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Q-SYSTEM COMPLETENESS OF UNITARY CONNECTIONS

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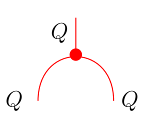
Abstract. A Q-system is a unitary version of a separable Frobenius algebra object in a C*-tensor category. In a recent joint work with P. Das, S. Ghosh and C. Jones, the author has categorified Bratteli diagrams and unitary connections by building a 2-category **UC**. We prove that every Q-system in **UC** splits.

1. Introduction

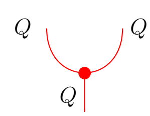
V. Jones' groundbreaking results on index for subfactors [J83] has led to remarkable progress in the development of the theory of subfactors. The standard invariant of a finite index subfactor of a II_1 factor was first defined as a λ-lattice [P95]. In [M03], a Q-system which is a unitary version of a Frobenius algebra object in a C*-tensor category or C*-2-category, is exhibited as an alternative axiomatization of the standard invariant of a finite index subfactor [O88, P95, J99]. This further fostered classification of small index subfactors [JMS14, AMP15]. Q-systems were first introduced in [L94] to characterize canonical endomorphism associated to a finite index subfactor of an infinite factor.

Given any rigid, semisimple, C*-tensor category $\mathcal{C}$ with simple tensor unit $\mathbb{1}$, an indecomposable Q-system $Q \in \mathcal{C}$ (that is, $\mathrm{End}_{Q-Q}(Q) \simeq \mathbb{C}$) and a fully-faithful unitary tensor functor $H : \mathcal{C} \to \mathrm{Bim}(N)$ for some II_1 factor N, we can apply realization procedure [JP19, JP20] to construct a II_1 factor M containing N as a generalized crossed product $N \rtimes_H Q$. Also, every irreducible finite index extension of N is of this form.

In the context of C*-2-categories, a Q-system is a 1-cell ${}_bQ_b \in \mathcal{C}_1(b, b)$ along with two 2-cells $m : Q \boxtimes Q \to Q$ (multiplication) and $i : 1_b \to Q$ (unit), which are graphically denoted by the following:

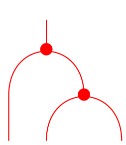
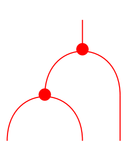
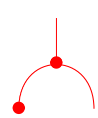
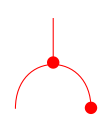
$m =$ 

$i =$ 

$m^* =$ 

$i^* =$ 

These 2-cells satisfy the following:

 $=$  (Associativity)
 $=$  $=$  (Unitality)

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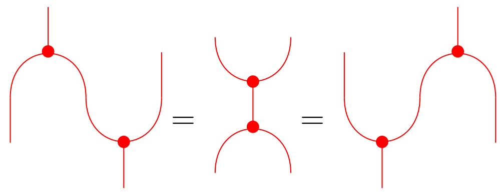
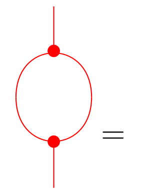

(Frobenius condition)


(Separability)

Recently [CPJP22] introduced the notion of *Q-system completion* for C*/W*-2-categories which is another version of a higher idempotent completion for C*/W*-2-categories in comparison with 2-categories of separable monads [DR18] and condensation monads in [GJF19]. Given a C*/W*-2-category $\mathcal{C}$, its *Q-system completion* is the 2-category $\mathbf{QSys}(\mathcal{C})$ of Q-systems, bimodules and intertwiners in $\mathcal{C}$. There is a canonical inclusion *-2-functor $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathbf{QSys}(\mathcal{C})$ which is always an equivalence on all hom categories. $\mathcal{C}$ is said to be *Q-system complete* if $\iota_{\mathcal{C}}$ is a *-equivalence of *-2-categories. We study *Q-system completeness* in the context of pre-C*-2-categories. We call a pre-C*-2-category $\mathcal{C}$ to be *Q-system complete* if every Q-system in $\mathcal{C}$ 'splits'.

In our recent joint paper [DGGJ22], we gave a higher categorical interpretation of Bratteli diagrams and unitary connections in terms of a larger W*-2-category $\mathbf{UC}^{\mathrm{tr}}$. The 0-cells of $\mathbf{UC}^{\mathrm{tr}}$ are Bratteli diagrams with tracial weighting data. These generalize the Bratteli diagrams appearing from taking the tower of relative commutants of a finite-index subfactor. 1-cells of $\mathbf{UC}^{\mathrm{tr}}$ are *unitary connections* between Bratteli diagrams which are compatible with the tracial data. Finally the 2-cells are defined as certain fixed points of a ucp (unital completely positive) map. To define $\mathbf{UC}^{\mathrm{tr}}$, we had to first consider a purely algebraic category $\mathbf{UC}$. The 0-cells of $\mathbf{UC}$ are Bratteli diagrams (without the tracial data). 1-cells of $\mathbf{UC}$ are unitary connections and 2-cells are natural intertwiners between connections which we call flat sequences. $\mathbf{UC}$ has a close resemblance to the 2-category studied in [CPJ22] in the context of fusion category actions on AF-C*-algebras, with minor differences at the level of 0-cells and 2-cells only.

We investigate Q-system completeness of $\mathbf{UC}$. The following is the main theorem of the paper.

1.1. Theorem. $\mathbf{UC}$ *is Q-system complete.*

Given a Q-system in $\mathbf{UC}$, to exhibit its 'splitting' one needs to construct a suitable 0-cell and a suitable dualizable 1-cell from the initial 0-cell to the newly constructed one which enables the splitting. The idea to construct our suitable 0-cell in $\mathbf{UC}$ comes from [CPJP22] and we use subfactor theoretic ideas [B97, EK98, P89, P94] to build our appropriate 1-cell in $\mathbf{UC}$.

There at least two natural questions appearing from our investigations. Bi-faithfulness of functors (that is, both the functor and its adjoint are faithful) plays a major role in achieving our results. So the first question is, if we drop the bi-faithfulness condition of 0-cells and 1-cells in $\mathbf{UC}$ (see Definition 2.11), then will the modifed 2-category be still Q-system complete. Second, is $\mathbf{UC}^{\mathrm{tr}}$ Q-system complete ? We will try to answer these questions in our future work.

The outline of the paper is as follows. In Section 2, we will quickly go through some basic results and definitions and set up some pictorial notations. In Section 3, we explore Q-systems in **UC** and prove some results that will be useful to construct our appropriate 0-cell in **UC**. In Section 4, we build the 0-cell and the dualizable 1-cell and then proceed to prove our main theorem.

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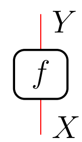
2. Preliminaries

In this section we will furnish the necessary background on *Q-system completion* and the 2-category of *Unitary connections* **UC**.

2.1. NOTATIONS RELATED TO 2-CATEGORIES. We refer the reader to [JY21] for basics of 2-categories.

Suppose $\mathcal{C}$ is a 2-category and $a, b \in \mathcal{C}_0$ be two 0-cells. A 1-cell from $a \xrightarrow{X} b$ is denoted by ${}_bX_a$. Pictorially, a 1-cell will be denoted by a red strand and a 2-cell will be denoted by a box with strings with passing through it. Suppose we have two 1-cells $X, Y \in \mathcal{C}_1(a, b)$ and $f \in \mathcal{C}_2(X, Y)$ be a 2-cell. Then we will denote f as 

We write tensor product $\boxtimes$ of 1-cells from right to left ${}_cY \underset{b}{\boxtimes} X_a$.

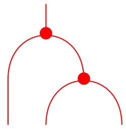
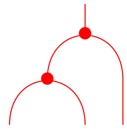
The notion of C*-2-categories is believed to first appear in [LR97]. For basics of C*/W*-2-categories we refer the reader to [CPJP22, GLR85]. For a detailed study about graphical calculus, we refer the reader to [HV19].

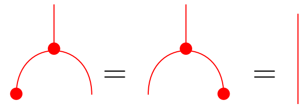
2.2. Q-SYSTEM COMPLETION.

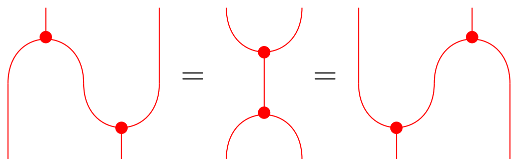
2.3. DEFINITION. *A pre-C*-2-category is a 2-category such that the hom-1-categories satisfies all the conditions of a C*-category except that the 2-cell spaces need not be complete with repsect to the given norm.*

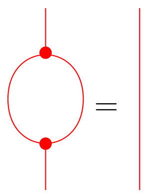
Let $\mathcal{C}$ be a pre-C*-2-category.

2.4. DEFINITION. *A Q-system in $\mathcal{C}$ is a 1-cell ${}_bQ_b \in \mathcal{C}_1(b, b)$ along with multiplication map $m \in \mathcal{C}_2(Q \boxtimes_b Q, Q)$ and unit map $i \in \mathcal{C}_2(1_b, Q)$, as mentioned in Section 1, satisfying the following properties:*

(Q1)  $=$  (associativity)

(Q2)  (unitality)

(Q3)  (Frobenius condition)

(Q4)  (Separability)

2.5. Definition. *[CPJP22] Given a Q-system (Q, m, i), we define*

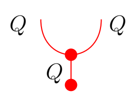
$d_Q :=$  $\in End_{\mathcal{C}}(1_b)^+$

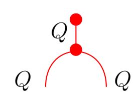
.

- *If d_Q is invertible, we call Q non-degenerate or an extension of 1_b.*
- *If d_Q is an idempotent, we call Q a summand of 1_b.*

We recall some facts about Q-systems in C*-tensor categories already mentioned in [CPJP22, Z07].

2.6. Fact.

(F1) Q is a self-dual 1-cell with $ev_Q :=$ 

and $coev_Q :=$ 

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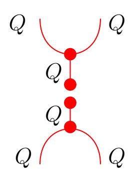
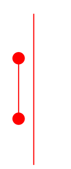
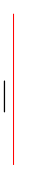
(F2) Using (F1) and [Z07, Lemma 1.16] we have the following inequalitites:



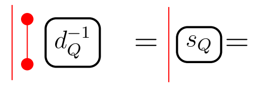
$\leq$  $\leq$ $\|d_Q\|$ 

(F3) By [Z07, Corollary 1.19] either d_Q is invertible, or zero is an isolated point in $Spec(d_Q)$. Define , $f : Spec(d_Q) \to \mathbb{C}$ by

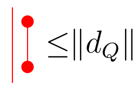
$$f(x) = \begin{cases} 0 & x = 0 \\ x^{-1} & x \neq 0 \end{cases}$$

By abuse of notation, set $d_Q^{-1} := f(d_Q)$. By continuous functional calculus, set $s_Q := d_Q d_Q^{-1}$. Then we have the following :

(a)



(b)



2.7. Definition. *Suppose $\mathcal{C}$ is a pre-C*-2-category and ${}_bX_a \in \mathcal{C}_1(a,b)$. A unitarily separable left dual for ${}_bX_a$ is a dual $\left({}_a\overline{X}_b, ev_X, coev_X\right)$ such that $ev_X \circ ev_X^* = id_{1_a}$ (cf. [CPJP22, Example 3.9]).*

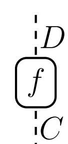
Given a unitarily separable left dual for ${}_bX_a \in \mathcal{C}_1(a,b)$, ${}_bX \underset{a}{\boxtimes} \overline{X}_b \in \mathcal{C}_1(b,b)$ is a Q-system with multiplication map $m := \mathrm{id}_X \boxtimes ev_X \boxtimes \mathrm{id}_{\overline{X}}$ and unit map $i := coev_X$.

