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FORMAL CATEGORY THEORY IN AUGMENTED VIRTUAL DOUBLE CATEGORIES

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Abstract. In this article we develop formal category theory within augmented virtual double categories. Notably we formalise the classical notions of Kan extension, Yoneda embedding $\mathrm{y}_A\colon A \to \widehat{A}$, exact square, total category and 'small' cocompletion; the latter in an appropriate sense. Throughout we compare our formalisations to their corresponding 2-categorical counterparts. Our approach has several advantages. For instance, the structure of augmented virtual double categories naturally allows us to isolate conditions that ensure small cocompleteness of formal presheaf objects $\widehat{A}$.

Given a monoidal augmented virtual double category $\mathcal{K}$ with a Yoneda embedding $\mathrm{y}_I\colon I \to \widehat{I}$ for its monoidal unit I we prove that, for any 'unital' object A in $\mathcal{K}$ that has a 'horizontal dual' A°, the Yoneda embedding $\mathrm{y}_A\colon A \to \widehat{A}$ exists if and only if the 'inner hom' $[A^\circ, \widehat{I}]$ exists. This result is a special case of a more general result that, given a functor $F\colon \mathcal{K} \to \mathcal{L}$ of augmented virtual double categories, allows a Yoneda embedding in $\mathcal{L}$ to be "lifted", along a pair of 'universal morphisms' in $\mathcal{L}$, to a Yoneda embedding in $\mathcal{K}$.

Contents

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Introduction

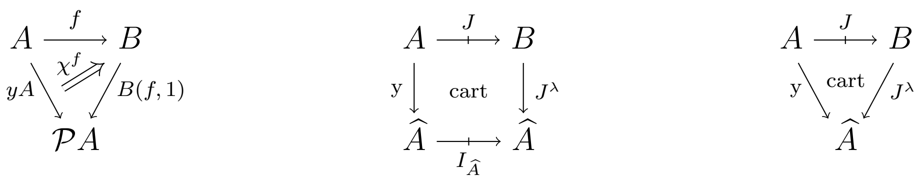
In this work we take a "double-dimensional" approach to formal category theory by taking augmented virtual double categories, which have been recently introduced in [Kou20], as a setting. The author's motivation for doing so is twofold. Firstly he considers double categorical structures to be a natural setting for the formalisation of classical categorical results that involve both profunctors and Yoneda embeddings. Consider for instance the classical result by Day ([Day70]) asserting that any promonoidal category A embeds into a monoidal category P. Denoting by T the 'free strict monoidal category' 2-monad, the promonoidal structure on A can be regarded as given by a profunctor $\alpha\colon A \nrightarrow TA$ satisfying certain conditions, while the Yoneda embedding $\mathrm{y}\colon A \to \mathsf{Set}^{A^{\mathrm{op}}} =: P$ underlies the promonoidal embedding $A \hookrightarrow P$, with the monoidal structure on P given by 'Day convolution'[1] with respect to α. Formalisation of Day's result potentially allows us to apply it to other category-like objects, such as posets, double categories and double 2-categories ([CLPS22]), that are equipped with promonoidal-like structures. Similarly it allows for generalisations to other 2-monads T, such as the ultrafilter monad on the 2-category of posets. In more detail, the latter generalisation isolates conditions on any 'modular topological space' A ([Tho09]), analogous to those satisfied by the profunctor α, ensuring that A embeds into an ordered compact Hausdorff space ([Tho09]).

Another relevant classical categorical result is Adámek and Rosický's Theorem 2.6 of [AR01]. Given a copresheaf $d\colon A \to \mathsf{Set}$ one of its assertions is that the left Kan extension $\mathrm{lan}_{\mathrm{y}}\, d\colon P \to \mathsf{Set}$ of d along the Yoneda embedding $\mathrm{y}\colon A \to P$ preserves finite products if and only if the category of elements $\int d$ is 'cosifted'. Writing S for the extension of the 'free category with finite products' 2-monad to profunctors, with unit transformation $\iota\colon \mathrm{id} \Rightarrow S$, and by regarding d as a profunctor $D\colon 1 \nrightarrow A$, the cosiftedness of $\int d$ can be equivalently expressed as a 'Beck-Chevalley'-like condition on the transformation of profunctors $\iota_D\colon D \Rightarrow SD$. In [Kou14b] this observation is used to formalise Adámek and Rosický's result in terms of any 'double monad', acting on some double category, whose vertical part is a colax-idempotent 2-monad; the latter in the sense of e.g. [KL97].

Before describing the second part of the author's motivation we pause to describe the main difference between our formal notion of Yoneda embedding (Definition 4.5 below) and the 2-categorical approach taken by Street and Walters in [SW78]. To do so we partly recall their notion of *Yoneda structure* on a 2-category $\mathcal{C}$, which consists of a 'right ideal' $\mathcal{A}$ of *admissible* morphisms in $\mathcal{C}$ and, for each admissible object A (that is $\mathrm{id}_A \in \mathcal{A}$), a formal Yoneda embedding $yA\colon A \to \mathcal{P}A$ that is itself admissible. The collection of these formal Yoneda embeddings is required to satisfy three axioms. We recall only Axiom 2

[1]For a formalisation, in augmented virtual double categories, of Day convolution when restricted to structure morphisms $A \nrightarrow TA$ that are representable, see Section 8 of [Kou15b].

here: for each admissible morphism $f\colon A \to B$ a cell χ^f as on the left below, which exhibits f as the absolute left lifting of yA through $B(f, 1)$ in $\mathcal{C}$, is required to exist. For the prototypical example of the classical Yoneda embeddings $yA\colon A \to \mathsf{Set}^{A^{\mathrm{op}}}$, one for each locally small category A, take admissible functors $f\colon A \to B$ to be those with all hom-sets $B(fa, b)$ small and set $B(f, 1)(b) := B(f-, b)$.

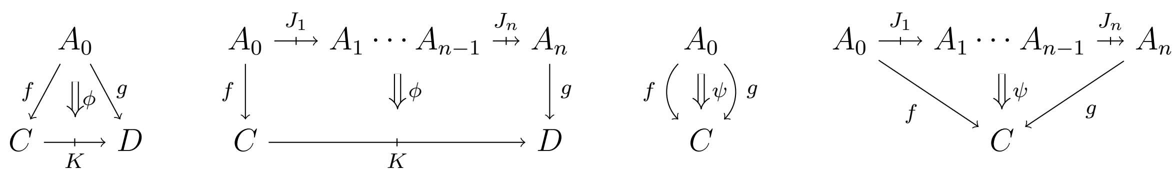


In our double-dimensional approach we take vertical morphisms $f\colon A \to C$ to represent abstract functors and horizontal morphisms $J\colon A \nrightarrow B$ to represent abstract profunctors. In contrast to Street and Walters' approach our formalisation of the Yoneda lemma does not require a notion of admissibility: instead we consider *all* horizontal morphisms to be admissible. For instance, to recover the classical Yoneda embeddings $\mathrm{y}\colon A \to \mathsf{Set}^{A^{\mathrm{op}}} =: \widehat{A}$, with A locally small and $\widehat{A}$ large in general (see [FS95]), we take the horizontal morphisms $J\colon A \nrightarrow B$ to be Set-profunctors $J\colon A^{\mathrm{op}} \times B \to \mathsf{Set}$: similar to the assignment $f \mapsto B(f, 1)$ for admissible functors f above, *every* such profunctor J induces a functor $J^\lambda\colon B \to \widehat{A}$ given by $J^\lambda(b) := J(-, b)$. Notice that a functor $f\colon A \to B$ is admissible precisely if it induces a representable Set-profunctor $f_*\colon A \nrightarrow B$ (the *companion* of f), and in that case $(f_*)^\lambda$ recovers $B(f, 1)$ above.

Regarding all horizontal morphisms as being admissible is the main feature of our approach, and we consider next the requirements that this imposes on our double-categorical setting. First notice that in the prototypical example, of Set-profunctors between large categories (such as $\widehat{A}$), we are not able to compose Set-profunctors in general. Thus, in general, we cannot require the vertical and horizontal morphisms to combine into a pseudo double category (see e.g. [GP99]), which is equipped with composition for both vertical and horizontal morphisms. Instead we require them to form a virtual double category (see e.g. [CS10] or the '**fc**-multicategories' of [Lei04]). This is a weaker structure that does not require horizontal composition; instead its cells are 'multicells' $\phi\colon (J_1, \dots, J_n) \Rightarrow K$, as on the left below, that have (possibly empty) paths as horizontal sources. Writing Set' for the category of large sets, the prototypical example is the virtual double category $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ of Set-profunctors between large categories (i.e. categories internal in Set').

Secondly notice that the classical Yoneda lemma supplies, for each Set-profunctor $J\colon A \nrightarrow B$, natural isomorphisms $J(a, b) \cong \widehat{A}(\mathrm{y}a, J^\lambda b)$. In the pseudo double category Set'-Prof of Set'-profunctors between large categories these isomorphisms combine into a *cartesian* cell of the form as in the middle above, where $I_{\widehat{A}}$ denotes the *horizontal unit* profunctor given by the hom-sets of $\widehat{A}$; for the universal properties of cartesian cells and horizontal units see e.g. Section 4 of [Kou20] or Definition 1.16 below. It is natural to axiomatise the Yoneda lemma as the requirement that this cartesian cell exists for every horizontal morphism $J\colon A \nrightarrow B$. However, since the unit profunctor $I_{\widehat{A}}$

is not a Set-profunctor in general, we cannot do so in the prototypical virtual double category $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$. We are thus led to the notion of augmented virtual double category [Kou20], which extends that of virtual double category by adding in *nullary* cells $\psi\colon (J_1, \dots, J_n) \Rightarrow C$ of the form as on the right below, with empty horizontal targets. The virtual double category $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ naturally extends to an augmented virtual double category whose nullary cells ψ below are transformations that map into the (possibly large) hom-sets of C. In particular we can, in the augmented virtual double category $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$, consider nullary cartesian cells as on the right above. Let now $\mathrm{y}\colon A \to \widehat{A}$ be any morphism in any augmented virtual double category: the *Yoneda axiom* of Definition 4.5 below requires that, for every horizontal morphism $J\colon A \nrightarrow B$, there exists a vertical morphism $J^\lambda\colon B \to \widehat{A}$ equipped with a nullary cartesian cell as on the right above.



Returning to the author's motivation for this article, its second part is to contribute to formal approaches to higher dimensional category theory, as follows. In Chapter 9 of [RV22] Riehl and Verity employ formal category theory in virtual double categories to define pointwise Kan extensions of functors between ∞-categories; in fact their definition is recovered by one of the notions of Kan extension that we consider (see Example 1.10 below). Their formal approach does not however include a formal notion of Yoneda embedding, and the author believes that the theory of the present paper is likely to be of help towards obtaining such a notion, as is explained shortly. In the case of double categories, Grandis and Paré in [GP07] introduce pointwise Kan extensions of lax double functors between pseudo double categories, as an instance of their formal notion of pointwise Kan extension in pseudo double categories [GP08]. It is currently unclear to the author whether the former notion can be reconciled with the notions considered in the present article. The author is aware of three approaches to a notion of Yoneda embedding for double categories: the original approach by Paré in [Par11]; Street's formal approach for strict double categories [Str17], which uses the main result of [Web07]; and a formal approach using "generalised Day convolution", by applying the formalisation of Day's result, as described previously, to the 'free strict double category' 2-monad. The precise relationship between these three approaches is currently unclear to the author; in particular he does not know if Paré's Yoneda embeddings satisfy any of the formal notions of Yoneda embedding.

With the main aim of this work being the formalisation of category theory, notably that of the notions of Kan extension and Yoneda embedding, in augmented virtual double categories, our second aim is to provide necessary and sufficient conditions for the existence of formal Yoneda embeddings. This gives us a handle on the points raised in the preceding paragraph: the sufficent condition allows us to obtain formal Yoneda embeddings (such as for ∞-categories) while, given a family of "ad hoc" Yoneda embeddings y_A (such as Paré's Yoneda embeddings for double categories), the necessary condition facilitates constructing an augmented virtual double category in which the y_A satisfy our formal notion of Yoneda embedding. The aforementioned conditions generalise as well as recover the following fact for finitely complete categories $\mathcal{E}$ with subobject classifier Ω: $\mathcal{E}$ has power objects if and only if Ω is exponentiable; see e.g. Section A2.1 of [Joh02]. In some more detail, they apply to a monoidal augmented virtual double category $(\mathcal{K}, \otimes, I)$ whose monoidal unit I admits a Yoneda embedding $y_I \colon I \to \widehat{I}$, as follows. Given any *unital* object A in $\mathcal{K}$, i.e. A admits a horizontal unit (Definition 1.16), we prove in Theorem 8.21 below that, under mild conditions, the Yoneda embedding $y_A \colon A \to \widehat{A}$ exists if and only if the 'inner hom' $[A^\circ, \widehat{I}]$ does, with A° the unital 'horizontal dual' of A (formalising the notion of dual category), and in that case $\widehat{A} \cong [A^\circ, \widehat{I}]$.

The horizontal dual A° here is defined by a 'horizontal copairing' $\iota \colon I \nrightarrow A^\circ \otimes A$ (Definition 8.11), which induces an assignment that maps every horizontal morphism $J \colon A \nrightarrow B$ to its 'adjunct' $J^\flat \colon I \nrightarrow A^\circ \otimes B$. Theorem 8.21 also applies in the cases where the assignment $J \mapsto J^\flat$ is not essentially surjective onto the collection of morphisms of the form $I \nrightarrow A^\circ \otimes B$, e.g. in the case of *small profunctors* between large categories in the sense of [DL07]; see also Example 2.8 of [Kou20]. In such cases the universal property of the inner-hom $[A^\circ, \widehat{I}]$ is restricted to morphisms $A^\circ \otimes B \to \widehat{I}$ in the essential image of the composite below, where $J \mapsto J^\lambda$ is given by the formal Yoneda axiom as described previously; see Definition 8.14 for the details.

$$\{A \nrightarrow B\} \xrightarrow{(-)^\flat} \{I \nrightarrow A^\circ \otimes B\} \xrightarrow{(-)^\lambda} \{A^\circ \otimes B \to \widehat{I}\}$$

Moreover Theorem 8.21 is obtained as a corollary of the following more general result, which combines the main results of Section 8: Theorems 8.33 and 8.36. Given a functor $F \colon \mathcal{K} \to \mathcal{L}$ of augmented virtual double categories and, in $\mathcal{L}$, a Yoneda embedding $y_A \colon A \to P$ and a 'universal morphism' $\iota \colon A \nrightarrow FA'$ (generalising the notion of copairing), under mild conditions these theorems show that a Yoneda embedding $y_{A'} \colon A' \to P'$ exists in $\mathcal{K}$ if and only if there exists a 'universal morphism' $\varepsilon \colon FP' \to P$ whose universal property is "restricted" like that of $[A^\circ, \widehat{I}]$ above. The author believes that these results will be useful in obtaining formal Yoneda embeddings.

The third aim of this work is to compare, throughout, its double-dimensional approach to formal category to the classical 2-categorical approaches that use Yoneda structures, as taken by Street and Walters in [SW78] and by Weber in [Web07]. We close this introduction by outlining one of the advantages of our approach: it allows for isolating conditions that ensure the 'small cocompleteness' of formal presheaf objects $\widehat{A}$, as follows. Weber in Definition 3.17 of [Web07] defines an object C, of a 2-category equipped with a 'good Yoneda structure', to be 'cocomplete' whenever it admits pointwise left Kan extensions of all diagrams $C \xleftarrow{d} X \xrightarrow{h} Z$ such that X, the presheaf object $\widehat{X}$, Z and h are admissible. All of our notions of left Kan extension, including the pointwise variant, that are defined in Section 1 below and used throughout this work, are along a (path of) horizontal morphism(s); see also the Overview below. Thus, as a consequence of our viewpoint of all horizontal morphisms being 'admissible', we regard our notions of left Kan extension as being along 'admissible' morphisms only. Analogous to Weber's definition we define an object M of an augmented virtual double category to be *cocomplete* (Definition 7.2) whenever it admits pointwise Kan extensions (Definition 1.24) of all (or some pre-specified class of) diagrams $M \xleftarrow{d} A \xrightarrow{J} B$.

Consider a formal Yoneda embedding $\mathrm{y}\colon M \to \widehat{M}$ in an augmented virtual double category $\mathcal{K}$. An advantage of our approach is that the existence of pointwise Kan extensions into $\widehat{M}$ is related to the existence of 'pointwise horizontal composites' in $\mathcal{K}$, and this can be used to obtain a condition that ensures cocompleteness of $\widehat{M}$. In some more detail, *pointwise composites* of horizontal morphisms are defined by 'pointwise cocartesian cells' (see Definition 9.1 of [Kou20] or Remark 2.13 below), and in Definition 2.12 below the latter notion is weakened in two ways, resulting in that of 'pointwise right unary-cocartesian cell'. Given a diagram $\widehat{M} \xleftarrow{d} A \xrightarrow{J} B$ it is shown in Corollary 5.7 below that the pointwise left Kan extension of d along J exists if and only if there exists a pointwise right unary-cocartesian cell of the form

$$\begin{array}{ccccc} M & \xrightarrow{\widehat{M}(\mathrm{y},d)} & A & \xrightarrow{J} & B \\ \| & & \Downarrow & & \| \\ M & & \xrightarrow[K]{\qquad\qquad} & & B, \end{array}$$

where $\widehat{M}(\mathrm{y}, d)$ denotes the *restriction* of $\widehat{M}$ along y and d (Definition 1.16) and where K is any horizontal morphism. Using this result Theorem 7.6, the main theorem of Section 7, isolates conditions that ensure that $\mathrm{y}\colon M \to \widehat{M}$ defines $\widehat{M}$ as the 'free cocompletion' of M, in the sense of Definition 7.2 (see also the Overview below); in particular they ensure that $\widehat{M}$ is cocomplete. In contrast, consider the analogous Theorem 3.20 of [Web07] which concerns a Yoneda embedding $y\colon C \to \widehat{C}$ of a good Yoneda structure. Like our Theorem 7.6 it proves that y defines $\widehat{C}$ as the free cocompletion of C, but it does not isolate conditions ensuring cocompleteness of $\widehat{C}$; it instead assumes cocompleteness of $\widehat{C}$. Further differences between Weber's result and our Theorem 7.6 are described in Remark 7.7. Given a monoidal augmented virtual double category $(\mathcal{K}, \otimes, I)$, Theorem 8.9 uses Theorem 7.6 to describe conditions ensuring that the Yoneda embedding $\mathrm{y}\colon I \to \widehat{I}$ defines $\widehat{I}$ as the 'free cocompletion' of the monoidal unit I.

Overview

We start in Section 1 by introducing four notions of left Kan extension in an augmented virtual double category $\mathcal{K}$: a notion of *weak Kan extension* (Definition 1.2); a notion of *Kan extension* (Definition 1.9), which formalises enriched Kan extension (Example 1.13); a notion of *pointwise weak Kan extension* (Definition 1.24), which is reminiscent of Street's

2-categorical notion of pointwise Kan extension [Str74b]; and a notion of *pointwise Kan extension* (Definition 1.24) which combines the latter two notions. Each of these defines the extension of a vertical morphism $d\colon A_0 \to M$ along a path $\underline{J}\colon A_0 \nrightarrow A_n$ of horizontal morphisms, with the resulting left Kan extension $l\colon A_n \to M$ being exhibited by a nullary cell

$$\begin{array}{ccc} A_0 \xrightarrow{J_1} A_1 \cdots A_{n-1} \xrightarrow{J_n} A_n \\ {\scriptstyle d}\searrow \quad \Downarrow\eta \quad \swarrow{\scriptstyle l} \\ M; \end{array}$$

such cells we call *(pointwise) (weak) left Kan*. Weakly left Kan extending along companions in $\mathcal{K}$ recovers the classical 2-categorical notion of left Kan extension in the vertical 2-category $V(\mathcal{K})$ of objects, vertical morphisms and 'vertical cells' of $\mathcal{K}$ (Proposition 1.7). If $\mathcal{K}$ admits all horizontal units and restrictions on the right then the notions of left Kan extension and pointwise left Kan extension coincide (Remark 1.26). Using the results of [Kou18], in Example 1.33 we construct pointwise left Kan extensions of morphisms of modular closure spaces [Tho09]. Given a functor $F\colon \mathcal{K} \to \mathcal{L}$ of augmented virtual double categories the notion of a universal morphism $\varepsilon\colon FC' \to C$, from F to an object $C \in \mathcal{L}$, is defined in Definition 1.39, analogously to the classical notion. Proposition 1.42 shows that if a nullary cell of the form $\phi\colon F\underline{J} \Rightarrow C$ defines a left Kan extension in $\mathcal{L}$ then so does its "ε-adjunct" $\phi^\sharp\colon \underline{J} \Rightarrow C'$ in $\mathcal{K}$.

In Section 2 we prove two pasting lemmas for left Kan cells that are used throughout this work. The horizontal pasting lemma (Lemma 2.2) concerns the horizontal composite of two left Kan cells. It recovers the classical result for enriched iterated Kan extensions, and it forms the main reason for our choice of Kan extending along *paths* of horizontal morphisms: without doing so the horizontal pasting lemma cannot be stated (Remark 2.4). The vertical pasting lemma (Lemma 2.17) concerns the vertical composite $\eta \circ \underline{\phi}$ of a left Kan cell η and a 'cocartesian path of cells' $\underline{\phi}$. In fact considering the weakest requirements on the path $\underline{\phi}$ such that, for each left Kan cell η composable with $\underline{\phi}$, the composite $\eta \circ \underline{\phi}$ is again left Kan, leads to the weakened notion of *right nullary-cocartesian* path (Definition 2.6). The vertical pasting lemma for this weakened notion of cocartesian path is used throughout. The remainder of Section 2 consists of consequences of the pasting lemmas. Given a path $(J_1, \ldots, J_n)\colon A_0 \nrightarrow A_n$ and a 'full and faithful morphism' $f\colon A_n \to B$, Proposition 2.26 for instance shows that if $l\colon B \to M$ is the pointwise left Kan extension of some $d\colon A_0 \to M$ along the concatenation $(J_1, \ldots, J_n, f_*)\colon A_0 \nrightarrow B$, where $f_*\colon A_n \nrightarrow B$ is the companion of f, then $l \circ f$ forms the left Kan extension of d along $\underline{J}$; this generalises the classical result on (enriched) left Kan extending along a full and faithful functor.

The main theorem of Section 3, Theorem 3.20, shows that the notions of pointwise weak left Kan extension and pointwise left Kan extension coincide in augmented virtual double categories $\mathcal{K}$ that have all restrictions on the right as well as all 'cocartesian tabulations'. The notion of tabulation (Definition 3.5) formalises that of graph of a functor. Proposition 3.22 then shows that pointwise left Kan extension along companions in such $\mathcal{K}$ coincides with pointwise left Kan extension in the vertical 2-category $V(\mathcal{K})$, the latter in the classical sense of [Str74b].

Using the notion of (weak) left Kan extension Section 4 starts by introducing the notions of *density* and *weak density* for vertical morphisms (Definition 4.3). Definition 4.5 then defines a *(weak) Yoneda morphism* $\mathrm{y}\colon A \to \widehat{A}$ to be a (weakly) dense morphism that satisfies the Yoneda axiom, as described in the Introduction above. These conditions on y do not imply that it is full and faithful, which instead is a consequence of the existence of the horizontal unit $I_A\colon A \nrightarrow A$ (Lemma 4.6); a full and faithful y is called a *(weak) Yoneda embedding*. Several of our results do not depend on the full and faithfulness of Yoneda morphisms (Remark 4.7). A Yoneda morphism y such that all restrictions $\widehat{A}(\mathrm{y}, f)$ exist, for any $f\colon B \to \widehat{A}$, induces, for every object B, an equivalence between the category of horizontal morphisms $A \nrightarrow B$ and that of vertical morphisms $B \to \widehat{A}$ (Proposition 4.24). Our notion of Yoneda embedding recovers that of enriched Yoneda embedding, that of enriched Yoneda embedding for small enriched presheaves in the sense of [DL07], that of power object in a finitely complete category, in the sense of Section A2.1 of [Joh02], and that of upper Vietoris space of downsets in a closed-ordered closure space, the latter in the sense of [Tho09]; see Examples 4.9–4.17 and Example 4.33. The Yoneda embeddings of the good Yoneda structure associated to a 2-topos [Web07] are instances of our notion too (Example 4.30). Given an augmented virtual double category $\mathcal{K}$, with vertical 2-category $V(\mathcal{K})$, in Theorem 4.35 we compare our notion of Yoneda embedding in $\mathcal{K}$ to the notion of Yoneda structure on $V(\mathcal{K})$ ([SW78]) and to the notion of good Yoneda structure on $V(\mathcal{K})$ ([Web07]).

In Section 5 the classical notion of exact square of functors, as considered by e.g. Guitart [Gui80], is formalised as follows. Given a cell ϕ with horizontal target $K\colon C \nrightarrow D$ and a morphism $d\colon C \to M$ we call ϕ *left d-exact* if, for every left Kan cell η defining the left Kan extension of d along K, the composite $\eta \circ \phi$ is again left Kan (Definition 5.2). In the presence of a Yoneda morphism $\mathrm{y}\colon C \to \widehat{C}$ this notion relates to that of cocartesianness and that of cocompleteness of $\widehat{C}$ as follows. If ϕ has the identity morphism id_D as vertical target then it is left y-exact if and only if it is right unary-cocartesian (Proposition 5.6). Moreover the left Kan extension of any morphism $d\colon A_0 \to \widehat{C}$ along a path $(J_1, \dots, J_n)\colon A_0 \nrightarrow A_n$ exists if and only if there exists a horizontal left y-exact cell with the concatenation $(\widehat{C}(\mathrm{y}, d), J_1, \dots, J_n)\colon C \nrightarrow A_n$ as horizontal source (Proposition 5.5). Theorem 5.14 describes left exactness in terms of a *Beck-Chevalley condition* (Definition 5.10); the latter condition, in turn, is used in Theorem 5.16 to characterise *absolute* left Kan extensions (Definition 1.36).

In Definition 6.2 a morphism $f\colon M \to N$ is defined to be *total* if the pointwise left Kan extension of f along every $J\colon M \nrightarrow B$ exists. An object M is total if its identity morphism id_M is total; this recovers the classical notion of totality for enriched categories ([DS86] and [Kel86]). Given a Yoneda morphism $\mathrm{y}\colon M \to \widehat{M}$ consider the morphism $f_*^\lambda\colon N \to \widehat{M}$ corresponding to the companion $f_*\colon M \nrightarrow N$, as given by the Yoneda axiom: Theorem 6.8 shows that the totality of f is equivalent to the existence of a left adjoint to f_*^λ. The latter condition is analogous to that of the classical 2-categorical definition of totality introduced in [SW78] (Example 6.9). Any presheaf object $\widehat{M}$ is total whenever the companion $y_*\colon M \nrightarrow \widehat{M}$ of its Yoneda morphism exists (Example 6.3) (the analogous result Corollary 14 of [SW78] requires both M and $\widehat{M}$ to be admissible). Under mild conditions any morphism $f\colon A \to C$ induces a morphism $\widehat{f}\colon \widehat{C} \to \widehat{A}$ of presheaf objects (Definition 6.13); this formalises the classical functor $\widehat{f}$ given by restricting presheaves along f. Proposition 6.15 describes the relation between the assignments $f \mapsto \widehat{f}$ and $J \mapsto J^\lambda$, the latter given by the Yoneda axiom; in Corollary 6.16 this is used to describe the uniqueness of Yoneda embeddings. Using the notion of totality, Corollaries 6.18 and 6.20 describe the right and left adjoints of $\widehat{f}$.

The aim of Section 7 is to isolate conditions ensuring that a Yoneda embedding $y\colon M \to \widehat{M}$ defines $\widehat{M}$ as the free 'small' cocompletion of M. The appropriate notion of 'smallness' here depends on the augmented virtual double category under consideration: while small cocompleteness in $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ most naturally means "admits all pointwise left Kan extensions along Set-profunctors $J\colon A \nrightarrow B$ with A a small category" (Example 7.9), in the pseudo double category $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{sProf}$ of small Set-profunctors between large categories the notion of "admitting pointwise left Kan extensions along *all* small Set-profunctors" turns out to be more appropriate (Example 7.11). This is why in Definition 7.2 we assume specified an 'ideal' $\mathcal{S}$ of left diagrams (d, J), consisting of pairs of morphisms $M \xleftarrow{d} A \overset{J}{\nrightarrow} B$, and then define an object N to be *$\mathcal{S}$-cocomplete* whenever, for every $(d, J) \in \mathcal{S}$ such that d has N as target, the pointwise left Kan extension of d along J exists. Given an ideal $\mathcal{S}$ of left diagrams, the main result Theorem 7.6 uses the notion of pointwise right unary-cocartesian cell to give conditions that ensure that a Yoneda embedding $y\colon M \to \widehat{M}$ defines $\widehat{M}$ as the free $\mathcal{S}$-cocompletion of M, as described previously at the end of the Introduction.

The main results of the final section (Section 8) have already been described in the Introduction. Its Theorem 8.21 is used in obtaining some of the examples of Yoneda embedding in Section 4. Section 8 depends on Sections 1–4 only, except for Theorem 8.9 which depends on Section 7.

References to the prequel. This work is a sequel to the paper [Kou20], which introduces the notion of augmented virtual double category. The results of the latter are used throughout this work. To save space we, when referring to such results, do not cite [Kou20] but instead refer to them by prefixing their numbering with the capital letter 'A'; e.g. "Definition A1.2" and "Lemma A8.1" in this text refer to Definition 1.2 and Lemma 8.1 of [Kou20]. References to sections of the prequel use the same prefix, e.g. "Section A7" refers to Section 7 of [Kou20].

Like the prequel this work is based on parts of the draft [Kou15b], specifically its Sections 4 and 5. The material presented here is significantly more streamlined and expanded in many ways; in particular the material of the present Section 5 is new. The author encourages readers to consult the present article rather than the corresponding draft material of [Kou15b].

NOTATION. We continue using the notation of [Kou20]. In particular:

- for any integer $n \geq 1$ we write $n' \coloneqq n - 1$ for its predecessor;
- given composable paths $\underline{J} = (J_1, \dots, J_n)$ and $\underline{H} = (H_1, \dots, H_m)$ of horizontal morphisms we write $\underline{J} \frown \underline{H} \coloneqq (J_1, \dots, J_n, H_1, \dots, H_m)$ for their concatenation;
- for a path $\underline{J} = (J_1, \dots, J_n)$ of horizontal morphisms we write $\mathrm{id}_{\underline{J}} \coloneqq (\mathrm{id}_{J_1}, \dots, \mathrm{id}_{J_n})$ for the corresponding path of identity cells;
- most cartesian and cocartesian cells (Definition A4.1 and Section A7; see also Definition 1.16 below) are left unnamed, and instead denoted by "cart" and "cocart";
- we assume fixed a category Set' of large sets and a subcategory $\mathsf{Set} \subsetneq \mathsf{Set}'$ of small sets, such that the collection of morphisms of Set forms an object in Set'.

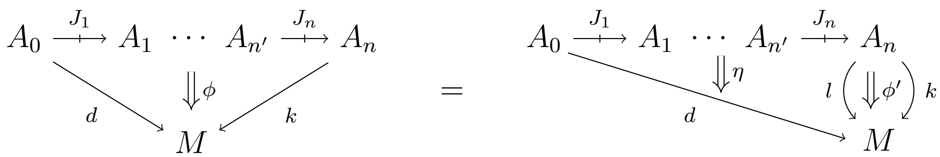
1. Left Kan extension

We begin by introducing four notions of left Kan extension in augmented virtual double categories. The first of these, defined below, is that of 'weak left Kan extension'. In Proposition 1.7 we will see that weak left Kan extension along companions (see Definition A5.1 or Definition 1.20 below) in an augmented virtual double category $\mathcal{K}$ corresponds to left Kan extension in the vertical 2-category $V(\mathcal{K})$ contained in $\mathcal{K}$ (Example A1.5), the latter in the classical sense. The stronger notion of 'left Kan extension', introduced in Definition 1.9 below, recovers the classical notions of 'weighted colimit' and 'enriched left Kan extension', as introduced by Borceux and Kelly in [BK75] (see also Sections 3 and 4 of [Kel82]), as we will see in Examples 1.12 and 1.13. On the other hand the notion of 'pointwise weak left Kan extension' of Definition 1.24 is reminiscent of that of pointwise left Kan extension in a 2-category, as introduced by Street in [Str74b]. The same definition also introduces the notion of 'pointwise left Kan extension', which combines the latter two strengthenings. In Section 3 we will see that the notions of 'pointwise weak left Kan extension' and 'pointwise left Kan extension' coincide in augmented virtual double categories $\mathcal{K}$ that have 'cocartesian tabulations' as well as restrictions on the right (Theorem 3.20); moreover in that case they recover Street's 2-categorical notion of pointwise left Kan extension in $V(\mathcal{K})$ (Proposition 3.22).

1.1. WEAK LEFT KAN EXTENSION. The notion of weak left Kan extension below generalises the notion of 'Kan extension' in a double category that was introduced in Definition 3.1 of [Kou14a], by allowing extensions of a vertical morphism $d \colon A \to M$ along a path of horizontal morphisms $J_1 \colon A \nrightarrow A_1, \dots, J_n \colon A_{n'} \nrightarrow A_n$ instead of a single morphism $J \colon A \nrightarrow B$. The latter notion in turn specialises that of Kan extension given in Section 2 of [GP08], which allows extension of $d \colon A \to M$ both at its source, along a morphism $J \colon A \nrightarrow B$, as well as at its target, along some $K \colon M \nrightarrow N$.

Recall from Lemma A1.3 the notion of horizontal composition $\phi \odot \psi$ of horizontally composable cells ϕ and ψ in an augmented virtual double category, which is defined whenever ϕ or ψ is nullary.

1.2. Definition. *Consider the nullary cell η in the composite on the right-hand side below. It is said to define $l\colon A_n \to M$ as the* weak left Kan extension *of $d\colon A_0 \to M$ along the (possibly empty) path $\underline{J} = (J_1, \dots, J_n)$ if any nullary cell ϕ, as on the left-hand side, factors uniquely through η as a vertical cell ϕ' as shown. In that case η is called* weakly left Kan.



As usual any two nullary cells defining the same weak left Kan extension factor through each other as invertible vertical cells. In Example 2.3 we will see that (weak) left Kan extensions along a path $\underline{J} = (J_1, \dots, J_n)$ can be obtained by extending along each of the $J_1, \dots, J_n$ recursively.

1.3. Example. A vertical cell is weakly left Kan if and only if it is invertible; in fact in that case it defines a left Kan extension in the sense of Definition 1.9.

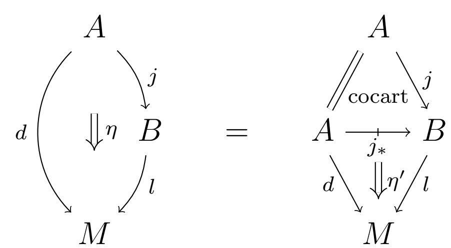
1.4. Example. In $Q(\mathcal{C})$, the double category of quintets in a 2-category $\mathcal{C}$ (see Example A6.3), the notion of weak Kan extension coincides with the usual 2-categorical notion of Kan extension in $\mathcal{C}$, as given in Section 2 of [Str72].

1.5. Remark. The definition above induces a notion of weak left Kan extension for unital virtual double categories, that is virtual double categories $\mathcal{K}$ that admit all horizontal units (see Section A4 or Definition 1.16 below), as follows. Recall from Example A1.7 and Section A10 that any such $\mathcal{K}$ induces an augmented virtual double category $N(\mathcal{K})$ which has the same objects and morphisms as $\mathcal{K}$ while its nullary cells $\underline{J} \Rightarrow M$ are precisely the unary cells $\underline{J} \Rightarrow I_M$ of $\mathcal{K}$, where $I_M\colon M \nrightarrow M$ is a chosen horizontal unit for M. Regarding the definition above for $N(\mathcal{K})$ in terms of $\mathcal{K}$ we obtain a notion of weak left Kan extension for $\mathcal{K}$, defined by universal unary cells $\eta\colon \underline{J} \Rightarrow I_M$ whose horizontal targets are horizontal units. Likewise all definitions and results of this article can be applied to unital virtual double categories.

1.6. Remark. Recall from Definition A1.8 that every augmented virtual double category $\mathcal{K}$ has a horizontal dual $\mathcal{K}^{\text{co}}$. Horizontally dual to the definition above, a nullary cell $\varepsilon\colon (J_1, \dots, J_n) \Rightarrow M$ of $\mathcal{K}$ is called *weakly right Kan*, thus defining a weak right Kan extension, whenever the corresponding cell $\varepsilon^{\text{co}}\colon (J_n^{\text{co}}, \dots, J_1^{\text{co}}) \Rightarrow M$ of $\mathcal{K}^{\text{co}}$ is weakly left Kan. Horizontal duals of the notions of 'left Kan cell' and 'pointwise left Kan cell', introduced in Definitions 1.9 and 1.24 below, are obtained analogously. This article only concerns left Kan extensions.

Remember from Example A1.5 that any augmented virtual double category $\mathcal{K}$ contains a 2-category $V(\mathcal{K})$ consisting of its objects, vertical morphisms and vertical cells. Weak left Kan extension along companions (see Definition A5.1 or Definition 1.20 below) in $\mathcal{K}$ corresponds to left Kan extension in $V(\mathcal{K})$ as in the following proposition. In Proposition 3.18 we will see that (weak) left Kan extension along a path $\underline{J}$ of horizontal morphisms reduces to (weak) left Kan extension along a single companion morphism whenever the 'cocartesian path of $(0,1)$-ary cells for $\underline{J}$' exists. In the right-hand side below "cocart" denotes the cocartesian cell that defines the companion j_* of j; see Section A5 or Definition 1.20 below.

1.7. Proposition. *In an augmented virtual double category $\mathcal{K}$ consider a vertical cell η, as on the left-hand side below, and its factorisation η' through the companion j_*, as shown.*

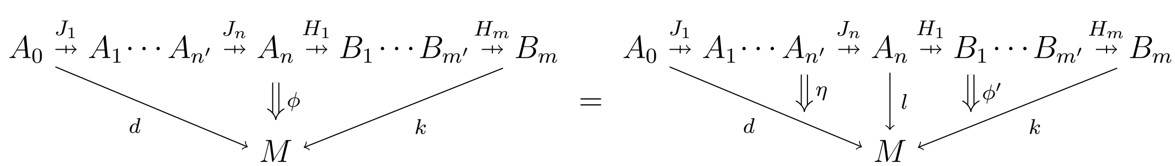


The factorisation η' is weakly left Kan in $\mathcal{K}$ if and only if η defines l as the left Kan extension of d along j in $V(\mathcal{K})$, in the sense of Section 2 of [Str72].

Proof (Sketch). It is straightforward to show that, by factorising through the cocartesian cell defining j_* (see Section A5 or Definition 1.20 below), the universal property of η in $\mathcal{K}$ is equivalent to that of η' in $V(\mathcal{K})$. ∎

1.8. Left Kan extension. Definition 1.2 strengthens to give a notion of left Kan extension in augmented virtual double categories as follows. This generalises the corresponding notion for double categories, that was given in Definition 3.10 of [Kou14a] under the name 'pointwise left Kan extension'; see Example 2.21 below.

1.9. Definition. *Consider the nullary cell η in the composite on the right-hand side below, where $\underline{J} = (J_1, \dots, J_n)$ is possibly empty. It is said to define $l\colon A_n \to M$ as the* left Kan extension *of $d\colon A_0 \to M$ along $\underline{J}$ if any nullary cell ϕ as on the left-hand side below, where $\underline{H} = (H_1, \dots, H_m)$ is any (possibly empty) path, factors uniquely through η as a nullary cell ϕ', as shown. In that case η is called* left Kan.



Clearly every left Kan extension is a weak left Kan extension, by restricting the universal property above to cells ϕ with $\underline{H} = (A_n)$ empty.

1.10. Example. In Section 8 of [RV22] Riehl and Verity introduce the unital virtual equipment (see Definition 1.19 below) $\mathbb{M}\mathrm{od}(\mathcal{K})$ of 'modules in an ∞-cosmos $\mathcal{K}$'. In $\mathbb{M}\mathrm{od}(\mathcal{K})$ consider a factorisation $\eta = \eta' \circ \mathrm{cocart}$ as in Proposition 1.7. The vertical cell $\eta\colon d \Rightarrow l \circ j$ corresponds to a '∞-natural transformation' $l \circ j \Rightarrow d$ in the 'homotopy 2-category associated to $\mathcal{K}$'; see Proposition 8.4.11 of [RV22]. Using Theorem 8.4.4 and Definition 9.1.2 of the latter it is straightforward to see that this transformation defines l as a 'pointwise right extension', in the sense of its Theorem 9.3.3(iii), precisely if η' is left Kan in $\mathbb{M}\mathrm{od}(\mathcal{K})$, in our sense above.

1.11. Weighted colimits and enriched left Kan extension. The notion of left Kan extension specialises to the classical notions of weighted colimit and enriched left Kan extension as follows. As noted in the introduction to [Str74b] recall that the 2-categorical notion of pointwise left Kan extension, as introduced therein, is too strong to recover the notion of enriched Kan extension; see Example 3.24 below.

1.12. Example. Let $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a monoidal category and η a cell in the unital virtual equipment $\mathcal{V}$-Prof of $\mathcal{V}$-profunctors (Examples A2.4 and A4.2) that is of the form as in the composite on the right-hand side below. Here I denotes the unit $\mathcal{V}$-category with single object $*$ and hom-object $I(*, *) = I$; we identify $\mathcal{V}$-functors $f\colon I \to M$ with objects in M and $\mathcal{V}$-profunctors $H\colon I \nrightarrow I$ with $\mathcal{V}$-objects.

$$\begin{array}{c} A_0 \xrightarrow{J_1} A_1 \cdots A_{n'} \xrightarrow{J_n} I \xrightarrow{H} I \\ {\scriptstyle d}\searrow \quad \Downarrow\phi \quad \swarrow{\scriptstyle k} \\ M \end{array} \quad = \quad \begin{array}{c} A_0 \xrightarrow{J_1} A_1 \cdots A_{n'} \xrightarrow{J_n} I \xrightarrow{H} I \\ {\scriptstyle d}\searrow \quad \Downarrow\eta \quad {\scriptstyle l}\downarrow \ \Downarrow\phi' \ \swarrow{\scriptstyle k} \\ M \end{array}$$

One checks that the universal property defining η as a left Kan cell in $\mathcal{V}$-Prof reduces to the unique factorisations through η of the cells ϕ of the form as on the left-hand side above; see Proposition 2.24 of [Kou15a] for the horizontally dual result in the case that $\mathcal{V}$ has large colimits that are preserved by $\otimes$ on both sides, so that $\mathcal{V}$-Prof is a pseudo double category (Example A9.2).

Unpacking the reduced universal property above for a $(1, 0)$-ary cell $\eta\colon J_1 \Rightarrow M$ we recover the notion of a 'couniversal $\mathcal{V}$-natural pair' (l, η) that defines $l \in M$ as the *tensor product of* J_1 *with* d in the sense of Definition 3.5 of [Lin81]. Next consider the $(n, 0)$-ary cell $\eta\colon \underline{J} \Rightarrow M$ above in a unital virtual equipment $\mathcal{V}'$-Prof where $\mathcal{V}'$ is a closed symmetric monoidal category, so that each $\mathcal{V}'$-profunctor J_i can be regarded as a $\mathcal{V}'$-functor $J_i\colon A_{i'}^{\mathrm{op}} \otimes A_i \to \mathcal{V}'$. It is straightforward to check that η satisfies the reduced universal property above if and only if the adjuncts of the composites

$$J_1(x_0, x_1) \otimes' \cdots \otimes' J_n(x_{n'}, *) \otimes' M(l, k) \xrightarrow{\phi \otimes' \mathrm{id}} M(dx_0, l) \otimes' M(l, k) \xrightarrow{\bar{M}} M(dx_0, k)$$

define $M(l, k)$ as the iterated end

$$\int_{x_0 \in A_0} \cdots \int_{x_{n'} \in A_{n'}} [J_1(x_0, x_1) \otimes' \cdots \otimes' J_n(x_{n'}, *), M(dx_0, k)]'.$$

If $n = 1$, so that the end reduces to the $\mathcal{V}'$-object $[A_0^{\text{op}}, \mathcal{V}'](J_1, M(d-, k))$ of $\mathcal{V}'$-functors $J_1 \to M(d-, k)$, this recovers the notion of η defining l as the J_1*-weighted colimit of* d, in the usual sense of equation (3.5) of [Kel82] (where the colimit is said to be 'indexed by J_1') and as originally introduced in [BK75].

In light of the previous we call paths $\underline{J}: A_0 \nrightarrow I$ of $\mathcal{V}$-profunctors $\mathcal{V}$*-weights* and say that the left Kan cell $\eta: \underline{J} \Rightarrow M$ above defines $l \in M$ as the $\underline{J}$*-weighted colimit of* d. We also use the term $\underline{J}$-weighted colimit d for left Kan cells $\underline{J} \Rightarrow M$ as above in the unital virtual double category $\mathcal{V}$-sProf of small $\mathcal{V}$-profunctors (Examples A2.8 and A4.7). Notice that the reduced universal property above, for $\eta \in \mathcal{V}$-sProf, is the same whether considered in $\mathcal{V}$-sProf or in $\mathcal{V}$-Prof, since all $\mathcal{V}$-profunctors of the form $H: I \nrightarrow I$ are small. Together with Lemma 1.14 we conclude that the embedding $\mathcal{V}$-sProf $\hookrightarrow \mathcal{V}$-Prof both preserves and reflects cells η defining weighted colimits.

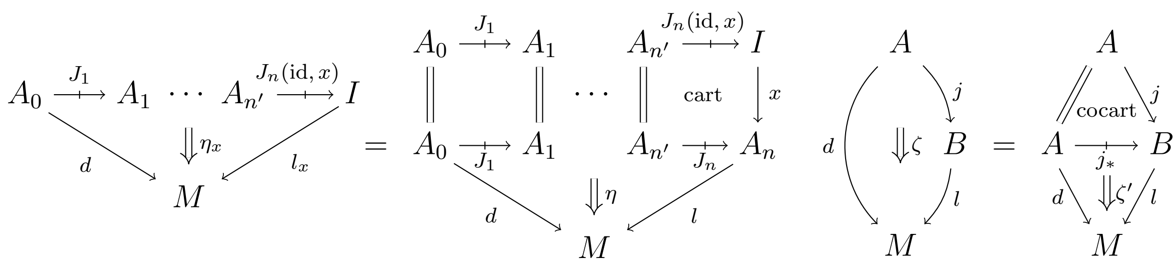
Next let $\mathcal{V} \subset \mathcal{V}'$ be a *universe enlargement* in the sense of Section 3.12 of [Kel82] (see also Example A2.7), that is a monoidal, limit-preserving and full embedding of $\mathcal{V}$ into a closed monoidal and locally large category $\mathcal{V}'$ that is both large complete and large cocomplete. Consider the sub-augmented virtual equipment $(\mathcal{V}, \mathcal{V}')$-Prof $\subseteq \mathcal{V}'$-Prof of $\mathcal{V}$*-profunctors* $J: A \nrightarrow B$ between $\mathcal{V}'$-categories, with $J(x, y) \in \mathcal{V}$ for all $x \in A$, $y \in B$; see Examples A2.7 and A4.6. Applying Lemma 1.14 we find that a cell $\eta: \underline{J} \Rightarrow M$ in $(\mathcal{V}, \mathcal{V}')$-Prof, of the form as in the right-hand side above, is left Kan in $(\mathcal{V}, \mathcal{V}')$-Prof whenever it defines l as the $\underline{J}$-weighted colimit of d in $\mathcal{V}'$-Prof. Notice however that the reduced universal property for η above is in general weaker when considered in $(\mathcal{V}, \mathcal{V}')$-Prof (where $H \in \mathcal{V}$) than when considered in $\mathcal{V}'$-Prof (where $H \in \mathcal{V}'$). One checks however that the two properties coincide if the iterated end above, which is known to exist in the large complete $\mathcal{V}'$, is (isomorphic to) a $\mathcal{V}$-object. Recalling that $\mathcal{V} \subset \mathcal{V}'$ preserves limits, to ensure the latter it suffices that $\mathcal{V}$ is closed symmetric monoidal and small complete, $\mathcal{V} \subset \mathcal{V}'$ is a closed symmetric monoidal functor, and all of the A_i are small $\mathcal{V}'$-categories.

1.13. Example. Let $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a monoidal category and let $\mathcal{K}$ denote either the unital virtual double category $\mathcal{V}$-Prof of $\mathcal{V}$-profunctors or that of small $\mathcal{V}$-profunctors $\mathcal{V}$-sProf. Notice that in either case $\mathcal{K}$ has all horizontal units and restrictions on the right (Examples A4.2 and A4.7). Given a path $\underline{J}: A_0 \nrightarrow A_n$ of (small) $\mathcal{V}$-profunctors and a $\mathcal{V}$-functor $d: A_0 \to M$, assume that for each $x \in A_n$ the $\big(J_1, \dots, J_n(\text{id}, x)\big)$-weighted colimit of d exists in $\mathcal{K}$, in the sense of the previous example. Here $J_n(\text{id}, x)$ is the restriction (see Definition A4.1 or Definition 1.16 below) of J_n along $x: I \to A_n$, the $\mathcal{V}$-functor that picks out $x \in A_n$. Denote each of these colimits by l_x and its defining cell, of the form as on the left below, by η_x. It is straightforward to show that the universal property of the η_x ensures that the objects l_x and cells η_x uniquely combine into a $\mathcal{V}$-functor $l: A_n \to M$ and a left Kan cell η such that the equation on the left below is satisfied in $\mathcal{K}$ for each $x \in A_n$; here "cart" denotes the cartesian cell that defines the restriction $J_n(\text{id}, x)$.

Using Corollary 1.22 below we conclude that a cell η in $\mathcal{K}$, of the form as in the right-hand side on the left below, is left Kan precisely if for each $x \in A_n$ the composite η_x on the left below defines $l_x = lx$ as the $\big(J_1, \dots, J_n(\text{id}, x)\big)$-weighted colimit of d in $\mathcal{K}$.

In particular it follows that $\mathcal{V}$-sProf $\hookrightarrow$ $\mathcal{V}$-Prof both preserves and reflects left Kan cells.

Given a universe enlargement $\mathcal{V} \subset \mathcal{V}'$ (Example 1.12) recall that the augmented virtual equipment $(\mathcal{V}, \mathcal{V}')$-Prof of $\mathcal{V}$-profunctors between $\mathcal{V}'$-categories need not have all horizontal units (Example A4.6). It follows that we cannot apply Corollary 1.22 to $(\mathcal{V}, \mathcal{V}')$-Prof in general, so that its left Kan cells may not be "pointwise"; see Example 1.34. Analogous to the above however any family of cells η_x in $(\mathcal{V}, \mathcal{V}')$-Prof as on the left below, defining weighted colimits l_x in $\mathcal{V}'$-Prof, uniquely combine into a left Kan cell η in $(\mathcal{V}, \mathcal{V}')$-Prof that satisfies the equation on the left below for each $x \in A_n$; see also the last paragraph of the previous example. In that case applying Lemma 1.28 below to η we find that it is pointwise left Kan in $(\mathcal{V}, \mathcal{V}')$-Prof, in the sense of Definition 1.24 below.



Next assume that $\mathcal{V}$ is symmetric monoidal and consider the factorisation ζ' of a vertical cell (i.e. a $\mathcal{V}$-natural transformation) $\zeta \in \mathcal{V}$-Prof as on the right above. Applying the previous to ζ', together with the observations of the previous example we find that ζ' is left Kan in $\mathcal{V}$-Prof precisely if (l, ζ) is the 'pointwise left Kan extension of d along j' in the sense of Section 4 of [Lin81]. If $\mathcal{V}$ is moreover closed symmetric monoidal then this recovers the usual $\mathcal{V}$-enriched notion of ζ 'exhibiting l as the left Kan extension of d along j' in the sense of Section 4 of [Kel82], as originally introduced in [BK75].

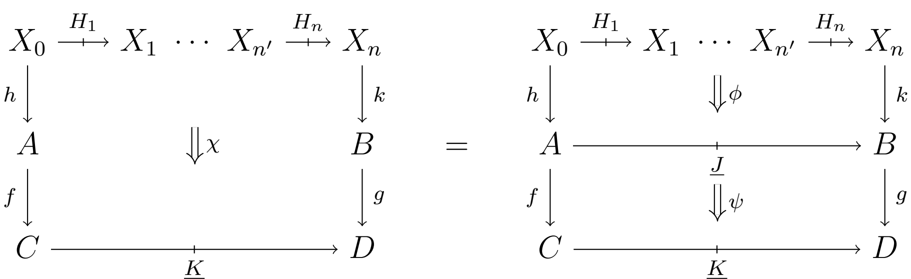
Finally assume that A and B are small $\mathcal{V}$-categories and that $\mathcal{V}$ is small complete, so that the $\mathcal{V}$-categories $[A, M]$ and $[B, M]$ of $\mathcal{V}$-functors $A \to M$ and $B \to M$ exist; see Section 2.2 of [Kel82]. In that case the existence of the left Kan extension $l \colon B \to M$ of d along j implies the following weaker condition: there exists a $\mathcal{V}$-natural isomorphism $[A, M](d, k \circ j) \cong [B, M](l, k)$ for any $\mathcal{V}$-functor $k \colon B \to M$; see Section 4.3 of [Kel82]. In particular if all left Kan extensions along j exist then the $\mathcal{V}$-functor $[j, M] \colon [B, M] \to [A, M]$ given by precomposition with j admits a left adjoint (Theorem 4.50 of [Kel82]); the latter condition, in the case of $\mathcal{V} = \mathsf{Set}$, was originally studied by Kan in [Kan58].

The following straightforward lemma is useful for obtaining (weak) left Kan extensions in locally full sub-augmented virtual double categories. For the notion of (locally) full and faithful functor between augmented virtual double categories see Definition A3.6.

1.14. Lemma. *Any locally full and faithful functor $F \colon \mathcal{K} \to \mathcal{L}$ reflects (weakly) left Kan cells, that is a cell $\eta \in \mathcal{K}$ is (weakly) left Kan whenever its image $F\eta$ is so in $\mathcal{L}$. If F is full and faithful then it preserves weakly left Kan cells as well: $\eta \in \mathcal{K}$ is weakly left Kan if and only if $F\eta$ is so in $\mathcal{L}$.*

1.15. Cartesian cells and restrictions of left Kan extensions. In Example 1.13 we used Corollary 1.22 below. The latter uses the notion of cartesian cell (Definition A4.1) which we recall here for convenience, together with the pasting lemma for cartesian cells (Lemma A4.15) and the notion of augmented virtual equipment (Definition A4.10). We also recall the notion of full and faithful morphism (Definition A4.12) and those of companion and conjoint (Definition A5.1), which are related to that of cartesian cell.

1.16. Definition. *A cell* $\psi\colon \underline{J} \Rightarrow \underline{K}$ *with* $\underline{J}$ *of length* $|\underline{J}| \leq 1$*, as in the right-hand side below, is called* cartesian *if any cell* χ*, as on the left-hand side, factors uniquely through* ψ *as a cell* ϕ *as shown.*



Vertically dual, provided that $|\underline{J}| = 1$*, the cell* ϕ *is called* weakly cocartesian *if any cell* χ *factors uniquely through* ϕ *as a cell* ψ *as shown.*

If a $(1,n)$-ary cartesian cell ψ of the form above exists then its horizontal source $J\colon A \nrightarrow B$ is called the *restriction* of $\underline{K}\colon C \nrightarrow D$ along f and g, and denoted $\underline{K}(f,g) \coloneqq J$. If $\underline{K} = (C \overset{K}{\nrightarrow} D)$ then we call $K(f,g)$ *unary*; in the case that $\underline{K} = C$ is an empty path we call $C(f,g)$ *nullary*. Restrictions of the form $\underline{K}(f,\mathrm{id})$ and $\underline{K}(\mathrm{id},g)$ are called restrictions *on the left* and *right*. We call the nullary restriction $C(\mathrm{id},\mathrm{id})\colon C \nrightarrow C$ the *(horizontal) unit* of the object C and denote it $I_C \coloneqq C(\mathrm{id},\mathrm{id})$; if I_C exists then we call C *unital*.

1.17. Lemma. [Pasting lemma for cartesian cells] *If the cell* ψ *in the composite on the right-hand side above is cartesian then the composite* $\psi \circ \phi$ *is cartesian if and only if* ϕ *is.*

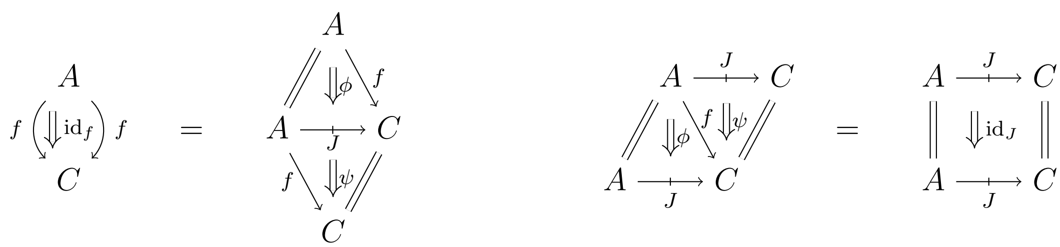
1.18. Definition. *A vertical morphism* $f\colon A \to C$ *is called* full and faithful *if its identity cell* id_f *is cartesian.*

1.19. Definition. *An augmented virtual double category* $\mathcal{K}$ *is said to have* restrictions on the left (resp. right) *if it has all unary restrictions of the form* $K(f,\mathrm{id})$ *(resp.* $K(\mathrm{id},g)$*). We call* $\mathcal{K}$ *a* unital virtual double category *if it has all horizontal units (see Section A10). An* augmented virtual equipment *is an augmented virtual double category that has all unary restrictions* $K(f,g)$*. A* unital virtual equipment *is an unital virtual double category that has all restrictions* $\underline{K}(f,g)$*.*

1.20. Definition. *Let* $f\colon A \to C$ *be a vertical morphism in an augmented virtual double category. The nullary restriction* $C(f, \mathrm{id})\colon A \nrightarrow C$ *is called the* companion *of* f *and denoted* f_*. *Likewise* $C(\mathrm{id}, f)\colon C \nrightarrow A$ *is called the* conjoint *of* f *and denoted* f^*.

Factorising the identity cell $\mathrm{id}_f\colon (A) \Rightarrow (C)$ through the cartesian cell defining the companion f_* we obtain a cocartesian cell, in the sense of Definition A7.1 (see also Definition 2.6 below), as described by the following lemma which combines Lemmas A5.4, A5.9 and A7.6 and Corollary A8.3. The identities below are called the *companion identities*. A horizontal dual result similarly applies to the conjoint f^*.

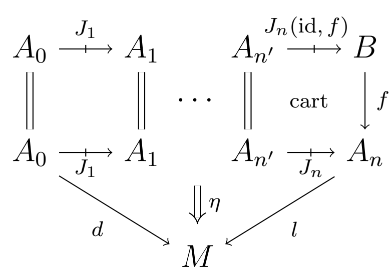
1.21. Lemma. *Consider the factorisation of a vertical identity cell on the left below. The following conditions are equivalent:* (a) ψ *is cartesian;* (b) *the identity on the right below holds;* (c) ϕ *is weakly cocartesian;* (d) ϕ *is cocartesian (Definition A7.1 or Definition 2.6). In that case* J *is the companion of* f.



If $f = \mathrm{id}_A$ *then each of the previous conditions is further equivalent to each of the following ones:* (e) ψ *is weakly cocartesian;* (f) ψ *is cocartesian;* (g) ϕ *is cartesian. In that case* J *is the horizontal unit of* A.

The following is an immediate consequence of Proposition 2.25 below.

1.22. Corollary. *In an augmented virtual double category that has restrictions on the right consider a cell* η *as in the composite below, with* $n \geq 1$ *and where the object* A_n *is unital. It is left Kan precisely if, for each vertical morphism* $f\colon B \to A_n$, *the composite is left Kan.*

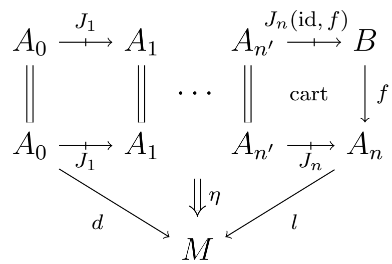


Proof. For the 'if'-part take $f = \mathrm{id}_{A_n}$ and use that $J_n(\mathrm{id}, \mathrm{id}) \cong J_n$. For the converse remember that A_n being unital ensures that all conjoints $f^*\colon A_n \nrightarrow B$ exist, by Corollary A4.16, and apply Proposition 2.25 below. ■

1.23. Pointwise left Kan extension. The previous result leads us to the following "pointwise" strengthening of the notion of (weak) left Kan extension.

1.24. Definition. *Consider a path of horizontal morphisms* $\underline{J}\colon A_0 \nrightarrow A_n$ *of length* $n \geq 1$ *as well as vertical morphisms* $d\colon A_0 \to M$ *and* $f\colon B \to A_n$. *We say that the (weak) left Kan extension of* d *along* $\underline{J}$ *(Definitions 1.2 and 1.9)* restricts along f *if the restriction* $J_n(\mathrm{id}, f)$ *exists and, for any (weakly) left Kan cell* η *of the form below, the composite below is again (weakly) left Kan. In that case we also say that* η restricts along f.

We call a (weakly) left Kan cell η pointwise *if it restricts along any* $f\colon B \to A_n$ *such that the restriction* $J_n(\mathrm{id}, f)$ *exists; in that case we say that* η *defines* l *as the* pointwise *(weak) left Kan extension of* d *along* $(J_1, \dots, J_n)$.



Restrictions of pointwise (weak) left Kan extensions are again pointwise (weak) left Kan extensions as follows.

1.25. Lemma. *Consider the composite above. If* η *is pointwise (weakly) left Kan then so is the composite.*

Proof. A consequence of the pasting lemma for cartesian cells (Lemma 1.17). ∎

1.26. Remark. As a consequence of Corollary 1.22 the notions of left Kan extension and pointwise left Kan extension coincide in any unital virtual double category that has all restrictions on the right (Definition 1.19).

1.27. Remark. The notion of 'Yoneda morphism' introduced in Definition 4.5 below formalises the classical notion of Yoneda embedding, and it is shown in Lemma 4.20 that all four notions of left Kan extension along a Yoneda morphism coincide.

The following is an easy consequence of Lemma 1.14.

