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ON THE CONSTRUCTION OF NOETHERIAN FORMS FOR ALGEBRAIC STRUCTURES

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ABSTRACT. A Noetherian form is a self-dual axiomatic context in which the Noether isomorphism theorems and other homomorphism theorems can be established. These theorems for group-like algebraic structures (for example groups, rings without unity and vector spaces) can be obtained by choosing a Noetherian form based on lattices of subalgebras. In this paper we show that by replacing lattices of subalgebras with some other lattices, it becomes possible to move beyond group-like structures and encompass all types of algebraic structures (including sets, monoids, lattices). Moreover, we show that in a suitable sense, existence of a Noetherian form for a give type of mathematical structure is intimately linked with algebraicity of structures. The isomorphism theorems resulting from applying these Noetherian forms recover the isomorphism theorems known for general algebraic structures in the literature.

1. Introduction

For groups, the Noether isomorphism theorems are results on quotient groups. One way of establishing these theorems, is by only using properties of the subgroup lattices together with homomorphisms (see for example [5]). A Noetherian form (see [1] and [6]) is a self-dual generalization of the context of groups, subgroup lattices and group homomorphisms. The axioms for a Noetherian form are a sufficient set of conditions that allows us to establish homomorphism theorems for groups, such as the Noether isomorphism theorems and homological diagram lemmas. Some of these conditions are:

- for any group homomorphism $f\colon X \to Y$ and subgroup $A \leqslant X$, we have

$$f^{-1}fA = A \vee \operatorname{Ker} f;$$

- surjective group homomorphisms p have the property that if $\operatorname{Ker} g \geqslant \operatorname{Ker} p$ for any homomorphism g, then there exists unique h such that $g = hp$;
- the join of normal subgroups is normal.

For group-like varieties (those varieties whose category is a semi-abelian category), their subalgebras lattices together with the usual direct- and inverse images gives rise to a

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Noetherian form. But for the non group-like varieties, the substructure lattices do not necessarily have these properties. For example, the variety of sets together with functions and subset lattices. In a Noetherian form, the kernel is defined as the inverse image of the least subobject. So for sets and subset lattices, $\operatorname{Ker} f = f^{-1}\emptyset = \emptyset$ for any function f. So immediately the first point will fail.

Noetherian forms are general enough, that they allow freedom for selecting different subobject lattices (as long as the axioms are satisfied). Together with Zurab Janelidze, suitable "subobject lattices" for sets were found (see [7]): For each set X, its new subobject lattice is

$$\begin{aligned}\operatorname{sub} X = \{(A,R) \mid & R \text{ is equivalence relation on } X, \text{ and} \\ & A \text{ is either an equivalence class of } R \text{ or empty subset}\},\end{aligned}$$

where the ordering is component-wise (refer to Section 6 for details). Both the lattice of subsets and the lattice of equivalence relations of X are sublattices of this new lattice.

For convenience, we can capture this new subobject lattice (together with the direct- and inverse image maps) as a functor $F\colon \mathbb{A} \to \mathsf{Set}$ to the category of sets and functions. More explicitly, we can construct category $\mathbb{A}$ and functor F, such that $\operatorname{sub} X \cong \{S \in \mathbb{A} \mid FS = X\}$, for any set X. Likewise, any Noetherian form can be conveniently defined as a functor having suitable properties. An advantage of representing a Noetherian form as a functor, is that it simplifies construction of Noetherian forms out of old ones. For example, if $\mathbb{V}$ is the category of a variety together with its homomorphisms, then the pullback of the Noetherian form $F\colon \mathbb{A} \to \mathsf{Set}$ along the forget $U\colon \mathbb{V} \to \mathsf{Set}$ turns out to also be a Noetherian form (see Corollary 6.2). The resulting subobject lattices for each algebra will then again be lattices which contain both the subalgebras as well as the congruences. The Noetherian isomorphism theorems that arise form these Noetherian forms are almost precisely equivalent to the known isomorphism theorems in universal algebra. These details are discussed in Section 6.

Since the forgetful functor $U\colon \mathbb{V} \to \mathsf{Set}$ from the category of any variety to sets is a right adjoint, the goal of this paper is then to study more generally when a pullback of a Noetherian form along a right adjoint results in a Noetherian form (Sections 2 – 4). One of the main results is that if this pullback is Noetherian, then the right adjoint U has to be monadic. So in particular, this construction of Noetherian forms for varieties will not hold for topological spaces (whether topological spaces has a Noetherian form, is still an open question).

2. Noetherian forms

2.1. Definitions and axioms. In this section, we introduce Noetherian and related forms and all the results which will be used throughout this paper. No further knowledge of Noetherian forms is required for this paper. Essentially an "orean" form is a category where every object is equipped with a bounded lattice and every morphism induces a direct and inverse image between these lattices. While a "Noetherian" form is an orean form satisfying three further axioms. The prime example of a Noetherian form is the category of groups together with its subgroup lattices. Keeping this example in mind, the new abstract concepts are other ways to generalize the known classical concepts.

2.2. Definition. *A* form *over a category* $\mathbb{C}$ *is a faithful amnestic functor* $F\colon \mathbb{B} \to \mathbb{C}$*, for some category* $\mathbb{B}$*.*

Recall that a functor is amnestic when the only isomorphisms mapping to the identity morphisms by the functor are the identity morphisms.

In a form $F\colon \mathbb{B} \to \mathbb{C}$, we can define the following structure, for every $A \in \mathbb{C}$ and $f\colon A \to B$ in $\mathbb{C}$:

- $\operatorname{sub} A = \{S \in \mathbb{B} \mid FS = A\}$;
- $R \geqslant_f S$ if and only if there is $k\colon S \to R$ in $\mathbb{B}$ such that $Fk = f$.

Elements of $\operatorname{sub} A$ are called *subobjects* of A.

2.3. Definition. *An* orean *form* F *over a category* $\mathbb{C}$ *is a form satisfying the following, for any* $f\colon A \to B$ *and* $g\colon B \to C$*:*

- $\operatorname{sub} A$ *under the ordering* $\geqslant_{1_A}$ *forms a bounded lattice with top element* $\top^A$ *and bottom element* $\bot^A$*;*
- $\{R \in \operatorname{sub} B \mid R \geqslant_f S\}$ *has a minimum element, which is called the* direct image *of* S *under* f *and is denoted by* $f \cdot^F S$*;*
- $\{S \in \operatorname{sub} A \mid R \geqslant_f S\}$ *has a maximum element, which is called the* inverse image *of* R *under* f *and is denoted by* $R \cdot^F f$*;*
- *we have the following identities, where* 1_A *is the identity morphism:*

$$1_A \cdot^F S = S = S \cdot^F 1_A, \qquad (gf) \cdot^F S = g \cdot^F f \cdot^F S, \qquad R \cdot^F (gf) = R \cdot^F g \cdot^F f.$$

In an orean form, the direct and inverse images form a galois connection. In particular, we have the following identity:

$$R \cdot^F f \geqslant^F S \quad\Longleftrightarrow\quad R \geqslant^F_f S \quad\Longleftrightarrow\quad R \geqslant^F f \cdot^F S.$$

We will typically drop the superscripts of F when it is clear in which form F we are working. Also, $\leqslant_{1_A}$ will typically be denoted by just $\leqslant$.

