Theory and Applications of Categories, Vol. 40, No. 3, 2024, pp. 63–79.

CARTESIAN CLOSED DOUBLE CATEGORIES

SUSAN NIEFIELD

Abstract. We consider two approaches to cartesian closed double categories generalizing two definitions which are equivalent for 1-categories, and give examples to show that the two differ in the double category case. One approach, previously considered in [N20], requires the lax functor $(-) \times Y$ on $\mathbb{D}$ to have a right adjoint $(-)^Y$, for every object Y, while the other supposes that the exponentials are given by a lax bifunctor $\mathbb{D}^{\mathrm{op}} \times \mathbb{D} \longrightarrow \mathbb{D}$ also involving vertical (i.e., loose) morphisms of $\mathbb{D}$. Examples include the double categories $\mathbb{C}\mathrm{at}, \mathbb{P}\mathrm{os}, \mathbb{T}\mathrm{op}, \mathbb{L}\mathrm{oc}$ and $\mathbb{Q}\mathrm{uant}$, whose objects are small categories, posets, topological spaces, locales, and commutative quantales, respectively; as well as, the double categories $\mathbb{S}\mathrm{pan}(\mathcal{D})$ and Q-$\mathbb{R}\mathrm{el}$, whose vertical morphisms are spans in a category $\mathcal{D}$ with pullback and relations valued in a locale Q, respectively.

1. Introduction

In [N20], we considered exponentiable objects and cartesian closure for a (lax) cartesian double category $\mathbb{D}$, i.e., objects Y such that the lax functor $(-) \times Y \colon \mathbb{D} \longrightarrow \mathbb{D}$ has a right adjoint in the 2-category **LxDbl** of double categories and lax functors. We showed that Y is exponentiable in a "glueing category" $\mathbb{D}$ if and only if Y is exponentiable in $\mathbb{D}_0$ and $(-) \times Y$ is oplax. Applications included the double categories $\mathbb{C}\mathrm{at}$, $\mathbb{P}\mathrm{os}$, $\mathbb{T}\mathrm{op}$, $\mathbb{L}\mathrm{oc}$, and $\mathbb{T}\mathrm{opos}$, whose objects are small categories, posets, topological space, locales, and Grothendieck toposes, respectively. We restricted to right adjoints in **LxDbl** because the right adjoints in some of these examples were not pseudo even though $(-) \times Y$ was.

Interest in exponentiability is related to the study of suitable topologies on function spaces, for if Y is exponentiable in Top, then taking $Y = 1$ in

$$\mathrm{Top}(X \times Y, Z) \cong \mathrm{Top}(X, Z^Y)$$

one sees that Z^Y can be identified with the set $\mathrm{Top}(Y, Z)$ of continuous maps from X to Z. Perhaps the first definitive result in this area appeared in [F45], the 1945 paper "On topologies for function spaces" by R. H. Fox, where he clearly stated the problem of finding an appropriate topology for $\mathrm{Top}(Y, Z)$, noted that it had been long known to be possible for locally compact Y, and showed that local compactness was also necessary for separable metrizable spaces.

For a 1-category $\mathcal{D}$, one can define cartesian closure via a pointwise or a 2-variable adjunction, and obtain equivalent definitions. The pointwise approach requires the existence

Received by the editors 2023-03-27 and, in final form, 2024-02-26.
Published on 2024-03-26 in the Bunge Festschrift.
2020 Mathematics Subject Classification: 18N10, 18D15, 18B10, 18F75, 54C35.
Key words and phrases: double categories, cartesian closed, spans/cospans, quantales, relations.

of an exponential Z^Y, for every object Y of $\mathcal{D}$, whereas for the latter, the exponentials are given by a bifunctor $\mathcal{D}^{\rm op} \times \mathcal{D} \longrightarrow \mathcal{D}$. We will see that these two approaches differ for a double categories $\mathbb{D}$. In particular, a bifunctor $\mathbb{D}^{\rm op} \times \mathbb{D} \longrightarrow \mathbb{D}$ would involve not only objects of $\mathbb{D}_0$, but also those of $\mathbb{D}_1$, i.e., vertical morphisms of $\mathbb{D}$.

After a review of adjoints on double categories in Section 3, we introduce the more general definition of a cartesian closed double category in Section 4. In addition to $\mathbb{P}$os and $\mathbb{C}$at, examples include the cartesian double category $\mathbb{S}$pan$(\mathcal{D})$, whose objects and horizontal morphisms are those of $\mathcal{D}$ and vertical morphisms are spans in $\mathcal{D}$, when $\mathcal{D}$ is a cartesian closed category with pullbacks and equalizers; as well as the double category Q-$\mathbb{R}$el of sets, functions, and Q-valued relations (in the sense of [MT14]), when Q is a locale. In Sections 5 and 6, we turn to exponentiable objects in non-cartesian closed double categories. Starting with $\mathbb{C}$ospan$(\mathcal{D})$, for a cartesian closed category $\mathcal{D}$ with pushouts and coequalizers, we show that every object is exponentiable, but $\mathbb{C}$ospan$(\mathcal{D})$ is not a cartesian closed double category, even though both $\mathbb{D}_0$ and $\mathbb{D}_1$ are cartesian closed. Using the characterization of oplax/lax adjunctions

$$\mathbb{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$$

from [N12b], we indicate how one can prove the above mentioned exponentiability results for $\mathbb{C}$at, $\mathbb{P}$os, $\mathbb{T}$op, and $\mathbb{L}$oc using only the assumption that $\mathbb{D}$ has cotabulators, companions, and conjoints (in the sense of [GP99]) rather than the full definition of a glueing category. We also obtain the analogous coexponentiability result for the double category $\mathbb{Q}$uant, whose objects and horizontal morphisms are commutative unital quantales and their usual morphisms, and vertical morphisms are lax morphisms. We conclude in Section 7, with examples of locally cartesian closed double categories, namely, $\mathbb{S}$pan$(\mathcal{D})$ and Q-$\mathbb{R}$el, but not $\mathbb{C}$at or $\mathbb{P}$os.

We begin with a tribute to Marta Bunge and her influence that led to this work.

1.1. In Memory of Marta Bunge. This section is in honor of Marta Bunge, and an account of how our joint research during my 1998 sabbatical at McGill played a role in my subsequent interest in double categories. The visit led to a paper [BN00] which used my previously unpublished notion of "model-generated categories" [N78] to show that the category UFL$/B$ of unique factorization lifting functors over B is a topos, when B is a linearly ordered small category, or more generally, satisfies the (IG) property (in the sense of [BF00]). Along the way, we considered open and closed subcategories of B, i.e., sieves and cosieves of B, and showed that the UFL inclusions are the locally closed ones and correspond precisely to the locally closed subtoposes of the presheaf topos Sets^B. At that time, she also introduced me to Street's note [St01] using Bénabou's equivalence $\mathrm{Lax}(B^{op}, \mathrm{Prof}) \simeq \mathrm{Cat}/B$ to show a functor is exponentiable in Cat if and only if it satisfies the Giraud-Conduché condition [G64, C72].