1.28. Lemma. *Any locally full and faithful functor* $F\colon \mathcal{K} \to \mathcal{L}$ *that preserves cartesian cells defining restrictions on the right reflects pointwise (weakly) left Kan cells, that is a cell* $\eta \in \mathcal{K}$ *is pointwise (weakly) left Kan whenever its image* $F\eta$ *is so in* $\mathcal{L}$.

1.29. Examples. We will consider formal category theory in the augmented virtual double categories $\mathcal{V}$-Prof, $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ and $\mathcal{V}$-sProf of (small) $\mathcal{V}$-enriched profunctors (Examples A2.4, A2.7 and A2.8; their left Kan extensions were described in Example 1.13); in the unital virtual equipment $\mathsf{Prof}(\mathcal{E})$ of internal profunctors in a category $\mathcal{E}$ (Examples A2.9 and A4.9); in the unital virtual equipment $\mathsf{dFib}(\mathcal{C})$ of discrete two-sided fibrations in a

$\mathcal{K}$	$V(\mathcal{K})$	**Equivalence of notions of Kan extension**
$(\mathcal{V}, \mathcal{V}')$-Prof	$\mathcal{V}'$-Cat	None in general
$\mathcal{V}$-Prof $\mathcal{V}$-sProf ClModRel	$\mathcal{V}$-Cat $\mathcal{V}$-Cat ClOrdCls	LK ⇔ PLK
ModRel($\mathcal{E}$) Prof($\mathcal{E}$) dFib($\mathcal{C}$)	PreOrd($\mathcal{E}$) Cat($\mathcal{E}$) $\mathcal{C}$	LK ⇔ PLK ⇔ PWLK (PLK along f_* in $\mathcal{K}$) = (PLK along f in $V(\mathcal{K})$)

Table 1.1: Augmented virtual double categories $\mathcal{K}$ grouped according to the equivalences in $\mathcal{K}$ of the notions of left Kan extension (LK) (Definition 1.9) and pointwise (weak) left Kan extension (P(W)LK) (Definition 1.24). The equivalence LK ⇔ PLK follows from applying Remark 1.26 to $\mathcal{K}$, while PLK ⇔ PWLK follows from applying Theorem 3.20 to $\mathcal{K}$. That pointwise left Kan extensions along companions f_* in $\mathcal{K}$ coincide with pointwise left Kan extensions along f in the vertical 2-category $V(\mathcal{K})$, the latter in the usual 2-categorical sense of [Str74b], follows from applying Proposition 3.22 to $\mathcal{K}$.

2-category $\mathcal{C}$ (Example 1.30 below); in the unital virtual equipment ModRel($\mathcal{E}$) of internal modular relations in a category $\mathcal{E}$ (Example 1.31 below); and in the strict double category ClModRel of closed modular relations between closed-ordered closure spaces (Example 1.32 below). Table 1.1 shows the relations between the different notions of left Kan extension in each of these augmented virtual double categories. After considering left Kan extensions in ClModRel, in Example 1.33, Example 1.34 gives an example of a left Kan extension that is not pointwise.

1.30. Example. Let $\mathcal{C}$ be a finitely complete 2-category and consider the unital virtual equipment spFib($\mathcal{C}$) of split two-sided fibrations $J: A \nrightarrow B$ in $\mathcal{C}$ (Examples A2.11 and A4.9). Recall that we consider such J to be internal profunctors $J: A^2 \nrightarrow B^2$ in the underlying category $\mathcal{C}_0$, where A^2 denotes the internal category that is the cotensor of A with the arrow category $2 = (0 \to 1)$; see Example A2.11 where A^2 was denoted by ΦA. The assignment $A \mapsto A^2$ extends to a locally full embedding of spFib($\mathcal{C}$) into the unital virtual equipment Prof($\mathcal{C}_0$) of internal profunctors in $\mathcal{C}_0$ (Examples A2.9 and A4.9), which restricts to the identity on horizontal morphisms.

A split two-sided fibration $J: A \nrightarrow B$, with underlying span $A \xleftarrow{j_A} J \xrightarrow{j_B} B$, is called *discrete* ([Str80a]) if for any cell $\phi: u \Rightarrow v: X \to J$ in $\mathcal{C}$ the following holds: if $j_A \circ \phi$ and $j_B \circ \phi$ are identity cells then so is ϕ (in particular $u = v$). We denote by dFib($\mathcal{C}$) ⊂ spFib($\mathcal{C}$) the full sub-augmented virtual double category generated by the discrete two-sided fibrations in $\mathcal{C}$. In Proposition 2 of [Str74b] it is shown that the assignment $A \mapsto A^2$ extends to a locally full and faithful 2-functor $(-)^2: \mathcal{C} \to \mathsf{Cat}(\mathcal{C}_0) = V(\mathsf{Prof}(\mathcal{C}_0))$, with the correspondence between the cells given by the universal property of the 2-cotensors. It follows that $V(\mathsf{spFib}(\mathcal{C})) \cong \mathcal{C}$. Since $\mathsf{dFib}(\mathcal{C}) \subset \mathsf{spFib}(\mathcal{C})$ is full with respect to vertical cells we likewise have $V(\mathsf{dFib}(\mathcal{C})) \cong \mathcal{C}$.

It is straightforward to check that the horizontal units $I_A = (A \leftarrow A^2 \rightarrow A)$ of $\mathsf{spFib}(\mathcal{C})$ are discrete two-sided fibrations, and that any restriction $K(f,g)$ in $\mathsf{spFib}(\mathcal{C})$ of a discrete two-sided fibration K is again discrete; for the latter recall that $K(f,g)$ is the iterated strict 2-pullback of $A \xrightarrow{f} C \leftarrow K \rightarrow D \xleftarrow{g} B$ (Example A4.9) and use its 2-dimensional universal property. Since the full and faithful inclusion $\mathsf{dFib}(\mathcal{C}) \subset \mathsf{spFib}(\mathcal{C})$ reflects cartesian cells (Lemma A4.5) we conclude that $\mathsf{dFib}(\mathcal{C})$, like $\mathsf{spFib}(\mathcal{C})$, is a unital virtual equipment (Definition 1.19).

Recall that $\mathsf{spFib}(\mathcal{C})$ has all horizontal composites as soon as $\mathcal{C}$ has reflexive coequalisers preserved by pullback (see Example A7.5). The horizontal composite $(J \odot H)$, in $\mathsf{spFib}(\mathcal{C})$, of two discrete two-sided fibrations $J\colon A \nrightarrow B$ and $H\colon B \nrightarrow E$, is not discrete in general however. Indeed in order for all horizontal composites in $\mathsf{dFib}(\mathcal{C})$ to exist further conditions on $\mathcal{C}$ are needed; compare [CJSV94] where discrete two-sided fibrations in a finitely complete bicategory $\mathcal{K}$ are shown to form a bicategory $\mathsf{DFib}(\mathcal{K})$ as soon as $\mathcal{K}$ is 'faithfully conservational'.

1.31. Example. Recall that a relation J in a category $\mathcal{E}$ (Example A2.10) is a span $A \xleftarrow{j_0} J \xrightarrow{j_1} B$ in $\mathcal{E}$ whose legs j_0 and j_1 are jointly monic. If $\mathcal{E}$ has pullbacks then relations in $\mathcal{E}$ form the horizontal morphisms of a unital virtual equipment $\mathsf{Rel}(\mathcal{E})$ whose vertical morphisms are the morphisms of $\mathcal{E}$; see Examples A2.10 and A5.8. $\mathsf{Rel}(\mathcal{E})$ is locally thin (Example A2.5), that is its cells are uniquely determined by their boundaries.

Consider the unital virtual double category $\mathsf{ModRel}(\mathcal{E}) \coloneqq (N \circ \mathsf{Mod})\big(\mathsf{Rel}(\mathcal{E})\big)$ of bimodules in $\mathsf{Rel}(\mathcal{E})$, as defined in Examples A2.1 and A2.2. The objects of $\mathsf{ModRel}(\mathcal{E})$ are *internal preorders* in $\mathcal{E}$: they consist of objects A of $\mathcal{E}$ equipped with a relation $\alpha = (A \xleftarrow{\alpha_0} \alpha \xrightarrow{\alpha_1} A)$ such that the (unique) horizontal multiplication and unit cells $\bar{\alpha}\colon (\alpha, \alpha) \Rightarrow \alpha$ and $\tilde{\alpha}\colon A \Rightarrow \alpha$ exist in $\mathsf{Rel}(\mathcal{E})$; in [CS86] these are called the 'ordered objects' of $\mathcal{E}$. The vertical morphisms of $\mathsf{ModRel}(\mathcal{E})$ are the morphisms of $\mathcal{E}$ that preserve order, that is morphisms $f\colon A \rightarrow C$ for which there exists a cell in $\mathsf{Rel}(\mathcal{E})$ as on the left below, while its horizontal morphisms are *internal modular relations* ('ideals' in [CS86]), that is relations $J\colon A \nrightarrow B$ in $\mathcal{E}$ such that there exist horizontal action cells $\lambda\colon (\alpha, J) \Rightarrow J$ and $\rho\colon (J, \beta) \Rightarrow J$ in $\mathsf{Rel}(\mathcal{E})$. $\mathsf{ModRel}(\mathcal{E})$ like $\mathsf{Rel}(\mathcal{E})$ is locally thin. In particular a nullary cell of the form $\underline{J} \Rightarrow (C, \gamma)$ exists in $\mathsf{ModRel}(\mathcal{E})$ if and only if the underlying cell $\underline{J} \Rightarrow \gamma$ exists in $\mathsf{Rel}(\mathcal{E})$. The locally thin vertical 2-category $\mathsf{PreOrd}(\mathcal{E}) \coloneqq V\big(\mathsf{ModRel}(\mathcal{E})\big)$ contained in $\mathsf{ModRel}(\mathcal{E})$ coincides with the 2-category of ordered objects of [CS86]. Notice that $\mathsf{Rel}(\mathcal{E})$ embeds into $\mathsf{ModRel}(\mathcal{E})$ by equipping each object $A \in \mathcal{E}$ with the *internal discrete ordering* $I_A = (A \xleftarrow{\mathrm{id}} A \xrightarrow{\mathrm{id}} A)$.

$\mathsf{ModRel}(\mathcal{E})$ is a unital virtual equipment by Examples A4.8 and A5.8: unary restrictions are created as in $\mathsf{Span}(\mathcal{E})$ (Example A4.3) while the nullary restrictions $C(f,g)$ of $\mathsf{ModRel}(\mathcal{E})$ are created as the unary restrictions $\gamma(f,g)$ in $\mathsf{Span}(\mathcal{E})$. Moreover the forgetful functor $U\colon \mathsf{ModRel}(\mathcal{E}) \rightarrow \mathsf{Rel}(\mathcal{E})$ creates pointwise composites (Definition A9.1), so that $\mathsf{ModRel}(\mathcal{E})$ is an equipment (Proposition A7.8) whenever $\mathsf{Rel}(\mathcal{E})$ is, e.g. in the case that

$\mathcal{E}$ is regular (Example A7.4). Indeed consider a path $\underline{J}\colon A_0 \nrightarrow A_n$ of internal modular relations in $\mathcal{E}$ and a horizontal cocartesian cell $\phi\colon \underline{J} \Rightarrow K$ in $\mathsf{Rel}(\mathcal{E})$, defining K as the horizontal composite of $\underline{J}$. Then the actions $\lambda\colon (\alpha_0, J_1) \Rightarrow J_1$ and $\rho\colon (J_n, \alpha_n) \Rightarrow J_n$ induce actions of α_0 and α_n on K, making K into an internal modular relation and ϕ into a cell in $\mathsf{ModRel}(\mathcal{E})$. That ϕ is pointwise cocartesian in $\mathsf{ModRel}(\mathcal{E})$ follows from Lemmas A9.4 and A9.8.

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\downarrow & \Downarrow \bar{f} & \downarrow{\scriptstyle f} \\ C & \xrightarrow[\gamma]{} & C \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\tilde{\alpha}} & \alpha \\ {\scriptstyle \tilde{\alpha}}\downarrow & & \downarrow{\scriptstyle (\alpha_1, \alpha_0)} \\ \alpha & \xrightarrow[(\alpha_0, \alpha_1)]{} & A \times A \end{array}$$

Next assume that $\mathcal{E}$ has all finite limits. An internal preorder A in $\mathcal{E}$ is an *internal partial order* (Examples B2.3.8 of [Joh02]) whenever the square on the right above is a pullback square in $\mathcal{E}$. Using the joint monicity of α_0 and α_1 one easily checks that the latter is equivalent to the following condition, which we will mostly use: for any parallel pair $\phi, \psi\colon X \to \alpha$ of morphisms, $(\alpha_0, \alpha_1) \circ \phi = (\alpha_1, \alpha_0) \circ \psi$ implies $\phi = \psi$. Notice that the latter implies the following for a pair of parallel morphisms f and $g\colon X \to A$: if there exists a vertical isomorphism $f \cong g$ in $\mathsf{ModRel}(\mathcal{E})$ then $f = g$.

1.32. EXAMPLE. By a *closure space* $A = (A, \mathrm{Cl}\, A)$ we will mean a set A equipped with a set $\mathrm{Cl}\, A$ of closed subsets of A, such that $A \in \mathrm{Cl}\, A$ and $\mathrm{Cl}\, A$ is closed under arbitrary intersections. As introduced by Tholen in [Tho09], a *closed-ordered closure space* $A = (A, \mathrm{Cl}\, A, \leq)$ is a closure space $(A, \mathrm{Cl}\, A)$ equipped with a preordering $\leq$ satisfying the closedness axiom ('preservation condition' in [Tho09])

(C) $V \in \mathrm{Cl}\, A \quad \Rightarrow \quad \uparrow V \in \mathrm{Cl}\, A,$

where $\uparrow V \coloneqq \{x \in A \mid \exists v \in V \colon v \leq x\}$ is the upset generated by V. A morphism $f\colon A \to C$ of closed-ordered closure spaces is an order preserving continuous map. Given a relation $J\colon A \nrightarrow B$ between sets, that is a subset $J \subseteq A \times B$, we will abbreviate $(x, y) \in J$ by xJy and write $JS \coloneqq \{y \in B \mid \exists s \in S \colon sJy\}$ for the image of a subset $S \subseteq A$ under J. A *closed modular relation* $J\colon A \nrightarrow B$ between closed-ordered closure spaces is a relation that satisfies the modularity and closedness axioms

(M) $x_1 \leq x_2, \quad x_2 J y_1, \quad y_1 \leq y_2 \quad \Rightarrow \quad x_1 J y_2,$

(C) $V \in \mathrm{Cl}\, A \quad \Rightarrow \quad JV \in \mathrm{Cl}\, B.$

Closedness is equivalent to the reverse relation $J^\circ\colon B \nrightarrow A$ being *upper hemi-continuous*, see e.g. Section 17.2 of [AB06]; see also Section 6 of [Kou18].

Closed modular relations and morphisms of closed-ordered closure spaces form a locally thin (Example A2.5) strict double category (Proposition A7.8) $\mathsf{ClModRel}$ in which a cell of the form below exists, and is unique, if and only if xJy implies $(fx)K(gy)$ for all $x \in A$ and $y \in B$. The composite $J \odot H$ of closed modular relations $J\colon A \nrightarrow B$ and $H\colon B \nrightarrow E$ is defined as usual: $x(J \odot H)z$ if and only if there exists $y \in B$ with xJy and yHz. The horizontal unit $I_A\colon A \nrightarrow A$ is the order relation $xI_Ay :\Leftrightarrow x \leq y$, which is closed because $I_AS = {\uparrow}S$ for all $S \subseteq A$.

$$\begin{array}{ccc} A & \xrightarrow{J} & B \\ {\scriptstyle f}\big\downarrow & \rotatebox{90}{$\subseteq$} & \big\downarrow{\scriptstyle g} \\ C & \xrightarrow[K]{} & D \end{array}$$

Consider the locally thin strict equipment $\mathsf{ModRel} \coloneqq 2\text{-}\mathsf{Prof}$ (Example A2.5) of modular relations between preorders which, by definition, are profunctors enriched in the quantale $2 \coloneqq (\bot \leq \top)$ of truth values (see Example 1.9 of [Kou18]). Notice that the forgetful functor $U\colon \mathsf{ClModRel} \to \mathsf{ModRel}$ creates all restrictions $K(\mathrm{id}, g)$ on the right (Lemma A4.5) but that restrictions $K(f, \mathrm{id})$ on the left do not exist in $\mathsf{ClModRel}$ in general. In particular the companion f_* of a morphism $f\colon A \to C$ exists if and only if ${\uparrow}fV$ is closed in C for every $V \in \mathrm{Cl}\,A$.

Modular closure spaces, introduced by Tholen in [Tho09], are closed-ordered closure spaces $A = (A, \mathrm{Cl}\,A, \leq)$ satisfying the modularity axiom

(M) $V \in \mathrm{Cl}\,A \quad \Rightarrow \quad {\uparrow}V = V,$

which strengthens the closedness axiom above. This notion coincides with that of *modular* $(P, 2)$*-categories* in the sense of Section 4 of [Kou18], where P is the powerset monad; see Example 3.2 of [Kou18]. We denote by $\mathsf{ClModRel}_{\mathrm{m}} \subset \mathsf{ClModRel}$ the full sub-double category generated by modular closure spaces.

1.33. Example. Consider morphisms $M \xleftarrow{d} A \overset{J}{\nrightarrow} B$ in the locally thin strict double category $\mathsf{ClModRel}_{\mathrm{m}}$ of closed modular relations between modular closure spaces and assume that for each $y \in B$ the maximum on the right-hand side below exists in M; for sufficient conditions see Theorem 8.1 of [Kou18]. These maxima combine to form an order preserving map $l\colon B \to M$ given by

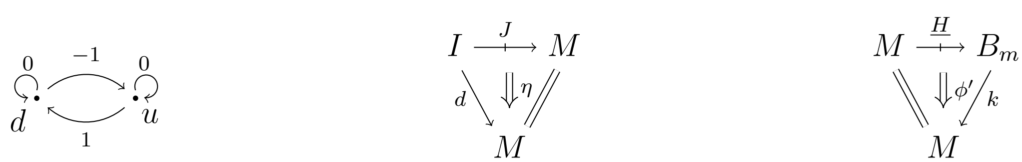
$$ly = \max_{x \in J^\circ y} dx$$

which forms the left Kan extension of d along J in the locally thin strict equipment ModRel of modular relations (Example 1.32). In fact, as a left Kan extension l satisfies the *left Beck-Chevalley* condition in ModRel, in the sense of Definition 2.4 of [Kou18]; see Example 2.7 of [Kou18] or Example 5.17 below.

Regarding d and J as morphisms of modular $(P, 2)$-categories (Example 1.32), the Beck-Chevalley condition for l, closedness of J and the fact that P preserves composites of modular relations allows us to apply Theorem 7.9 of [Kou18] to $\mathsf{Mod}(P)$ (see Section 4 of [Kou18]), which asserts that l is continuous and thus a morphism in $\mathsf{ClModRel}_{\mathrm{m}}$. The latter result partly generalises the "maximum theorem", a classical result in analysis; see e.g. Lemma 17.30 of [AB06]. Applying Lemma 1.14 to the locally full and faithful functors $\mathsf{ClModRel}_{\mathrm{m}} \hookrightarrow \mathsf{ClModRel} \xrightarrow{U} \mathsf{ModRel}$ we conclude that l forms the left Kan extension of d along J both in $\mathsf{ClModRel}_{\mathrm{m}}$ and in $\mathsf{ClModRel}$. Since the latter are strict double categories that have restrictions on the right l is in fact a pointwise left Kan extension by Remark 1.26.

1.34. Example. In order to construct a Kan extension (Definition 1.9) that fails to be a pointwise weak Kan extension (Definition 1.24) consider the locally thin equipment $[-\infty, \infty]$-Prof of categories and profunctors enriched in the quantale $[-\infty, \infty]$ of extended real numbers, with reversed order $\geq$ and addition $(+, 0)$ as monoid structure; see Examples 1.3 and 1.8 of [Kou18]. As is customary ([Law73]) we think of $[-\infty, \infty]$-categories as generalised metric spaces with distances in $[-\infty, \infty]$. Let $\mathcal{K}$ denote the full sub-augmented virtual double category of $[-\infty, \infty]$-Prof generated by those $[-\infty, \infty]$-profunctors $J \colon A \nrightarrow B$ with images $J(x, y) \geq 0$ for all $x \in A$ and $y \in B$. Notice that $\mathcal{K}$ is closed under horizontal composition of $[-\infty, \infty]$-profunctors (Example 1.3 of [Kou18]) so that $\mathcal{K}$ has all horizontal composites by Lemma A9.4.

Let M be the generalised metric space with two points d and u as pictured on the left below and let I be the 'unit $[-\infty, \infty]$-category', consisting of a single point $*$ with $I(*, *) = 0$. We claim that *any* cell η in $\mathcal{K}$ of the form as in the middle below is left Kan. To see this first notice that, by Example 2.21 and using that $\mathcal{K}$ is locally thin, it suffices to show that all cells of the form as on the right below exist in $\mathcal{K}$, where $\underline{H}$ is any path of $[-\infty, \infty]$-profunctors of length $m \leq 1$ in $\mathcal{K}$ and $k \colon B_m \to M$ is any morphism. It is not hard to show that the latter follows from the fact that k and $\underline{H}$ are non-expanding (Examples 1.10 and 1.11 of [Kou18]).



For a left Kan cell in $\mathcal{K}$ that is not pointwise weakly left Kan take $J \colon I \nrightarrow M$ to be given by $J(*, d) = 1$ and $J(*, u) = 0$. Then J is non-expanding and the cell η of the form above exists. By the previous η is left Kan; to arrive at a contradiction let us assume that η is pointwise weakly left Kan as well. Let $u \colon I \to M$ denote the morphism that picks out u in M and consider the composition $J(\mathrm{id}, u) \Rightarrow M$ of η and the cartesian cell defining the restriction $J(\mathrm{id}, u)$. By the assumption and Definition 1.24 the latter defines u as the weak left Kan extension of d along $J(\mathrm{id}, u) \colon I \nrightarrow I$. By the definition of J the latter equals the horizontal unit of I so that, by Example 2.19, $u \cong d \colon I \to M$ follows. But that is impossible as there exists no vertical cell $d \Rightarrow u$ in $\mathcal{K}$ (nor in $[-\infty, \infty]$-Prof).

1.35. Internal left adjoints are cocontinuous. The two remaining results of this section describe the interaction between adjunctions and left Kan extensions. The first of these, Proposition 1.37 below, shows that left adjoints in an augmented virtual double category are cocontinuous in the sense of the definition below. Analogous results, for left adjoints in 2-categories (see e.g. Proposition 2.19(1) of [Web07]) and enriched left adjoints (see e.g. Section 4.1 of [Kel82]), are well known. The second result, Proposition 1.42, instead considers 'external' adjunctions $F \dashv G\colon \mathcal{L} \to \mathcal{K}$, between augmented virtual double categories and $\mathcal{K}$ and $\mathcal{L}$, and shows that the 'adjunct' of a left Kan cell in $\mathcal{L}$ is again left Kan in $\mathcal{K}$.

1.36. Definition. *Consider a (pointwise) (weakly) left Kan cell η of the form below.*

$$\begin{array}{ccc} A_0 \xrightarrow{J_1} A_1 & \cdots & A_{n'} \xrightarrow{J_n} A_n \\ & \Downarrow \eta & \\ {\scriptstyle d} \searrow & M & \swarrow {\scriptstyle l} \end{array}$$

(a) *A morphism $f\colon M \to N$ is said to* preserve η, *and to preserve the (pointwise) (weak) left Kan extension of d along $(J_1, \dots, J_n)$, if the composite $f \circ \eta$ is again (pointwise) (weakly) left Kan.*

(b) *The cell η is called* absolutely *(pointwise) (weakly) left Kan if it is preserved by all morphisms $f\colon M \to N$; in that case the (pointwise) (weak) left Kan extension of d along $(J_1, \dots, J_n)$ is called* absolute.

(c) *A morphism $f\colon M \to N$ is called* (weakly) cocontinuous *if it preserves any (weakly) left Kan cell ζ with horizontal target M.*

Notice that (weakly) cocontinuous morphisms preserve pointwise (weakly) left Kan cells as well. In Theorem 5.16 below absolutely left Kan cells are characterised in terms of a 'left Beck-Chevalley condition' (Definition 5.10).

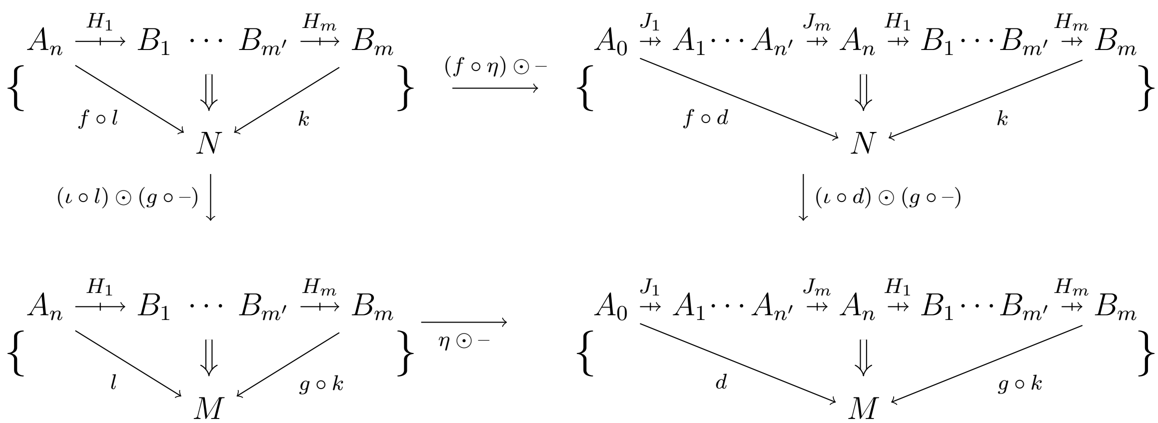
By a left adjoint in an augmented virtual double category $\mathcal{K}$ we mean a left adjoint in the vertical 2-category $V(\mathcal{K})$ (Example A1.5), in the usual 2-categorical sense; see e.g. Lemma A5.16.

1.37. Proposition. *Left adjoints are both cocontinuous and weakly cocontinuous.*

Proof. Let $f\colon M \to N$ be left adjoint to $g\colon N \to M$, with unit and counit vertical cells $\iota\colon \mathrm{id}_M \Rightarrow g \circ f$ and $\varepsilon\colon f \circ g \Rightarrow \mathrm{id}_N$. To show that f is cocontinuous consider any left Kan cell

$$\begin{array}{ccc} A_0 \xrightarrow{J_1} A_1 & \cdots & A_{n'} \xrightarrow{J_n} A_n \\ & \Downarrow \eta & \\ {\scriptstyle d} \searrow & M; & \swarrow {\scriptstyle l} \end{array}$$

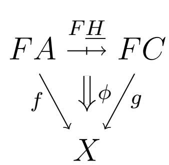
we have to show that $f \circ \eta$ is again left Kan. To this end consider the commuting diagram of assignments, between collections of cells of the forms as shown, below. The vertically drawn assignments are bijections. Indeed, the triangle identities for ι and ε imply that their inverses are given by composition with ε on the right. The bottom assignment is a bijection as well, because η is left Kan. We conclude that the top assignment is a bijection too, showing that $f \circ \eta$ is left Kan as required.



Restricting the previous to the empty path $\underline{H} = (A_n)$ shows that f is weakly cocontinuous too. ■

1.38. External adjunctions and left Kan extension. Consider an adjunction between augmented virtual double categories, that is an adjunction $F \dashv G \colon \mathcal{L} \to \mathcal{K}$ in the 2-category AugVirtDblCat of augmented virtual double categories, their functors and the transformations between them; see Section A3. Proposition 1.42 below asserts that a cell $\phi \colon \underline{J} \Rightarrow GM$ is left Kan in $\mathcal{K}$ whenever its 'adjunct' $\phi^\flat \colon F\underline{J} \Rightarrow M$ is left Kan in $\mathcal{L}$. More generally it applies to pairs of cells that are adjunct with respect to a 'locally universal morphism', as we will now define. The latter is related to (the vertical dual of) the notion of 'universal 2-cell' considered in Proposition 8.6 of [Shu08], for functors between pseudo double categories. Proposition 1.42 and the notion of 'relative universal morphism', also defined below, will be crucial in Section 8.

Given a functor $F \colon \mathcal{K} \to \mathcal{L}$ between augmented virtual double categories (Definition A3.1) and an object X in $\mathcal{L}$ we define the *vertical slice category* $F /_{\mathrm{v}} X$ as follows. Its objects are pairs (A, f) consisting of an object A in $\mathcal{K}$ and a morphism $f \colon FA \to X$ in $\mathcal{L}$. Its morphisms $(\underline{H}, \phi) \colon (A, f) \to (C, g)$ are pairs consisting of a path $\underline{H} \colon A \nrightarrow C$ of horizontal morphisms in $\mathcal{K}$, of any length, and a nullary cell ϕ in $\mathcal{L}$ that is of the form below. Composition in $F /_{\mathrm{v}} X$ is given by horizontal composition in $\mathcal{L}$; notice that invertible morphisms $(\underline{H}, \phi)$ of $F /_{\mathrm{v}} X$ necessarily have $\underline{H}$ empty and ϕ vertical. We abbreviate $\mathcal{L} /_{\mathrm{v}} X \coloneqq \mathrm{id}_{\mathcal{L}} /_{\mathrm{v}} X$. In Section 4 we shall consider 'horizontal slice categories'; see Proposition 4.24 below.



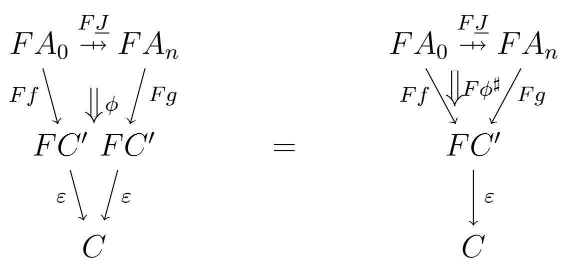
1.39. DEFINITION. *Let* $F\colon \mathcal{K} \to \mathcal{L}$ *be a functor of augmented virtual double categories (Definition A3.1) and* $C \in \mathcal{L}$ *an object. We call a morphism* $\varepsilon\colon FC' \to C$ locally universal from F to C *if the functor*

$$\varepsilon \circ F\,-\colon \mathcal{K} /_{\mathrm{v}} C' \to F /_{\mathrm{v}} C$$

is full and faithful; if it is an equivalence then we call ε universal from F to C.

Let $\mathcal{J} \subseteq F /_{\mathrm{v}} C$ *be a full subcategory. A locally universal morphism* $\varepsilon\colon FC' \to C$ *is called* universal relative to $\mathcal{J}$ *if the full and faithful functor* $\varepsilon \circ F\,-$ *above factors through the inclusion* $\mathcal{J} \hookrightarrow F /_{\mathrm{v}} C$ *as an equivalence* $\mathcal{K} /_{\mathrm{v}} C' \xrightarrow{\simeq} \mathcal{J}$.

Unpacking the definition of a locally universal morphism $\varepsilon\colon FC' \to C$, notice that any nullary cell ϕ as on the left-hand side below factors through ε as the F-image of a unique nullary cell $\phi^\sharp\colon \underline{J} \Rightarrow C'$, as shown. The cells ϕ and $\phi^\sharp$ are said to be *adjuncts* of each other. If moreover ε is universal relative to $\mathcal{J}$ then for any morphism $h\colon FA \to C$ in $\mathcal{J}$ we can choose an *adjunct* $h^\sharp\colon A \to C'$ in $\mathcal{K}$ as well, such that $h \cong \varepsilon \circ Fh^\sharp$ in $\mathcal{L}$.



Notice that the uniqueness of the adjuncts $\phi^\sharp$ implies that the assignment $\phi \mapsto \phi^\sharp$ is functorial with respect to horizontal composition of nullary cells, i.e. $(\phi \odot \psi)^\sharp = \phi^\sharp \odot \psi^\sharp$. Also notice that any morphism universal relative to $\mathcal{J}$ is itself contained in $\mathcal{J}$. It follows that, for any two morphisms $\varepsilon\colon FC' \to C$ and $\zeta\colon FC'' \to C$ that are universal relative to $\mathcal{J}$, the objects C' and C'' are equivalent in the vertical 2-category $V(\mathcal{K})$ (Example A1.5), with the equivalence being the chosen morphism $\zeta^\sharp\colon C'' \to C'$ such that $\varepsilon \circ F\zeta^\sharp \cong \zeta$.

1.40. EXAMPLE. If $F\colon \mathcal{K} \to \mathcal{L}$ is locally full and faithful (Definition A3.6) then any full and faithful morphism $\varepsilon\colon FC' \to C$ in $\mathcal{L}$ (Definition 1.18) is locally universal from F to C.

1.41. EXAMPLE. Let $F \dashv G\colon \mathcal{L} \to \mathcal{K}$ be an adjunction between augmented virtual double categories, with unit $\zeta\colon \mathrm{id}_{\mathcal{K}} \Rightarrow GF$ and counit $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{L}}$ transformations (Definition A3.2). For any object $C \in \mathcal{L}$ the vertical morphism component $\varepsilon_C\colon FGC \to C$ of the counit is a universal morphism from F to C. Indeed the triangle identities for ζ and ε imply that $\varepsilon_C \circ F\,-\colon \mathcal{K} /_{\mathrm{v}} GC \to F /_{\mathrm{v}} C$ has an inverse $(-)^\sharp$ given by $h^\sharp := Gh \circ \zeta_A$ on objects $h\colon FA \to C$ and $\phi^\sharp := G\phi \circ \zeta_{\underline{J}}$ on morphisms $\phi\colon F\underline{J} \Rightarrow C$, where $\zeta_{\underline{J}} := (\zeta_{J_1}, \dots, \zeta_{J_n})$ if $\underline{J} = (J_1, \dots, J_n)$ and $\zeta_{\underline{J}} := \zeta_{A_0}$ if $\underline{J} = (A_0)$. The notion of adjunction between augmented virtual double categories is used in Definition 8.16 below in order to define closed monoidal augmented virtual double categories; see also the subsequent examples.

1.42. Proposition. *Consider the functor $F\colon \mathcal{K} \to \mathcal{L}$, the locally universal morphism $\varepsilon\colon FC' \to C$ and the adjuncts ϕ and $\phi^\sharp$ above. If $\phi\colon F\underline{J} \Rightarrow C$ is (weakly) left Kan in $\mathcal{L}$ then so is $\phi^\sharp\colon \underline{J} \Rightarrow C'$ in $\mathcal{K}$. If moreover ϕ restricts (Definition 1.24) along all F-images Fk of morphisms $k\colon B \to A_n$ for which the restriction $J_n(\mathrm{id}, k)$ exists, and F preserves such restrictions, then $\phi^\sharp$ is pointwise (weakly) left Kan.*

Proof. To prove that $\phi^\sharp$ is left Kan whenever ϕ is so we have to show that any nullary cell $\psi\colon \underline{J} \mathbin{^\frown} \underline{H} \Rightarrow C'$ in $\mathcal{K}$, with vertical source f, factors uniquely as $\psi = \phi^\sharp \odot \psi'$ with $\psi'\colon \underline{H} \Rightarrow C'$, as in Definition 1.9. Since $\phi = \varepsilon \circ F\phi^\sharp$ is left Kan there exists a unique cell $\theta\colon F\underline{H} \Rightarrow C$ in $\mathcal{L}$ such that $\varepsilon \circ F\psi = \phi \odot \theta$, and we claim that $\psi' \coloneqq \theta^\sharp$ is the unique factorisation that we seek. Indeed $\psi = (\varepsilon \circ F\psi)^\sharp = (\phi \odot \theta)^\sharp = \phi^\sharp \odot \psi'$ and, to show that ψ' is unique, consider any other cell ψ'' with $\psi = \phi^\sharp \odot \psi''$. Applying $\varepsilon \circ F-$ gives $\varepsilon \circ F\psi = \phi \odot (\varepsilon \circ F\psi'')$, where we use that F (like any functor of augmented virtual double categories) preserves horizontal composition of cells (see Definition A3.1). By the uniqueness of factorisations through the left Kan cell ϕ it follows that $\varepsilon \circ F\psi'' = \theta$, and hence $\psi'' = (\varepsilon \circ F\psi'')^\sharp = \theta^\sharp = \psi'$.

When restricted to the empty path $\underline{H} = (A_n)$ the argument above reduces to the weakly left Kan case. To prove the pointwise (weakly) left Kan case apply the above to composites of the form $\phi^\sharp \circ (\mathrm{id}, \dots, \mathrm{id}, \mathrm{cart})$, where cart is any cartesian cell that defines a right restriction $J_n(\mathrm{id}, k)$ of J_n along some $k\colon B \to A_n$, as in Definition 1.24, using the fact that $\phi \circ (\mathrm{id}, \dots, \mathrm{id}, F\mathrm{cart}) = \varepsilon \circ F\big(\phi^\sharp \circ (\mathrm{id}, \dots, \mathrm{id}, \mathrm{cart})\big)$ is (weakly) left Kan if ϕ restricts along Ff and F preserves the cartesian cell. ∎

1.43. Example. Let $F \dashv G\colon \mathcal{D} \to \mathcal{C}$ be a strict 2-adjunction between 2-categories, $d\colon FA \to M$ a morphism in $\mathcal{D}$ and $j\colon A \to B$ a morphism in $\mathcal{C}$. The left Kan extension of d along Fj exists in $\mathcal{D}$ whenever that of the adjunct $d^\sharp\colon A \to GM$ along j does in $\mathcal{C}$, and in that case the extensions are adjuncts. To see this consider the induced adjunction $Q(F) \dashv Q(G)$ between the strict double categories of quintets $Q(\mathcal{C})$ and $Q(\mathcal{D})$; see Proposition A6.4. By Example 1.41 the component $\varepsilon_M\colon Q(F)Q(G)M \to M$ of the counit is universal from $Q(F)$ to M, so that the result follows from the previous proposition and Example 1.4.

2. Pasting lemmas

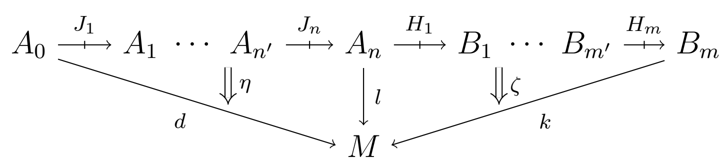
This section describes two useful pasting lemmas for left Kan cells together with some of their consequences. The first of these, below, is the 'horizontal' pasting lemma, which concerns horizontal compositions of left Kan cells and whose proof is straightforward. Applying it to the unital virtual double category $\mathcal{V}$-Prof of $\mathcal{V}$-profunctors (Example 1.13) recovers the classical result on iterated enriched Kan extension (see page 42 of [Dub70] or Theorem 4.47 of [Kel82]).

The 'vertical' pasting lemma, Lemma 2.17 below, concerns (weakly) left Kan cells vertically composed with (weakly) cocartesian paths of cells; the latter in the sense of

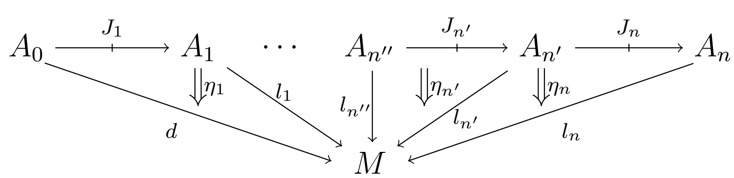
Definition A7.1 (see also Definition 2.6 below). It also applies to weaker variants of the notion of (weakly) cocartesian path, which are introduced in Definition 2.6 below.

2.1. THE HORIZONTAL PASTING LEMMA.

2.2. LEMMA. [Horizontal pasting lemma for left Kan cells] *Assume that the cell* η *in the composite below is left Kan. Then* $\eta \odot \zeta$ *is (weakly) left Kan precisely if* ζ *is so. In that case* $\eta \odot \zeta$ *restricts along* $f : C \to B_m$ *(Definition 1.24) precisely if* ζ *does so. In particular* $\eta \odot \zeta$ *is pointwise (weakly) left Kan precisely if* ζ *is so.*



2.3. EXAMPLE. By iterating the horizontal pasting lemma we see that (pointwise) (weak) left Kan extensions along a path $\underline{J} = (J_1, \dots, J_n)$ can be obtained by extending along each of the $J_1, \dots, J_n$ recursively: if for each $1 \leq i < n$ the cell η_i below is left Kan and η_n is (pointwise) (weak) left Kan then so is their horizontal composite $\eta_1 \odot \cdots \odot \eta_n$.



2.4. REMARK. Notice that we would not have been able to state the horizontal pasting lemma if our notions of left Kan extension had been defined along a *single* horizontal morphism, rather than a path of horizontal morphisms.

2.5. RIGHT COCARTESIAN PATHS. Throughout the remainder of this article it will become clear that the notions of weakly cocartesian path of cells and cocartesian path of cells, as defined in Definition A7.1, are too strong. The vertical pasting lemma (Lemma 2.17 below) for instance, is most naturally stated using the weaker notions defined below. The first of these weakens the univeral property of weakly cocartesian paths by restricting it to either nullary cells or unary cells only, while the second introduces weaker, "right-sided" and "left-sided" variants of the notion of cocartesian path.

2.6. DEFINITION. *A path of cells* $\underline{\phi} = (\phi_1, \dots, \phi_n)$*, as in the right-hand side below, is called* weakly nullary-cocartesian *if any nullary cell* χ*, as on the left-hand side and with* $\underline{L} = (C)$ *empty, factors uniquely through* $\underline{\phi}$ *as shown. Analogously* $\underline{\phi}$ *is called* weakly unary-cocartesian *if any unary cell* χ*, as on the left-hand side below and with* $|\underline{L}| = 1$*, factors uniquely through* $\underline{\phi}$*. The path* $\underline{\phi}$ *is* weakly cocartesian*, in the sense of Definition 1.16, if it is both weakly nullary-cocartesian and weakly unary-cocartesian.*

$$\begin{array}{ccccc}
X_{10} \overset{H_{11}}{\nrightarrow} X_{11} \cdots X_{1m'_1} \overset{H_{1m_1}}{\nrightarrow} X_{1m_1} & & \cdots & & X_{n0} \overset{H_{n1}}{\nrightarrow} X_{n1} \cdots X_{nm'_n} \overset{H_{nm_n}}{\nrightarrow} X_{nm_n} \\
{\scriptstyle h \circ f_0}\downarrow \qquad & & \Downarrow \chi & & \qquad \downarrow {\scriptstyle k \circ f_n} \\
C & & \xrightarrow[\underline{L}]{\qquad\qquad} & & D
\end{array}$$

$$= \begin{array}{ccccccc}
X_{10} \overset{H_{11}}{\nrightarrow} X_{11} \cdots X_{1m'_1} \overset{H_{1m_1}}{\nrightarrow} X_{1m_1} & & & & X_{n0} \overset{H_{n1}}{\nrightarrow} X_{n1} \cdots X_{nm'_n} \overset{H_{nm_n}}{\nrightarrow} X_{nm_n} \\
{\scriptstyle f_0}\downarrow \qquad \Downarrow \phi_1 \qquad \downarrow {\scriptstyle f_1} & & \cdots & & {\scriptstyle f_{n'}}\downarrow \qquad \Downarrow \phi_n \qquad \downarrow {\scriptstyle f_n} \\
A_0 \xrightarrow[\underline{J_1}]{\qquad} A_1 & & & & A_{n'} \xrightarrow[\underline{J_n}]{\qquad} A_n \\
{\scriptstyle h}\downarrow \qquad & & \Downarrow \chi' & & \qquad \downarrow {\scriptstyle k} \\
C & & \xrightarrow[\underline{L}]{\qquad\qquad} & & D
\end{array}$$

A weakly nullary-cocartesian path $(\phi_1, \ldots, \phi_n)$ *of the form below is called* right nullary-cocartesian *if, for any non-empty path* $(K_1, \ldots, K_q)$ *of horizontal morphisms such that the restriction* $K_1(f_n, \mathrm{id})$ *exists, the composite path below is weakly nullary-cocartesian.* Right unary-cocartesian *paths are defined analogously; a path is* right cocartesian *whenever it is both right nullary-cocartesian and right unary-cocartesian.*

$$\begin{array}{ccccccccccc}
X_{10} \overset{H_{11}}{\nrightarrow} X_{11} \cdots X_{1m'_1} \overset{H_{1m_1}}{\nrightarrow} X_{1m_1} & & X_{n0} \overset{H_{n1}}{\nrightarrow} X_{n1} \cdots X_{nm'_n} \overset{H_{nm_n}}{\nrightarrow} X_{nm_n} & \overset{K_1(f_n, \mathrm{id})}{\nrightarrow} & Y_1 & \overset{K_2}{\nrightarrow} & Y_2 & & Y_{q'} & \overset{K_q}{\nrightarrow} & Y_q \\
{\scriptstyle f_0}\downarrow \quad \Downarrow \phi_1 \quad \downarrow {\scriptstyle f_1} & \cdots & {\scriptstyle f_{n'}}\downarrow \quad \Downarrow \phi_n \quad \downarrow {\scriptstyle f_n} & \mathrm{cart} & \Vert & & \Vert & \cdots & \Vert & & \Vert \\
A_0 \xrightarrow[\underline{J_1}]{\qquad} A_1 & & A_{n'} \xrightarrow[\underline{J_n}]{\qquad} A_n & \underset{K_1}{\nrightarrow} & Y_1 & \underset{K_2}{\nrightarrow} & Y_2 & & Y_{q'} & \underset{K_q}{\nrightarrow} & Y_q
\end{array}$$

Horizontally dual, the weakly nullary-cocartesian path $(\phi_1, \ldots, \phi_n)$ *above is called* left nullary-cocartesian *if, for any non-empty path* $(K'_1, \ldots, K'_p)\colon Y'_0 \nrightarrow A_0$ *such that the restriction* $K'_p(\mathrm{id}, f_0)$ *exists, the composite path* $(\mathrm{id}_{K'_1}, \ldots, \mathrm{id}_{K'_{p'}}, \mathrm{cart}, \phi_1, \ldots, \phi_n)$ *is weakly nullary-cocartesian, where* cart *defines* $K'_p(\mathrm{id}, f_0)$*.* Left unary-cocartesian paths *are defined analogously; a path is* left cocartesian *whenever it is both left nullary-cocartesian and left unary-cocartesian.*

A right nullary-cocartesian path $(\phi_1, \ldots, \phi_n)$ *is called* nullary-cocartesian *if each of the composite paths* $(\phi_1, \ldots, \phi_n, \mathrm{cart}, \mathrm{id}_{K_2}, \ldots, \mathrm{id}_{K_q})$ *above are left nullary-cocartesian.* Unary-cocartesian *paths are defined analogously. A path is* cocartesian *in the sense of Definition A7.1*[1] *precisely if it is both nullary-cocartesian and unary-cocartesian.*

Notice that the universal property of a right (respectively weakly) nullary-cocartesian cell does not determine its horizontal target up to isomorphism, in contrast to the universal property of (right) (respectively weakly) (unary-)cocartesian cells. While we

[1] Definition 7.1 of the original version of [Kou20] (published 2020-02-24), defining the notion of cocartesian path $\underline{\phi}$, contains a typo: the paths of identity cells that $\underline{\phi}$ is concatenated with are allowed to be of any lengths p and q, that is $p, q \geq 0$ (and not $p, q \geq 1$ as originally printed).

will mostly use the nullary variant of right cocartesian paths of cells, the unary variant will be important in our study of 'exact cells' in Section 5 below. Proposition 5.6 for instance shows that, under mild conditions, horizontal cells are 'left exact' with respect to a 'Yoneda morphism' (Definition 4.5) if and only if they are right unary-cocartesian. The notion of left nullary-cocartesian paths is used in Section 3.

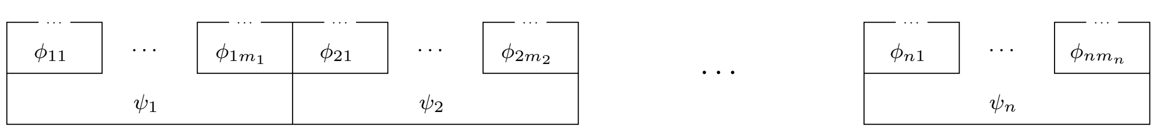
2.7. Example. Assume that the horizontal unit $I_C : C \nrightarrow C$ (Definition 1.16) of the object C exists. It follows from the horizontal unit identities (see Lemma A5.9 or Lemma 1.21) that the factorisations for the nullary cells χ above, with empty horizontal target C, correspond precisely to factorisations of unary cells χ with horizontal target I_C. We conclude that in unital virtual double categories (Definition 1.19) the notions of weakly unary-cocartesian path and weakly cocartesian path coincide, as well as those of right (respectively left) unary-cocartesian path and right (respectively left) cocartesian path.

Similarly the notions of right (respectively left or weakly) nullary-cocartesian path and right (respectively left or weakly) cocartesian path coincide in augmented virtual double categories $\mathcal{K}$ that admit, for each horizontal morphism $L : C \nrightarrow D$, a nullary cartesian cell $L \Rightarrow X$ where X is any object; see also Definition 3.2 below. Examples of such $\mathcal{K}$ are the augmented virtual equipments of enriched profunctors $\mathcal{V}$-Prof and $(\mathcal{V}, \mathcal{V}')$-Prof (Examples A2.4 and A2.7); see Example 3.10.

2.8. Example. Clearly cocartesian paths are both left and right cocartesian. The converse however does not hold in general since cocartesian paths $\underline{\phi}$ are required to be preserved under two-sided concatenations of the form $(\mathrm{id}, \dots, \mathrm{id}, \mathrm{cart}) \frown \underline{\phi} \frown (\mathrm{cart}, \mathrm{id}, \dots, \mathrm{id})$.

The following is a straightforward variation on the pasting lemma for cocartesian paths (Lemma A7.7). Likewise Corollary 2.10 is a variation on Corollary A8.5.

2.9. Lemma. [Pasting lemma for cocartesian paths] *In the configuration of cells below denote by* $\underline{\phi_j}$ *the path* $\underline{\phi_j} := (\phi_{j1}, \dots, \phi_{jn_j})$*, for each* $1 \leq j \leq n$*, and assume that the path* $(\phi_{11}, \dots, \phi_{nm_n})$ *is weakly nullary-cocartesian. Then the path* $\underline{\psi} := (\psi_1, \dots, \psi_n)$ *is weakly nullary-cocartesian if and only if the path of composites* $\big(\psi_1 \circ \underline{\phi_1}, \dots, \psi_n \circ \underline{\phi_n}\big)$ *is so.*



Next denote the vertical targets of ϕ_{nm_n} *and* ψ_n *by* $f_{nm_n} : X_{nm_nk_{m_n}} \to A_{nm_n}$ *and* $h_n : A_{nm_n} \to C_n$*. If the path* $(\phi_{11}, \dots, \phi_{nm_n})$ *is right nullary-cocartesian then the following hold:*

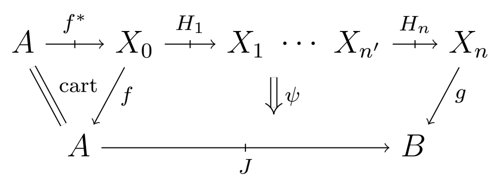
(a) *if* $\underline{\psi}$ *is right nullary-cocartesian then so is* $\big(\psi_1 \circ \underline{\phi_1}, \dots, \psi_n \circ \underline{\phi_n}\big)$ *provided that for any horizontal morphism* $K : C_n \nrightarrow C'$ *the following holds: if the restriction* $K(h_n \circ f_{nm_n}, \mathrm{id})$ *exists then so does* $K(h_n, \mathrm{id})$*;*

(b) *if* $(\psi_1 \circ \underline{\phi}_1, \dots, \psi_n \circ \underline{\phi}_n)$ *is right nullary-cocartesian then so is* $\underline{\psi}$ *provided that for any horizontal morphism* $K : C_n \nrightarrow C'$ *the following holds: if the restriction* $K(h_n, \mathrm{id})$ *exists then so does* $K(h_n \circ f_{nm_n}, \mathrm{id})$.

Analogous assertions hold for right (respectively weakly) (unary-)cocartesian paths; horizontally dual assertions hold for left (nullary- or unary-)cocartesian paths.

2.10. Corollary. *The cell* ψ *below is right (respectively weakly) nullary-cocartesian if and only if the composite is so.*

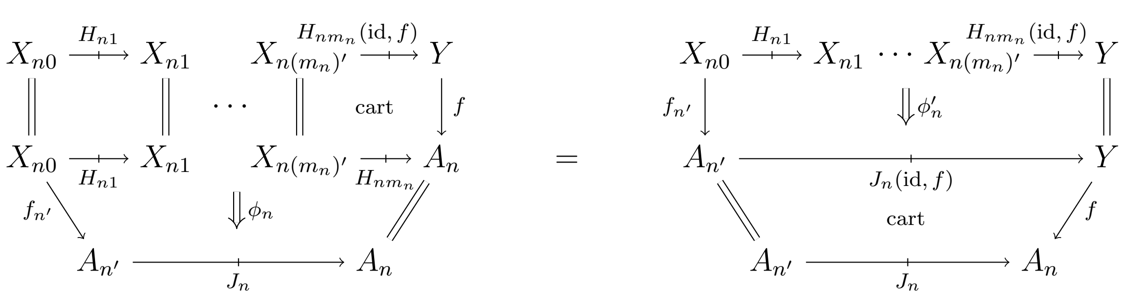
An analogous assertion holds for the composite $\psi \odot \mathrm{cart}$, *where* cart *defines the companion* $g_* : X_n \nrightarrow B$. *Analogous assertions hold for right (respectively weakly) (unary-)cocartesian cells and left (nullary- or unary-)cocartesian cells.*



Proof. Apply the pasting lemma to $\psi_1 = \mathrm{cart} \odot \psi$ and $\underline{\phi}_1 = (\mathrm{cocart}, \mathrm{id}_{H_1}, \dots, \mathrm{id}_{H_n})$ where cocart denotes the cocartesian cell corresponding to the cartesian cell above (Lemma 1.21), so that $\psi_1 \circ \underline{\phi}_1 = \psi$. ■

2.11. Pointwise right cocartesian paths. The pointwise variants of the notions of right (respectively weakly) nullary- and unary-cocartesian path are similar to the pointwise variant of cocartesian path given in Definition A9.1, as follows; see Remark 2.13 below for the differences. Recall that $n' := n - 1$ for any postive integer n.

2.12. Definition. *Consider a right (respectively weakly) nullary-cocartesian path* $\underline{\phi} = (\phi_1, \dots, \phi_n)$ *whose last cell* ϕ_n *has trivial vertical target and arity* $(m_n, 1)$ *with* $m_n \geq 1$, *as in the left-hand side below.*



(a) *Let* $f : Y \to A_n$ *be any morphism. We say that* $\underline{\phi}$ restricts along f *if both restrictions* $H_{nm_n}(\mathrm{id}, f)$ *and* $J_n(\mathrm{id}, f)$ *exist and the path* $(\phi_1, \dots, \phi_{n'}, \phi'_n)$ *is again right (respectively weakly) nullary-cocartesian, where* ϕ'_n *is the unique factorisation in the right-hand side above.*

(b) *We call* $\underline{\phi}$ pointwise *if it restricts along any morphism* $f\colon Y \to A_n$ *such that* $H_{nm_n}(\mathrm{id}, f)$ *exists.*

The notion of restriction *for right (respectively weakly) unary-cocartesian and right (respectively weakly) cocartesian cells is defined analogously, as is the notion of* pointwise right (respectively weakly) unary-cocartesian *path and that of* pointwise right (respectively weakly) cocartesian *path.*

Unpacking the definition of a pointwise right (respectively weakly) nullary-cocartesian path $\underline{\phi}$ above we find that it requires the following for every morphism $f\colon Y \to A_n$: if the restriction $H_{nm_n}(\mathrm{id}, f)$ exists then so does $J_n(\mathrm{id}, f)$ and, in that case, the path $(\phi_1, \dots, \phi_{n'}, \phi'_n)$, with ϕ'_n as above, is right (respectively weakly) nullary-cocartesian. A single horizontal cell $\phi\colon (H_1, \dots, H_n) \Rightarrow J$ is called *(pointwise) right cocartesian* whenever the singleton path (ϕ) is (pointwise) right cocartesian; in that case we call J the *(pointwise) right composite* of $(H_1, \dots, H_n)$. If (ϕ) is cocartesian (Definition 2.6) then J is the *horizontal composite* $(H_1 \odot \cdots \odot H_n)$ in the sense of Definition A7.1; in this article we will use the notation $(H_1 \odot \cdots \odot H_n)$ for both horizontal composites and right composites. Notice that the pasting lemma (Lemma 2.9) only allows us to combine right composites on the left, that is given a right composite $(H_1 \odot \cdots \odot H_n)\colon A_0 \nrightarrow A_n$ and a path $\underline{J}\colon A_n \nrightarrow B_m$, the right composite $\big((H_1 \odot \cdots \odot H_n) \odot J_1 \odot \cdots \odot J_m\big)$ exists if and only if the right composite $(H_1 \odot \cdots \odot H_n \odot J_1 \odot \cdots \odot J_m)$ does, and in that case they are canonically isomorphic. In particular the associator $\big((H \odot J) \odot K)\big) \cong \big(H \odot (J \odot K)\big)$ of right composites need not exist in general, unless $(J \odot K)$ is a horizontal composite.

2.13. Remark. Consider a path $\underline{\phi} = (\phi_1, \dots, \phi_n)$ of the form as in the definition above, but which is not necessarily nullary-cocartesian. In Definition A9.1 $\underline{\phi}$ is called *right pointwise cocartesian* if, for every morphism $f\colon Y \to A_n$ for which both $H_{nm_n}(\mathrm{id}, f)$ and $J_n(\mathrm{id}, f)$ exist, the path $(\phi_1, \dots, \phi_{n'}, \phi'_n)$, with ϕ'_n as above, is cocartesian. Since every cocartesian path is right cocartesian (Example 2.8) it follows that any right pointwise cocartesian path, in the sense of Definition A9.1, is pointwise right cocartesian, in the above sense, whenever, for each $f\colon Y \to A_n$, if the restriction $H_{nm_n}(\mathrm{id}, f)$ exists then so does the restriction $J_n(\mathrm{id}, f)$.

We conclude that in augmented virtual double categories that have restrictions on the right (Definition 1.19) any right pointwise cocartesian path is pointwise right cocartesian and, in particular, any pointwise composite $(H_1 \odot \cdots \odot H_n)$ in the sense of Definition A9.1 is a pointwise right composite in the sense above. Under some conditions on the base category $\mathcal{V}$ such pointwise composites exist in the augmented virtual double categories $\mathcal{V}$-Prof, $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ and $\mathcal{V}$-sProf of (small) $\mathcal{V}$-profunctors, where they are given by $\mathcal{V}$-coends; see Examples A9.2 and A9.3.

Pointwise cocartesian paths are coherent in the sense of the following two lemmas. Taking into account the slight difference in the notions of pointwise right cocartesian path and right pointwise cocartesian path regarding the existence of the relevant restrictions, as described in the previous remark, the proofs of these lemmas are essentially the same as those of Lemmas A9.5 and A9.6.

2.14. Lemma. *If the path* $(\phi_1, \ldots, \phi_n)$ *is pointwise right (respectively weakly) (nullary- or unary-)cocartesian then any path of the form* $(\phi_1, \ldots, \phi'_n)$ *as in Definition 2.12 is again pointwise right (respectively weakly) (nullary- or unary-)cocartesian.*

Let $(H_1 \odot \cdots \odot H_n)$ be a pointwise right composite and f a morphism such that the restriction $H_n(\mathrm{id}, f)$ exists. Applying the lemma above to the singleton path consisting of the pointwise right cocartesian cell defining the composite we find that the restriction $(H_1 \odot \cdots \odot H_n)(\mathrm{id}, f)$ forms the pointwise right composite $\big(H_1 \odot \cdots \odot H_n(\mathrm{id}, f)\big)$.

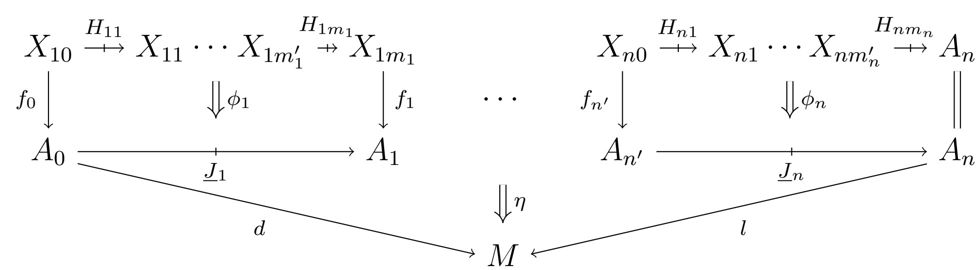
2.15. Lemma. [Pasting lemma for pointwise cocartesian paths] *Consider the configuration of cells of Lemma 2.9. Assume that all its cells* ψ_i *and* ϕ_{ij} *are unary, that* ϕ_{nm_n} *has non-empty horizontal source, and that the vertical targets of the final cells* ψ_n *and* ϕ_{nm_n} *are both the identity morphism on the object* C_n*. Denote by* J *the horizontal target of* ϕ_{nm_n} *and by* H *the final morphism of its horizontal source. Assume that the path* $(\phi_{11}, \ldots, \phi_{nm_n})$ *is pointwise right (respectively weakly) (nullary- or unary-)cocartesian.*

If the path $(\psi_1, \ldots, \psi_n)$ *pointwise right (respectively weakly) (nullary- or unary-)cocartesian then so is the path of composites* $(\psi_1 \circ \underline{\phi}_1, \ldots, \psi_n \circ \underline{\phi}_n)$*. The converse holds whenever the following holds for all morphisms* $f\colon Y \to C_n$*: if the restriction* $J(\mathrm{id}, f)$ *exists then so does* $H(\mathrm{id}, f)$*.*

2.16. The vertical pasting lemma. We are now ready to state the vertical pasting lemma for left Kan cells.

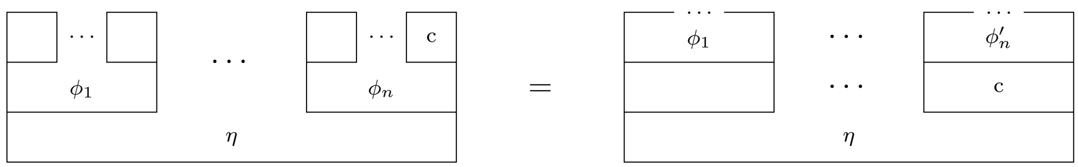
2.17. Lemma. [Vertical pasting lemma for left Kan cells] *Consider the composite below. If the path* $\underline{\phi} = (\phi_1, \ldots, \phi_n)$ *is right (respectively weakly) nullary-cocartesian then the cell* η *is (weakly) left Kan precisely if the composite* $\eta \circ \underline{\phi}$ *is so. If moreover* $\underline{\phi}$ *restricts along* $f\colon Y \to A_n$ *then* η *restricts along* f *(Definition 1.24) precisely if the composite does so.*

Next assume that $\underline{\phi}$ *is pointwise right (respectively weakly) nullary-cocartesian. If* η *is pointwise (weakly) left Kan (Definition 1.24) then so is the composite. The converse holds whenever the restrictions* $H_{nm_n}(\mathrm{id}, f)$ *exist for all* $f\colon Y \to A_n$*.*



Proof. The main assertion follows from applying Lemma 2.20 below to η and $\underline{\chi} = \underline{\phi}$, while $\underline{\zeta}$ ranges over all (including the empty) paths of horizontal identity cells. Indeed, the assertion of that lemma implies that the unique factorisations showing that η is (weakly) left Kan correspond precisely to those showing that $\eta \circ \underline{\phi}$ is so.