This additional structure from an orean form allows us to define the following concepts:

- the *kernel* and *image* of a morphism $f\colon A \to B$ are respectively defined as

$$\operatorname{Ker} f = \bot^B \cdot^F f \quad\text{and}\quad \operatorname{Im} f = f \cdot^F \top^A;$$

- a subobject is *normal* (or F-normal) if it is a kernel of some morphism, while a subobject is *conormal* (or F-conormal) if is the image of some morphism.

Of course, in the example of groups the normal subobjects are precisely the normal subgroups. The initial goal of Noetherian forms was to find a self-dual context in which one could establish the Noether isomorphism theorems. In the example of groups, all subgroups are conormal, but for the sake of self-duality, we will not require all subobjects to be conormal for a general Noetherian form.

Lastly, we generalize the concept of quotienting out with a normal subgroup and embedding a (conormal) subgroup as follows:

- morphism m is an *embedding* of conormal subobject C if $\operatorname{Im} m = C$ and for any f such that $\operatorname{Im} f \leqslant C$, there is a unique h such that $mh = f$;

- morphism p is a *projection* of normal subobject N if $\operatorname{Ker} p = N$ and for any g such that $\operatorname{Ker} g \geqslant N$, there is a unique k such that $kp = g$.

A diagrammatic display of the embedding m and the projection p:

$$\begin{tikzcd} \bullet \arrow[r, "m"] & \bullet \\ \bullet \arrow[u, dashed, "h"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "p"] \arrow[dr, "g"'] & \bullet \arrow[d, dashed, "k"] \\ & \bullet \end{tikzcd}$$

2.4. Definition. *A* Noetherian *form F over a category $\mathbb{C}$ is an orean form satisfying:*

(N1) for every morphism $f\colon A \to B$ and subobjects $S \in \operatorname{sub} A$ and $R \in \operatorname{sub} B$, we have

$$f \cdot (R \cdot f) = R \wedge \operatorname{Im} f \qquad \textit{and} \qquad (f \cdot S) \cdot f = S \vee \operatorname{Ker} f;$$

(N2) every morphism f factorizes as $f = mp$, where m is an embedding of $\operatorname{Im} f$ *and p is a projection of* $\operatorname{Ker} f$*;*

(N3) the meet of conormal subobjects is conormal, and the join of normal subobjects is normal.

2.5. Preliminary results. Most of the proofs in this subsection could be found in [1].

The dual of a form F is the opposite functor F^{op}. In particular, $R \leqslant_f S$ is dual to $R \geqslant_f S$, meet is dual to join, top element is dual to bottom element, inverse images are dual to direct images, kernel is dual to image, projection is dual to embedding, normal subobject is dual to conormal subobject. Note an orean form as well as (N1), (N2), (N3) are self-dual. Consequently, the duals of all the results stated in this subsection will also be true.

2.6. Lemma. *Consider an orean form satisfying (N1) and (N2). A morphism f is a projection if and only if* $\operatorname{Im} f = \top$.

Here are some immediate consequences of this lemma and (N2), and also from the definition.

2.7. Lemma. *In an orean form satisfying (N2),*

- *any projection is an epimorphism;*
- *the composite of any two projections is a projection;*
- *if gf is a projection, then g is a projection, for any two composable morphisms;*
- *any split epimorphism is a projection;*
- *any morphism is an isomorphism if and only if it is both an embedding and a projection.*

2.8. Lemma. *In an orean form F, consider two conormal subobjects S and R of the same object A, which have respective embeddings ι_S and ι_R and whose meet $R \wedge S$ is conormal. The commutative diagram*

$$\begin{tikzcd} \bullet \arrow[r, "\iota_S"] & A \\ \bullet \arrow[u, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[ur, "d"] & \bullet \arrow[u, "\iota_R"'] \end{tikzcd}$$

is a pullback if and only if the diagonal d is an embedding of $R \wedge S$.

Proof. Suppose d is an embedding of $R \wedge S$. Consider any two morphism f and g with the same domain, such that $\iota_S f = \iota_R g$. We readily have that $\mathrm{Im}(\iota_S f) \leqslant S \wedge R$. Since d is an embedding of $R \wedge S$, there is a unique h such that $dh = \iota_S f = \iota_R g$. We have

$$\iota_S f = dh = \iota_S \pi_1 h \quad \Longrightarrow \quad f = \pi_1 h,$$

since any embedding is a mono (dual of the first point of Lemma 2.6). Likewise, $g = \pi_2 h$. If there was another morphism k such that $f = \pi_1 k$ and $g = \pi_2 k$, then

$$dk = \iota_S \pi_1 k = \iota_S f = \iota_S \pi_1 h = dh \quad \Longrightarrow \quad k = h,$$

since any embedding is a mono. Consequently, the diagram in the lemma statement is a pullback.

Conversely, suppose the given diagram is a pullback. Since d factors through ι_S and ι_R, $\mathrm{Im}\, d \leqslant R \wedge S$. Consider any morphism f into A such that $\mathrm{Im}\, f \leqslant R \wedge S$. Then in particular, $\mathrm{Im}\, f \leqslant S$, and thus there exists a unique k such that $f = \iota_S k$. Likewise, there exists a unique l such that $f = \iota_R l$. Since the given square is a pullback, there exists a unique h such that $\pi_1 h = k$ and $\pi_2 h = l$. Consequently

$$f = \iota_S k = \iota_S \pi_1 h = dh.$$

So f factors through d. Since embeddings are monos, and π_1 and π_2 are jointly monic, it follows that f uniquely factors through d. Thus, if we can conclude that $\mathrm{Im}\, d = R \wedge S$, then d is an embedding of $R \wedge S$. From the assumptions, we assumed that $R \wedge S$ is conormal, thus there is some morphism g such that $\operatorname{Im} g = R \wedge S$. Then, from the previous arguments, g factors through d, and thus

$$R \wedge S = \operatorname{Im} g \leqslant \operatorname{Im} d \leqslant R \wedge S,$$

and thus $\operatorname{Im} d = R \wedge S$. ∎

3. Lifting orean forms along functors

Consider an orean form $N_{\mathsf{X}} \colon \mathbb{B} \to \mathbb{X}$ over a category $\mathbb{X}$ and a functor $G \colon \mathbb{C} \to \mathbb{X}$. We can construct a pullback

$$\begin{tikzcd} \mathbb{A} \arrow[r, "N_{\mathsf{C}}"] \arrow[d, "F"'] & \mathbb{C} \arrow[d, "G"] \\ \mathbb{B} \arrow[r, "N_{\mathsf{X}}"'] & \mathbb{X} \end{tikzcd}$$

where the objects of $\mathbb{A}$ are pairs of objects $(S, A) \in \mathbb{B} \times \mathbb{C}$, such that $N_{\mathsf{X}}(S) = GA$ and the morphisms in $\mathbb{A}$ are pairs of morphisms (k, f) in $\mathbb{B} \times \mathbb{C}$ such that $N_{\mathsf{X}}(k) = Gf$, and the functors F and N_{C} projects on to the first and second component respectively. Throughout this paper, we will call this specifically constructed N_{C} as "the" pullback of N_{X} along functor G.