In the decade that followed, I used variations of Bénabou's equivalence to characterize exponentiable morphisms in a number of slice categories. Following a related talk, Bob

Paré commented that I was actually working in a double category. In the latter setting, I then defined a locally closed inclusion using cotabulators (i.e., collages of categories and Artin-Wraith glueing of toposes) to construct the exponentials of these inclusions in glueing categories, and obtained applications for small categories, posets, topological space, locales, and toposes. Their existence was not new (see [BN00, N01, N78, N81]), but the construction via a single theorem was.

2. Examples of Double Categories

Recall that a *double category* is an internal pseudo category

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1 \underset{t}{\overset{s}{\underset{\longleftarrow\,\mathrm{id}^\bullet\,}{\rightrightarrows}}} \mathbb{D}_0$$

in the 2-category **CAT** of locally small categories. It consists of *objects* (those of $\mathbb{D}_0$), two types of morphisms: *horizontal* (those of $\mathbb{D}_0$) and *vertical* (objects of $\mathbb{D}_1$ with domain and codomain given by s and t), and *cells* (morphisms of $\mathbb{D}_1$)

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\big\downarrow\bullet & \varphi & \bullet\big\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array} \tag{$\star$}$$

sometimes denoted by $\varphi\colon v \xrightarrow[f_t]{f_s} w$. Composition and identity morphisms are given horizontally in $\mathbb{D}_0$ and vertically via $\odot$ and $\mathrm{id}^\bullet$, respectively. Horizontal and vertical morphisms are called *strict* and *loose*, respectively, by some authors.

Note that when w is the vertical $\mathrm{id}^\bullet_Y$, we often denote the cell ($\star$) by

$$\begin{array}{ccc} X_s & & \\ {\scriptstyle v}\big\downarrow\bullet & \varphi \;\; \overset{\searrow^{f_s}}{\underset{\nearrow_{f_t}}{}} & Y \\ X_t & & \end{array}$$

2.1. Example. For a category $\mathcal{D}$ with pullbacks, the double category $\mathbb{S}\mathrm{pan}(\mathcal{D})$ has objects and horizontal morphisms in $\mathcal{D}$, and vertical morphisms spans in $\mathcal{D}$, with composition defined via pullback and identities $\mathrm{id}^\bullet\colon X \mathrel{\multimap\!\!\!\to} X$ given by $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$. The cells ($\star$) are commutative diagrams in $\mathcal{D}$ of the form

$$\begin{array}{ccccc} & \overset{v_s}{\nearrow} & X_s & \xrightarrow{f_s} & Y_s \\ X & \xrightarrow{f} & Y & \overset{w_s}{\nearrow} \; \underset{w_t}{\searrow} & \\ & \underset{v_t}{\searrow} & X_t & \xrightarrow[f_t]{} & Y_t \end{array}$$

In particular, $\mathbb{S}\mathrm{pan}(\mathrm{Sets})$ is the double category $\mathbb{S}\mathrm{et}$ considered by Paré in [P11].

2.2. Example. $\mathbb{C}\mathrm{at}$ has small categories as objects and functors as horizontal morphisms. Vertical morphisms $v\colon X_s \to X_t$ are profunctors $v\colon X_s^{\mathrm{op}} \times X_t \to \mathrm{Sets}$, and cells $\varphi\colon v \xrightarrow[f_t]{f_s} w$ are natural transformations $v \to w(f_s-, f_t-)$.

2.3. Example. $\mathbb{P}\mathrm{os}$ has partially-ordered sets as objects and order-preserving maps as horizontal morphisms. Vertical morphisms $v\colon X_s \to X_t$ are order ideals $v \subseteq X_s^{\mathrm{op}} \times X_t$, and there is a cell $\varphi\colon v \xrightarrow[f_t]{f_s} w$ if and only if $(x_s, x_t) \in v$ implies $(f_s(x_s), f_t(x_t)) \in w$.

2.4. Example. For a unital quantale Q, the double category Q-$\mathbb{R}\mathrm{el}$ has sets and functions as objects and horizontal morphisms. Vertical morphisms $v\colon X_s \to X_t$ are Q-valued relations, in the sense of monoidal topology [MT14], i.e., functions $v\colon X_s \times X_t \to Q$. There is a cell of the form $(\star)$ if and only if $v(x_s, x_t) \leq w(f_s(x_s), f_t(x_t))$. Vertical composition with $w\colon X_t \to X_u$ is given by

$$(w \odot v)(x_s, x_u) = \bigvee_{x_t \in X_t} w(x_t, x_u)\, v(x_s, x_t)$$

and the identity $\mathrm{id}_X^\bullet\colon X \to X$ by

$$\mathrm{id}_X^\bullet(x, x') = \begin{cases} e & \text{if } x = x' \\ 0 & \text{if } x \neq x' \end{cases}$$

where e is the unit and 0 is the bottom element of Q. In particular, 2-$\mathbb{R}\mathrm{el} \cong \mathbb{R}\mathrm{el}$ is the double category $\mathbb{S}\mathrm{et}$ considered by Aleiferi in [A18].

2.5. Example. $\mathbb{T}\mathrm{op}$ has topological spaces as objects and continuous maps as horizontal morphisms. Vertical morphisms $X_s \to X_t$ are finite meet-preserving maps

$$\mathcal{O}(X_s) \to \mathcal{O}(X_t)$$

on the open set lattices, and there is a cell $\varphi\colon v \xrightarrow[f_t]{f_s} w$ if and only if $f_t^{-1} w \subseteq v f_s^{-1}$.

2.6. Example. $\mathbb{L}\mathrm{oc}$ has locales as objects, locale morphisms (in the sense of [J82]) as horizontal morphisms, and finite meet-preserving maps as vertical morphisms. There is a cell $\varphi\colon v \xrightarrow[f_t]{f_s} w$ if and only if $f_t^* w \leq v f_s^*$.

2.7. Example. $\mathbb{Q}\mathrm{uant}$ has commutative unital quantales as objects, quantale homomorphisms as horizontal morphisms, and lax maps as vertical morphisms, i.e., order-preserving $v\colon X_s \to X_t$ such that $v(x_s)v(x_s') \leq v(x_s x_s')$ and $e_t \leq v(e_s)$. There is a cell $\varphi\colon v \xrightarrow[f_t]{f_s} w$ if and only if $f_t v \leq w f_s$.

3. Cartesian Double Categories

In this section, we recall the definitions of oplax/lax adjunctions and cartesian double categories, in the sense of [GP04] and [A18], respectively.

A *lax functor* $F\colon \mathbb{D} \longrightarrow \mathbb{E}$ consists of functors $F_0\colon \mathbb{D}_0 \longrightarrow \mathbb{E}_0$ and $F_1\colon \mathbb{D}_1 \longrightarrow \mathbb{E}_1$, compatible with s and t, together with comparison cells

$$\mathrm{id}^\bullet_{FX} \xrightarrow{\rho_X} F(\mathrm{id}^\bullet_X) \quad \text{and} \quad Fw \odot Fv \xrightarrow{\rho_{v,w}} F(w \odot v)$$

for every object X and every composition $w \odot v$ in $\mathbb{D}$, and satisfying naturality and coherence conditions. Note that we drop the subscripts on F when the context is clear. An *oplax* functor is defined with the comparison cells in the opposite direction. A lax functor F is called *normal* if ρ_X is invertible, for all X, and it is a *pseudo functor* if the cells $\rho_{v,w}$ are also invertible.