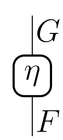
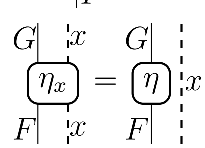
Given a Q-system $Q \in \mathcal{C}_1(b,b)$, if it is of the above form then we say that the Q-system Q 'splits'.

2.8. Definition. *A pre-C*-2-category $\mathcal{C}$ is said to be Q-system complete if every Q-system in $\mathcal{C}$ 'splits', that is, given a Q-system $Q \in \mathcal{C}_1(b,b)$ there is an object $c \in \mathcal{C}_0$ and a dualizable 1-cell $X \in \mathcal{C}_1(c,b)$ which admits a unitary separable dual $\left(\overline{X}, ev_X, coev_X\right)$ such that (Q,m,i) is isomorphic to ${}_bX \underset{c}{\boxtimes} \overline{X}_b$ as Q-systems.*

2.9. Remark. In [CPJP22], Q-system completion has been treated in the context of C*/W*-2-categories. It has been proved that Definition 2.8 is equivalent to their definition of *Q-system completeness* (see [CPJP22, Theorem 3.36]) of C*/W*-2-categories.

2.10. Unitary connections. **Pictorial notations.** We will apply the graphical calculus as mentioned in Section 2.1 to the 2-category of *Categories* (cf. [HV19]).

(i) Let $\mathcal{C}$ be a category and let $f \in \mathcal{C}(C,D)$. It will be denoted by  and composition of two morphisms will be represented by two vertically stacked labelled boxes.

(ii) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. Then a natural transformation $\eta : F \to G$ will be denoted by  . For an object x in $\mathcal{C}$, the morphism $\eta_x : Fx \to Gx$ will be denoted by  .

(iii) For a $*$-linear functor $F : \mathcal{C} \to \mathcal{D}$ between two semisimple C*-categories categories, we will denote a solution to conjugate equation by

$$\rho = \; F \smile F' : \mathrm{id}_{\mathcal{D}} \longrightarrow FF' \quad \text{and} \quad \rho' = \; F' \smile F : \mathrm{id}_{\mathcal{C}} \longrightarrow F'F$$

$$\rho^* = \; F \frown F' : FF' \longrightarrow \mathrm{id}_{\mathcal{D}} \quad \text{and} \quad [\rho']^* = \; F' \frown F : F'F \longrightarrow \mathrm{id}_{\mathcal{C}}$$

where $F' : \mathcal{D} \to \mathcal{C}$ is an adjoint functor of F.

We will extend the above dictionary (between things appearing in the category world and pictures) in an obvious way. For instance, composition of morphisms and natural transformations will be pictorially represented by stacking the boxes vertically whereas tensor product (resp., composition) of objects (resp., functors) by parallel vertical strings. For simplicity, sometimes we will not label all of the strings (with any object or functor) emanating from a box (labelled with a morphism or a natural transformation) when it can be read off from the context. To distinguish between a functor arising in 0-cell and a functor arising in 1-cell, we will denote the former by a black strand and the latter by a red strand unless otherwise mentioned.

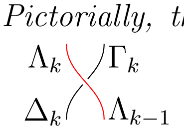
Let us recall the definition of the pre-C*-2-category of *unitary connections* **UC** described in [DGGJ22].

2.11. Definition. *The 2-category* **UC** *consists of the following :*

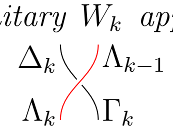
(1) 0-cells are $*$-linear, bi-faithful functors $\Gamma_k : \mathcal{M}_{k-1} \to \mathcal{M}_k$ (where $\mathcal{M}_k$ is a finite, semisimple, C-category whose isomorphism classes of the simple objects are indexed by the vertex set $V_{\mathcal{M}_k}$). We will denote such a 0-cell by $\left\{ \mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k \right\}_{k \geq 1}$ or sometimes simply $\Gamma_\bullet$.*

(2) A 1-cell from the 0-cell $\left\{ \mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k \right\}_{k \geq 1}$ to the 0-cell $\left\{ \mathcal{N}_{k-1} \xrightarrow{\Delta_k} \mathcal{N}_k \right\}_{k \geq 1}$ consists of a sequence of $*$-linear bi-faithful functors $\{\Lambda_k : \mathcal{M}_k \to \mathcal{N}_k\}_{k \geq 0}$ and natural unitaries $W_k : \Delta_k \Lambda_{k-1} \to \Lambda_k \Gamma_k$ for $k \geq 1$. Such a 1-cell will be denoted by $(\Lambda_\bullet, W_\bullet)$ or simply by $\Lambda_\bullet$, and $W_\bullet$ will be referred as a unitary connection associated to $\Lambda_\bullet$. Denote the set of 1-cells from $\Gamma_\bullet$ to $\Delta_\bullet$ by $\mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$.

Pictorially, the natural unitary W_k appearing in the 1-cell will be represented by

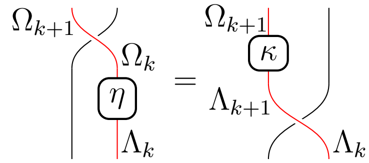


and W_k^* by



(3) Let $\Lambda_\bullet, \Omega_\bullet \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$. For describing 2-cells we need the following definition:

2.12. Definition. A pair $(\eta, \kappa) \in \mathrm{NT}(\Lambda_k, \Omega_k) \times \mathrm{NT}(\Lambda_{k+1}, \Omega_{k+1})$ is said to satisfy *exchange relation* if the condition



holds.

2.13. Remark. *The exchange relation pair is unique separately in each variable, that is, if (η, κ_1) and (η, κ_2) (resp., (η_1, κ) and (η_2, κ)) both satisfy exchange relation, then $\kappa_1 = \kappa_2$ (resp., $\eta_1 = \eta_2$); this is because the connections are unitary and the functors Γ_k and Δ_k are bi-faithful.*

Let $Ex(\Lambda_\bullet, \Omega_\bullet)$ denote the space of sequences $\{\eta^{(k)} \in \mathrm{NT}(\Lambda_k, \Omega_k)\}_{k\geq 0}$ such that there exists an N such that (η_k, η_{k+1}) satifies the exchange relation for all $k \geq N$. Consider the subspace

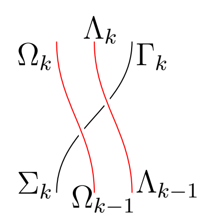
$$Ex_0(\Lambda_\bullet, \Omega_\bullet) := \{\{\eta_k\}_{k\geq 0} \in Ex(\Lambda_\bullet, \Omega_\bullet) : \ \eta_k = 0 \ \ \textit{for all} \ \ k \geq N \ \ \textit{for some} \ \ N \in \mathbb{N}\}$$

We define the space of 2-cells

$$\mathbf{UC}_2(\Lambda_\bullet, \Omega_\bullet) := \frac{Ex(\Lambda_\bullet, \Omega_\bullet)}{Ex_0(\Lambda_\bullet, \Omega_\bullet)}$$

(4) *For $\Omega_\bullet \in \mathbf{UC}_1(\Delta_\bullet, \Sigma_\bullet)$ and $\Lambda_\bullet \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$, define*

$$\Omega_\bullet \boxtimes \Lambda_\bullet := \left(\{\Omega_k \Lambda_k\}_{k\geq 0} \, , \, \left\{ \quad \right\}_{k\geq 1} \right) . \qquad (1)$$



For notational convenience, instead of denoting a 2-cell by an equivalence class of sequences, we simply use a sequence in the class and truncate upto a level after which the exchange relation holds for every consecutive pair, namely, $\{\eta^{(k)}\}_{k\geq N} \in \mathbf{UC}_2(\Lambda_\bullet, \Omega_\bullet)$ where $(\eta^{(k)}, \eta^{(k+1)})$ satisfies the exchange relation for all $k \geq N$.

2.14. Remark. *From the definition of $\mathbf{UC}_2(\Lambda_\bullet, \Omega_\bullet)$, we observe that two 2-cells $\{\eta^{(k)}\}_{k\geq N}$, $\{\tau^{(k)}\}_{k\geq L} \in \mathbf{UC}_2(\Lambda_\bullet, \Omega_\bullet)$ are equal if and only if $\eta^{(k)} = \tau^{(k)}$ eventually. So, two 1-cells $\Lambda_\bullet$ and $\Omega_\bullet$ are isomorphic in $\mathbf{UC}$ if there is a sequence of natural transformations $U_k : \Lambda_k \to \Omega_k$ which satisfies exchange relation from some level l and which implements isomorphism between Λ_k and Ω_k eventually.*

For horizontal and vertical composition of 2-cells we refer the reader to [DGGJ22].

Given a 0-cell $\Gamma_\bullet \in \mathbf{UC}_0$, we fix an object $m_0 \coloneqq \bigoplus_{v \in V_0} v \in \mathrm{ob}(\mathcal{M}_0)$. Consider the sequence of finite dimensional C*-algebras $\{A_k \coloneqq \mathrm{End}(\Gamma_k \cdots \Gamma_1 m_0)\}_{k \geq 0}$ (assuming $A_0 = \mathrm{End}(m_0)$) along with the unital $*$-algebra inclusions given by

$$A_{k-1} \ni \alpha \hookrightarrow \Gamma_k \, \alpha \in A_k \ . \tag{2}$$

Indeed, the Bratteli diagram of A_{k-1} inside A_k is given by the graph Γ_k. To the 0-cell $\Gamma_\bullet$, we associate the $*$-algebra $A_\infty \coloneqq \bigcup_{k \geq 0} A_k$

To each 1-cell $(\Lambda_\bullet, W_\bullet) \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$, we will associate an A_∞-B_∞ right correspondence where n_0 and B_k's are related to $\left\{ \mathcal{N}_{k-1} \xrightarrow{\Delta_k} \mathcal{N}_k \right\}_{k \geq 1}$ exactly the way m_0 and A_k's are related to $\left\{ \mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k \right\}_{k \geq 1}$ respectively. For $k \geq 0$, set

$$H_k \coloneqq \mathcal{N}_k \left(\Delta_k \cdots \Delta_1 n_0, \Lambda_k \Gamma_k \cdots \Gamma_1 m_0 \right) .$$

We have an obvious A_k-B_k-bimodule structure on H_k in the following way:

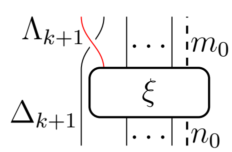
$$A_k \times H_k \times B_k \ni (\alpha, \xi, \beta) \longmapsto \Lambda_k(\alpha) \circ \xi \circ \beta \ \in H_k \ . \tag{3}$$

Again, there is a B_k-valued inner product on H_k given by

$$H_k \times H_k \ni (\xi, \zeta) \overset{\langle \cdot, \cdot \rangle_{B_k}}{\longmapsto} \langle \xi, \zeta \rangle_{B_k} \coloneqq \zeta^* \circ \xi \ \in B_k \ . \tag{4}$$

Next, observe that H_k sits inside H_{k+1} via the map

$$H_k \ni \xi \overset{I_{k+1}}{\longmapsto} \left[(W_{k+1})_{\Gamma_k \cdots \Gamma_1 m_0} \right] \circ \left[\Delta_{k+1} \xi \right] \ = \tag{5}$$



$\in H_{k+1}$.

2.15. Lemma. *([DGGJ22]) The inclusions $H_k \hookrightarrow H_{k+1}$, $A_k \hookrightarrow A_{k+1}$, $B_k \hookrightarrow B_{k+1}$ and the corresponding actions are compatible in the obvious sense.*

Set $H_\infty \coloneqq \bigcup_{k \geq 0} H_k$ which clearly becomes an A_∞-B_∞ right correspondence. To the 1-cell $(\Lambda_\bullet, W_\bullet)$ we associate the A_∞-B_∞ right correspondence H_∞.