For simplicity, when N_{C} appears as a superscript, we will instead use C. That is, instead of $X \cdot^{N_{\mathsf{C}}} f$, we write $X \cdot^{\mathsf{C}} f$. And instead of $X \geqslant_f^{N_{\mathsf{C}}} Y$, we write $X \geqslant_f^{\mathsf{C}} Y$. Similarly, for N_{X} we will use X.

3.1. Lemma. *If N_{X} is a form over a category $\mathbb{X}$ and $G \colon \mathbb{C} \to \mathbb{X}$ is a functor, then the pullback N_{C} of N_{X} along G is a form where*

- $\operatorname{sub}^{\mathsf{C}} A = \{(S, A) \in \mathbb{A} \mid S \in \operatorname{sub}^{\mathsf{X}}(GA)\}$, *and*
- $(S, B) \geqslant_f^{\mathsf{C}} (R, A) \iff S \geqslant_{Gf}^{\mathsf{X}} R,$

for any object $A \in \mathbb{C}$, morphism $f \colon A \to B$ in $\mathbb{C}$ and subobjects (R, A) and (S, B).

Proof. That N_{C} is a form, readily follows from how N_{C} is defined and that N_{X} is a form.

For any object $A \in \mathbb{C}$, by definition of the subobject lattice and N_{C}, we have

$$\operatorname{sub}^{\mathsf{C}} A = \{(S, A) \in \mathbb{A} \mid N_{\mathsf{X}}(S) = GA\} = \{(S, A) \in \mathbb{A} \mid S \in \operatorname{sub}^{\mathsf{X}}(GA)\}.$$

Further, by definition

$$(S, B) \geqslant_f^{\mathsf{C}} (R, A) \iff \exists_{k \colon R \to S}\big((k, f) \in \mathbb{A}\big) \iff \exists_{k \colon R \to S}(N_{\mathsf{X}} k = Gf) \iff S \geqslant_{Gf}^{\mathsf{X}} R.$$

∎

3.2. Notation. For simplicity, we will identify each subobject (S, A) in $\mathrm{sub}^{\mathsf{C}} A$ with S, so that

$$\mathrm{sub}^{\mathsf{C}} A = \mathrm{sub}^{\mathsf{X}}(GA).$$

Also then, for any morphism f in $\mathbb{C}$ and subobjects R and S, we have

$$S \geqslant^{\mathsf{C}}_{f} R \quad \iff \quad S \geqslant^{\mathsf{X}}_{Gf} R.$$

From this notational convention, we readily have the following result.

3.3. Proposition. *If N_{X} is an orean form, then its pullback N_{C} is also orean. In particular, the following identities are satisfied:*

- $f \cdot^{\mathsf{C}} R = Gf \cdot^{\mathsf{X}} R$ *and* $S \cdot^{\mathsf{C}} f = S \cdot^{\mathsf{X}} Gf$;
- $R \wedge^{\mathsf{C}} S = R \wedge^{\mathsf{X}} S$ *and* $R \vee^{\mathsf{C}} S = R \vee^{\mathsf{X}} S$;
- $\top^{\mathsf{C}} = \top^{\mathsf{X}}$ *and* $\bot^{\mathsf{C}} = \bot^{\mathsf{X}}$.

Further, if N_{X} satisfies (N1), then N_{C} also satisfies (N1).

We also readily have the following consequence.

3.4. Corollary. *If N_{X} is an orean form satisfying (N1), then N_{C} is also an orean form satisfying (N1).*

The other two axioms of a Noetherian form does not necessarily carry over from N_{X} to its pullback N_{C}. For example, a subobject $N \in \mathrm{sub}^{\mathsf{X}}(GA) = \mathrm{sub}^{\mathsf{C}} A$ that is normal in N_{X} might not be normal in N_{C}, since normality of a subobject depends on the morphisms that are in the category, and not just the orean structure. Consequently, (N3) does not carry over so nicely as (N1). We will explore conditions under which (N2) and (N3) carry over.

4. Lifting orean forms along right adjoints

Throughout this section we will be working with

- an adjunction $\langle F, G, \eta, \varepsilon\rangle \colon \mathbb{X} \to \mathbb{C}$, and $T = GF$,
- where $\mathbb{X}$ has an orean form N_{X},
- and N_{C} denotes the pullback of N_{X} along G.

The proposition below is only for interest sake. Unlike the other propositions, this proposition will have no further use in this paper.

4.1. Proposition. *For any N_{X}-projection $p \colon X \to Y$ in $\mathbb{X}$, the smallest N_{C}-normal subobject in $\mathrm{sub}^{\mathsf{C}}(FX) = \mathrm{sub}^{\mathsf{X}}(GFX)$ which contains $\eta_X \cdot^{\mathsf{X}} \mathrm{Ker}^{\mathsf{X}} p$ is $\mathrm{Ker}^{\mathsf{C}}(Fp) = \mathrm{Ker}^{\mathsf{X}}(GFp)$.*

Proof. From the naturality of η and that inverse images form a monotone Galois connection, we see that

$$\operatorname{Ker}^{\mathsf{X}}(GF(p)\eta_X) = \operatorname{Ker}^{\mathsf{X}}(\eta_Y p) \geqslant \operatorname{Ker}^{\mathsf{X}} p \quad \Longrightarrow \quad \operatorname{Ker}^{\mathsf{C}}(Fp) = \operatorname{Ker}^{\mathsf{X}}(GFp) \geqslant \eta_X \cdot^{\mathsf{X}} \operatorname{Ker}^{\mathsf{X}} p.$$

Consider any N_{C}-normal subobject N above $\eta_X \cdot^{\mathsf{X}} \operatorname{Ker} p$. Since N is normal, there is some f in $\mathbb{C}$ such that $\operatorname{Ker}^{\mathsf{C}} f = \operatorname{Ker}^{\mathsf{X}}(Gf) = N$. We have the following diagram

$$\begin{tikzcd}
GFX \arrow[rr, "GFp"] \arrow[dr, "Gf"] & & GFY \arrow[dl, dashed, "Gh"] \\
& GM & \\
X \arrow[uu, "\eta_X"] \arrow[rr, "p"] & & Y \arrow[uu, "\eta_Y"'] \arrow[ul, dashed, "g"']
\end{tikzcd}$$