Next assume that $\underline{\phi}$ restricts along a morphism $f\colon Y \to A_n$ so that, in particular, ϕ_n is of arity $(m_n, 1)$ with $m_n \geq 1$. The schematically drawn identity below, where the cartesian cells defining $H_{nm_n}(\mathrm{id}, f)$ and $J_n(\mathrm{id}, f)$ are both denoted by 'c', follows from the identity of Definition 2.12. By Definition 2.12(a) the top row of the right-hand side is again right (respectively weakly) nullary-cocartesian.



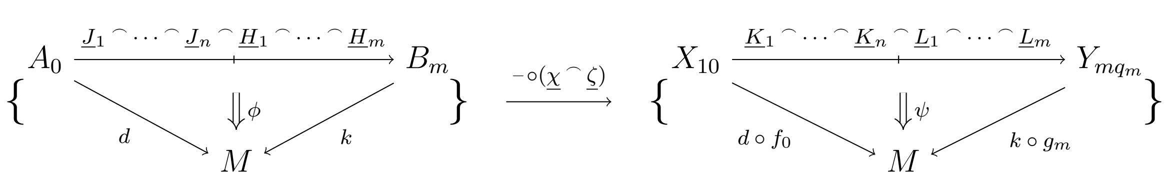
Notice that, by Definition 1.24, $\eta \circ \underline{\phi}$ restricts along f as soon as the left-hand side above is (weakly) left Kan while η restricts along f whenever the composite of the bottom two rows of the right-hand side is so. That the latter are equivalent follows from applying the main assertion to $(\phi_1, \dots, \phi'_n)$.

Finally assume that $\underline{\phi}$ is pointwise right (respectively weakly) nullary-cocartesian. If η is pointwise (weakly) left Kan then, for each $f\colon Y \to A_n$ such that $H_{nm_n}(\mathrm{id}, f)$ exists, both $\underline{\phi}$ and η restrict along f so that by the previous the composite $\eta \circ \underline{\phi}$ restricts along f too. We conclude that the composite is pointwise (weakly) left Kan. The converse similarly follows from the previous provided that all restrictions $H_{nm_n}(\mathrm{id}, f)$ exist so that, by the assumption on $\underline{\phi}$, so do all restrictions $J_n(\mathrm{id}, f)$. ■

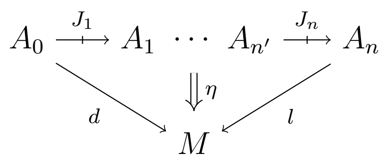
2.18. Example. Consider a path $(J_1, \dots, J_n)\colon A_0 \nrightarrow A_n$ of horizontal morphisms as well as a vertical morphism $d\colon A_0 \to M$ and assume that the horizontal composite $(J_1 \odot \cdots \odot J_n)$ exists (Definition 2.12). Applying the vertical pasting lemma to the cocartesian cell defining the composite we find that the (weak) left Kan extension of d along $(J_1, \dots, J_n)$ exists precisely if that of d along $(J_1 \odot \cdots \odot J_n)$ does, and in that case they are isomorphic. If $(J_1 \odot \cdots \odot J_n)$ is a pointwise composite (Remark 2.13) and all restrictions on the right exist (Definition 1.19) then the analogous equivalence holds for pointwise (weak) left Kan extensions.

2.19. Example. Consider parallel morphisms d and $l\colon A \to M$ whose source admits a horizontal unit $I_A\colon A \nrightarrow A$ (Definition 1.16). Applying the vertical pasting lemma to the cocartesian cell defining I_A (Lemma 1.21), combined with Example 1.3, we find that l is the (weak) left Kan extension of d along I_A if and only if $l \cong d$.

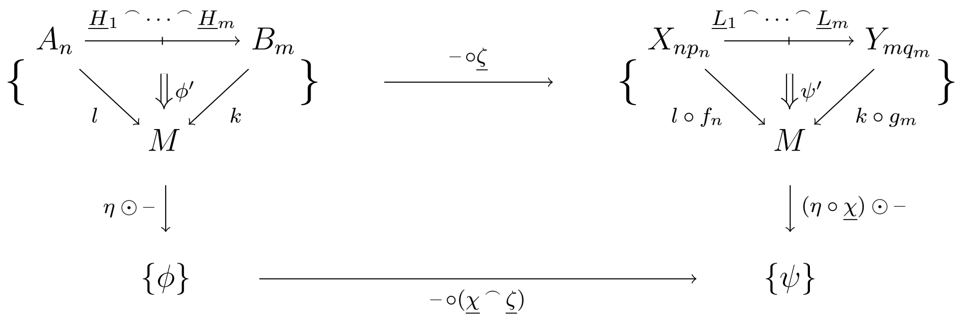
2.20. Lemma. *Consider composable paths* $\underline{\chi} = (\chi_1, \dots, \chi_n)$ *and* $\underline{\zeta} = (\zeta_1, \dots, \zeta_m)$ *of cells* $\chi_i\colon \underline{K}_i \Rightarrow \underline{J}_i$ *and* $\zeta_j\colon \underline{L}_j \Rightarrow \underline{H}_j$*, and assume that their concatenation* $\underline{\chi} \frown \underline{\zeta}$ *is weakly nullary-cocartesian so that the assignment of cells below, given by composition with* $\underline{\chi} \frown \underline{\zeta}$*, is a bijection; here* $f_0\colon X_{10} \to A_0$ *and* $g_m\colon Y_{mq_m} \to B_m$ *denote the vertical source of* χ_1 *and the vertical target of* ζ_m*.*



Consider a nullary cell η of the form below. If $\underline{\zeta}$ is weakly nullary-cocartesian too then cells of the form ϕ above factor uniquely through η, i.e. each $\phi = \eta \odot \phi'$ for a unique nullary cell $\phi' \colon \underline{H}_1 \frown \cdots \frown \underline{H}_m \Rightarrow M$, precisely if cells of the form ψ above factor uniquely through $\eta \circ \underline{\chi}$, i.e. each $\psi = (\eta \circ \underline{\chi}) \odot \psi'$ for a unique nullary cell $\psi' \colon \underline{L}_1 \frown \cdots \frown \underline{L}_m \Rightarrow M$.



Proof. Consider the following assignments between collections of nullary cells that are of the form as shown, where $f_n \colon X_{np_n} \to A_n$ denotes the common vertical boundary of χ_n and ζ_1. The diagram commutes by one of the interchange axioms (Lemma A1.3).



The horizontally drawn assignments are bijective because the paths $\underline{\zeta}$ and $\underline{\chi} \frown \underline{\zeta}$ are assumed to be weakly nullary-cocartesian, so that the proof follows. ■

2.21. Example. Consider an augmented virtual double category $\mathcal{K}$ that admits all horizontal composites (Definition 2.12). Applying the lemma to the cocartesian cells $\zeta =$ cocart defining these composites we find that the universal property of a left Kan cell $\eta \colon \underline{J} \Rightarrow M$ in $\mathcal{K}$, as given in Definition 1.9, reduces to the requirement that cells of the form $\phi \colon \underline{J} \frown \underline{H} \Rightarrow M$ with $\underline{H}$ of length $|\underline{H}| = 0$ or 1 factor uniquely through η.

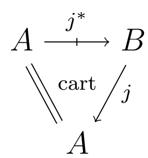
If $\mathcal{K}$ has horizontal units too, i.e. it is induced by a pseudo double category (see Proposition A7.8), then the universal property need only be checked for cells of the form $\phi \colon \underline{J} \frown \underline{H} \Rightarrow M$ with $|\underline{H}| = 1$. It follows that the present notion of left Kan extension in $\mathcal{K}$ coincides with that of 'pointwise left Kan extension' in $\mathcal{K}$ regarded as a pseudo double category, in the sense of Definition 3.10 of [Kou14a].

Finally consider a 'proarrow equipment' $(-)_* \colon \mathcal{K} \to \mathcal{M}$ in the sense of Wood [Woo85] (where $\mathcal{M}$ need not be biclosed, as is required in [Woo82]). As is shown in Proposition C.3 of [Shu08], if $\mathcal{K}$ is a strict 2-category then $(-)_*$ induces a pseudo double category $\mathcal{D}$ that has all companions and conjoints (a 'framed bicategory'). The objects and vertical morphisms of $\mathcal{D}$ are those of $\mathcal{K}$, and the horizontal morphisms are those of $\mathcal{M}$. In such proarrow equipments Wood's notion of 'indexed limit', given in Section 2 of [Woo82], coincides with the notion of 'pointwise left Kan extension' in the corresponding pseudo double category

$\mathcal{D}$, in the sense of [Kou14a] (see its Section 3.5), and hence coincides with our notion of left Kan extension.

2.22. Consequences of the pasting lemmas. The remainder of this section consists of consequences of the pasting lemmas.

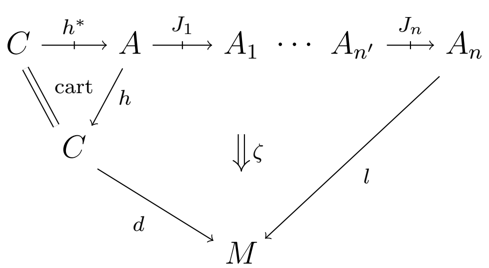
2.23. Corollary. *Let $j\colon B \to A$ be a vertical morphism. The nullary cartesian cell below, that defines the conjoint of j, is absolutely pointwise left Kan (Definition 1.36).*

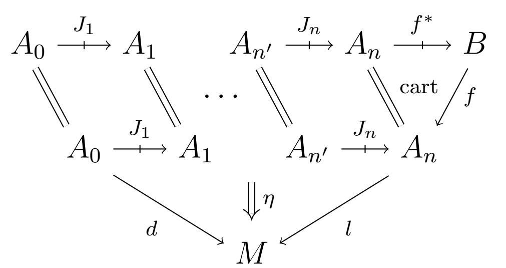


Proof. To see that the cartesian cell above is absolutely left Kan we have to show that $g \circ \text{cart}$ is left Kan for any $g\colon A \to N$. This follows from applying the vertical pasting lemma to the identity $g \circ \text{cart} \circ \text{cocart} = \text{id}_{g\circ j}$ which is the conjoint identity for j^* (Lemma 1.21) composed with g, and where $\text{id}_{g\circ j}$ is left Kan by Example 1.3. That the cartesian cell is in fact absolutely pointwise left Kan follows from the fact that $j^*(\text{id}, f) \cong (j \circ f)^*$ for any $f\colon C \to B$; see Lemma 1.17. ■

Combined with the horizontal pasting lemma the previous result implies the following.

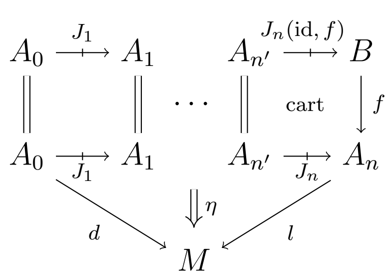
2.24. Corollary. *The composite on the left below is (pointwise) (weak) left Kan precisely if the cell ζ is so. The composite on the right is pointwise left Kan as soon as the cell η is left Kan.*





The following result was used in the proof of Corollary 1.22.

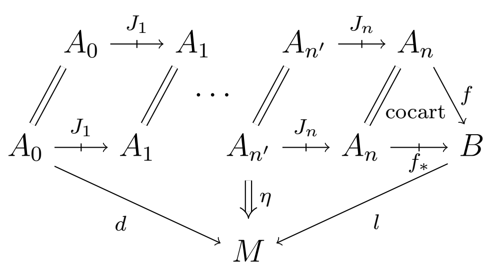
2.25. Proposition. *The composite below is left Kan as soon as the cell η is so and the conjoint $f^*\colon A_n \nrightarrow B$ exists.*



Proof. Lemma A8.1 supplies a cocartesian horizontal cell $\phi\colon (J_n, f^*) \Rightarrow J_n(\mathrm{id}, f)$ which, when composed with the composite above, gives the composite on the right of Corollary 2.24. The latter is left Kan because η is so and the proof follows from applying the vertical pasting lemma to the path $(\mathrm{id}_{J_1}, \dots, \mathrm{id}_{J_{n'}}, \phi)$. ■

The following propositions generalise well known results for Kan extensions internal in a 2-category and enriched Kan extensions. The first of these generalises Proposition 22 of [Str74b] (for internal Kan extensions) and Proposition 4.23 of [Kel82] (for enriched Kan extensions); see also Proposition 14 of [Woo82] for the analogous result in proarrow equipments. Remember that a vertical morphism $f\colon A \to C$ is called full and faithful (Definition 1.18) whenever its identity cell id_f is cartesian.

2.26. Proposition. *Consider a (weakly) left Kan cell η as in the composite below and assume it restricts along the morphism $f\colon A_n \to B$ (Definition 1.24). If f is full and faithful and the restriction $B(f, f)$ exists then the composite is again (weakly) left Kan; in particular if $n = 0$ then the composite, in that case a vertical cell, is invertible.*

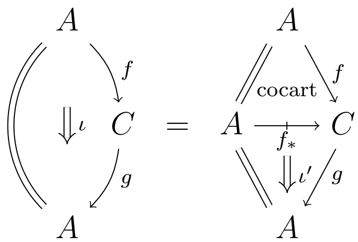


If f is full and faithful then $B(f, f)$ is the horizontal unit of A_n by Lemma A5.14 so that, in an augmented virtual double category with restrictions on the right, the composite above is in fact pointwise (weakly) left Kan by Corollary 1.22.

Proof. Assume that f is full and faithful. Writing $\chi\colon B(f, f) \Rightarrow B$ for the cartesian cell defining $B(f, f)$ consider the unique factorisation $\mathrm{id}_f = \chi \circ \mathrm{id}'_f$, where the horizontal cell $\mathrm{id}'_f\colon A_n \Rightarrow B(f, f)$ is cartesian and cocartesian by Lemma 1.21. Factorising both sides of the latter identity through the cartesian cell $f_* \Rightarrow B$ corresponding to the cocartesian cell above, in the sense of Lemma 1.21, we obtain $\mathrm{cocart} = \chi' \circ \mathrm{id}'_f$ where $\chi'\colon B(f, f) \Rightarrow f_*$ is cartesian by the pasting lemma (Lemma 1.17) and id'_f is cocartesian. The main assertion now follows from the assumption that η restricts along f, so that $\eta \circ (\mathrm{id}_{\underline{J}} \frown \chi')$ is (weakly) left Kan, and the application of the vertical pasting lemma (Lemma 2.17) to the path $\mathrm{id}_{\underline{J}} \frown \mathrm{id}'_f$ in $\eta \circ (\mathrm{id}_{\underline{J}} \frown \chi') \circ (\mathrm{id}_{\underline{J}} \frown \mathrm{id}'_f) = \eta \circ (\mathrm{id}_{\underline{J}} \frown \mathrm{cocart})$. That the composite is invertible in the case of $n = 0$ follows immediately from Example 1.3. ■

When applied to the unital virtual double categories $Q(\mathcal{C})$, of quintets in a 2-category $\mathcal{C}$ (see Example 1.4), or $\mathcal{V}$-Prof, of $\mathcal{V}$-profunctors (see Example 1.13), the following result reduces to the classical description of right adjoints as left Kan extensions, see e.g. Example 2.17 of [Web07] and Theorem 4.81 of [Kel82] respectively.

2.27. Proposition. *In an augmented virtual double category $\mathcal{K}$ consider the factorisation below.*



The following are equivalent:

(a) *ι is the unit of an adjunction $f \dashv g$ in the 2-category $V(\mathcal{K})$ (Example A1.5);*

(b) *ι' is cartesian in $\mathcal{K}$ (defining f_* as the conjoint of g);*

(c) *ι' is weakly left Kan in $\mathcal{K}$ and is preserved by f (Definition 1.36);*

(d) *ι' is absolutely pointwise left Kan in $\mathcal{K}$ (Definition 1.36).*

Under these conditions f is full and faithful (Definition 1.18) if ι is invertible; the converse holds whenever the restriction $C(f, f)$ exists.

Proof. We have (a) $\Rightarrow$ (b) by Lemma A5.16 and (b) $\Rightarrow$ (d) by Corollary 2.23 while clearly (d) $\Rightarrow$ (c). That (c) $\Rightarrow$ (a) follows from combining Proposition 1.7 with Example 2.17 of [Web07]. For the final assertion notice that the invertible unit cell ι is cartesian (Example A4.4) so that, composing with the counit $\varepsilon\colon f \circ g \Rightarrow \mathrm{id}_C$, the identity cell $\mathrm{id}_f = (f \circ \iota) \odot (\varepsilon \circ f)$ is cartesian too by Lemma A4.17. The converse follows from Proposition 2.26. ∎

2.28. Example. Applying the first assertion of Corollary 2.24 to the cell ι' of the previous proposition we obtain the following. Consider an adjoint pair $f \dashv g\colon A_0 \to C$, a morphism $d\colon C \to M$ and a path $\underline{J}\colon A_0 \nrightarrow A_n$. If the companion $f_*\colon C \nrightarrow A_0$ exists then the (pointwise) (weak) left Kan extension of $d \circ g$ along $\underline{J}$ exists if and only if that of d along $f_* \frown \underline{J}$ does, and in that case they are isomorphic.

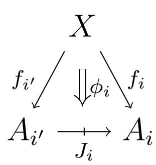
3. Pointwise Kan extension in terms of pointwise weak Kan extension

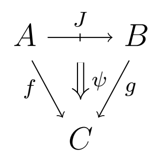
In Theorem 3.20 below we prove that the notions of pointwise weak left Kan extension and pointwise left Kan extension (Definition 1.24) coincide in augmented virtual double categories $\mathcal{K}$ that have restrictions on the right (Definition 1.19) as well as 'left nullary-cocartesian paths of (0, 1)-ary cells'; the latter in the sense of the definition below, which is a strengthening of Definition A7.10. Together with Remark 1.26 and Example 2.21 this result recovers Theorem 5.11 of [Kou14a]. In most of our examples left nullary-cocartesian paths of (0, 1)-ary cells can be obtained by "concatenating" cocartesian universal cells that define 'tabulations', in the sense of Definition 3.5 below; this is explained in Corollary 3.6.

Proposition 3.22 shows that if $\mathcal{K}$ has such 'cocartesian tabulations' then pointwise left Kan extension along a companion j_* in $\mathcal{K}$ coincides with pointwise left Kan extension along j in the vertical 2-category $V(\mathcal{K})$ (Example A1.5); the latter in the classical sense of [Str74b].

3.1. COCARTESIAN PATHS OF $(0,1)$-ARY CELLS.

3.2. DEFINITION. *Let* $\underline{J} = (J_1, \dots, J_n)\colon A_0 \nrightarrow A_n$ *be a path of horizontal morphisms. A* (left, right or weakly) (nullary-)cocartesian path of $(0,1)$-ary cells *for* $\underline{J}$ *consists of an object* X *together with a (left, right or weakly) (nullary-)cocartesian path (Definition 2.6)* $\underline{\phi} = (\phi_1, \dots, \phi_n)$ *of* $(0,1)$*-ary cells* ϕ_i *as on the left below. An augmented virtual double category* $\mathcal{K}$ *is said to* have (left, right or weakly) (nullary-)cocartesian paths of $(0,1)$-ary cells *if every path* $\underline{J} \in \mathcal{K}$ *admits a (left, right or weakly) (nullary-)cocartesian path* $\underline{\phi}$ *of* $(0,1)$*-ary cells.*





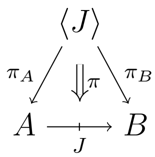
Vertically dual, a cartesian nullary cell *for a horizontal morphism* $J\colon A \nrightarrow B$ *is a cartesian cell* ψ *as on the right above. An augmented virtual double category* $\mathcal{K}$ *is said to* have cartesian nullary cells *if every horizontal morphism* $J \in \mathcal{K}$ *admits a cartesian nullary cell.*

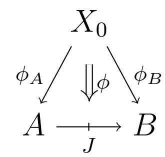
Notice that the cartesian nullary cell ψ on the right above defines J as the nullary restriction $C(f, g)$ (Definition 1.16). Conversely nullary restrictions, including companions, conjoints and horizontal units, admit cartesian nullary cells, by definition.

3.3. EXAMPLE. Given a '(weak) Yoneda morphism' $\mathrm{y}\colon A \to \widehat{A}$, in the sense of Definition 4.5 below, any horizontal morphism $J\colon A \nrightarrow B$ admits a cartesian nullary cell $J \Rightarrow \widehat{A}$; this is a direct consequence of the 'Yoneda axiom' that is satisfied by y.

3.4. COCARTESIAN TABULATIONS. Many augmented virtual double categories $\mathcal{K}$ admit a universal $(0,1)$-ary cell among all $(0,1)$-ary cells $\phi\colon X \Rightarrow J$ into any fixed horizontal morphism J, in the sense of the following definition, which is a direct translation of the double categorical notion of tabulation that was introduced by Grandis and Paré in [GP99]. Examples 3.8–3.13 below give examples of (co)tabulations. Often this universal cell is cocartesian so that, in that case, $\mathcal{K}$ admits all cocartesian paths of $(0,1)$-ary cells that are of length 1. In Corollary 3.6 below we will see that the latter can be "concatenated" to form cocartesian paths of $(0,1)$-ary cells of any length, provided that $\mathcal{K}$ is an augmented virtual equipment (Definition 1.19).

3.5. Definition. *The* tabulation $\langle J \rangle$ *of a horizontal morphism* $J\colon A \nrightarrow B$ *consists of an object* $\langle J \rangle$ *equipped with a* $(0,1)$*-ary cell* π *as on the left below, satisfying the following* 1*-dimensional and* 2*-dimensional universal properties.*

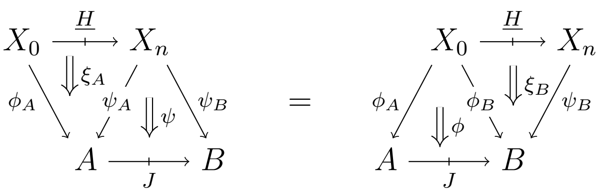


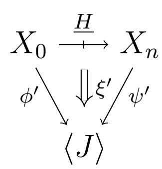


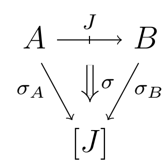
Given another $(0,1)$*-ary cell* ϕ *as on the right above, the* 1*-dimensional property states that there exists a unique morphism* $\phi'\colon X_0 \to \langle J \rangle$ *such that* $\pi \circ \phi' = \phi$.

The 2*-dimensional property is the following. Suppose we are given another* $(0,1)$*-ary cell* ψ *as in the identity below, which factors through* π *as* $\psi'\colon X_n \to \langle J \rangle$*, like* ϕ *factors as* ϕ'*. Then for any pair of cells* ξ_A *and* ξ_B *as in the identity on the left below there exists a unique cell* ξ' *as in the middle below such that* $\pi_A \circ \xi' = \xi_A$ *and* $\pi_B \circ \xi' = \xi_B$.

We call the tabulation $\langle J \rangle$ (left) (nullary-)cocartesian *whenever its defining cell* π *is (left) (nullary-)cocartesian (Definition 2.6).*







Vertically dual, the cotabulation $[J]$ *of* J *is defined by a nullary cell* σ *as on the right above, satisfying 1-dimensional and 2-dimensional universal properties that are vertical dual to those for* $\langle J \rangle$*. We call* $[J]$ cartesian *whenever* σ *is cartesian.*

The following is a direct consequence of the pasting lemma for cocartesian paths of $(0,1)$-ary cells, Lemma 3.16 below.

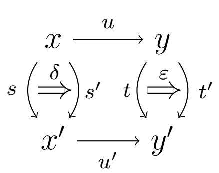
3.6. Corollary. *An augmented virtual double category has all left (nullary-)cocartesian paths of* $(0,1)$*-ary cells (Definition 3.2) whenever it has all left (nullary-)cocartesian tabulations and all restrictions on the right (Definition 1.19). An augmented virtual equipment (Definition 1.19) has all (nullary-)cocartesian paths of* $(0,1)$*-ary cells whenever it has all (nullary-)cocartesian tabulations.*

3.7. Examples of tabulations.

3.8. Example. In the unital virtual equipment Set-Prof (Example A2.4), of Set-profunctors between locally small categories, the tabulation $\langle J \rangle$ is the well known *graph* of $J\colon A \nrightarrow B$ as follows. It has triples (x, u, y) as objects, where $(x, y) \in A \times B$ are objects and $u \in J(x, y)$, while a morphism $(x, u, y) \to (x', u', y')$ is a pair $(s, t)\colon (x, y) \to (x', y')$ in $A \times B$ such that $\lambda(s, u') = \rho(u, t)$ in $J(x, y')$, where λ and ρ denote the actions of A and B on J. The functors π_A and π_B are the projections while the cell $\pi\colon \langle J \rangle \Rightarrow J$ maps

(x, u, y) to $u \in J(x, y)$. It is straightforward to check that π satisfies the universal properties above, and that it is cocartesian. Cocartesian tabulations in the augmented virtual equipment $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$, of Set-profunctors between Set'-categories (Example A2.6), are constructed as graphs in the same way. We conclude that Set-Prof and $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ have all cocartesian paths of $(0, 1)$-ary cells.

3.9. Example. Let $\mathsf{Cat} := \mathsf{Cat}(\mathsf{Set})$ (Example A2.9) denote the category of small categories. In the unital virtual equipment Cat-Prof (Example A2.4) the tabulation $\langle J \rangle$ of a 2-profunctor (that is a Cat-enriched profunctor) $J : A \nrightarrow B$, where A and B are locally small 2-categories, is constructed as follows. It has as underlying category $\langle J \rangle_0$ the graph of the profunctor J_0 underlying J, whose images $J_0(x, y)$ are the sets of objects of the categories $J(x, y)$, for all $x \in A$ and $y \in B$. The cells $(s, t) \Rightarrow (s', t')$ of $\langle J \rangle$ are pairs (δ, ε) of cells $\delta : s \Rightarrow s'$ in A and $\varepsilon : t \Rightarrow t'$ in B as in the diagram on the left below such that $\lambda(\delta, u') = \rho(u, \varepsilon)$ in $J(x, y')$. Tabulations in the augmented virtual equipment $(\mathsf{Cat}, \mathsf{Cat}')$-$\mathsf{Prof}$, of 2-profunctors between (possibly locally large) 2-categories (Example A2.7), are constructed in the same way.



$$J(*, *) = \big(\; u \to v \;\big) \qquad\qquad K(*, *) = \big(\; u' \qquad v' \;\big)$$

Tabulations of 2-profunctors fail to be cocartesian in general. As an example consider the 2-profunctors J and $K : 1 \nrightarrow 1$, where 1 denotes the terminal 2-category with single object $*$, whose images $J(*, *)$ and $K(*, *)$ are the 'interval category' and the discrete category with two objects respectively, as shown above. The tabulation $\langle J \rangle$ is discrete with objects $(*, u, *)$ and $(*, v, *)$, so that the assignments $(*, u, *) \mapsto u'$ and $(*, v, *) \mapsto v'$ define a cell $\phi : \langle J \rangle \Rightarrow K$. It is easily checked that ϕ does not factor through $\pi : \langle J \rangle \Rightarrow J$, showing that π is not weakly cocartesian. As a consequence Proposition 3.22 below fails to hold in $(\mathsf{Cat}, \mathsf{Cat}')$-$\mathsf{Prof}$; see Example 3.24.

3.10. Example. Let $\mathcal{V}' = (\mathcal{V}', \otimes, I)$ be a monoidal category with initial object $\emptyset$ preserved by the functors $x \otimes -$ and $- \otimes x$, for all $x \in \mathcal{V}'$. In the unital virtual equipment $\mathcal{V}'$-Prof (Example A2.4) the cotabulation $[J]$ of a $\mathcal{V}'$-profunctor $J : A \nrightarrow B$ is the *cograph* of J, as follows. Its collection of objects is the disjoint union $\operatorname{ob}[J] := \operatorname{ob} A \sqcup \operatorname{ob} B$ while its hom-objects are given by

$$[J](x, y) := \begin{cases} A(x, y) & \text{if } x, y \in A; \\ J(x, y) & \text{if } x \in A \text{ and } y \in B; \\ B(x, y) & \text{if } x, y \in B; \\ \emptyset & \text{otherwise.} \end{cases}$$

Composition in $[J]$ is induced by composition in A and B as well as the actions of A and B on J. Taking σ_A and σ_B to be the embeddings of A and B into $[J]$, the universal cell $\sigma : J \Rightarrow [J]$ is simply given by the identities on the $\mathcal{V}'$-objects $J(x, y)$. It is straightforward to check that σ satisfies the universal properties and that it is cartesian. We conclude that $\mathcal{V}'$-Prof has all cartesian cotabulations and that, applying Lemma 3.14 below to $(\mathcal{V}, \mathcal{V}')$-Prof $\hookrightarrow \mathcal{V}'$-Prof, so does the augmented virtual equipment $(\mathcal{V}, \mathcal{V}')$-Prof (Example A2.7). In greater generality, in [Str80b] cographs of "$\mathcal{V}$-gamuts" are used to characterise $\mathcal{V}$-profunctors; see paragraph 6.10 and Corollary 6.16 therein.

3.11. Example. Let $\mathcal{E}$ be a category with pullbacks. The unital virtual equipment $\mathsf{Prof}(\mathcal{E})$ of internal profunctors in $\mathcal{E}$ (Examples A2.9 and A4.9) has cocartesian tabulations as follows, and hence has cocartesian paths of $(0,1)$-ary cells. In Proposition 5.15 of [Kou14a], where $\mathcal{E}$ is assumed to have coequalisers preserved by pullback so that $\mathsf{Prof}(\mathcal{E})$ has all horizontal composites (Example A7.5), the tabulation $\langle J \rangle$ of $J\colon A \nrightarrow B$ in $\mathsf{Prof}(\mathcal{E})$, with underlying span $A \leftarrow J \rightarrow B$, was constructed as follows. It is the internal category that has J as its object of objects and, as its object of morphisms, the pullback $\langle J \rangle$ below, where λ and ρ denote the actions of the objects of morphisms α and β, of the internal categories A and B, on J. The source and target morphisms of $\langle J \rangle$ are the composite projections $d_0 = \big[\langle J \rangle \to J \times_B \beta \to J\big]$ and $d_1 = \big[\langle J \rangle \to \alpha \times_A J \to J\big]$. If $\mathcal{E} = \mathsf{Set}$ then the latter recovers $\langle J \rangle$ as the graph of J (Example 3.8).

$$\begin{array}{ccc} \langle J \rangle & \longrightarrow & J \times_B \beta \\ \downarrow & \lrcorner & \downarrow \rho \\ \alpha \times_A J & \xrightarrow[\lambda]{} & J \end{array}$$

The argument given in [Kou14a], proving that $\langle J \rangle$ is an internal category which forms the tabulation of J, carries over to the general case, with $\mathcal{E}$ not necessarily having coequalisers, with minor adjustments. In particular the universal cell $\pi\colon \langle J \rangle \Rightarrow J$, as a $(0,1)$-ary cell, is given by the identity on J and it is straightforward to show that this makes π cocartesian in the unital virtual double category $\mathsf{Prof}(\mathcal{E})$.

3.12. Example. Let $\mathcal{E}$ be a category with finite limits. The unital virtual equipment $\mathsf{ModRel}(\mathcal{E})$ of internal modular relations in $\mathcal{E}$ (Example 1.31) has cocartesian tabulations as follows, and hence has cocartesian paths of $(0,1)$-ary cells. By applying the composite 2-functor $N \circ \mathsf{Mod}$ of Example A2.2 to the embedding $\mathsf{Rel}(\mathcal{E}) \hookrightarrow \mathsf{Span}(\mathcal{E})$ we obtain an embedding $\mathsf{ModRel}(\mathcal{E}) \hookrightarrow \mathsf{Prof}(\mathcal{E})$. We claim that the latter creates cocartesian tabulations so that, like $\mathsf{Prof}(\mathcal{E})$ by the previous example, $\mathsf{ModRel}(\mathcal{E})$ has cocartesian tabulations. Since the embedding is full and faithful and preserves cartesian cells, to prove the claim it suffices, using Lemma 3.14 below, to show that for any internal modular relation $J\colon A \nrightarrow B$ in $\mathsf{ModRel}(\mathcal{E})$ the tabulation $\langle J \rangle$ in $\mathsf{Prof}(\mathcal{E})$, as constructed in the previous example, is an internal preorder in $\mathcal{E}$ (Example 1.31). We do so below; as an aside we remark that one can also show that if A and B are internal partial orders (Example 1.31) then so is $\langle J \rangle$.

We have to show that the span $J \xleftarrow{d_0} \langle J \rangle \xrightarrow{d_1} J$ underlying $\langle J \rangle$, with legs as described in the previous example, is a relation in $\mathcal{E}$, that is d_0 and d_1 are jointly monic. To do so consider any parallel pair ϕ, $\psi\colon X \to \langle J\rangle$ of morphisms in $\mathcal{E}$ such that $d_i \circ \phi = d_i \circ \psi$ for $i = 0, 1$. Writing $p_\alpha^{\langle J\rangle}$ for the composite projection $p_\alpha^{\langle J\rangle} := \big[\langle J\rangle \to \alpha \times_A J \to \alpha\big]$ one checks that $\alpha_0 \circ p_\alpha^{\langle J\rangle} \circ \phi = \alpha_0 \circ p_\alpha^{\langle J\rangle} \circ \psi$ follows from the fact that λ and ρ are morphisms of spans, the definition of $\langle J\rangle$ as a pullback, the definition of d_0, and the assumption on ϕ and ψ. Similarly $\alpha_1 \circ p_\alpha^{\langle J\rangle} \circ \phi = \alpha_1 \circ p_\alpha^{\langle J\rangle} \circ \psi$ so that $p_\alpha^{\langle J\rangle} \circ \phi = p_\alpha^{\langle J\rangle} \circ \psi$ follows from the joint monicity of α_0 and α_1. Together with $d_1 \circ \phi = d_1 \circ \psi$ we conclude that $p_{\alpha\times_A J}^{\langle J\rangle} \circ \phi = p_{\alpha\times_A J}^{\langle J\rangle} \circ \psi$ where $p_{\alpha\times_A J}^{\langle J\rangle}\colon \langle J\rangle \to \alpha \times_A J$ is the projection. Similarly also $p_{J\times_B \beta}^{\langle J\rangle} \circ \phi = p_{J\times_B \beta}^{\langle J\rangle} \circ \psi$; together the latter imply $\phi = \psi$ as required.

3.13. Example. The unital virtual equipment $\mathsf{dFib}(\mathcal{C})$ of discrete two-sided fibrations in a finitely complete 2-category $\mathcal{C}$ (Example 1.30) has cocartesian tabulations that are preserved by the embeddings $\mathsf{dFib}(\mathcal{C}) \hookrightarrow \mathsf{spFib}(\mathcal{C}) \hookrightarrow \mathsf{Prof}(\mathcal{C}_0)$, as follows. Given any discrete two-sided fibration $J\colon A \nrightarrow B$ in $\mathsf{dFib}(\mathcal{C})$ one can show that the square in $\mathcal{C}$ on the left below commutes and that it is a pullback square; compare the definition of internal discrete two-sided fibration given in Section 8 of [Str17].

$$\begin{array}{ccc} J^2 & \xrightarrow{(d_0, j_B^2)} & J \times_B B^2 \\ {\scriptstyle (j_A^2, d_1)}\downarrow & & \downarrow{\scriptstyle \rho} \\ A^2 \times_A J & \xrightarrow[\lambda]{} & J \end{array} \qquad \begin{array}{ccc} & J & \\ {\scriptstyle j_A}\swarrow & \Downarrow{\scriptstyle \pi} & \searrow{\scriptstyle j_B} \\ A & \underset{J}{\nrightarrow} & B \end{array} \qquad \begin{array}{ccc} & X^2 & \\ {\scriptstyle \phi_A^2}\swarrow & \Downarrow{\scriptstyle \phi} & \searrow{\scriptstyle \phi_B^2} \\ A^2 & \underset{J}{\nrightarrow} & B^2 \end{array}$$

It follows that, when regarding J as an internal profunctor $J\colon A^2 \nrightarrow B^2$ in $\mathsf{Prof}(\mathcal{C}_0)$, we can take its tabulation, as described in Example 3.11, to have $\langle J\rangle := J^2$ as its object of morphisms. Going through the proof of Proposition 5.15 of [Kou14a] it is then easy to check that this choice implies that $\langle J\rangle = J^2$ as an internal category, and that the projections $\pi_A\colon J^2 \to A^2$ and $\pi_B\colon J^2 \to B^2$ are the internal functors j_A^2 and j_B^2. Hence the $(0,1)$-ary cell $\pi\colon J^2 \Rightarrow J$ in $\mathsf{Prof}(\mathcal{C}_0)$, that defines J^2 as the tabulation of J, forms a $(0,1)$-ary cell π in $\mathsf{dFib}(\mathcal{C})$ that is of the form as in the middle above. Next consider any $(0,1)$-ary cell $\phi\colon X \Rightarrow J$ in $\mathsf{dFib}(\mathcal{C})$, i.e. any cell ϕ in $\mathsf{Prof}(\mathcal{C}_0)$ of the form as on the right above. One readily checks that the latter factors through π (in $\mathsf{Prof}(\mathcal{C}_0)$) as the internal functor $\phi^2\colon X^2 \to J^2$, showing that in $\mathsf{dFib}(\mathcal{C})$ the cell ϕ factors through π as the morphism $\phi\colon X \to J$. We conclude that the 1-dimensional universal property of π (Definition 3.5) in $\mathsf{Prof}(\mathcal{C}_0)$ restricts to its sub-unital virtual equipments $\mathsf{dFib}(\mathcal{C})$ and $\mathsf{spFib}(\mathcal{C})$. That the 2-dimensional universal property does so too follows from the fact that the embeddings $\mathsf{dFib}(\mathcal{C}) \hookrightarrow \mathsf{spFib}(\mathcal{C}) \hookrightarrow \mathsf{Prof}(\mathcal{C}_0)$ are locally full and faithful; we conclude that π defines the object J as the tabulation of $J\colon A \nrightarrow B$ both $\mathsf{dFib}(\mathcal{C})$ and $\mathsf{spFib}(\mathcal{C})$. Because the embeddings preserve cartesian cells π is cocartesian in $\mathsf{dFib}(\mathcal{C})$ and $\mathsf{spFib}(\mathcal{C})$ too by Lemma A9.4. In particular $J \cong j_A^* \odot j_{B*}$ in $\mathsf{dFib}(\mathcal{C})$ by Lemma 3.17 below; compare Proposition 4.25 of [CJSV94] which shows that $E \cong p_* \circ q^*$ for any ‘discrete fibration’ $A \xleftarrow{p} E \xrightarrow{q} B$ in a ‘faithfully conservational bicategory’.

The following lemma was used in the examples above in obtaining (co)tabulations in full sub-virtual double categories. The proof of its main assertion is straightforward; for the other assertions use Lemmas A4.5 and A9.4.

3.14. LEMMA. *Any full and faithful functor $F\colon \mathcal{K} \to \mathcal{L}$ (Definition A3.6) reflects tabulations, that is $\pi\colon X \Rightarrow J$ defines X as the tabulation of J in $\mathcal{K}$ whenever $F\pi$ defines FX as the tabulation of FJ in $\mathcal{L}$. Similarly F reflects (cartesian) cotabulations. If moreover F preserves cartesian cells then it reflects cocartesian tabulations as well.*

3.15. PROPERTIES OF COCARTESIAN PATHS OF $(0,1)$-ARY CELLS. Before stating the main theorem of this section we record some useful properties of cocartesian paths of $(0,1)$-ary cells. Recall that $n' \coloneqq n - 1$ for any positive integer n.

3.16. LEMMA. [Pasting lemma for cocartesian paths of $(0,1)$-ary cells] *Consider composable paths of horizontal morphisms $\underline{J}\colon A_0 \nrightarrow A_n$ and $\underline{H}\colon A_n \nrightarrow B_m$. Let $\underline{\psi} = (\psi_1, \dots, \psi_m)$ be a left cocartesian path of $(0,1)$-ary cells for $\underline{H}$ and assume that the restriction $J_n(\mathrm{id}, g_0)\colon A_{n'} \nrightarrow Y$ exists, where $g_0\colon Y \to A_n$ is the vertical source of ψ_1. If $\underline{\phi} = (\phi_1, \dots, \phi_n)$ is a left cocartesian path of $(0,1)$-ary cells for $(J_1, \dots, J_{n'}, J_n(\mathrm{id}, g_0))$ then the concatenation $(\phi_1, \dots, \phi_{n'}, \mathrm{cart} \circ \phi_n, \psi_1, \dots, \psi_m)$ is a left cocartesian path of $(0,1)$-ary cells for $\underline{J} \frown \underline{H}$, where* cart *denotes the cartesian cell defining $J_n(\mathrm{id}, g_0)$.*

Denoting the vertical targets of ϕ_n and ψ_n by $f_n\colon X \to Y$ and $g_m\colon Y \to B_m$, the latter concatenation is a cocartesian path of $(0,1)$-ary cells for $\underline{J} \frown \underline{H}$ whenever $\underline{\phi}$ and $\underline{\psi}$ are cocartesian and, for every $K\colon B_m \nrightarrow C$, if the restriction $K(g_m \circ f_n, \mathrm{id})$ exists then so does $K(g_m, \mathrm{id})$.

Analogous assertions hold for left nullary-cocartesian paths of $(0,1)$-ary cells; horizontally dual assertions hold for right (nullary-)cocartesian paths of $(0,1)$-ary cells.

The next lemma shows that a $(0,1)$-ary cocartesian cell $\phi_1\colon X \Rightarrow J_1$ can be used to write J_1 as a composite of a conjoint followed by a companion. This formalises the classical fact (see e.g. Proposition 2.3.2 of [Bén73]) that any profunctor $J\colon A \nrightarrow B$ can be written as $J \cong \pi_A^* \odot \pi_{B*}$ where $A \xleftarrow{\pi_A} \langle J \rangle \xrightarrow{\pi_B} B$ are the projections of the graph $\langle J \rangle$ of J (Example 3.8).

3.17. LEMMA. *Let $\phi_1\colon X \Rightarrow J_1$ be a cocartesian $(0,1)$-ary cell for $J_1\colon A_0 \nrightarrow A_1$ as in Definition 3.2. If the conjoint $f_0^*\colon A_0 \nrightarrow X$ and the companion $f_{1*}\colon X \nrightarrow A_1$ exist then the composite* $\mathrm{cart} \odot \phi_1 \odot \mathrm{cart}\colon (f_0^*, f_{1*}) \Rightarrow J_1$*, where the cartesian cells defining f_0^* and f_{1*} are denoted by* cart*, is cocartesian. In particular $J_1 \cong f_0^* \odot f_{1*}$.*

PROOF. Denote by cocart the cocartesian cells corresponding to the cartesian cells that define f_0^* and f_{1*} (Lemma 1.21). By the pasting lemma for cocartesian paths (Lemma A7.7) the path $(\mathrm{cocart}, \mathrm{cocart})\colon X \Rightarrow (f_0^*, f_{1*})$ is cocartesian. Applying the same lemma to $(\mathrm{cart} \odot \phi_1 \odot \mathrm{cart}) \circ (\mathrm{cocart}, \mathrm{cocart}) = \phi_1$ shows that $\mathrm{cart} \odot \phi_1 \odot \mathrm{cart}$ is cocartesian too. ■

Cocartesian paths of $(0,1)$-ary cells can be used to reduce left Kan extension along a path $\underline{J}$ of horizontal morphisms to left Kan extension along a single companion morphism as follows.

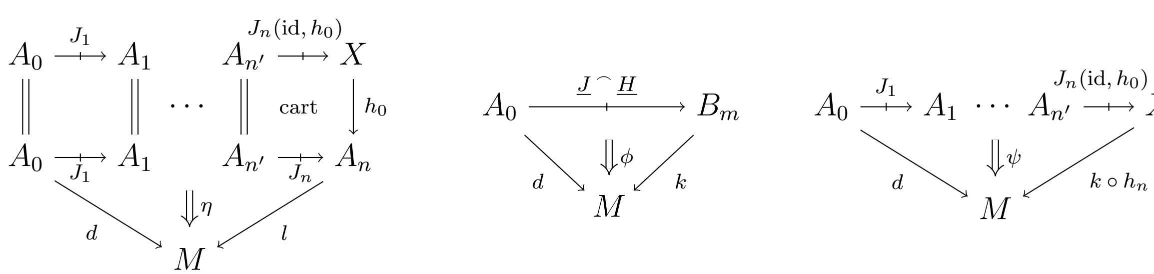
3.18. Proposition. *Consider the right (respectively weakly) nullary-cocartesian path* $\underline{\phi}$ *of* $(0,1)$*-ary cells for the path* $\underline{J}$ *in Definition 3.2 and assume that the companion* $f_{n*}\colon X \to A_n$ *exists. For any morphism* $d\colon A_0 \to M$ *the (weak) left Kan extension of* d *along* $\underline{J}$ *exists if and only if the (weak) left Kan extension of* $d \circ f_0$ *along* f_{n*} *does so, and in that case they are isomorphic.*

Proof. Let cart and cocart denote the cartesian and cocartesian cells that define the companion f_{n*}, as in Lemma 1.21. Applying the pasting lemma for nullary-cocartesian paths (Lemma 2.9) to the identity $(\phi_1, \dots, \phi_n, \text{cart}) \circ \text{cocart} = (\phi_1, \dots, \phi_n)$, which follows from the companion identities (Lemma 1.21), we find that the path $(\phi_1, \dots, \phi_n, \text{cart})\colon f_{n*} \Rightarrow \underline{J}$, which has vertical source f_0, is right (respectively weakly) nullary-cocartesian because $\underline{\phi}$ and cocart are so. The result follows from applying the vertical pasting lemma for left Kan extensions (Lemma 2.17) to the path $(\phi_1, \dots, \phi_n, \text{cart})$. ■

3.19. Pointwise Kan extensions in terms of pointwise weak Kan extensions. The following theorem is the main result of this section.

3.20. Theorem. *In an augmented virtual double category that has restrictions on the right (Definition 1.19) as well as left nullary-cocartesian paths of* $(0,1)$*-ary cells (Definition 3.2), all pointwise weakly left Kan cells are pointwise left Kan (Definition 1.24).*

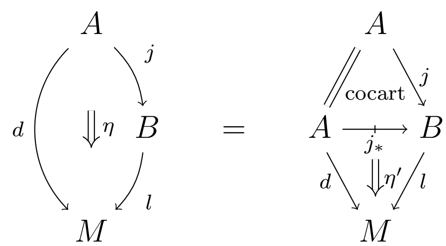
Proof. Consider a pointwise weakly left Kan cell η as in the composite on the left below. We will first prove that η is left Kan (Definition 1.9), that is any cell ϕ as in the middle below factors uniquely through η. To do so let $\underline{\zeta} = (\zeta_1, \dots, \zeta_m)$ be a left nullary-cocartesian path of $(0,1)$-ary cells for the path $\underline{H} = (H_1, \dots, H_m)$ and denote by $h_0\colon X \to A_n$ and $h_m\colon X \to B_m$ the vertical source of ζ_1 and the vertical target of ζ_m. Under precomposition with the weakly cocartesian path $(\text{id}_{J_1}, \dots, \text{id}_{J_{n'}}, \text{cart}, \zeta_1, \dots, \zeta_n)$, where cart defines $J_n(\text{id}, h_0)$, cells ϕ as in the middle below correspond to cells ψ as on the right. Applying Lemma 2.20 to $\underline{\chi} = (\text{id}_{J_1}, \dots, \text{id}_{J_{n'}}, \text{cart})$ and $\underline{\zeta}$ we find that, under this correspondence, the cells ϕ factor uniquely through η precisely if the cells ψ factor uniquely through the composite on the the left below, as vertical cells $l \circ h_0 \Rightarrow k \circ h_n$. The latter factorisations exist by the assumption that η is pointwise weakly left Kan, so that the existence of the factorisations of the cells ϕ through η follows as required.



Finally, to prove that η is in fact pointwise left Kan, we have to show that any composite $\eta \circ (\text{id}_{J_1}, \dots, \text{id}_{J_{n'}}, \text{cart})$, where cart defines the restriction $J_n(\text{id}, f)$ along any morphism $f\colon B \to A_n$, is again left Kan. Since the latter composite is again pointwise weakly left Kan by Lemma 1.25 this follows immediately from the argument above. ■

3.21. Pointwise left Kan extension along a companion. We can use the previous theorem to extend Proposition 1.7 to the pointwise case. This generalises the corresponding result for pseudo double categories, Proposition 5.12 of [Kou14a].

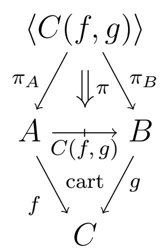
3.22. Proposition. *Consider the following factorisation in an augmented virtual double category $\mathcal{K}$ that has restrictions on the right (Definition 1.19) as well as left nullary-cocartesian tabulations (Definition 3.5).*



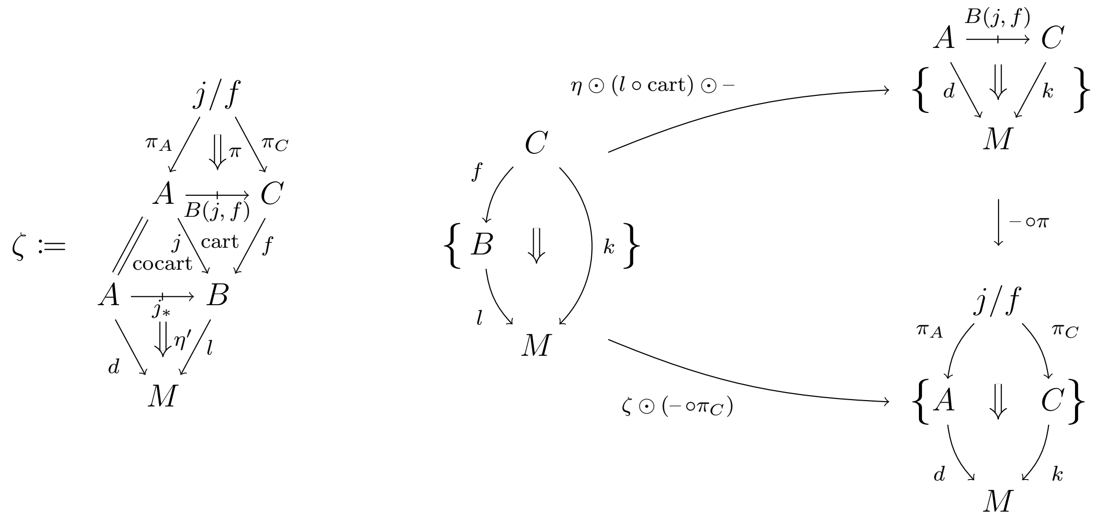
The cell η defines l as the pointwise left Kan extension of d along j in the 2-category $V(\mathcal{K})$ (Example A1.5), in the sense of Section 4 of [Str74b], precisely if its factorisation η' is pointwise left Kan in $\mathcal{K}$ (Definition 1.24).

Street's notion of pointwise Kan extension in a 2-category uses the well known notion of *comma object*; see e.g. Section 1 of [Str74b]. Instead of recalling the definition of comma object we record the following straightforward lemma, which relates it to that of tabulation.

3.23. Lemma. *Consider morphisms $f\colon A \to C$ and $g\colon B \to C$ in an augmented virtual double category $\mathcal{K}$. If both the cartesian cell and the tabulation below exist then their composite defines $\langle C(f,g)\rangle$ as the comma object f/g of f and g in $V(\mathcal{K})$.*



Proof of Proposition 3.22. First notice that the assumptions imply that $\mathcal{K}$ has left nullary-cocartesian paths of $(0,1)$-ary cells by Corollary 3.6, so that Theorem 3.20 applies. Moreover notice that all restrictions $B(j,f)$, where $f\colon C \to B$, exist in $\mathcal{K}$ as the restrictions $j_*(\mathrm{id}, f)$ (compare Lemma A5.11). Next consider composites ζ as on the left below, where $f\colon C \to B$ varies, $\mathrm{cart} \circ \pi$ defines the comma object j/f as in the previous lemma, and $\eta = \eta' \circ \mathrm{cocart}$. In each ζ the composite $\mathrm{cocart} \odot \mathrm{cart}\colon B(j,f) \Rightarrow j_*$ is cartesian: using the pasting lemma for cartesian cells (Lemma 1.17) this follows from the fact that composing it with the cartesian cell defining j_*, that corresponds to cocart (Lemma 1.21), results in the cartesian cell defining $B(j,f)$.

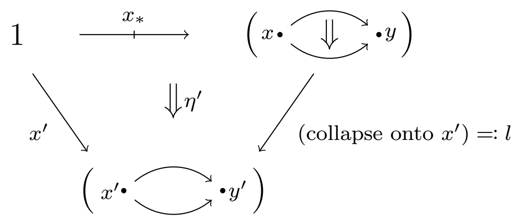


It follows that the top assignment in the commutative diagram on the right above, between collections of cells in $\mathcal{K}$ as shown, is a bijection, for every f, precisely if η' is pointwise weakly left Kan in $\mathcal{K}$. By Theorem 3.20 the latter is equivalent to η' being pointwise left Kan. Because the cell π is weakly nullary-cocartesian the assignment on the right is a bijection, and we conclude that η' is pointwise left Kan precisely if the bottom assignment is a bijection, for every f. Since cart $\circ\, \pi$ defines the comma object j/f the latter, by definition, precisely means that η exhibits l as the pointwise left Kan extension in the 2-category $V(\mathcal{K})$, which completes the proof. ■

The following is Example 2.24 of [Kou13].

3.24. Example. Recall from Example 3.9 that tabulations of 2-profunctors are in general not cocartesian. As a consequence the equivalence of Proposition 3.22 fails to hold in the unital virtual equipment $\mathsf{Cat\text{-}Prof}$. For a counterexample consider the cell η' below, where 1 is the terminal 2-category and the collapsing 2-functor l on the right has the 'free living' cell as source and the free living parallel pair of arrows as target; η' is uniquely determined by its boundary. Using Example 1.13 it is straightforward to check that η' is left Kan in $\mathsf{Cat\text{-}Prof}$, so that the corresponding 2-natural transformation $\eta\colon x' \Rightarrow l \circ x$ defines l as the enriched left Kan extension of x' along x, in the sense of e.g. Section 4.1 of [Kel82]. In contrast the pointwise left Kan extension of x' along x in the 2-category 2-$\mathsf{Cat} = V(\mathsf{Cat\text{-}Prof})$, of 2-categories, 2-functors and 2-natural transformations, and in the sense of Section 4 of [Str74b], does not exist. To see this one checks that the composite of

η with the cell defining the comma object x/y does not define ly as a left Kan extension.



4. Yoneda morphisms

Having introduced notions of left Kan extension in augmented virtual double categories we now turn to formalising the classical notion of Yoneda embedding. Informally, in Definition 4.5 below we will take a 'Yoneda morphism' in an augmented virtual double category to be a vertical morphism that is 'dense' (as defined below) and that satifies an axiom that formalises the classical Yoneda's lemma (see Example 4.10 below). These two conditions are closely related to the axioms satisfied by the formal Yoneda embeddings comprising a 'Yoneda structure' on a 2-category, as introduced by Street and Walters in [SW78]. In fact given an augmented virtual double category $\mathcal{K}$ the main theorem of this section, Theorem 4.35, describes relations between supplying a family of (weak) Yoneda morphisms in $\mathcal{K}$, in our sense (Definition 4.5), and equipping the vertical 2-category $V(\mathcal{K})$ (Example A1.5) with a Yoneda structure, in the sense of [SW78], as well as equipping $V(\mathcal{K})$ with two strengthenings of the latter that were introduced in [SW78] and [Web07].

An advantage of our approach to formalising Yoneda embeddings is a consequence of our viewpoint of regarding all horizontal morphisms as being 'admissible' (informally these are to be thought of as "small in size"; see the Introduction): this enables us to give a relatively simple definition of a *single* Yoneda morphism, which satisfies a formalisation of Yoneda's lemma "with respect to all horizontal morphisms". In contrast the notions of Yoneda structure of [SW78] and [Web07] consist of a *collection* of Yoneda embeddings satisfying a formalisation of Yoneda's lemma with respect to a specified collection of 'admissible' morphisms.

4.1. Density. Using the notion of (weak) left Kan extension, we start by defining the notion of (weak) density as one of the three equivalent conditions below. For augmented virtual double categories $\mathcal{K}$ satisfying the assumptions of Proposition 3.22 condition (c) below coincides with the original 2-categorical notion of density, given in Section 3 of [Str74a], when applied to the vertical 2-category $V(\mathcal{K})$; in particular density in $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ (Example A2.6) recovers the classical notion of density for functors (see e.g. Section X.6 of [ML98]). Similarly it follows from Example 1.13 that, when considered in the unital virtual equipment $\mathcal{V}$-Prof of $\mathcal{V}$-profunctors, condition (c) below coincides with the classical notion of density for enriched functors; see e.g. Section 5.1 of [Kel82].

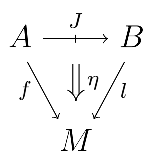
4.2. Lemma. *For a morphism $f\colon A \to M$ the following conditions are equivalent:*

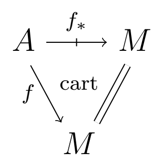
(a) *if a cell η, as on the left below, is cartesian then it is (weakly) left Kan (Definitions 1.2 and 1.9);*

(b) *if a cell η as below is cartesian then it is pointwise (weakly) left Kan (Definition 1.24).*

If the companion $f_*\colon A \nrightarrow M$ exists then the following condition is equivalent too:

(c) *the cartesian cell defining f_* on the right below is pointwise (weakly) left Kan.*

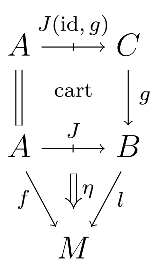


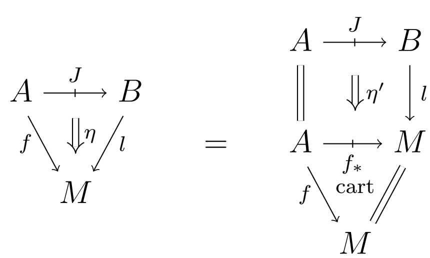


4.3. Definition. *A morphism $f\colon A \to M$ is* (weakly) dense *if the equivalent conditions above are satisfied.*

Notice that the notions of weak density and density coincide in augmented virtual double categories that satisfy the assumptions of Theorem 3.20, as do the notions of weak Yoneda morphism and Yoneda morphism below.

Proof of Lemma 4.2. (b) $\Rightarrow$ (a) is clear. For the converse assume that (a) holds: we have to show that any composite as on the left-hand side below is (weakly) left Kan. Since η is cartesian the composite is cartesian too by the pasting lemma (Lemma 1.17), so that it is (weakly) left Kan by (a).

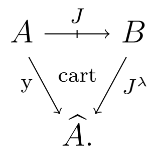




(b) $\Rightarrow$ (c) is clear. For the converse consider a cartesian cell η as on the right above and let η' be its factorisation as shown; η' is cartesian because η is so, by the pasting lemma. Assuming (c) it follows from Lemma 1.25 that η is pointwise (weakly) left Kan. ∎

4.4. Yoneda morphisms. (Weak) Yoneda morphisms in an augmented virtual double category are (weakly) dense morphisms satisfying a 'Yoneda axiom' as follows.

4.5. Definition. *A (weakly) dense morphism* $\mathrm{y}\colon A \to \widehat{A}$ *is called a* (weak) Yoneda morphism *if it satisfies the* Yoneda axiom*: for every horizontal morphism* $J\colon A \nrightarrow B$ *there exists a vertical morphism* $J^\lambda\colon B \to \widehat{A}$ *equipped with a cartesian cell*



Notice that the (weak) density of $\mathrm{y}\colon A \to \widehat{A}$ implies that the vertical morphism J^λ is unique up to vertical isomorphism. In particular, if the companion $\mathrm{y}_*\colon A \nrightarrow \widehat{A}$ exists then its defining cartesian cell implies that $\mathrm{y}_*^\lambda \cong \mathrm{id}_{\widehat{A}}$. Since density implies weak density, any Yoneda morphism is a weak Yoneda morphism. We call the target $\widehat{A}$ of y the *object of presheaves on* A, or *presheaf object* for short. (Weak) Yoneda morphisms $\mathrm{y}\colon A \to \widehat{A}$ such that all nullary restrictions $\widehat{A}(\mathrm{y}, f)$ exist, for any $f\colon B \to \widehat{A}$, are especially pleasant to work with; in that case we will say that y *admits nullary restrictions*. Notice that, when considered in an augmented virtual double category with restrictions on the right (Definition 1.19), the latter condition reduces to the existence of the companion y_*, since $\widehat{A}(\mathrm{y}, f) \cong \mathrm{y}_*(\mathrm{id}, f)$ by the pasting lemma for cartesian cells (Lemma 1.17).

Before giving examples of Yoneda morphisms in Examples 4.9–4.17 below we first consider full and faithfulness of Yoneda morphisms $\mathrm{y}\colon A \to \widehat{A}$, that is the cartesianness of their identity cell id_y (Definition 1.18). Full and faithfulness of y does not hold in general; it is instead related to the unitality of A as described by the lemma below. This is in contrast to the situation for Yoneda structures, in the sense of Street and Walters [SW78], whose Axiom 3 ensures that its Yoneda embeddings are full and faithful, in their sense.

By a *(weak) Yoneda embedding* we will mean a (weak) Yoneda morphism that is full and faithful. All examples of Yoneda morphisms below are Yoneda embeddings. In Corollary 6.16 below we will see that (weak) Yoneda embeddings are unique up to equivalence provided that they admit nullary restrictions.

4.6. Lemma. *Let* $\mathrm{y}\colon A \to \widehat{A}$ *be a weak Yoneda morphism. The object* A *is unital (Definition 1.16) if and only if both the restriction* $\widehat{A}(\mathrm{y}, \mathrm{y})$ *exists and* y *is full and faithful (Definition 1.18). In that case* $I_A \cong \widehat{A}(\mathrm{y}, \mathrm{y})$ *so that* $I_A^\lambda \cong \mathrm{y}$.