There is a double category $\mathbb{D}\mathrm{bl}$, introduced in [GP04], whose objects are double categories, horizontal morphisms are lax functors, and vertical morphisms are oplax functors, with suitable cells. An *oplax/lax adjunction* is an orthogonal adjunction in $\mathbb{D}\mathrm{bl}$. If the left adjoint F above is also lax, then it is an adjunction in the 2-category **LxDbl** whose morphisms are lax functors. These adjunctions are characterized in [GP04] as follows.

3.1. Proposition. *The following are equivalent for functors $F_n\colon \mathbb{D}_n \longrightarrow \mathbb{E}_n$ and $G_n\colon \mathbb{E}_n \longrightarrow \mathbb{D}_n$, where $n = 0, 1$, compatible with s and t.*

(a) $F\colon \mathbb{D} \longrightarrow \mathbb{E}$ is oplax and $G\colon \mathbb{E} \longrightarrow \mathbb{D}$ is a lax right adjoint.

(b) $F_0 \dashv G_0$, $F_1 \dashv G_1$, and F is oplax.

(c) $F_0 \dashv G_0$, $F_1 \dashv G_1$, and G is lax.

A double category $\mathbb{D}$ is *lax cartesian*, called pre-cartesian in [A18], if the pseudo functors $\Delta\colon \mathbb{D} \longrightarrow \mathbb{D} \times \mathbb{D}$ and $!\colon \mathbb{D} \longrightarrow \mathbb{1}$ have lax right adjoints, denoted by $\times$ and 1, respectively. If $\times$ and 1 are pseudo functors, we say $\mathbb{D}$ is a *cartesian double category*.

One can show that $\mathbb{S}\mathrm{pan}(\mathcal{D})$, $\mathbb{R}\mathrm{el}$, $\mathbb{C}\mathrm{at}$, and $\mathbb{P}\mathrm{os}$ are cartesian double categories; as is Q-$\mathbb{R}\mathrm{el}$, for any locale Q. Also, $\mathbb{T}\mathrm{op}$ and $\mathbb{L}\mathrm{oc}$ are lax cartesian, and $\mathbb{Q}\mathrm{uant}$ is oplax cocartesian. Proofs for $\mathbb{S}\mathrm{pan}(\mathcal{D})$, $\mathbb{R}\mathrm{el}$, $\mathbb{C}\mathrm{at}$, and $\mathbb{P}\mathrm{os}$ can be found in [A18]. The latter two, as well as $\mathbb{T}\mathrm{op}$ and $\mathbb{L}\mathrm{oc}$, also appeared in [N20], and $\mathbb{Q}\mathrm{uant}$ is a generalization of $\mathbb{L}\mathrm{oc}$. For Q-$\mathbb{R}\mathrm{el}$, define

$$(u \wedge v)((x_s, y_s), (x_t, y_t)) = u(x_s, x_t) \wedge v(y_s, y_t)$$

for $u\colon X_s \times X_t \longrightarrow Q$ and $v\colon Y_s \times Y_t \longrightarrow Q$.

4. Cartesian Closed Double Categories

In this section, we introduce the notion of a cartesian closed double category, and show that Examples 2.1–2.4, with appropriate assumptions, are cartesian closed. We are restricting to **LxDbl** because the right adjoints in our examples are not pseudo even though the left adjoints are.

Suppose $\mathcal{D}$ is a category with finite products. Recall that $\mathcal{D}$ is called cartesian closed if the functor $(-) \times Y : \mathcal{D} \longrightarrow \mathcal{D}$ has a right adjoint, for every object Y, often denoted by $(-)^Y$; or equivalently, there is a bifunctor $[-,-] : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{D}$ with natural bijections $\mathcal{D}(X \times Y, Z) \longrightarrow \mathcal{D}(X, [Y, Z])$, denoted by $f \mapsto \hat{f}$.

Replacing $\mathcal{D}$ by $\mathbb{D}$ in the bifunctor approach doesn't make sense, unless we replace the natural bijections with ones in $\mathbb{D}_0$ and $\mathbb{D}_1$. On the other hand, we can generalized the object-wise definition to double categories in [N20], by interpreting $(-) \times Y$ as $(-) \times \mathrm{id}^\bullet_Y$ on $\mathbb{D}_1$, and obtained examples of object-wise cartesian closed double categories using Proposition 3.1 for glueing categories. We will return to this definition in Section 5 with additional examples and different proofs than those in [N20].

4.1. Definition. *A lax cartesian double category $\mathbb{D}$ is called cartesian closed if there is a lax bifunctor $[-,-] : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \longrightarrow \mathbb{D}$ with natural bijections $\mathbb{D}_0(X \times Y, Z) \cong \mathbb{D}_0(X, [Y, Z])$ and $\mathbb{D}_1(u \times v, w) \cong \mathbb{D}_1(u, [v, w])$, compatible with s and t.*

4.2. Theorem. *A cartesian double category $\mathbb{D}$ is cartesian closed, as a lax cartesian category, if and only if $\mathbb{D}_0$ and $\mathbb{D}_1$ are cartesian closed categories satisfying $s[v, w] = Z_s^{Y_s}$ and $t[v, w] = Z_t^{Y_t}$, for all $v : Y_s \longrightarrow\!\!\!\!\!\!\bullet\;\; Y_t$ and $w : Z_s \longrightarrow\!\!\!\!\!\!\bullet\;\; Z_t$.*

Proof. The forward direction is clear. For the converse, suppose $\mathbb{D}_0$ and $\mathbb{D}_1$ are cartesian closed categories with exponentials compatible with s and t. Then, by the remarks above, there are bifunctors $[-,-]_i : \mathbb{D}_i^{\mathrm{op}} \times \mathbb{D}_i \longrightarrow \mathbb{D}_i$, for $i = 0, 1$, and so it suffices to show that $[-,-]$ is lax on $\mathbb{D}^{\mathrm{op}} \times \mathbb{D}$. We have

$$([v', w'] \odot [v, w]) \times (v' \odot v) \xrightarrow{\alpha} ([v', w'] \times v') \odot ([v, w] \times v) \xrightarrow{ev \odot ev} w' \odot w$$

$$\mathrm{id}^\bullet_{[Y,Z]} \times \mathrm{id}^\bullet_Y \xrightarrow{\beta} \mathrm{id}^\bullet_{[Y,Z] \times Y} \xrightarrow{\mathrm{id}^\bullet_{ev}} \mathrm{id}^\bullet_Z$$

where α and β exist since $\times$ is oplax, and so their transposes in $\mathbb{D}_1$ give rise to

$$[v', w'] \odot [v, w] \longrightarrow [v' \odot v, w' \odot w] \quad \text{and} \quad \mathrm{id}^\bullet_{[Y,Z]} \longrightarrow [\,\mathrm{id}^\bullet_Y, \mathrm{id}^\bullet_Z]$$

and so $[-,-]$ is lax, as desired. ∎

4.3. Example. Suppose $\mathcal{D}$ is a cartesian closed category with equalizers. To see that $\mathbb{S}\mathrm{pan}(\mathcal{D})$ is a cartesian closed double category, we first consider the case where $\mathcal{D} = \mathrm{Sets}$.