Notice that

$$\operatorname{Ker}^{\mathsf{X}}(Gf) = N \geqslant \eta_X \cdot^{\mathsf{X}} \operatorname{Ker}^{\mathsf{X}} p \quad \Longrightarrow \quad \operatorname{Ker}^{\mathsf{X}}\big(G(f)\eta_X\big) \geqslant \operatorname{Ker}^{\mathsf{X}} p.$$

Since p is a projection, there exists a unique g such that the left lower triangle commutes. Since η_Y is a universal arrow to G, there exists a unique $h\colon FY \to M$ such that the right triangle commutes. And lastly, notice that

$$G(hFp)\eta_X = G(h)GF(p)\eta_X = G(h)\eta_Y p = gp = G(f)\eta_X.$$

And since η_X is a universal arrow, $hF(p) = f$, and consequently the upper triangle commutes as well. From the commutativity of the upper triangle, it follows that $\operatorname{Ker}^{\mathsf{X}}(GFp) \leqslant \operatorname{Ker}^{\mathsf{X}}(Gf) = N$, demonstrating that $\operatorname{Ker}^{\mathsf{X}}(GFp) = \operatorname{Ker}^{\mathsf{C}}(Fp)$ is the smallest N_{C}-normal subobject above $\eta_X \cdot^{\mathsf{X}} \operatorname{Ker}^{\mathsf{X}} p$. ■

4.2. Proposition. *Suppose N_{C} has embeddings for any conormal subobject and G preserves embeddings. For any morphism $f\colon X \to Y$, the smallest N_{C}-conormal in*

$$\operatorname{sub}^{\mathsf{C}}(FY) = \operatorname{sub}^{\mathsf{X}}(GFY)$$

which contains $\operatorname{Im}^{\mathsf{X}}(\eta_Y f)$, *is* $\operatorname{Im}^{\mathsf{C}}(Ff) = \operatorname{Im}^{\mathsf{X}}(GFf)$.

Proof. From the naturality of η, we have

$$\operatorname{Im}^{\mathsf{X}}(\eta_Y f) = \operatorname{Im}^{\mathsf{X}}(GF(f)\eta_X) \leqslant \operatorname{Im}^{\mathsf{X}}(GFf) = \operatorname{Im}^{\mathsf{C}}(Ff).$$

Consider any N_{C}-conormal subobject C above $\operatorname{Im}^{\mathsf{X}}(\eta_Y f)$, whose embedding is ι_C. We have the following diagram

$$\begin{tikzcd}
GFX \arrow[rr, "GFf"] \arrow[dr, dashed, "Gh"] & & GFY \\
& GC \arrow[ur, "G\iota_C"] & \\
X \arrow[uu, "\eta_X"] \arrow[rr, "f"] \arrow[ur, dashed, "g"] & & Y \arrow[uu, "\eta_Y"]
\end{tikzcd}$$

Since G preserves embeddings, $G\iota_C$ is an embedding, and moreover, $\mathrm{Im}^{\mathsf{X}}(G\iota_C) = \mathrm{Im}^{\mathsf{C}}\,\iota_C = C$. Further, since C is above above $\mathrm{Im}^{\mathsf{X}}(\eta_Y f)$, there exists a $g\colon X \to GC$ such that the bottom triangle commutes. Since η_X is a universal arrow to G, there exists a unique arrow $h\colon FX \to C$ making the left triangle commute. Again it follows from the universality of η_X that the top triangle commutes. Consequently, $C \geqslant \mathrm{Im}^{\mathsf{X}}(GFf)$. ∎

4.3. Corollary. *If N_{C} has embeddings for any conormal subobject and G preserves embeddings, then the smallest N_{C}-subobject above $\mathrm{Im}^{\mathsf{X}}\,\eta_X$ is $\top^{\mathsf{C}} = \top^{\mathsf{X}}$.*

Proof. Apply the above proposition with $f = 1_X$. ∎

4.4. Proposition. *If both N_{X} and N_{C} satisfy (N1) and (N2), then*

- *G preserves and reflects projections and embeddings;*
- *F preserves projections.*

In particular, f is an N_{C}-embedding of S if and only if Gf is an N_{X}-embedding of S. Likewise, also for projections.

Proof. From the construction of N_{C} from N_{X}, $\mathrm{Im}^{\mathsf{C}}\,f = \mathrm{Im}^{\mathsf{X}}(Gf)$ and $\top^{\mathsf{C}} = \top^{\mathsf{X}}$. Thus, from Lemma 2.6, it follows that f is an N_{C}-projection if and only if Gf is an N_{X}-projection. In other words, G preserves and reflects projections. Likewise, G will preserve and reflect embeddings. The last line of the proposition follows likewise from the fact that $\mathrm{Im}^{\mathsf{C}}\,f = \mathrm{Im}^{\mathsf{X}}(Gf)$ and $\mathrm{Ker}^{\mathsf{C}}\,f = \mathrm{Im}^{\mathsf{X}}(Gf)$.

Consider a projection $p\colon X \to Y$ in $\mathbb{X}$. Then $\mathrm{Im}^{\mathsf{C}}(Fp)$ is the smallest N_{C}-conormal subobject which contains $\mathrm{Im}^{\mathsf{X}}(\eta_Y p)$ by Proposition 4.2. Using the above corollary, we have

$$\mathrm{Im}^{\mathsf{C}}(Fp) \geqslant \mathrm{Im}^{\mathsf{X}}(\eta_Y p) = \eta_Y \cdot^{\mathsf{X}} p \cdot^{\mathsf{X}} \top^{\mathsf{X}} = \eta_Y \cdot^{\mathsf{X}} \top^{\mathsf{X}} = \top^{\mathsf{X}} = \top^{\mathsf{C}}.$$

Thus $\mathrm{Im}^{\mathsf{C}}(Fp) = \top^{\mathsf{C}}$, and thus from Lemma 2.6, Fp is a projection. ∎

4.5. Proposition. *Suppose both N_{X} and N_{C} satisfy (N1) and (N2). If N_{X} satisfies the meet part of (N3), then N_{C} also satisfies the meet part of (N3).*