We also have a Pimsner-Popa (PP) basis of the right-B_∞-module H_∞ with respect to the B_∞-valued inner product.

2.16. Lemma. *([DGGJ22]) There exists a finite subset $\mathscr{S}$ of H_0 such that $\sum_{\sigma \in \mathscr{S}} \sigma \circ \sigma^* = 1_{\Lambda_0 m_0}$; moreover, any such $\mathscr{S}$ is a PP-basis for the right B_∞-module H_∞.*

2.17. Theorem. *([DGGJ22]) Starting from a 2-cell* $\left\{\eta^{(k)} \in \mathrm{NT}\left(\Lambda_k, \Omega_k\right)\right\}_{k \geq K}$, *we have an intertwiner* $\Phi_{\eta^{(k)}_{\Gamma_k \cdots \Gamma_1 m_0}} \in {}_{A_\infty}\mathcal{L}_{B_\infty}(H_\infty, G_\infty)$ *which is independent of* $k \geq K$.

Conversely, for every $T \in {}_{A_\infty}\mathcal{L}_{B_\infty}(H_\infty, G_\infty)$ *(= the space of* A_∞*-*B_∞*-linear adjointable operator) and for all* $k \geq K_T := \min\{l : TH_0 \subset G_l\}$, *there exists unique* $\eta^{(k)} \in \mathrm{NT}(\Lambda_k, \Omega_k)$ *such that* $T = \Phi_{\eta^{(k)}_{\Gamma_k \cdots \Gamma_1 m_0}}$. *Further,* $\left(\eta^{(k)}, \eta^{(k+1)}\right)$ *satisfies exchange relation for all* $k \geq K_T$.

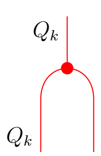
Clearly **UC** becomes a pre-C*-2-category.

2.18. Remark. We will denote the object m_0 by dashed lines and any other object by dotted lines in (ii) of the pictorial notations mentioned at the beginning of Section 2.10.

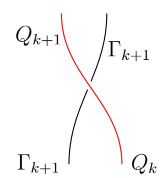
3. Q-systems in UC

In this section, given a Q-system in **UC** for a 0-cell, we explore certain structural properties of the associated bimodules that will further enable us to construct new 0-cells and a new dualizable 1-cell in the next section, that will implement Q-system completion of **UC**.

Let $(\Gamma_\bullet, \mathcal{M}_\bullet)$ be a 0-cell in **UC** and $(Q_\bullet, W^Q_\bullet, m_\bullet, i_\bullet)$ be a Q-system in $\mathbf{UC}_1((\Gamma_\bullet, \mathcal{M}_\bullet), (\Gamma_\bullet, \mathcal{M}_\bullet))$. Graphically, each natural transformation m_k, i_k and W^Q_{k+1} will be represented by the following respective diagrams:

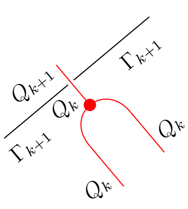
$m_k :=$ 

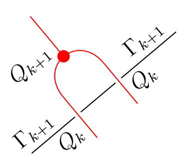
, $i_k :=$ 

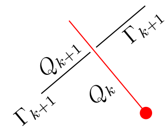
and $W^Q_{k+1} :=$ 

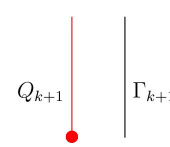
$\forall k \geq 0$

Pictorially, exchange relation of m_k's and i_k's with respect to $W_\bullet$ will be denoted as follows:



$=$ 

and 

$=$ 

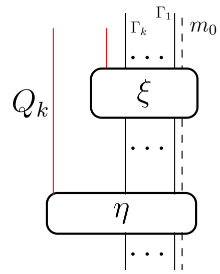
eventually for all k.

3.1. Remark. From Remark 2.14, we observe that for our Q-system $(Q_\bullet, m_\bullet, i_\bullet)$ in **UC** the natural transformations m_k and i_k satisfy (Q1)-(Q4) as in Definition 2.4 eventually for all k . For the rest of the paper we fix a natural number l such that m_k and i_k satisfy (Q1)-(Q4) and the exchange relations for $k \geq l$.

Consider the filtration of finite dimensional C*-algebras $\{A_k := \mathrm{End}(\Gamma_k \cdots \Gamma_1 m_0)\}_{k\geq 1}$ associated to the 0-cell $\Gamma_\bullet$ where m_0 is direct sum of a maximal set of mutually non-isomorphic simple objects in $\mathcal{M}_0$. Let $\{H_k := \mathcal{M}_k(\Gamma_k \cdots \Gamma_1 m_0, Q_k \Gamma_k \cdots \Gamma_1 m_0)\}_{k\geq 1}$ be the right correspondence associated to $Q_\bullet$. By construction (Equation (3) and Equation (4)), H_k is a right A_k-A_k correspondence. Thus, one may view H_k as a 1-cell in the 2-category **C*Alg** of right correspondence bimodules over pairs of C*-algebras.

We will further establish that each H_k is a Q-system in $\mathbf{C^*Alg}(A_k, A_k)$ for $k \geq l$. In order to do this, we will use the following identification.

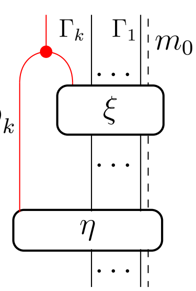
3.2. Remark. Let $\{Y_k := \mathcal{M}_k(\Gamma_k \cdots \Gamma_1 m_0, Q_k^2 \Gamma_k \cdots \Gamma_1 m_0)\}_{k\geq 1}$ denote the right correspondence associated to the 1-cell $Q_\bullet \boxtimes Q_\bullet$ in **UC**. The proof of [DGGJ22, Proposition 3.12] tells us that the map $\xi \boxtimes \eta \longmapsto$



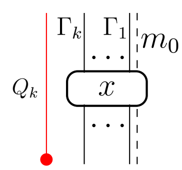
is an isomorphism between $H_k \underset{A_k}{\boxtimes} H_k$ and Y_k as right A_k-A_k correspondence.

Via the above identification, the multiplication 2-cell $m_\bullet$ and the unit 2-cell $i_\bullet$ in **UC** corresponds to the maps $\widetilde{m}_k : H_k \underset{A_k}{\boxtimes} H_k \to H_k$ and $\widetilde{i}_k : A_k \to H_k$ respectively at the level of bimodules; more explicitly

$\widetilde{m}_k(\xi \boxtimes \eta) :=$



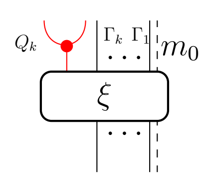
and $\widetilde{i}_k(x) :=$



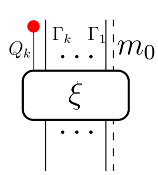
.

3.3. Proposition. *For each $k \geq l$, $\widetilde{m}_k$ and $\widetilde{i}_k$ are adjointable maps and hence 2-cells in* **C*Alg**. *Moreover, $(H_k, \widetilde{m}_k, \widetilde{i}_k)$ becomes a Q-system in $\mathbf{C^*Alg}(A_k, A_k)$ for each $k \geq l$.*

Proof. Using the identification in Remark 3.2, the adjoint of $\widetilde{m}_k$ is given by



and that of $\widetilde{i}_k$ is given by



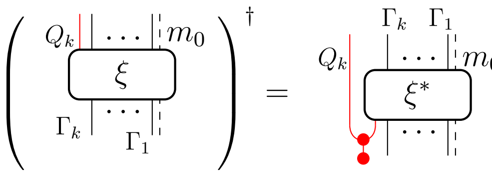
. Now using $\widetilde{m}_k$ and $\widetilde{i}_k$ and their adjoints, and the properties (Q1-Q4) of $m_\bullet$ and $i_\bullet$ mentioned at in preliminaries , associativity, unitality, frobenius property and separability of $\left(H_k, \widetilde{m}_k, \widetilde{i}_k\right)$ easily follows. ∎

We now explore certain structural properties of H_k.

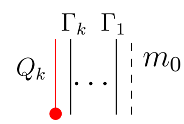
We prove the following proposition using ideas from [CPJP22].

3.4. Proposition. *For each $k \geq l$, the space H_k is a unital* C*-*algebra with multiplication, adjoint and unit given by*

$\xi \cdot \eta := \widetilde{m}_k(\xi \boxtimes \eta)$,



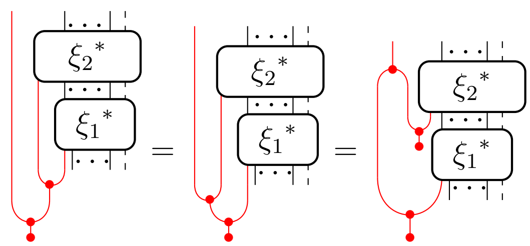
and



respectively for $\xi, \eta \in H_k$.

Proof. Indeed, $\xi^{\dagger\dagger} = \xi$. Again

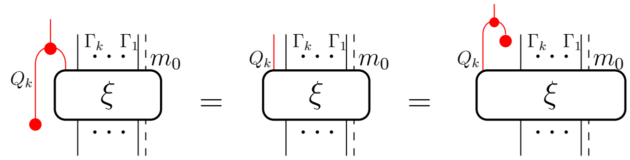
$(\xi_1.\xi_2)^\dagger =$



$= \xi_2^\dagger.\xi_1^\dagger$

where the second equality follows from associativity and the third comes from Frobenius and unitality conditions. Also,

$1_{H_k} \cdot \xi =$

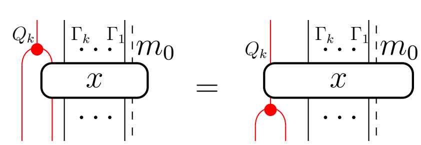


$= \xi \cdot 1_{H_k}.$

Hence, H_k becomes a unital *-algebra.

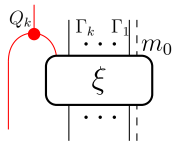
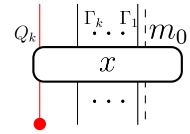
To prove that H_k is a C*-algebra, we show that it is isomorphic to a *-subalgebra of a finite dimensional C*-algebra. Define

$S_k := \left\{ x \in \mathrm{End}(Q_k \Gamma_k \cdots \Gamma_1 m_0) \;\middle|\; \right.$

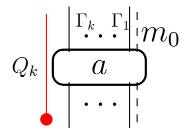
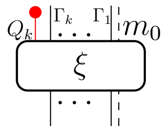
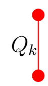
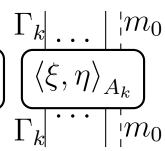


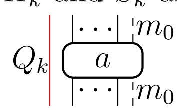
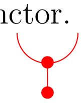
$\left. \right\}$

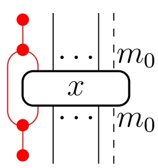
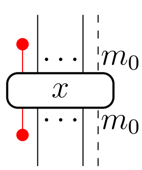
sitting inside the finite dimensional C*-algebra $\mathrm{End}(Q_k \Gamma_k \cdots \Gamma_1 m_0)$. Clearly S_k is closed under multiplication, as well as *-closed (using Frobenius property and unitality). Define $\phi_1^{(k)} : H_k \to S_k$ and $\phi_2^{(k)} : S_k \to H_k$ as follows:

$\phi_1^{(k)}(\xi) :=$  $\in S_k$ (by associativity of Q_k) and $\phi_2^{(k)}(x) :=$  .