One can show that $\mathbb{S}\mathrm{pan}(\mathrm{Sets})$ is cartesian closed as follows. Let $[Y, Z]_0 = Z^Y$ and $[v, w]_1$ be given by the span

$$\begin{array}{ccc} & & Z_s^{Y_s} \\ & \nearrow^{[v,w]_s} & \\ [v,w] & & \\ & \searrow_{[v,w]_t} & \\ & & Z_t^{Y_t} \end{array} \qquad \text{where} \qquad [v,w] = \left\{ \begin{array}{ccccc} & \overset{v_s}{\nearrow} & Y_s & \xrightarrow{\sigma_s} & Z_s \\ Y & \xrightarrow{\sigma} & Z & \overset{w_s}{\nearrow} \underset{w_t}{\searrow} & \\ & \underset{v_t}{\searrow} & Y_t & \xrightarrow[\sigma_t]{} & Z_t \end{array} \right\}$$

Then

$$\begin{array}{ccccc} & \overset{u_s \times v_s}{\nearrow} & X_s \times Y_s & \xrightarrow{f_s} & Z_s \\ X \times Y & \xrightarrow{f} & Z & \overset{w_s}{\nearrow} \underset{w_t}{\searrow} & \\ & \underset{u_t \times v_t}{\searrow} & X_t \times Y_t & \xrightarrow[f_t]{} & Z_t \end{array} \quad \text{corresponds to} \quad \begin{array}{ccccc} & \overset{u_s}{\nearrow} & X_s & \xrightarrow{\hat{f}_s} & Z_s^{Y_s} \\ X & \xrightarrow{\hat{f}} & [v,w] & \overset{[v,w]_s}{\nearrow} \underset{[v,w]_t}{\searrow} & \\ & \underset{u_t}{\searrow} & X_t & \xrightarrow[\hat{f}_t]{} & Z_t^{Y_t} \end{array}$$

where $\hat{f}_s(x_s)(y_s) = f_s(x_s, y_s)$, $\hat{f}_t(x_t)(y_t) = f_t(x_t, y_t)$, and $\hat{f}(x)$ is obtained by fixing x in the diagram on the left.

To generalize to $\mathbb{S}\mathrm{pan}(\mathcal{D})$, one can define $[v,w]$, by the equalizer

$$[v,w] \rightarrowtail Z_s^{Y_s} \times Z^Y \times Z_t^{Y_t} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Z_s^Y \times Z_t^Y$$

in $\mathcal{D}$, where $\alpha = \langle w_s^Y \pi_2, w_t^Y \pi_2 \rangle$ and $\beta = \langle Z_s^{v_s} \pi_1, Z_t^{v_t} \pi_3 \rangle$; and proceed as above. Moreover, since $\mathbb{S}\mathrm{pan}(\mathcal{D})_0 \cong \mathcal{D}$, one can show that $\mathbb{S}\mathrm{pan}(\mathcal{D})$ is cartesian closed if and only if $\mathcal{D}$ is.

Using the equivalences $\mathbb{P}\mathrm{os}_1 \simeq \mathbb{P}\mathrm{os}_0/\mathbb{2}$ and $\mathbb{C}\mathrm{at}_1 \simeq \mathbb{C}\mathrm{at}_0/\mathbb{2}$, one can show that $\mathbb{P}\mathrm{os}$ and $\mathbb{C}\mathrm{at}$ are cartesian closed double categories (see [N12a]). Instead, we give an explicit description of the exponentials, rather than introduce glueing categories here.

4.4. Example. For $\mathbb{P}\mathrm{os}$, first note that $\mathbb{P}\mathrm{os}_0$ is cartesian closed with $[Y,Z] = Z^Y$, the poset of order-preserving maps $\sigma \colon Y \longrightarrow Z$ with the pointwise order. To show $\mathbb{P}\mathrm{os}_1$ is cartesian closed, recall $((x_s, y_s), (x_t, y_t)) \in u \times v$ if and only if $(x_s, x_t) \in u$ and $(y_s, y_t) \in v$, where $u \colon X_s \multimap X_t$ and $v \colon Y_s \multimap Y_t$. Given $w \colon Z_s \multimap Z_t$, consider $[v,w] \colon Z_s^{Y_s} \multimap Z_t^{Y_t}$ defined by $(\sigma_s, \sigma_t) \in [v,w]$ if and only if $(\sigma_s(y_s), \sigma_t(y_t)) \in w, \forall (y_s, y_t) \in v$. Then

$$u \times v \underset{f_t}{\overset{f_s}{\to}} w \quad \text{corresponds to} \quad u \underset{\hat{f}_t}{\overset{\hat{f}_s}{\to}} [v,w]$$

since

$$((x_s, y_s), (x_t, y_t)) \in u \times v \ \text{ implies } \ (f_s(x_s, y_s), f_t(x_t, y_t)) \in w$$

if and only if

$$(x_s, x_t) \in u \ \text{ implies } \ (f_s(x_s, y_s), f_t(x_t, y_t)) \in w, \forall (y_s, y_t) \in v$$

if and only if

$$(x_s, x_t) \in u \ \text{ implies } \ (\hat{f}_s(x_s), \hat{f}_t(x_t) \in [v,w]$$

Therefore, $\mathbb{P}\mathrm{os}$ is cartesian closed.

4.5. Example. For $\mathbb{C}\mathrm{at}$, first note that $\mathbb{C}\mathrm{at}_0$ is cartesian closed with $[Y, Z] = Z^Y$, the category of functors $\sigma: Y \longrightarrow Z$ with natural transformations as morphisms. To show $\mathbb{C}\mathrm{at}_1$ is cartesian closed, recall that

$$(u \times v)((x_s, y_s), (x_t, y_t)) = u(x_s, x_t) \times v(y_s, y_t)$$

where $u: X_s \xrightarrow{\bullet} X_t$ and $v: Y_s \xrightarrow{\bullet} Y_t$. Given $w: Z_s \xrightarrow{\bullet} Z_t$, consider $[v, w]: Z_s^{Y_s} \xrightarrow{\bullet} Z_t^{Y_t}$, where $[v, w](\sigma_s, \sigma_t)$ is the set of cells of the form $\varphi: v \xrightarrow[\sigma_t]{\sigma_s} w$ in $\mathbb{C}\mathrm{at}$. Then, one can show that $[v, w]$ is a profunctor; and cells

$$u \times v \xrightarrow[f_t]{f_s} w \quad \text{correspond bijectively with} \quad u \xrightarrow[\hat{f}_t]{\hat{f}_s} [v, w]$$

since functions $u(x_s, x_t) \times v(y_s, y_t) \longrightarrow w(f_s(x_s, y_s), f_t(x_t, y_t))$ correspond naturally to

$$u(x_s, x_t) \longrightarrow \mathrm{Sets}(v(y_s, y_t), w(f_s(x_s, y_s), f_t(x_t, y_t)))$$

in Sets. Because

$$\mathrm{Sets}(v(y_s, y_t), w(f_s(x_s, y_s), f_t(x_t, y_t))) = \mathrm{Sets}(v(y_s, y_t), w(\hat{f}_s(x_s)(y_s), \hat{f}_t(x_t)(y_s)))$$

we get

$$u(x_s, x_t) \longrightarrow [v, w](\hat{f}_s(x_s), \hat{f}_t(x_t))$$

as desired.