Proof. Consider two N_{C}-conormal subobjects R and S of an object C in $\mathbb{C}$, where ι_R and ι_S are their respective embeddings. By Proposition 4.4, both $G\iota_R$ and $G\iota_S$ are embeddings. Then, by Lemma 2.8 there exists a pullback diagram in $\mathbb{X}$

$$\begin{tikzcd}
 & GA \arrow[r, "G\iota_R"] & GC \\
 & P \arrow[u, "p"] \arrow[r, "q"] & GB \arrow[u, "G\iota_S"'] \\
GFP \arrow[uur, bend left, "G(\varepsilon_A Fp)"] \arrow[ur, dashed, "h"] \arrow[urr, bend right, "G(\varepsilon_B Fq)"'] & &
\end{tikzcd}$$

where the diagonal of the square is an embedding of $R \wedge S$ in N_{X}. By the naturality of ε, we have

$$\begin{aligned}\iota_R \varepsilon_A F p &= \varepsilon_C F G(\iota_R) F p \\ &= \varepsilon_C F\big(G(\iota_R) p\big) \\ &= \varepsilon_C F\big(G(\iota_S) q\big) \\ &= \dots \\ &= \iota_S \varepsilon_B F q.\end{aligned}$$

Thus, $G(\iota_R)G(\varepsilon_A F p) = G(\iota_S)G(\varepsilon_B F q)$. Since the square is a pullback, there exists a unique $h\colon GFP \to P$ making the above diagram commute. We further have

$$p h \eta_P = G(\varepsilon_A F p)\eta_P = G(\varepsilon_A) GF(p)\eta_P = G(\varepsilon_A)\eta_{GA} p = p.$$

Since p is mono, h is a split epi, thus a projection. Thus

$$\mathrm{Im}^{\mathsf{C}}\left(\iota_R \varepsilon_A F(p)\right) = \mathrm{Im}^{\mathsf{X}}\left(G(\iota_R \varepsilon_A F(p)\right) = \mathrm{Im}^{\mathsf{X}}\left(G(\iota_R) p h\right) = \mathrm{Im}^{\mathsf{X}}\left(G(\iota_R) p\right) = R \wedge S,$$

which is therefor conormal in N_{C}, and thus, N_{C} satisfies the meet part of (N3). ∎

Notice that, when N_{C} satisfies the meet part of (N3) [together with (N1) and (N2)], then $\mathbb{C}$ will have pullbacks of embeddings. If we assume $\mathbb{C}$ had pullbacks of embeddings as well in the above proposition, the proof becomes much simpler:

Proof Alternative proof of Proposition 4.5. Consider any two N_{C}-conormal subobjects R and S of an object $C \in \mathbb{C}$. Construct their pullback (the left diagram):

$$\begin{tikzcd} A \arrow[r, "\iota_S"] & C \\ P \arrow[u, "a"] \arrow[r, "b"'] & B \arrow[u, "\iota_R"'] \end{tikzcd} \qquad \begin{tikzcd} GA \arrow[r, "G\iota_S"] & GC \\ GP \arrow[u, "Ga"] \arrow[r, "Gb"'] & B \arrow[u, "G\iota_R"'] \end{tikzcd}$$

Since G is a right adjoint, G preserves pullbacks. By Proposition 4.4, G preserves embeddings as well. Then from Lemma 2.8 it follows that the diagonal $G(\iota_S a)$ of the right diagram is an embedding of $R \wedge S$. In particular,

$$R \wedge S = \mathrm{Im}^{\mathsf{X}}\left(G(\iota_S a)\right) = \mathrm{Im}^{\mathsf{C}}(\iota_S a).$$

Thus, N_{C} satisfies the meet part of (N3). ∎

In Example 5.2, we construct an example where N_{X} is Noetherian, and N_{C} satisfies (N1) and (N2), but not the join part of (N3).

4.6. Proposition. *Suppose N_{X} is Noetherian and N_{C} satisfies (N1) and (N2). Then N_{C} is Noetherian if and only if $\mathbb{C}$ has pushouts of projections and G preserves pushouts of projections.*

PROOF. Suppose $\mathbb{C}$ has pushouts of projections and G preserves pushouts of projections. By Proposition 4.5, N_{C} satisfies the meet part of (N3). The alternative proof of Proposition 4.5 can be dualized to show that N_{C} satisfies the join part of (N3). Thus, N_{C} is Noetherian.

Suppose N_{C} is a Noetherian form. Then $\mathbb{C}$ has pushouts of projections by the dual of Lemma 2.8. Further, from Lemma 2.8 and Proposition 4.4 it follows that these pushouts are preserved. ∎

The Crude Tripleability Theorem in [2] states that a right adjoint $G\colon \mathbb{A} \to \mathbb{X}$ is monadic if

- G reflects isomorphims,
- $\mathbb{A}$ has coequalizers of reflexive pairs, and
- G preserves the coequalizers of reflexive pairs.

A reflexive pair is a parallel pair of morphisms $f, g\colon A \to B$ such that there exists $s\colon B \to A$ such that $fs = 1_B = gs$.

4.7. THEOREM. *Given a functor $G\colon \mathbb{C} \to \mathbb{X}$ which has a right adjoint and a Noetherian form N_{X} over $\mathbb{X}$. If the pullback of N_{X} along G is a Noetherian form, then G has to be monadic.*

PROOF. Note that any split epimorphism is a projection (Lemma 2.7). Further, the coequalizer of a reflexive pair is essentially the same as the pushout of the reflexive pair. Then by Proposition 4.6, $\mathbb{C}$ will have coequalizers of reflexive pairs and G will preserve them. Thus, by the Crude Tripleability Theorem above, G is monadic. ∎

5. Characterization theorems

5.1. THEOREM. *Consider an adjunction $\langle F, G, \eta, \varepsilon\rangle\colon \mathbb{X} \to \mathbb{C}$, where the counit ε is an isomorphism and $\mathbb{X}$ has an orean form N_{X} satisfying (N1), (N2) and the meet part of (N3). The pullback N_{C} of N_{X} along G satisfies (N1), (N2) and the meet part of (N3) if and only if $T = GF$ preserves projections.*

PROOF. By Proposition 4.4, we already have that GF preserves projections if N_{C} satisfies (N1) and (N2).