It follows that if y admits nullary restrictions (Definition 4.5) then the unitality of A is equivalent to the full and faithfulness of y.

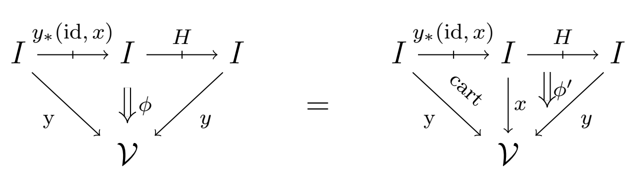
Proof. The 'if'-part follows immediately from Lemma A5.14. To prove the 'only if'-part assume that A is unital with horizontal unit $I_A\colon A \nrightarrow A$ and consider the cartesian cell ε that defines I_A as the restriction of $\widehat{A}$ along y and $I_A^\lambda\colon A \to \widehat{A}$, as supplied by the Yoneda axiom (Definition 4.5). Weak density of y (Definition 4.3) implies that ε is weakly left Kan so that the composite $\varepsilon \circ \mathrm{cocart}$, where cocart denotes the weakly cocartesian cell defining I_A (Lemma 1.21), is an invertible vertical cell $\mathrm{y} \cong I_A^\lambda$ by Example 2.19. Composing ε with the inverse $I_A^\lambda \cong \mathrm{y}$ we thus obtain a cartesian cell that defines I_A as the restriction $\widehat{A}(\mathrm{y}, \mathrm{y})$. Moreover cocart is cartesian by Lemma 1.21 so that $\varepsilon \circ \mathrm{cocart}\colon \mathrm{y} \cong I_A^\lambda$ is cartesian by the pasting lemma (Lemma 1.17). Composing the latter with its own inverse we find that the identity cell id_y is cartesian, that is y is full and faithful. ■

4.7. Remark. We remark that several of our results concerning Yoneda morphisms do not require full and faithfulness. None of the results on exact cells (Section 5) do. Several results on totality (Section 6) do, but not the main result, Theorem 6.8, of that section. Theorem 7.6, which describes the sense in which a Yoneda embedding $\mathrm{y}\colon A \to \widehat{A}$ defines $\widehat{A}$ as the 'free small cocompletion' of A, requires A to be unital.

4.8. Examples of enriched Yoneda embeddings. The following two examples show that the classical Yoneda embeddings y_A for $\mathcal{V}$-enriched categories A satisfy Definition 4.5. The first treates the simple case y_I of the unit $\mathcal{V}$-category I (Example 1.12). The second summarises how y_I and the closed monoidal structure for $\mathcal{V}$-categories can be used to "generate" Yoneda embeddings for other $\mathcal{V}$-categories, by using the results of Section 8. We will see in Example 4.33 that the Yoneda embeddings of the unital virtual double category $\mathcal{V}$-sProf of small $\mathcal{V}$-profunctors (Example A2.8), in the sense of Definition 4.5, can be obtained by corestricting of the Yoneda embeddings y_A to $\mathcal{V}$-categories of small $\mathcal{V}$-presheaves.

4.9. Example. Let $\mathcal{V}$ be a monoidal category and let $\mathcal{V} \subset \mathcal{V}'$ be a universe enlargement (Example 1.12). The Yoneda embedding $\mathrm{y}\colon I \to \widehat{I}$ for the unit $\mathcal{V}$-category I (Example 1.12) in the augmented virtual equipment $(\mathcal{V}, \mathcal{V}')$-Prof, of $\mathcal{V}$-profunctors between $\mathcal{V}'$-categories, is defined as follows. Since $\mathcal{V}'$ is closed monoidal $\mathcal{V}$ can be enriched in $\mathcal{V}'$, with hom $\mathcal{V}'$-objects $\mathcal{V}(x, y) := [x, y]'$, and we take $\widehat{I} := \mathcal{V}$ to be the object of presheaves on I. As expected the Yoneda embedding $\mathrm{y}\colon I \to \mathcal{V}$ is the $\mathcal{V}'$-functor that maps the single object $* \in I$ to $\mathrm{y}(*) := I$, the monoidal unit of $\mathcal{V}$: we will show that y satisfies Definition 4.5.

First notice that the companion $\mathrm{y}_*\colon I \nrightarrow \mathcal{V}$ exists in $(\mathcal{V}, \mathcal{V}')$-Prof because for all $x \in \mathcal{V}$ the image $\mathrm{y}_*(*, x) = [I, x]' \cong x$ is (isomorphic to) a $\mathcal{V}$-object (Example A4.6). Hence to prove that y is dense it suffices by Definition 4.3 to show that the cartesian cell defining y_* is pointwise left Kan. By Example 1.13 in turn it suffices to show that, for all $x \in \mathcal{V}$, the cartesian cell in the right-hand side below defines $x\colon I \to \mathcal{V}$ as the $y_*(\mathrm{id}, x)$-weighted colimit of y in $\mathcal{V}'$-Prof which, by Example 1.12, means that any cell ϕ below, with $H \in \mathcal{V}'$, factors uniquely as shown. It is easy to check that, under the isomorphisms $[I, x]' \cong x$ and $[I, y]' \cong y$, the factorisation $\phi'\colon H \to [x, y]'$ is the adjunct of $\phi\colon [I, x]' \otimes' H \to [I, y]'$.



Finally to prove that $\mathrm{y}\colon I \to \mathcal{V}$ satisfies the Yoneda axiom define, for any $\mathcal{V}$-profunctor $J\colon I \nrightarrow B$, the $\mathcal{V}'$-functor $J^\lambda\colon B \to \mathcal{V}$ as follows. Set $J^\lambda(y) \coloneqq J(*, y)$ on objects and let the action $J^\lambda\colon B(y_1, y_2) \to [J(*, y_1), J(*, y_2)]'$ on hom-objects be the adjunct of the action of B on J. That J is the restriction of $\mathcal{V}$ along y and J^λ, as required, follows from the isomorphisms $[I, J^\lambda(y)]' \cong J^\lambda(y) = J(*, y)$ which are natural in $y \in B$.

4.10. Example. Consider a universe enlargement $\mathcal{V} \subset \mathcal{V}'$ (Example 1.12) that is *symmetric*, i.e. such that $\mathcal{V}$, $\mathcal{V}'$ and the embedding itself are symmetric monoidal; see Section 3.11 of [Kel82]. Writing $[A^{\mathrm{op}}, \mathcal{V}]'$ for the $\mathcal{V}'$-category of $\mathcal{V}$-presheaves on a $\mathcal{V}$-category A (see e.g. Section 2.4 of [Kel82]) consider the classical enriched Yoneda embedding $\mathrm{y}_A\colon A \to [A^{\mathrm{op}}, \mathcal{V}]'$ given by $\mathrm{y}_A x = A(-, x)$. In Example 8.23 below we will see that, using the closed monoidal structure of $\mathcal{V}'$-Prof, the Yoneda embedding $\mathrm{y}\colon I \to \mathcal{V}$ of the previous example "generates", for each $\mathcal{V}$-category A, y_A as a Yoneda embedding in $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ and that y_A admits nullary restrictions, both in the sense of Definition 4.5. In particular any $\mathcal{V}$-profunctor $J\colon A \nrightarrow B$ induces a $\mathcal{V}'$-functor $J^\lambda\colon B \to [A^{\mathrm{op}}, \mathcal{V}]'$ given by $J^\lambda y = J(-, y)$, which is equipped with a cartesian cell (Example A4.6) consisting of isomorphisms $J(x, y) \cong [A^{\mathrm{op}}, \mathcal{V}]'(\mathrm{y}x, J^\lambda y)$ natural in $x \in A$ and $y \in B$. Restricted to $B = I$, so that $\mathcal{V}$-profunctors $J\colon A \nrightarrow I$ can be identified with $\mathcal{V}$-presheaves $J\colon A^{\mathrm{op}} \to \mathcal{V}$, these isomorphisms recover the classical enriched Yoneda lemma; see e.g. Section 2.4 of [Kel82]. Instantiating $\mathcal{V} \subset \mathcal{V}'$ by $\mathsf{Set} \subset \mathsf{Set}'$ recovers the original (unenriched) Yoneda lemma; see e.g. Proposition I.1.4 of [GV72] or Section III.2 of [ML98]. Returning to general $\mathcal{V} \subset \mathcal{V}'$ and B, it follows from Proposition 4.24 below that the assignment $J \mapsto J^\lambda$ extends to an equivalence between $\mathcal{V}$-profunctors $A \nrightarrow B$ and $\mathcal{V}'$-functors $B \to [A^{\mathrm{op}}, \mathcal{V}]'$.

In the case that the universe enlargement $\mathcal{V} \subset \mathcal{V}'$ does not admit a symmetric monoidal structure the Yoneda embedding $\mathrm{y}_A\colon A \to \widehat{A}$ in $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ can still be constructed, directly, for any $\mathcal{V}$-category A. Indeed one can take $\mathcal{V}$-presheaves on A, which are the objects of $\widehat{A}$, to be $\mathcal{V}$-profunctors of the form $p\colon A \nrightarrow I$, while taking the hom $\mathcal{V}'$-object $\widehat{A}(p, q)$ to be the "end-like" limit of the diagram in $\mathcal{V}'$ that consists of all cospans of the form $[px, qx]' \to [A(x, y) \otimes py, qx]' \leftarrow [py, qy]'$, one for each pair $x, y \in A$, whose legs are induced by the actions of A on p and q. For details see Proposition 5.5 of [Kou15b].

4.11. Example. Consider the universe enlargement $\mathsf{2} \subset \mathsf{Set}'$ of the category of truth values $\mathsf{2} = (\bot \to \top)$, that maps $\bot$ to the empty set $\emptyset$ and $\top$ to the terminal set 1. Applying Lemma 4.13 below to the full inclusion $\mathsf{ModRel} \coloneqq \mathsf{2}$-$\mathsf{Prof} \hookrightarrow (\mathsf{2}, \mathsf{Set}')$-$\mathsf{Prof}$ we find that the Yoneda embeddings in $(\mathsf{2}, \mathsf{Set}')$-$\mathsf{Prof}$, as described in the previous example, reflect as Yoneda embeddings in the locally thin strict equipment ModRel of modular relations between preorders (Example 1.32) as follows. The presheaf object $\widehat{A}$ of a preorder $(A, \leq)$ is the set $\mathrm{Dn}\, A$ of *downsets* $X \subseteq A$ in A, satisfying

$$x \leq y \quad \text{and} \quad y \in X \quad \Rightarrow \quad x \in X$$

for all $x, y \in A$. The preorder on $\widehat{A} \coloneqq \mathrm{Dn}\, A$ is given by inclusion and the Yoneda embedding $\mathrm{y}\colon A \to \mathrm{Dn}\, A$ is given by $\mathrm{y}(x) = {\downarrow} x$, the downset generated by x. Under the equivalence of Proposition 4.24 below any modular relation $J\colon A \nrightarrow B$ corresponds to the order preserving morphism $J^\lambda\colon B \to \operatorname{Dn} A$ given by $J^\lambda(y) = J^\circ y \coloneqq \{x \in A \mid xJy\}$, the preimage of $y \in B$ under J, as supplied by the Yoneda axiom (Definition 4.5).

4.12. Example. The Yoneda embedding for a closed-ordered closure space A (Example 1.32) reflects along the forgetful functor $U\colon \mathsf{ClModRel} \to \mathsf{ModRel}$ as follows. The closed subsets $\operatorname{Cl} A$ of A induce a set $\operatorname{Cl}(\operatorname{Dn} A)$ of closed subsets of the preorder $\operatorname{Dn} A$ of downsets in A, making $\operatorname{Dn} A$ into a modular closure space (Example 1.32), as follows. We take $\operatorname{Cl}(\operatorname{Dn} A)$ to be generated by (arbitrary intersections of) subsets of the form

$$V^+ \coloneqq \{X \in \operatorname{Dn} A \mid X \cap V \neq \emptyset\}$$

where $V \in \operatorname{Cl} A$; compare the *upper Vietoris topology* on the powerset PA of a topological space A, see e.g. Section 1 of [CT97]. Since $\uparrow(V^+) = V^+$ for all $V \in \operatorname{Cl} A$, with respect to $\subseteq$ on $\operatorname{Dn} A$, it follows that each $W \in \operatorname{Cl}(\operatorname{Dn} A)$ is an upset, i.e. $\uparrow W = W$, showing that $\operatorname{Dn}^+ A \coloneqq \big(\operatorname{Dn} A, \operatorname{Cl}(\operatorname{Dn} A), \subseteq\big)$ is a modular closure space. We call $\operatorname{Dn}^+ A$ the *upper Vietoris space of downsets in* A.

Next consider the Yoneda embedding $\mathrm{y}\colon A \to \operatorname{Dn} A$ in ModRel and, for each closed modular relation $J\colon A \nrightarrow B$ (Example 1.32), its corresponding morphism $J^\lambda\colon B \nrightarrow \operatorname{Dn} A$; both as described in the previous example. Notice that y as well as all J^λ are continuous morphisms with respect to the closed subsets of $\operatorname{Dn}^+ A$: this follows from the closedness axioms satisfied by A and J (Example 1.32) together with the fact that $\mathrm{y}^{-1}(V^+) = \uparrow V$ and $(J^\lambda)^{-1}(V^+) = JV$ for all $V \in \operatorname{Cl} A$. Notice too that $\uparrow(\mathrm{y}V) = V^+$ for $V \in \operatorname{Cl} A$, so that the companion $\mathrm{y}_*\colon A \nrightarrow \operatorname{Dn}^+ A$ exists in $\mathsf{ClModRel}$. Using Lemma 4.13 below we conclude that $\mathrm{y}\colon A \to \operatorname{Dn}^+ A$ forms a Yoneda embedding in $\mathsf{ClModRel}$. In particular, using Proposition 4.24 below, we obtain a correspondence between closed modular relations $A \nrightarrow B$ and continuous maps $B \to \operatorname{Dn}^+ A$. Compare Proposition 3.1 of [CT97] which describes the related correspondence for (unordered) topological spaces, between closed relations $A \nrightarrow B$ and continuous maps $B \to P^+A$, where P^+A is the powerset PA equipped with the upper Vietoris topology.

Finally notice that, if A itself is a modular closure space too then, again by the lemma below, we find that $\mathrm{y}\colon A \to \operatorname{Dn}^+ A$ also forms a Yoneda embedding in the full sub-double category $\mathsf{ClModRel}_{\mathrm{m}}$ of $\mathsf{ClModRel}$ that is generated by all modular closure spaces.

Let $F\colon \mathcal{K} \to \mathcal{L}$ be a functor and let $A \xrightarrow{f} C \overset{K}{\nrightarrow} D \xleftarrow{g} B$ be morphisms in $\mathcal{K}$, with $|\underline{K}| \leq 1$. We say that F *creates* the restriction $\underline{K}(f, g)$ in $\mathcal{K}$ if, given a cartesian cell ψ defining $(F\underline{K})(Ff, Fg)$ in $\mathcal{L}$, there exists a unique cartesian cell ϕ in $\mathcal{K}$ that defines $\underline{K}(f, g)$ such that $F\phi = \psi$. Notice that if $(F\underline{K})(Ff, Fg)$ exists then this means that F preserves any cartesian cell that defines $\underline{K}(f, g)$. The following lemma is a straightforward consequence of Lemma 1.14.

4.13. Lemma. *A locally full and faithful functor (Definition A3.6) $F\colon \mathcal{K} \to \mathcal{L}$ reflects any (weak) Yoneda morphism $\mathrm{y}\colon A \to \widehat{A}$ in $\mathcal{K}$, that is y is a (weak) Yoneda morphism in $\mathcal{K}$ if $F\mathrm{y}$ is so in $\mathcal{L}$, whenever the following conditions hold:*

(a) *F preserves and creates restrictions of the form $\widehat{A}(\mathrm{y}, f)$, for any $f\colon B \to \widehat{A}$;*

(b) *F is essentially full on morphisms of the form $(FJ)^\lambda\colon FB \to F\widehat{A}$, that is for every $J\colon A \nrightarrow B$ in $\mathcal{K}$ there exists $J^\lambda\colon B \to \widehat{A}$ in $\mathcal{K}$ such that $F(J^\lambda) \cong (FJ)^\lambda$, where $(FJ)^\lambda$ is given by the Yoneda axiom for $F\mathrm{y}$ (Definition 4.5).*

4.14. Generic subobjects as Yoneda embeddings. Let $\mathcal{E}$ be a category with finite limits. The next two examples show that a generic subobject in $\mathcal{E}$ (see e.g. Section A1.6 of [Joh02]) is the same as a Yoneda embedding $\mathrm{y}_1\colon 1 \to \widehat{1}$ in $\mathsf{ModRel}(E)$ (Example 1.31) for the terminal internal preorder 1 whose presheaf object $\widehat{1}$ is an internal partial order (Example 1.31). Assuming that $\mathcal{E}$ is cartesian closed Example 4.17 then describes how y_1 generates the other Yoneda embeddings of $\mathsf{ModRel}(\mathcal{E})$, using the results of Section 8.

4.15. Example. Consider the unital virtual equipment $\mathsf{ModRel}(\mathcal{E})$ of internal modular relations in a category $\mathcal{E}$ with finite limits (Example 1.31) and write $1 := (1, I_1)$ for the terminal internal preorder in $\mathcal{E}$. Given any internal preorder $B = (B, \beta)$ we call internal modular relations $J\colon 1 \nrightarrow B$ *modular subobjects* of B. They consist of monomorphisms $j\colon J \rightarrowtail B$ equipped with a right action $\rho\colon (J, \beta) \Rightarrow J$, that is a (necessarily unique) morphism $\rho\colon J \times_B \beta \to J$ over B. Modular subobjects of a discrete internal preorder (B, I_B) are precisely the subobjects of B in the usual sense; see e.g. Section A1.3 of [Joh02]. Given another modular subobject $K\colon 1 \nrightarrow (D, \delta)$ and an order preserving morphism $g\colon (B, \beta) \to (D, \delta)$ notice that a cell $J \Rightarrow K$, as labelled (a) below, exists in $\mathsf{ModRel}(\mathcal{E})$ if and only if there exists a morphism $J \to K$ in $\mathcal{E}$ that makes the square labelled (b) below commute. Recalling from Example 1.31 the way restrictions are created in $\mathsf{ModRel}(\mathcal{E})$ we find that the cell $J \Rightarrow K$ is cartesian if and only if the latter square is a pullback square.

$$
\begin{array}{ccc}
1 & \xrightarrow{J} & B \\
\| & \Downarrow & \downarrow g \\
1 & \xrightarrow[K]{} & D
\end{array}
\qquad
\begin{array}{ccc}
J & \overset{j}{\rightarrowtail} & B \\
\downarrow & & \downarrow g \\
K & \underset{k}{\rightarrowtail} & D
\end{array}
\qquad
\begin{array}{ccc}
J & \overset{j}{\rightarrowtail} & B \\
\downarrow & \lrcorner & \downarrow J^\lambda \\
\mathrm{y}_* & \underset{\mathrm{y}_{*1}}{\rightarrowtail} & \widehat{1}
\end{array}
\qquad
\begin{array}{ccc}
J & \overset{j}{\rightarrowtail} & B \\
\downarrow & & \downarrow g \\
\mathrm{y}_* & \underset{\mathrm{y}_{*1}}{\rightarrowtail} & \widehat{1}
\end{array}
$$

(a) (b) (c) (d)

Next consider an order preserving morphism of the form $\mathrm{y}\colon 1 \to (\widehat{1}, \omega)$. Its companion $\mathrm{y}_* = (1 \xleftarrow{!} \mathrm{y}_* \xrightarrow{\mathrm{y}_{*1}} \widehat{1})$ is a modular subobject of $\widehat{1}$; recall that y_* is the pullback of $1 \xrightarrow{\mathrm{y}} \widehat{1} \xleftarrow{\omega_0} \omega$ and that y_{*1} is the composite $\mathrm{y}_* \to \omega \xrightarrow{\omega_1} \widehat{1}$. Using the previous we can translate the Yoneda and density axioms for y (Definitions 4.3 and 4.5) in terms of (pullback) squares in $\mathcal{E}$, by factoring the (cartesian) cells considered in these axioms through y_* and by appying the pasting lemma for cartesian cells (Lemma 1.17). We find that y is a weak Yoneda embedding (and hence a Yoneda embedding, by Definition 4.3 and Example 3.12) if and only if for each modular subobject $j\colon J \rightarrowtail B$ there exists an order preserving morphism $J^\lambda\colon B \to \widehat{1}$ equipped with a pullback square in $\mathcal{E}$ as labelled (c) above, that is universal as follows: for every commuting square as labelled (d) above, with $g\colon B \to \widehat{1}$ order preserving, there exists a (unique) vertical cell $J^\lambda \Rightarrow g$ in $\mathsf{ModRel}(\mathcal{E})$.

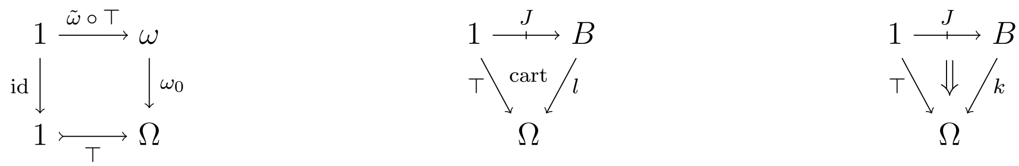
Using the weak density of y this implies that J^λ is unique up to isomorphism among order preserving morphisms $g\colon B \to \widehat{1}$ for which there exists a pullback square of the form as on the right above.

In particular if $\widehat{1}$ is an internal partial order (Example 1.31) then J^λ is uniquely determined by $J\colon 1 \nrightarrow B$ so that, restricting the previous to discrete $B = (B, I_B)$ and thus to ordinary subobjects $J \rightarrowtail B$ in $\mathcal{E}$, we find that $\mathrm{y}_{*1}\colon \mathrm{y}_* \rightarrowtail \widehat{1}$ is a *generic subobject* in $\mathcal{E}$, defining $\widehat{1}$ as the *subobject classifier* for $\mathcal{E}$; see e.g. Section A1.6 of [Joh02]. As discussed there it follows in this case that $\mathrm{y}_* \cong 1$; since the cocartesian cell defining $\mathrm{y}_*\colon 1 \nrightarrow \widehat{1}$ necessarily consists of this isomorphism we conclude that $\mathrm{y} = \big[1 \cong \mathrm{y}_* \xrightarrow{\mathrm{y}_{*1}} \widehat{1}\big]$ is itself a generic subobject in $\mathcal{E}$, whose companion in $\mathsf{ModRel}(\mathcal{E})$ can be taken to be $\mathrm{y}_* \cong (1 \xleftarrow{\mathrm{id}} 1 \xrightarrow{\mathrm{y}} \widehat{1})$.

4.16. Example. In the previous example we saw that a (weak) Yoneda embedding $\mathrm{y}\colon 1 \to \widehat{1}$ in $\mathsf{ModRel}(\mathcal{E})$ is a generic subobject in $\mathcal{E}$ whenever $\widehat{1}$ is an internal partial order. Here we will show the converse: a generic subobject $\top\colon 1 \to \Omega$ in $\mathcal{E}$ lifts as a Yoneda embedding $\top\colon (1, I_1) \to (\Omega, \omega)$ in $\mathsf{ModRel}(\mathcal{E})$. To start, first recall from e.g. Lemma A1.6.3(a) of [Joh02] that Ω admits the structure of an internal Heyting semilattice in $\mathcal{E}$, which equips Ω with an internal partial ordering $\omega = (\Omega \xleftarrow{\omega_0} \omega \xrightarrow{\omega_1} \Omega)$; see also Examples B2.3.8(a) therein. That $\top$ is an order preserving morphism $\top\colon (1, I_1) \to (\Omega, \omega)$ (Example 1.31) follows immediately from the definition of ω; see the proof of Lemma A1.6.3(i) of [Joh02], where the ordering is denoted Ω_1. The latter contains another assertion that we will use: any morphism $(f, g)\colon A \to \Omega \times \Omega$ in $\mathcal{E}$, where f and $g\colon A \to \Omega$ classify subobjects $f^*(\top)$ and $g^*(\top)$ of A, factors through $(\omega_0, \omega_1)\colon \omega \to \Omega \times \Omega$ if and only if $f^*(\top) \leq g^*(\top)$, that is there exists a morphism $f^*(\top) \to g^*(\top)$ in the slice category $\mathcal{E}/A$ of $\mathcal{E}$ over A. If f and g are order preserving morphisms $(A, \alpha) \to (\Omega, \omega)$ in $\mathsf{ModRel}(\mathcal{E})$ then the former means that there exists a vertical cell $f \Rightarrow g$ in $\mathsf{ModRel}(\mathcal{E})$. For instance notice that $\mathrm{id}_\Omega^*(\top) = \top \leq \mathrm{id}_\Omega = (\top \circ !)^*(\top)$ so that there exists a vertical cell $\phi\colon \mathrm{id}_\Omega \Rightarrow \top \circ !$ in $\mathsf{ModRel}(\mathcal{E})$.

We can use the cell ϕ to prove that the companion $\top_*$ is the modular relation $(1 \xleftarrow{\mathrm{id}} 1 \xrightarrow{\top} \Omega)$ in $\mathcal{E}$, as follows. Analogous to the construction of y_* in the previous example recall that $\top_*$ is the pullback of $1 \xrightarrow{\top} \Omega \xleftarrow{\omega_0} \omega$, with non-trivial leg $\top_{*1}$ the composite $\top_* \to \omega \xrightarrow{\omega_1} \Omega$. Hence it suffices to show that the commuting square on the left below is a pullback. To do so consider any morphism $f\colon X \to \omega$ with $\omega_0 \circ f = \top \circ !$; we have to show that $f = \tilde{\omega} \circ \top \circ !$. Regarding $\omega_1 \circ f$ as an order preserving morphism $\omega_1 \circ f\colon (X, I_X) \to (\Omega, \omega)$, the assumption on f implies that there exists a vertical cell $\top \circ ! \Rightarrow \omega_1 \circ f$ in $\mathsf{ModRel}(\mathcal{E})$. Together with the vertical cell $\phi \circ \omega_1 \circ f\colon \omega_1 \circ f \Rightarrow \top \circ !$ we conclude that $\omega_1 \circ f \cong \top \circ !$ so that, because Ω is an internal partial order, $\omega_1 \circ f = \top \circ !$ follows; see Example 1.31. But this means that $(\omega_0, \omega_1) \circ f = (\omega_1, \omega_0) \circ \tilde{\omega} \circ \top \circ !$ from which, as required, $f = \tilde{\omega} \circ \top \circ !$ follows, again by using that Ω is an internal partial order.

We can now show that $\top$ is weakly dense in $\mathsf{ModRel}(\mathcal{E})$ so that, since $\mathsf{ModRel}(\mathcal{E})$ has cocartesian tabulations (Example 3.12), $\top$ is in fact dense; see Definition 4.3. Weak density of $\top$ means that any nullary cartesian cell in $\mathsf{ModRel}(\mathcal{E})$ as in the middle below is weakly left Kan (Definition 1.2), that is any nullary cell as in the right below induces a vertical cell $l \Rightarrow k$ in $\mathsf{ModRel}(\mathcal{E})$. As explained in the previous example, the factorisations of these nullary cells through $\top_* = (1 \xleftarrow{\text{id}} 1 \xrightarrow{\top} \Omega)$ correspond to commutative squares in $\mathcal{E}$: the cartesian cell corresponds to the pullback square $l \circ j = \top \circ !$ while the other cell corresponds to the commuting square $k \circ j = \top \circ !$. The pullback square implies $l^*(\top) = j$ while the other square factors through the pullback that defines the subobject $k^*(\top)$. We conclude that $l^*(\top) = j \leq k^*(\top)$ so that the existence of the cell $l \Rightarrow k$ follows as required.



It remains to show that $\top$ satisfies the Yoneda axiom in $\mathsf{ModRel}(\mathcal{E})$ (Definition 4.5). To do so let $J\colon (1, I_1) \nrightarrow (B, \beta)$ be a modular subobject and consider the morphism $l\colon B \to \Omega$ that classifies $J \rightarrowtail B$. It suffices to show that l is an order preserving morphism $(B, \beta) \to (\Omega, \omega)$: as explained in the previous example the pullback square defining l then corresponds to the cartesian cell in the middle above in $\mathsf{ModRel}(\mathcal{E})$. That l is order preserving means that $(l \circ \beta_0, l \circ \beta_1)\colon \beta \to \Omega \times \Omega$ factors through (ω_0, ω_1). Composing the pullback square that defines l with the action $\rho\colon J \times_B \beta \to J$ we obtain the commuting square $\big[J \times_B \beta \to \beta \xrightarrow{\beta_1} B \xrightarrow{l} \Omega\big] = \top \circ !$. The latter factors through the pullback square that defines $(l \circ \beta_1)^*(\top)$ so that $(l \circ \beta_0)^*(\top) = \big[J \times_B \beta \to \beta\big] \leq (l \circ \beta_1)^*(\top)$; hence $(l \circ \beta_0, l \circ \beta_1)$ factors through (ω_0, ω_1) as required.

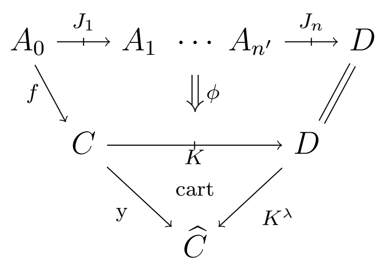
4.17. Example. Assume that $\mathcal{E}$ is a cartesian closed category with finite limits. A Yoneda embedding $\mathrm{y}\colon 1 \to \widehat{1}$ in $\mathsf{ModRel}(\mathcal{E})$ (Example 4.15) induces Yoneda embeddings for all internal preorders $A = (A, \alpha)$ in $\mathcal{E}$ as follows. Writing A° for the 'horizontal dual' $A^\circ := (A, \alpha^\circ)$ of A, with reversed internal order $\alpha^\circ = (A \xleftarrow{\alpha_1} \alpha \xrightarrow{\alpha_0} A)$, we will see in Example 8.12 below that the internal order α can be considered as a modular subobject of $A^\circ \times A$ which, under the correspondence of Example 4.15, induces an order preserving morphism $A^\circ \times A \to \widehat{1}$. The cartesian closed structure on $\mathcal{E}$ induces a cartesian closed structure on the locally thin 2-category $\mathsf{PreOrd}(\mathcal{E}) = V\big(\mathsf{ModRel}(\mathcal{E})\big)$ (Examples 1.31 and 8.19) of internal preorders, under which the latter corresponds to an order preserving morphism $\mathrm{y}_A\colon A \to [A^\circ, \widehat{1}]$. In Example 8.24 we will see that y_A forms a Yoneda embedding in $\mathsf{ModRel}(\mathcal{E})$. In particular, using Proposition 4.24 below, internal modular relations $A \nrightarrow B$ correspond to order preserving morphisms $B \to [A^\circ, \widehat{1}]$.

Next assume that $\widehat{1}$ is an internal partial order so that y defines $\widehat{1}$ as the subobject classifier $\Omega := \widehat{1}$ for $\mathcal{E}$ (Example 4.15). In this case the latter correspondence recovers the natural equivalence considered on page 283 of [CS86]. Moreover, using that $I_A \cong [A^\circ, \widehat{1}](\mathrm{y}, \mathrm{y})$ by Lemma 4.6 we find that our $\mathrm{y}_A\colon A \to [A^\circ, \widehat{1}]$ coincides with the Yoneda embedding $y_A\colon A \to \mathcal{P}A$ considered there. In our terms the 'membership ideal', considered on the same page, is the companion $\in_A := \mathrm{y}_{A*}\colon A \nrightarrow [A^\circ, \widehat{1}]$. Finally restrict to a discrete internal preorder $A = (A, I_A)$ in the previous, so that $A^\circ = A$ and, as we will see in Example 8.19, $[A, \widehat{1}]$ is an internal partial order with the exponential $\widehat{1}^A$ as underlying $\mathcal{E}$-object. Using arguments similar to the ones used in Example 4.15 it is straightforward to see that, in this case, y_A being a Yoneda embedding in $\mathsf{ModRel}(\mathcal{E})$ implies that the internal relation underlying $\in_A = \mathrm{y}_{A*}$ defines $\widehat{1}^A$ as the *power object* of A in $\mathcal{E}$ in the classical sense; see e.g. Section A2.1 of [Joh02].

4.18. Properties of Yoneda morphisms. The results below record some basic properties of Yoneda morphisms. The first of these follows immediately from the density of Yoneda morphisms (Definition 4.3) and the vertical pasting lemma (Lemma 2.17).

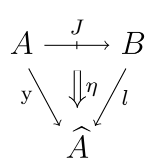
4.19. Corollary. *Let* $\mathrm{y}\colon C \to \widehat{C}$ *be a (weak) Yoneda morphism. Consider the composite below where* K^λ *is supplied by the Yoneda axiom (Definition 4.5). If the cell* ϕ *is right (respectively weakly) nullary-cocartesian (Definition 2.6) then the composite is (weakly) left Kan (Definitions 1.2 and 1.9).*

If moreover ϕ *restricts along* $g\colon X \to D$ *(Definition 2.12) then so does the (weakly) left Kan composite (Definition 1.24). If* ϕ *is pointwise right (respectively weakly) nullary-cocartesian (Definition 2.12) then the composite is pointwise (weakly) left Kan (Definition 1.24).*



The lemma below is a variation of Lemma 3.2 of [Web07]. It implies that the notions of weak left Kan extension and pointwise weak left Kan extension coincide when extending along a weak Yoneda morphism, and that all four notions of left Kan extension coincide when extending along a Yoneda morphism.

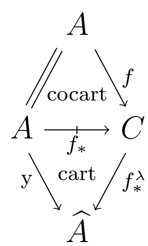
4.20. Lemma. *Let* $\mathrm{y}\colon A \to \widehat{A}$ *be a (weak) Yoneda morphism. The following are equivalent for the cell* η *below:* (a) η *is cartesian;* (b) η *is pointwise (weakly) left Kan (Definition 1.24);* (c) η *is weakly left Kan (Definition 1.2).*



Proof. (a) $\Rightarrow$ (b) follows from the (weak) density of y (Definition 4.3) and (b) $\Rightarrow$ (c) is clear. To prove (c) $\Rightarrow$ (a) consider the morphism $J^\lambda\colon B \to \widehat{A}$ supplied by the Yoneda axiom (Definition 4.5), which comes equipped with a cartesian cell that defines J as the nullary restriction of $\widehat{A}$ along y and J^λ. By (weak) density of y this cartesian cell is weakly left Kan so that, assuming (c) and using the uniqueness of left Kan extensions, η factors through it as a vertical isomorphism $J^\lambda \cong l$. Since cartesian cells are preserved by horizontal composition with vertical isomorphisms we conclude that the cell η is cartesian too, as required. ■

The following is a variation of Corollary 3.5(2) of [Web07].

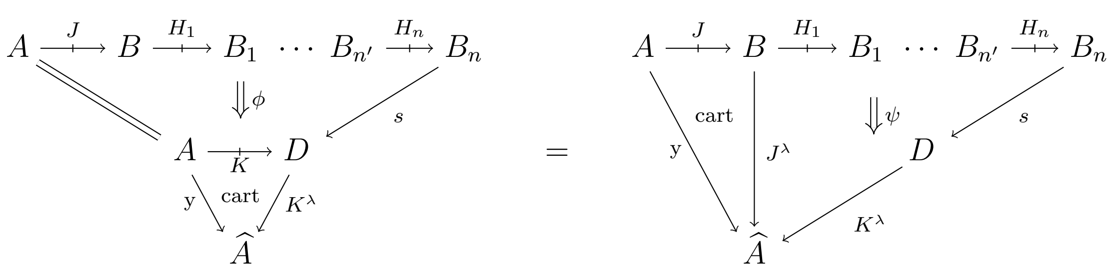
4.21. Proposition. *Let* $\mathrm{y}\colon A \to \widehat{A}$ *be a weak Yoneda embedding. If the composite below is invertible then* $f\colon A \to C$ *is full and faithful (Definition 1.18). The converse holds whenever the restriction* $C(f, f)$ *exists.*



Proof. Assume that the composite is invertible. Since the identity cell id_y is cartesian and because cartesian cells are preserved under horizontal composition with invertible vertical cells, it follows that the composite is cartesian as well. Hence the cocartesian cell cocart is cartesian by the pasting lemma for cartesian cells (Lemma 1.17). Since id_f factors through the latter as a cartesian cell $f_* \Rightarrow C$ (see Lemma 1.21), the same pasting lemma implies that id_f is cartesian too, so that f is full and faithful. The converse follows from the fact that the cartesian cell defining f_*^λ is weakly left Kan, by the weak density of y (Definition 4.3), and Proposition 2.26. ■

The following lemma is important to our study of Yoneda morphisms: its corollary Proposition 5.5, for instance, is used in Theorem 7.6 to give a condition that ensures the 'cocompleteness' (Definition 7.2) of presheaf objects. Its restriction to empty paths $\underline{H} = (B)$ is analogous to Proposition 7 of [SW78] for Yoneda structures.

4.22. Lemma. *Let* $\mathrm{y}\colon A \to \widehat{A}$ *be a Yoneda morphism. The equality below determines a bijection between cells* ϕ *and* ψ *of the forms as shown.*



If y *is merely a weak Yoneda morphism then the equality above determines a bijection between cells* ϕ *and* ψ *with* $\underline{H} = (B)$ *empty. Under the latter restriction* ϕ *is cartesian if and only if the corresponding* ψ *(in this case a vertical cell) is invertible.*

Proof. The bijection is given by the assignment $\phi \mapsto \psi$ obtained by factorising the left-hand side above through the cartesian cell that defines J^λ (Definition 4.5), which is (weakly) left Kan by density of y (Definition 4.3), and the assignment $\psi \mapsto \phi$ obtained by factorising the right-hand side through the cartesian cell that defines K^λ. Restricting to empty paths $\underline{H} = (B)$, notice that ψ is invertible if and only if the right-hand side, and hence both sides, are weakly left Kan. By the previous lemma this is equivalent to the left-hand side being cartesian which, by the pasting lemma for cartesian cells (Lemma 1.17), is in turn equivalent to ϕ being cartesian. ∎

4.23. Equivalence of morphisms $A \nrightarrow B$ and morphisms $B \to \widehat{A}$. Let $\mathrm{y}\colon A \to \widehat{A}$ be a weak Yoneda morphism. The proposition below uses the previous lemma to describe the functoriality of the assignment $J \mapsto J^\lambda$ (Definition 4.5), showing also that the resulting functor is an equivalence if and only if y admits nullary restrictions (Definition 4.5).

In stating the proposition we use the following two notions of slice category in an augmented virtual double category $\mathcal{K}$. Given an object A of $\mathcal{K}$ the *horizontal slice category* $A \mathbin{/_{\mathrm{h}}} \mathcal{K}$ has as objects horizontal morphisms $J\colon A \nrightarrow B$ and as morphisms $J \to K$ cells ϕ of the form as on the left below. Fixing a target B in $\mathcal{K}$ we denote by $H(\mathcal{K})(A, B) \subseteq A \mathbin{/_{\mathrm{h}}} \mathcal{K}$ the subcategory generated by morphisms $J \to K$ with vertical target $s = \mathrm{id}_B$. Next recall that $\mathcal{K}$ contains a 2-category $V(\mathcal{K})$ of vertical morphisms (Example A1.5). Given an object P of $\mathcal{K}$ we denote by $V(\mathcal{K})/P$ the *lax slice category* consisting of vertical morphisms $g\colon A \to P$ as objects and cells ψ of the form as on the right below as morphisms $g \to h$; see e.g. Section I,2.5 of [Gra74] for the more general notion of "lax comma 2-category" (therein called "2-comma category"). Notice that $V(\mathcal{K})/P$ contains the hom-categories $V(\mathcal{K})(A, P)$ as subcategories.

$$\begin{array}{ccc} A & \xrightarrow{J} & B \\ \Vert & \Downarrow\phi & \downarrow s \\ A & \xrightarrow[K]{} & D \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{s} & B \\ & {\scriptstyle g}\searrow \; \overset{\psi}{\Rightarrow} \; \swarrow{\scriptstyle h} & \\ & P & \end{array}$$

4.24. Proposition. *Let* $\mathrm{y}\colon A \to \widehat{A}$ *be a weak Yoneda morphism in an augmented virtual double category* $\mathcal{K}$*. Choosing for each* $J\colon A \nrightarrow B$ *a morphism* $J^\lambda\colon B \to \widehat{A}$ *as in the Yoneda axiom (Definition 4.5) induces a full and faithful functor*

$$(\text{–})^\lambda\colon A \mathbin{/_{\mathrm{h}}} \mathcal{K} \to V(\mathcal{K})/\widehat{A}$$

that maps a cell $\phi\colon J \Rightarrow K$ *as on the left above to the vertical cell* $\phi^\lambda\colon J^\lambda \Rightarrow K^\lambda \circ s$ *that corresponds to* ϕ *under the bijection of Lemma 4.22 (with* $\underline{H} = (B)$ *empty). Fixing the target* B *restricts* $(\text{–})^\lambda$ *to a full and faithful functor*

$$(\text{–})^\lambda\colon H(\mathcal{K})(A, B) \to V(\mathcal{K})(B, \widehat{A}).$$

Moreover $(-)^\lambda$ *is an equivalence of categories* $A\,/_{\mathrm{h}}\,\mathcal{K} \simeq V(\mathcal{K})/\widehat{A}$ *if and only if* y *admits nullary restrictions (Definition 4.5).*

Proof. To show that $(-)^\lambda$ is functorial consider composable morphisms in $A\,/_{\mathrm{h}}\,\mathcal{K}$, that is cells $\phi\colon J \Rightarrow K$ and $\psi\colon K \Rightarrow L$ both with vertical source id_A; let us denote their vertical targets by $s\colon B \to D$ and $t\colon D \to E$ respectively. Using the bijection of Lemma 4.22 and the interchange axioms (Lemma A1.3) we obtain the following equality, where the cartesian cells define the morphisms J^λ, K^λ and L^λ.

$$\begin{aligned}\mathrm{cart}_{J^\lambda} \odot (\psi \circ \phi)^\lambda = \mathrm{cart}_{L^\lambda} \circ \psi \circ \phi &= (\mathrm{cart}_{K^\lambda} \odot \psi^\lambda) \circ \phi \\ &= (\mathrm{cart}_{K^\lambda} \circ \phi) \odot (\psi^\lambda \circ s) = \mathrm{cart}_{J^\lambda} \odot \phi^\lambda \odot (\psi^\lambda \circ s)\end{aligned}$$

By weak density of y (Definition 4.3) the cartesian cell $\mathrm{cart}_{J^\lambda}$ is weakly left Kan so that, using the uniqueness of factorisations through left Kan cells, the equality above implies $(\psi \circ \phi)^\lambda = \phi^\lambda \odot (\psi^\lambda \circ s)$ as required. That the functor $(-)^\lambda$ is full and faithful is because the correspondence of Lemma 4.22 is a bijection.

To show the final assertion it suffices to prove that y admits nullary restrictions if and only if $(-)^\lambda$ is essentially surjective (see e.g. Theorem 1 of Section IV.4 of [ML98]). For the 'only if'-part consider any morphism $g\colon B \to \widehat{A}$ and assume that the restriction $\widehat{A}(\mathrm{y}, g)\colon A \nrightarrow B$ exists; we will show that $g \cong \widehat{A}(\mathrm{y}, g)^\lambda$ (Definition 4.5). Consider the cartesian cells that define $\widehat{A}(\mathrm{y}, g)$ and $\widehat{A}(\mathrm{y}, g)^\lambda$. By weak density of y (Definition 4.3) both define the weak left Kan extension of y along $\widehat{A}(\mathrm{y}, g)$ so that the required isomorphism exists by the uniqueness of Kan extensions. To show the converse assume that $(-)^\lambda$ is essentially surjective: for every $g\colon B \to \widehat{A}$ there exists $J\colon A \nrightarrow B$ with $J^\lambda \cong g$. Composing the latter isomorphism with the cartesian cell defining J^λ (Definition 4.5) we obtain a cartesian cell that defines J as the nullary restriction $\widehat{A}(\mathrm{y}, g)$. ■

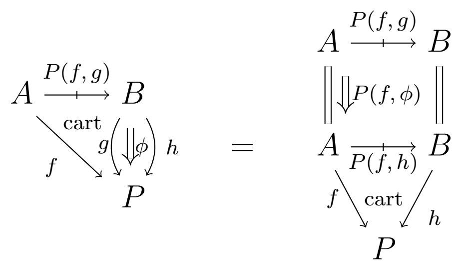
4.25. Remark. Consider a Yoneda structure on a 2-category $\mathcal{C}$, in the sense of [SW78], consisting of a right ideal $\mathcal{A}$ of 'admissible' morphisms in $\mathcal{C}$ and a 'Yoneda embedding' $yA\colon A \to \mathcal{P}A$ for each admissible object A (i.e. with $\mathrm{id}_A \in \mathcal{A}$). Similar to the above result such a structure induces full and faithful functors $\mathcal{C}_{\mathcal{A}}(A, B) \to \mathcal{C}(B, \mathcal{P}A)$, given by $f \mapsto B(f, 1)$ (see Axiom 1 and Proposition 7 of [SW78]), where $\mathcal{C}_{\mathcal{A}}(A, B) \subseteq \mathcal{C}(A, B)$ denotes the full subcategory of admissible morphisms. These functors however are not essentially surjective in general, as can be easily seen by taking $\mathcal{C} = \mathsf{Cat}(\mathsf{Set}')$ the 2-category of large categories and $A = B = 1$ the terminal category.

The following result is a partial converse to the previous proposition.

4.26. Proposition. *Let* $f\colon A \to P$ *be a morphism in an augmented virtual double category* $\mathcal{K}$ *and let* $B \in \mathcal{K}$ *be an object. Choosing nullary restrictions* $P(f, g)\colon A \nrightarrow B$ *for all* $g\colon B \to P$ *induces a functor*

$$P(f, \text{-})\colon V(\mathcal{K})(B, P) \to H(\mathcal{K})(A, B)$$

which maps a vertical cell $\phi\colon g \Rightarrow h$ to the unique factorisation $P(f,\phi)\colon P(f,g) \Rightarrow P(f,h)$ in the right-hand side below.



The following hold:

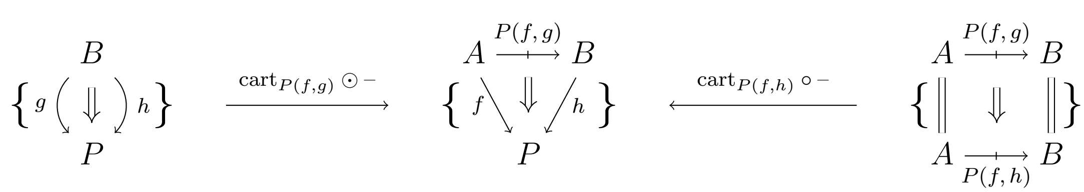
(a) *f is weakly dense (Definition 4.3) if and only if the functors $P(f,–)$ are full and faithful for each object $B \in \mathcal{K}$;*

(b) *f satisfies the Yoneda axiom (Definition 4.5) if and only if the functors $P(f,–)$ are essentially surjective for each $B \in \mathcal{K}$;*

(c) *f is a weak Yoneda morphism if and only if the functors $P(f,–)$ are equivalences of categories $V(\mathcal{K})(B,P) \simeq H(\mathcal{K})(A,B)$ for each $B \in \mathcal{K}$.*

Moreover if f is a weak Yoneda morphism then the functors $P(f,–)$ extend to a pseudo-inverse to the equivalence $(–)^\lambda\colon A\,/_{\mathrm{h}}\,\mathcal{K} \simeq V(\mathcal{K})/P$ of Proposition 4.24, that is induced by f.

Proof. To see part (b) notice that the Yoneda axiom asserts that, for every horizontal morphism $J\colon A \nrightarrow B$, there exists a vertical morphism $J^\lambda\colon B \to P$ such that $J \cong P(f, J^\lambda)$. That part (c) follows from parts (a) and (b) is well-known (see e.g. Theorem 1 of Section IV.4 of [ML98]). To prove part (a) consider the assignments below between cells of $\mathcal{K}$ of the form as shown, given by composition with the cartesian cells that define the chosen nullary restrictions $P(f,g)$ and $P(f,h)$ respectively. By definition (Definition 1.16) the assignment on the right is a bijection.



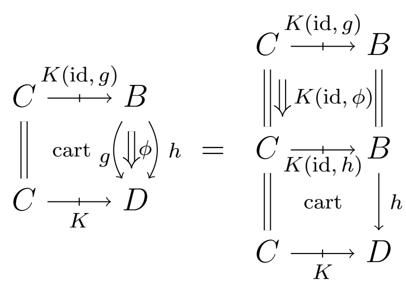
By definition the image under $P(f,-)$ of a vertical cell $\phi\colon g \Rightarrow h$ as on the left above is the horizontal cell $P(f,\phi)\colon P(f,g) \Rightarrow P(f,h)$ that corresponds to $\mathrm{cart}_{P(f,g)} \odot \phi$ under the bijection on the right. Hence $P(f,-)$ is full and faithful precisely if the assignment $\mathrm{cart}_{P(f,g)} \odot -$ on the left is a bijection for each $g\colon B \to P$. By Definition 1.2 the latter means that any nullary cartesian cell with f as vertical source is weakly left Kan, that is f is weakly dense (Definition 4.3) as required.

For the final assertion assume that $f\colon A \to P$ is a weak Yoneda morphism, thus inducing the an equivalence of categories $(-)^\lambda\colon A/_{\mathrm{h}}\mathcal{K} \simeq V(\mathcal{K})/P$ as in Proposition 4.24. As explained in proof of the latter $(-)^\lambda$ is essentially surjective because $g \cong P(f,g)^\lambda$ for each $g\colon A \to P$. It follows that the assignment $g \mapsto P(f,g)$ uniquely extends to a pseudo-inverse to $(-)^\lambda$, with the latter isomorphims forming the unit (see e.g. Theorem 1 of Section IV.4 of [ML98]). That the action of this pseudo-inverse on vertical cells coincides with that of the functors $P(f,-)$ of the statement is straightforward to check. ∎

4.27. Remark. Given a morphism $y\colon C \to \bar{C}$ in a 'bicategory $\mathcal{C}$ equipped with proarrows', in the sense of [Woo82], Melliès and Tabareau consider in [MT08] functors analogous to the functors $P(f,-)$ of Proposition 4.26; they define y to be a 'Yoneda situation' if both y and each of these functors are full and faithful.

4.28. Yoneda embeddings from 2-toposes. Weber shows in [Web07] that every '2-topos' $\mathcal{C}$ admits a 'good Yoneda structure', the construction of which he attributes to Street ([Str74a] and [Str80a]). Given a finitely complete cartesian closed category $\mathcal{E}$, in Section 7 of [Str17] this result is used to obtain a good Yoneda structure on the 2-category $\mathsf{Cat}(\mathcal{E})$ of categories internal to $\mathcal{E}$. Similarly to Weber's result, Example 4.30 below shows that a 2-topos structure on a 2-category $\mathcal{C}$ induces a collection of Yoneda embeddings, in our sense, in a certain full sub-augmented virtual equipment of the unital virtual equipment $\mathsf{dFib}(\mathcal{C})$ of discrete two-sided fibrations in $\mathcal{C}$ (Example 1.30). We will use the following lemma, which generalises the functors $P(f,-)$ described in Proposition 4.26, to functors $K(\mathrm{id},-)$ given by restriction of a fixed horizontal morphism K on the right.

4.29. Lemma. *Let $K\colon C \nrightarrow D$ be a morphism and B an object in an augmented virtual double category $\mathcal{K}$. Choosing a restriction $K(\mathrm{id},g)\colon C \nrightarrow B$ for each morphism $g\colon B \to D$ induces a functor $K(\mathrm{id},-)\colon V(\mathcal{K})(B,D) \to H(\mathcal{K})(C,B)$ which maps a cell $\phi\colon g \Rightarrow h$ to the unique factorisation $K(\mathrm{id},\phi)\colon K(\mathrm{id},g) \Rightarrow K(\mathrm{id},h)$ in the right-hand side below:*



4.30. Example. Let $\mathcal{C}$ be a finitely complete 2-category with terminal object denoted by 1. A *discrete opfibration* (Section 2 of [Web07]) is, in our terms, a discrete two-sided fibration $J\colon 1 \nrightarrow B$ in $\mathsf{dFib}(\mathcal{C})$ (Example 1.30). In Definition 4.1 of [Web07] a discrete opfibration $\tau\colon 1 \nrightarrow \Omega$ is called *classifying* if, in our terms, for each $B \in \mathcal{C}$ the functor $\tau(\mathrm{id},-)\colon \mathcal{C}(B,\Omega) \to H(\mathsf{dFib}(\mathcal{C}))(1,B)$ of the previous lemma, given by pulling back τ, is full and faithful. Definition 4.10 of [Web07] defines a 2-*topos* $(\mathcal{C},(-)^\circ,\tau)$ to consist of a finitely complete cartesian closed 2-category $\mathcal{C}$ equipped with a 'duality involution' $(-)^\circ\colon \mathcal{C}^{\text{co}} \to \mathcal{C}$ (Definition 2.14 of [Web07]) and a classifying discrete opfibration $\tau\colon 1 \nrightarrow \Omega$.

Given an object A in a 2-topos $\mathcal{C}$, Section 5 of [Web07] sets $\widehat{A} \coloneqq [A^\circ, \Omega]$ and considers, for each $B \in \mathcal{C}$, the composite functor

$$\mathcal{C}(B, \widehat{A}) \cong \mathcal{C}(A^\circ \times B, \Omega) \xrightarrow{\tau(\text{id}, -)} H(\mathsf{dFib}(\mathcal{C}))(1, A^\circ \times B) \simeq H(\mathsf{dFib}(\mathcal{C}))(A, B),$$

where the isomorphism is given by the cartesian closed structure and the equivalence is given by the involution structure; this composite is full and faithful and pseudonatural in A and B. A discrete two-sided fibration $J\colon A \nrightarrow B$ is then called an *attribute* if it is contained in the essential image of this composite. We write $\mathsf{Attr}(\mathcal{C}) \subseteq \mathsf{dFib}(\mathcal{C})$ for the full sub-augmented virtual double category generated by the attributes. The pseudonaturality of the composite above implies that $\mathsf{Attr}(\mathcal{C})$ is closed under taking restrictions so that it, like $\mathsf{dFib}(\mathcal{C})$ (Example 1.30), is an augmented virtual equipment. Moreover by Lemma 3.14 $\mathsf{Attr}(\mathcal{C})$ has all cocartesian tabulations, that are created as in $\mathsf{dFib}(\mathcal{C})$ (Example 3.13).

A morphism $f\colon A \to C$ of $\mathcal{C}$ is defined to be *admissible* in Section 5 of [Web07] whenever, in our terms, the companion $f_*\colon A \nrightarrow C$ of f (in $\mathsf{dFib}(\mathcal{C})$) is an attribute; i.e. f admits a companion in $\mathsf{Attr}(\mathcal{C})$. In particular an object A is admissible if and only if its horizontal unit $I_A\colon A \nrightarrow A$ (Definition 1.16) is an attribute, that is A admits a horizontal unit in $\mathsf{Attr}(\mathcal{C})$. Consider an admissible object A. By definition there exists a morphism $\text{y}_A\colon A \to \widehat{A}$ whose image, under the composite above, is isomorphic to I_A and we will show that y_A forms a Yoneda embedding (Definition 4.5) in the augmented virtual equipment $\mathsf{Attr}(\mathcal{C})$ of attributes in $\mathcal{C}$.

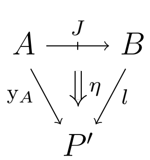
To prove the Yoneda axiom for y_A consider any attribute $J\colon A \nrightarrow B$ and let $J^\lambda\colon B \to \widehat{A}$ be any morphism whose image under the composite above is isomorphic to J. By Proposition 5.2 of [Web07] we have $J \cong \text{y}_A/J^\lambda$, the *comma object* of y_A and J^λ in $\mathcal{C}$; see e.g. Section 1 of [Str74b]. In our terms, by Proposition 1 of the latter, $\text{y}_A/J^\lambda \cong \text{y}_{A*} \odot (\widehat{A})^2 \odot J^{\lambda *}$ as spans in $\mathsf{Span}(\mathcal{C}_0)$ (Example A2.9), so that $J \cong \widehat{A}(\text{y}_A, J^\lambda)$ in $\mathsf{dFib}(\mathcal{C})$ (see Example 1.30 and Example A4.9) and hence in $\mathsf{Attr}(\mathcal{C})$, which proves the Yoneda axiom for y_A. It remains to show that y_A is dense in $\mathsf{Attr}(\mathcal{C})$. We use that y_A is admissible, which is a consequence of Proposition 5.2 of [Web07]. It follows that the companion y_{A*} exists in $\mathsf{Attr}(\mathcal{C})$ so that, by Definition 4.3, it suffices to show that the cartesian cell cart defining y_{A*} is pointwise left Kan in $\mathsf{Attr}(\mathcal{C})$. By applying Proposition 3.22 to the companion identity $\text{id}_{\text{y}_A} = \text{cart} \circ \text{cocart}$ (Lemma 1.21), we may equivalently show that the identity cell id_{y_A} defines $\text{id}_{\widehat{A}}$ as the pointwise left Kan extension of y_A along y_A in the vertical 2-category $V\big(\mathsf{Attr}(\mathcal{C})\big) \cong \mathcal{C}$. Since id_{y_A} trivially defines y_A as an absolute left lifting of y_A along $\text{id}_{\widehat{A}}$ the latter follows from Theorem 5.3(2) of [Web07].

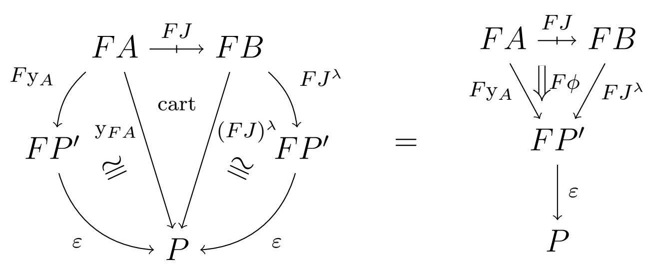
4.31. Lifting Yoneda morphisms along universal morphisms. Given a functor $F\colon \mathcal{K} \to \mathcal{L}$ and an object $P \in \mathcal{L}$ consider a universal morphism $\varepsilon\colon FP' \to P$ from F to P (Definition 1.39). Given $A \in \mathcal{K}$ and a (weak) Yoneda morphism $\text{y}_{FA}\colon FA \to P$ in $\mathcal{L}$, the following theorem gives conditions ensuring that y_{FA} induces a (weak) Yoneda embedding $\text{y}_A\colon A \to P'$ in $\mathcal{K}$. In Example 4.33 below we use this to obtain Yoneda embeddings in the unital virtual double category $\mathcal{V}$-sProf of small $\mathcal{V}$-profunctors (Example A2.8) from those in $(\mathcal{V}, \mathcal{V}')$-Prof (Example 4.10). A related result is Theorem 4.1 of [Her01], which allows one to transfer a Yoneda structure ([SW78]) on a 2-category $\mathcal{L}$ along a biadjunction $F \dashv G\colon \mathcal{L} \to \mathcal{K}$ with full and faithful unit.

To state the theorem consider the vertical slice category $F /_{\mathrm{v}} P$ (Definition 1.39) and let $(F /_{\mathrm{v}} P)^\lambda \subseteq F /_{\mathrm{v}} P$ denote the full subcategory generated by all objects $(B, f\colon FB \to P)$ such that $f \cong (FJ)^\lambda$ for some $J\colon A \nrightarrow B$ in $\mathcal{K}$, where $(FJ)^\lambda$ is supplied by the Yoneda axiom for y_{FA} (Definition 4.5). In other words $(B, f) \in (F /_{\mathrm{v}} P)^\lambda$ if and only if the restriction $P(\mathrm{y}_A, f)$ exists in $\mathcal{L}$, and it is contained in the essential image of F. Assume that $\varepsilon\colon FP' \to P$ is universal from F to P relative to $(F /_{\mathrm{v}} P)^\lambda$ (Definition 1.39) and that A is unital, with horizontal unit $I_A\colon A \nrightarrow A$. It follows that $FA \in \mathcal{L}$ is unital, with horizontal unit FI_A (see Corollary A5.5), and that $FI_A \cong P(\mathrm{y}_{FA}, \mathrm{y}_{FA})$ by Lemma 4.6. Comparing the cartesian cell defining the latter restriction with the cartesian cell defining $(FI_A)^\lambda$ (Definition 4.5), using that both cartesian cells are (weakly) left Kan by the (weak) density of y_{FA} (Definition 4.3), we find that $(FI_A)^\lambda \cong \mathrm{y}_{FA}$. We conclude that $\mathrm{y}_{FA} \in (F /_{\mathrm{v}} P)^\lambda$ so that, by universality of ε, there exists a morphism $\mathrm{y}_A \coloneqq (\mathrm{y}_{FA})^\sharp\colon A \to P'$ in $\mathcal{K}$ such that $\mathrm{y}_{FA} \cong \varepsilon \circ F\mathrm{y}_A$.

4.32. Theorem. *Let the functor $F\colon \mathcal{K} \to \mathcal{L}$, the unital object $A \in \mathcal{K}$, the (weak) Yoneda morphism $\mathrm{y}_{FA}\colon FA \to P$ in $\mathcal{L}$ and the universal morphism $\varepsilon\colon FP' \to P$ relative to $(F /_{\mathrm{v}} P)^\lambda$ be as above. The morphism $\mathrm{y}_A\colon A \to P'$, as obtained above, is a (weak) Yoneda embedding in $\mathcal{K}$ as long as the functor $\varepsilon \circ F -\colon \mathcal{K} /_{\mathrm{v}} P' \to F /_{\mathrm{v}} P$ (Definition 1.39) preserves and reflects all cartesian cells in $\mathcal{K} /_{\mathrm{v}} P'$ that define restrictions of the form $P'(\mathrm{y}_A, g)$, where $g\colon B \to P'$ is any morphism.*

Proof. Denote the invertible vertical cell $\mathrm{y}_{FA} \Rightarrow \varepsilon \circ F\mathrm{y}_A$ by σ. To show that y_A is (weakly) dense we have to show that any cartesian cell η in $\mathcal{K}$ as on the left below is (weakly) left Kan (Definition 4.3). By assumption $\varepsilon \circ F\eta$ is cartesian in $\mathcal{L}$ so that $\sigma \odot (\varepsilon \circ F\eta)$ is (weakly) left Kan by the (weak) density of y_{FA}. We conclude that $\varepsilon \circ F\eta$ is (weakly) left Kan so that η, being the adjunct of $\varepsilon \circ F\eta$, is (weakly) left Kan by Proposition 1.42.