4.6. Example. Consider Q-$\mathbb{R}\mathrm{el}$, where Q is a locale. Given $v: Y_s \xrightarrow{\bullet} Y_t$ and $w: Z_s \xrightarrow{\bullet} Z_t$, define $[v, w]: Z_s^{Y_s} \xrightarrow{\bullet} Z_t^{Y_t}$ by

$$[v, w](\sigma_s, \sigma_t) = \bigwedge_{(y_s, y_t)} v(y_s, y_t) \Rightarrow w(\sigma_s y_s, \sigma_t y_t)$$

where $a \Rightarrow (-)$ is the right adjoint to $a \wedge (-)$ in the locale Q. Then, there is a cell

$$\begin{array}{ccc} X_s \times Y_s & \xrightarrow{f_s} & Z_s \\ {\scriptstyle u \times v}\downarrow\bullet & \leq & \bullet\downarrow{\scriptstyle w} \\ X_t \times Y_t & \xrightarrow[f_t]{} & Z_t \end{array}$$

if and only if

$$u(x_s, x_t) \wedge v(y_s, y_t) \leq w(f_s(x_s, y_s), f_t(x_t, y_t))$$

if and only if

$$u(x_s, x_t) \leq \big(v(y_s, y_t) \Rightarrow w(f_s(x_s, y_s), f_t(x_t, y_t))\big), \forall (y_s, y_t)$$

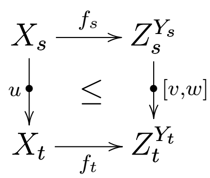
if and only if

$$u(x_s, x_t) \leq \bigwedge_{(y_s, y_t)} \big(v(y_s, y_t) \Rightarrow w(f_s(x_s, y_s), f_t(x_t, y_t)) \big)$$

if and only if

$$u(x_s, x_t) \leq [v, w](\hat{f}_s(x_s), \hat{f}_t(x_t))$$

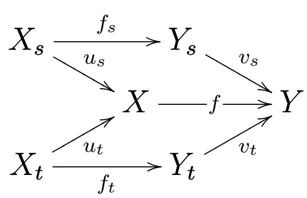
if and only if there is a cell



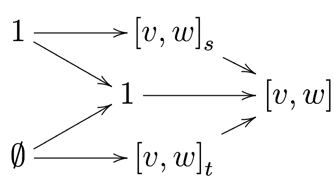
Therefore, Q-$\mathbb{R}$el is cartesian closed when Q is a locale.

We conclude this section with an example of a double category $\mathbb{D}$ such that $\mathbb{D}_0$ and $\mathbb{D}_1$ are cartesian closed but $\mathbb{D}$ is not cartesian closed as a double category, showing that compatibility with s and t is necessary in Defininition 4.1. But, we will see that this example is object-wise cartesian closed in the sense of Section 5.

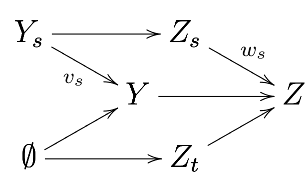
4.7. Example. Let $\mathbb{C}$ospan denote the double category whose objects and horizontal morphisms are sets and functions, vertical morphisms are cospans with composition via pushout, and cells are diagrams of form



One can show that $\mathbb{C}$ospan is a cartesian double category by Proposition 3.1. We know $\mathbb{C}\text{ospan}_0$ is cartesian closed, since Sets is, and $\mathbb{C}\text{ospan}_1$ is cartesian closed, since it is a presheaf topos. But, the exponentials are not compatible with s, i.e., $[v, w]_s \neq Z_s^{Y_s}$, where $[v, w]_s \longleftarrow [v, w] \longrightarrow [v, w]_t$ denotes the exponential in $\mathbb{C}\text{ospan}_1$, since elements of $[v, w]_s$ correspond to diagrams



or equivalently



5. Object-Wise Cartesian Closed Double Categories

In this section, we introduce object-wise cartesian closure in **LxDbl**, called "pre-cartesian closure" in [N20]; and compare this notion to cartesian closure introduced in Section 4.

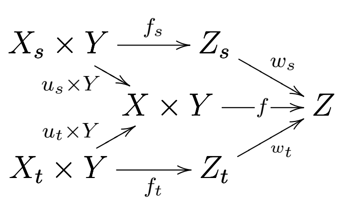
Suppose Y is an object of $\mathbb{D}$ such that the product $X \times Y$ exists in $\mathbb{D}_0$, for all X; and the product $u \times \mathrm{id}^\bullet_Y$ exists in $\mathbb{D}_1$ and satisfies $s(u \times \mathrm{id}^\bullet_Y) = su \times Y$ and $t(u \times \mathrm{id}^\bullet_Y) = tu \times Y$ in $\mathbb{D}_0$, for all u. Taking $u \times Y = u \times \mathrm{id}^\bullet_Y$ and applying Proposition 3.1, we get a lax functor $(-) \times Y : \mathbb{D} \to \mathbb{D}$. In this case, we say $(-) \times Y$ exists in **LxDbl**.

5.1. Definition. *An object Y is called exponentiable in $\mathbb{D}$ if $(-) \times Y$ exists and has a right adjoint in* **LxDbl**; *and $\mathbb{D}$ is called object-wise cartesian closed if every object Y is exponentiable in $\mathbb{D}$.*

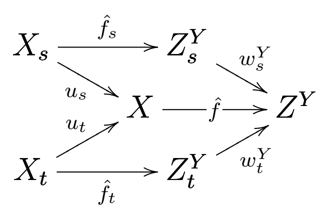
Every cartesian closed double category, in the sense of Definition 4.1, is clearly object-wise cartesian closed. Thus, $\mathbb{S}\mathrm{pan}(\mathcal{D})$, $\mathbb{C}\mathrm{at}$, $\mathbb{P}\mathrm{os}$, and Q-$\mathbb{R}\mathrm{el}$ are object-wise cartesian closed, for every cartesian closed category $\mathcal{D}$ with equalizers and every locale Q (see Examples 4.3–4.6). The following generalization of Example 4.7 gives object-wise cartesian closed double categories which are not cartesian closed.