Conversely, suppose that $T = GF$ preserves projections. From the construction of N_{C}, it satisfies (N1). To show (N2), consider any morphism $f\colon A \to C$ in $\mathbb{C}$. Suppose $Gf = em$, where $m\colon X \to GC$ is an N_{X}-embedding of $\mathrm{Im}^{\mathsf{X}} Gf$ and $e\colon GA \to X$ is a N_{X}-projection of $\mathrm{Ker}^{\mathsf{X}} Gf$. Since the counit ε is an isomorphism, f factorizes as

$$f = \varepsilon_C F(m) F(e) \varepsilon_A^{-1}.$$

We have the following commutative diagram

$$\begin{tikzcd} TGA \arrow[r, "Te"] & TX \arrow[r, "Tm"] & TGC \\ GA \arrow[u, "\eta_{GA}"] \arrow[r, "e"'] & X \arrow[u, "\eta_X"] \arrow[r, "m"'] & GC \arrow[u, "\eta_{GC}"] \end{tikzcd}$$

Since $G\varepsilon$ is a natural isomorphism, η_G must also be a natural isomorphism. By assumption Te is a projection, thus $\eta_X e$ is a projection, and thus η_X is a projection as well. Furthermore, $\eta_{GC} m = T(m)\eta_X$ is an embedding, thus η_X is an embedding. Consequently, η_X is an isomorphism. Thus $G\big(F(e)\varepsilon_A^{-1}\big) = T(e)\eta_{GA}$ is an N_{X}-projection of $\mathrm{Ker}^{\mathsf{X}}(Gf)$ and $G\big(\varepsilon_C F(m)\big) = \eta_{GC}^{-1}T(m)$ is an N_{X}-embedding of $\mathrm{Im}^{\mathsf{X}}(Gf)$. Consider any $g\colon A \to B$ in $\mathbb{C}$ such that $\mathrm{Ker}^{\mathsf{C}}\, g \geqslant \mathrm{Ker}^{\mathsf{C}}\, f$, that is $\mathrm{Ker}^{\mathsf{X}}(Gg) \geqslant \mathrm{Ker}^{\mathsf{X}}(Gf)$. Thus, there exists a unique $h\colon GB \to TX$ such that the left diagram commutes:

$$\begin{tikzcd} GA \arrow[d, "Gg"'] \arrow[dr, "G\left(F(e)\varepsilon_A^{-1}\right)"] & \\ GB \arrow[r, "h"'] & GFX \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "g"'] \arrow[dr, "F(e)\varepsilon_A^{-1}"] & \\ B \arrow[r, dashed, "k"'] & FX \end{tikzcd}$$

The right adjoint G is full and faithful, since the counit is an isomorphism (Theorem 1 on page 90 in [4]). Thus, there exists a unique $k\colon B \to FX$ such that $Gk = h$, and this k will be the unique morphism making the right diagram commute. Thus $F(e)\varepsilon_A^{-1}$ is a projection of $\mathrm{Ker}^{\mathsf{C}}\, f$. We can likewise proof that $\varepsilon_C F(m)$ is an embedding of $\mathrm{Im}^{\mathsf{C}}\, f$. Thus, N_{C} satisfies (N2). We can now use Proposition 4.5 to conclude that N_{C} also satisfies the meet part of (N3). ∎

If N_{X} is Noetherian and GF preserves projection, then it is not necessarily true that N_{C} is a Noetherian form, as demonstrated in Example 5.2.

5.2. Example. In this example, we will construct an adjunction $\langle F, G, \eta, \varepsilon\rangle\colon \mathbb{X} \to \mathbb{C}$, where the counit is an isomorphism, $\mathbb{X}$ has a Noetherian form N_{X}, but the pullback N_{C} of N_{X} along G is not a Noetherian form.

Consider the following lattices $\mathbb{A}$ and $\mathbb{X}$ seen as categories.

$$\mathbb{C}\colon \begin{tikzcd} & b \arrow[dr, "f"] & \\ a \arrow[ur, "p"] \arrow[dr, "q"'] & & d \\ & c \arrow[ur, "g"'] & \end{tikzcd} \qquad \mathbb{X}\colon \begin{tikzcd} & x \arrow[dr] & & \\ w \arrow[ur] \arrow[dr] & & z \arrow[r] & t \\ & y \arrow[ur] & & \end{tikzcd}$$

Define functors $G\colon \mathbb{C} \to \mathbb{X}$ and $F\colon \mathbb{X} \to \mathbb{C}$ as follows:

$$\begin{array}{llll} Fa = w, & Fb = x, & Gw = a, & Gx = b, \\ Fc = y, & Fd = t, & Gy = c, & Gz = Gt = d. \end{array}$$

Since both categories are preorders, functors are the same as order preserving functions on the objects, which the above definitions of F and G are. The functor G is the right adjoint of F, such that the counit is an isomorphism (all components of the counit are identity morphisms). We can construct a Noetherian form N_{X} over $\mathbb{X}$ such that

$$\mathrm{sub}^{\mathsf{X}}(u) = \uparrow u, \qquad f \cdot^{\mathsf{X}} s = s \vee v, \quad r \cdot^{\mathsf{X}} f = r.$$

for any object u, morphism $f\colon u \to v$, and subobjects $s \in \mathrm{sub}^{\mathsf{X}}(u)$ and $r \in \mathrm{sub}^{\mathsf{X}}(v)$. Let N_{C} be the pullback of N_{X} along G. By construction of N_{C}, $\mathrm{sub}^{\mathsf{C}}(u) = \mathrm{sub}^{\mathsf{X}}(Gu)$, and $\mathrm{Ker}^{\mathsf{C}}(h) = \mathrm{Ker}^{\mathsf{X}}(Gh)$. Then all possible normal subobjects of a are

$$\begin{array}{ll} \mathrm{Ker}^{\mathsf{C}}\, p = x, & \mathrm{Ker}^{\mathsf{C}}\, 1_a = w, \\ \mathrm{Ker}^{\mathsf{C}}\, q = y, & \mathrm{Ker}^{\mathsf{C}}(fp) = \mathrm{Ker}^{\mathsf{C}}(gq) = t. \end{array}$$

But $x \vee^{\mathsf{C}} y = z$, which is not normal. Thus N_{C} is not Noetherian. Lastly, all morphisms in both $\mathbb{C}$ and $\mathbb{X}$ projections, thus GF will trivially preserve projections.

5.3. Proposition. *If $\langle F, G, \eta, \varepsilon\rangle\colon \mathbb{X} \to \mathbb{C}$ is an equivalence of categories, then $\mathbb{X}$ has a Noetherian form if and only if $\mathbb{C}$ has a Noetherian form. In particular, the pullback of a Noetherian form over $\mathbb{X}$ along G is a Noetherian form over $\mathbb{C}$.*

Proof. Suppose $\mathbb{X}$ has a Noetherian form N_{X}, and N_{C} is the pullback of N_{X} along G. Consider an N_{X}-projection $p\colon X \to Y$ in $\mathbb{X}$. Since η is an isomorphism

$$GF(p) = \eta_Y p \eta_X^{-1}.$$

is a projection as well. Then, by Proposition 5.1, N_{C} satisfies (N1), (N2) and the meet part of (N3).