To prove the Yoneda axiom (Definition 4.5) for $\mathrm{y}_A\colon A \to P'$ we have to supply, for every $J\colon A \nrightarrow B$ in $\mathcal{K}$, a morphism $J^\lambda\colon B \to P'$ and a cartesian cell that defines J as the restriction $P'(\mathrm{y}_A, J^\lambda)$. By the Yoneda axiom for y_{FA} there exists a morphism $(FJ)^\lambda\colon FB \to P$ such that $FJ \cong P\big(\mathrm{y}_{FA}, (FJ)^\lambda\big)$, and we take $J^\lambda \coloneqq \big((FJ)^\lambda\big)^\sharp$ to be its adjunct, satisfying $(FJ)^\lambda \cong \varepsilon \circ FJ^\lambda$, which exists by the universality of ε. Consider the composite cartesian cell on the left-hand side above, of the cartesian cell that defines $(FJ)^\lambda$ and the invertible vertical cells that equip the adjuncts y_A and J^λ. Its adjunct $\phi\colon J \Rightarrow P'$, which exists by the local universality of ε (Definition 1.39), satisfies the identity above. By assumption the cartesianness of the left-hand side above implies the cartesianness of ϕ, which thus defines J as the restriction $P'(\mathrm{y}_A, J^\lambda)$ as required. ■

4.33. Example. Let $\mathcal{V} \subset \mathcal{V}'$ be a symmetric universe enlargement (Example 4.10), and assume that $\mathcal{V}$ is small complete. By applying the previous theorem to the full embedding $F\colon \mathcal{V}\text{-}\mathsf{sProf} \hookrightarrow (\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}$ we will see that the Yoneda embeddings $\mathrm{y}_A\colon A \to [A^{\mathrm{op}}, \mathcal{V}]'$ of $(\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}$ (Example 4.10), for $\mathcal{V}$-categories A, can be corestricted to $\mathcal{V}$-categories of 'small $\mathcal{V}$-presheaves on A', as introduced by Lindner in [Lin74], to form Yoneda embeddings in the unital virtual double category $\mathcal{V}\text{-}\mathsf{sProf}$ of small $\mathcal{V}$-profunctors (Example A2.8).

We denote by $[A^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}} \subseteq [A^{\mathrm{op}}, \mathcal{V}]'$ the full sub-$\mathcal{V}'$-category of $\mathcal{V}$-presheaves $p\colon A^{\mathrm{op}} \to \mathcal{V}$ that are *small* in the sense of [Lin74] and [DL07]: these are precisely the $\mathcal{V}$-presheaves $A^{\mathrm{op}} \to \mathcal{V}$ that correspond to small $\mathcal{V}$-profunctors $A \nrightarrow I$ in the sense of Example A2.8. It is straightforward to check that $\mathcal{V}$ being small complete implies that $[A^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$ is a $\mathcal{V}$-category; see also Corollary 2.3 of [Lin74]. In fact, using that any small $\mathcal{V}$-presheaf $p\colon A \nrightarrow I$ is "generated" by its restriction to a small sub-$\mathcal{V}$-category $A_* \subseteq A$, in the sense of Example A2.8, one checks that the hom $\mathcal{V}'$-objects $[A^{\mathrm{op}}, \mathcal{V}]'(p, q)$ are computed by the small $\mathcal{V}$-ends $\int_{x \in A_*}[px, qx]$.

Regarding $[A^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$ as an object of $\mathcal{V}\text{-}\mathsf{sProf}$ let $\varepsilon\colon F[A^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}} \hookrightarrow [A^{\mathrm{op}}, \mathcal{V}]'$ denote the embedding in $(\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}$. Because ε is full and faithful in $(\mathcal{V}, \mathcal{V})'\text{-}\mathsf{Prof}$ and F is a full and faithful functor, ε is locally universal by Example 1.40. To show that ε is universal relative to $(F \mathbin{/_{\mathrm{v}}} [A^{\mathrm{op}}, \mathcal{V}]')^\lambda$ (Definition 1.39) let $f\colon FB \to [A^{\mathrm{op}}, \mathcal{V}]'$ be any $\mathcal{V}'$-functor, with B a $\mathcal{V}$-category. Using the correspondence of $\mathcal{V}$-profunctors $A \nrightarrow B$ and $\mathcal{V}'$-functors $B \to [A^{\mathrm{op}}, \mathcal{V}]'$ in $(\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}$ (Example 4.10) one checks that $(B, f) \in \big(F \mathbin{/_{\mathrm{v}}} [A^{\mathrm{op}}, \mathcal{V}]'\big)^\lambda$ precisely if f, regarded as a $\mathcal{V}$-profunctor $A \nrightarrow B$, is small in the sense of Example A2.8, that is $f(y)\colon A^{\mathrm{op}} \to \mathcal{V}$ is a small $\mathcal{V}$-presheaf for every $y \in B$ or, equivalently, the $\mathcal{V}'$-functor $A^{\mathrm{op}} \to [B, \mathcal{V}]'$ corresponding to f is *pointwise small* in the sense of [DL07]. Hence $(B, f) \in \big(F \mathbin{/_{\mathrm{v}}} [A^{\mathrm{op}}, \mathcal{V}]'\big)^\lambda$ if and only if there exists a $\mathcal{V}$-functor $f'\colon B \to [A^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$ such that $f = \varepsilon \circ F f'$ and we conclude that $\varepsilon \circ F{-}\colon \mathcal{V}\text{-}\mathsf{sProf} \mathbin{/_{\mathrm{v}}} [A^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}} \to F \mathbin{/_{\mathrm{v}}} [A^{\mathrm{op}}, \mathcal{V}]'$ factors as an equivalence through $\big(F \mathbin{/_{\mathrm{v}}} [A^{\mathrm{op}}, \mathcal{V}]'\big)^\lambda \hookrightarrow F \mathbin{/_{\mathrm{v}}} [A^{\mathrm{op}}, \mathcal{V}]'$, so that ε is universal relative to $\big(F \mathbin{/_{\mathrm{v}}} [A^{\mathrm{op}}, \mathcal{V}]'\big)^\lambda$.

Moreover since F preserves and reflects cartesian cells (use Example A4.7), so does $\varepsilon \circ F{-}\colon \mathcal{V}\text{-}\mathsf{sProf} \mathbin{/_{\mathrm{v}}} [A^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}} \to F \mathbin{/_{\mathrm{v}}} [A^{\mathrm{op}}, \mathcal{V}]'$, by combining Example A4.11 and the pasting lemma (Lemma 1.17). Thus all hypotheses of the theorem are satisfied. We conclude that the Yoneda embedding $\mathrm{y}_A\colon A \to [A^{\mathrm{op}}, \mathcal{V}]'$ in $(\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}$ factors through ε as a $\mathcal{V}$-functor $\mathrm{y}_A\colon A \to [A^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$ that is a Yoneda embedding in $\mathcal{V}\text{-}\mathsf{sProf}$. Finally notice that the companion $\mathrm{y}_{A*}\colon A \nrightarrow [A^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$ is a small $\mathcal{V}$-profunctor so that, because $\mathcal{V}\text{-}\mathsf{sProf}$ has restrictions on the right (Example A4.7), y_A admits nullary restrictions in the sense of Definition 4.5. To see this consider, for any $p \in [A, \mathcal{V}]_{\mathrm{s}}$, the small sub-$\mathcal{V}$-category $A_p \coloneqq A_* \subseteq A$ as in the above. The cascade of isomorphisms below, for each $x \in A$, shows that y_* is small. The first and last isomorphisms here are induced by the 'strong

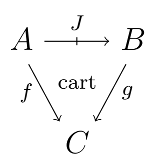
Yoneda lemma' (see e.g. Formula 2.31 of [Kel82]) while the middle isomorphism exhibits the smallness of p.

$$\int^{x' \in A_p} A(x, x') \otimes \mathrm{y}_*(x', p) \cong \int^{x' \in A_p} A(x, x') \otimes px' \cong px \cong \mathrm{y}_*(x, p)$$

4.34. Yoneda morphisms compared to Yoneda structures and powers. We next compare our notion of (weak) Yoneda embedding to the notions of *Yoneda structure* and *good Yoneda structure* of [SW78] and [Web07] respectively, in the theorem below, as well as to the notion of *power* of [Lam22], in Proposition 4.40 below. Recall that part of a (good) Yoneda structure is a *right ideal* $\mathcal{A}$ of *admissible* morphisms in $V(\mathcal{K})$: any composite $g \circ f$ in $V(\mathcal{K})$ is admissible as soon as g is so. An object $A \in \mathcal{K}$ is called *admissible* whenever its identity morphism id_A is admissible.

4.35. Theorem. *Let $\mathcal{K}$ be an augmented virtual double category and let $\mathcal{A}$ be a right ideal of admissible morphisms in $V(\mathcal{K})$. Assume that for every admissible $f: A \to C$ the companion $f_*: A \nrightarrow C$ exists and consider given an admissible morphism $\mathrm{y}_A: A \to \widehat{A}$ for each admissible object A.*

The implications (gys) $\Rightarrow$ (ys*) $\Rightarrow$ (ys) *and* (ye) $\Rightarrow$ (wye) *hold among the conditions below. If $\mathcal{K}$ has restrictions on the right (Definition 1.19) then* (wye) $\Rightarrow$ (ys) *holds too. If $\mathcal{K}$ has weakly cocartesian paths of* $(0, 1)$*-ary cells (Definition 3.2) then* (wye) $\Rightarrow$ (ys*). *If $\mathcal{K}$ has left nullary-cocartesian tabulations (Definition 3.5) and restrictions on the right then* (ye) $\Leftrightarrow$ (wye) $\Rightarrow$ (gys). *If $\mathcal{K}$ has left cocartesian tabulations, restrictions on the right and, for each $J: A \nrightarrow B$ with A admissible, a cartesian nullary cell (Definition 3.2) below exists with f admissible, then* (gys) $\Rightarrow$ (ye) *holds too.*



(ys) *The morphisms y_A form a Yoneda structure on $V(\mathcal{K})$ in the sense of [SW78];*

(ys*) *the morphisms y_A form a Yoneda structure on $V(\mathcal{K})$ that satisfies Axiom 3* of [SW78];*

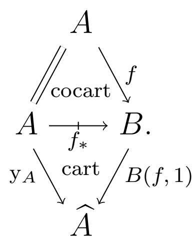
(gys) *the morphisms y_A form a good Yoneda structure on $V(\mathcal{K})$ in the sense of [Web07];*

(wye) *each morphism y_A is a weak Yoneda embedding (Definition 4.5);*

(ye) *each morphism y_A is a Yoneda embedding (Definition 4.5).*

PROOF. That (gys) $\Rightarrow$ (ys*) and (ye) $\Rightarrow$ (wye) is easily checked. Proposition 11 of [SW78] proves (ys*) $\Rightarrow$ (ys). Suppose that $\mathcal{K}$ has restrictions on the right. To show (wye) $\Rightarrow$ (ys) we have to show that the family of weak Yoneda embeddings y_A satisfies Axioms 1, 2 and 3 of [SW78]. Axioms 1 and 2 ask, for each $f\colon A \to B$ with both A and f admissible, a morphism $B(f,1)\colon B \to \widehat{A}$ equipped with a cell $\chi^f\colon \mathrm{y}_A \Rightarrow B(f,1) \circ f$ that simultaneously defines, in $V(\mathcal{K})$, $B(f,1)$ as the left Kan extension of y_A along f and f as the absolute left lifting of y_A along $B(f,1)$. By assumption f_* exists and we can take $B(f,1) := f_*^\lambda$ as supplied by the Yoneda axiom (Definition 4.5); let χ^f be the composite

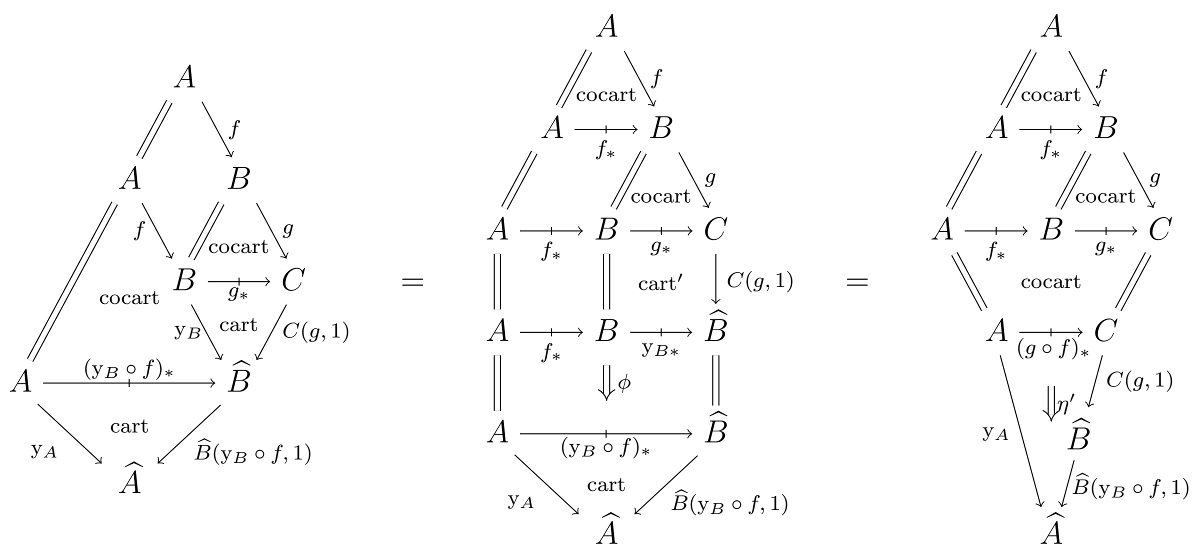
$\chi^f :=$



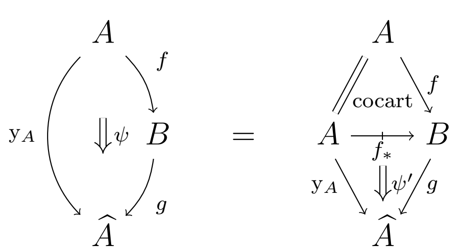
It follows from weak density of y_A (Definition 4.3) and Proposition 1.7 that χ^f defines $B(f,1)$ as the left Kan extension of y_A along f in $V(\mathcal{K})$; that it defines f as the absolute left lifting of y_A along $B(f,1)$ too follows from Lemma A5.17. Notice that Axioms 1 and 2 hold without requiring that $\mathcal{K}$ has restrictions on the right.

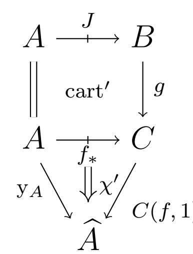
The first part of Axiom 3 requires that the identity cell $\mathrm{id}_{\mathrm{y}_A}\colon \mathrm{y}_A \Rightarrow \mathrm{id}_{\widehat{A}} \circ \mathrm{y}_A$ defines id_A as the left Kan extension of y_A along y_A. By assumption y_A is admissible so that y_{A*} exists and, by weak density of y_A (Definition 4.3), it follows that the defining cell $\mathrm{cart}\colon \mathrm{y}_{A*} \Rightarrow \widehat{A}$ is weakly left Kan so that $\mathrm{id}_{\mathrm{y}_A} = \mathrm{cart} \circ \mathrm{cocart}$ (Lemma 1.21) defines $\mathrm{id}_{\widehat{A}}$ as the required left Kan extension by Proposition 1.7. Next consider a composable pair of morphisms $f\colon A \to B$ and $g\colon B \to C$ with A, B and g admissible; notice that admissibility of B implies that of f. The second part of Axiom 3 requires that the composite $\chi^{\mathrm{y}_B \circ f} \odot (\widehat{B}(\mathrm{y}_B \circ f, 1) \circ \chi^g \circ f)$, which by definition of $\chi^{\mathrm{y}_B \circ f}$ and χ^g is the left-hand side of the equality below, defines $\widehat{B}(\mathrm{y}_B \circ f, 1) \circ C(g,1)$ as the left Kan extension of y_A along $g \circ f$ in $V(\mathcal{K})$.

In the middle composite below the cell cart' denotes the factorisation of the cartesian cell that defines $C(g,1)$ through the cartesian cell that defines y_{B*} while the cell ϕ is the composite of the cocartesian cell defining $(\mathrm{y}_B \circ f)_*$ with the cartesian cells defining f_* and y_{B*}. Notice that the first identity below follows from the companion identity for f_* (Lemma 1.21) and the definition of cart', and that the latter is again cartesian by the pasting lemma for cartesian cells (Lemma 1.17). Moreover it follows easily from the companion identities that, together with the cell $\psi\colon (\mathrm{y}_B \circ f)_* \Rightarrow \mathrm{y}_{B*}$ obtained by composing the cartesian cell defining $(\mathrm{y}_B \circ f)_*$ with the cocartesian cell defining y_{B*}, the cell ϕ satisfies both identities of Lemma A8.1, so that ϕ is pointwise right cocartesian by Lemma A9.7 and Remark 2.13. By weak density of y_A (Definition 4.3) the cartesian cell cart defining $\widehat{B}(\mathrm{y}_B \circ f, 1)$ in the middle composite below is pointwise weakly left Kan, so that by the vertical pasting lemma (Lemma 2.17) the composite $\mathrm{cart} \circ \phi$ is pointwise weakly left Kan too. By Definition 1.24 it follows that the composite $\eta := \text{cart} \circ \phi \circ (\text{id}, \text{cart}')$ of the bottom three rows of the middle composite below is weakly left Kan. Finally consider, as in the second identity below, the unique factorisation η' of η through the cocartesian cell $(f_*, g_*) \Rightarrow (g \circ f)_*$ that is defined analogously to the cocartesian cell ϕ. By the vertical pasting lemma η' is weakly left Kan. By definition of cocart: $(f_*, g_*) \Rightarrow (g \circ f)_*$ the composite of the top three rows in the right-hand side below is the cocartesian cell that defines the companion $(g \circ f)_*$. Applying Proposition 1.7 we conclude that the right-hand side, and hence the left-hand side, defines $\widehat{B}(\text{y}_B \circ f, 1) \circ C(g, 1)$ as the left Kan extension of y_A along $g \circ f$ in $V(\mathcal{K})$. This completes the proof of (wye) ⇒ (ys).



Next, assume that $\mathcal{K}$ has weakly cocartesian paths of $(0, 1)$-ary cells (Definition 3.2). To show (wye) ⇒ (ys*) we have to prove that the weak Yoneda embeddings y_A satisfy Axioms 1, 2 and 3* of [SW78] in $V(\mathcal{K})$. That they satisfy Axioms 1 and 2 follows from the argument given above. Axiom 3* is equivalent to the assertion that any cell ψ as on the left below, with A and f admissible, defines g as a left Kan extension of y_A along f in $V(\mathcal{K})$ as soon as it defines f as an absolute left lifting of y_A along g. Considering the factorisation ψ' of ψ through f_* as shown, this is a consequence of the assumption on $\mathcal{K}$ as follows: ψ defining f as an absolute left lifting in $V(\mathcal{K})$ means that ψ' is cartesian in $\mathcal{K}$, by Lemma A5.17 and Proposition A7.12, which implies that ψ' is weakly left Kan by weak density of y_A, so that ψ defines g as a left Kan extension in $V(\mathcal{K})$ by Proposition 1.7.





Next we will show that (ye) ⇔ (wye) ⇒ (gys) whenever $\mathcal{K}$ has left nullary-cocartesian tabulations and restrictions on the right. By Corollary 3.6 $\mathcal{K}$ has left nullary-cocartesian paths of $(0,1)$-ary cells so that by Theorem 3.20 the notions of pointwise weak Kan extension and pointwise Kan extension coincide in $\mathcal{K}$. As remarked after Definition 4.3 this implies (ye) ⇔ (wye). Assuming (ye) we will show that the Yoneda embeddings y_A, together with the cells χ^f defined above, satisfy the two axioms of a good Yoneda structure on $V(\mathcal{K})$, in the sense of Definition 3.1 of [Web07], which proves (gys). The first axiom of a good Yoneda structure coincides with Axiom 2 of a Yoneda structure, which we proved previously. The second axiom strengthens Axiom 3* as follows: any cell ψ as on the left above, with A and f admissible, defines g as a pointwise left Kan extension in $V(\mathcal{K})$ as soon as it defines f as an absolute left lifting. To see this notice that, as before, the assumption on ψ implies that its factorisation ψ' is cartesian in $\mathcal{K}$, so that ψ' is pointwise left Kan by density of y_A (Definition 4.3) and hence ψ defines g as a pointwise left Kan extension in $V(\mathcal{K})$ by Proposition 3.22.

For the converse (gys) ⇒ (ye) assume that, besides left cocartesian tabulations and restrictions on the right, $\mathcal{K}$ has cartesian nullary cells for all $J\colon A \nrightarrow B$ with A admissible, in the sense described in the statement. By Corollary 3.6 $\mathcal{K}$ has all left cocartesian paths of $(0,1)$-ary cells. That each y_A is dense follows easily from Proposition 3.4(1) of [Web07], which shows that the identity cell $\mathrm{id}_{\mathrm{y}_A}$ defines $\mathrm{id}_{\widehat{A}}$ as the pointwise left Kan extension of y_A along y_A in $V(\mathcal{K})$: applying Proposition 3.22 to the companion identity $\mathrm{id}_{\mathrm{y}_A} = \mathrm{cart} \circ \mathrm{cocart}$ for y_{A*} it follows that cart is pointwise left Kan in $\mathcal{K}$ so that y_A is dense by Definition 4.3. It remains to show that y_A satisfies the Yoneda axiom (Definition 4.5), that is for each $J\colon A \nrightarrow B$ there exists a morphism $J^\lambda\colon B \to \widehat{A}$ such that J is the restriction $\widehat{A}(\mathrm{y}_A, J^\lambda)$. Consider a cartesian nullary cell $\mathrm{cart}\colon J \Rightarrow C$ as in the statement, with admissible vertical source $f\colon A \to C$ and vertical target $g\colon B \to C$, and set $J^\lambda := C(f,1) \circ g$, where $C(f,1)\colon C \to \widehat{A}$ is supplied by the good Yoneda structure (Definition 3.1 of [Web07]). To show that J is the restriction $\widehat{A}(\mathrm{y}_A, J^\lambda)$ consider the composite cell on the right above, where cart' is the factorisation of $\mathrm{cart}\colon J \Rightarrow C$ through f_* and χ' is the factorisation of the vertical cell $\chi^f\colon \mathrm{y}_A \Rightarrow C(f,1) \circ f$, that defines $C(f,1)$, through f_*. By the pasting lemma (Lemma 1.17) cart' is cartesian and, since χ^f defines f as an absolute left lifting, χ' is cartesian by Lemma A5.17 and Proposition A7.12. Applying the pasting lemma again we find that $\chi' \circ \mathrm{cart}'$ is cartesian, thus defining J as $\widehat{A}(\mathrm{y}_A, J^\lambda)$. This completes the proof. ∎

4.36. Example. Applying the theorem to the Yoneda embeddings of the augmented virtual equipment $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ (Example 4.10) we recover the Yoneda structure on the 2-category $\mathcal{V}'$-Cat of (large) $\mathcal{V}'$-categories, as described in Section 7 of [SW78], whose admissible morphisms are the $\mathcal{V}'$-functors $f\colon A \to C$ for which all hom objects $C(fx, z)$ are (isomorphic to) $\mathcal{V}$-objects (Example A5.6).

Instantiating $\mathcal{V} \subset \mathcal{V}'$ by $\mathsf{Cat} \subset \mathsf{Cat}'$ (Example A2.7) recall that the augmented virtual equipment $(\mathsf{Cat}, \mathsf{Cat}')$-$\mathsf{Prof}$ of 2-profunctors does not have cocartesian tabulations (see Example 3.9), so that we cannot apply the previous theorem to obtain a good Yoneda structure on the 2-category Cat'-$\mathsf{Cat} = V\big((\mathsf{Cat}, \mathsf{Cat}')$-$\mathsf{Prof}\big)$ of (locally large) 2-categories, in the sense of [Web07]. In fact it is shown in Remark 9 of [Wal18] that the Yoneda embedding $\mathrm{y}\colon 1 \to \mathsf{Cat}(\mathsf{Set})$ for the terminal 2-category 1 (Example 4.9), with presheaf object $\mathsf{Cat}(\mathsf{Set})$ the 2-category of small categories (Example A2.9), cannot be part of a Yoneda structure on the 2-category Cat'-Cat that satisfies Axiom 3* of [SW78].

4.37. Example. In an augmented virtual equipment $\mathcal{K}$ (Definition 1.19) call a morphism $f\colon A \to C$ admissible if its companion $f_*\colon A \nrightarrow C$ exists. Notice that the class $\mathcal{A}$ of admissible morphisms in $V(\mathcal{K})$ is a right ideal: for a composite $g \circ f$ with g admissible the companion $(g \circ f)_* \cong g_*(f, \mathrm{id})$ (Lemma A5.11) exists as well because $\mathcal{K}$ has restrictions on the left, so that $g \circ f$ is admissible too. An object $A \in \mathcal{K}$ is admissible precisely if it is unital, that is its horizontal unit $I_A \cong (\mathrm{id}_A)_*$ (Definition 1.16) exists. Using the assumption that $\mathcal{K}$ has restrictions on the right the previous theorem shows that choosing an admissible weak Yoneda embedding $\mathrm{y}_A\colon A \to \widehat{A}$ in $\mathcal{K}$, for each unital object A, induces a Yoneda structure on $V(\mathcal{K})$, in the sense of [SW78].

4.38. Example. Consider a right ideal $\mathcal{A}$ of admissible morphisms in a 2-category $\mathcal{C}$. Recall from Examples A6.3 and A7.9 the strict double category $Q(\mathcal{C})$ of quintets in $\mathcal{C}$, whose vertical and horizontal morphisms both are the morphisms of $\mathcal{C}$. Let $Q_{\mathcal{A}}(\mathcal{C}) \subseteq Q(\mathcal{C})$ be the full sub-augmented virtual double category generated by those horizontal morphisms $j\colon A \nrightarrow B$ that are admissible. Supplying a family of admissible weak Yoneda embeddings $\mathrm{y}\colon A \to \widehat{A}$ in $Q_{\mathcal{A}}(\mathcal{C})$, one for each admissible object A, is equivalent to equipping $\mathcal{C} = V\big(Q_{\mathcal{A}}(\mathcal{C})\big)$ with a Yoneda structure that satisfies Axiom 3* of [SW78]. Indeed notice that all horizontal morphisms of $Q_{\mathcal{A}}(\mathcal{C})$, like those of $Q(\mathcal{C})$, are companions, so that they admit cocartesian $(0, 1)$-ary cells as well as restrictions on the left. It follows from the pasting lemma for cocartesian paths of $(0, 1)$-ary cells (Lemma 3.16) that $Q_{\mathcal{A}}(\mathcal{C})$ has right cocartesian paths of $(0, 1)$-ary cells so that (wye) $\Rightarrow$ (ys*) holds in the previous theorem. Using Lemma A5.17 and Proposition A7.12 it is straightforward to check that the converse (ys*) $\Rightarrow$ (wye) holds as well.

4.39. Example. Let $\mathcal{A}$ be a right ideal of admissible morphisms in a finitely complete 2-category $\mathcal{C}$ and consider the unital virtual equipment $\mathsf{dFib}(\mathcal{C})$ of discrete two-sided fibrations in $\mathcal{C}$ (Example 1.30). Write $\mathsf{dFib}_{\mathcal{A}}(\mathcal{C}) \subseteq \mathsf{dFib}(\mathcal{C})$ for the full sub-augmented virtual double category generated by those discrete two-sided fibrations $J\colon A \nrightarrow B$ that admit a cartesian nullary cell $\psi\colon J \Rightarrow C$ (Definition 3.2) whose vertical source $A \to C$ is admissible. Using the pasting lemma for cartesian cells (Lemma 1.17) and the fact that $\mathcal{A}$ is a right ideal notice that $\mathsf{dFib}_{\mathcal{A}}(\mathcal{C})$ is closed under taking restrictions so that it is an augmented virtual equipment by Lemma A4.5. By Lemma 3.14 $\mathsf{dFib}_{\mathcal{A}}(\mathcal{C})$ has all cocartesian tabulations too, which are created as in $\mathsf{dFib}(\mathcal{C})$ (Example 3.13), so that by Corollary 3.6 $\mathsf{dFib}_{\mathcal{A}}(\mathcal{C})$ has left cocartesian paths of $(0, 1)$-ary cells. Applying the previous theorem we find firstly that equipping $\mathcal{C} = V\big(\mathsf{dFib}_{\mathcal{A}}(\mathcal{C})\big)$ with a good Yoneda structure, in the sense of [Web07], is equivalent to supplying a family of admissible Yoneda embeddings $\mathrm{y}_A\colon A \to \widehat{A}$ in $\mathsf{dFib}_{\mathcal{A}}(\mathcal{C})$, in the sense of Definition 4.5, one for each admissible object A. Secondly the theorem shows that a family of admissible weak Yoneda embeddings y_A in $\mathsf{dFib}_{\mathcal{A}}(\mathcal{C})$, one for each admissible object A, induces a Yoneda structure on $\mathcal{C}$ that satisfies Axiom 3* of [SW78].

Finally consider an equipment $\mathcal{K}$ (Proposition A7.8), that is $\mathcal{K}$ is a pseudo double category that has all restrictions. The horizontal dual of Definition 11.4 of [Lam22] defines $\mathcal{K}$ to have *powers* whenever each object $A \in \mathcal{K}$ admits a horizontal morphism $\in_A : A \nrightarrow PA$ satisfying the following property: for each $J : A \nrightarrow B$ there is a unique $f : B \to PA$ equipped with a cartesian cell of the form below. Having powers is one of several conditions required for $\mathcal{K}$ to be equivalent to the equipment $\mathsf{Rel}(\mathcal{E})$ of relations (Examples A2.10, A5.8 and A7.4) internal in some topos $\mathcal{E}$; this is proved in Theorem 11.5 of [Lam22].

$$\begin{array}{ccc} A & \xrightarrow{J} & B \\ \| & \text{cart} & \downarrow f \\ A & \xrightarrow[\in_A]{} & PA \end{array}$$

Let us call the vertical 2-category $V(\mathcal{K})$ (Example A1.5) *locally skeletal* whenever any isomorphic pair $f \cong g$ of morphisms in $V(\mathcal{K})$ is equal: $f = g$. Notice that in that case the morphisms $J^\lambda : B \to \widehat{A}$ supplied by the Yoneda axiom (Definition 4.5) are uniquely determined by the horizontal morphisms $J : A \nrightarrow B$ that induce them. The following result is a straightforward consequence of the pasting lemma for cartesian cells (Lemma 1.17).

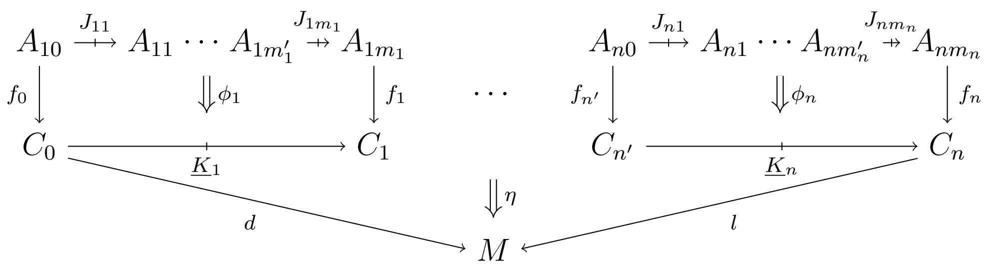
4.40. Proposition. *Let $\mathcal{K}$ be an equipment with $V(\mathcal{K})$ locally skeletal. If $\mathcal{K}$ has weak Yoneda morphisms $\mathrm{y}_A : A \to \widehat{A}$ (Definition 4.5) for all objects A then it has all powers $PA := \widehat{A}$, defined as such by the companions $\in_A := \mathrm{y}_{A*} : A \nrightarrow \widehat{A}$.*

5. Exact cells

In this sections we use the formal notions of left Kan extension and Yoneda morphism to consider in augmented virtual double categories the classical notions of exact squares, as studied by Guitart [Gui80]. In Section 6 we likewise consider total morphisms and objects, originally introduced by Street and Walters [SW78], and in Section 7 we consider cocomplete objects.

5.1. Left exact paths of cells. The following definition generalises Guitart's notion of 'carré exact' [Gui80] to various notions of exactness for paths of cells in augmented virtual double categories. In Example 5.11 we will see that, when considered in the augmented virtual double category $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ (Example A2.6), the notion of pointwise left exactness below coincides with the classical notion of exactness. Analogous to the classical situation Theorem 5.14 below characterises exactness in terms of a 'Beck-Chevalley condition', in the sense of Definition 5.10. The latter condition is used in Theorem 5.16 to characterise absolute left Kan extensions (Definition 1.36).

5.2. Definition. *Consider a path $\underline{\phi} = (\phi_1, \dots, \phi_n)$ as in the composite below, with $n \geq 1$, and let $d\colon C_0 \to M$ be any vertical morphism. The path $\underline{\phi}$ is called* (weakly) left d-exact *if for any (weakly) left Kan cell η (Definitions 1.2 and 1.9) as below, with d as vertical source, the composite $\eta \circ \underline{\phi}$ is again (weakly) left Kan. If $\underline{\phi}$ is (weakly) left d-exact for all morphisms $d\colon C_0 \to M$, where M varies, then it is called* (weakly) left exact.



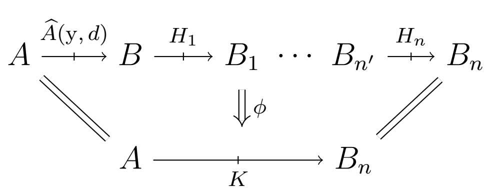
Analogously $\underline{\phi}$ is called pointwise *(weakly) left d-exact if for any pointwise (weakly) left Kan cell η (Definition 1.24) as above the composite above is pointwise (weakly) left Kan too. A path that is pointwise (weakly) left d-exact for all $d\colon C_0 \to M$ is called* pointwise *(weakly) left exact.*

5.3. Example. Consider a (pointwise) right (respectively weakly) nullary-cocartesian path $\underline{\phi} = (\phi_1, \dots, \phi_n)$ (Definitions 2.6 and 2.12) such that the vertical target of ϕ_n is trivial. By the vertical pasting lemma (Lemma 2.17) $\underline{\phi}$ is (pointwise) (weakly) left exact.

5.4. Cells that are left exact with respect to Yoneda morphisms. Given a Yoneda morphism $\mathrm{y}\colon A \to \widehat{A}$ the following two results concern left y-exact cells with a trivial vertical target. The first of these is used in Example 6.3 below to show that, under mild conditions, presheaf objects $\widehat{A}$ are 'total', in the sense of Definition 6.2 below. In Theorem 7.6 it is also used, to give a condition that ensures the 'cocompleteness' (Definition 7.2) of presheaf objects. The second result below is used in Theorem 5.14.

5.5. Proposition. *Let $\mathrm{y}\colon A \to \widehat{A}$ be a Yoneda morphism (Definition 4.5) and let ϕ and ψ be cells that correspond under the bijection of Lemma 4.22. The cell ϕ is (pointwise) (weakly) left y-exact if and only if the cell ψ is (pointwise) (weakly) left Kan.*

In particular consider a morphism $d\colon B \to \widehat{A}$ such that the restriction $\widehat{A}(\mathrm{y}, d)$ exists as well as a path $\underline{H}\colon B \nrightarrow B_n$. If there exists a (pointwise) (weakly) horizontal left y-exact cell ϕ of the form below then the (pointwise) (weak) left Kan extension $l\colon B_n \to \widehat{A}$ of d along $\underline{H}$ exists. The converse holds whenever the restriction $\widehat{A}(\mathrm{y}, l)$ exists.



Proof. We only treat the left y-exact and left Kan case; the proofs for the (pointwise) (weakly) cases are the same. Recall from Lemma 4.22 that the unary cell $\phi\colon J \frown \underline{H} \Rightarrow K$ and the nullary cell $\psi\colon \underline{H} \Rightarrow \widehat{A}$ correspond via the equality

$$\mathrm{cart}_{K^\lambda} \circ \phi = \mathrm{cart}_{J^\lambda} \odot \psi,$$

where the cartesian cells define K^λ and J^λ (Definition 4.5). By density of y (Definition 4.3) these cartesian cells define K^λ and J^λ as pointwise left Kan extensions of y. Hence, by definition, ϕ is left y-exact if and only if the left-hand side above, and hence both sides, is left Kan which, by the horizontal pasting lemma (Lemma 2.2), is equivalent to ψ being left Kan.

For the second assertion take $J \coloneqq \widehat{A}(\mathrm{y}, d)$ so that $J^\lambda \cong d$ by uniqueness of J^λ (Definition 4.5). By the main assertion any horizontal left y-exact cell $\phi\colon \widehat{A}(\mathrm{y}, d) \frown \underline{H} \Rightarrow K$ corresponds to a nullary cell $\psi\colon \underline{H} \Rightarrow \widehat{A}$ that defines K^λ as the left Kan extension of d along $\underline{H}$. For the converse set $K \coloneqq \widehat{A}(\mathrm{y}, l)$ so that $K^\lambda \cong l$. By the main assertion any nullary cell $\psi\colon \underline{H} \Rightarrow \widehat{A}$ that defines l as the left Kan extension of $\underline{H}$ along d corresponds to a horizontal left y-exact cell $\phi\colon \widehat{A}(\mathrm{y}, d) \frown \underline{H} \Rightarrow \widehat{A}(\mathrm{y}, l)$. ∎

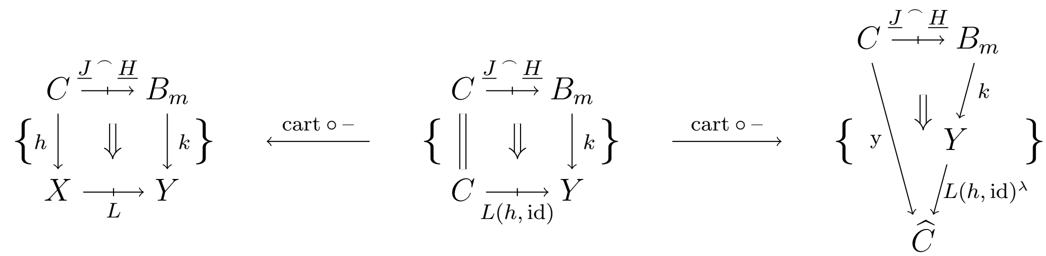
5.6. Proposition. *Let $\mathrm{y}\colon C \to \widehat{C}$ be a (weak) Yoneda morphism (Definition 4.5) such that the restrictions $L(h, \mathrm{id})$ exist for all morphisms $L\colon X \nrightarrow Y$ and $h\colon C \to X$. Consider the cell ϕ below. Assuming that either $f = \mathrm{id}_C$ or the conjoint $f^*\colon C \nrightarrow A_0$ exists, ϕ is (weakly) y-left exact if and only if ϕ is right (respectively weakly) unary-cocartesian (Definition 2.6). If moreover the restrictions $K(\mathrm{id}, g)$ exist for all $g\colon X \to D$ then the analogous assertion for the pointwise (weak) case (Definition 2.12) holds too.*

$$\begin{array}{ccccccc} A_0 & \xrightarrow{J_1} & A_1 & \cdots & A_{n'} & \xrightarrow{J_n} & D \\ {\scriptstyle f}\downarrow & & & \Downarrow\phi & & & \| \\ C & & & \xrightarrow[K]{\qquad\qquad} & & & D \end{array}$$

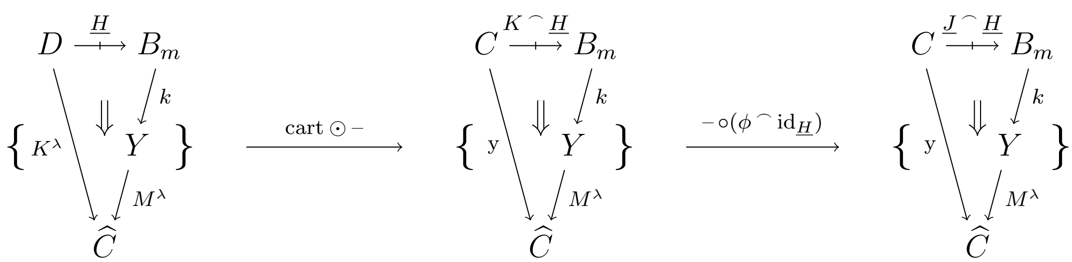
Notice that together with Example 5.3 this proposition implies that, under the conditions above, if the cell ϕ above is (pointwise) right (respectively weakly) nullary-cocartesian then it is (pointwise) right (respectively weakly) cocartesian.

Proof. We will first prove the non-weak case in which $f = \mathrm{id}_C$. Afterwards we generalise to the case where f^* exists as well as consider the pointwise case. For the weak case take $\underline{H}$ to be empty in the following. Given any path $\underline{H}\colon D \nrightarrow B_m$ of horizontal morphisms consider the assignments below of cells of the forms as shown, that are given by composition with the cartesian cells defining respectively the restriction $L(h, \mathrm{id})$ and the morphism $L(h, \mathrm{id})^\lambda$ (Definition 4.5). By definition ϕ is right unary-cocartesian if all unary cells $\underline{J} \frown \underline{H} \Rightarrow L$ on the left below factor uniquely through the path $\phi \frown \mathrm{id}_{\underline{H}}$, where

$\mathrm{id}_{\underline{H}} \coloneqq (\mathrm{id}_{H_1}, \dots, \mathrm{id}_{H_q})$.



By the uniqueness of factorisations through cartesian cells the assignments above are bijections, so that the latter is equivalent to all nullary cells $\underline{J} \frown \underline{H} \Rightarrow \widehat{C}$ on the right above factoring uniquely through $\phi \frown \mathrm{id}_{\underline{H}}$. Abbreviating $M \coloneqq L(h, \mathrm{id})$, spelled out this means that ϕ is right unary-cocartesian if and only if the second assignment below is a bijection. The first assignment below is given by horizontal composition with the cartesian cell cart defining K^λ; it is a bijection because cart is pointwise left Kan, by the density of y (Definition 4.3). We conclude that ϕ is right unary-cocartesian if and only if the composite assignment below, given by horizontal composition with $\mathrm{cart} \circ \phi$, is a bijection. But the latter means that $\mathrm{cart} \circ \phi$ again left Kan which, since cart has y as vertical source, by definition is equivalent to ϕ being left y-exact.



This completes the proof of the non-pointwise case with $f = \mathrm{id}_C$. In the general case, where the conjoint f^* exists, apply the previous to the composite horizontal cell $\mathrm{cart} \odot \phi \colon f^* \frown \underline{J} \Rightarrow K$ where cart defines f^*. The proof follows by noticing that ϕ is (weakly) left y-exact if and only if $\mathrm{cart} \odot \phi$ is so, by Corollary 2.24, and likewise that ϕ is right (respectively weakly) unary-cocartesian if and only if $\mathrm{cart} \odot \phi$ is so, by Corollary 2.10.

Finally assume that the restrictions $K(\mathrm{id}, g)$ exist for all $g \colon X \to D$. Restricting to those g for which the restriction $J_n(\mathrm{id}, g)$ holds too, consider the unique factorisations ϕ'_g in

$$\phi \circ (\mathrm{id}_{J_1}, \dots, \mathrm{id}_{J_{n'}}, \mathrm{cart}_{J_n(\mathrm{id}, g)}) = \mathrm{cart}_{K(\mathrm{id}, g)} \circ \phi'_g$$

as in Definition 2.12, where the cartesian cells define the restrictions $J_n(\mathrm{id}, g)$ and $K(\mathrm{id}, g)$ respectively. Still denoting the cartesian cell defining K^λ by cart, recall that it is pointwise (weakly) left Kan. It follows that ϕ is pointwise (weakly) y-exact precisely if for each g the left-hand side above, and thus either side, postcomposed with cart is (weakly) left Kan.

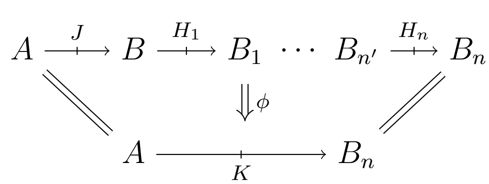
Since the composites $\text{cart} \circ \text{cart}_{K(\text{id},g)}$ are cartesian by the pasting lemma (Lemma 1.17), and thus (weakly) left Kan by (weak) density of y (Definition 4.3), the latter is equivalent to the factorisations ϕ'_g being (weakly) y-exact. By the main assertion this is equivalent to ϕ'_g being right (respectively weakly) unary-cocartesian, for each g such that both $J_n(\text{id}, g)$ and $K(\text{id}, g)$ exist. But that means precisely that ϕ is pointwise right (respectively weakly) unary cocartesian. This completes the proof. ∎

Combining Propositions 5.3 and 5.6 we obtain the following.

5.7. Corollary. *In an augmented virtual double category $\mathcal{K}$ consider a horizontal morphism $J\colon A \nrightarrow B$ as well as a path $\underline{H}\colon B \nrightarrow B_n$ of length $n \geq 1$, and assume that the Yoneda morphism $\text{y}_A\colon A \to \widehat{A}$ exists (Definition 4.5). Among the following conditions the implications* (a) ⇒ (b) ⇒ (c) *and* (b) ⇒ (d) ⇒ (e) *hold. If $\mathcal{K}$ is unital (Definition 1.19) then* (b) ⇔ (c)*. If $\mathcal{K}$ has restrictions on the left (Definition 1.19) then* (c) ⇔ (d)*. If y_A admits nullary restrictions (Definition 4.5) then* (d) ⇔ (e)*. The same implications hold for the pointwise variants of the conditions (Definitions 1.24 and 2.12 and Definition A9.1), except for* (a) ⇒ (b) *and* (c) ⇔ (d) *which moreover require the restrictions $K(\text{id}, g)$ to exist for all $g\colon X \to B_n$.*

- (a) *The horizontal composite $(J \odot H_1 \odot \cdots \odot H_n)$ exists (Definition 2.12);*
- (b) *a right (respectively weakly) cocartesian cell of the form below exists (Definition 2.6);*
- (c) *a right (respectively weakly) unary-cocartesian cell of the form below exists;*
- (d) *a (weakly) left y_A-exact cell of the form below exists;*
- (e) *the (weak) left Kan extension of $J^\lambda\colon B \to \widehat{A}$ (Definition 4.5) along $\underline{H}$ exists (Definitions 1.2 and 1.9).*

Finally restrict to paths $\underline{H} = (H_1)$ of length $n = 1$ and assume that the Yoneda morphism $\text{y}_B\colon B \to \widehat{B}$ exists. If the pointwise (weak) left Kan extension of $J^\lambda\colon B \to \widehat{A}$ along y_{B*} exists then the pointwise variant of condition (e) *is satisfied.*



Notice that if y_A admits nullary restrictions then condition (e) above also applies to (pointwise) (weak) left Kan extensions along $\underline{H}$ of any morphism $d\colon B \to \widehat{A}$, by taking $J \coloneqq \widehat{A}(\text{y}_A, d)$ so that, by uniqueness of J^λ (Definition 4.5), $J^\lambda \cong d$.

PROOF. (a) ⇒ (b) follows from Example 2.8, Definition 2.6 and Remark 2.13. (b) ⇒ (c) and (c) ⇒ (b) follow from Definitions 2.6 and 2.12, and Example 2.7. (b) ⇒ (d) follows from Example 5.3. Proposition 5.6 shows that (c) ⇔ (d). Applying Proposition 5.5 to $d := J^\lambda \colon B \to \widehat{A}$, so that $\widehat{A}(\mathrm{y}_A, J^\lambda) \cong J$ by the Yoneda axiom (Definition 4.5), shows that (d) ⇒ (e) and (e) ⇒ (d). For the final assertion write $\eta\colon \mathrm{y}_{B*} \Rightarrow \widehat{A}$ for the nullary cell that defines the pointwise (weak) left Kan extension of J^λ along y_{B*}, and write $\mathrm{cart}'\colon H_1 \Rightarrow \mathrm{y}_{B*}$ for the factorisation of the cartesian cell defining $H_1^\lambda\colon B_1 \to \widehat{B}$ (Definition 4.5) through the cartesian cell defining the companion y_{B*}. Since cart' is cartesian by the pasting lemma (Lemma 1.17), the composite $\eta \circ \mathrm{cart}'\colon H_1 \Rightarrow \widehat{A}$ defines the pointwise (weak) left Kan extension of J^λ along H_1, by Definition 1.24. ■

In an augmented virtual equipment (Definition 1.19) let us for a moment call an object A 'admissible' if it admits a Yoneda morphism $\mathrm{y}_A\colon A \to \widehat{A}$ that has nullary restrictions (Definition 4.5). Fixing an object B it follows from the previous corollary that the following conditions are equivalent where, for condition (c) only, we assume that B itself is admissible.

(a) For any path $A \xrightarrow{J} B \xrightarrow{H} E$ with A admissible a horizontal pointwise right unary-cocartesian cell (Definition 2.12) with horizontal source (J, H) exists;

(b) for any admissible object A and any morphisms $\widehat{A} \xleftarrow{d} B \xrightarrow{H} E$ the pointwise left Kan extension (Definition 1.24) of d along H exists;

(c) for any admissible object A and any morphism $d\colon B \to \widehat{A}$ the pointwise left Kan extension of d along y_{B*} exists.

The next example shows that for a locally small category B condition (a) implies essential smallness. This is why we think of the conditions above as "restricting the size of the object B".

5.8. EXAMPLE. Consider a locally small category B in the augmented virtual equipment $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ (Example A2.6) and assume that for any path $1 \xrightarrow{J} B \xrightarrow{H} 1$ of Set-profunctors a weakly unary-cocartesian horizontal cell (Definition 2.6) with horizontal source (J, H) exists. Using an argument similar to the one used in Example A4.18, together with the fact that the embedding $\mathsf{Set} \hookrightarrow \mathsf{Set}'$ preserves colimits, this assumption implies that the coend $\int^{y \in B} J(*, y) \times H(y, *)$ is small. We can use the latter to show that the category $\widehat{B}$ of presheaves $p\colon B^{\mathrm{op}} \to \mathsf{Set}$ is locally small too, so that B is essentially small by [FS95]. To do so let p and q be any presheaves on B and consider the composite below, where the injection is induced by the unit of the double-powerset monad $2^{(2^{-})}$ on Set (see e.g. Example 1 of [Tho09]). Applying the assumption to $J = 2^q$ and $H = p$ we find that the right-hand side below is small, so that the hom-set $\widehat{B}(p, q)$ on the left-hand side is small too, as required.

$$\widehat{B}(p, q) = \int_{y \in B} (qy)^{py} \hookrightarrow \int_{y \in B} \big(2^{(2^{qy})}\big)^{py} \cong \int_{y \in B} 2^{2^{qy} \times py} \cong 2^{\int^{y \in B} 2^{qy} \times py}$$

5.9. The left Beck-Chevalley condition. In Example 5.3, we saw that nullary-cocartesian paths with trivial vertical target are exact. The following definition, of the 'left Beck-Chevalley condition', allows us to use this to obtain exact paths with any vertical target (Corollary 5.13). In Example 5.11 we will see that this condition recovers the classical notion of exactness for transformations of functors, as used by Guitart in [Gui80].

5.10. Definition. *Consider a non-empty path $\underline{\phi} = (\phi_1, \dots, \phi_n)$ of cells. Denote the (possibly empty) horizontal target of each ϕ_i by $\underline{K}_i \colon C_{i'} \nrightarrow C_i$ and assume that ϕ_n is of the form as on the left-hand side below. We say that $\underline{\phi}$ satisfies the* (weak) left Beck-Chevalley condition *if the restriction $\underline{K}_n(\mathrm{id}, f_n)$ exists and the path $\underline{\phi}' := (\phi_1, \dots, \phi_{n'}, \phi'_n)$ is right (respectively weakly) nullary-cocartesian (Definition 2.6), where ϕ'_n is the unique factorisation below. Here if $\underline{K}_n = (C_{n'})$ is empty then $\underline{K}_n(\mathrm{id}, f_n) = f_n^*$ is the conjoint of f_n.*

$$
\begin{array}{ccc}
A_{n0} \xrightarrow{J_{n1}} A_{n1} \cdots A_{nm'_n} & \xrightarrow{J_{nm_n}} & A_{nm_n} \\
{\scriptstyle f_{n'}}\downarrow & \Downarrow \phi_n & \downarrow {\scriptstyle f_n} \\
C_{n'} & \xrightarrow[\underline{K}_n]{} & C_n
\end{array}
\quad = \quad
\begin{array}{ccc}
A_{n0} \xrightarrow{J_{n1}} A_{n1} \cdots A_{nm'_n} & \xrightarrow{J_{nm_n}} & A_{nm_n} \\
{\scriptstyle f_{n'}}\downarrow & \Downarrow \phi'_n & \parallel \\
C_{n'} & \xrightarrow[\underline{K}_n(\mathrm{id}, f_n)]{} & A_{nm_n} \\
\parallel & \text{cart} & \downarrow {\scriptstyle f_n} \\
C_{n'} & \xrightarrow[\underline{K}_n]{} & C_n
\end{array}
$$

Likewise we say that $\underline{\phi}$ satisfies the pointwise (weak) left Beck-Chevalley condition *if $\underline{\phi}'$ is pointwise right (respectively weakly) nullary-cocartesian (Definition 2.12). A single cell ϕ is said to satisfy the (pointwise) (weak) left Beck-Chevalley condition whenever the single path (ϕ) does so.*

Notice that if the vertical target $f_n = \mathrm{id}_{C_n}$ of ϕ_n is trivial then the left Beck-Chevalley conditions above reduce to $\underline{\phi}$ being (pointwise) right (respectively weakly) nullary-cocartesian.

5.11. Example. Let $\mathcal{V} \subset \mathcal{V}'$ be a universe enlargement and consider the cell ϕ below in the augmented virtual equipment $(\mathcal{V}, \mathcal{V}')$-Prof of $\mathcal{V}$-profunctors between $\mathcal{V}'$-categories (Example A2.7). Assume that the conjoint $f^* \colon C \nrightarrow A$ exists (Example A4.6).

$$
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
{\scriptstyle f}\downarrow & \Downarrow \phi & \downarrow {\scriptstyle g} \\
C & \xrightarrow[K]{} & D
\end{array}
$$

Combining Corollary 2.10, Example A9.2 and Remark 2.13 we find that ϕ satisfies the pointwise left Beck-Chevalley condition whenever the morphisms

$$C(z, fx) \otimes J(x, y) \xrightarrow{\mathrm{id} \otimes \phi} C(z, fx)K(fx, gy) \xrightarrow{\lambda} K(z, gy)$$

define each $K(z, gy)$ as the coend $\int^{x \in A} C(z, fx) \otimes J(x, y)$ in $\mathcal{V}'$ (or, equivalently, in $\mathcal{V}$, provided that $\mathcal{V} \subset \mathcal{V}'$ preserves large colimits). If the latter coend, which exists in $\mathcal{V}'$, is known to be (isomorphic to) a $\mathcal{V}$-object then by Theorem 5.14 below the afore-mentioned sufficient condition is equivalent to ϕ satisfying the pointwise left Beck-Chevalley condition, as well as to ϕ being pointwise left exact (Definition 5.2).

In particular, specialising to $\mathsf{Set} \subset \mathsf{Set}'$ and assuming that A is small, if $J = j_*$ and $K = k_*$ for functors $j\colon A \to B$ and $k\colon C \to D$, so that ϕ corresponds to a transformation $k \circ f \Rightarrow g \circ j$ of functors, we find that our notion of pointwise left exactness coincides with the notion of 'carré exact' given in Definition 1.1 of [Gui80]; see condition BC' of Theorem 1.2 therein.

Paths satisfying the left Beck-Chevalley condition can be concatenated as follows.

5.12. Lemma. *Consider the path $\underline{\phi}$ of Definition 5.10 and assume that it satisfies the left Beck-Chevalley condition. A concatenation $\underline{\phi} \frown \underline{\psi}$ of paths satisfies the (pointwise) (weak) left Beck-Chevalley condition if and only if the path $\mathrm{id}_{\underline{K}'} \frown \mathrm{cart} \frown \underline{\psi}$ does so, where $\underline{K}' \coloneqq \underline{K}_1 \frown \cdots \frown \underline{K}_{n'}$ and cart defines the restriction $\underline{K}_n(\mathrm{id}, f_n)$.*

Proof (Sketch). Factorise $\phi_n = \mathrm{cart} \circ \phi'_n$ and $\psi_m = \mathrm{cart} \circ \psi'_m$ as in Definition 5.10 and apply the pasting lemma for cocartesian paths (Lemma 2.9) to $(\phi_1, \dots, \phi'_n, \mathrm{id}_{\underline{H}_1}, \dots, \mathrm{id}_{\underline{H}_m})$ and $\mathrm{id}_{\underline{K}'} \frown (\mathrm{cart}, \psi_1, \dots, \psi'_m)$, where each $\underline{H}_j$ denotes the horizontal source of ψ_j and cart defines $\underline{K}_n(\mathrm{id}, f_n)$. ■

The following result, which is a straightforward consequence of the vertical pasting lemma for left Kan extensions (Lemma 2.17), shows that the left Beck-Chevalley conditions imply left exactness. Recall from Definition 1.24 the notion of a left Kan extension restricting along a vertical morphism.

5.13. Corollary. *Consider the path $\underline{\phi}$ of Definition 5.10 and a morphism $d\colon C_0 \to M$. The path $\underline{\phi}$ is (weakly) left d-exact (Definition 5.2) if $\underline{\phi}$ satisfies the (weak) left Beck-Chevalley condition and the (weak) left Kan extension of d along $\underline{K}_1 \frown \cdots \frown \underline{K}_n$ restricts along f_n (and $|\underline{K}_n| \geq 1$).*

Likewise $\underline{\phi}$ is pointwise (weakly) left exact if $\underline{\phi}$ satisfies the pointwise (weak) left Beck-Chevalley condition (and $|\underline{K}_n| \geq 1$).

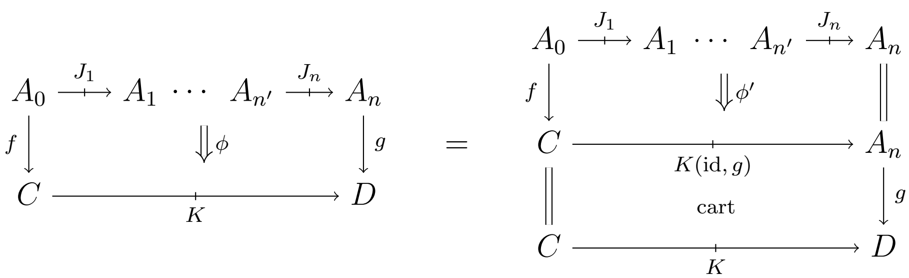
Proof. Write $\underline{K} \coloneqq \underline{K}_1 \frown \cdots \frown \underline{K}_n$ and consider the factorisation $\phi_n = \mathrm{cart} \circ \phi'_n$ and the path $\underline{\phi}' = (\phi_1, \dots, \phi_{n'}, \phi'_n)$ of Definition 5.10, where cart defines the restriction $\underline{K}_n(\mathrm{id}, f_n)$. We first consider the case where $\underline{K}_n$ is non-empty. Given a (weakly) left Kan cell $\eta\colon \underline{K} \Rightarrow M$ that defines the (weak) left Kan extension of d along $\underline{K}$ we have to show that $\eta \circ \underline{\phi} = \big(\eta \circ (\mathrm{id}, \dots, \mathrm{id}, \mathrm{cart})\big) \circ \underline{\phi}'$ is again (weakly) left Kan, assuming that $\underline{\phi}'$ is right (respectively weakly) nullary-cocartesian and that η restricts along f_n. By Definition 1.24 the latter means that the composite $\eta \circ (\mathrm{id}, \dots, \mathrm{id}, \mathrm{cart})$ is (weakly) left Kan, so that the result follows by applying the vertical pasting lemma (Lemma 2.17) to the nullary-cocartesian path $\underline{\phi}'$.

For the second assertion assume that $\eta\colon \underline{K} \Rightarrow M$ is pointwise (weakly) left Kan (Definition 1.24). By Lemma 1.25 the composite $\eta \circ (\mathrm{id}, \dots, \mathrm{id}, \mathrm{cart})$ is pointwise (weakly) left Kan as well so that $\eta \circ \underline{\phi} = \big(\eta \circ (\mathrm{id}, \dots, \mathrm{id}, \mathrm{cart})\big) \circ \underline{\phi}'$ is so too by applying the vertical pasting lemma to the path $\underline{\phi}'$, which is pointwise right (respectively weakly) nullary-cocartesian by assumption.

Finally consider the non-weak case where $\underline{K}_n = (C_{n'})$ is empty, so that cart in $\phi_n = \mathrm{cart} \circ \phi'_n$ defines the conjoint $f_n^*\colon C_{n'} \nrightarrow A_{nm_n}$ of f_n. The same arguments apply except for the claim that $\eta \circ (\mathrm{id}, \dots, \mathrm{id}, \mathrm{cart})$ is (pointwise) left Kan, which in this case follows from the assumption that η is left Kan by Corollary 2.24. ∎

Combining Proposition 5.6 and Corollary 5.13 the following theorem relates the notions of nullary- and unary-cocartesianness, the notion of left exactness and the left Beck-Chevalley condition.