5.2. Example. Suppose $\mathcal{D}$ is a category with pushouts and finite products, and let $\mathbb{C}\mathrm{ospan}(\mathcal{D})$ denote the double category whose objects and horizontal morphisms are those of $\mathcal{D}$, vertical morphisms are cospans in $\mathcal{D}$ with composition via pushout, and cells are commutative diagrams in $\mathcal{D}$. One can show that $\mathbb{C}\mathrm{ospan}(\mathcal{D})$ is lax cartesian by Proposition 3.1, but we know it is not, in general, cartesian closed by Example 4.7.

Now, if Y is exponentiable in $\mathcal{D}$, then $(-) \times Y$ is a pseudo functor on $\mathbb{C}\mathrm{ospan}(\mathcal{D})$, since $(-) \times Y$ preserves pushouts on $\mathcal{D}$ being a left adjoint, and so Y is exponentiable in $\mathbb{C}\mathrm{ospan}(\mathcal{D})$ by Proposition 3.1, since



corresponds to



by the naturality in the definition of the adjunction $(-) \times Y \dashv (-)^Y$ on $\mathcal{D}$.

Therefore, $\mathbb{C}\mathrm{ospan}(\mathcal{D})$ is object-wise cartesian closed whenever $\mathcal{D}$ is cartesian closed.

6. Exponentiable Objects in Double Categories

In [N20], we showed that the exponentiable objects of $\mathbb{D}$ are those of $\mathbb{D}_0$, for glueing categories. Examples included $\mathbb{C}\mathrm{at}$, $\mathbb{P}\mathrm{os}$, $\mathbb{L}\mathrm{oc}$, and $\mathbb{T}\mathrm{op}$. See Section 4 above for a direct construction of exponentials in $\mathbb{C}\mathrm{at}$ and $\mathbb{P}\mathrm{os}$. Here, we use Theorem 5.5 from [N12b] to present a more general construction which applies to these double categories, as well as $\mathbb{Q}\mathrm{uant}^{\mathrm{op}}$. The latter example was not considered in [N20], since we were unable to determine if $\mathbb{Q}\mathrm{uant}^{\mathrm{op}}$ is a glueing category. After reviewing the definitions of companions, conjoints, and cotabulators, we present this general construction using Theorem 5.5 from [N12b], which we then apply directly to $\mathbb{T}\mathrm{op}$ and $\mathbb{Q}\mathrm{uant}$. The construction for $\mathbb{L}\mathrm{oc}$ is similar to that of $\mathbb{T}\mathrm{op}$. For more on companions, conjoints, and cotabulators, see [GP04] or [GP17].

A *companion* for $f\colon X \longrightarrow Y$ is a vertical morphism $f_*\colon X \longrightarrow\!\!\!\!\!\bullet\;\, Y$ together with cells

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \mathrm{id}^\bullet_X}\,\big\downarrow\bullet & \alpha & \bullet\big\downarrow\,{\scriptstyle f_*} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f_*}\,\big\downarrow\bullet & \beta & \bullet\big\downarrow\,{\scriptstyle \mathrm{id}^\bullet_Y} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

whose horizontal and vertical compositions are identity cells. A *conjoint* for f is a vertical morphism $f^*\colon Y \longrightarrow\!\!\!\!\!\bullet\;\, X$ together with cells

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}^\bullet_X}\,\big\downarrow\bullet & \rho & \bullet\big\downarrow\,{\scriptstyle f^*} \\ X & \xrightarrow[\mathrm{id}_X]{} & X \end{array} \qquad\qquad \begin{array}{ccc} Y & \xrightarrow{\mathrm{id}_Y} & Y \\ {\scriptstyle f^*}\,\big\downarrow\bullet & \sigma & \bullet\big\downarrow\,{\scriptstyle \mathrm{id}^\bullet_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

whose horizontal and vertical compositions are identity cells.

A *cotabulator* of a vertical morphism $u\colon X_s \longrightarrow\!\!\!\!\!\bullet\;\, X_t$ consists an object Γu and a cell

$$\begin{array}{ccc} X_s & & \\ & \searrow^{i_s} & \\ {\scriptstyle u}\,\big\downarrow\bullet \;\; \gamma_u & & \Gamma u \\ & \nearrow_{i_t} & \\ X_t & & \end{array} \qquad (\star\star)$$

such that for any other cell

$$\begin{array}{ccc} X_s & & \\ & \searrow^{f_s} & \\ {\scriptstyle u}\,\big\downarrow\bullet \;\; \varphi & & Y \\ & \nearrow_{f_t} & \\ X_t & & \end{array}$$

there exists a unique horizontal morphism $f\colon \Gamma u \longrightarrow Y$ such that $\mathrm{id}^\bullet_f \gamma_u = \varphi$. The cotabulator is called *strong* if the following cell obtained from $(\star\star)$ by the universal properties of the companion $(i_s)_*$ and conjoint $(i_t)^*$

$$\begin{array}{ccc} X_s & \xrightarrow{\mathrm{id}} & X_s \\ & & \bullet\big\downarrow\,{\scriptstyle (i_s)_*} \\ {\scriptstyle u}\,\big\downarrow\bullet & \hat{\gamma}_u & \Gamma u \\ & & \bullet\big\downarrow\,{\scriptstyle (i_t)^*} \\ X_t & \xrightarrow[\mathrm{id}]{} & X_t \end{array}$$

is invertible. One can show that the cotabulator Γu exists, for all u, if and only if $\mathrm{id}^\bullet\colon \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ has a left adjoint (induced by Γ) [GP99]. *Tabulators*

$$\begin{array}{ccc} & & X_s \\ & \overset{\pi_s}{\nearrow} & \\ \Sigma u & \tau_u & \big\downarrow{\scriptstyle\bullet}\, u \\ & \underset{\pi_t}{\searrow} & \\ & & X_t \end{array} \qquad (\star\star\star)$$

for u in $\mathbb{D}$ are defined dually.

We know that $\mathbb{C}\mathrm{at}$, $\mathbb{P}\mathrm{os}$, $\mathbb{L}\mathrm{oc}$, and $\mathbb{T}\mathrm{op}$ have companions, conjoints, and strong cotabulators (see [N12b] or [N20]), and $\mathbb{Q}\mathrm{uant}$ has strong tabulators by a construction similar to that of $\mathbb{L}\mathrm{oc}^{\mathrm{op}}$.

6.1. Theorem. [N12b] *Suppose $\mathbb{D}$ is a double category such that $\mathbb{D}_0$ has pushouts. Then $\mathbb{D}$ has companions, conjoints, and cotabulators if and only if there is an oplax/lax adjunction*

$$\mathbb{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$$

such that G is normal and restricts to the identity on $\mathbb{D}_0$ Moreover, if cotabulators are strong in $\mathbb{D}$, then the unit $\eta\colon \mathrm{id}_{\mathbb{D}} \longrightarrow GF$ is invertible.

Proof. The proof of the first part is in [N12b]. In particular, it is shown that the left adjoint F is induced by the cotabulator diagram $(\star\star)$ and the right adjoint G is given by

$$G(X_s \xrightarrow{f_s} X \xleftarrow{f_t} X_t) = (f_t)^*(f_s)_*$$

It easily follows that $\eta\colon \mathrm{id}_{\mathbb{D}} \longrightarrow GF$ is invertible, when cotabulators are strong in $\mathbb{D}$.