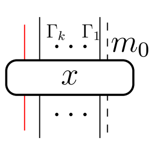
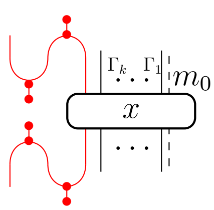
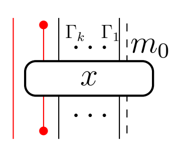
Now, it is routine to check using the axioms of Q-systems that $\phi_1^{(k)}$ and $\phi_2^{(k)}$ are unital, *-homomorphisms. Also, $\phi_1^{(k)}$ and $\phi_2^{(k)}$ are mutually inverse to each other, hence they are isomorphisms. ∎

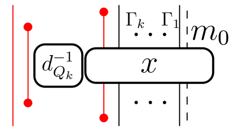
3.5. Lemma. *The map* $\widetilde{i}_k : A_k \to H_k$ *defined by* $\widetilde{i}_k(a) :=$  *is a unital inclusion of* C*-*algebras. In the reverse direction, the map* $E_k : H_k \to A_k$ *defined by* $E_k(\xi) :=$ $\boxed{d_{Q_k}^{-1}}$  *(where* $d_{Q_k} =$  *) is a finite index, faithful conditional expectation satisfying* $E_k\left(\eta^\dagger \cdot \xi\right) =$ $\boxed{d_{Q_k}^{-1}}$  *(where* $\langle\cdot,\cdot\rangle_{A_k}$ *is the right* A_k*-valued inner product on* H_k *as defined in Equation* (4)*) for each* $k \geq l$.

Proof. We make use of the *-algebra isomorphisms ϕ_1, ϕ_2 between H_k and S_k and find that the map $\widetilde{i}_k : A_k \to H_k$ corresponds to $A_k \ni a \xmapsto{\phi_1^{(k)} \circ \widetilde{i}_k} Q_k\, a =$  $\in S_k$ which is indeed an inclusion since Q_k is a bi-faithful functor. Now, Q_k is symmetrically self-dual with the solution to conjugate equation given by  . Thus, we have a conditional expectation given by

$S_k \ni x \xmapsto{E'}$ $\boxed{d_{Q_k}^{-1}}$  $=$ $\boxed{d_{Q_k}^{-1}}$  $\in A_k$

where the equality follows from the definition of S_k and separability axiom. This conditional expectation is automatically faithful and translates into E_k (defined in the statement) via the *-isomorphism $\phi_2^{(k)}$. Now, for $x \in S_k^+$, we have

 $=$  $\leq \|d_{Q_k}\|$ 

where the first equality follows from (F1) of Fact 2.6 and the second inequality from (F2) of Fact 2.6 and the definition of S_k. We rewrite the last term as  $\leq$ $\|d_{Q_k}\|\, Q_k\,(E'(x))$. Hence, the conditional expectation E', and thereby E_k has finite index. ∎

Next, we will test the compatibility of the countable family of finite dimensional C*-algebras $\{H_k\}_{k\geq 0}$ and the inclusions $H_k \overset{I_{k+1}}{\hookrightarrow} H_{k+1}$ for $k \geq 0$ (as described in Equation (5)).

3.6. LEMMA. *The inclusion* $H_k \overset{I_{k+1}}{\hookrightarrow} H_{k+1}$ *is a* $*$*-algebra homomorphism eventually for all* k*. Further, the unital filtration* $\{A_k\}_{k\geq 0}$ *of finite dimensional C*-algebras (as described 2) sits inside* $H_\infty = \underset{k}{\cup} H_k$ *via the inclusions* $\widetilde{i}_k : A_k \to H_k$ *eventually for all* k*. In particular, the above conditions commence when* (m_k, m_{k+1}) *and* (i_k, i_{k+1}) *start satisfying the exchange relation.*

PROOF. This easily follows from the exchange relation of m_k and i_k, and the definitions of $\widetilde{m}_k$ and $\widetilde{i}_k$. ∎

3.7. REMARK. We can obtain $\mathscr{S}_k \subset H_k$ such that $\sum\limits_{\sigma\in\mathscr{S}_k} \sigma\sigma^* = 1_{Q_k\Gamma_k\cdots\Gamma_1 m_0}$ using Lemma 2.16 and Equation (5).

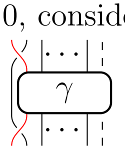
4. Splitting of $(Q_\bullet, m_\bullet, i_\bullet)$

In this section we will first construct a suitable 0-cell in **UC** using results from the previous section. Then move on to construct a dualizable 1-cell $X_\bullet$ from $\Gamma_\bullet$ to the newly constructed 0-cell. Subsequently we build a unitary from $\overline{X}_\bullet \boxtimes X_\bullet$ to $Q_\bullet$ which intertwine the algebra maps as well as satisfy exchange relations eventually.

Notation: Thoughout this section, given a finite dimensional C*-algebra A, we will use the notation $\mathcal{R}_A$ for the category of finite-dimensional (as a complex vector space) right A-correspondences. Note that $\mathcal{R}_A$ is a finite, semisimple C*-category.

4.1. NEW 0-CELLS IN **UC**.

Let $l \in \mathbb{N}$ be as in Remark 3.1.

For each $k \geq 0$, consider the C*-algebra inclusions $A_k \overset{Q_k}{\hookrightarrow} C_k \coloneqq \mathrm{End}(Q_k\Gamma_k\cdots\Gamma_1 m_0)$ and $C_k \ni \gamma \longmapsto$  $\in C_{k+1}$. Note that $Q_k\,(A_k) \subset S_k \subset C_k$ for all $k \geq l$. Consider the filtration of C*-algebras $\{B_k\}_{k\geq 0}$ defined as follows:

$$B_k = \begin{cases} H_k & \text{if } k \geq l \\ S_l \cap C_k & \text{if } 0 \leq k \leq l-1 \end{cases}$$

where the inclusion $B_k \hookrightarrow B_{k+1}$ is given by I_k for $k \geq l$, set inclusions for $0 \leq k \leq l-2$ and the remaining inclusion $B_{l-1} \hookrightarrow B_l$ is $\phi_2^{(l)}\Big|_{S_l \cap C_{l-1}} : S_l \cap C_{l-1} \to H_l$ (where $\phi_2^{(l)} : S_l \to H_l$ is the isomorphism defined in Proposition 3.4).

Define $\Delta_{k+1} := \bullet \underset{B_k}{\boxtimes} B_{k+1} : \mathcal{R}_{B_k} \to \mathcal{R}_{B_{k+1}}$ for $k \geq 0$. Each Δ_k is a bi-faithful functor (which follows from the unital inclusion $B_k \hookrightarrow B_{k+1}$ of finite dimensional C*-algebra for $k \geq 0$). Thus, we have a 0-cell $(\Delta_\bullet, \mathcal{R}_{B_\bullet}) \in \mathbf{UC}_0$.

Similarly, using the unital filtration $\{A_k\}_{k\geq 0}$ (resp., $\{C_k\}_{k\geq 0}$) of finite dimensional C*-algebras, we define another 0-cell $\Sigma_\bullet$ (resp. $\Psi_\bullet$) defined by $\Sigma_k := \bullet \underset{A_{k-1}}{\boxtimes} A_k \ : \ \mathcal{R}_{A_{k-1}} \to \mathcal{R}_{A_k}$ (resp., $\Psi_k := \bullet \underset{C_{k-1}}{\boxtimes} C_k \ : \ \mathcal{R}_{C_{k-1}} \to \mathcal{R}_{C_k}$) for $k \geq 1$.

4.2. Construction of dualizable 1-cell from $(\Gamma_\bullet, \mathcal{M}_\bullet)$ to $(\Delta_\bullet, \mathcal{R}_{B_\bullet})$.

Our strategy is to build two dualizable 1-cells $\left(F_\bullet, W_\bullet^F\right) : \Gamma_\bullet \to \Sigma_\bullet$ and $\left(\Lambda_\bullet, W_\bullet^\Lambda\right) : \Sigma_\bullet \to \Delta_\bullet$ and define $(X_\bullet, W_\bullet)$ to be their composition in $\mathbf{UC}$ as depicted in Equation (1) of Definition 2.11 and thereby obtaining our desired dualizable 1-cell $X_\bullet : \Gamma_\bullet \to \Delta_\bullet$ in $\mathbf{UC}$. We first prove the following easy fact.

4.3. Proposition. *Given a finite semisimple C*-category $\mathcal{M}$ and an object m which contains every simple object as a sub-object, the functor $F := \mathcal{M}(m, \bullet) : \mathcal{M} \to \mathcal{R}_A$ is an equivalence where $A = \mathrm{End}(m)$ and $\mathcal{R}_A$ is the category of right A-correspondences.*

Proof. For $x \in \mathrm{ob}(\mathcal{M})$, $F(x)$ becomes a right A-correspondence with the A-action and A-valued inner product defined in the following way

$$F(x) \times A \ni (u, a) \longmapsto u\, a \in F(x) \quad \text{and} \quad \langle u, v\rangle := v^* u \ .$$

For $f \in \mathcal{M}(x, y), F(f)(u) = f\, u \in F(y)$ for each $u \in F(x)$. Indeed, F-action on any morphism of $\mathcal{M}$ is adjointable ($F(f)^* = F(f^*)$) and A-linear. Clearly, F is a faithful functor.

Let $T \in \mathcal{R}_A(F(x), F(y))$. Since every simple appears as a sub-object in m, we can find a finite set $\mathscr{S}_x \subseteq F(x)$ such that $\sum\limits_{u\in\mathscr{S}_x} u\, u^* = 1_x$. Define $f := \sum\limits_{u\in\mathscr{S}_x} T(u) u^* \in \mathcal{M}(x, y)$.

For $v \in F(x)$,we have,

$$\begin{aligned} T(v) &= T\left(\sum_{u\in\mathscr{S}_x} u\, u^* v\right) \\ &= \sum_{u\in\mathscr{S}_x} T(u)\, u^* v \quad \text{(since } T \text{ is right } A\text{-linear)} \\ &= F(f)(v) \quad \text{(since } F(f) = f \circ - \text{ and by definition of } f) \end{aligned}$$

Thus, F is full.

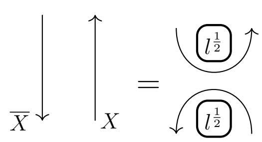
Now, we show that F is essentially surjective. Since F is fully faithful by Schur's lemma, we have, $F(x)$ is simple if x is simple. We show that for simple $H \in \mathcal{R}_A$ there is a simple x in $\mathcal{M}$ such that $F(x)_A \simeq H_A$. Choose, $\xi \in H \setminus \{0\}$ such that $\langle \xi, \xi \rangle_A \neq 0$. By spectral decomposition of $\langle \xi, \xi \rangle_A$, there is a minimal projection p in A such that $\langle \xi, \xi \rangle_A p \in \mathbb{C}p \setminus \{0\}$. Now ,since p is minimal, $\langle \xi p, \xi p \rangle_A = p \langle \xi, \xi \rangle_A p \in \mathbb{C}p \setminus \{0\}$. Without loss of generality, we assume $\xi p = \xi$. Now, H being irreducible, we have, $H = \xi A$. Now, by semi-simplicity of $\mathcal{M}$ there is a simple $x \in \mathcal{M}$ and an isometry $\alpha : x \to m$ such that $p = \alpha\alpha^*$. Observe that, $\alpha^* \in F(x)$ and $F(x) = \alpha^* A$. Define $T' : F(x)_A \to H_A$ as $T'(\alpha^* a) = \xi a$ for all $a \in A$. Clearly, T' is well-defined, right-A linear and onto. Thus, T' is an isomorphism. Hence, F is an equivalence. ■

4.3.1. Construction of $\left(F_\bullet, W^F_\bullet\right) \in \mathbf{UC}_1\left(\Gamma_\bullet, \Sigma_\bullet\right)$.