5.14. Theorem. *Consider the factorisation below. Assume that the (weak) Yoneda morphism* $\mathrm{y}\colon C \to \widehat{C}$ *exists and that either* $f = \mathrm{id}_C$ *or the conjoint* f^* *exists.*



Among the following conditions the implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Leftrightarrow$ (e) *hold. All conditions are equivalent whenever a right (respectively weakly) cocartesian cell with horizontal source* $\underline{J}$ *and vertical morphisms* f *and* id_{A_n} *is known to exist, or when they are considered in a unital virtual double category (Definition 1.19).*

(a) ϕ' *is right (respectively weakly) nullary-cocartesian (Definition 2.6);*

(b) ϕ *satisfies the (weak) left Beck-Chevalley condition (Definition 5.10);*

(c) ϕ *is (weakly) left* d*-exact (Definition 5.2) for any morphism* $d\colon C \to M$ *whose (weak) left Kan extension along* K *restricts along* g *(Definition 1.24);*

(d) ϕ *is (weakly) left* y*-exact;*

(e) ϕ' *is right (respectively weakly) unary-cocartesian.*

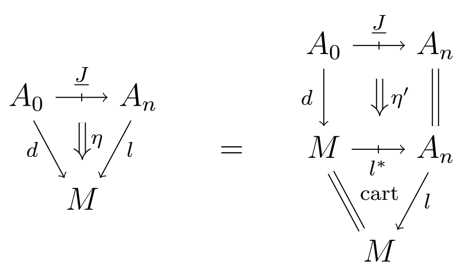
If moreover the restrictions $K(\mathrm{id}, g \circ h)$ *exist for all* $h\colon X \to A_n$ *then the same implications hold among the pointwise variants (Definition 2.12) of the conditions above, with the pointwise variant of* (c) *being*

(c') ϕ *is pointwise (weakly) left exact.*

Proof. (a) $\Leftrightarrow$ (b) by Definition 5.10. (b) $\Rightarrow$ (c) follows from Corollary 5.13. Let $\mathrm{cart}_{K^\lambda}\colon K \Rightarrow \widehat{C}$ denote the cartesian cell that defines $K^\lambda\colon D \to \widehat{C}$ (Definition 4.5). By (weak) density of y (Definition 4.3) it defines K^λ as the pointwise (weak) left Kan extension of y along K, proving (c) $\Rightarrow$ (d). Next consider the equality $\mathrm{cart}_{K^\lambda} \circ \phi = \mathrm{cart}_{K^\lambda} \circ \mathrm{cart} \circ \phi'$ where both $\mathrm{cart}_{K^\lambda}$ and $\mathrm{cart}_{K^\lambda} \circ \mathrm{cart}$ define pointwise left Kan extensions of y, the latter by Lemma 1.25. It follows that condition (d) is equivalent to ϕ' being (pointwise) (weakly) left y-exact which, by Proposition 5.6, is equivalent to (e). Finally assume either that all horizontal units exist or that a (pointwise) right (respectively weakly) cocartesian cell with horizontal source $\underline{J}$ and vertical morphisms f and id_{A_n} exists. In both cases (e) $\Rightarrow$ (a) follows, either by Example 2.7 or by using that weakly unary-cocartesian cells uniquely determine their horizontal targets up to isomorphism. ■

5.15. The left Beck-Chevalley condition and absolute left Kan extension. The second theorem of this section describes the relation between absolutely left Kan cells (Definition 1.36) and the left Beck-Chevalley condition for nullary cells. Specialising (c) $\Leftrightarrow$ (d) to Set-Prof (Example A2.4) recovers Example 1.14(9) of [Gui80]; specialising (a) $\Leftrightarrow$ (b) to a locally thin equipment (see Proposition A7.8 and Example A2.5) recovers Theorem 2.6 of [Kou18]. When considered in a pseudo double category (Proposition A7.8) condition (f) below is equivalent to the nullary cell η exhibiting l as an 'absolute left Kan extension' in the sense of Section 2.2 of [GP08]. The latter notion is more general however as it allows for unary cells η as well.

5.16. Theorem. *Consider the cell η on the left-hand side below.*



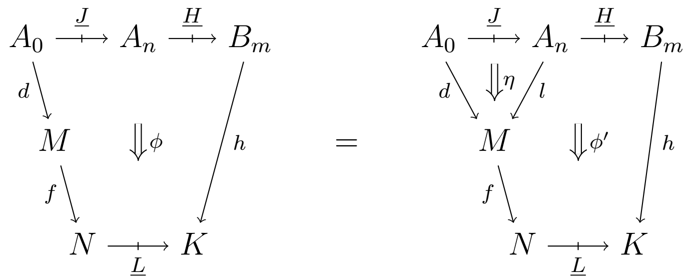
Among the conditions below the implication (a) $\Rightarrow$ (d) *holds. If the conjoint l^* exists then the implications* (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) *hold. If moreover all cartesian nullary cells (Definition 3.2) exist then all four conditions are equivalent.*

(a) *Any cell ϕ as on the left-hand side below (with $\underline{H}$ empty) factors uniquely as shown;*

(b) *the factorisation η' in the right-hand side above is right (respectively weakly) cocartesian (Definition 2.6);*

(c) *η satisfies the (weak) left Beck-Chevalley condition (Definition 5.10);*

(d) *η is absolutely (weakly) left Kan (Definition 1.36).*



If the (weak) Yoneda morphism $\mathrm{y}\colon M \to \widehat{M}$ *exists then, among the conditions above and those below, the implications* (a) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g) *hold. If all companions exist then* (g) $\Rightarrow$ (d).

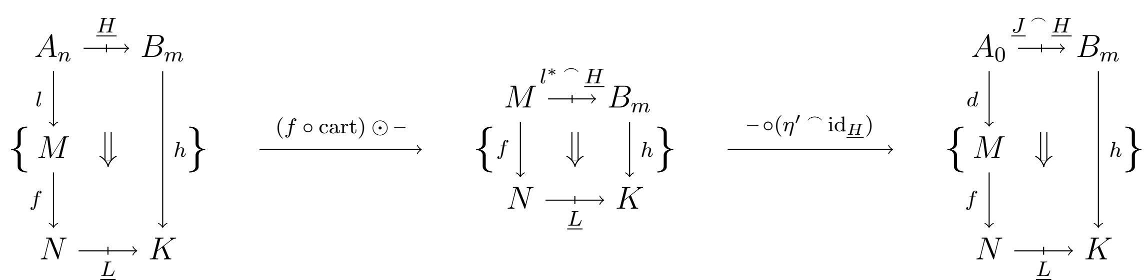
(e) *η is a (weakly) left Kan cell that is preserved by* y *(Definition 1.36);*

(f) *every cell ϕ above with $f = \mathrm{id}_M$ (and with $\underline{H}$ empty) factors as shown;*

(g) *η is a (weakly) left Kan cell that is preserved by any $g\colon M \to N$ whose companion g_* exists.*

If the conjoint $(l \circ k)^*$ exists for all $k\colon X \to A_n$ then the same implications hold for the analogous conditions in the pointwise (weak) case where, in (a) *and* (f)*, the unique factorisations are through composites of the form $f \circ \eta \circ (\mathrm{id}_{J_1}, \dots, \mathrm{id}_{J_{n'}}, \mathrm{cart})$, with* cart *defining any restriction of the form $J_n(\mathrm{id}, k)$.*

Proof. For the weak case take $\underline{H} = (A_n)$ empty in the following. The implication (a) $\Rightarrow$ (d) simply follows from the fact that restricting the unique factorisation above to cells ϕ that are nullary, that is $\underline{L} = (N)$ empty, gives the universal property of η asserted by (d). (b) $\Rightarrow$ (c) immediately follows from Definitions 5.10 and 2.6 while (c) $\Rightarrow$ (b) follows from Example 2.7 provided that all cartesian nullary cells exist. To see that (a) $\Leftrightarrow$ (b) consider the assignments below, of cells of the form as shown; here cart denotes the cartesian cell defining the conjoint l^*. It follows from the conjoint identities (Lemma 1.21) that the left assignment below is a bijection. Hence the assignment on the right below is a bijection, that is (b) holds, precisely if the composite assignment $(f \circ \eta) \odot -$ below is so, that is (a) holds. This proves (a) $\Leftrightarrow$ (b). That (c) $\Leftrightarrow$ (d) is similar: restricting the assignments below to nullary cells, that is $\underline{L} = (N)$ empty, it follows that the assignment on the right is a bijection, that is η' is right (respectively weakly) nullary-cocartesian or, equivalently, (c) holds, if and only if the composite is a bijection, that is (d) holds.



Next assume that the (weak) Yoneda morphism $\mathrm{y}\colon M \to \widehat{M}$ exists. Clearly (d) $\Rightarrow$ (e). To prove (e) $\Rightarrow$ (f) first notice that the required unique factorisation for unary cells $\phi\colon \underline{J}\frown\underline{H} \Rightarrow L$, as in the statement and where $f = \mathrm{id}_M$, reduces to that of the corresponding nullary cells $\mathrm{cart} \circ \phi\colon \underline{J} \frown \underline{H} \Rightarrow \widehat{M}$, with $f = \mathrm{y}$ and where cart defines $L^\lambda\colon K \to \widehat{M}$ (Definition 4.5); here we use the uniqueness of factorisations through cart. Thus it suffices to prove that nullary cells of the form either $\underline{J} \frown \underline{H} \Rightarrow M$, with $f = \mathrm{id}_M$, or $\underline{J} \frown \underline{H} \Rightarrow \widehat{M}$, with $f = \mathrm{y}$, factorise uniquely as in the statement. But that is precisely the assertion that η and $\mathrm{y} \circ \eta$ are (weakly) left Kan, which is condition (e). To see that (f) $\Rightarrow$ (g) first notice that the universal property of η being (weakly) left Kan is the unique factorisation of nullary cells $\underline{J} \frown \underline{H} \Rightarrow M$ as in the statement, with $f = \mathrm{id}_M$, which is asserted by (f). Next consider any $g\colon M \to N$ whose companion g_* exists. The unique factorisations that exhibit $g \circ \eta$ as being left Kan factor precisely, through the cartesian cell $g_* \Rightarrow N$ that defines g_*, as the unique factorisations of unary cells $\underline{J} \frown \underline{H} \Rightarrow g_*$ as in the statement, with $f = \mathrm{id}_M$: these too exist by (f). This proves (f) $\Rightarrow$ (g). Clearly (g) $\Rightarrow$ (d) if all companions exist.

Notice that the pointwise (weak) variants of (a) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g) follow from applying these implications to the composites $\eta \circ (\mathrm{id}_{J_1}, \dots, \mathrm{id}_{J_{n'}}, \mathrm{cart}_{J_n(\mathrm{id},k)})$, where $\mathrm{cart}_{J_n(\mathrm{id},k)}$ defines any restriction of the form $J_n(\mathrm{id}, k)$. The pointwise (weak) variants of (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) follow from Definitions 5.10 and 2.12. For the pointwise (weak) variants of (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (d) consider, for each $k\colon X \to A_n$ such that the restriction $J_n(\mathrm{id}, k)$ exists and instead of the two assignments above, the analogous assignments $(f \circ \mathrm{cart} \circ \mathrm{cart}_{(l\circ k)^*}) \odot -$, where $\mathrm{cart}_{(l\circ k)^*}\colon (l \circ k)^* \Rightarrow l^*$ defines $(l \circ k)^*$ as a restriction of l^* (see Lemma A5.11), and $- \circ (\eta'' \frown \mathrm{id}_{\underline{H}})$, where η'' is the unique factorisation in

$$\eta' \circ (\mathrm{id}_{J_1}, \dots, \mathrm{id}_{J_{n'}}, \mathrm{cart}_{J_n(\mathrm{id},k)}) = \mathrm{cart}_{(l\circ k)^*} \circ \eta'',$$

as in Definition 2.12. Notice that the composite of these two assignments is

$$\begin{aligned}(f \circ \mathrm{cart} \circ \mathrm{cart}_{(l\circ k)^*} \circ \eta'') \odot - &= \big(f \circ \mathrm{cart} \circ \eta' \circ (\mathrm{id}_{J_1}, \dots, \mathrm{id}_{J_{n'}}, \mathrm{cart}_{J_n(\mathrm{id},k)})\big) \odot - \\ &= \big(f \circ \eta \circ (\mathrm{id}_{J_1}, \dots, \mathrm{id}_{J_{n'}}, \mathrm{cart}_{J_n(\mathrm{id},k)})\big) \odot - .\end{aligned}$$

Using arguments similar to those used above one then concludes that, for each morphism $k\colon X \to A_n$ such that the restriction $J_n(\mathrm{id}, k)$ exists, the cell η'' is right (respectively weakly) nullary-cocartesian if and only if unique factorisations through the composite $f \circ \eta \circ (\mathrm{id}_{J_1}, \dots, \mathrm{id}_{J_{n'}}, \mathrm{cart}_{J_n(\mathrm{id},k)})$ exist, analogous to the factorisations of condition (a). From this the pointwise (weak) analogue of (a) $\Leftrightarrow$ (b) follows and, as before, that of (c) $\Leftrightarrow$ (d) follows by restricting the factorisations to nullary cells. ∎

5.17. Example. In the locally thin strict equipment ModRel of modular relations between preorders (see Example 1.9 of [Kou18]) consider an order preserving map $d\colon A \to M$ and a modular relation $J\colon A \nrightarrow B$. The left Kan extension $l\colon B \to M$ of d along J in ModRel, if it exists, is given by the suprema

$$ly = \sup_{x \in J^\circ y} dx \qquad \text{where} \qquad J^\circ y = \{x \in A \mid xJy\};$$

see Example 2.3 of [Kou18]. Example 2.7 therein describes the left Beck-Chevalley condition for the left Kan cell defining l: it requires that for every $y \in B$ there is $x \in J^\circ y$ with $dx = ly$, that is the suprema above are attained as maxima.

As an immediate corollary of the previous theorem we find that an absolute left Kan extension l is preserved by any functor of augmented virtual double categories (Definition A3.1) that preserves right cocartesian cells, provided that the conjoint of l exists, as follows.

5.18. Corollary. *Let $F\colon \mathcal{K} \to \mathcal{L}$ be a functor of augmented double categories. In $\mathcal{K}$ consider the factorisation $\eta = \text{cart} \circ \eta'$ of Theorem 5.16 and assume that η satisfies the (pointwise) (weak) left Beck-Chevalley condition (Definition 5.10). The following are equivalent:*

(a) *F preserves η, that is $F\eta$ satisfies the (pointwise) (weak) left Beck-Chevalley condition;*

(b) *F preserves η', that is $F\eta'$ is (pointwise) right (respectively weakly) cocartesian.*

Proof. This follows from Definition 5.10 and the fact that the image $F\eta = F\text{cart} \circ F\eta'$ of $\eta = \text{cart} \circ \eta'$ under F is again of the form as that of Theorem 5.16. This is because functors of augmented virtual double categories preserve conjoints (Corollary A5.5) and vertical identity cells. ∎

6. Totality

In this section we study the notions of total morphism and total object in augmented virtual double categories, which are introduced in the definition below. It follows from Examples 1.12 and 1.13 that, when applied to the unital virtual equipment $\mathcal{V}$-Prof of profunctors enriched in a closed symmetric monoidal category $\mathcal{V}$ (Example A2.4), the notion of total object below coincides with that of total $\mathcal{V}$-category as considered in [DS86] and [Kel86]. In contrast to Street and Walters' original 2-categorical notion of total morphism given in Section 6 of [SW78], which assumes a Yoneda structure, the notion of (weak) total morphism below does not require the existence of (weak) Yoneda morphisms.

In order to make precise the relation between our notion of weakly total object below and that of total object in the sense of Street and Walters, consider an augmented virtual equipment $\mathcal{K}$ (Definition 1.19) that has weak Yoneda embeddings $\mathrm{y}_A\colon A \to \widehat{A}$ admitting nullary restrictions (Definition 4.5) for all unital objects A. We will see in Example 6.9 below that a unital object A is weakly total in the vertical 2-category $V(\mathcal{K})$ (Example A1.5) in the sense of [SW78], that is the weak Yoneda embedding y_A admits a left adjoint, if and only if A is weakly total in $\mathcal{K}$ in our sense.

6.1. Total morphisms and objects.

6.2. Definition. *A morphism* $f\colon M \to N$ *is called* (weakly) total *if, for any horizontal morphism* $J\colon M \nrightarrow B$*, the pointwise (weak) left Kan extension of* f *along* J *(Definition 1.24) exists. An object* M *is called* (weakly) total *if its identity morphism* id_M *is (weakly) total.*

6.3. Example. In an augmented virtual equipment $\mathcal{K}$ (Definition 1.19) any presheaf object $\widehat{M}$, defined by a Yoneda morphism $\mathrm{y}\colon M \to \widehat{M}$ (Definition 4.5), is total provided that the companion y_* exists. To see this let $J\colon \widehat{M} \nrightarrow B$ be any horizontal morphism. Since the restriction $J(\mathrm{y}, \mathrm{id})$ exists in $\mathcal{K}$ so does the pointwise horizontal composite $(\mathrm{y}_* \odot J)$, by Lemmas A8.1 and A9.7. By Remark 2.13 the cell defining $(\mathrm{y}_* \odot J)$ is pointwise right cocartesian so that, by Example 5.3, it is pointwise left exact. Hence the existence of the pointwise left Kan extension of $\mathrm{id}_{\widehat{M}}$ along J follows from Proposition 5.5.

Notice that besides assuming that y_* exists we do not need to require any size restriction on $\widehat{M}$. This is in contrast to Corollary 14 of [SW78] which, in order to prove that a presheaf object $\mathcal{P}A$, given by a Yoneda structure on a 2-category, is total, in the sense of [SW78], requires $\mathcal{P}A$ to be admissible.

The following result is analogous to the first assertion of Proposition 27 of [SW78] for Yoneda structures.

6.4. Proposition. *Consider an adjunction* $f \dashv g\colon M \to N$ *with* g *full and faithful (Definition 1.18). If* g *is (weakly) total then so is* M*.*

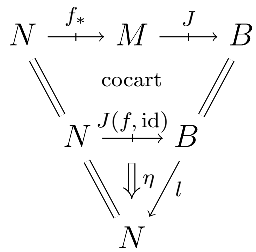
Proof. The counit $\varepsilon\colon f \circ g \Rightarrow \mathrm{id}_M$ of the adjunction is invertible because g is full and faithful, by Lemma A4.14 and (b) $\Rightarrow$ (c) of Proposition 10 of [SW78]. We have to show that the pointwise (weak) left Kan extension of any $J\colon M \nrightarrow B$ along id_M exists. If g is (weakly) total then the pointwise (weak) left Kan extension of g along J exists; let $\eta\colon J \Rightarrow N$ denote its defining nullary cell. Using the fact that the left adjoint f is cocontinuous (Lemma 1.37), $f \circ \eta$ defines the pointwise (weak) left Kan extension of $f \circ g$ along J so that, composing on the left with the inverse $\varepsilon^{-1}\colon \mathrm{id}_M \Rightarrow f \circ g$, we obtain a nullary cell defining the pointwise (weak) left Kan extension of id_M along J, as required. ■

The following formalises Corollary 6.2 of [Kel86], which is recovered by specialising to the unital virtual equipment $\mathcal{V}$-Prof (Example A2.4).

6.5. Proposition. *Consider an adjunction* $f \dashv g\colon M \to N$ *with* g *full and faithful. Assume that the companion* $f_*\colon N \nrightarrow M$ *exists as well as all restrictions of the form* $J(f, \mathrm{id})$*, for all* $J\colon M \nrightarrow B$*. If* N *is (weakly) total then so are* g *and* M*.*

Proof. The counit $\varepsilon\colon f \circ g \Rightarrow \mathrm{id}_M$ of the adjunction is invertible because g is full and faithful, by Lemma A4.14 and (b) $\Rightarrow$ (c) of Proposition 10 of [SW78]. Notice that (weak) totality of M follows from that of g by the previous proposition. To show that g is total we have to show that the pointwise (weak) left Kan extension of g along any $J\colon M \nrightarrow B$ exists. To this end consider the composite below where η defines $l\colon B \to N$

as the pointwise (weak) left Kan extension of id_N along $J(f, \mathrm{id})$, which exists because N is (weakly) total, and where cocart defines $J(f, \mathrm{id})$ as the pointwise horizontal composite of (f_*, J); see Lemmas A8.1 and A9.7.



Notice that for each morphism $h\colon X \to B$, if the restriction $J(\mathrm{id}, h)$ exists then so does $J(f, \mathrm{id})(\mathrm{id}, h) \cong J(\mathrm{id}, h)(f, \mathrm{id})$ (Lemma 1.17), by assumption. Hence cocart is pointwise right cocartesian by Remark 2.13, so that the composite above defines l as the pointwise (weak) left Kan extension of id_N along (f_*, J) by the vertical pasting lemma (Lemma 2.17). By Example 2.28 l is the pointwise (weak) left Kan extension of g along J, as required. ∎

The following 'adjoint functor theorem' is an immediate corollary of Proposition 2.27. The analogous result for Yoneda structures is mentioned on page 372 of [SW78].

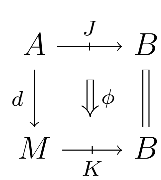
6.6. Corollary. *Consider a morphism $f\colon M \to N$ whose companion f_* exists and whose source M is weakly total, so that the weak left Kan extension $l\colon N \to M$ of id_M along f_* exists. If l is preserved by f (Definition 1.36) then $f \dashv l$.*

6.7. Totality in the presence of Yoneda morphisms. The following theorem is the main result of this section. Its implication (c) ⇒ (d) is analogous to Lemma 3.18 of [Web07] for Yoneda structures. The latter requires $\widehat{M}$ to be admissible; here only the existence of the companion $\mathrm{y}_*\colon M \nrightarrow \widehat{M}$ is assumed.

6.8. Theorem. *For a morphism $f\colon M \to N$ the following are equivalent:*

(a) *f is (weakly) total;*

(b) *if a pointwise (weakly) left f-exact cell (Definition 5.2) of the form below exists then so does the pointwise (weak) left Kan extension of $f \circ d$ along J (Definition 1.24).*



If the (weak) Yoneda morphism $\mathrm{y}\colon M \to \widehat{M}$ (Definition 4.5) as well as the companions y_* and f_* exist then the conditions above and those below are all equivalent.

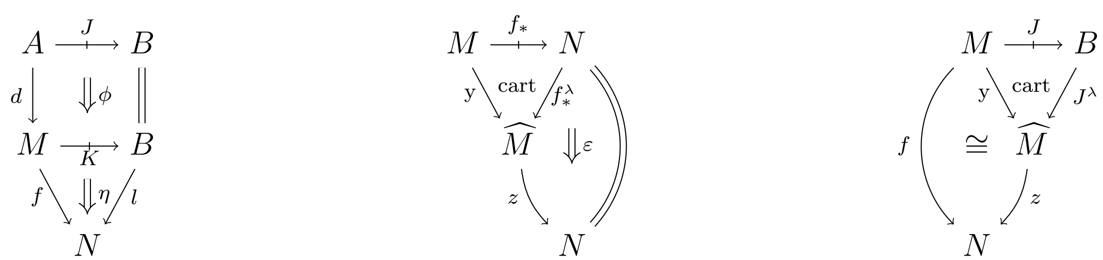
(c) *The weak left Kan extension of f along y_* exists and restricts along $f_*^\lambda\colon N \to \widehat{M}$ (Definitions 1.2, 1.24 and 4.5);*

(d) *f_*^λ admits a left adjoint $z\colon \widehat{M} \to N$.*

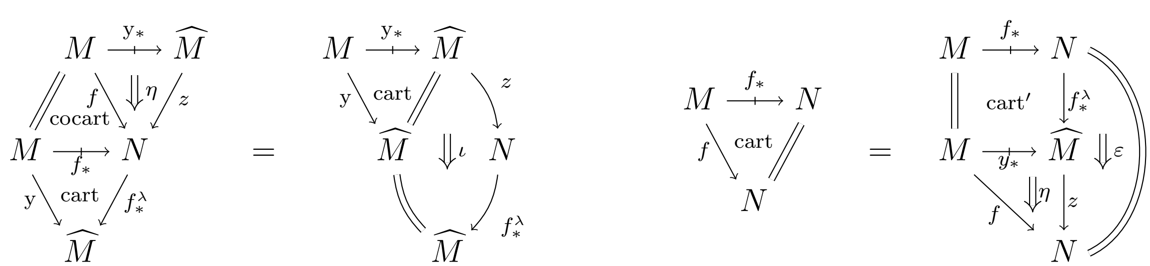
Moreover in that case the weak left Kan extension of condition (c) *and the left adjoint z of condition* (d) *coincide.*

PROOF. (a) $\Rightarrow$ (b) follows from applying the Definition 5.2 to the composites as on the left below, where the cell η defines $l\colon B \to N$ as the pointwise left Kan extension of f along K. (b) $\Rightarrow$ (a) simply follows from applying (b) to the identity cell id_J. (a) $\Rightarrow$ (c) is clear, but notice that the restriction $\mathrm{y}_*(\mathrm{id}, f_*^\lambda)$ exists and is equal to f_*. Indeed this follows from the fact that the unique factorisation $\mathrm{cart}'\colon f_* \Rightarrow y_*$, of the cartesian cell defining f_*^λ (Definition 4.5) through that defining y_*, is cartesian by the pasting lemma (Lemma 1.17).

(d) $\Rightarrow$ (a). Consider the composite in the middle below, of the cartesian cell that defines f_*^λ and the counit ε of $z \dashv f_*^\lambda$. It is cartesian by Lemma A4.17, showing that $(z \circ \mathrm{y})_* \cong f_*$. Since taking companions extends to an equivalence $(-)_*$ of augmented virtual double categories (Theorem A6.5) we conclude that $z \circ \mathrm{y} \cong f$. The pointwise (weak) left Kan extension of f along any $J\colon M \nrightarrow B$ can now be constructed as $z \circ J^\lambda$, with the composite on the right below as the defining pointwise (weakly) left Kan cell. Here we use the fact that the left adjoint z is cocontinuous; see Proposition 1.37.

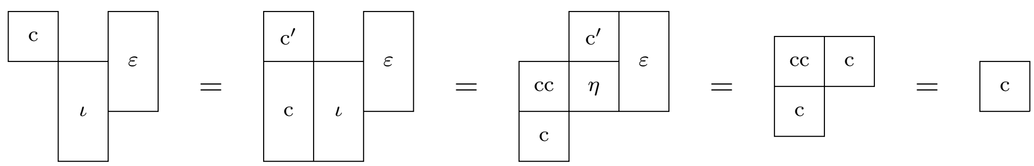


(c) $\Rightarrow$ (d). Let $z\colon \widehat{M} \to N$ denote the weak left Kan extension of f along y_* and let η denote its defining cell, as in the left-hand side of the identity on the left below; we claim that z forms the left adjoint of f_*^λ.

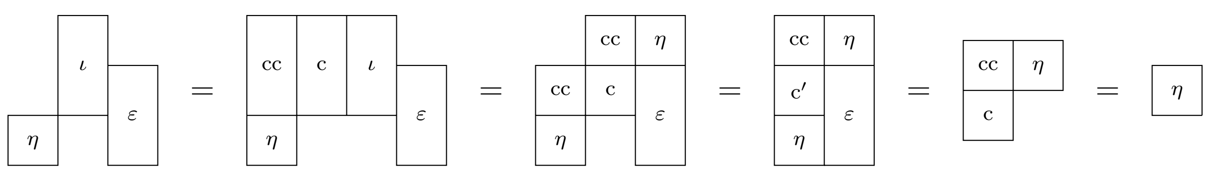


As the unit cell $\iota\colon \mathrm{id}_{\widehat{M}} \Rightarrow f_*^\lambda \circ z$ we take the unique factorisation as in the identity on the left above; recall that the cartesian cell defining y_* is weakly left Kan because y is weakly dense (Definition 4.3). The counit $\varepsilon\colon z \circ f_*^\lambda \Rightarrow \mathrm{id}_N$ we take to the be unique factorisation in the identity on the right above, where cart' is the cartesian cell defining f_* as the restriction $\mathrm{y}_*(\mathrm{id}, f_*^\lambda)$, as described previously, so that $\eta \circ \mathrm{cart}'$ is weakly left Kan because η restricts along f_*^λ by assumption.

To prove the triangle identity $(\iota \circ f_*^\lambda) \odot (f_*^\lambda \circ \varepsilon) = \mathrm{id}_{f_*^\lambda}$ consider the equality of composite cells that are drawn schematically below where, for ease of drawing, vertical and nullary cells are drawn as rectangles. Here the cartesian cells defining f_*^λ, y_* and f_* are denoted by 'c', the cocartesian cells corresponding to the latter two cartesian cells by 'cc', and the cartesian cell $\mathrm{cart}'\colon f_* \Rightarrow \mathrm{y}_*$ by 'c′'. The identities follow from the definitions of cart', ι and ε, as well as the horizontal companion identity for f_* (Lemma 1.21). Notice that the left and right-hand side below are the two sides of the triangle identity composed with the cartesian cell that defines f_*^λ. Since the latter is weakly left Kan, by the weak density of y, and because factorisations through weakly left Kan cells are unique, the triangle identity itself follows.



The schematically drawn equality below likewise shows that the sides of the second triangle identity $(z \circ \iota) \odot (\varepsilon \circ z) = \mathrm{id}_z$ coincide after composition on the left with η. Since η is weakly left Kan the triangle identity itself follows.



The third identity above follows from the claim that $\mathrm{cocart}_{\mathrm{y}_*} \odot \mathrm{cart}_{f_*^\lambda} = \mathrm{cart}'$, where the subscripts in the left-hand side denote the companions defined by the cocartesian and cartesian cells. Indeed notice that the two sides of the latter identity coincide after composing them with the cartesian cell corresponding to $\mathrm{cocart}_{\mathrm{y}_*}$, by the vertical companion identity for y_* and the definition of cart'. The claim then follows from the uniqueness of factorisations through cartesian cells. The other identities above follow from the horizontal companion identity for y_*, the definition of ι, that of ε, and the vertical companion identity for f_*. This completes the proof. ■

6.9. Example. In an augmented virtual equipment $\mathcal{K}$ (Definition 1.19) consider chosen weak Yoneda embeddings $\mathrm{y}_A\colon A \to \widehat{A}$ that admit nullary restrictions (Definition 4.5), one for each unital object A. By Example 4.37 they induce a Yoneda structure on $V(\mathcal{K})$ whose admissible morphisms are those morphisms admitting companions. Fixing a unital object M notice that $(\mathrm{id}_M)_*^\lambda \cong I_M^\lambda \cong \mathrm{y}_M$ by Proposition 4.24 and Lemma 4.6. Together with (a) ⇔ (d) above it follows that M is total in the sense of Section 6 of [SW78], that is y_M admits a left adjoint, if and only if M is weakly total in our sense of Definition 6.2.

Similarly an admissible morphism $f\colon M \to N$, with M unital, is total in the sense of [SW78] if and only if it is weakly total in our sense such that the left adjoint $z\colon \widehat{M} \to N$ of (d) above admits a companion.

Applying the previous theorem to an identity morphism gives the following corollary. Specialising (a) ⇔ (b) below to a $\mathcal{V}$-category M in the augmented virtual equipment $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ (Example A2.7) recovers Theorem 5.2 of [Kel86], while (a) ⇔ (c) is similar to its Theorem 5.3.

6.10. Corollary. *Consider the following conditions for a (weak) Yoneda embedding* $\mathrm{y}\colon M \to \widehat{M}$ *(Definition 4.5). The implication* (b) ⇒ (a) *holds. If the companion* y_* *and the horizontal unit* I_M *exist then all conditions are equivalent. In either case the counit* $\varepsilon\colon z \circ \mathrm{y} \Rightarrow \mathrm{id}_M$ *of the adjunction of condition* (b) *is invertible.*

(a) M *is (weakly) total;*

(b) y *admits a left adjoint* $z\colon \widehat{M} \to M$*;*

(c) *if a pointwise (weakly) left* id_M*-exact (Definition 5.2) of the form below exists then so does the pointwise (weak) left Kan extension of* d *along* J *(Definition 1.24).*

$$\begin{array}{ccc} A & \xrightarrow{\;J\;} & B \\ {\scriptstyle d}\big\downarrow & \Downarrow\phi & \big\| \\ M & \xrightarrow[\;K\;]{} & B \end{array}$$

Proof. The proof of (b) ⇒ (a) is similar to that of the implication (d) ⇒ (a) of the previous theorem, as follows. Write $z\colon \widehat{M} \to M$ for the left adjoint and $\varepsilon\colon z \circ \mathrm{y} \Rightarrow \mathrm{id}_M$ for the counit of the adjunction. That y is full and faithful implies that ε is invertible, by Lemma A4.14 and (b) ⇒ (c) of Proposition 10 of [SW78]. Using the fact that z is (weakly) cocontinuous (Proposition 1.37) we can now show that M is (weakly) total: for any $J\colon M \nrightarrow B$ the pointwise (weak) left Kan extension of id_M along J is $z \circ J^\lambda$, defined as such by the composite of the inverse of ε with the cartesian cell that defines J^λ (Definition 4.5), the latter of which is pointwise (weakly) left Kan by (weak) density of y (Definition 4.3).

If the companion y_* and the horizontal unit I_M exist then the equivalence of the conditions above follows from applying the previous theorem to $f = \mathrm{id}_M$: remember that $(\mathrm{id}_M)_* \cong I_M$ is the horizontal unit of M so that $(\mathrm{id}_M)_*^\lambda \cong I_M^\lambda \cong \mathrm{y}$ by Proposition 4.24 and Lemma 4.6. ∎

The following is analogous to the first assertion of Proposition 25 of [SW78] for Yoneda structures.

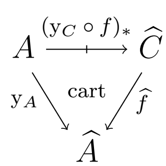
6.11. Corollary. *Let* $f\colon M \to N$ *be a morphism and let* $\mathrm{y}\colon N \to \widehat{N}$ *be a (weak) Yoneda embedding such that the companion* y_* *exists and* N *is unital. If* N *and* $\mathrm{y} \circ f$ *are (weakly) total then so is* f*.*

Corollary 6.20 below gives conditions ensuring that $\mathrm{y} \circ f$ is (weakly) total.

Proof. By the previous corollary y has a left adjoint $z\colon \widehat{N} \to N$ with invertible counit $\varepsilon\colon z \circ \mathrm{y} \Rightarrow \mathrm{id}_N$. To prove that f is (weakly) total we have to show that the pointwise (weak) left Kan extension of f along any $J\colon M \nrightarrow B$ exists. If $\mathrm{y} \circ f$ is (weakly) total then the pointwise (weakly) left Kan extension of $\mathrm{y} \circ f$ along J exists; let $\eta\colon J \Rightarrow \widehat{N}$ denote its defining nullary cell. Using that the left adjoint z is cocontinuous (Proposition 1.37), it follows that the composite $(\varepsilon^{-1} \circ f) \odot (z \circ \eta)$ defines the pointwise (weakly) left Kan extension of f along J, as required. ∎

6.12. Formal restriction of presheaves. Given a morphism $f\colon A \to C$, the following definition introduces the induced morphism $\widehat{f}\colon \widehat{C} \to \widehat{A}$ that formalises the restriction of presheaves along f; this is analogous to the definition of '$\mathcal{P}f\colon \mathcal{P}C \to \mathcal{P}A$' of Section 2 of [SW78] and that of '$\mathrm{res}_f\colon \widehat{C} \to \widehat{A}$' of Section 3 of [Web07], for (good) Yoneda structures.

6.13. Definition. *Consider weak Yoneda morphisms* $\mathrm{y}_A\colon A \to \widehat{A}$ *and* $\mathrm{y}_C\colon C \to \widehat{C}$ *and let* $f\colon A \to C$ *be a morphism such that the companion* $(\mathrm{y}_C \circ f)_*\colon A \nrightarrow \widehat{C}$ *exists. We set* $\widehat{f} \coloneqq (\mathrm{y}_C \circ f)_*^{\lambda}$*, as defined by the cartesian cell below (Definition 4.5).*



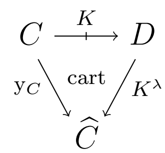
Recall that $(\mathrm{y}_C \circ f)_* \cong \mathrm{y}_{C*}(f, \mathrm{id})$ by Lemma A5.11, so that in an augmented virtual double category with restrictions on the left (Definition 1.19) all companions $(\mathrm{y}_C \circ f)_*$ exist, for any $f\colon A \to C$, as soon as the companion y_{C*} exists. Recall that the cell above defines $\widehat{f}$ uniquely up to isomorphism (Definition 4.5).

6.14. Example. As in Example 4.10 consider $\mathcal{V}$-categories A and C as well as their $\mathcal{V}'$-enriched presheaf categories $[A^{\mathrm{op}}, \mathcal{V}]'$ and $[C^{\mathrm{op}}, \mathcal{V}]'$. For any $\mathcal{V}$-functor $f\colon A \to C$ the $\mathcal{V}'$-functor $\widehat{f}\colon [C^{\mathrm{op}}, \mathcal{V}]' \to [A^{\mathrm{op}}, \mathcal{V}]'$ exists in $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ and can be taken to be given by precomposition with f, that is $\widehat{f}(p) = p \circ f^{\mathrm{op}}$ for every $\mathcal{V}$-presheaf $p \in [C^{\mathrm{op}}, \mathcal{V}]'$.

Proposition 12 of [SW78], for Yoneda structures, is analogous to the second assertion below.

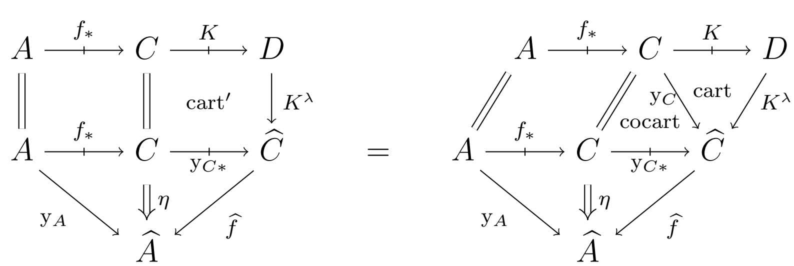
6.15. Proposition. *Let* $\mathrm{y}_A\colon A \to \widehat{A}$ *be a weak Yoneda morphism and* $\mathrm{y}_C\colon C \to \widehat{C}$ *a (weak) Yoneda morphism. Consider morphisms* $f\colon A \to C$ *and* $K\colon C \nrightarrow D$ *as well as* $K^\lambda\colon D \to \widehat{C}$*, as defined by the cartesian cell below (Definition 4.5). If the companions* y_{C*}*,* f_* *and* $(\mathrm{y}_C \circ f)_*$ *exist, as well as the restriction* $\widehat{C}(\mathrm{y}_C \circ f, K^\lambda)$*, then* $\widehat{f} \circ K^\lambda \cong K(f, \mathrm{id})^\lambda$.

If moreover y_A *is a Yoneda morphism and* C *is unital (Definition 1.16) then* $\widehat{f}\colon \widehat{C} \to \widehat{A}$ *preserves the (weak) left Kan cell below (Definitions 4.3 and 1.36). If the restrictions* $\widehat{C}(\mathrm{y}_C, g)$ *and* $\widehat{C}(\mathrm{y}_C \circ f, g)$ *exist, for all* $g\colon X \to \widehat{C}$*, then* $\widehat{f}$ *preserves the pointwise (weak) left Kan cell below.*



Applying the first assertion above to the companion $K = h_*$ of a morphism $h\colon C \to E$ we obtain $\widehat{f} \circ h_*^\lambda \cong (h \circ f)_*^\lambda$ (use Lemma A5.11). Analogous isomorphisms are required to exist in a Yoneda structure; see Axiom 3 of [SW78]. If C is unital, so that $I_C^\lambda \cong \mathrm{y}_C$ by Lemma 4.6, then taking $K = I_C$ gives $\widehat{f} \circ \mathrm{y}_C \cong f_*^\lambda$ (use Corollary A4.16).

Proof. Write $\mathrm{cart}'\colon K \Rightarrow \mathrm{y}_{C*}$ for the factorisation of the cartesian cell defining K^λ above through the cartesian cell defining y_{C*}, as in the left-hand side below; cart' is cartesian by the pasting lemma (Lemma 1.17). Next notice that $(\mathrm{y}_C \circ f)_* \cong \mathrm{y}_{C*}(f, \mathrm{id})$ (Lemma A5.11) so that, by (Lemmas A8.1 and A9.7), there exists a pointwise horizontal cocartesian cell $\mathrm{cocart}\colon (f_*, \mathrm{y}_{C*}) \Rightarrow (\mathrm{y}_C \circ f)_*$. Since the restrictions $\mathrm{y}_{C*}(\mathrm{id}, K^\lambda) \cong K$ and $(\mathrm{y}_C \circ f)_*(\mathrm{id}, K^\lambda) \cong \widehat{C}(\mathrm{y}_C \circ f, K^\lambda)$ (Lemma 1.17) exist, it follows from Definition A9.1 that cocart is a right cocartesian cell that restricts along K^λ (Definition 2.12). Hence by Corollary 4.19 there exists a weakly left Kan cell η as in the composites below that restricts along K^λ, so that the composite on the left-hand side is weakly left Kan too. Similarly, by factorising the latter composite through the cocartesian cell $(f_*, K) \Rightarrow K(f, \mathrm{id})$ (Lemma A8.1) we obtain a nullary cell $K(f, \mathrm{id}) \Rightarrow \widehat{A}$ that defines $\widehat{f} \circ K^\lambda$ as the weak left Kan extension of y_A along $K(f, \mathrm{id})$, by the vertical pasting lemma (Lemma 2.17). Since $K(f, \mathrm{id})^\lambda$ too is the weak left Kan extension of y_A along $K(f, \mathrm{id})$ (Definitions 4.3 and 4.5), the assertion $\widehat{f} \circ K^\lambda \cong K(f, \mathrm{id})^\lambda$ follows from the uniqueness of Kan extensions.



Next assume that y_A is a Yoneda morphism so that, by the same argument as above, the cell η and the left-hand side above are left Kan. Consider the right-hand side above, obtained by substituting $\text{cart}' = \text{cocart} \odot \text{cart}$, which follows immediately from the definition of cart' and the vertical companion identity for y_{C*} (Lemma 1.21). To prove that $\widehat{f} \circ \text{cart}$ is again left Kan it suffices by the horizontal pasting lemma (Lemma 2.2) to show that the composite $f_* \Rightarrow \widehat{A}$ of the first column of the right-hand side above is left Kan. Assuming that C is unital, so that y_C is full and faithful and the restriction $\widehat{C}(\mathrm{y}_C, \mathrm{y}_C)$ exists by Lemma 4.6, this follows from Proposition 2.26. Finally if for all $g\colon X \to \widehat{C}$ the restrictions $\widehat{C}(\mathrm{y}_C, g)$ and $\widehat{C}(\mathrm{y}_C \circ f, g)$ exist then $\text{cocart}\colon (f_*, \mathrm{y}_{C*}) \Rightarrow (\mathrm{y}_C \circ f)_*$ is pointwise right cocartesian by Remark 2.13, so that η and the left-hand side above are pointwise left Kan by Corollary 4.19 and Lemma 1.25. Applying the horizontal pasting lemma we conclude that $\widehat{f} \circ \text{cart}$ is pointwise left Kan as well. This completes the proof. ∎

The previous proposition can be used to describe the uniqueness of (weak) Yoneda embeddings, as follows. In an augmented virtual double category $\mathcal{K}$ consider weak Yoneda embeddings $\mathrm{y}\colon A \to P$ and $\mathrm{y}'\colon A \to P'$ that admit nullary restrictions (Definition 4.5), for the same object A. Given any horizontal morphism $J\colon A \nrightarrow B$ we denote by $J^\lambda\colon B \to P$ and $J^{\lambda'}\colon B \to P'$ the morphisms associated to J by the Yoneda axiom (Definition 4.5) for y and y' respectively. Consider the morphisms $(\mathrm{y}'_*)^\lambda\colon P' \to P$ and $(\mathrm{y}_*)^{\lambda'}\colon P \to P'$.

6.16. Corollary. *The morphisms $(\mathrm{y}'_*)^\lambda$ and $(\mathrm{y}_*)^{\lambda'}$ defined above form an equivalence $P \simeq P'$ in $V(\mathcal{K})$. Moreover the pointwise weakly left Kan cartesian cells that define the morphisms $J^{\lambda'}\colon B \to P'$, as defined above, are preserved by postcomposition with $(\mathrm{y}'_*)^\lambda\colon P' \to P$ (Definition 1.36). Finally y is dense (Definition 4.3) if and only if y' is so.*

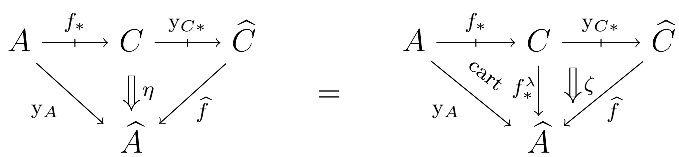
Proof. Using weak density of y and y' (Definition 4.3) the cartesian cells defining $(\mathrm{y}'_*)^\lambda$ and $(\mathrm{y}_*)^{\lambda'}$ are pointwise weakly left Kan so that, applying Proposition 2.26, we obtain isomorphisms $(\mathrm{y}'_*)^\lambda \circ \mathrm{y}' \cong \mathrm{y}$ and $(\mathrm{y}_*)^{\lambda'} \circ \mathrm{y} \cong \mathrm{y}'$. Next notice that $\widehat{\mathrm{id}_A} = (\mathrm{y}'_*)^\lambda\colon P' \to P$ by Definition 6.13 so that $(\mathrm{y}'_*)^\lambda \circ (\mathrm{y}_*)^{\lambda'} = \widehat{\mathrm{id}_A} \circ (\mathrm{y}_*)^{\lambda'} \cong (\mathrm{y}_*)^\lambda \cong \mathrm{id}_P$ where the first isomorphism follows from the proposition above; for the second one see Definition 4.5. By symmetry $(\mathrm{y}_*)^{\lambda'} \circ (\mathrm{y}'_*)^\lambda \cong \mathrm{id}_{P'}$ too, and we conclude that $(\mathrm{y}'_*)^\lambda$ and $(\mathrm{y}_*)^{\lambda'}$ form an equivalence $P \simeq P'$ in $V(\mathcal{K})$.

To prove the second assertion, for any $J\colon A \nrightarrow B$ we denote by $\eta_J\colon J \Rightarrow P'$ the pointwise weakly left Kan cartesian cell that defines $J^{\lambda'}\colon B \to P'$ (Definitions 4.3 and 4.5). Using that both $(\mathrm{y}'_*)^\lambda$ and $(\mathrm{y}_*)^{\lambda'}$ are left adjoints (see e.g. Proposition 1.5.7 of [Lei04]) and Proposition 1.37 it follows that $(\mathrm{y}'_*)^\lambda \circ \eta_J$ is pointwise weakly left Kan. Composing $(\mathrm{y}'_*)^\lambda \circ \eta_J$ with $\mathrm{y} \cong (\mathrm{y}'_*)^\lambda \circ \mathrm{y}'$ and using Lemma 4.20 we find that $(\mathrm{y}'_*)^\lambda \circ \eta_J$ is cartesian too, proving the second assertion. Finally notice that if y' is dense (Definition 4.3) then the latter composite defines $(\mathrm{y}'_*)^\lambda \circ J^{\lambda'}$ as the pointwise left Kan extension of y along J, by combining Proposition 1.37, Lemma 1.3 the horizontal pasting lemma (Lemma 2.2). Using the uniqueness of left Kan extensions we conclude that y is dense too. ∎

6.17. Adjoints of $\widehat{f}$. The remaining two results of this section ensure the existence of left and right adjoints to the induced morphism $\widehat{f}\colon \widehat{C} \to \widehat{A}$ of Definition 6.13. In both cases the existence of the adjoint is a consequence of a related morphism being (weakly) total; in particular both results are corollaries of Theorem 6.8. The first of these, below, ensures the existence of the right adjoint. It is analogous to Proposition 13 of [SW78] for Yoneda structures.

6.18. Corollary. *Let* $\mathrm{y}_A\colon A \to \widehat{A}$ *be a Yoneda morphism,* $\mathrm{y}_C\colon C \to \widehat{C}$ *a (weak) Yoneda morphism and* $f\colon A \to C$ *a morphism. If the companions* y_{C*}*,* f_**,* $(\mathrm{y}_C \circ f)_*$ *and* $(f_*^\lambda)_*$ *exist as well as the restriction* $\widehat{C}\big(\mathrm{y}_C \circ f, (f_*^\lambda)_*^\lambda\big)$ *then* $f_*^\lambda\colon C \to \widehat{A}$ *is (weakly) total and* $\widehat{f} \dashv (f_*^\lambda)_*^\lambda\colon \widehat{A} \to \widehat{C}$*.*

Proof. Using (c) ⇒ (a) of Theorem 6.8 applied to f_*^λ it suffices to construct the weak left Kan extension of f_*^λ along y_{C*} so that it restricts along $(f_*^\lambda)_*^\lambda\colon \widehat{A} \to \widehat{C}$. To do so apply the proof of the first assertion of Proposition 6.15 to $f\colon A \to C$ and $K \coloneqq (f_*^\lambda)_*\colon C \nrightarrow \widehat{A}$, thus obtaining a weakly left Kan cell η as on the left-hand side below that restricts along $K^\lambda = (f_*^\lambda)_*^\lambda$.



Next consider the unique factorisation ζ in the right-hand side above, through the cartesian cell that defines f_*^λ, which is left Kan by the density of y_A (Definition 4.3). Applying the horizontal pasting lemma (Lemma 2.2) we find that ζ defines $\widehat{f}$ as the weak left Kan extension of f_*^λ along y_{C*}, which restricts along $(f_*^\lambda)_*^\lambda$ as required. ■

The following example is a variation of Corollary 14 of [SW78].

6.19. Example. Let $\mathrm{y}_M\colon M \to \widehat{M}$ and $\mathrm{y}_{\widehat{M}}\colon \widehat{M} \to \widehat{\widehat{M}}$ be Yoneda morphisms. Assume that $\widehat{M}$ is unital and that the companions y_{M*}, $\mathrm{y}_{\widehat{M}*}$ and $(\mathrm{y}_{\widehat{M}} \circ \mathrm{y}_M)_*$ exist, as well as the restriction $\widehat{\widehat{M}}(\mathrm{y}_{\widehat{M}} \circ \mathrm{y}_M, \mathrm{y}_{\widehat{M}})$. Using that $\mathrm{y}_{M*}^\lambda \cong \mathrm{id}_{\widehat{M}}$ (Definition 4.5), so that $(\mathrm{y}_{M*}^\lambda)_*^\lambda \cong I_{\widehat{M}}^\lambda \cong \mathrm{y}_{\widehat{M}}$ (Lemma 4.6), it follows from the previous corollary that $\widehat{\mathrm{y}_M} \dashv \mathrm{y}_{\widehat{M}}$. As an alternative to Example 6.3, by Corollary 6.10 it now follows that $\widehat{M}$ is total.

The following result ensures the existence of a left adjoint to $\widehat{f}\colon \widehat{C} \to \widehat{A}$; its implication (b) ⇒ (a) is analogous to Theorem 3.20(2) of [Web07] for good Yoneda structures.

6.20. Corollary. *Let* $\mathrm{y}_A\colon A \to \widehat{A}$ *be a (weak) Yoneda morphism and let* $\mathrm{y}_C\colon C \to \widehat{C}$ *be a weak Yoneda morphism. Given a morphism* $f\colon A \to C$ *assume that the pointwise weak left Kan extension* $f^\sharp\colon \widehat{A} \to \widehat{C}$ *of* $\mathrm{y}_C \circ f$ *along* y_{A*} *exists (Definition 1.24). Consider the conditions below. If* (b) *holds then* $\mathrm{y}_C \circ f$ *is (weakly) total and* $f^\sharp \dashv \widehat{f}\colon \widehat{C} \to \widehat{A}$*, so that* (a) *holds too.*

If y_A *admits nullary restrictions (Definition 4.5) and is full and faithful (Definition 1.18) then* (a) ⇒ (b)*; if moreover all restrictions on the right exist (Definition 1.19) then* (b) ⇒ (c) *as well. If the horizontal composites* $(J \odot \mathrm{y}_{C*})$ *(Definition 2.12) exist for all* $J\colon A \nrightarrow C$ *then* (c) ⇒ (b)*.*

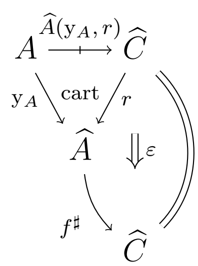
(a) *$f^\sharp$ admits a right adjoint;*

(b) *the companion $(\mathrm{y}_C \circ f)_*\colon A \nrightarrow \widehat{C}$ exists;*

(c) *the companion $f_*\colon A \nrightarrow C$ exists.*

PROOF. Assuming that the companion $(\mathrm{y}_C \circ f)_*$ exists it follows from (c) $\Rightarrow$ (d) of Theorem 6.8 that $f^\sharp$ has $\widehat{f} \coloneqq (\mathrm{y}_C \circ f)_*^\lambda$ as a right adjoint, while $\mathrm{y}_C \circ f$ is (weakly) total by (c) $\Rightarrow$ (a) of the same theorem. This proves the first assertion.

To prove (a) $\Rightarrow$ (b) denote the right adjoint to $f^\sharp$ by $r\colon \widehat{C} \to \widehat{A}$; we claim that the restriction $\widehat{A}(\mathrm{y}_A, r)\colon A \nrightarrow \widehat{C}$, which exists because y_A admits nullary restrictions, forms the companion of $y_C \circ f$. To see this consider the composite below where ε is the counit of $f^\sharp \dashv r$; it is cartesian by Lemma A4.17. Because $f^\sharp$ is the pointwise weak Kan extension along the companion of the Yoneda embedding y_A we have $\mathrm{y}_C \circ f \cong f^\sharp \circ \mathrm{y}_A$ by Proposition 2.26. Composing the composite with this isomorphism we obtain a cartesian cell that defines $\widehat{A}(\mathrm{y}_A, r)$ as the companion of $\mathrm{y}_C \circ f$.



If all restrictions on the right exist then the existence of $(\mathrm{y}_C \circ f)_*$ implies that of $(\mathrm{y}_C \circ f)_*(\mathrm{id}, \mathrm{y}_C) \cong \widehat{C}(\mathrm{y}_C \circ f, \mathrm{y}_C) \cong f_*$, where the isomorphisms follow from Lemma 1.17 and Lemma A5.12; this proves (b) $\Rightarrow$ (c). Using Lemmas A8.1 and A5.11 the converse follows from $(f_* \odot \mathrm{y}_{C*}) \cong \mathrm{y}_{C*}(f, \mathrm{id}) \cong (\mathrm{y}_C \circ f)_*$. ∎

6.21. EXAMPLE. Let $\mathcal{V}$ be a closed symmetric monoidal category that is small complete and small cocomplete. In the pseudo double category $\mathcal{V}$-sProf of small $\mathcal{V}$-profunctors (Examples A2.8 and A9.3) consider a $\mathcal{V}$-functor $f\colon A \to C$ as well as the Yoneda embeddings $\mathrm{y}_A\colon A \to [A^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$ and $\mathrm{y}_C\colon C \to [C^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$, as in Example 4.33. The $\mathcal{V}$-functor $f^\sharp\colon [A^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}} \to [C^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$ of the corollary exists and is given by left Kan extending small $\mathcal{V}$-presheaves on A along f: to see this notice that, using Example 5.3 and Proposition 5.5, $f^\sharp$ corresponds to the pointwise horizontal composite $\big([C^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}(\mathrm{y}_C, \mathrm{y}_C \circ f) \odot \mathrm{y}_{A*}\big) \cong (f^* \odot \mathrm{y}_{A*})$ (Lemma A5.12). All assumptions of the corollary are satisfied and we conclude that $f^\sharp$ has a right adjoint if and only if the companion $f_*\colon A \nrightarrow C$ exists in $\mathcal{V}$-sProf, that is f_* is a small $\mathcal{V}$-profunctor (Example A2.8). This recovers Proposition 3.3 of [DL07].

7. Cocompleteness

In this section we study the sense in which a Yoneda embedding $\mathrm{y}\colon M \to \widehat{M}$ (Definition 4.5) exhibits the presheaf object $\widehat{M}$ as the free 'small' cocompletion of M. As described at the end of the Overview and as is clear from Examples 7.9–7.15 below, the right notion of smallness here depends on the augmented virtual double category under consideration; it is defined in terms of 'left diagrams' as follows. Recall that every augmented virtual double category $\mathcal{K}$ contains a vertical 2-category $V(\mathcal{K})$ (Example A1.5).

7.1. Cocompleteness, cocontinuity and free cocompletion.

7.2. Definition. *Let $\mathcal{K}$ be an augmented virtual double category. By a* left diagram *in $\mathcal{K}$ we mean a span of the form $M \xleftarrow{d} A \overset{J}{\nrightarrow} B$. A collection $\mathcal{S}$ of left diagrams is called an* ideal *if $(f \circ d, J) \in \mathcal{S}$ for all $(d, J) \in \mathcal{S}$ and $f \in \mathcal{K}$ composable with d; given such an ideal and an object $N \in \mathcal{K}$ we write $\mathcal{S}(N) \subset \mathcal{S}$ for the subcollection of spans of the form $N \xleftarrow{d} A \overset{J}{\nrightarrow} B$. We make the following definitions.*

- *An object N is called* $\mathcal{S}$-cocomplete *if for any $(d, J) \in \mathcal{S}(N)$ the pointwise left Kan extension of d along J exists (Definition 1.24);*
- *a morphism $f\colon M \to N$ is called* $\mathcal{S}$-cocontinuous *if, for any $(d, J) \in \mathcal{S}(M)$ and any pointwise left Kan cell η that defines the pointwise left Kan extension of d along J, the composite $f \circ \eta$ is again pointwise left Kan;*
- *a morphism $w\colon M \to \bar{M}$ is said to define $\bar{M}$ as the* free $\mathcal{S}$-cocompletion *of M if $\bar{M}$ is $\mathcal{S}$-cocomplete and, for any $\mathcal{S}$-cocomplete N, the composite*

$$V_{\mathcal{S}\text{-cocts}}(\mathcal{K})(\bar{M}, N) \hookrightarrow V(\mathcal{K})(\bar{M}, N) \xrightarrow{V(\mathcal{K})(w,N)} V(\mathcal{K})(M, N)$$

is an equivalence, where $V_{\mathcal{S}\text{-cocts}}(\mathcal{K}) \subseteq V(\mathcal{K})$ denotes the locally full sub-2-category of $\mathcal{S}$-cocontinuous morphisms.

The following result describes the relation between cocompleteness and totality; it is a direct consequence of the definitions involved.

7.3. Corollary. *Let $\mathcal{S}$ be an ideal of left diagrams and M an object. If $(\mathrm{id}_M, J) \in \mathcal{S}$ for all $J\colon M \nrightarrow B$ then M is total (Definition 6.2) whenever it is $\mathcal{S}$-cocomplete. The converse holds if, for each $(d, J) \in \mathcal{S}(M)$, there exists a pointwise left id_M-exact cell (Definition 5.2)*

$$\begin{array}{ccc} A & \overset{J}{\nrightarrow} & B \\ {\scriptstyle d}\downarrow & \Downarrow\phi & \| \\ M & \underset{K}{\nrightarrow} & B. \end{array}$$

The proof of the following result is similar to that of Proposition 6.4.

7.4. Proposition. *Let $\mathcal{S}$ be an ideal of left diagrams and $f \dashv g\colon M \to N$ an adjunction with g full and faithful (Definition 1.18). If N is $\mathcal{S}$-cocomplete then so is M.*

7.5. Presheaf objects as free cocompletions. The following theorem is the main result of this section. Using the notion of left exactness (Definition 5.2) it gives conditions ensuring that a Yoneda embedding $\mathrm{y}\colon M \to \widehat{M}$ defines $\widehat{M}$ as the free $\mathcal{S}$-cocompletion of M. Recall that, for condition (e) below to be satisfied, it suffices that the cell ϕ defines K as the pointwise right composite $\big(\widehat{M}(\mathrm{y}, d) \odot J\big)$ (Definition 2.12); see Example 5.3.

7.6. Theorem. *Let $\mathrm{y}\colon M \to \widehat{M}$ be a Yoneda embedding in an augmented virtual double category $\mathcal{K}$ and let $\mathcal{S}$ be an ideal of left diagrams in $\mathcal{K}$. Assume that y admits nullary restrictions (Definition 4.5). The presheaf object $\widehat{M}$ is $\mathcal{S}$-cocomplete if and only if*

(e) *for every $(d, J) \in \mathcal{S}(\widehat{M})$ there exists a pointwise left y-exact cell*

$$\begin{array}{ccccc} M & \xrightarrow{\widehat{M}(\mathrm{y},d)} & A & \xrightarrow{J} & B \\ \| & & \Downarrow\phi & & \| \\ M & & \xrightarrow[K]{\hspace{4em}} & & B \end{array}$$

and, in that case, y defines $\widehat{M}$ as the free $\mathcal{S}$-cocompletion of M if moreover

(y) *$(f, \mathrm{y}_*) \in \mathcal{S}$ for all $f\colon M \to N$.*

It follows from Lemma 5.6 that if $\mathcal{K}$ is an augmented virtual equipment (Definition 1.19) then pointwise left y-exactness of the cell ϕ in condition (e) is equivalent to pointwise right unary-cocartesianness (Definition 2.12). We think of condition (y) as "restricting the size of the object M", as explained after Corollary 5.7.

Proof. That condition (e) is equivalent to $\widehat{M}$ being $\mathcal{S}$-cocomplete follows from Proposition 5.5. That, for any $\mathcal{S}$-cocomplete N, precomposition with y induces an equivalence $V_{\mathcal{S}\text{-cocts}}(\mathcal{K})(\widehat{M}, N) \simeq V(\mathcal{K})(M, N)$ is shown in Lemma 7.18 below. ∎

Before proving the main lemma used in its proof, in the remark below we compare Theorem 7.6 to an analogous result for Yoneda structures and, in Examples 7.9–7.15, describe some of its applications.

7.7. Remark. The above theorem is similar to Theorem 3.20(1) of [Web07] which, given a small object C in a 2-category $\mathcal{C}$ equipped with a good Yoneda structure, asserts the existence of equivalences $\mathcal{C}_{\text{cocts}}(\widehat{C}, X) \simeq \mathcal{C}(C, X)$ given by precomposition with the Yoneda embedding $y_C\colon C \to \widehat{C}$, one for each admissible and cocomplete object X. The latter result differs from the theorem above in the following ways.

- It assumes that C is small, that is both C and $\widehat{C}$ are admissible; the size restrictions that we require are that $\mathrm{y}\colon M \to \widehat{M}$ admits nullary restrictions and that it is full and faithful, which imply that M is unital by Lemma 4.6.

- It proves the existence of the equivalences only for cocomplete objects X that are admissible. In contrast our notion of free $\mathcal{S}$-cocompletion (Definition 7.2) does not restrict the size of the $\mathcal{S}$-cocomplete objects N.
- It assumes that the presheaf object $\widehat{C}$ itself is cocomplete. In contrast, the theorem above asserts that $\mathcal{S}$-cocompleteness of $\widehat{M}$ is equivalent to its condition (e). In some cases the latter is trivially satisfied, e.g. when $\mathcal{K}$ is a pseudo double category with all restrictions on the right (Definition 1.19); see Example 7.10 below.

7.8. Examples of free cocompletions.

7.9. Example. In the augmented virtual equipment $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ of $\mathcal{V}$-profunctors between $\mathcal{V}'$-categories (Example A2.7) consider the ideal of left diagrams

$$\mathcal{S} = \{(d, J) \mid J\colon A \nrightarrow B \text{ is a } \mathcal{V}\text{-profunctor with } A \text{ a small } \mathcal{V}\text{-category}\}.$$

Assuming that $\mathcal{V} \subset \mathcal{V}'$ is symmetric in the sense of Example 4.10, consider the Yoneda embedding $\mathrm{y}\colon M \to [M^{\mathrm{op}}, \mathcal{V}]'$ for a $\mathcal{V}$-category M as described there. If moreover $\mathcal{V}$ is small cocomplete then the pointwise composites $\widehat{M}(\mathrm{y}, d) \odot J$ exist in $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ for all $(d, J) \in \mathcal{S}(\widehat{M})$ by Example A9.2, so that condition (e) above is satisfied by Remark 2.13. Condition (y) is satisfied if moreover M is small, so that in that case y defines the $\mathcal{V}'$-category $[M^{\mathrm{op}}, \mathcal{V}]'$ of $\mathcal{V}$-presheaves on M as the free $\mathcal{S}$-cocompletion of M in $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$.