■

6.2. Corollary. *Suppose $\mathbb{D}$ is a double category such that $\mathbb{D}_0$ has pushouts; and $(-)\times Y$ exists in* **LxDbl**. *If Y is exponentiable in $\mathbb{D}$, then Y is exponentiable in $\mathbb{D}_0$ and $(-)\times Y$ is oplax. The converse holds if $\mathbb{D}$ has companions, conjoints, strong cotabulators, and $\Gamma(u \times Y) \cong \Gamma u \times Y$, for all u.*

Proof. The first part follows from Proposition 3.1. For the converse, we know Y is exponentiable in $\mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$ by Example 5.2. Consider $[[Y,-]]$ defined on $\mathbb{D}$ by the composite

$$\mathbb{D} \xrightarrow{F} \mathbb{C}\mathrm{ospan}(\mathbb{D}_0) \xrightarrow{[Y,-]} \mathbb{C}\mathrm{ospan}(\mathbb{D}_0) \xrightarrow{G} \mathbb{D}$$

Given $u\colon X_s \xrightarrow{\bullet} X_t$ and $v\colon Z_s \xrightarrow{\bullet} Z_t$, we get

$$\begin{aligned} \mathbb{D}_1(u \times Y, v) &\cong \mathbb{D}_1(u \times Y, GFv) \\ &\cong \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)_1\big(F(u \times Y), Fv\big) \\ &\cong \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)_1\big(Fu \times Y, Fv\big) \end{aligned}$$

$$\begin{aligned}&\cong \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)_1\big(Fu,[Y,Fv]\big)\\&\cong \mathbb{D}_1(u,G([Y,Fv]))\\&\cong \mathbb{D}_1(u,[[Y,v]])\end{aligned}$$

where the first equivalence holds since cotabulators are strong, and the second since $\Gamma(u\times Y)\cong\Gamma u\times Y$. Therefore, $[[Y,-]]$ is right adjoint to $(-)\times Y$ by Proposition 3.1, since $(-)\times Y$ is oplax, $s[[Y,v]]=Z_s^Y$, and $t[[Y,v]]=Z_t^Y$. ∎

6.3. Example. From [DK70], we know Y is exponentiable in Top if and only if $\mathcal{O}(Y)$ is a continuous lattice, in the sense of [Sc72]. Since $\mathbb{T}\mathrm{op}$ has companions, conjoints, and strong cotabulators, to see that such a Y is exponentiable in $\mathbb{T}\mathrm{op}$, applying Corollary 6.2, it suffices to show that $(-)\times Y$ exists in **LxDbl** and $(-)\times Y\colon\mathbb{T}\mathrm{op}\longrightarrow\mathbb{T}\mathrm{op}$ is oplax; and $\Gamma(u\times Y)\cong\Gamma u\times Y$, for all $u\colon X_s \,{-\!\!\bullet\!\!\!\rightarrow}\, X_t$.

Recall (from [J82]), since Y is exponentiable, we know $\mathcal{O}(X\times Y)\cong\mathcal{O}(X)\otimes\mathcal{O}(Y)$, for all X, where $\otimes$ denotes the product of locales, and so $\mathcal{O}(X\times Y)$ is the product of $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ in Loc. Taking $u\times Y\colon\mathcal{O}(X_s\times Y) \,{-\!\!\bullet\!\!\!\rightarrow}\, \mathcal{O}(X_t\times Y)$ induced by $U_s\times V\mapsto u(U_s)\times V$, one can show that $(-)\times Y$ exists and defines a pseudo functor $(-)\times Y\colon\mathbb{T}\mathrm{op}\longrightarrow\mathbb{T}\mathrm{op}$. In particular, $(-)\times Y$ is oplax.

To see that $\Gamma(u\times Y)\cong\Gamma u\times Y$, recall (from [N12b]) that Γu is the disjoint union $X_s\sqcup X_t$, with basic opens $U_s\sqcup U_t$ such that $U_t\subseteq u(U_s)$, and so the map $f\colon\Gamma(u\times Y)\longrightarrow\Gamma u\times Y$ is the continuous bijection $(X_s\times Y)\sqcup(X_t\times Y)\longrightarrow(X_s\sqcup X_t)\times Y$. We claim that this is an open map. Suppose $W\subseteq\Gamma(u\times Y)$ is open. If $(x_s,y)\in W$, then

$$(x_s,y)\in(U_s\times V)\sqcup(u(U_s)\times V)\subseteq W$$

for some $U_s\times V\in\mathcal{O}(X_s\times Y)$, and $f((U_s\times V)\sqcup(u(U_s)\times V))=(U_s\sqcup u(U_s))\times V$ which is open in $\Gamma u\times Y$. If $(x_t,y)\in W$, then $(x_t,y)\in(U_s\times V_s)\sqcup(U_t\times V_t)\subseteq W$, for some $U_s\times V_s\in\mathcal{O}(X_s\times Y)$ and $U_t\times V_t\in\mathcal{O}(X_t\times Y)$ such that $U_t\subseteq u(U_s)$ and $V_t\subseteq V_s$, Then $(x_t,y)\in(U_s\times V_t)\sqcup(U_t\times V_t)\subseteq W$, which is open in $\Gamma(u\times Y)$, and

$$f((U_s\times V_t)\sqcup(U_t\times V_t))=(U_s\sqcup U_t)\times V_t$$

is open in $\Gamma u\times Y$. Therefore, Y is exponenitable in $\mathbb{T}\mathrm{op}$ if and only if Y is exponenitable in Top if and only if $\mathcal{O}(Y)$ is a continuous lattice.

Next, we apply the dual of Corollary 6.2 to the double category $\mathbb{Q}\mathrm{uant}$ of commutative unital quantales.

6.4. Example. We know coproducts in Quant are given by the tensor product $\otimes$, since Quant is the category of commutative monoids in the symmetric monoidal category Sup of suplattices, i.e., complete lattices and sup-preserving maps (see [JT84]). In [N16], we showed that Y is coexponentiable in Quant if and only if it is projective in Sup if and only if it is a totally continuous lattice, or equivalently, a constructively complete distributive (CCD) lattice, in the sense of [FW90].

Now, Y is projective in Sup if and only if $Y \otimes Z \cong \mathrm{Sup}(\mathrm{Sup}(Y, 2), Z)$, for all suplattices Z (see [JT84]). Since $\mathrm{Sup}(\mathrm{Sup}(Y, 2), Z) \cong \mathrm{Pos}(Y, Z)$, it follows that $Y \otimes Z \cong \mathrm{Pos}(Y, Z)$, whenever Y is coexponentiable in Quant. Moreover, under this isomorphism, $\mathrm{Pos}(Y, Z)$ is a quantale via $(\alpha\beta)(y) = \alpha(y)\beta(y)$ with unit given by the constant e-valued map; and $Y \otimes v \colon Y \otimes Z_s \to Y \otimes Z_t$ is given by $\mathrm{Pos}(Y, v) \colon \mathrm{Pos}(Y, Z_s) \to \mathrm{Pos}(Y, Z_t)$. Thus, $Y \otimes (-)$ defines a pseudo functor on $\mathbb{Q}\mathrm{uant}$, and so to apply the dual of Corollary 6.2, it suffices to show that $\mathbb{Q}\mathrm{uant}$ has companions, conjoints, strong tabulators Σ, and $Y \otimes \Sigma v \cong \Sigma(Y \otimes v)$, or equivalently, $\mathrm{Pos}(Y, \Sigma v) \cong \Sigma\mathrm{Pos}(Y, v)$, for all v.