For each $k \geq 0$, setting $m = \Gamma_k \cdots \Gamma_1 m_0$ in Proposition 4.3, we obtain the functor $F_k := \mathcal{M}_k(\Gamma_k \cdots \Gamma_1 m_0, \bullet) : \mathcal{M}_k \to \mathcal{R}_{A_k}$ which is an equivalence.

4.4. Proposition. *Suppose $\mathcal{C}$ is a C*-2-category. Let $X \in \mathcal{C}_1(a,b)$ be dualizable with dual $\overline{X} \in \mathcal{C}_1(b,a)$ such that each component in the solution $\left(R, \overline{R}\right)$ to the conjugate equations for $\left(X, \overline{X}\right)$ are invertible. Then, there exists another solution $\left(R', \overline{R}'\right)$ to the conjugate equations for $\left(X, \overline{X}\right)$ such that R' and $\overline{R}'$ are unitaries.*

Proof. Without loss of generality, we may assume that $\mathcal{C}$ is strict. Since R and R^* are invertible, so R^*R is also invertible. Let $l := R^*R \in \mathrm{End}(1_b)$. Define $R' := R \circ l^{-\frac{1}{2}} \in \mathcal{C}_2(1_b, X \boxtimes \overline{X})$. Clearly, R' is invertible and $R'^*R' = \mathrm{id}_{1_b}$ which further implies $R'R'^* = 1_X \boxtimes 1_{\overline{X}}$. In terms of graphical calculus, the last equality can be expressed as the following identity using the conjugate equations satisfied by $(R, \overline{R})$



(6)

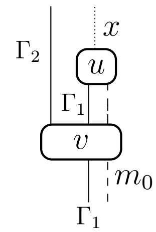
Now, define $\overline{R}' := (1_{\overline{X}} \boxtimes l^{\frac{1}{2}} \boxtimes 1_X)\overline{R}$. It is easy to verify that $\left(R', \overline{R}'\right)$ satisfy the conjugate equations for $\left(X, \overline{X}\right)$. Equation (6) ensures that $\overline{R}'$ is a unitary. ■

4.5. Remark. F_k being an equivalence is a part of an adjoint equivalence [JY21], so we may obtain an adjoint $\overline{F}_k$ of F_k, and by Proposition 4.4, we assume evaluation and coevaulation implementing the duality are both natural unitaries. Thus, for each $k \geq 0$, bi-faithfulness of F_k is immediate.

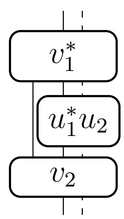
Before we describe the unitary connections for F_k's, we digress a bit to prove some results which will be useful in the construction.

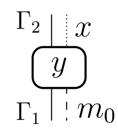
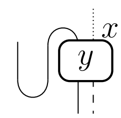
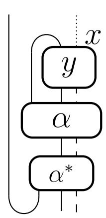
Suppose $\mathcal{N}$ is a C*-semisimple category. For $x, y \in \mathrm{Ob}(\mathcal{N})$, consider the morphism space $\mathcal{N}(x,y)$ and consider the C*-algebra $A = \mathrm{End}(x)$. Then, $\mathcal{N}(x,y)$ becomes a right-A correspondence with A-valued inner product, $\langle u, v \rangle_A = u^*v$.

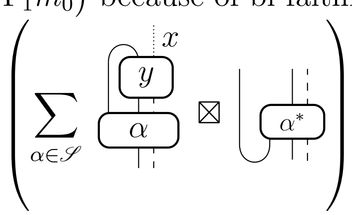
We proceed with the following lemma.

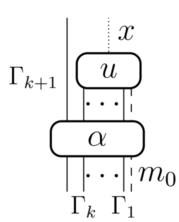
4.6. Lemma. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are finite, C^*-semisimple categories. Let $\Gamma_1 : \mathcal{M} \to \mathcal{N}$ and $\Gamma_2 : \mathcal{N} \to \mathcal{N}$ be bi-faithful, $*$-linear functors. Then the map $T : \mathcal{N}(\Gamma_1 m_0, x) \underset{\mathrm{End}(\Gamma_1 m_0)}{\boxtimes} \mathcal{N}(\Gamma_1 m_0, \Gamma_2\Gamma_1 m_0) \to \mathcal{N}(\Gamma_1 m_0, \Gamma_2 x)$ given by $u \boxtimes v \overset{T}{\longmapsto}$*  *is a unitary as a right- $\mathrm{End}(\Gamma_1 m_0)$-linear map.*

Proof. Let $A = \mathrm{End}(\Gamma_1 m_0)$. Clearly, T is middle A-linear. Now,

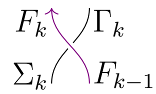
$\langle T(u_1 \boxtimes v_1), T(u_2 \boxtimes v_2)\rangle_A =$  $= \langle v_1, \langle u_1, u_2\rangle_A v_2\rangle_A = \langle u_1 \boxtimes v_1, u_2 \boxtimes v_2\rangle_A.$

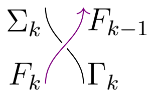
Hence, T is an isometry. If we can show that T is surjective then we get our desired result from [L95]. Now, let $y \in \mathcal{N}(\Gamma_1 m_0, \Gamma_2 x)$. Then,  $=$  $=$ $\sum_{\alpha \in \mathscr{S}}$ . Last equality follows from the fact that, we can find such a set $\mathscr{S} \subseteq \mathrm{End}\left(\overline{\Gamma}_2 \Gamma_1 m_0\right)$ because of bi-faithfulness of Γ_2, Γ_1 and m_0 contains all irreducibles of $\mathcal{M}$.

Now, T  $= y$. Hence, T is surjective. So, T is an unitary. ∎

4.7. Corollary. *The maps $T^k_x : F_k(x) \underset{A_k}{\boxtimes} A_{k+1\,A_{k+1}} \to \mathcal{M}_k(\Gamma_{k+1}\Gamma_k \cdots \Gamma_1 m_0, \Gamma_{k+1} x)_{A_{k+1}}$ given by $u \boxtimes \alpha \overset{T^k_x}{\longmapsto}$*  *are unitaries and they are natural in x, for each $x \in \mathcal{M}_k$ and $k \geq 0$.*

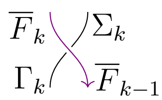
PROOF. Clearly, T_x^k are right-A_{k+1} linear. Unitarity of T_x^k follows from Lemma 4.6. Naturality of T^k follows from the definition of F_k acting on morphism spaces as in Proposition 4.3. ∎

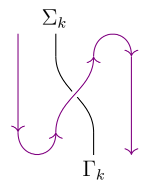
We now define the unitary connections for $\{F_k\}_{k\geq 0}$ as $W_{k+1}^F := T^k : \Sigma_{k+1}F_k \to F_{k+1}\Gamma_{k+1}$ as defined in Corollary 4.7, for each $k \geq 0$. Pictorially we denote, for each $k \geq 0$, F_k by  and $\overline{F}_k$ by  and for each $k \geq 1$, W_k^F by 

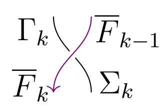
and $\left(W_k^F\right)^*$ by 

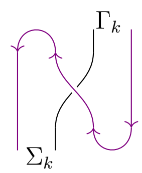
. Hence, we have a 1-cell $\left(F_\bullet, W_\bullet^F\right) \in \mathbf{UC}_1(\Gamma_\bullet, \Sigma_\bullet)$.

For each $k \geq 1$, define

$\overline{W}_k^F :=$ 

$:=$ 

$\left(\overline{W}_k^F\right)^* :=$ 

$:=$ 

Since the evaluation and coevaluation are chosen (in Remark 4.5) to be unitaries, therefore $\overline{W}_k^F$'s are also so. We claim that $F_\bullet$ is a dualizable 1-cell in $\mathbf{UC}$ with dual $\left(\overline{F}_\bullet, \overline{W}_\bullet^F\right)$. For this, we verify that solutions to conjugate equations (as in Remark 4.5) satisfy exchange relations for $k \geq 0$, which is equivalent to the equations by which W_k^F's and $\overline{W}_k^F$'s become unitaries.

4.8. REMARK. Observe that by Proposition 4.3, we have an adjoint equivalence $G_k : \mathcal{M}_k \to \mathcal{R}_{C_k}$ using the fact that $Q_k\Gamma_k \cdots \Gamma_1 m_0$ contains every simple of $\mathcal{M}_k$ as a subobject for each $k \geq 0$. Further, the square

$$\begin{tikzcd} \mathcal{M}_{k+1} \arrow[r, "G_{k+1}"] & \mathcal{R}_{C_{k+1}} \\ \mathcal{M}_k \arrow[u, "\Gamma_k"] \arrow[r, "G_k"'] & \mathcal{R}_{C_k} \arrow[u, "\bullet \underset{C_k}{\boxtimes} C_{k+1}"'] \end{tikzcd}$$

commutes up to a natural unitary, say W_{k+1}^G, which can be proven exactly the same was done for F_k's, and thereby yeilding a dualizable 1-cell $\left(G_\bullet, W_\bullet^G\right)$ in $\mathbf{UC}$ from $\Gamma_\bullet$ to $\Psi_\bullet$.

Picking a dual $\overline{G}_\bullet \in \mathbf{UC}_1(\Psi_\bullet, \Gamma_\bullet)$ of $G_\bullet$, we set $\left(R_\bullet, W_\bullet^R\right) := F_\bullet \boxtimes \overline{G}_\bullet \in \mathbf{UC}_1(\Psi_\bullet, \Sigma_\bullet)$. That is, $R_k = F_k\overline{G}_k : \mathcal{R}_{C_k} \to \mathcal{R}_{A_k}$ for $k \geq 0$ which along with the unitary connections are compatible with the Σ_k's and Ψ_k's.

4.8.1. Construction of $\left(\Lambda_\bullet, W^\Lambda_\bullet\right) \in \mathbf{UC}_1\left(\Sigma_\bullet, \Delta_\bullet\right)$ and its dual.

Observe that in Section 4.1, for each $k \geq 0$, we have unital inclusions $A_k \hookrightarrow B_k$ of C*-algebras; in particular, for $k \geq l$, this is given in Lemma 3.5. As a result, the functor $\Lambda_k \coloneqq \bullet \underset{A_k}{\boxtimes} B_k : \mathcal{R}_{A_k} \to \mathcal{R}_{B_k}$ turns out to bi-faithful for each $k \geq 0$. Next, we need to define the unitary connection for $\Lambda_\bullet$. We acheive this using the following easy fact.