Next assume that $\mathcal{V}$ is closed symmetric monoidal and both small complete and small cocomplete, and that the embedding $\mathcal{V} \subset \mathcal{V}'$ is a closed symmetric monoidal functor (Example A2.7). From Examples 1.12 and 1.13 it follows that for $\mathcal{V}$-categories $\mathcal{S}$-cocompleteness coincides with the classical notion of small cocompleteness, the latter in the sense of e.g. of Section 3.2 of [Kel82]. The theorem in this case recovers the fact that, for a small $\mathcal{V}$-category M, the $\mathcal{V}$-category of $\mathcal{V}$-presheaves on M forms the free small cocompletion of M; see Theorem 4.51 of [Kel82].

7.10. Example. In any augmented virtual double category $\mathcal{K}$ consider the ideal $\mathcal{A}$ of all left diagrams below. By Corollary 7.3 $\mathcal{A}$-cocompleteness implies totality.

$$\mathcal{A} = \{(d, J) \mid (d, J) \text{ is any left diagram}\}$$

If $\mathcal{K}$ is a pseudo double category (Proposition A7.8) that has restrictions on the right (Definition 1.19) then any Yoneda embedding $\mathrm{y}\colon M \to \widehat{M}$ in $\mathcal{K}$ satisfies the conditions of the theorem above, and thus defines $\widehat{M}$ as the free $\mathcal{A}$-cocompletion of M, as long as its companion y_* exists. Indeed condition (e) follows from the fact that $\mathcal{K}$ has all pointwise right composites (Remark 2.13). Moreover by combining Corollary 7.3 and Corollary 6.10 we find that the following are equivalent: M is $\mathcal{A}$-cocomplete; M is total (Definition 6.2); $\mathrm{y}\colon M \to \widehat{M}$ admits a left adjoint. Indeed we can use Corollary A8.5, Lemma A9.8 and Remark 2.13 to construct pointwise right cocartesian cells ϕ of the form as in Corollary 7.3.

7.11. Example. In the unital virtual double category $\mathcal{V}$-sProf of small $\mathcal{V}$-profunctors (Example A2.8) consider the ideal $\mathcal{A}$ of all left diagrams as in the previous example. We claim that if $\mathcal{V}$ is closed symmetric monoidal then $\mathcal{A}$-cocompleteness coincides with the classical notion of small cocompleteness for $\mathcal{V}$-categories, in the sense of e.g. of Section 3.2 of [Kel82], as we will show below. If moreover $\mathcal{V}$ is small complete then for any (possibly large) $\mathcal{V}$-category M the Yoneda embedding $\mathrm{y}\colon M \to [M^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$ exists in $\mathcal{V}$-sProf, with $[M^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$ the $\mathcal{V}$-category of small $\mathcal{V}$-presheaves on M; see Example 4.33. If $\mathcal{V}$ is also small cocomplete, so that $\mathcal{V}$-sProf is a pseudo double category (Example A9.3), then by the previous example y defines $[M^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$ as the free $\mathcal{A}$-cocompletion of M; this recovers Theorem 2.11 of [Lin74]. As in the previous example we find that the following are equivalent: M is small cocomplete, in the classical sense; M is $\mathcal{A}$-cocomplete (Definition 7.2); M is total in $\mathcal{V}$-sProf (Definition 6.2); the Yoneda embedding $\mathrm{y}\colon M \to [M^{\mathrm{op}}, \mathcal{V}]_{\mathrm{s}}$ admits a left adjoint.

To prove the claim first recall from Example 1.12 that small $\mathcal{V}$-weighted colimits can be equivalently defined as left Kan extensions in $\mathcal{V}$-sProf along $\mathcal{V}$-profunctors of the form $J\colon A \nrightarrow I$ with A small: this shows that $\mathcal{A}$-cocompleteness implies small cocompleteness. For the converse consider any left diagram $M \xleftarrow{d} A \xrightarrow{J} B$ with J small and M small cocomplete: we will show that the left Kan extension of d along J exists in $\mathcal{V}$-Prof and hence in $\mathcal{V}$-sProf (Example 1.13), so that it is pointwise by Remark 1.26. Smallness of J (Example A2.8) can be rephrased as follows: for every $y \in B$ there exists a small sub-$\mathcal{V}$-category $\iota_y\colon A_y \subseteq A$ and a cocartesian cell cocart: $\big(\iota_y^*, J(\iota_y, y)\big) \Rightarrow J(\mathrm{id}, y)$ in $\mathcal{V}$-Prof. For each $y \in B$ let $ly \in M$ denote the $J(\iota_y, y)$-weighted colimit of $d \circ \iota_y$ (Example 1.12), which exists by assumption. We assert that each ly coincides with the $J(\mathrm{id}, y)$-weighted colimit of d, so that together the ly combine into a $\mathcal{V}$-functor $l\colon B \to M$ that forms the left Kan extension of d along J, as described in Example 1.13. To prove this assertion denote by $\eta_y\colon J(\iota_y, y) \Rightarrow M$ the nullary cell in $\mathcal{V}$-Prof that defines ly and consider the composite $(d \circ \mathrm{cart}) \odot \eta_y\colon \big(\iota_y^*, J(\iota_y, y)\big) \Rightarrow M$, where cart defines ι_y^*. The latter is left Kan by Corollary 2.24 so that, using the vertical pasting lemma (Lemma 2.17), by factorising it through cocart: $\big(\iota_y^*, J(\iota_y, y)\big) \Rightarrow J(\mathrm{id}, y)$ we obtain a left Kan cell that defines ly as the $J(\mathrm{id}, y)$-weighted colimit of d. This completes the proof of the claim.

7.12. Example. Let $\mathcal{E}$ be a cartesian closed regular category, so that $\mathsf{ModRel}(\mathcal{E})$ is an equipment (Proposition A7.8) by Example 1.31. Example 7.10 applies to the Yoneda embedding $\mathrm{y}\colon M \to [M^\circ, \widehat{1}]$ of an internal preorder M, as constructed in Example 4.17, so that it defines $[M^\circ, \widehat{1}]$ as the $\mathcal{A}$-cocompletion of M.

7.13. Example. Example 7.10 applies to the Yoneda embedding $\mathrm{y}\colon M \to \mathrm{Dn}^+ M$ of a closed-ordered closure space M (Example 4.12) in the locally thin strict double category $\mathsf{ClModRel}$ of closed modular relations, so that y defines $\mathrm{Dn}^+ M$ as the free $\mathcal{A}$-cocompletion of M therein. Likewise if M is a modular closure space (Example 1.32) then y defines $\mathrm{Dn}^+ M$ as the free $\mathcal{A}$-completion in the full sub-double category $\mathsf{ClModRel}_{\mathrm{m}} \subset \mathsf{ClModRel}$ generated by modular closure spaces.

We claim that any $\mathcal{A}$-cocomplete closed-ordered closure space N has all suprema.

Indeed consider the singleton closed-ordered closure space $* := \{*\}$ with closed subsets $\mathrm{Cl}\,* := \{\emptyset, \{*\}\}$ and notice that downsets $X \subseteq N$ correspond precisely to closed modular relations $X : N \nrightarrow *$. In fact it is straightforward to see that $l \in N$ is a supremum of X if and only if the morphism $l : * \to N$, that picks out l, forms the pointwise left Kan extension of $X : N \nrightarrow *$ along id_N in $\mathsf{ClModRel}$, so that the claim follows. Having all suprema is however unlikely to be sufficient for $\mathcal{A}$-cocompleteness in general, as follows. Consider the full sub-augmented virtual double category $\mathsf{CptClModRel}_{\mathrm{m}} \subset \mathsf{ClModRel}_{\mathrm{m}}$ generated by those closed modular relations $J : A \nrightarrow B$ for which, for each $y \in B$, the preimage $J^{\circ}y$ is both compact and up-directed in A in the sense of Section 8 of [Kou18]. Combining Theorem 8.1 therein with Example 1.33 it follows that a modular closure space N is $\mathcal{A}$-cocomplete in $\mathsf{CptClModRel}_{\mathrm{m}}$ whenever N has all suprema and is *normalised*, that is $\overline{\{x\}} = {\uparrow}x$ for all $x \in N$, where $\overline{\{x\}}$ denotes the closure of the singleton set; see Section 4 of [Kou18].

7.14. EXAMPLE. To give a useful ideal $\mathcal{C}$ of left diagrams in a general augmented virtual double category $\mathcal{K}$ let us call a horizontal morphism $J : A \nrightarrow B$ *left composable* when the pointwise right composite of (H, J) exists for any $H : C \nrightarrow A$ (Definition 2.12); we set

$$\mathcal{C} = \{(d, J) \mid J \text{ is left composable}\}.$$

Using Example 5.3 a Yoneda embedding $\mathrm{y} : M \to \widehat{M}$ in $\mathcal{K}$ satisfies the conditions of the theorem, so that $\widehat{M}$ forms the free $\mathcal{C}$-cocompletion of M, as soon as y admits nullary restrictions and its companion y_* is left composable. Notice that if $\mathcal{K}$ has restrictions on the right (Definition 1.19) then the latter is equivalent to the existence of the pointwise right composites of all $H : A \nrightarrow M$ and $J : M \nrightarrow B$, which follows from Lemma 2.14 and the fact that $J \cong \mathrm{y}_*(\mathrm{id}, J^{\lambda})$ (Definition 4.5).

Consider a 'monoidal augmented virtual double category' $(\mathcal{K}, \otimes, I)$ in the sense of Definition 8.2 below. In Theorem 8.9 we will apply the previous to obtain conditions ensuring that the Yoneda embedding $\mathrm{y} : I \to \widehat{I}$ for the monoidal unit defines $\widehat{I}$ as the free $\mathcal{C}$-cocompletion of I.

7.15. EXAMPLE. Let T be a monad on an augmented virtual double category $\mathcal{K}$, in the sense of Section 6 of [Kou15b]. As explained there 'colax T-algebras', 'lax vertical T-morphisms' and 'horizontal T-morphisms' in $\mathcal{K}$ form an augmented virtual double category $T\text{-}\mathsf{Alg}_{(\mathrm{c,l})}$ that comes equipped with a forgetful functor $U : T\text{-}\mathsf{Alg}_{(\mathrm{c,l})} \to \mathcal{K}$. Given a colax T-algebra A and a Yoneda embedding $\mathrm{y} : UA \to \widehat{A}$ in $\mathcal{K}$ one of the main results of [Kou15b], its Theorem 8.1, gives conditions ensuring that y can be lifted along U as an "algebraic" Yoneda embedding in $T\text{-}\mathsf{Alg}_{(\mathrm{c,l})}$. This formalises equipping the category $[A^{\mathrm{op}}, \mathsf{Set}]$ of presheaves on a monoidal category $(A, \otimes)$ with the monoidal structure given by *Day-convolution* induced by $\otimes$, as introduced in [Day70].

Next consider any ideal $\mathcal{S}$ of left diagrams in $\mathcal{K}$ and assume that $\mathrm{y} : UA \to \widehat{A}$ satisfies the conditions of the theorem above, so that $\widehat{A}$ is the free $\mathcal{S}$-cocompletion of UA in $\mathcal{K}$. As described in Section 7.4 of [Kou15b] the ideal $\mathcal{S}$ induces an ideal $\mathcal{S}_{(\mathrm{c,l})}$ of left diagrams in $T\text{-}\mathsf{Alg}_{(\mathrm{c,l})}$, and its Theorem 8.5 gives conditions on $\mathcal{S}$ ensuring that the lift of y in

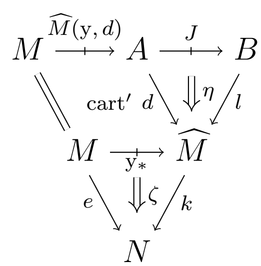
T-$\mathsf{Alg}_{(\mathrm{c,l})}$ satisfies the theorem above with respect to $\mathcal{S}_{(\mathrm{c,l})}$, so that it defines the free $\mathcal{S}_{(\mathrm{c,l})}$-cocompletion of A in T-$\mathsf{Alg}_{(\mathrm{c,l})}$. Theorems 8.1 and 8.5 of [Kou15b] also treat the lifting of Yoneda embeddings along the forgetful functor for the augmented virtual double category T-$\mathsf{Alg}_{(\mathrm{c,ps,lbc})}$ of colax T-algebras, 'pseudo vertical T-morphisms' and horizontal T-morphisms satisfying a 'left Beck-Chevalley condition'.

7.16. Lemmas used in the proof of Theorem 7.6. In the proof of Lemma 7.18 below the following lemma is used, which is a weakening of the implication (c) ⇒ (d) of Theorem 6.8. Given a Yoneda morphism $\mathrm{y}\colon M \to \widehat{M}$ the latter asserts that a left Kan extension of a morphism $f\colon M \to N$ along the companion y_* is a left adjoint but, unlike the result below, it requires the companion f_* to exist. The result below allows us to prove in Theorem 7.6 that $\widehat{M}$ is the free $\mathcal{S}$-cocompletion of M among all $\mathcal{S}$-cocomplete objects N (see Definition 7.2), without having to restrict to those $\mathcal{S}$-cocomplete objects that are unital.

7.17. Lemma. *Let $\mathrm{y}\colon M \to \widehat{M}$ be a Yoneda morphism that admits all nullary restrictions (Definition 4.5) and let $\mathcal{S}$ be an ideal of left diagrams that satisfies condition* (e) *of Theorem 7.6. Any pointwise left Kan extension along the companion y_* is $\mathcal{S}$-cocontinuous.*

Proof. Suppose that the cell ζ in the composite below defines k as the pointwise left Kan extension of e along y_*. We have to show that, for any left diagram $(d, J) \in \mathcal{S}(\widehat{M})$ and any pointwise left Kan cell η as in the composite θ below, the composite $k \circ \eta$ is again pointwise left Kan.

$\theta :=$



To see this consider the pointwise left y-exact cell $\phi\colon \big(M(\mathrm{y}, d), J\big) \Rightarrow K$ that exists by condition (e) of Theorem 7.6. By the uniqueness of Kan extensions we may without loss of generality assume that $l = K^\lambda\colon B \to \widehat{M}$ (Definition 4.5) and that η corresponds to ϕ as in Proposition 5.5; that is

$$\mathrm{cart}_{K^\lambda} \circ \phi = \mathrm{cart}_{\widehat{M}(\mathrm{y},d)} \odot \eta$$

as in Lemma 4.22, where the cartesian cells define K^λ and $\widehat{M}(\mathrm{y}, d)$ respectively. We may take the top row of θ above to be the factorisation of the right-hand side immediately above through the cartesian cell that defines y_* so that, by the equation above, $\theta = \zeta \circ \mathrm{cart}'_{K^\lambda} \circ \phi$ where $\mathrm{cart}'_{K^\lambda}$ is the factorisation of $\mathrm{cart}_{K^\lambda}$ through y_*. The factorisations cart' and $\mathrm{cart}'_{K^\lambda}$ are cartesian by the pasting lemma (Lemma 1.17) so that, using Lemma 1.25 and the assumption that ϕ is pointwise left y-exact, we conclude that both θ and $\zeta \circ \mathrm{cart}'$ are pointwise left Kan. By the horizontal pasting lemma (Lemma 2.2) it follows that the second column $k \circ \eta$ of θ is also pointwise left Kan, as required. ∎

7.18. LEMMA. *Let* $\mathrm{y}\colon M \to \widehat{M}$ *and* $\mathcal{S}$ *be as in Theorem 7.6. For any object* N *the top leg of the diagram*

$$\begin{array}{ccc} V(\mathcal{K})(\widehat{M}, N) & \xrightarrow{V(\mathcal{K})(\mathrm{y}, N)} & V(\mathcal{K})(M, N) \\ \uparrow & & \uparrow \\ V_{\mathcal{S}\text{-cocts}}(\mathcal{K})(\widehat{M}, N) & \underset{\simeq}{\dashrightarrow} & V(\mathcal{K})(M, N)' \end{array}$$

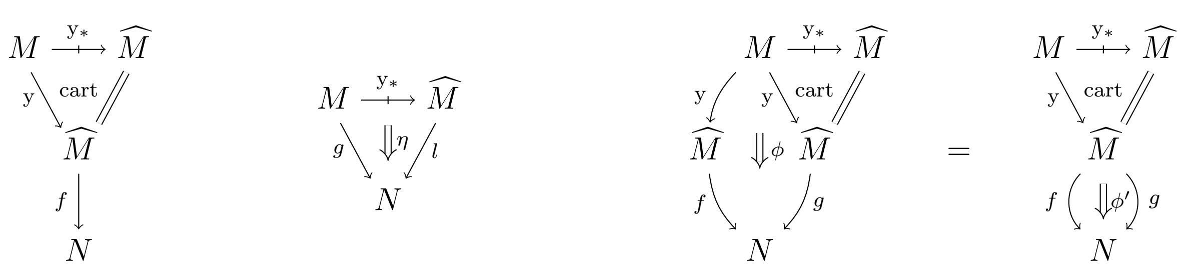
factors through the full subcategory $V(\mathcal{K})(M, N)'$ *of* $V(\mathcal{K})(M, N)$*, that is generated by all* $g\colon M \to N$ *whose pointwise left Kan extension along* y_* *exists, as an equivalence as shown.*

In particular if N *is* $\mathcal{S}$*-cocomplete then the factorisation above reduces to an equivalence* $V_{\mathcal{S}\text{-cocts}}(\mathcal{K})(\widehat{M}, N) \simeq V(\mathcal{K})(M, N)$.

PROOF. Firstly, for the final assertion, simply notice that condition (y) of Theorem 7.6 ensures that $V(\mathcal{K})(M, N)' = V(K)(M, N)$ for $\mathcal{S}$-cocomplete N. Next to see that the top leg of the diagram above factors through $V(\mathcal{K})(M, N)'$ consider any $\mathcal{S}$-cocontinuous $f\colon \widehat{M} \to N$; we have to show that the pointwise left Kan extension of $f \circ \mathrm{y}$ along y_* exists. Since $(\mathrm{y}, \mathrm{y}_*) \in \mathcal{S}$ by the same condition it clearly does: it is f itself, defined by the composite on the left below, where cart is pointwise left Kan by the density of y (Definition 4.3).

To prove that the factorisation is an equivalence we will show that it is essentially surjective and full and faithful. For the former consider any $g \in V(\mathcal{K})(M, N)'$, so that the pointwise left Kan extension $l\colon \widehat{M} \to N$ of g along y_* exists; we denote its defining cell by η, as in the middle below. By Lemma 7.17 l is $\mathcal{S}$-cocontinuous while, by precomposing η with the weakly cocartesian cell defining y_*, we obtain a vertical isomorphism $g \cong l \circ \mathrm{y}$, as follows from the assumption that y is full and faithful and Proposition 2.26. This shows essential surjectivity.

To prove full and faithfulness, consider any vertical cell $\phi\colon f \circ \mathrm{y} \Rightarrow g \circ \mathrm{y}$ with f and g $\mathcal{S}$-cocontinuous. We have to show that there exists a unique vertical cell $\phi'\colon f \Rightarrow g$ such that $\phi = \phi' \circ \mathrm{y}$.



Since y is dense, the cartesian cell in the right-hand side of the equation above is pointwise left Kan by Definition 4.3. It follows that its composition with f is too, by $\mathcal{S}$-cocontinuity of f and condition (y) of Theorem 7.6, so that the composite on the left-hand side above factors uniquely as a cell ϕ' as shown. Composing both sides with the weakly cocartesian cell corresponding to the cartesian cell cart above, using the vertical companion identity (Lemma 1.21) we conclude that ϕ' is unique such that $\phi = \phi' \circ \mathrm{y}$, as required. This completes the proof. ∎

8. Yoneda embeddings in monoidal augmented virtual double categories

Let $\mathcal{E}$ be a finitely complete category with a subobject classifier Ω. Recall from Example 4.16 that Ω induces a Yoneda embedding $\mathrm{y}\colon 1 \to \Omega$ in $\mathsf{ModRel}(\mathcal{E})$. Also recall, from e.g. Section A2 of [Joh02], that the following are equivalent for $\mathcal{E}$: (a) $\mathcal{E}$ has power objects; (b) $\mathcal{E}$ has all exponentials of the form Ω^A; (c) $\mathcal{E}$ is cartesian closed. In this final section we generalise the implications (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c) to augmented virtual double categories as follows. Given a monoidal augmented virtual double category $\mathcal{K} = (\mathcal{K}, \otimes, I)$ in the sense below consider a Yoneda embedding $\mathrm{y}\colon I \to \widehat{I}$ for the monoidal unit as well as a unital object A in $\mathcal{K}$. The main result of this section, Theorem 8.21, shows that under mild conditions the Yoneda embedding $\mathrm{y}_A\colon A \to \widehat{A}$ exists in $\mathcal{K}$ if and only if the 'internal hom' $[A^\circ, \widehat{I}]$ (Definition 8.14) does, where A° denotes the 'horizontal dual' of A (Definition 8.11), and, in that case, $\widehat{A} \cong [A^\circ, \widehat{I}]$. This result was used in Examples 4.10 and 4.17 to obtain Yoneda embeddings in $\mathcal{V}$-Prof and $\mathsf{ModRel}(\mathcal{E})$, with the latter example recovering the classical implication (b) $\Rightarrow$ (a) above.

Instead of proving the main result Theorem 8.21 directly we will prove a generalisation in Theorems 8.33 and 8.36 as follows; this generalisation is more widely applicable and allows for simpler diagrams in its proof. Given a functor $F\colon \mathcal{K} \to \mathcal{L}$ of augmented virtual double categories, a Yoneda embedding $\mathrm{y}_A\colon A \to \widehat{A}$ in $\mathcal{L}$, a 'locally universal horizontal morphism' $\iota\colon A \nrightarrow FA'$ from A to F in $\mathcal{L}$ (Definition 8.26) and an object $P \in \mathcal{K}$, the generalisation asserts the equivalence of the existence of a Yoneda embedding $\mathrm{y}_{A'}\colon A' \to P$ in $\mathcal{K}$ and that of a relative universal morphism $\varepsilon\colon FP \to \widehat{A}$ in $\mathcal{L}$ (Definition 1.39). Applying the generalisation to the endofunctor $F \coloneqq A^\circ \otimes -\colon \mathcal{K} \to \mathcal{K}$ recovers Theorem 8.21.

The main result and its generalisation only depend on Sections 1–4. Theorem 8.9 below, which describes conditions ensuring that the Yoneda embedding $\mathrm{y}\colon I \to \widehat{I}$ defines $\widehat{I}$ as a free cocompletion of I, depends on Section 7.

8.1. Monoidal augmented virtual double categories. We start by introducing the notion of monoidal augmented virtual double category. Notice that the 2-category $\mathsf{AugVirtDblCat}$ of augmented virtual double categories, their functors and the transformations between them (Section A3) has finite products. In particular the square $\mathcal{K} \times \mathcal{K}$ of an augmented virtual double category $\mathcal{K}$ can be taken to have as objects, vertical morphisms, horizontal morphisms and cells, ordered pairs of those in $\mathcal{K}$ where, in the case of a pair of cells (ϕ, ϕ'), the arities of ϕ and ϕ' coincide. Likewise the terminal object in $\mathsf{AugVirtDblCat}$ is the strict double category 1 (see Proposition A7.8) generated by a single object $*$. Analogous to the fact that monoidal categories are pseudomonoids (see

Section 3 of [DS97]) in the 2-category of categories, monoidal augmented virtual double categories are pseudomonoids in the 2-category $\mathsf{AugVirtDblCat}$ as follows.

8.2. Definition. *A* monoidal augmented virtual double category *is an augmented virtual double category $\mathcal{K}$ equipped with a* monoidal product $\otimes\colon \mathcal{K}\times\mathcal{K}\to\mathcal{K}$ *and a* monoidal unit *object $I\in\mathcal{K}$, with specified horizontal unit $I_I\colon I \nrightarrow I$, as well as invertible* associator *and* unitor *transformations*

$$\mathfrak{a}\colon \otimes\circ(\otimes\times\mathrm{id}) \overset{\cong}{\Rightarrow} \otimes\circ(\mathrm{id}\times\otimes), \quad \mathfrak{l}\colon \otimes\circ(I\times\mathrm{id}) \overset{\cong}{\Rightarrow} \mathrm{id} \quad \textit{and} \quad \mathfrak{r}\colon \otimes\circ(\mathrm{id}\times I) \overset{\cong}{\Rightarrow} \mathrm{id}$$

satisfying the usual axioms.

We call $(\mathcal{K},\otimes,I)$ cartesian monoidal *if $\otimes\colon\mathcal{K}\times\mathcal{K}\to\mathcal{K}$ and $I\colon 1\to\mathcal{K}$ form right adjoints to the diagonal functor $\Delta\colon\mathcal{K}\to\mathcal{K}\times\mathcal{K}$ and the terminal functor $!\colon\mathcal{K}\to 1$ respectively, in the* 2*-category* $\mathsf{AugVirtDblCat}$ *(see Example 8.6 below).*

By the definition of functor of augmented virtual double categories and that of transformation of such functors (Section A3), a monoidal structure on an augmented virtual double category $\mathcal{K}$ restricts to a monoidal structure $(\otimes_{\mathrm{v}}, I, \mathfrak{a}_{\mathrm{v}}, \mathfrak{l}_{\mathrm{v}}, \mathfrak{r}_{\mathrm{v}})$ on the category $\mathcal{K}_{\mathrm{v}}$ of its objects and vertical morphisms. Notice that the invertible cells of the associator and unitor transformations are of the forms below, where $J\colon A\nrightarrow B$, $J'\colon A'\nrightarrow B'$ and $J''\colon A''\nrightarrow B''$ are any horizontal morphisms in $\mathcal{K}$. To prevent confusing the monoidal unit *object* I with horizontal unit *morphisms* $I_A\colon A\nrightarrow A$, throughout this section the latter are consistently denoted with their object A as subscript.

$$\begin{array}{ccc} A\otimes(A'\otimes A'') & \xrightarrow{J\otimes(J'\otimes J'')} & B\otimes(B'\otimes B'') \\ {\scriptstyle\mathfrak{a}}\downarrow & \Downarrow\mathfrak{a} & \downarrow{\scriptstyle\mathfrak{a}} \\ (A\otimes A')\otimes A'' & \xrightarrow[(J\otimes J')\otimes J'']{} & (B\otimes B')\otimes B'' \end{array} \qquad \begin{array}{ccc} I\otimes A & \xrightarrow{I_I\otimes J} & I\otimes B \\ {\scriptstyle\mathfrak{l}}\downarrow & \Downarrow\mathfrak{l} & \downarrow{\scriptstyle\mathfrak{l}} \\ A & \xrightarrow[J]{} & B \end{array} \qquad \begin{array}{ccc} A\otimes I & \xrightarrow{J\otimes I_I} & B\otimes I \\ {\scriptstyle\mathfrak{r}}\downarrow & \Downarrow\mathfrak{r} & \downarrow{\scriptstyle\mathfrak{r}} \\ A & \xrightarrow[J]{} & B \end{array}$$

Given a unital object X and a path $\underline{J}=(J_1,\dots,J_n)\colon A_0\nrightarrow A_n$ in $\mathcal{K}$ it will be useful to abbreviate $X\otimes\underline{J} \coloneqq (I_X\otimes J_1,\dots,I_X\otimes J_n)\colon X\otimes A_0\nrightarrow X\otimes A_n$.

8.3. Example. A symmetry $\mathfrak{s}\colon X\otimes Y\xrightarrow{\cong} Y\otimes X$ for a monoidal category $\mathcal{V}$ induces a monoidal structure on the unital virtual equipment $\mathcal{V}$-Prof of $\mathcal{V}$-profunctors, with the usual monoidal product $A\otimes A'$ of $\mathcal{V}$-categories A and A' (see e.g. Section 1.4 of [Kel82]) and the monoidal product $J\otimes J'\colon A\otimes A'\nrightarrow B\otimes B'$ of $\mathcal{V}$-profunctors J and J' defined by $(J\otimes J')\big((x,x'),(y,y')\big) = J(x,y)\otimes J'(x',y')$. Likewise a symmetric universe enlargement $\mathcal{V}\subset\mathcal{V}'$ (Example 4.10) induces a monoidal structure on the augmented virtual equipment $(\mathcal{V},\mathcal{V}')$-$\mathsf{Prof}$ of $\mathcal{V}$-profunctors between $\mathcal{V}'$-categories.

8.4. Example. As long as the monoidal product $\otimes$ of a symmetric monoidal category $\mathcal{V}$ preserves small colimits on either side the monoidal structure on $\mathcal{V}$-Prof restricts to the unital virtual double category $\mathcal{V}$-$\mathsf{sProf}\subset\mathcal{V}$-$\mathsf{Prof}$ of small $\mathcal{V}$-profunctors. Indeed if $J\colon A\nrightarrow B$ and $J'\colon A'\nrightarrow B'$ are small $\mathcal{V}$-profunctors, with each $J(-,y)$ and $J'(-,y')$

"generated" by their actions on small sub-$\mathcal{V}$-categories $A_y \subseteq A$ and $A'_{y'} \subseteq A'$ in the sense of Example A2.8, then by using the 'Fubini formula' for coends (see Section 2.1 of [Kel82] for its dual) one easily sees that each $(J \otimes J')\big(-, (y, y')\big)$ is generated by its action on $A_y \otimes A_{y'}$, regarded as a small sub-$\mathcal{V}$-category of $A \otimes A'$.

8.5. Example. Let $\mathsf{DblCat}_{\mathrm{nl}}$ denote the 2-category of pseudo double categories, normal lax double functors and transformations; see e.g. Section 6 of [Shu08], and let DblCat denote its locally full sub-2-category generated by pseudo double functors. Let $\mathsf{VirtDblCat}_{\mathrm{u}}$ denote the 2-category of unital virtual double categories, normal functors and transformations; see Section A10. We have embeddings of 2-categories

$$\mathsf{DblCat} \hookrightarrow \mathsf{DblCat}_{\mathrm{nl}} \hookrightarrow \mathsf{VirtDblCat}_{\mathrm{u}} \xrightarrow[\simeq]{N} \mathsf{AugVirtDblCat}_{\mathrm{u}} \hookrightarrow \mathsf{AugVirtDblCat};$$

see Section 2 of [DPP06] (where virtual double categories are called 'lax double categories') for $\mathsf{DblCat}_{\mathrm{nl}} \hookrightarrow \mathsf{VirtDblCat}_{\mathrm{u}}$ and Section A10 for the 2-equivalence N. Under this composite a pseudo double category $\mathcal{K}$, with horizontal composition denoted $\odot$, is mapped to the augmented virtual double category $N(\mathcal{K})$ with the same objects and morphisms as $\mathcal{K}$, whose unary cells $(J_1, \ldots, J_n) \Rightarrow K$ are cells $J_1 \odot \cdots \odot J_n \Rightarrow K$ in $\mathcal{K}$ and whose nullary cells $(J_1, \ldots, J_n) \Rightarrow C$ are cells $J_1 \odot \cdots \odot J_n \Rightarrow I_C$ in $\mathcal{K}$, where I_C denotes the horizontal unit of the object C. Except for $\mathsf{DblCat} \hookrightarrow \mathsf{DblCat}_{\mathrm{nl}}$ the embeddings above are full and faithful.

Analogous to the definition above, in Definition 2.9 of [Shu10] a *monoidal pseudo double category* is defined to be a pseudomonoid $(\mathcal{K}, \otimes, I)$ in the 2-category DblCat. Since the composite above preserves finite products any monoidal pseudo double category $\mathcal{K}$ can be regarded as a monoidal augmented virtual double category $N(\mathcal{K})$ in our sense.

8.6. Example. In Definition 4.2.1 of [Ale18] a pseudo double category $\mathcal{K}$ is defined to be *cartesian* if it forms a *cartesian object* in the 2-category DblCat, that is the diagonal pseudo double functor $\Delta\colon \mathcal{K} \to \mathcal{K} \times \mathcal{K}$ and the terminal pseudo double functor $!\colon \mathcal{K} \to 1$ admit right adjoints $\times\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and $1\colon 1 \to \mathcal{K}$ in DblCat. The pseudo double category $\mathsf{Span}(\mathcal{E})$ of a spans in a category $\mathcal{E}$ with finite limits (Example A2.9) is cartesian under the cartesian product of objects and spans; see Proposition 4.2.7 of [Ale18].

Using that the embedding $\mathsf{DblCat} \hookrightarrow \mathsf{AugVirtDblCat}$ above preserves finite products, any cartesian pseudo double category can be regarded as a cartesian object in $\mathsf{AugVirtDblCat}$ and thus, since cartesian objects canonically are pseudomonoids (see Remark 2.11 of [Shu10]), as a cartesian monoidal augmented virtual double category in the sense of Definition 8.2. We conclude that $\mathsf{Span}(\mathcal{E})$, as a unital virtual equipment, admits a cartesian monoidal structure $(\times, 1)$. Notice that the latter restricts to a cartesian monoidal structure on the unital virtual equipment $\mathsf{Rel}(\mathcal{E})$ of relations in $\mathcal{E}$ (Example 1.31).

8.7. Example. Recall the 2-functor $\mathsf{Mod}\colon \mathsf{VirtDblCat} \to \mathsf{VirtDblCat}_{\mathrm{u}}$ that maps a virtual double category $\mathcal{K}$ to the unital virtual double category $\mathsf{Mod}(\mathcal{K})$ of monoids and bimodules in $\mathcal{K}$; see Definition A2.1. Mod preserves finite products and hence the endo-2-functor

$$\mathsf{AugVirtDblCat}_{\mathrm{u}} \xrightarrow{U} \mathsf{VirtDblCat} \xrightarrow{\mathsf{Mod}} \mathsf{VirtDblCat}_{\mathrm{u}} \xrightarrow[\simeq]{N} \mathsf{AugVirtDblCat}_{\mathrm{u}},$$

where U forgets the nullary cells (Proposition A3.3), preserves (cartesian) monoidal structures on unital virtual double categories. Hence the cartesian monoidal structure on $\mathsf{Rel}(\mathcal{E})$ of the previous example induces a cartesian monoidal structure $(\times, 1)$ on the unital virtual equipment $\mathsf{ModRel}(\mathcal{E}) := (N \circ \mathsf{Mod})\big(\mathsf{Rel}(\mathcal{E})\big)$ of internal modular relations in $\mathcal{E}$ (Example 1.31).

8.8. Free cocompletion of the monoidal unit. Having introduced the notion of monoidal augmented virtual double category $\mathcal{K} = (\mathcal{K}, \otimes, I)$, we now pause to describe conditions ensuring that the Yoneda embedding $\mathrm{y}\colon I \to \widehat{I}$ for the monoidal unit defines $\widehat{I}$ as the free cocompletion of I, in the sense of Definition 7.2. More precisely we will show that $\widehat{I}$ is the free $\mathcal{C}$-cocompletion, with $\mathcal{C}$ the ideal of left diagrams

$$\mathcal{C} = \{(d, J) \mid J \text{ is left composable}\}$$

where $J\colon A \nrightarrow B$ is left composable in the sense of Example 7.14: the pointwise right composites $(H \odot J)$ (Definition 2.12) exist for all $H\colon C \nrightarrow B$. In order to state the conditions consider the nullary cartesian cell $\mathrm{cart}\colon I_I \Rightarrow I$ on the left below that defines the horizontal unit I_I (Definition 1.16) and, for any pair of horizontal morphisms $J\colon A \nrightarrow I$ and $H\colon I \nrightarrow B$, the monoidal product cell $\chi_{(J,H)} := (\mathrm{id}_J \odot \mathrm{cart}) \otimes (\mathrm{cart} \odot \mathrm{id}_H)$ on the right below. Because $\mathrm{cart}\colon I_I \Rightarrow I$ is cocartesian (Lemma 1.21), so are $\mathrm{id}_J \odot \mathrm{cart}$ and $\mathrm{cart} \odot \mathrm{id}_H$ by the pasting lemma (Lemma A7.7).

$$\begin{array}{ccc} I & \xrightarrow{I_I} & I \\ & \searrow\!\!\text{cart}\!\!\swarrow & \\ & I & \end{array} \qquad\qquad \chi_{(J,H)} := \begin{array}{ccccc} A\otimes I & \xrightarrow{J\otimes I_I} & I\otimes I & \xrightarrow{I_I\otimes H} & I\otimes B \\ \| & & \Downarrow (\mathrm{id}_J \odot \mathrm{cart}) \otimes (\mathrm{cart} \odot \mathrm{id}_H) & & \| \\ A\otimes I & & \xrightarrow[J\otimes H]{} & & I\otimes B \end{array}$$

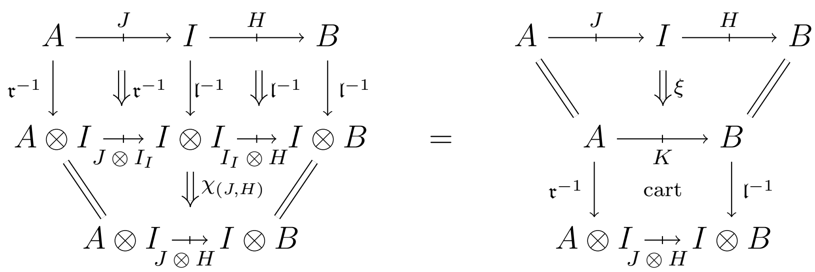
8.9. Theorem. *Let $\mathcal{K}$ and $\mathrm{y}\colon I \to \widehat{I}$ be as above. Assume that $\mathcal{K}$ has restrictions on the right as well as restrictions along vertical isomorphisms on the left (Definition 1.19). If the companion y_* exists then y defines $\widehat{I}$ as the free $\mathcal{C}$-cocompletion of I (Definition 7.2) whenever the following conditions hold for any triple of morphisms $J\colon A \nrightarrow I$, $H\colon I \nrightarrow B$ and $f\colon X \to B$:*

(a) *the cell $\chi_{(J,H)}$ on the right above is cocartesian;*

(b) *the restriction $H(\mathrm{id}, f)$ is preserved by the assignment $\mathrm{id}_J \otimes -$.*

Notice that condition (a) means that $\chi_{(J,H)}$ defines $J \otimes H$ as the horizontal composite $((J \otimes I_I) \odot (I_I \otimes H))$ (Definition 2.12).

Proof. By Example 7.14 it suffices to prove that the pointwise right composite $(J \odot H)$ exists for any pair $J\colon A \nrightarrow I$ and $H\colon I \nrightarrow B$. We will do so by showing that the unique factorisation ξ in the right-hand side below is right pointwise cocartesian (Definition A9.1), thus defining the restriction $K := (J \otimes H)(\mathfrak{r}^{-1}, \mathfrak{l}^{-1})\colon A \nrightarrow B$ as the required pointwise right composite $(J \odot H)$ by Remark 2.13. Notice that the top row of cells in the left-hand side below is composable because (in any monoidal category) $\mathfrak{r} = \mathfrak{l}\colon I \otimes I \to I$; see e.g. Section VII.1 of [ML98].



We will start by showing that ξ is cocartesian (Definition 2.6). First notice that the cartesian cell in the right-hand side above admits an inverse $\text{cart}^{-1}\colon J \otimes H \Rightarrow K$, which can be seen by factorising the identity cell $\text{id}_{J \otimes H}$ through cart. Next notice that the path of cells $(\mathfrak{r}^{-1}, \mathfrak{l}^{-1})$, that makes up the top row of the left-hand side, is cocartesian: this follows from the fact that $\mathfrak{r}^{-1}\colon J \Rightarrow J \otimes I_I$, $\mathfrak{l}^{-1}\colon H \Rightarrow I_I \otimes H$, as well as cartesian cells that define restrictions along $\mathfrak{r}^{-1}\colon A \to A \otimes I$ or $\mathfrak{l}^{-1}\colon B \to B \otimes I$, are invertible cells. Likewise cart^{-1} is cocartesian so that, if $\chi_{(J,H)}$ is cocartesian then $\xi = \text{cart}^{-1} \circ \chi_{(J,H)} \circ (\mathfrak{r}^{-1}, \mathfrak{l}^{-1})$ is so too by the pasting lemma (Lemma A7.7).

To show that ξ is in fact right pointwise cocartesian fix a vertical morphism $f\colon X \to B$ and consider the unique factorisation $\xi'\colon (J, H(\text{id}, f)) \Rightarrow K(\text{id}, f)$ in $\xi \circ (\text{id}_J, \text{cart}_{H(\text{id},f)}) = \text{cart}_{K(\text{id},f)} \circ \xi'$, as in Definition A9.1 and where the cartesian cells define the restrictions $H(\text{id}, f)$ and $K(\text{id}, f)$; we have to show that ξ' is again cocartesian. To do so consider the equality below, whose identities follow from the functoriality of $\otimes$, the naturality of the cells $\mathfrak{l}^{-1}$ (Definition A3.2), and the definitions of ξ and ξ'.

$$\begin{aligned}
&\big(\text{id}_J \otimes \text{cart}_{H(\text{id},f)}\big) \circ \chi_{(J,H(\text{id},f))} \circ \big(\mathfrak{r}^{-1}, \mathfrak{l}^{-1}\big) \\
&\qquad = \chi_{(J,H)} \circ \big(\text{id}_{J \otimes I_I}, \text{id}_{I_I} \otimes \text{cart}_{H(\text{id},f)}\big) \circ (\mathfrak{r}^{-1}, \mathfrak{l}^{-1}) \\
&\qquad = \chi_{(J,H)} \circ \big(\mathfrak{r}^{-1}, \mathfrak{l}^{-1}\big) \circ \big(\text{id}_J, \text{cart}_{H(\text{id},f)}\big) \\
&\qquad = \text{cart} \circ \xi \circ \big(\text{id}_J, \text{cart}_{H(\text{id},f)}\big) = \text{cart} \circ \text{cart}_{K(\text{id},f)} \circ \xi'
\end{aligned}$$

Now write $\text{cart}'\colon K(\text{id}, f) \Rightarrow J \otimes H(\text{id}, f)$ for the factorisation of $\text{cart} \circ \text{cart}_{K(\text{id},f)}$, in the right-hand side above, through the cartesian cell $\text{id}_J \otimes \text{cart}_{H(\text{id},f)}$; cart' is cartesian by the pasting lemma (Lemma 1.17). We thus obtain

$$\big(\text{id}_J \otimes \text{cart}_{H(\text{id},f)}\big) \circ \chi_{(J,H(\text{id},f))} \circ \big(\mathfrak{r}^{-1}, \mathfrak{l}^{-1}\big) = \big(\text{id}_J \otimes \text{cart}_{H(\text{id},f)}\big) \circ \text{cart}' \circ \xi',$$

so that $\chi_{(J,H(\text{id},f))} \circ (\mathfrak{r}^{-1}, \mathfrak{l}^{-1}) = \text{cart}' \circ \xi'$ by the uniqueness of factorisations through cartesian cells. Since cart' is cartesian the previous argument, proving the cocartesianness of ξ, applies to the latter identity so that, because $\chi_{(J,H(\text{id},f))}$ is cocartesian by assumption, ξ' is cocartesian too. This completes the proof. ■

8.10. Horizontal duals. In order to explain the notion of 'weak horizontal right dual' defined below consider the dual A° of a category A enriched in a symmetric monoidal category $\mathcal{V}'$, with hom-objects $A^\circ(x, y) = A(y, x)$ (see Section 1.4 of [Kel82]). Assuming that $\mathcal{V}'$ has large colimits preserved by $\otimes'$ on both sides, so that $\mathcal{V}'\text{-}\mathsf{Prof}$ is an equipment (Example A9.2), it is shown in Theorem 5.1 of [Shu10] that the monoidal structure on $\mathcal{V}'\text{-}\mathsf{Prof}$ restricts to a monoidal structure on the horizontal bicategory $H(\mathcal{V}'\text{-}\mathsf{Prof})$ that it contains, consisting of $\mathcal{V}'$-categories, $\mathcal{V}'$-profunctors and the horizontal cells (i.e. transformations) between them. In the monoidal bicategory $H(\mathcal{V}'\text{-}\mathsf{Prof})$ A° forms the 'right bidual' of A, in the sense of Definition 6 of [DS97], defined as such by the 'exact copairing' $\iota\colon I \nrightarrow A^\circ \otimes' A$ given by $\iota\big(*, (x, y)\big) = A(x, y)$ which, for any $\mathcal{V}'$-categories B and C, induces an equivalence of categories

$$\iota^\flat\colon H(\mathcal{V}'\text{-}\mathsf{Prof})(A \otimes' B, C) \to H(\mathcal{V}'\text{-}\mathsf{Prof})(B, A^\circ \otimes' C)$$

given by $\iota^\flat(J) \coloneqq (\iota \otimes' I_B) \odot (I_{A^\circ} \otimes' J)$ where I_B and I_{A° are unit profunctors. Using the 'Yoneda isomorphisms' (see e.g. Formula 3.71 of [Kel82]) we find that $\iota^\flat(J)$ can simply be defined as $\iota^\flat(J)\big(y, (x, z)\big) \coloneqq J\big((x, y), z\big)$. It follows that, for a symmetric universe enlargement $\mathcal{V} \subset \mathcal{V}'$ (Example 4.10), the equivalences above restrict to equivalences $H\big((\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}\big)(A \otimes' B, C) \simeq H\big((\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}\big)(B, A^\circ \otimes' C)$ of categories of $\mathcal{V}$-profunctors between $\mathcal{V}'$-categories. Taking $B = I$ and restricting A and C to be $\mathcal{V}$-categories, we obtain an equivalence between $\mathcal{V}$-profunctors $A \nrightarrow C$ and $\mathcal{V}$-profunctors $I \nrightarrow A^\circ \otimes C$. Notice that the latter (trivially) are small $\mathcal{V}$-profunctors (Example A2.8) while the former are not in general, so that these equivalences do not restrict to analogous equivalences for small $\mathcal{V}$-profunctors. They do however restrict to full and faithful functors between categories of small $\mathcal{V}$-profunctors, and we conclude that the copairing ι induces full and faithful functors $\iota^\flat\colon H(\mathcal{K})(A, C) \to H(\mathcal{K})(I, A^\circ \otimes' C)$ in each of the cases $\mathcal{K} = \mathcal{V}'\text{-}\mathsf{Prof}$, $\mathcal{K} = (\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}$ and $\mathcal{K} = \mathcal{V}\text{-}\mathsf{sProf}$. Since $\mathcal{V}'\text{-}\mathsf{Prof}$ is a pseudo double category (Example A9.2) it is straightforward to see that, in the cases $\mathcal{K} = \mathcal{V}'\text{-}\mathsf{Prof}$ and $\mathcal{K} = (\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}$, the full and faithfulness of $\iota^\flat$ is equivalent to condition (b) below. The same holds true for $\mathcal{K} = \mathcal{V}\text{-}\mathsf{sProf}$ whenever $\mathcal{V}$ is small cocomplete and $\otimes$ preserves colimits on both sides, so that $\mathcal{V}\text{-}\mathsf{sProf}$ is a pseudo double category (Example A9.3).

8.11. Definition. *Let A be an object of a monoidal augmented virtual double category $\mathcal{K} = (\mathcal{K}, \otimes, I)$. A* weak horizontal right dual *of A (shortly* weak horizontal dual *of A) is a unital object A° equipped with a horizontal* copairing *$\iota\colon I \nrightarrow A^\circ \otimes A$ satisfying the following conditions:*

(a) *for each $J\colon A \nrightarrow B$ the pointwise right composite (Definition 2.12) below exists, called the* adjunct *of J;*

$$J^\flat \coloneqq \iota \odot (A^\circ \otimes J)\colon I \nrightarrow A^\circ \otimes B$$

(b) *the assignments below, between collections of horizontal cells in $\mathcal{K}$ as shown and where the cell* cocart *defines $J^\flat$, is a bijection.*

$$\left\{\begin{array}{ccc} A & \xrightarrow{H} & B \\ \| & \Downarrow & \| \\ A & \xrightarrow[J]{} & B \end{array}\right\} \xrightarrow{\text{cocart} \circ (\text{id}_\iota, A^\circ \otimes -)} \left\{\begin{array}{ccccc} I & \xrightarrow{\iota} & A^\circ \otimes A & \xrightarrow{A^\circ \otimes H} & A^\circ \otimes B \\ & \diagdown\!\diagdown & \Downarrow & /\!/ & \\ & I & \xrightarrow[J^\flat]{} & A^\circ \otimes B & \end{array}\right\}$$

If moreover for every $K\colon I \nrightarrow A^\circ \otimes B$ in $\mathcal{K}$ there exists a morphism $J\colon A \nrightarrow B$ with $J^\flat \cong K$ then we call A° the horizontal dual *of A.*

8.12. Example. The horizontal dual A° of an internal preorder $A = (A, \alpha)$ in $\mathsf{ModRel}(\mathcal{E})$ (Example 8.7) is $A^\circ := (A, \alpha^\circ)$, where $\alpha^\circ = (A \xleftarrow{\alpha_1} \alpha \xrightarrow{\alpha_0} A)$ is the reverse of α. Its horizontal copairing $\iota\colon 1 \nrightarrow A^\circ \times A$ is the modular subobject $\alpha \xrightarrow{(\alpha_0, \alpha_1)} A \times A$ (Example 4.15), whose right action is induced by the multiplication $\bar{\alpha}\colon (\alpha, \alpha) \Rightarrow \alpha$ of A; similarly the adjunct $J^\flat\colon 1 \nrightarrow A^\circ \times B$ of an internal modular relation $J\colon A \nrightarrow B$ is J itself considered as the modular subobject $J^\flat := \big[J \xrightarrow{(j_0, j_1)} A \times B\big]$. Indeed consider the cell $(\iota, A^\circ \times J) \Rightarrow J^\flat$ induced by the action of A on J. It is a split epimorphism whose section is induced by the unit cell $\tilde{\alpha}\colon A \Rightarrow \alpha$ of A, so that it is (pointwise) cocartesian in $\mathsf{Rel}(\mathcal{E})$ (Example A7.4) and hence pointwise right cocartesian by Example 1.31 and Remark 2.13; this proves condition (a) above. Recall that $\mathsf{ModRel}(\mathcal{E})$ is locally thin (Example 1.31) so that, to prove (b), it suffices to show that the assignments above are surjective. But that follows immediately from the fact that there exists a cell $H_1 \odot \cdots \odot H_n \Rightarrow \iota \odot (A^\circ \times H_1) \odot \cdots \odot (A^\circ \times H_n)$ in $\mathsf{Span}(\mathcal{E})$, induced by the unit cell $\tilde{\alpha}$, combined with the fact that the forgetful functor $U\colon \mathsf{ModRel}(\mathcal{E}) \to \mathsf{Span}(\mathcal{E})$ is locally full and faithful.

8.13. Internal homs. If besides a weak horizontal dual A° the Yoneda embedding $\mathrm{y}\colon I \to \widehat{I}$ for the monoidal unit I exists then we can compose the assignment $(-)^\flat$ of Definition 8.11 with the assignment $(-)^\lambda$ given by the Yoneda axiom (Definition 4.5), thus obtaining a composite assignment of morphisms of the form

$$\{A \nrightarrow B\} \xrightarrow{(-)^\flat} \{I \nrightarrow A^\circ \otimes B\} \xrightarrow{(-)^\lambda} \{A^\circ \otimes B \to \widehat{I}\}$$

which, using Proposition 4.24, is essentially surjective if and only if A° is a horizontal dual and y admits nullary restrictions (Definition 4.5). In the definition below we define to be 'ι-small' those morphisms $A^\circ \otimes B \to \widehat{I}$ in its essential image, besides defining the internal hom for such morphisms. Alternative conditions equivalent to that of ι-smallness are given in Lemma 8.30 below.

8.14. Definition. *Let $\mathcal{K} = (\mathcal{K}, \otimes, I)$ be a monoidal augmented virtual double category. Assume that the Yoneda embedding $\mathrm{y}\colon I \to \widehat{I}$ exists and let A be any object equipped with a weak horizontal dual A°, defined by a copairing $\iota\colon I \nrightarrow A^\circ \otimes A$.*

- *A morphism $f\colon A^\circ \otimes B \to \widehat{I}$ is called* ι-small *if there exists a morphism $J\colon A \nrightarrow B$ such that $f \cong (J^\flat)^\lambda$.*

- *We denote by* $(A^\circ \otimes - /_{\mathrm{v}} \widehat{I})_\iota \subseteq A^\circ \otimes - /_{\mathrm{v}} \widehat{I}$ *(Definition 1.39) the full subcategory generated by all* ι*-small morphisms* $A^\circ \otimes B \to \widehat{I}$.
- *The* internal ι-small hom $[A^\circ, \widehat{I}]_\iota$*, if it exists, is an object* $[A^\circ, \widehat{I}]_\iota$ *of* $\mathcal{K}$ *equipped with a universal morphism* $\mathrm{ev}\colon A^\circ \otimes [A^\circ, \widehat{I}]_\iota \to \widehat{I}$ *from* $A^\circ \otimes -$ *to* $\widehat{I}$ *relative to* $(A^\circ \otimes - /_{\mathrm{v}} \widehat{I})_\iota$ *(Definition 1.39);* ev *is called* evaluation*.*

The next result follows from the discussion preceding the definition; it can also be obtained by instantiating the functor F of Proposition 8.31 below by $A^\circ \otimes -$.

8.15. Proposition. *Let* $\mathrm{y}\colon I \to \widehat{I}$ *and* $\iota\colon I \nrightarrow A^\circ \otimes A$ *be as above. Every morphism of the form* $A^\circ \otimes B \to \widehat{I}$ *is* ι*-small if and only if* A° *is the horizontal dual of* A *(Definition 8.11) and* y *admits nullary restrictions (Definition 4.5).*

Assume that every morphism of the form $A^\circ \otimes B \to \widehat{I}$ is ι-small, that is we have $(A^\circ \otimes - /_{\mathrm{v}} \widehat{I})_\iota = A^\circ \otimes - /_{\mathrm{v}} \widehat{I}$ in Definition 8.14, and that the functor $A^\circ \otimes -\colon \mathcal{K} \to \mathcal{K}$ admits a right adjoint $[A^\circ, -]$. It then follows from Example 1.41 that the evaluation $\varepsilon_{\widehat{I}}\colon A^\circ \otimes [A^\circ, \widehat{I}] \to \widehat{I}$ defines $[A^\circ, \widehat{I}]$ as the internal ι-small hom $[A^\circ, \widehat{I}]_\iota$. We are thus interested in monoidal augmented virtual double categories that are 'closed' in the following sense.

8.16. Definition. *Let* $\mathcal{K} = (\mathcal{K}, \otimes, I)$ *be a (cartesian) monoidal augmented virtual double category (Definition 8.2). We call* $\mathcal{K}$ closed (cartesian) monoidal *if, for every unital object* $A \in \mathcal{K}$*, the functor* $A \otimes -\colon \mathcal{K} \to \mathcal{K}$ *admits a right adjoint* $[A, -]$ *in the 2-category* $\mathsf{AugVirtDblCat}$ *(Section A3).*

8.17. Example. Let $\mathcal{K}$ be a cartesian pseudo double category (Example 8.6). Using that the embedding $N\colon \mathsf{DblCat}_{\mathrm{nl}} \to \mathsf{AugVirtDblCat}$ of 2-categories of Example 8.5 is full and faithful, it follows that $N(\mathcal{K})$ is closed cartesian monoidal in the above sense if and only if $\mathcal{K}$ is *precartesian closed* in the sense of Definition 4.2 of [Nie20] such that, for each $A \in \mathcal{K}$, the right adjoint $(-)^A$ to $- \times A\colon \mathcal{K} \to \mathcal{K}$ is a normal lax functor (see e.g. Definition 6.1 of [Shu08]).

8.18. Example. Let $\mathcal{V}' = (\mathcal{V}', \otimes', I')$ be a closed symmetric monoidal category that is large complete and let A be any (large) $\mathcal{V}'$-category. Recall from Sections 2.2 and 2.3 of [Kel82] that, restricted to the 2-category $\mathcal{V}'\text{-}\mathsf{Cat} = V(\mathcal{V}'\text{-}\mathsf{Prof})$ of $\mathcal{V}'$-categories, the endo-2-functor $A \otimes' -$ (Example 8.3) admits a right adjoint $[A, -]'\colon \mathcal{V}'\text{-}\mathsf{Cat} \to \mathcal{V}'\text{-}\mathsf{Cat}$, whose image $[A, C]'$ of a $\mathcal{V}'$-category C is the $\mathcal{V}'$-category of $\mathcal{V}'$-functors $p\colon A \to C$.

It is straightforward to check that the adjoint functor pairs $A \otimes' - \dashv [A, -]'$ on $\mathcal{V}'\text{-}\mathsf{Cat}$ extend to adjoint functor pairs $A \otimes' - \dashv [A, -]'$ on $\mathcal{V}'\text{-}\mathsf{Prof}$, thus making $\mathcal{V}'\text{-}\mathsf{Prof}$ into a closed monoidal unital virtual equipment. The extension of $[A, -]'$ maps a $\mathcal{V}'$-profunctor $K\colon C \nrightarrow D$ to the $\mathcal{V}'$-profunctor $[A, K]'\colon [A, C]' \nrightarrow [A, D]'$ given by the ends

$$[A, K]'(p, q) \coloneqq \int_{x \in A} K(px, qx) \qquad\qquad (p \in [A, C]', q \in [A, D]'),$$

where K is regarded as a $\mathcal{V}'$-profunctor $K\colon C^{\mathrm{op}} \otimes' D \to \mathcal{V}'$; these ends exist because $\mathcal{V}'$ is assumed to be large complete. Notice that if $K = I_C$ is a unit $\mathcal{V}'$-profunctor then $[A, I_C]'$ consists of the hom objects of the functor $\mathcal{V}'$-category $[A, C]'$.

Taking $\mathcal{V}' = \mathsf{Set}'$ in previous we recover Corollary 4.8 of [Nie20], which proves that the pseudo double category Set'-Prof (Example A2.4) of Set'-profunctors between (locally large) categories is precartesian closed; see the previous example.

8.19. Example. Let $\mathcal{E}$ be a cartesian closed category with pullbacks. As is shown in Section 2 of [CS86], the locally thin 2-category $\mathsf{PreOrd}(\mathcal{E}) = V\big(\mathsf{ModRel}(\mathcal{E})\big)$ of internal preorders in $\mathcal{E}$ (Example 1.31) inherits a closed cartesian monoidal structure from that of $\mathcal{E}$ as follows. The exponential $[A, C]$ of internal preorders $A = (A, \alpha)$ and $C = (C, \gamma)$ is constructed as the pullback on the left below, and its internal ordering is obtained by pulling the internal ordering $(C^A \xleftarrow{\gamma_0^A} \gamma^A \xrightarrow{\gamma_1^A} C^A)$ on C^A back along $[A, C] \times [A, C] \rightarrowtail C^A \times C^A$. Notice that $[A, C]$ is an internal partial order (Example 1.31) whenever C is so and that $[A, C] \cong C^A$ whenever $A = (A, I_A)$ is discrete. Underlying the order preserving evaluation $\varepsilon\colon A \times [A, C] \to C$ in $\mathsf{PreOrd}(\mathcal{E})$ is the composite $A \times [A, C] \rightarrowtail A \times C^A \xrightarrow{\varepsilon} C$ in $\mathcal{E}$.

$$\begin{array}{ccc} [A, C] & \rightarrowtail & C^A \\ \downarrow & & \downarrow{\scriptstyle (C^{\alpha_0}, C^{\alpha_1})} \\ \gamma^\alpha & \xrightarrow[(\gamma_0^\alpha, \gamma_1^\alpha)]{} & C^\alpha \times C^\alpha \end{array} \qquad\qquad \begin{array}{ccc} [A, K] & \rightarrowtail & K^A \\ {\scriptstyle ([A,K]_0, [A,K]_1)}\downarrow & & \downarrow{\scriptstyle (k_0^A, k_1^A)} \\ [A, C] \times [A, D] & \rightarrowtail & C^A \times D^A \end{array}$$

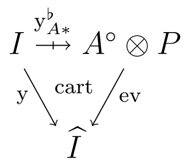
It is straightforward to show that the adjoint functor pairs $A \times - \dashv [A, -]$ on $\mathsf{PreOrd}(\mathcal{E})$ extend to adjoint functor pairs $A \times - \dashv [A, -]$ on $\mathsf{ModRel}(\mathcal{E})$, thus making $\mathsf{ModRel}(\mathcal{E})$ into a closed cartesian monoidal unital virtual equipment. The image $[A, K]\colon [A, C] \nrightarrow [A, D]$ of an internal modular relation $K\colon C \nrightarrow D$ is given by the left morphism in the pullback square on the right above.

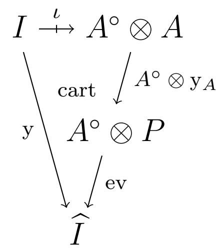
8.20. Yoneda embeddings and internal homs. The following theorem is the main result of this section. As promised in its introduction, for an object A in a monoidal augmented virtual double category $\mathcal{K} = (\mathcal{K}, \otimes, I)$ it relates the existence of a Yoneda embedding $\mathrm{y}_A\colon A \to \widehat{A}$ to that of the internal ι-small hom $[A^\circ, \widehat{I}]_\iota$.