To show that the join part of (N3) holds in N_{C}, consider any two N_{C}-normal subobjects R and S of $C \in \mathbb{C}$. We will use a very similar argument as in the proof of Proposition 4.5 to conclude that $R \vee S$ is N_{C}-normal. Since N_{C} satisfies (N2), R and S have N_{C}-projections π_R and π_S. By Proposition 4.4, both $G\pi_R$ and $G\pi_S$ are projections. Then by the dual of Lemma 2.8, there exists a pushout diagram in $\mathbb{C}$

$$\begin{tikzcd}
& GA \arrow[d, "p"'] \arrow[dl, bend right, "G\left(F(p)\varepsilon_A^{-1}\right)"'] & GC \arrow[l, "G\pi_R"'] \arrow[d, "G\pi_S"] \\
& P \arrow[dl, dashed, "h"'] & GB \arrow[l, "q"'] \arrow[dll, bend left, "G\left(F(p)\varepsilon_B^{-1}\right)"] \\
GFP & &
\end{tikzcd}$$

where the diagonal of the square is a projection of $R \vee S$ in N_{X}. By the naturality of ε^{-1}, we have

$$\begin{aligned} F(p)\varepsilon_A^{-1}\pi_R &= F(p)FG(\pi_R)\varepsilon_C^{-1} \\ &= F\big(pG(\pi_R)\big)\varepsilon_C^{-1} \\ &= F\big(qG(\pi_S)\big)\varepsilon_C^{-1} \\ &= \dots \\ &= F(p)\varepsilon_B^{-1}\pi_S. \end{aligned}$$

Thus $G\big(F(p)\varepsilon_A^{-1}\big)G(\pi_R) = G\big(F(p)\varepsilon_B^{-1}\big)G(\pi_S)$. Since the square is a pushout, there exists a unique h making the diagram commute. We further have

$$\eta_P^{-1}hp = \eta_P^{-1}GF(p)G(\varepsilon_A^{-1}) = p\eta_{GA}^{-1}G(\varepsilon_A^{-1} = p.$$

Since p is an epi, h is a split mono (in fact iso), thus an embedding. Thus

$$\mathrm{Ker}^{\mathsf{C}}\left(F(p)\varepsilon_A^{-1}\pi_R\right) = \mathrm{Ker}^{\mathsf{X}}\left(GF(p)G(\varepsilon_A^{-1})G\pi_R\right) = \mathrm{Ker}^{\mathsf{X}}(hpG\pi_R) = \mathrm{Ker}^{\mathsf{X}}(pG\pi_R) = R\vee S,$$

which is therefor normal in N_{C}, and thus, N_{C} satisfies the join part of (N3). ∎

For a monad T in a category $\mathbb{X}$, $\mathbb{X}^T$ will denote the category of T-algebras and $G^T\colon \mathbb{X}^T \to \mathbb{X}$ will denote the forgetful functor.

5.4. Theorem. *Consider a monad $\langle T, \mu, \eta\rangle$ in a category $\mathbb{X}$ which has a Noetherian form N_{X}. The pullback N_{T} of N_{X} along the forgetful functor $G^T\colon \mathbb{X}^T \to \mathbb{X}$ is an orean form satisfying (N1) and (N2) if and only if T preserves projections. Furthermore, N_{T} is Noetherian if and only if T preserves pushouts of projections.*

Proof. By construction N_{T} will be orean satisfying (N1). If N_{T} satisfy (N2), then by Proposition 4.4 both G^T and its left adjoint F^T will preserve projections, consequently $T = G^T F^T$ preserves projections. Furthermore, by Proposition 4.6, we have that if N_{T} is Noetherian, then G^T preserves projections and preserves pushouts of projections. Since F^T preserves pushouts as well as projections, the composite $T = G^T F^T$ will preserve pushouts of projections.

Conversely, suppose T preserves projections. By the construction of N_{T},

$$\mathrm{sub}^{\mathsf{T}}\langle X, h\rangle = \mathrm{sub}^{\mathsf{X}} X, \qquad f \cdot^{\mathsf{T}} S = f \cdot^{\mathsf{X}} S, \quad R \cdot^{\mathsf{T}} f = R \cdot^{\mathsf{X}} f.$$

So readily N_{T} satisfies (N1). To show (N2), consider any morphism $f\colon \langle X, h_X\rangle \to \langle Y, h_Y\rangle$ of T-algebras. In N_{X}, f decomposes as projection-embedding $f = mp$. We have the following diagram

$$\begin{tikzcd} TX \arrow[r, "Tp"] \arrow[d, "h_X"'] & TZ \arrow[r, "Tm"] \arrow[d, dashed, "h_Z"] & TY \arrow[d, "h_Y"] \\ X \arrow[r, "p"'] & Z \arrow[r, "m"'] & Y \end{tikzcd}$$

By assumption, Tp is a projection. We have

$$\mathrm{Ker}(ph_X) = \mathrm{Ker}(mph_X) = \mathrm{Ker}(h_Y T(m)T(p)) \geqslant \mathrm{Ker}(Tp).$$

Thus there exists a unique morphism h_Z such that the left square commutes. Since Tp is in particular an epimorphism, the right square also commutes. Since m is a mono and $\langle Y, h_Y\rangle$ is a T-algebra, it follows that $\langle Z, h_Z\rangle$ is a T-algebra. Thus both p and m are morphisms in $\mathbb{X}^T$. Further, similar arguments could be used to demonstrate that if kp is in $\mathbb{X}^T$, then so is k, and if ml is in $\mathbb{X}^T$, then so is l. From this, it readily follows that p is an

N_{T}-projection and m an N_{T}-embedding. Thus (N2) holds in N_{T}. From Proposition 4.5, N_{T} satisfies the meet part of (N3).

Suppose that T preserves pushouts of projections. To show the join part of (N3), consider any object $\langle X, h_X\rangle$ in $\mathbb{X}^T$ and any two normal subobjects B and C in N_{T} with N_{T}-projections $b\colon X \to Y$ and $c\colon X \to Z$. Consider the following diagram in $\mathbb{X}$, where the inner square is the pushout of b and c in $\mathbb{X}$:

$$\begin{tikzcd}
TX \arrow[rrr, "Tb"] \arrow[dr, "h_X"] \arrow[ddd, "Ta"'] & & & TZ \arrow[dl, "h_Z"'] \arrow[ddd, "Tq"] \\
& X \arrow[r, "b"] \arrow[d, "a"'] & Z \arrow[d, "q"] & \\
& Y \arrow[r, "p"] & W & \\
TY \arrow[rrr, "Tp"] \arrow[ur, "h_Y"] & & & TW \arrow[ul, dashed, "h_W"']
\end{tikzcd}$$

Since the outside square is a pushout, there exists a unique h_W making the lower and right rectangles commute. From T^2p and Tp being epimorphisms, it follows that $\langle W, h_W\rangle$ is a T-algebra such that p and q are morphisms of T-algebras. Consequently, $A \vee B = \mathrm{Ker}(pa)$ is normal in N_{T}. ■

Combining, Theorem 4.7,Theorem 5.4 and Proposition 5.3, we get the following theorem.

5.5. Theorem. *Consider a category $\mathbb{X}$ which has a Noetherian form N_{X}, and consider a functor $G\colon \mathbb{A} \to \mathbb{X}$ which has a left adjoint F. The pullback N_{A} of N_{X} along G is a Noetherian form over $\mathbb{A}$ if and only if G is monadic and GF preserves pushouts of projections.*

6. Varieties of Universal Algebras

We start of with the most important result in this section.