4.9. Fact. *Suppose A, B, C, D are finite dimensional C*-algebras such that we have a square of unital inclusions* $\begin{array}{ccc} C & \hookrightarrow & D \\ \uparrow & & \uparrow \\ A & \hookrightarrow & B \end{array}$. *This induces a square of categories and functors* $\begin{array}{ccc} \mathcal{R}_C & \xrightarrow{\bullet \underset{C}{\boxtimes} D} & \mathcal{R}_D \\ {\scriptstyle \bullet \underset{A}{\boxtimes} C}\uparrow & & \uparrow{\scriptstyle \bullet \underset{B}{\boxtimes} D} \\ \mathcal{R}_A & \xrightarrow[\bullet \underset{A}{\boxtimes} B]{} & \mathcal{R}_B \end{array}$. *Corresponding to this last square, there exists a unitary natural transformation between the functors $\bullet \underset{A}{\boxtimes} B \underset{B}{\boxtimes} D$ and $\bullet \underset{A}{\boxtimes} C \underset{C}{\boxtimes} D$.*

For $0 \leq k \leq l-1$, the unitaries W^Λ_{k+1} may be obtained by applying Fact 4.9 to the squares $\begin{array}{ccc} \mathcal{R}_{A_{k+1}} & \xrightarrow{\Lambda_{k+1}} & \mathcal{R}_{B_{k+1}} \\ {\scriptstyle \Sigma_{k+1}}\uparrow & & \uparrow{\scriptstyle \Delta_{k+1}} \\ \mathcal{R}_{A_k} & \xrightarrow[\Lambda_k]{} & \mathcal{R}_{B_k} \end{array}$

We now explicitly describe the unitaries $W^\Lambda_k : \Delta_k \Lambda_{k-1} \to \Lambda_k \Sigma_k$ for each $k \geq l+1$. For $V \in \operatorname{Ob}\left(\mathcal{R}_{A_{k-1}}\right)$, define $\left(W^\Lambda_k\right)_V : V \underset{A_{k-1}}{\boxtimes} H_{k-1} \underset{H_{k-1}}{\boxtimes} H_{k H_k} \to V \underset{A_{k-1}}{\boxtimes} A_k \underset{A_k}{\boxtimes} H_{k H_k}$ as follows :

$$V \underset{A_{k-1}}{\boxtimes} H_{k-1} \underset{H_{k-1}}{\boxtimes} H_k \ni q \underset{A_{k-1}}{\boxtimes} \xi_1 \underset{H_{k-1}}{\boxtimes} \xi_2 \overset{\left(W^\Lambda_k\right)_V}{\longmapsto} q \underset{A_{k-1}}{\boxtimes} 1_{A_k} \underset{A_k}{\boxtimes} \xi_1 \cdot \xi_2 \quad \text{for each} \quad q \in V \ .$$

It is easy to see that each $\left(W^\Lambda_k\right)_V$ is a unitary and natural in V, and $\left(W^\Lambda_k\right)^*_V$ is given as follows:

$$V \underset{A_{k-1}}{\boxtimes} A_k \underset{A_k}{\boxtimes} H_k \ni q \underset{A_{k-1}}{\boxtimes} \alpha \underset{A_k}{\boxtimes} \xi \overset{\left(W^\Lambda_k\right)^*_V}{\longmapsto} q \underset{A_{k-1}}{\boxtimes} 1_{H_{k-1}} \underset{A_k}{\boxtimes} \xi_1 \cdot \xi_2 \quad \text{for each} \quad q \in V \ .$$

Thus, we get a 1-cell $(\Lambda_\bullet, W_\bullet) : \Sigma_\bullet \to \Delta_\bullet$ in $\mathbf{UC}$.

We now define $\left(\overline{\Lambda}_\bullet, \overline{W}^\Lambda_\bullet\right) \in \mathbf{UC}_1\left(\Delta_\bullet, \Sigma_\bullet\right)$ so that it becomes dual to $(\Lambda_\bullet, W_\bullet)$ in $\mathbf{UC}$. For $0 \leq k \leq l-1$, define $\overline{\Lambda}_k \coloneqq R_k \circ \left(\bullet \underset{B_k}{\boxtimes} C_k\right) : \mathcal{R}_{B_k} \to \mathcal{R}_{A_k}$ where $R_k : \mathcal{R}_{C_k} \to \mathcal{R}_{A_k}$ is the equivalence given in Remark 4.8.

For $k \geq l$, define $\overline{\Lambda}_k \coloneqq \bullet \underset{H_k}{\boxtimes} H_k : \mathcal{R}_{H_k} \to \mathcal{R}_{A_k}$. Here the right action of A_k on H_k is given by the inclusion $A_k \hookrightarrow H_k$ (as in Lemma 3.5) and the multiplication in C*-algebra H_k;

however, the right A_k-valued inner product is the one defined in Equation (4) (and not the one coming from conditional expectation).

4.10. Remark. Although the functors $\overline{\Lambda}_k$ may not be adjoint to Λ_k for $0 \leq k \leq l-1$, we will need these functors to define an adjoint of $\left(\Lambda_\bullet, W^\Lambda_\bullet\right)$ in **UC** .

Our next job is to define the unitary connections $\left\{\overline{W}^\Lambda_k\right\}_{k\geq 1}$ for $\overline{\Lambda}_\bullet$. This will be divide into three different ranges for k, namely $\{1,\ldots,l-1\}$, $\{l\}$ and $\{l+1, l+2, \ldots\}$; the choice of the natural unitaries in the first two ranges could be arbitrary

Case $0 \leq k \leq l-2$: For the unitary connection $\overline{W}^\Lambda_{k+1}$, we look at the following horizontally stacked squares of functors.

$$\begin{tikzcd}
\mathcal{R}_{B_{k+1}} \arrow[r, "\bullet \underset{B_{k+1}}{\boxtimes} C_{k+1}"] & \mathcal{R}_{C_{k+1}} \arrow[r, "R_{k+1}"] & \mathcal{R}_{A_{k+1}} \\
\mathcal{R}_{B_k} \arrow[u, "\Delta_{k+1}"] \arrow[r, "\bullet \underset{B_k}{\boxtimes} C_k"'] & \mathcal{R}_{C_k} \arrow[u, "\bullet \underset{C_k}{\boxtimes} C_{k+1}"'] \arrow[r, "R_k"'] & \mathcal{R}_{A_k} \arrow[u, "\Sigma_{k+1}"']
\end{tikzcd} .$$

Both the squares are commutative up to natural unitaries; the left one follows from Fact 4.9 and the right comes from Remark 4.8. $\overline{W}^\Lambda_{k+1}$ is defined as the appropriate composition of above two natural unitaries.

Case $k = l$: To define the natural unitary $\overline{W}^\Lambda_l : \Sigma_l \; \overline{\Lambda}_{l-1} \to \overline{\Lambda}_l \; \Delta_l$, it is enough to check whether the square

$$\begin{tikzcd}
\mathcal{R}_{H_l} \arrow[rr, "\bullet \underset{H_l}{\boxtimes} H_l", "\overline{\Lambda}_l"'] & & \mathcal{R}_{A_l} \\
\mathcal{R}_{S_l \cap C_{l-1}} \arrow[u, "\bullet \underset{B_{l-1}}{\boxtimes} B_l", "\Delta_l"'] \arrow[rr, "\overline{\Lambda}_{l-1}", "R_{l-1} \circ \left(\bullet \underset{B_{l-1}}{\boxtimes} C_{l-1}\right)"'] & & \mathcal{R}_{A_{l-1}} \arrow[u, "\Sigma_l", "\bullet \underset{A_{l-1}}{\boxtimes} A_l"']
\end{tikzcd}$$

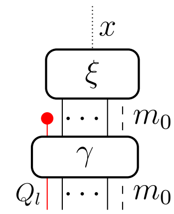
commutes up to a natural isomorphism; let us call this square $\mathbb{S}$. Consider the horizontal pair of squares

$$\begin{tikzcd}
\mathcal{R}_{S_l} \arrow[r, "\bullet \underset{S_l}{\boxtimes} C_l"] & \mathcal{R}_{C_l} \arrow[r, "R_l"] & \mathcal{R}_{A_l} \\
\mathcal{R}_{S_l \cap C_{l-1}} \arrow[u, "\bullet \underset{S_l \cap C_{l-1}}{\boxtimes} S_l"] \arrow[r, "\bullet \underset{S_l \cap C_{l-1}}{\boxtimes} C_{l-1}"'] & \mathcal{R}_{C_{l-1}} \arrow[u, "\Psi_l"'] \arrow[r, "R_{l-1}"'] & \mathcal{R}_{A_{l-1}} \arrow[u, "\Sigma_l"']
\end{tikzcd}$$

referred as $\mathbb{S}_1$; the first square of $\mathbb{S}_1$ commutes by Fact 4.9 and the second follows from Remark 4.8. Note that the bottom and the right sides of $\mathbb{S}$ matches with that of $\mathbb{S}_1$.

We next claim that the top side of $\mathbb{S}_1$ is naturally isomorphic to $\bullet \underset{S_l}{\boxtimes} S_l : \mathcal{R}_{S_l} \to \mathcal{R}_{A_l}$.

To see this, consider the square $\begin{array}{ccc} \mathcal{R}_{C_l} & \xleftarrow{G_l} & \mathcal{M}_l \\ {\scriptstyle \bullet \underset{C_l}{\boxtimes} C_l}\big\downarrow & & \big\downarrow{\scriptstyle F_l} \\ \mathcal{R}_{S_l} & \xleftarrow[\bullet \underset{A_l}{\boxtimes} S_l]{} & \mathcal{R}_{A_l} \end{array}$ referred as $\mathbb{S}_2$. For $x \in \mathrm{Ob}\,(\mathcal{M}_l)$, the map

$$F_l(x) \underset{A_l}{\boxtimes} S_l \ni \xi \underset{A_l}{\boxtimes} \gamma \longmapsto$$  $$\underset{C_l}{\boxtimes} 1_{C_l} \in G_l(x) \underset{C_l}{\boxtimes} C_l$$

is S_l-linear and natural in x. To show that the map is surjective, pick a basic tensor $\zeta \underset{C_l}{\boxtimes} 1_{C_l} \in G_l(x) \underset{C_l}{\boxtimes} C_l$; note that it can be expressed as the image of $\displaystyle\sum_{\sigma \in \mathscr{S}_l} \zeta \circ \sigma \underset{A_l}{\boxtimes} \phi_1^{(l)}\left(\sigma^\dagger\right)$ where $\mathscr{S}_l$ is as in Remark 3.7 and $\phi_1^{(l)} : H_l \to S_l$ is the isomorphism mentioned in Proposition 3.4. This concludes natural commutativity of $\mathbb{S}_2$. Now, the adjoint of the functors $\bullet \underset{C_l}{\boxtimes} C_l : \mathcal{R}_{C_l} \to \mathcal{R}_{S_l}$ and $\bullet \underset{A_l}{\boxtimes} S_l : \mathcal{R}_{A_l} \to \mathcal{R}_{S_l}$ (appearing in the square $\mathbb{S}_1$) are given by $\bullet \underset{S_l}{\boxtimes} C_l : \mathcal{R}_{S_l} \to \mathcal{R}_{C_l}$ and $\bullet \underset{S_l}{\boxtimes} A_l : \mathcal{R}_{S_l} \to \mathcal{R}_{A_l}$ respectively; this can be achieved by solving the conjugate equations using the set $\mathscr{S}_l$ again and the conditional expectations. Thus, dualizing the square $\mathbb{S}_2$, we get $\overline{F}_l \circ \left(\bullet \underset{S_l}{\boxtimes} {S_l}_{A_l}\right) \cong \overline{G}_l \circ \left(\bullet \underset{S_l}{\boxtimes} {C_l}_{C_l}\right)$. Now, using the fact that F_l is an adjoint equivalence and using $R_l = F_l \overline{G}_l$, we get $R_l \circ \left(\bullet \underset{S_l}{\boxtimes} {C_l}_{C_l}\right) \cong \left(\bullet \underset{S_l}{\boxtimes} {S_l}_{A_l}\right)$. Using this natural isomorphism and natural commutativity of the square $\mathbb{S}_1$, we obtain natural commutativity of
$$\begin{array}{ccc} \mathcal{R}_{S_l} & \xrightarrow{\bullet \underset{S_l}{\boxtimes} S_l} & \mathcal{R}_{A_l} \\ {\scriptstyle \bullet \underset{S_l \cap C_{l-1}}{\boxtimes} S_l}\big\uparrow & & \big\uparrow{\scriptstyle \Sigma_l} \\ \mathcal{R}_{S_l \cap C_{l-1}} & \xrightarrow[R_{l-1} \circ \left(\bullet \underset{S_l \cap C_{l-1}}{\boxtimes} C_{l-1}\right)]{} & \mathcal{R}_{A_{l-1}} \end{array}.$$

Finally, using the isomorphism $\phi_2^{(l)} : S_l \to H_l$ (as in Proposition 3.4), we get our desired natural commutativity of $\mathbb{S}$. Set $\overline{W}_l^\Lambda$ to be a natural unitary implementing commutativity of $\mathbb{S}$.