8.21. Theorem. *In a monoidal augmented virtual double category $\mathcal{K} = (\mathcal{K}, \otimes, I)$ that has restrictions on the right (Definition 1.19) let $\mathrm{y}\colon I \to \widehat{I}$ be a Yoneda embedding (Definition 4.5). Let A be a unital object (Definition 1.16) whose weak horizontal dual A° exists, with copairing $\iota\colon I \nrightarrow A^\circ \otimes A$ (Definition 8.11), such that the functor $A^\circ \otimes -\colon \mathcal{K} \to \mathcal{K}$ preserves restrictions on the right. Consider any object P equipped with morphisms $\mathrm{y}_A\colon A \to P$ and $\mathrm{ev}\colon A^\circ \otimes P \to \widehat{I}$. If the companion $\mathrm{y}_{A*}\colon A \nrightarrow P$ exists then the following are equivalent:*

(ye) *$\mathrm{y}_A\colon A \to P$ is a Yoneda embedding that defines P as the object of presheaves on A (Definition 4.5) and there exists a cartesian cell as on the left below;*

(ih) *$\mathrm{ev}\colon A^\circ \otimes P \to \widehat{I}$ is the evaluation that defines P as the internal ι-small hom $[A^\circ, \widehat{I}]_\iota$ (Definition 8.14) and there exists a cartesian cell as on the right below.*





For reference we summarise the main definitions used in the proof of the implication (ih) $\Rightarrow$ (ye) above; for further details apply Definition 8.35 below to the endofunctor $F \coloneqq A^\circ \otimes -$. Let $\mathcal{K}$, $\mathrm{y}\colon I \to \widehat{I}$, A and $\iota\colon I \nrightarrow A^\circ \otimes A$ be as in the theorem, and let $\mathrm{ev}\colon A^\circ \otimes [A^\circ, \widehat{I}]_\iota \to \widehat{I}$ be the evaluation defining the ι-small hom $[A^\circ, \widehat{I}]_\iota$ (Definition 8.14). Like any functor $A^\circ \otimes -$ preserves the horizontal unit I_A (Corollary A5.5), that is $A^\circ \otimes I_A \cong I_{A^\circ \otimes A}$. It follows that $I_A^\flat \cong \iota$ so that $(I_A^\flat)^\lambda \cong \iota^\lambda\colon A^\circ \otimes A \to \widehat{I}$ by the functoriality of $(-)^\lambda$ (Proposition 4.24); hence ι^λ is ι-small. Using the universality of the evaluation we obtain a morphism $\mathrm{y}_A\colon A \to [A^\circ, \widehat{I}]_\iota$ such that $\mathrm{ev} \circ (A^\circ \otimes \mathrm{y}_A) \cong \iota^\lambda$. Composing the latter isomorphism with the cartesian cell defining ι^λ (Definition 4.5) we obtain a nullary cartesian cell $\iota \Rightarrow \widehat{I}$ as on the right above, so that condition (ih) above is satisfied. Next assume that the companion y_{A*} exists so that we can apply the theorem; we conclude that y_A forms a Yoneda embedding. The existence of y_{A*} also means that y_A has nullary restrictions (Definition 4.5) so that it induces an equivalence between horizontal morphisms $A \nrightarrow B$ and vertical morphisms $B \to [A^\circ, \widehat{I}]_\iota$ (Proposition 4.24). Under this equivalence a horizontal morphism $J\colon A \nrightarrow B$ corresponds to the vertical morphism $J^\lambda \coloneqq \big((J^\flat)^\lambda\big)^\sharp\colon B \to [A^\circ, \widehat{I}]_\iota$, that is $(J^\flat)^\lambda \cong \mathrm{ev} \circ \big(A^\circ \otimes J^\lambda\big)$ where $(J^\flat)^\lambda\colon A^\circ \otimes B \to \widehat{I}$ is induced by $J^\flat\colon I \nrightarrow A^\circ \otimes B$, using the Yoneda axiom for $\mathrm{y}\colon I \to \widehat{I}$ (Definition 4.5).

Proof of Theorem 8.21. As described in the introduction to this section, instead of giving a direct proof we will prove a generalisation of the present theorem in Theorems 8.33 and 8.36 below. Applied to $F = A^\circ \otimes -\colon \mathcal{K} \to \mathcal{K}$ the latter imply the equivalence (ye) $\Leftrightarrow$ (ih) above as follows. Applying Lemma 8.28 to the copairing $\iota\colon I \nrightarrow A^\circ \otimes A$ shows that ι is a locally universal morphism from I to the endofunctor $F \coloneqq A^\circ \otimes -$, in the sense of Definition 8.26 below. Assuming (ye), so that $\mathrm{y}_A\colon A \to P$ is a Yoneda embedding, we can use Theorem 8.33 to obtain an evaluation $\varepsilon\colon FP \to \widehat{I}$ that defines P as the internal ι-small hom $[A^\circ, \widehat{I}]_\iota$. From the definition of ε, the existence of the cartesian cell on the left above and the uniqueness of the cartesian cells provided by the Yoneda axiom (Definition 4.5), we conclude that $\varepsilon \cong \mathrm{ev}$. Using that A is assumed to be unital, Theorem 8.33 also supplies the cartesian cell on the right above. This proves (ye) $\Rightarrow$ (ih). For the converse (ih) $\Rightarrow$ (ye) notice that, similarly to the previous argument, the existence of the cartesian cell on the right above implies that $\mathrm{ev} \circ (A^\circ \otimes \mathrm{y}_A) \cong \iota^\lambda$. It follows that we can take $(\iota^\lambda)^\sharp \coloneqq \mathrm{y}_A$ in Definition 8.35, so that the (ih) ⇒ (ye) follows from Theorem 8.36. ∎

8.22. Remark. The theorem above is reminiscent of Weber's construction of a good Yoneda structure on any 2-topos $\mathcal{C} = (\mathcal{C}, (-)^\circ, \tau)$, as given in Section 5 of [Web07]; see also Example 4.30 above. Like the Yoneda embeddings $\mathrm{y}_A\colon A \to [A^\circ, \widehat{I}]_\iota$ obtained above, the Yoneda embeddings $y_A\colon A \to [A^\circ, \Omega]$ of Weber's Yoneda structure on $\mathcal{C}$ also map into an inner hom-object, where Ω is the target of the classifying discrete opfibration τ. Another similarity is the importance of the equivalence of the discrete two-sided fibrations $A \nrightarrow B$ in $\mathcal{C}$ and those of the form $1 \nrightarrow A^\circ \times B$, analogous to our notion of horizontal dual (Definition 8.11).

Differences include our construction requiring the monoidal unit I to be unital, while Weber's construction applies regardless of whether the cartesian unit 1 is admissible in $\mathcal{C}$; see Example 8.3 of [Web07]. Weber's construction applies to 2-topoi only; unlike our approach it cannot, for instance, be used to obtain enriched Yoneda embeddings (see Example 4.36 and Example 8.23). Our result, finally, generalises in the form of Theorems 8.33 and 8.36 below, as described in the introduction to this section.

8.23. Example. Let $\mathcal{V} \subset \mathcal{V}'$ be a symmetric universe enlargement (Example 4.10) and consider the Yoneda embedding $\mathrm{y}\colon I \to \mathcal{V}$ (Example 4.9) in the monoidal augmented virtual equipment $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ (Example 8.3). Let A be a $\mathcal{V}$-category and let the horizontal dual $A^\circ \coloneqq A^{\mathrm{op}}$ and the copairing $\iota\colon I \nrightarrow A^{\mathrm{op}} \otimes' A$ in $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ be as defined preceding Definition 8.11. Recall that we can take the image $J^\flat\colon I \nrightarrow A^{\mathrm{op}} \otimes' B$ of a $\mathcal{V}$-profunctor $J\colon A \nrightarrow B$, under the assignment $J \mapsto J^\flat$ of Definition 8.11, to be given by $J^\flat\big(*, (x, y)\big) \coloneqq J(x, y)$ for all $x \in A$ and $y \in B$. By Proposition 8.15 any $\mathcal{V}'$-functor $A^{\mathrm{op}} \otimes' B \to \mathcal{V}$ is ι-small (Definition 8.14).

Consider the adjunction $A^{\mathrm{op}} \otimes' - \dashv [A^{\mathrm{op}}, -]'$ of endofunctors on $\mathcal{V}'$-Prof of Example 8.18 and notice that, because A is a $\mathcal{V}$-category, $A^{\mathrm{op}} \otimes' -$ restricts to an endofunctor on $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$. As discussed before Definition 8.16 the component $\varepsilon_{\mathcal{V}}\colon A^{\mathrm{op}} \otimes' [A^{\mathrm{op}}, \mathcal{V}]' \to \mathcal{V}$ of the evaluation defines $[A^{\mathrm{op}}, \mathcal{V}]'$ as the internal ι-small hom in $\mathcal{V}'$-Prof. Moreover, while $[A^{\mathrm{op}}, -]'$ is unlikely to restrict along the embedding $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof} \subset \mathcal{V}'$-$\mathsf{Prof}$, the fact that this embedding is full (Definition A3.6) implies that $\varepsilon_{\mathcal{V}}$ is universal too from $A^{\mathrm{op}} \otimes' -$, as an endofunctor on $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$, to $\mathcal{V}$, which follows easily from the unpacking of Definition 1.39. We conclude that $\mathrm{ev} \coloneqq \varepsilon_{\mathcal{V}}\colon A^{\mathrm{op}} \otimes' [A^{\mathrm{op}}, \mathcal{V}]' \to \mathcal{V}$ defines the functor-$\mathcal{V}'$-category $[A^{\mathrm{op}}, \mathcal{V}]'$ as an internal ι-small hom in $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ too, in the sense of Definition 8.14.

The universality of ev induces a $\mathcal{V}'$-functor $\mathrm{y}_A\colon A \to [A^{\mathrm{op}}, \mathcal{V}]'$ as described after the statement of Theorem 8.21. Using that $\iota^\lambda\colon A^{\mathrm{op}} \otimes' A \to \mathcal{V}$ is given by $\iota^\lambda(x, y) = A(x, y)$, and that $\mathrm{y}_A \coloneqq (\iota^\lambda)^\sharp = [A^{\mathrm{op}}, \iota^\lambda]' \circ \mathrm{coev}_A$ by Example 1.41, where $\mathrm{coev}_A\colon A \to [A^{\mathrm{op}}, A^{\mathrm{op}} \otimes' A]'$ is the coevaluation, it follows that $\mathrm{y}_A\colon A \to [A^{\mathrm{op}}, \mathcal{V}]'$ recovers the classical enriched Yoneda embedding given by $\mathrm{y}_A x = A(-, x)$; see e.g. Section 2.4 of [Kel82].

Finally notice that the companion $\mathrm{y}_{A*}\colon A \nrightarrow [A^{\mathrm{op}}, \mathcal{V}]'$ exists in $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$: this follows from the well known isomorphisms $\mathrm{y}_*(x, p) = [A^{\mathrm{op}}, \mathcal{V}]'(\mathrm{y}_A x, p) \cong px \in \mathcal{V}$, for any $x \in A$ and $p \in [A^{\mathrm{op}}, \mathcal{V}]'$, that are given by the 'strong Yoneda lemma'; see e.g.

Formula 2.31 of [Kel82]. Applying Theorem 8.21 we conclude that, for each $\mathcal{V}$-category A, the classical enriched Yoneda embedding $\mathrm{y}_A\colon A \to [A^{\mathrm{op}}, \mathcal{V}]'$ forms a Yoneda embedding that admits nullary restrictions in the augmented virtual equipment $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$, in the sense of Definition 4.5.

8.24. Example. Let $\mathcal{E}$ be a cartesian closed category with finite limits. Let $\mathrm{y}\colon 1 \to \widehat{1}$ be a Yoneda embedding in $\mathsf{ModRel}(\mathcal{E})$ (Example 4.15) and let $A = (A, \alpha)$ be any internal preorder in $\mathcal{E}$. The modular subobject α of $A^\circ \times A$, that defines the horizontal dual A° (Example 8.12), corresponds to an order preserving morphism $A^\circ \times A \to \widehat{1}$ (Example 4.15) which, under the closed cartesian monoidal structure on $\mathsf{ModRel}(\mathcal{E})$ (Example 8.19), corresponds to an order preserving morphism $\mathrm{y}_A\colon A \to [A^\circ, \widehat{1}]$. That condition (ih) of the theorem above holds follows from Proposition 8.15 and the discussion following it, so that y_A forms a Yoneda embedding in $\mathsf{ModRel}(\mathcal{E})$ by condition (ye).

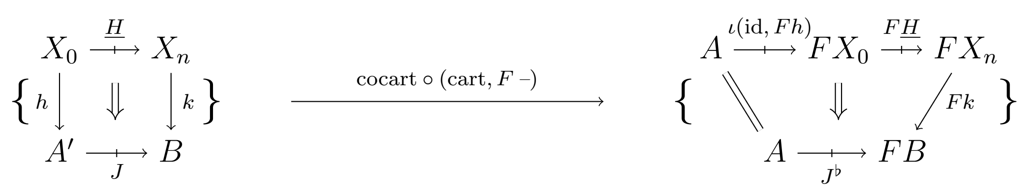
8.25. Universal horizontal morphisms. In the remainder of this section we state and prove a generalisation of Theorem 8.21, as described in the introduction to this section. We start by generalising the notion of horizontal dual (Definition 8.11) as a horizontal variant of the notion of locally universal vertical morphism (Definition 1.39).

8.26. Definition. *Let $F\colon \mathcal{K} \to \mathcal{L}$ be a functor of augmented virtual double categories (Definition A3.1) and let $A \in \mathcal{L}$ be an object. We call a morphism $\iota\colon A \nrightarrow FA'$* locally universal from A to F *if the following conditions are satisfied:*

(a) *for each $J\colon A' \nrightarrow B$ the pointwise right composite (Definition 2.12) below exists, called the* adjunct *of J;*
$$J^\flat \coloneqq \iota \odot FJ\colon A \nrightarrow FB$$

(b) *for each $h\colon X \to A'$ the restriction $\iota(\mathrm{id}, Fh)$ exists;*

(c) *the assigments below, between collections of cells in $\mathcal{K}$ and cells in $\mathcal{L}$ as shown and where the cells* cocart *and* cart *define $J^\flat$ and $\iota(\mathrm{id}, Fh)$, are bijections.*



If moreover every $H\colon A \nrightarrow FB$ in $\mathcal{L}$ admits a morphism $J\colon A' \nrightarrow B$ in $\mathcal{K}$ with $J^\flat \cong H$ then we say that $\iota\colon A \nrightarrow FA'$ is universal from A to F.

The assignment $J \mapsto J^\flat$ preserves restrictions as follows.

8.27. Lemma. *Consider $J\colon A' \nrightarrow B$ and $k\colon Y \to B$ such that the restrictions $J(\mathrm{id}, k)$ and $J^\flat(\mathrm{id}, Fk)$ exist. If $J(\mathrm{id}, k)$ is preserved by F then $\big(J(\mathrm{id}, k)\big)^\flat \cong J^\flat(\mathrm{id}, Fk)$.*

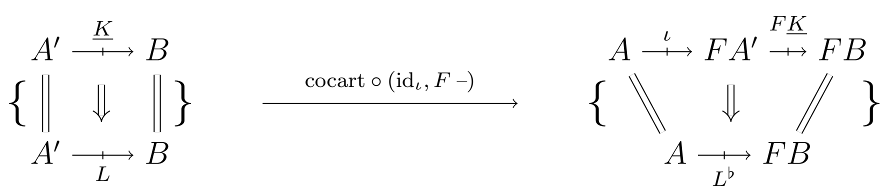
PROOF. Write $\mathrm{cart}_{J(\mathrm{id},k)}$, $\mathrm{cart}_{J^\flat(\mathrm{id},Fk)}$ and $\mathrm{cocart}_{J^\flat}$ for the (co)cartesian cells defining $J(\mathrm{id},k)$, $J^\flat(\mathrm{id},Fk)$ and $J^\flat$. Since $J^\flat$ is a pointwise right composite, by Definition 2.12 there exists a cocartesian cell $\mathrm{cocart}_{J^\flat(\mathrm{id},Fk)} \colon (\iota, FJ(\mathrm{id},k)) \Rightarrow J^\flat(\mathrm{id},Fk)$ unique such that

$$\mathrm{cocart}_{J^\flat} \circ (\mathrm{id}_\iota, F\mathrm{cart}_{J(\mathrm{id},k)}) = \mathrm{cart}_{J^\flat(\mathrm{id},Fk)} \circ \mathrm{cocart}_{J^\flat(\mathrm{id},Fk)},$$

so that the assertion follows from the uniqueness of horizontal composites. ∎

Under mild conditions condition (c) above simplifies as follows; compare the condition satisfied by a locally universal vertical morphism (Definition 1.39).

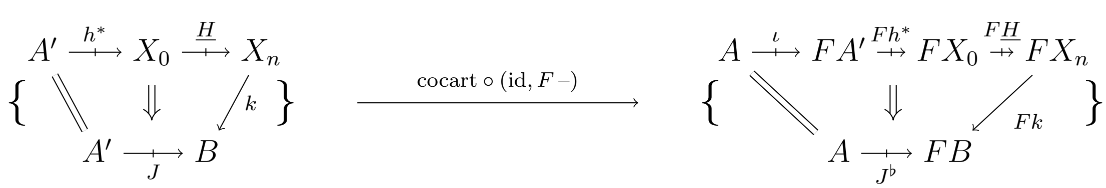
8.28. LEMMA. *If A' is unital and both $\mathcal{K}$ and $\mathcal{L}$ admit restrictions on the right, with those of $\mathcal{K}$ preserved by F, then condition* (c) *of Definition 8.26 simplifies to the assignments below, between collections of horizontal cells in $\mathcal{K}$ and in $\mathcal{L}$ and where* cocart *defines $L^\flat$, being bijections.*



PROOF. Since A' is unital and $\mathcal{K}$ has restrictions on the right the conjoint h^* of any morphism $h \colon X_0 \to A'$ exists by Corollary A4.16; we write cart_{h^*} for the defining cartesian cell. By Corollary A5.5 the functor F preserves cart_{h^*} and, because $\iota(\mathrm{id}, Fh)$ exists, by Lemma A8.1 there exists a cocartesian horizontal cell $\mathrm{cocart}_{\iota(\mathrm{id},Fh)} \colon (\iota, Fh^*) \Rightarrow \iota(\mathrm{id}, Fh)$ such that

$$\mathrm{id}_\iota \odot F\mathrm{cart}_{h^*} = \mathrm{cart}_{\iota(\mathrm{id},Fh)} \circ \mathrm{cocart}_{\iota(\mathrm{id},Fh)},$$

where $\mathrm{cart}_{\iota(\mathrm{id},Fh)}$ defines $\iota(\mathrm{id}, Fh)$. It follows that postcomposing the assignment of condition (c) of Definition 8.26 with the assignment $- \circ (\mathrm{cocart}_{\iota(\mathrm{id},Fh)}, \mathrm{id}, \dots, \mathrm{id})$ coincides with precomposing the assignment below with $\mathrm{cart}_{h^*} \odot -$. Since the assignments given by $\mathrm{cocart}_{\iota(\mathrm{id},Fh)}$ and cart_{h^*} are bijections, by Definition 2.6 and by using the conjoint identities for h^* (Lemma 1.21) respectively, it follows that condition (c) is equivalent to the assignments below being bijections.



Next it follows from the identity derived in the proof of the previous lemma that precomposing the assignment above with $\mathrm{cart}_{J(\mathrm{id},k)} \circ -$ coincides with postcomposing the assignment below, between horizontal cells in $\mathcal{K}$ and in $\mathcal{L}$, with the assignment

$\mathrm{cart}_{J^\flat(\mathrm{id},Fk)} \circ -$. Since vertical composition with cartesian cells gives bijective assignments we conclude that condition (c) is equivalent to the assignments below being bijections.

$$\mathrm{cocart}_{J^\flat(\mathrm{id},Fk)} \circ (\mathrm{id}, F-)\colon \big\{h^* \frown \underline{H} \Rightarrow J(\mathrm{id},k)\big\} \to \big\{(\iota, Fh^*) \frown F\underline{H} \Rightarrow J^\flat(\mathrm{id},Fk)\big\}$$

To complete the proof notice that, under the isomorphism $J^\flat(\mathrm{id},Fk) \cong \big(J(\mathrm{id},k)\big)^\flat$, the assignments above are of the form as in the statement, with $\underline{K} = h^* \frown \underline{H}$ and $L = J(\mathrm{id},k)$. ∎

8.29. Definition. *Let $F\colon \mathcal{K} \to \mathcal{L}$ be a functor between augmented virtual double categories (Definition A3.1) and let $\iota\colon A \nrightarrow FA'$ be a morphism in $\mathcal{L}$ that satisfies conditions (a) and (b) of Definition 8.26. Given a Yoneda morphism $\mathrm{y}_A\colon A \to \widehat{A}$ (Definition 4.5) we make the following definitions.*

- *A morphism $f\colon FB \to \widehat{A}$ in $\mathcal{L}$ is called* ι-small *if there exists $J\colon A' \nrightarrow B$ in $\mathcal{K}$ such that f is the left Kan extension (Definition 1.9) of y_A along $J^\flat\colon A \nrightarrow FB$.*
- *We denote by $(F /_{\mathrm{v}} \widehat{A})_\iota \subseteq F /_{\mathrm{v}} \widehat{A}$ (Definition 1.39) the full subcategory generated by all ι-small morphisms $FB \to \widehat{A}$.*

Theorem 8.33 below shows that, under mild conditions, the existence of a Yoneda morphism $\mathrm{y}_{A'}\colon A' \to \widehat{A'}$ induces a universal morphism $\varepsilon\colon F\widehat{A'} \to \widehat{A}$ from F to $\widehat{A}$ relative to $(F /_{\mathrm{v}} \widehat{A})_\iota$ (Definition 1.39). The lemma below lists conditions equivalent to ι-smallness: notice that the original notion above is condition (c) below for some $J\colon A' \nrightarrow B$. In Proposition 8.31 we use this to relate the universality of ι (Definition 8.26) to the ι-smallness of all morphisms in $\mathcal{L}$ of the form $FB \to \widehat{A}$.

8.30. Lemma. *Let $F\colon \mathcal{K} \to \mathcal{L}$, $\iota\colon A \nrightarrow FA'$ and $\mathrm{y}_A\colon A \to \widehat{A}$ be as in Definition 8.29. The following conditions are equivalent for morphisms $f\colon FB \to \widehat{A}$ in $\mathcal{L}$ and $J\colon A' \nrightarrow B$ in $\mathcal{K}$ and, in that case, the left Kan extensions of conditions* (c), (d) *and* (e) *are pointwise (Definition 1.24).*

(a) *$f \cong (J^\flat)^\lambda$ (Definition 4.5);*

(b) *$J^\flat\colon A \nrightarrow FB$ (Definition 8.26) is the restriction of $\widehat{A}$ along y_A and f;*

(c) *f is the left Kan extension of y_A along $J^\flat\colon A \nrightarrow FB$;*

(d) *f is the left Kan extension of y_A along (ι, FJ);*

(e) *f is the left Kan extension of $\iota^\lambda\colon FA' \to \widehat{A}$ along FJ.*

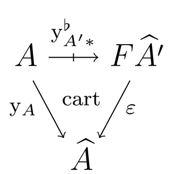
PROOF. (a) $\Leftrightarrow$ (c) follows from the fact that $(J^\flat)^\lambda$ is the left Kan extension of y_A along $J^\flat$, by the density of y_A (Definition 4.3), and the uniqueness of left Kan extensions. (b) $\Leftrightarrow$ (c) follows from Lemma 4.20. (c) $\Leftrightarrow$ (d) follows from applying the vertical pasting lemma (Lemma 2.17) to the pointwise right cocartesian cell $(\iota, FJ) \Rightarrow J^\flat$ and (d) $\Leftrightarrow$ (e) follows from applying the horizontal pasting lemma (Lemma 2.2) to the cartesian cell that defines ι^λ (Definition 4.5), which is left Kan by density of y_A. Finally notice that $f \cong (J^\flat)^\lambda$ is the pointwise left Kan extension of y_A along $J^\flat$ by density of y_A so that, because $J^\flat$ is defined by a pointwise right cocartesian cell and by the vertical and horizontal pasting lemmas, the previous arguments in fact obtain f as a pointwise left Kan extension in each of the conditions (c), (d) and (e). ∎

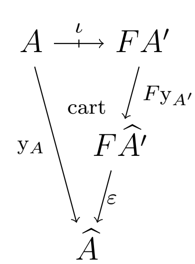
8.31. PROPOSITION. *Let $F\colon \mathcal{K} \to \mathcal{L}$, $\iota\colon A \nrightarrow FA'$ and $\mathrm{y}_A\colon A \to \widehat{A}$ be as in Definition 8.29. Every morphism in $\mathcal{L}$ of the form $FB \to \widehat{A}$ is ι-small if and only if ι is a universal morphism from A to F (Definition 8.26) and y_A admits nullary restrictions (Definition 4.5).*

PROOF. Using the equivalence (b) $\Leftrightarrow$ (c) of the previous lemma, here we take ι-smallness of any morphism $f\colon FB \to \widehat{A}$ in $\mathcal{L}$ to mean the existence of a horizontal morphism $J\colon A' \nrightarrow B$ in $\mathcal{K}$ such that $J^\flat \cong \widehat{A}(\mathrm{y}_A, f)$. For the 'if'-part, assume that for any $f\colon FB \to \widehat{A}$ the restriction $\widehat{A}(\mathrm{y}_A, f)\colon A \nrightarrow FB$ exists. Assuming moreover that ι is universal from A to F there exists $J\colon A' \nrightarrow B$ such that $J^\flat \cong \widehat{A}(\mathrm{y}_A, f)$, proving that f is ι-small. For the 'only if'-part assume that all morphisms of the form $f\colon FB \to \widehat{A}$ are ι-small. This immediately implies that y_A admits all restrictions $\widehat{A}(\mathrm{y}_A, f)$. To show that ι is universal too let $H\colon A \nrightarrow FB$ be any horizontal morphism in $\mathcal{L}$. We have $H \cong \widehat{A}(\mathrm{y}_A, H^\lambda)$ by the Yoneda axiom (Definition 4.5) so that, using that $H^\lambda\colon FB \to \widehat{A}$ is ι-small, there exists $J\colon A' \nrightarrow B$ such that $J^\flat \cong \widehat{A}(\mathrm{y}_A, H^\lambda) \cong H$ as required. ∎

8.32. UNIVERSAL VERTICAL MORPHISMS FROM YONEDA MORPHISMS. The following theorem generalises the implication (ye) $\Rightarrow$ (ih) of Theorem 8.21.

8.33. THEOREM. *Let $F\colon \mathcal{K} \to \mathcal{L}$ be a functor of augmented virtual double categories (Definition A3.1) and $\iota\colon A \nrightarrow FA'$ a locally universal morphism from A to F (Definition 8.26). Consider Yoneda morphisms $\mathrm{y}_A\colon A \to \widehat{A}$ and $\mathrm{y}_{A'}\colon A' \to \widehat{A'}$ (Definition 4.5) such that, for any $f\colon B \to \widehat{A'}$, the restrictions $\widehat{A'}(\mathrm{y}_{A'}, f)$ and $\mathrm{y}^\flat_{A'*}(\mathrm{id}, Ff)$ exist.*





If F preserves all $\widehat{A'}(\mathrm{y}_{A'}, f)$ then the morphism $\varepsilon \coloneqq (\mathrm{y}^\flat_{A'*})^\lambda\colon F\widehat{A'} \to \widehat{A}$, that is given by the Yoneda axiom (Definition 4.5) and that comes equipped with a cartesian cell as on *the left above, is universal from F to $\widehat{A}$ relative to $(F /_{\mathrm{v}} \widehat{A})_\iota$ (Definitions 1.39 and 8.29). If moreover A' is unital then there exists a cartesian cell as on the right.*

Proof. The proof consists of three steps: we need to show that the functor (Definition 1.39)

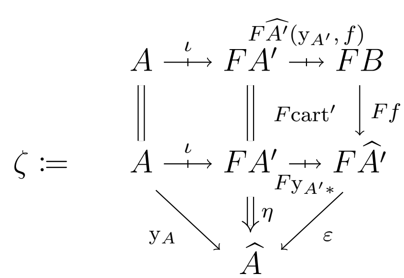
$$\varepsilon \circ F- : \mathcal{K} /_{\mathrm{v}} \widehat{A'} \to F /_{\mathrm{v}} \widehat{A}$$

(1) factors through $(F /_{\mathrm{v}} \widehat{A})_\iota \hookrightarrow F /_{\mathrm{v}} \widehat{A}$ (Definition 8.29), (2) is essentially surjective onto $(F /_{\mathrm{v}} \widehat{A})_\iota$ and (3) is full and faithful. To prove (1) consider any $f : B \to \widehat{A'}$; we have to prove that $\varepsilon \circ Ff$ is ι-small. To do so write $\mathrm{cart}' : \widehat{A'}(\mathrm{y}_{A'}, f) \Rightarrow \mathrm{y}_{A'*}$ for the unique factorisation of the cartesian cell defining $\widehat{A'}(\mathrm{y}_{A'}, f)$ through that defining the companion $\mathrm{y}_{A'*}$. By the pasting lemma (Lemma 1.17) cart' is the cartesian cell that defines $\widehat{A'}(\mathrm{y}_{A'}, f)$ as the restriction of $\mathrm{y}_{A'*}$ along f. Since F preserves the cartesian cells defining $\widehat{A'}(\mathrm{y}_{A'}, f)$ and $\mathrm{y}_{A'*}$ it follows that it preserves cart', again by the pasting lemma. We can thus apply Lemma 8.27 to find that

$$\widehat{A}(\mathrm{y}_A, \varepsilon \circ Ff) \cong \widehat{A}(\mathrm{y}_A, \varepsilon)(\mathrm{id}, Ff) \cong \mathrm{y}^\flat_{A'*}(\mathrm{id}, Ff) \cong (\mathrm{y}_{A'*}(\mathrm{id}, f))^\flat,$$

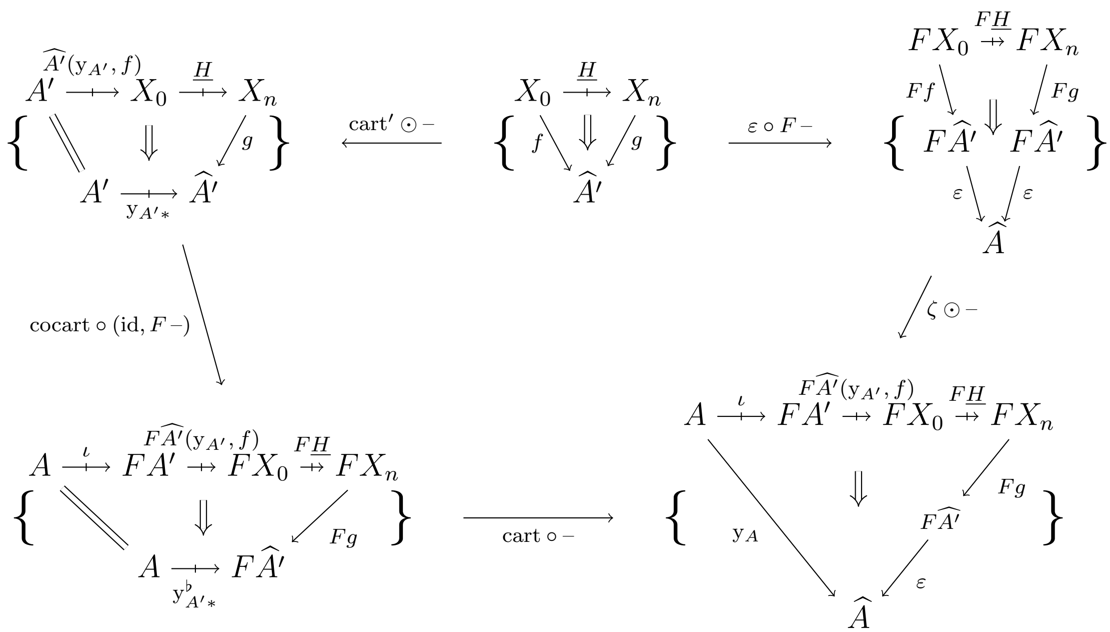
where the first isomorphism follows from the pasting lemma and the second from the cartesian cell defining ε, on the left above. Using Lemma 8.30(b) we conclude that $\varepsilon \circ Ff$ is ι-small.

In proving (2) and (3) we will use composites ζ of the form below, where $f : B \to \widehat{A'}$ is any morphism and where the cell η defines ε as the pointwise left Kan extension of y_A along $(\iota, F\mathrm{y}_{A'*})$. That η exists follows the fact that ε is ι-small and Lemma 8.30(d). Notice that the composite ζ is pointwise left Kan too, because $F\mathrm{cart}'$ is cartesian and by Definition 1.24. To prove the final assertion for the moment assume that A' is unital so that both $\mathrm{y}_{A'}$ and $F\mathrm{y}_{A'}$ are full and faithful, by Lemma 4.6 and Corollary A5.15 respectively. Precomposing η with the F-image of the cocartesian cell that defines $\mathrm{y}_{A'*}$ (Lemma 1.21), which is again cocartesian by Corollary A5.5, we obtain a nullary cell of the form as on the right above, that is left Kan by Proposition 2.26 and hence cartesian by Lemma 4.20, as required.

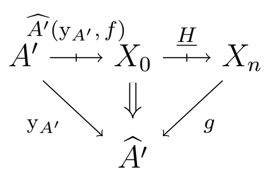


To show (2) consider any ι-small morphism $g : FB \to \widehat{A}$. By Lemma 8.30(d) g is the pointwise left Kan extension of y_A along (ι, FJ) for some $J : A' \nrightarrow B$. Taking $f := J^\lambda : B \to \widehat{A'}$, as supplied by the Yoneda axiom (Definition 4.5), in the composite above and composing it with $FJ \cong F\widehat{A'}(\mathrm{y}_{A'}, J^\lambda)$ we find that $\varepsilon \circ FJ^\lambda$ too is the pointwise left Kan extension of y_A along (ι, FJ). By uniqueness of Kan extensions we conclude that $g \cong \varepsilon \circ FJ^\lambda$, which proves (2).

To prove (3) consider the diagram of assignments between collections of cells in $\mathcal{K}$ and $\mathcal{L}$ below, where cart' is the cartesian cell $\widehat{A'}(\mathrm{y}_{A'}, f) \Rightarrow \mathrm{y}_{A'*}$; cocart denotes the cocartesian cell defining $\mathrm{y}^\flat_{A'*}$ (Definition 8.26); ζ is the composite above and cart is the cartesian cell defining $\varepsilon = (\mathrm{y}^\flat_{A'*})^\lambda$. That this diagram commutes follows from the equality $\eta = \mathrm{cart} \circ \mathrm{cocart}$ in definition of ζ above (see the proof of (b) $\Rightarrow$ (d) in Lemma 8.30), the axioms satisfied by horizontal composition (Lemma A1.3) and the fact that F, like any functor of augmented virtual double categories, preserves horizontal compositions of cells.



Notice that showing (3), that is $\varepsilon \circ F-$ is full and faithful, amounts to showing that the assignment $\varepsilon \circ F-$ in the right leg above is a bijection. Since ζ is left Kan the assignment $\zeta \odot -$ is a bijection; hence it suffices to prove that the composite assignment of the left leg above, and thus that of the right leg as well, is a bijection. In the left leg $\mathrm{cart} \circ -$ is a bijection by definition (Definition 1.16) and $\mathrm{cocart} \circ (\mathrm{id}, F-)$ is a bijection by Definition 8.26(c) (where we take $h = \mathrm{id}_{A'}$), which leaves the top assignment $\mathrm{cart}' \odot -$.

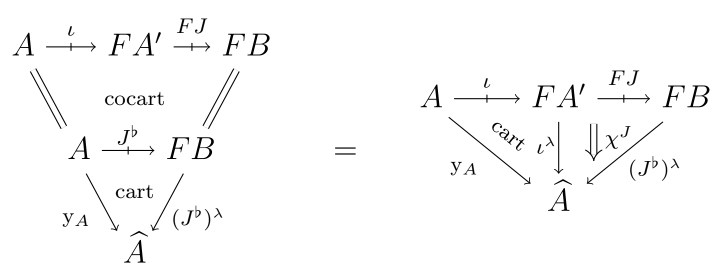


To see that it too is a bijection notice that $\text{cart} \circ (\text{cart}' \odot -) = (\text{cart} \circ \text{cart}') \odot -$, where cart denotes the cartesian cell defining $\text{y}_{A'*}$, is a bijective assignment onto the collection of nullary cells of the form above, because the cartesian cell $\text{cart} \circ \text{cart}'$ is left Kan by the density of $\text{y}_{A'}$ (Definition 4.3). Since $\text{cart} \circ -$ is a bijection too, from the collection of cells in the top left in the diagram above onto that of the cells of the form above, we conclude that $\text{cart}' \odot -$ is a bijection as well. This concludes the proof. ∎

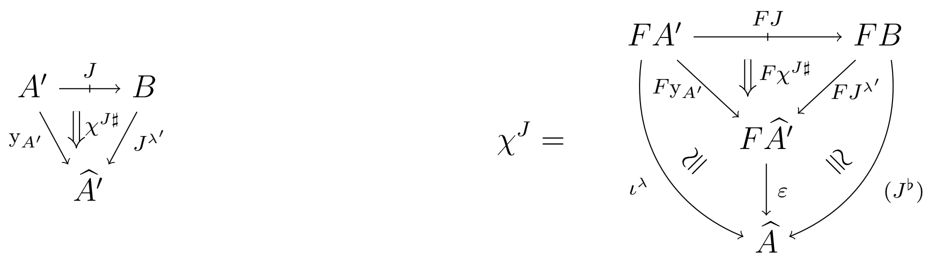
8.34. Lifting Yoneda embeddings along universal morphisms. The final theorem of this section — Theorem 8.36 below — is, roughly, a converse to the previous theorem: given a functor $F\colon \mathcal{K} \to \mathcal{L}$ and, in $\mathcal{L}$, a locally universal morphism $\iota\colon A \nrightarrow FA'$, a Yoneda morphism $\text{y}_A\colon A \to \widehat{A}$ and a universal morphism $\varepsilon\colon F\widehat{A'} \to \widehat{A}$, it gives conditions ensuring that the morphism $\text{y}_{A'}\colon A' \to \widehat{A'}$ in $\mathcal{K}$ constructed below is a Yoneda embedding.

8.35. Definition. *Let $F\colon \mathcal{K} \to \mathcal{L}$ be a functor of augmented virtual double categories (Definition A3.1) and $\text{y}_A\colon A \to \widehat{A}$ a Yoneda morphism in $\mathcal{L}$ (Definition 4.5). Let A' be a unital object in $\mathcal{K}$, with horizontal unit $I_{A'}$ (Definition 1.16), and $\iota\colon A \nrightarrow FA'$ a morphism satisfying conditions (a) and (b) of Definition 8.26. Given a morphism $\varepsilon\colon F\widehat{A'} \to \widehat{A}$ that is universal from F to $\widehat{A}$ relative to $(F /_{\text{v}} \widehat{A})_\iota$ (Definitions 1.39 and 8.29) we make the following definitions.*

- *The morphism $\iota^\lambda\colon FA' \to \widehat{A}$, given by the Yoneda axiom (Definition 4.5) and defined by the cartesian cell in the right-hand side below, is ι-small (Definition 8.29) because $I^\flat_{A'} \cong \iota$ by Lemma A8.1. We define $\text{y}_{A'}\colon A' \to \widehat{A'}$ in $\mathcal{K}$ to be $\text{y}_{A'} := (\iota^\lambda)^\sharp$, i.e. the chosen morphism $\text{y}_{A'}\colon A' \to \widehat{A'}$ such that $\varepsilon \circ F\text{y}_{A'} \cong \iota^\lambda$ (Definition 1.39).*
- *For every $J\colon A' \nrightarrow B$ in $\mathcal{K}$ we define $J^{\lambda'}\colon B \to \widehat{A'}$ to be $J^{\lambda'} := \big((J^\flat)^\lambda\big)^\sharp$, where $(J^\flat)^\lambda\colon FB \to \widehat{A}$ is defined by the cartesian cell in the left-hand side below, so that it is ι-small by Lemma 8.30(b).*



- *For every $J\colon A' \nrightarrow B$ in $\mathcal{K}$ we denote by χ^J the unique factorisation in the right-hand side above, which exists because the cartesian cell defining ι^λ is left Kan (Definition 4.3). Because $\varepsilon \circ F -$ is full and faithful onto $F /_{\text{v}} \widehat{A}$ (Definition 1.39) there exists a unique cell $\chi^{J\sharp}$, of the form as on the left below, that satisfies the identity* *on the right.*



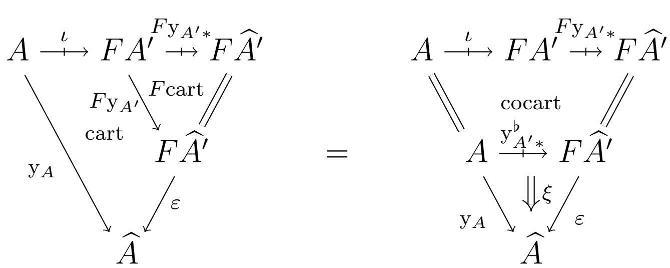
8.36. Theorem. *The morphism* $\mathrm{y}_{A'} \colon A' \to \widehat{A'}$ *defined above is a Yoneda embedding, with each of the cells* $\chi^{J\sharp}$ *being cartesian, if the following conditions are satisfied:*

(a) *the restrictions* $\mathrm{y}_{A'*}(\mathrm{id}, f)$ *and* $\mathrm{y}^\flat_{A'*}(\mathrm{id}, Ff)$ *exist for any* $f \colon B \to \widehat{A'}$ *in* $\mathcal{K}$ *and* F *preserves all* $\mathrm{y}_{A'*}(\mathrm{id}, f)$*;*

(b) $\iota \colon A \nrightarrow FA'$ *is locally universal from* A *to* F *(Definition 8.26).*

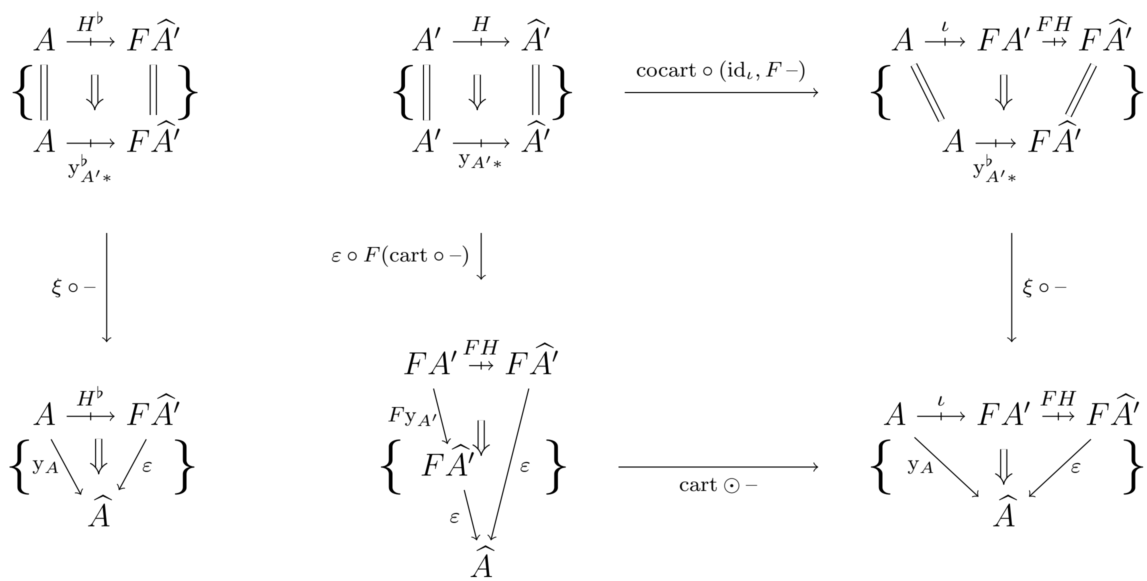
Moreover, in that case $\mathrm{y}^\flat_{A'*} \colon A \nrightarrow F\widehat{A'}$ *is the restriction of* $\widehat{A}$ *along* y_A *and* ε*.*

Proof. To show that $\mathrm{y}_{A'}$ is a Yoneda embedding we have to prove that it is dense (Definition 4.3) and that it satisfies the Yoneda axiom (Definition 4.5); that $\mathrm{y}_{A'}$ is full and faithful then follows from Lemma 4.6 as A' is assumed to be unital (Definition 8.35). That $\mathrm{y}_{A'}$ satisfies the Yoneda axiom is proved in Lemma 8.37 below; here we prove the density of $\mathrm{y}_{A'}$. To do so consider the equation below, where the left cartesian cell in the left-hand side defines $\iota^\lambda \cong \varepsilon \circ F\mathrm{y}_{A'}$ (Definition 8.35) and where the nullary cell $\xi \colon \mathrm{y}^\flat_{A'*} \Rightarrow \widehat{A}$ is the unique factorisation through the pointwise right cocartesian cell that defines $\mathrm{y}^\flat_{A'*}$ (Definition 8.26). We claim that ξ is cartesian, and hence pointwise left Kan by density of $\mathrm{y}_A \colon A \to \widehat{A}$. Before proving this claim let us use it to prove the density of $\mathrm{y}_{A'}$. Notice also that the final assertion above simply follows from the fact that ξ is cartesian.



Using that the restrictions $(F\mathrm{y}_{A'*})(\mathrm{id}, Ff) \cong F\big(\mathrm{y}_{A'*}(\mathrm{id}, f)\big)$ and $\mathrm{y}^\flat_{A'*}(\mathrm{id}, Ff)$ exist for any $f \colon B \to \widehat{A'}$ by condition (a), it follows from the vertical pasting lemma (Lemma 2.17) that the right-hand side above, and thus both sides, are left Kan cells that restrict along any F-image Ff. Since the left cartesian cell in the left-hand side is left Kan too, again by the density of y_A, it follows from the horizontal pasting lemma (Lemma 2.2) that the composite $\varepsilon \circ F\mathrm{cart}$ in the left-hand side is a left Kan cell that restricts along any Ff too. Using Proposition 1.42 we conclude that the adjunct of $\varepsilon \circ F\mathrm{cart}$, which is the cartesian cell cart defining $\mathrm{y}_{A'*}$ itself, is pointwise left Kan. By Definition 4.3 this means that $\mathrm{y}_{A'}$ is dense as required.

It remains to prove the claim that the nullary cell ξ in the right-hand side above is cartesian. Notice that $\varepsilon = \varepsilon \circ F(\mathrm{id}_{\widehat{A}'})$ is ι-small because $\varepsilon \circ F-$ factors through $(F /_{\mathrm{v}} \widehat{A})_\iota$ (Definitions 1.39 and 8.29). Hence, by Lemma 8.30(b), the restriction $\widehat{A}(\mathrm{y}_A, \varepsilon) \colon A \nrightarrow F\widehat{A}'$ exists and $\widehat{A}(\mathrm{y}_A, \varepsilon) \cong J^\flat$ for some $J \colon A' \nrightarrow \widehat{A}'$. To prove the claim we will show that the cartesian cell $J^\flat \Rightarrow \widehat{A}$, that defines $\widehat{A}(\mathrm{y}_A, \varepsilon)$, factors through ξ as an invertible horizontal cell $J^\flat \cong \mathrm{y}^\flat_{A'*}$. For this it suffices to prove that for any $H \colon A' \nrightarrow \widehat{A}'$ in $\mathcal{K}$ the assignment $\xi \circ -$ on the left below, between collections of cells in $\mathcal{L}$ as shown, is a bijection. Precomposing with the right cocartesian cells $(\iota, FH) \Rightarrow H^\flat$ (Definition 8.26) and using the uniqueness of factorisations through cocartesian cells, we may equivalently prove that the assigment $\xi \circ -$ on the right below is a bijection.

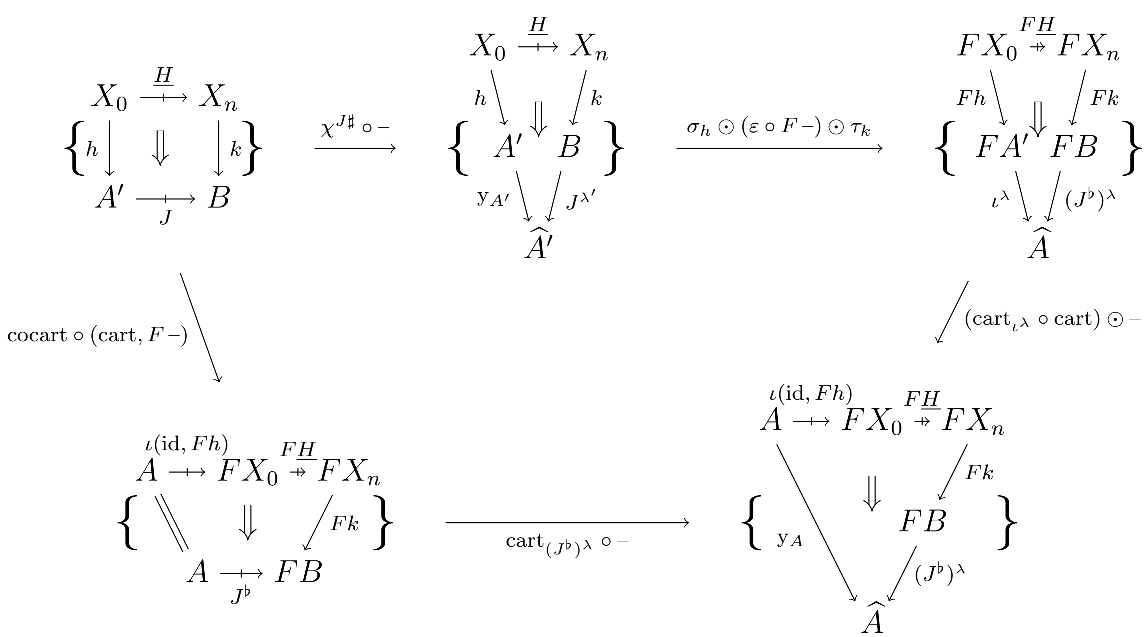


To do so, consider the whole diagram of assignments on the right above, where the cocartesian cell defines $\mathrm{y}^\flat_{A'*}$ and where the cartesian cells are those in the left-hand side of equality that defines ξ above; that the diagram commutes follows from the same equality and the interchange axioms (Lemma A1.3). We claim that, besides $\xi \circ -$ on the right, all assignments above are bijections, so that $\xi \circ -$ is a bijection as well. Indeed, $\mathrm{cart} \odot -$ is a bijection since cart is left Kan, by the density of y_A (Definition 4.3); $\varepsilon \circ F(\mathrm{cart} \circ -)$ is a bijection because of the companion identities for cart (Lemma 1.21) and the fact that $\varepsilon \circ F- \colon \mathcal{K} /_{\mathrm{v}} \widehat{A}' \to F /_{\mathrm{v}} \widehat{A}$ is full and faithful (Definition 1.39); $\mathrm{cocart} \circ (\mathrm{id}_\iota, F-)$ is a bijection by Definition 8.26(c), where $h = \mathrm{id}_{A'}$ and $k = \mathrm{id}_{\widehat{A}'}$. This completes the proof. ■

8.37. Lemma. *In Definition 8.35 the morphism $\iota \colon A \nrightarrow FA'$ satisfies condition* (c) *of Definition 8.26 (i.e. ι is locally universal) precisely if, for each $J \colon A' \nrightarrow B$ in $\mathcal{K}$, the cell*

$\chi^{J\sharp}$ *is cartesian (i.e.* $\mathrm{y}_{A'}\colon A' \to \widehat{A'}$ *satisfies the Yoneda axiom of Definition 4.5).*

Proof. Consider the diagram of assignments between collections of cells in $\mathcal{K}$ and $\mathcal{L}$ below, where cocart denotes the cocartesian cell definining $J^\flat$ (Definition 8.26) and the cartesian cells denoted by cart, $\mathrm{cart}_{(J^\flat)^\lambda}$ and $\mathrm{cart}_{\iota^\lambda}$ define the morphisms $\iota(\mathrm{id}, Fh)$, $(J^\flat)^\lambda$ (Definition 4.5) and ι^λ respectively. The vertical isomorphisms σ_h and τ_k in the middle assignment of the right leg denote the composites $\sigma_h \coloneqq \sigma \circ Fh$ and $\tau_k \coloneqq \tau \circ Fk$, where $\sigma\colon \iota^\lambda \cong \varepsilon \circ F\mathrm{y}_{A'}$ and $\tau\colon \varepsilon \circ FJ^{\lambda'} \cong (J^\flat)^\lambda$ are the isomorphisms that equip the chosen morphisms $\mathrm{y}_{A'}\colon A' \to \widehat{A'}$ and $J^{\lambda'}\colon B \to \widehat{A'}$ (Definition 8.35).



Notice that $\mathrm{cart}_{(J^\flat)^\lambda} \circ -$ is a bijection by the definition of cartesian cell, and that $(\mathrm{cart}_{\iota^\lambda} \circ \mathrm{cart}) \odot -$ is so too because $\mathrm{cart}_{\iota^\lambda}$ is pointwise left Kan, by the density of y_A (Definition 4.3). Since $\varepsilon \circ F-\colon \mathcal{K} /_{\mathrm{v}} \widehat{A'} \to F /_{\mathrm{v}} \widehat{A}$ is full and faithful, by Definition 1.39, and because σ_h and τ_k are isomorphisms, it follows that $\sigma_h \odot \big(\varepsilon \circ F-\big) \odot \tau_k$, in the right leg above, is a bijection too. Finally notice that condition (c) of Definition 8.26 states that the assignment $\mathrm{cocart} \circ (\mathrm{cart}, F-)$, in the left leg above, is a bijection for all $J\colon A' \nrightarrow B$, while the cells $\chi^{J\sharp}$ being cartesian amounts to the top assignment being a bijection for all $J\colon A' \nrightarrow B$. Hence the proof follows by showing that the diagram above commutes. That it does is shown by the equality below, whose left-hand side is the composite of the right leg above and whose right-hand side is that of the left leg.

$$
\begin{aligned}
(\mathrm{cart}_{\iota^\lambda} \circ \mathrm{cart}) \odot \sigma_h \odot \big(\varepsilon \circ F(\chi^{J\sharp} \circ -)\big) \odot \tau_k &= (\mathrm{cart}_{\iota^\lambda} \circ \mathrm{cart}) \odot \sigma_h \odot \big(\varepsilon \circ F\chi^{J\sharp} \circ F-\big) \odot \tau_k \\
= (\mathrm{cart}_{\iota^\lambda} \circ \mathrm{cart}) \odot \big[\big(\sigma \odot (\varepsilon \circ F\chi^{J\sharp}) \odot \tau\big) \circ F-\big] &= (\mathrm{cart}_{\iota^\lambda} \circ \mathrm{cart}) \odot (\chi^J \circ F-) \\
= (\mathrm{cart}_{\iota^\lambda} \odot \chi^J) \circ (\mathrm{cart}, F-) &= \mathrm{cart}_{(J^\flat)^\lambda} \circ \mathrm{cocart} \circ (\mathrm{cart}, F-)
\end{aligned}
$$

The identities follow from the functoriality of F, the definitions of σ_h and τ_k together with the interchange axiom (Lemma A1.3), the definition of $\chi^{J\sharp}$ (Definition 8.35), the interchange axiom, and the definition of χ^J (Definition 8.35). ■

References

References to results and sections with their numbering prefixed with the capital letter 'A', such as "Lemma A8.1" and "Section A7", refer to results and sections of the prequel to the present article [Kou20].

[AB06] C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer, third edition, 2006.

[Ale18] E. Aleiferi. *Cartesian double categories with an emphasis on characterizing spans*. PhD thesis, Dalhousie University, 2018.

[AR01] J. Adámek and J. Rosický. On sifted colimits and generalized varieties. *Theory and Applications of Categories*, 8:33–53, 2001.

[Bén73] J. Bénabou. Les distributeurs. *Séminaires de Mathématique Pure, Université Catholique de Louvain*, Rapport no. 33, 1973.

[BK75] F. Borceux and G. M. Kelly. A notion of limit for enriched categories. *Bulletin of the Australian Mathematical Society*, 12(1):49–72, 1975.

[CJSV94] A. Carboni, S. Johnson, R. Street, and D. Verity. Modulated bicategories. *Journal of Pure and Applied Algebra*, 94(3):229–282, 1994.

[CLPS22] G. S. H. Cruttwell, M. J. Lambert, D. A. Pronk, and M. Szyld. Double fibrations. *Theory and Applications of Categories*, 38(35):1326–1394, 2022.

[CS86] A. Carboni and R. Street. Order ideals in categories. *Pacific Journal of Mathematics*, 124(2):275–288, 1986.

[CS10] G. S. H. Cruttwell and M. A. Shulman. A unified framework for generalized multicategories. *Theory and Applications of Categories*, 24(21):580–655, 2010.

[CT97] M. M. Clementino and W. Tholen. A characterization of the Vietoris topology. *Topology Proceedings*, 22:71–95, 1997.

[Day70] B. Day. On closed categories of functors. In *Reports of the Midwest Category Seminar IV*, volume 137 of *Lecture Notes in Mathematics*, pages 1–38. Springer-Verlag, 1970.

[DL07] B. J. Day and S. Lack. Limits of small functors. *Journal of Pure and Applied Algebra*, 210(3):651–663, 2007.

[DPP06] R. J. MacG. Dawson, R. Paré, and D. A. Pronk. Paths in double categories. *Theory and Applications of Categories*, 16(18):460–521, 2006.

[DS86] B. Day and R. Street. Categories in which all strong generators are dense. *Journal of Pure and Applied Algebra*, 43:235–242, 1986.

[DS97] B. Day and R. Street. Monoidal bicategories and Hopf algebroids. *Advances in Mathematics*, 129(1):99–157, 1997.

[Dub70] E. J. Dubuc. *Kan Extensions in Enriched Category Theory*, volume 145 of *Lecture Notes in Mathematics*. Springer-Verlag, 1970.

[FS95] P. Freyd and R. Street. On the size of categories. *Theory and Applications of Categories*, 1(9):174–178, 1995.

[GP99] M. Grandis and R. Paré. Limits in double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 40(3):162–220, 1999.

[GP07] M. Grandis and R. Paré. Lax Kan extensions for double categories (On weak double categories, Part IV). *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 48(3):163–199, 2007.

[GP08] M. Grandis and R. Paré. Kan extensions in double categories (On weak double categories, Part III). *Theory and Applications of Categories*, 20(8):152–185, 2008.

[Gra74] J. W. Gray. *Formal Category Theory: Adjointness for 2-Categories*, volume 391 of *Lecture Notes in Mathematics*. Springer-Verlag, 1974.

[Gui80] R. Guitart. Relations et carrés exacts. *Les Annales des Sciences Mathématiques du Québec*, 4(2):103–125, 1980.

[GV72] A. Grothendieck and J. L. Verdier. *Théorie des topos et cohomologie étale des schémas (SGA 4). Tome 1: théorie des topos. Exposés I à IV*, volume 269 of *Lecture Notes in Mathematics*. Springer-Verlag, 1972. Séminaire de géométrie algébrique du Bois-Marie 1963–1964.

[Her01] C. Hermida. Fibrations and Yoneda structure for multicategories. Preprint, 2001.

[Joh02] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium. Volume 1*, volume 43 of *Oxford Logic Guides*. Oxford University Press, 2002.

[Kan58] D. M. Kan. Adjoint functors. *Transactions of the American Mathematical Society*, 87:294–329, 1958.

[Kel82] G. M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982.

[Kel86] G. M. Kelly. A survey of totality for enriched and ordinary categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 27(2):109–132, 1986.

[KL97] G. M. Kelly and S. Lack. On property-like structures. *Theory and Applications of Categories*, 3(9):213–250, 1997.

[Kou13] S. R. Koudenburg. *Algebraic weighted colimits.* PhD thesis, University of Sheffield, 2013. Available as `arXiv:1304.4079`.

[Kou14a] S. R. Koudenburg. On pointwise Kan extensions in double categories. *Theory and Applications of Categories*, 29(27):781–818, 2014.

[Kou14b] S. R. Koudenburg. Left Kan extensions that are algebraic over colax-idempotent 2-monads. Draft, available as `arXiv:1412.3760`, December 2014.

[Kou15a] S. R. Koudenburg. Algebraic Kan extensions in double categories. *Theory and Applications of Categories*, 30(5):86–146, 2015.

[Kou15b] S. R. Koudenburg. A double-dimensional approach to formal category theory. Draft, available as `arXiv:1511.04070`, November 2015.

[Kou18] S. R. Koudenburg. A categorical approach to the maximum theorem. *Journal of Pure and Applied Algebra*, 222(8):2099–2142, 2018.

[Kou20] S. R. Koudenburg. Augmented virtual double categories. *Theory and Applications of Categories*, 35(10):261–325, 2020. Revised in 2022.

[Lam22] M. Lambert. Double categories of relations. *Theory and Applications of Categories*, 38(33):1249–1283, 2022.

[Law73] F. W. Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matématico e Fisico di Milano*, 43:135–166, 1973.

[Lei04] T. Leinster. *Higher Operads, Higher Categories*, volume 298 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, 2004.

[Lin74] H. Lindner. Morita equivalences of enriched categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 15(4):377–397, 1974.

[Lin81] H. Lindner. Enriched categories and enriched modules. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 22(2):161–174, 1981. Third Colloquium on Categories (Amiens, 1980).

[ML98] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer, second edition, 1998.

[MT08] P.-A. Melliès and N. Tabareau. Free models of T-algebraic theories computed as Kan extensions. Preprint, 2008.

[Nie20] S. Niefield. Exponentiability in double categories and the glueing construction. *Theory and Applications of Categories*, 35(32):1208–1226, 2020.

[Par11] R. Paré. Yoneda theory for double categories. *Theory and Applications of Categories*, 25(17):436–489, 2011.

[RV22] E. Riehl and D. Verity. *Elements of ∞-Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2022.

[Shu08] M. Shulman. Framed bicategories and monoidal fibrations. *Theory and Applications of Categories*, 20(18):650–738, 2008.

[Shu10] M. A. Shulman. Constructing symmetric monoidal bicategories. Preprint, 2010.

[Str72] R. Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2(2):149–168, 1972.

[Str74a] R. Street. Elementary cosmoi I. In *Proceedings of the Sydney Category Theory Seminar 1972/1973*, volume 420 of *Lecture Notes in Mathematics*, pages 134–180. Springer-Verlag, 1974.

[Str74b] R. Street. Fibrations and Yoneda's lemma in a 2-category. In *Proceedings of the Sydney Category Theory Seminar 1972/1973*, volume 420 of *Lecture Notes in Mathematics*, pages 104–133. Springer-Verlag, 1974.

[Str80a] R. Street. Cosmoi of internal categories. *Transactions of the American Mathematical Society*, 258(2):271–318, 1980.

[Str80b] R. Street. Fibrations in bicategories. *Cahiers de Topologie et Géométrie Différentielle*, 21(2):111–160, 1980.

[Str17] R. Street. Categories in categories, and size matters. *Higher Structures*, 1(1):225–270, 2017.

[SW78] R. Street and R. Walters. Yoneda structures on 2-categories. *Journal of Algebra*, 50(2):350–379, 1978.

[Tho09] W. Tholen. Ordered topological structures. *Topology and its Applications*, 156(12):2148–2157, 2009.

[Wal18] C. Walker. Yoneda structures and KZ doctrines. *Journal of Pure and Applied Algebra*, 222(6):1375–1387, 2018.

[Web07] M. Weber. Yoneda structures from 2-toposes. *Applied Categorical Structures*, 15(3):259–323, 2007.

[Woo82] R. J. Wood. Abstract proarrows I. *Cahiers de Topologie et Géométrie Différentielle*, 23(3):279–290, 1982.

[Woo85] R. J. Wood. Proarrows II. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 26(2):135–168, 1985.

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