The companion and conjoint of $f \colon X \to Y$ are given by $f_* = f$ and its right adjoint f^*. Note that f^* is a lax map, since f preserves the quantale operation and unit. The tabulator of $u \colon X_s \to X_t$ is poset $\Sigma u = \{(x_s, x_t) | x_t \leq u x_s\}$, which is a quantale since it is closed under the product and unit, and hence, a subquantale of the product (see [NR88] or [R90]). Tabulators are strong, since π_s and π_t are the projections, and so $\pi_s^*(x_s) = (x_s, u x_s)$. To see that the induced map $\bar{f} \colon \mathrm{Pos}(Y, \Sigma v) \to \Sigma\mathrm{Pos}(Y, v)$ is invertible, consider the commutative diagram of inverters (in the sense of [CJSV94])

$$\begin{array}{ccccc}
\mathrm{Pos}(Y, \Sigma v) & \longrightarrow & \mathrm{Pos}(Y, Z_s \times Z_t) & \overset{\mathrm{Pos}(Y,\pi_t)}{\underset{\mathrm{Pos}(Y,v\pi_s)}{\rightrightarrows}} & \mathrm{Pos}(Y, Z_t) \\
{\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} \\
\Sigma(\mathrm{Pos}(Y, v)) & \longrightarrow & \mathrm{Pos}(Y, Z_s) \times \mathrm{Pos}(Y, Z_t) & \overset{\pi_t}{\underset{\mathrm{Pos}(Y,v)\pi_s}{\rightrightarrows}} & \mathrm{Pos}(Y, Z_t)
\end{array}$$

where $f = \langle \mathrm{Pos}(Y, \pi_s), \mathrm{Pos}(Y, \pi_t) \rangle$. Since $\mathrm{Pos}(Y, -)$ preserves products, being a right adjoint, we know f is invertible, and it follows that so is $\bar{f}$, as desired.

Thus, Y is coexponentiable in $\mathbb{Q}\mathrm{uant}$ if and only if it is coexponentiable in Quant if and only if it is a totally continuous lattice if and only if it is CCD.

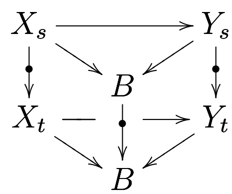
7. Locally Cartesian Closed Double Categories

In this section, we give the definition and examples of local cartesian closed double categories, but first we recall from [P11] the definition of the double slice category $\mathbb{D}//B$.

Objects of $\mathbb{D}//B$ are horizontal morphisms $X \to B$, horizontal arrows are commutative triangles, vertical arrows are cells

$$\begin{array}{ccc}
X_s & \xrightarrow{p_s} & B \\
{\scriptstyle u}\big\downarrow\!\bullet & \pi & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}^\bullet_B} \\
X_t & \xrightarrow[p_t]{} & B
\end{array}$$

and cells are commutative diagrams of cells



with the induced horizontal and vertical composition, that is

$$(\mathbb{D}/\!/B)_0 = \mathbb{D}/B \qquad \text{and} \qquad (\mathbb{D}_0/\!/B)_1 = \mathbb{D}_1/\operatorname{id}^\bullet_B$$

7.1. Definition. *A double category* $\mathbb{D}$ *is called locally cartesian closed if* $\mathbb{D}/\!/B$ *is cartesian closed, for every object* B.

Note that $\mathbb{C}\mathrm{at}$ and $\mathbb{P}\mathrm{os}$ are not locally cartesian closed by Theorem 4.2, since

$$(\mathbb{C}\mathrm{at}/\!/2)_1 \simeq \mathbb{C}\mathrm{at}/(2 \times 2) \quad \text{and} \quad (\mathbb{P}\mathrm{os}/\!/2)_1 \simeq \mathbb{P}\mathrm{os}/(2 \times 2)$$

which are not cartesian closed by [G64] and [N01], respectively. We will see that $\mathbb{S}\mathrm{pan}(\mathcal{D})$ is locally cartesian closed, for every locally cartesian closed category $\mathcal{D}$, as is Q-$\mathbb{R}\mathrm{el}$, for every locale Q.

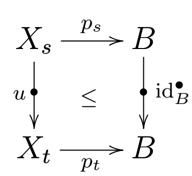
7.2. Example. Suppose B is an object of $\mathcal{D}$. Then one can show that

$$\mathbb{S}\mathrm{pan}(\mathcal{D})/\!/B \simeq \mathbb{S}\mathrm{pan}(\mathcal{D}/B)$$

and so $\mathbb{S}\mathrm{pan}(\mathcal{D})/\!/B$ is cartesian closed if $\mathcal{D}/B$ is a cartesian closed category with equalizers. Thus, $\mathbb{S}\mathrm{pan}(\mathcal{D})$ is locally cartesian closed whenever $\mathcal{D}$ is.

7.3. Example. Suppose Q is a locale. To see that Q-$\mathbb{R}\mathrm{el}/\!/B$ is cartesian closed, for every set B, we first show it is equivalent to (B^*Q)-$\mathbb{R}\mathrm{el}(\mathrm{Sets}/B)$, where B^*Q is the internal locale $\pi_1\colon B \times Q \longrightarrow B$ in the topos Sets/B. Since the construction for Q-$\mathbb{R}\mathrm{el}$ in Example 4.6 is valid for any internal locale in a topos $\mathcal{E}$, taking $\mathcal{E} = \mathrm{Sets}/\mathcal{B}$, the desired result will follow.

By definition of $\operatorname{id}^\bullet_B$ in Example 2.4, a vertical morphism



in Q-$\mathbb{R}\mathrm{el}/\!/B$ is given by a function $u\colon X_s \times X_t \longrightarrow Q$ such that $u(x_s, x_t) = 0$ if $x_s \neq x_t$, and hence, a morphism $\langle p, \bar{u}\rangle\colon X_s \times_B X_t \longrightarrow B \times Q$ in Sets/B, where $\bar{u}$ is the restriction of u to $X_s \times_B X_t$. Conversly, given $\bar{u}\colon X_s \times_B X_t \longrightarrow B$, define

$$u(x_s, x_t) = \begin{cases} \bar{u}(x_s, x_t) & \text{if } x_s = x_t \\ 0 & \text{if } x_s \neq x_t \end{cases}$$

and the desired equivalence follows.

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Union College
Department of Mathematics
Schenectady, NY 12308
Email: niefiels@union.edu

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