Case $k \geq l$: To define $\overline{W}_{k+1}^\Lambda$, we will need the solutions to conjugate equations for Λ_k and $\overline{\Lambda}_k$ for each $k \geq l$. We will use the following pictorial notations:

$$\Lambda_k := \big\uparrow \quad \text{and} \quad \overline{\Lambda}_k := \big\downarrow \quad \text{for each} \quad k \geq 0$$

4.11. Definition.

(i) $: Id_{\mathcal{R}_{H_k}} \to \Lambda_k \overline{\Lambda}_k$ *is the natural transformation defined as:*

$\Big|_V : V \to V \underset{H_k}{\boxtimes} H_k \underset{A_k}{\boxtimes} H_k$ *is given by* $q \longmapsto \sum_{\sigma \in \mathscr{S}_k} q \underset{H_k}{\boxtimes} \sigma \underset{A_k}{\boxtimes} \sigma^\dagger$ *where* $V \in \mathcal{R}_{H_k}$ *and* $\mathscr{S}_k$ *is as in Remark 3.7.*

(ii) $: \Lambda_k \overline{\Lambda}_k \to Id_{\mathcal{R}_{H_k}}$ *is the natural transformation defined as :*

$\Big|_V : V \underset{H_k}{\boxtimes} H_k \underset{A_k}{\boxtimes} H_k \to V$ *is given by* $q \underset{H_k}{\boxtimes} \xi_1 \underset{A_k}{\boxtimes} \xi_2 \longmapsto q.(\xi_1 \cdot \xi_2)$ *where* $V \in \mathcal{R}_{H_k}$.

(iii) $: Id_{\mathcal{R}_{A_k}} \to \overline{\Lambda}_k \Lambda_k$ *is the natural transformation defined as :*

$\Big|_V : V \to V \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_k$ *is given by* $q \longmapsto q \underset{A_k}{\boxtimes} 1_{H_k} \underset{H_k}{\boxtimes} 1_{H_k}$ *where* $V \in \mathcal{R}_{A_k}$.

(iv) $: \overline{\Lambda}_k \Lambda_k \to Id_{\mathcal{R}_{A_k}}$ *is the natural transformation defined as :*

$\Big|_V : V \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_k \to V$ *is given by* $q \underset{A_k}{\boxtimes} \xi_1 \underset{H_k}{\boxtimes} \xi_2 \longmapsto q.\langle \xi_2, \xi_1^\dagger \rangle_{A_k}$ *where* $V \in \mathcal{R}_{A_k}$.

4.12. Lemma.

(i) , , , *satisfy conjugate equations for* $\Lambda_k, \overline{\Lambda}_k$ *for each* $k \geq l$ *.*

(ii) Also, $\left(\quad \right)^* =$ *and* $\left(\quad \right)^* =$.

(iii) $= \mathrm{id}_{\mathrm{Id}_{\mathcal{R}_{H_k}}}$.

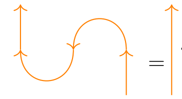
Proof. (i) We have, for every $V \in \mathcal{R}_{A_k}$, $q \in V$ and $\xi \in H_k$,

$$\Big|_V \left(q \boxtimes \xi \right) = \Big|_V \left(q \boxtimes \xi \boxtimes 1_{H_k} \boxtimes 1_{H_k} \right) = q \boxtimes \xi.$$

Therefore, we get $=$. We have, for every $V \in \mathcal{R}_{A_k}$, $q \in V$ and $\xi \in H_k$,

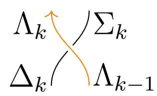
$$\Big|_V \left(q \underset{A_k}{\boxtimes} \xi \right) = \Big|_V \left(\sum_{\sigma \in \mathscr{S}_k} q \underset{A_k}{\boxtimes} \xi \underset{H_k}{\boxtimes} \sigma \underset{A_k}{\boxtimes} \sigma^\dagger \right)$$

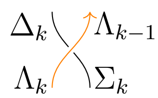
$$= \sum_{\sigma \in \mathscr{S}_k} q.\langle \sigma, \xi^\dagger \rangle_{A_k} \underset{A_k}{\boxtimes} \sigma^\dagger = \sum_{\sigma \in \mathscr{S}_k} q \underset{A_k}{\boxtimes} \langle \sigma, \xi^\dagger \rangle_{A_k} \sigma^\dagger = q \underset{A_k}{\boxtimes} \xi.$$

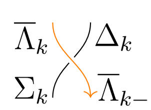
The last equality follows from Equation (4). Therefore, we get . The other equations can be proved similarly.

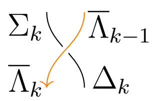
(ii) The proof is similar to that of (i).

(iii) It follows easily from Definition 4.11(i) and Definition 4.11(ii). ■

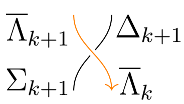
Pictorially, we denote W_k^Λ by 

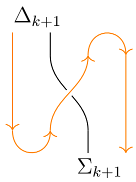
, $\left(W_k^\Lambda\right)^*$ by 

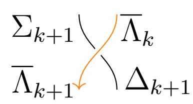
, $\overline{W}_k^\Lambda$ by 

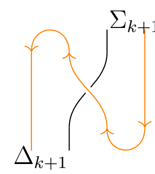
and $\left(\overline{W}_k^\Lambda\right)^*$ by 

for each $k \geq 1$. We have already defined all W_k^Λ's and $\overline{W}_k^\Lambda$ for $1 \leq k \leq l$ in the above two cases. Now, for $k \geq l$, we define

$\overline{W}_{k+1}^\Lambda =$ 

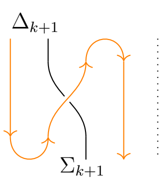
$:=$ 

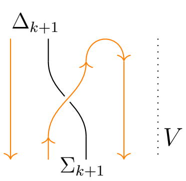
and $\left(\overline{W}_{k+1}^\Lambda\right)^* =$ 

$:=$ 

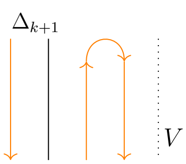
which turn out to be natural unitaries by the following remark.

4.13. Remark. For each $k \geq l$ and $V \in \mathcal{R}_{H_k}$, $q \in V$, $\xi \in H_k$, $\alpha \in A_{k+1}$, $\eta, \zeta \in H_{k+1}$ the element $\left(\overline{W}_{k+1}^\Lambda\right)_V \left(q \underset{H_k}{\boxtimes} \xi \underset{A_k}{\boxtimes} \alpha\right)$ can be expressed as



$\left(q \underset{H_k}{\boxtimes} \xi \underset{A_k}{\boxtimes} \alpha\right) =$ 

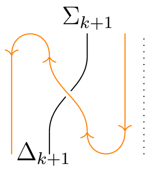
$\left(q \underset{H_k}{\boxtimes} \xi \underset{A_k}{\boxtimes} \alpha \underset{A_{k+1}}{\boxtimes} 1_{H_{k+1}} \underset{H_{k+1}}{\boxtimes} 1_{H_{k+1}}\right)$

$=$ 

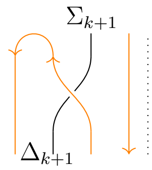
$\left(q \underset{H_k}{\boxtimes} \xi \underset{A_k}{\boxtimes} 1_{H_k} \underset{A_{k+1}}{\boxtimes} \alpha \underset{H_{k+1}}{\boxtimes} 1_{H_{k+1}}\right)$

$= q.\xi \underset{H_k}{\boxtimes} \alpha \underset{H_{k+1}}{\boxtimes} 1_{H_{k+1}}$

and $\left(\overline{W}^{\Lambda}_{k+1}\right)^*_V \left(q \underset{H_k}{\boxtimes} \eta \underset{H_{k+1}}{\boxtimes} \zeta\right)$ can be expressed as

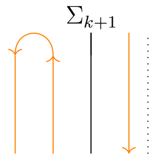


$$_V \left(q \underset{H_k}{\boxtimes} \eta \underset{H_{k+1}}{\boxtimes} \zeta\right) =$$



$$_V \left(\sum_{\sigma \in \mathscr{S}_k} q \underset{H_k}{\boxtimes} \sigma \underset{A_k}{\boxtimes} \sigma^\dagger \underset{H_k}{\boxtimes} \eta \underset{H_{k+1}}{\boxtimes} \zeta\right)$$

$$=$$



$$_V \left(\sum_{\sigma \in \mathscr{S}_k} q \underset{H_k}{\boxtimes} \sigma \underset{A_k}{\boxtimes} 1_{A_{k+1}} \underset{A_{k+1}}{\boxtimes} \sigma^\dagger \cdot \eta \underset{H_{k+1}}{\boxtimes} \zeta\right)$$

$$= \sum_{\sigma \in \mathscr{S}_k} q \underset{H_k}{\boxtimes} \sigma \underset{A_k}{\boxtimes} \langle \zeta, \eta^\dagger \cdot \sigma \rangle_{A_{k+1}} .$$

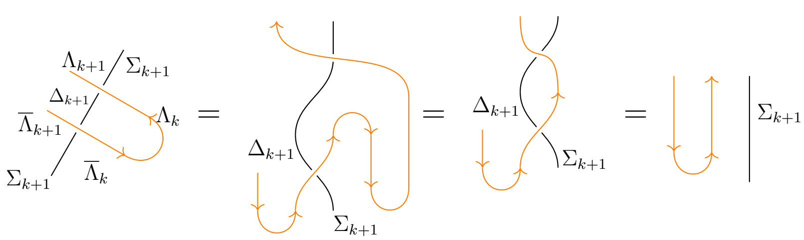
It is a straightforward verification that each $\left(\overline{W}^{\Lambda}_{k+1}\right)_V$ is a unitary and natural in V.

Thus, we have defined a 1-cell $\left(\overline{\Lambda}_\bullet, \overline{W}^{\Lambda}_\bullet\right)$ in **UC** from $\Delta_\bullet$ to $\Sigma_\bullet$. We need to prove that $\left(\overline{\Lambda}_\bullet, \overline{W}^{\Lambda}_\bullet\right)$ is dual to $\left(\Lambda_\bullet, W^{\Lambda}_\bullet\right)$. In order to define the solution to conjugate equation (which is in fact a pair of 2-cells in **UC**), we have the liberty to ignore finitely many terms and define them eventually (by Remark 2.14).

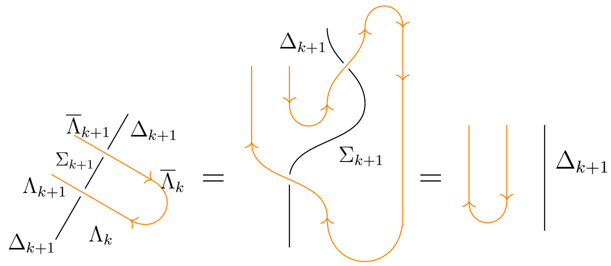
By Lemma 4.12, we see that there are solutions to conjugate equations for Λ_k and $\overline{\Lambda}_k$ for each $k \geq l$. So, we are only left with showing exchange relations of solutions eventually.

We now verify that  and  satisfy exchange relations for $k \geq l$.

4.14. Remark. The solutions to conjugate equations for Λ_k and Λ_{k+1} (as in Definition 4.11) satisfy exchange relation eventually for all k with respect to $W^{\Lambda}_\bullet$ and $\overline{W}^{\Lambda}_\bullet$. This directly follows from the fact W^{Λ}_k's and $\overline{W}^{\Lambda}_k$'s are unitaries. Nevertheless, we still furnish a proof below. Note that



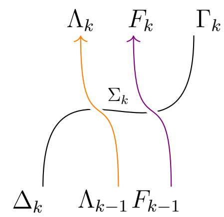
and



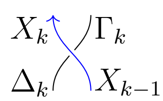
.