6.1. Theorem. *Consider a forgetful functor between two varieties $U\colon \mathbb{V} \to \mathbb{W}$. If N_{w} is a Noetherian from over $\mathbb{W}$ where the surjections are precisely the projections, then the pullback N_{v} is a Noetherian form over $\mathbb{V}$.*

Proof. It is known that U is a right adjoint. Therefore, by Theorem 5.5, it suffice to verify that U preserves pushouts of projections. But by construction of pushouts of surjections in any variety, U will preserve such pushouts, which completes the proof. ■

One Noetherian form over the category of abelian groups Ab is the form where $\operatorname{sub} A$ is the lattice of subgroups, and direct and inverse images are the usual ones. Then, by the theorem above rings with unity Rng_1 has a Noetherian form where the subobject lattices are the (additive) subgroups, since Ab is a reduct of Rng_1. Likewise, from the above theorem it follows that any pointed variety has a Noetherian form where the subobjects are equivalence relations.

The more important consequence is that if we can construct a Noetherian form over Set, then all varieties will have a Noetherian form. One Noetherian form is described as follows:

- subobjects of a set X are pairs (A, R), where R is an equivalence relation on X, and A is either an equivalence class of R or the empty set;
- for any function $f\colon X \to Y$, $(A, R) \in \operatorname{sub} X$ and $(B, S) \in \operatorname{sub} Y$,

$$(B, S) \geqslant_f (A, R) \quad \iff \quad A \subseteq f^{-1}(B), \quad R \leqslant f^{-1}(S).$$

For a given subset A of X, let αA denote the least equivalence relation on X where all of A is identified. Further, if R is an equivalence on X, let $A * R$ be the union of all equivalence classes of R which intersects with A. With not much effort, one can verify that N_{Set} is a Noetherian form, where

$$\begin{aligned} (A, R) \cdot f &= \big(f^{-1}A, f^{-1}R\big), \qquad (A, R) \wedge (B, S) = (A \cap B, R \cap S), \\ f \cdot (A, R) &= \big(fA * fR, fR \vee \alpha(fA)\big), \\ (A, R) \vee (B, S) &= \big((A \cup B) * (R \vee S), R \vee S \vee \alpha(A \cup B)\big). \end{aligned}$$

Further,

- the projections are exactly the surjections, while the embeddings are exactly the injections;
- $\bot_X = \big(\emptyset, \alpha(\emptyset)\big)$ and $\top_X = (X, \alpha X)$;
- $\operatorname{Im} f = \big(fX, \alpha(fX)\big)$ where f is any function $f\colon X \to Y$. Consequently, the conormals are of the form $(A, \alpha A)$, where A is a subset;
- $\operatorname{Ker} f = (\emptyset, \mathsf{K}f)$, where $\mathsf{K}f = \{(x, y) \in X^2 \mid f(x) = f(y)\}$. Consequently, the normal subobjects are of the form $(\emptyset, R)$, where R is an equivalence relation on X.

The conormals are essentially the subsets, while the normals are essentially the equivalence relations.

With this Noetherian form N_{Set}, by Theorem 6.1, we have the following corollary.

6.2. Corollary. *For any variety $\mathbb{V}$, the pullback N_{V} of N_{Set} along the forgetful functor is a Noetherian form. The subobjects of an algebra X are pairs (A, R), where R is an equivalence relation on X and A is an equivalence class of R or the empty subset. The conormals are of the form $(A, \alpha A)$, where A is a subalgebra, and the normals are $(\emptyset, R)$, where R is a congruence. Lastly, the projections in N_{V} are the surjective homomorphism, while the embeddings are the injective homomorphisms.*

The isomorphism theorems from [1] applied to these Noetherian forms N_{V}, gives almost the same known isomorphism theorems in Universal Algebra. First, we formalize what we mean by quotiening in a Noetherian form, but the definition is not surprising.

6.3. Notation. For an object X and normal subobject R in a Noetherian form, X/R is the codomain of the projection π_R of R.

In the Noetherian form N_{V}, the object X/R as defined above is (up to isomorphism) the same as quotiening the algebra X by the congruence R, that is, X/R is equivalent to its usual meaning in Universal Algebra.

The following theorem is a special case of Theorem 4.4 in [1].

6.4. Theorem. [Double-Quotient isomorphism theorems] *In a Noetherian form, consider two normal subobjects R and S of an object X, where $S \leqslant R$. Then, where π_S is the projection of S,*

$$G/R \cong (G/S)/\pi_S(R).$$

For convenience, denote the normal subobjects $(\emptyset, R)$ in N_{V} just by the congruence R. For two normal subobjects R and S of the same object, where $S \leqslant R$, we have

$$\pi_S(R) = \{(\pi_S x, \pi_S y) \mid (x, y) \in R\}.$$

In Universal Algebra, R/S is defined exactly as $\pi_S(R)$. With this observation, the above Double-Quotient Isomorphism Theorem in N_{V} is precisely the known Double-Quotient Isomorphism Theorem in Universal Algebra.

The Diamond Isomorphism Theorem for Noetherian forms, as stated in [1], for the form N_{V} is almost the same as the one in Universal Algebra; some initial assumptions will differ. We instead generalize the Diamond Isomorphism Theorem from [1]. The same proof for Theorem 4.3 in [1] will hold for the (slightly) generalized version.

6.5. Theorem. [Diamond Isomorphism Theorem] *In any Noetherian form, consider a conormal subobject B and a subobject R, such that*

- *there exists a largest conormal subobject $\underline{B \vee R}$ below $B \vee R$,*
- *$R \cdot \iota_{\underline{B \vee R}}$ is normal, where $\iota_{\underline{B \vee R}}$ is the embedding of $\underline{B \vee R}$,*
- *$B \vee (R \wedge (\underline{B \vee R})) = \underline{B \vee R}$.*

Then $R \cdot \iota_B$ *is normal, and*

$$B/(R \cdot \iota_B) \cong \underline{B \vee R}/(R \cdot \iota_{\underline{B \vee R}}).$$

Again, we look at what this is in N_{V}. For the special case when R is a congruence, the second point will immediately be satisfied. Further,

$$\underline{B \vee R} = B * R = B^R$$

is the union of all equivalence classes of R which intersects B,

$$R \cdot \iota_{B^R} = R \cap \left(B^R\right)^2 = R\restriction_{B^R} \quad \text{and} \quad R \cdot \iota_B = R \cap B^2 = R\restriction_B .$$

The conclusion of the theorem is then that $B/R\restriction_B \cong B^R/R\restriction_{B^R}$, which is the known Diamond Isomorphism Theorem in Universal Algebra.

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