Y : Y_2 \longrightarrow Y_1$ in $\mathbb{E}/X_0$ (and thus in $\mathbb{E}$) such that Burroni's Axioms 4, 7, 8 hold. The first part of Axioms 4 is $d_0^Y.d_1^Y = d_0^Y.d_0^Y$, while the second part is $\bar{d}_1.d_1^Y = \mu_{X_\bullet^T}(g_0).T_{X_\bullet^T}(\bar{d}_1).\bar{d}_2 = \delta_1^{g_0}.T_{X_\bullet^T}(\bar{d}_1).\bar{d}_2$. This second part is equivalent to $\text{“}\delta_1^Y\text{”}.d_1^Y = \text{“}\delta_1^Y\text{”}.\text{“}\delta_2^Y\text{”}$ and $g_1.d_1^Y = d_1.g_2$, which would complete the structure of a T-functor:

$$\begin{tikzcd}
Y_0 \arrow[r, shift left, "s_0^Y"] \arrow[d, "g_0"'] & Y_1 \arrow[l, shift right=2, "d_0^Y"'] \arrow[l, dotted, shift left=2, "\text{“}\delta_1^Y\text{”}"] \arrow[d, "g_1"] & Y_2 \arrow[l, shift right=2, "d_0^Y"'] \arrow[l, "d_1^Y" description] \arrow[l, dotted, shift left=2, "\text{“}\delta_2^Y\text{”}"] \arrow[d, "g_2"] \\
X_0 \arrow[r, shift left, "s_0"] & X_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, dotted, shift left=2, "\text{“}\delta_1\text{”}"] & X_2 \arrow[l, shift right=2, "d_0"'] \arrow[l, "d_1" description] \arrow[l, dotted, shift left=2, "\text{“}\delta_2\text{”}"]
\end{tikzcd}$$

provided that neutrality and associativity of the composition map $d_1^Y : Y_2 \longrightarrow Y_1$ hold, which is straightforward with Axioms 7 and 8. ■

So, the canonical inclusion is the following one:

$$\mathsf{T}_{\mathsf{X}_\bullet^\mathsf{T}}\sharp\mathsf{C} : \mathsf{Alg}\, T_{X_\bullet^T} = DisF(T{-}Cat\mathbb{E}/X_\bullet^T) \rightarrowtail T{-}Cat\mathbb{E}/X_\bullet^T$$

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