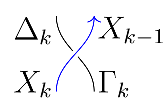
Hence, $\left(\Lambda_\bullet, W^\Lambda_\bullet\right) : \Sigma_\bullet \to \Delta_\bullet$ is a dualizable 1-cell in $\mathbf{UC}$ with dual $\left(\overline{\Lambda}_\bullet, \overline{W}^\Lambda_\bullet\right)$ as described above.

We are now in a position to describe our desired dualizable 1-cell $(X_\bullet, W_\bullet)$ which will split $(Q_\bullet, m_\bullet, i_\bullet)$ as Q-system.

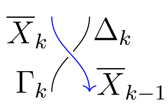
Define $(X_\bullet, W_\bullet) := \Lambda_\bullet \boxtimes F_\bullet = \left(\{\Lambda_k F_k\}_{k\geq 0}, \left\{ \right.\right.$ 

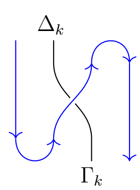
$\left.\left.\right\}_{k\geq 1}\right) \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$.

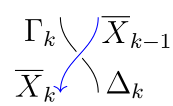
Pictorially, we denote X_k by , $\overline{X}_k$ by , W_k by 

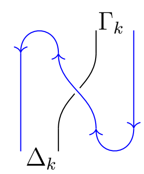
and W_k^* by 

.

Define $\overline{W}_k :=$ 

$:=$ 

and $\left(\overline{W}_k\right)^* :=$ 

$:=$ 

Thus, we arrive at our desired 1-cell $(X_\bullet, W_\bullet) \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$. We list some of the properties of $(X_\bullet, W_\bullet)$.

4.15. Lemma.

(i) $(X_\bullet, W_\bullet)$ *is a dualizable* 1*-cell in* $\mathbf{UC}$ *.*

(ii) $(X_\bullet, W_\bullet)$ *has a unitarily separable dual in* $\mathbf{UC}$ *.*

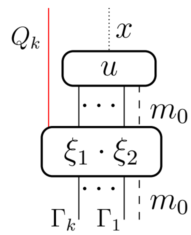
Proof. (i) $(X_\bullet, W_\bullet)$ being a composition of two dualizable 1-cells $\left(\Lambda_\bullet, W^\Lambda_\bullet\right)$ and $\left(F_\bullet, W^F_\bullet\right)$ concludes the result .
(ii) This is immediate from Remark 2.14 and (iii) of Lemma 4.12 . ∎

4.16. Isomorphism of Q-systems.

In this subsection, we build an isomorhpism between $\overline{X}_\bullet \boxtimes X_\bullet$ and $Q_\bullet$. We construct unitaries $\gamma^{(k)} : \overline{X}_k X_k \to Q_k$ for each $k \geq l$ which intertwines the mutliplication and unit maps. In the next subsection, we verify the exchange relation of $\gamma^{(k)}$ for each $k \geq l$, thus implementing isomorphism of the aforementioned Q-systems in **UC**.

For $k \geq l$ and for each $x \in \mathrm{Ob}(\mathcal{M}_k)$, define a map $\beta_x^{(k)} : \overline{\Lambda}_k \Lambda_k F_k(x) \to F_k Q_k(x)$ as follows :

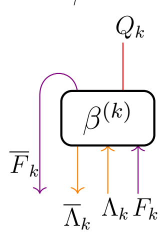
$$F_k(x) \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_{k A_k} \ni u \boxtimes \xi_1 \boxtimes \xi_2 \xmapsto{\beta_x^{(k)}}$$



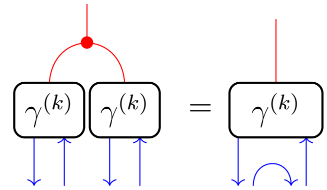
$$\in F_k Q_k(x)$$

It is easy to see that, each $\beta_x^{(k)}$ is an isometry. Since, ${}_{A_k}H_k \underset{H_k}{\boxtimes} H_{k A_k}$ is unitarily isomorhpic to ${}_{A_k}H_{k A_k}$ and by application of Lemma 4.6, we see that $\overline{\Lambda}_k \Lambda_k F_k(x)$ and $F_k Q_k(x)$ has same dimension (as a vector space). Hence, surjectiveness will follow. Thus, each $\beta_x^{(k)}$ is a unitary. Also, it easily follows that each $\beta_x^{(k)}$ is a natural in x. Thus, we get a unitary natural transformation $\beta^{(k)} : \overline{\Lambda}_k \Lambda_k F_k \to F_k Q_k$.

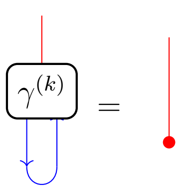
Define $\gamma^{(k)} :=$



$: \overline{X}_k X_k \to Q_k$. We show that $\gamma^{(k)}$ is an isomorphism of Q-systems $\overline{X}_k X_k$ and Q_k for $k \geq l$. Each $\gamma^{(k)}$ is a unitary because each $\beta^{(k)}$ is so and each F_k is an adjoint equivalence (see Remark 4.5). We need to show that $\gamma^{(k)}$ intertwines the multiplication and unit maps. We need to show that

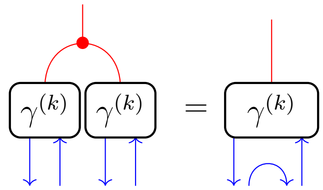


and

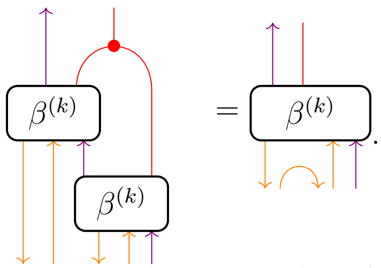


for $k \geq l$. This is what we prove next.

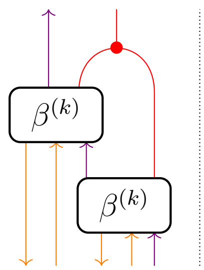
4.17. Proposition. *For $k \geq l$, $\gamma^{(k)} : \overline{X}_k X_k \to Q_k$ is an isomorphism of Q-systems.*

Proof. It easily follows that, 

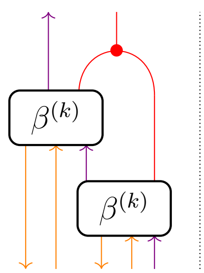
if and only if

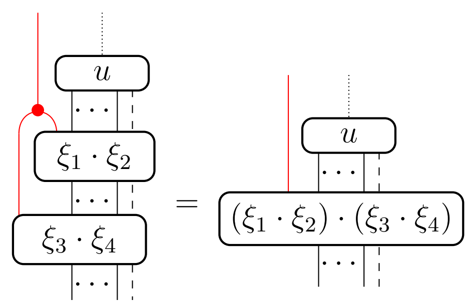


.

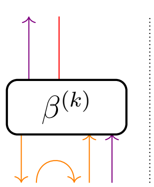
Now the map, 

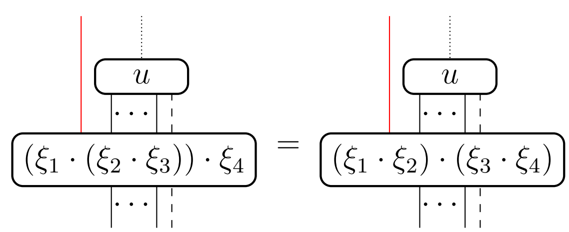
$: F_k(x) \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_k \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_k \to F_kQ_k(x)$ given as follows:



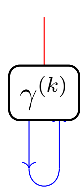
$\left(u \boxtimes \xi_1 \boxtimes \xi_2 \boxtimes \xi_3 \boxtimes \xi_4\right) =$ 

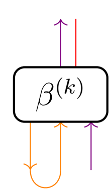
for every $u \in F_k(x)$ and $\xi_1, \xi_2, \xi_3, \xi_4 \in H_k$. It is straightforward to show that

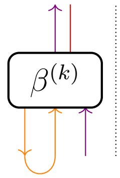


$\left(u \boxtimes \xi_1 \boxtimes \xi_2 \boxtimes \xi_3 \boxtimes \xi_4\right) =$ 

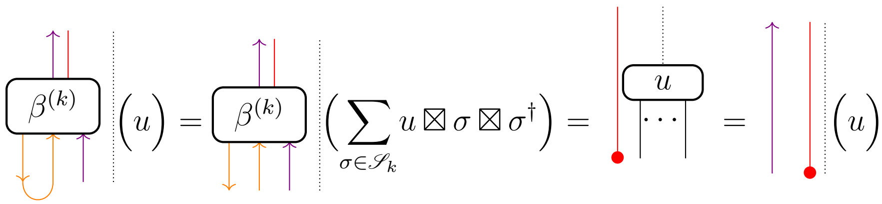
for every $u \in F_k(x)$ and $\xi_1, \xi_2, \xi_3, \xi_4 \in H_k$. The last equality follows because of associativity of H_k as shown in Proposition 3.4. Thus, $\gamma^{(k)}$ intertwines the multiplication maps for each $k \geq l$.

Also, it is easy to see that 

$=$  if and only if 

$=$  . Now, the map 

$: F_k(x) \to F_k Q_k(x)$ is given as follows :



for every $u \in F_k(x)$

where $\mathscr{S}_k \subset H_k$ is as given in Remark 3.7 . Thus, $\gamma^{(k)}$ intertwines the unit maps for each $k \geq l$. This concludes the proposition. ■

4.18. Exchange relation of $\gamma^{(k)}$'s.

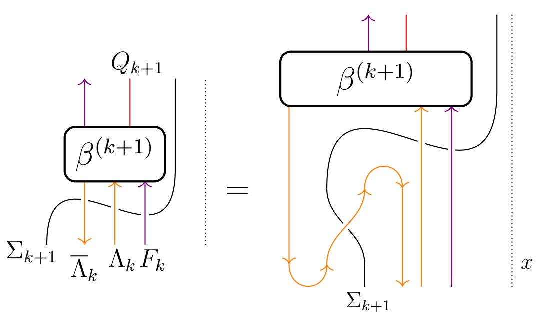
To achieve isomorphism in **UC**, we still have to show that $\gamma^{(k)}$'s satisfy exchange relation for $k \geq l$. This will establish 'splitting' of $(Q_\bullet, m_\bullet, i_\bullet) \in \mathbf{UC}_1(\Gamma_\bullet, \Gamma_\bullet)$ by $(X_\bullet, W_\bullet) \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$.

4.19. Remark. In order to show that $\gamma^{(k)}$'s will satisfy exchange relation for $k \geq l$, it is enough to show that $\beta^{(k)}$'s also does so because solutions to conjugate equations for F_k's and $\overline{F}_k$'s satisfy exchange relations for each $k \geq l$. So instead of showing exchange relation of $\gamma^{(k)}$'s we will show that $\beta^{(k)}$'s satisfy exchange relation for $k \geq l$.

We now proceed to show that $\beta^{(k)}$'s satisfy exchange relation for $k \geq l$.

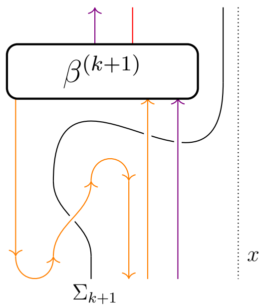
4.20. Proposition. *For $k \geq l$, $\beta^{(k)}$'s satisfy exchange relation.*

Proof. For $x \in \mathrm{Ob}(\mathcal{M}_k)$ the map,

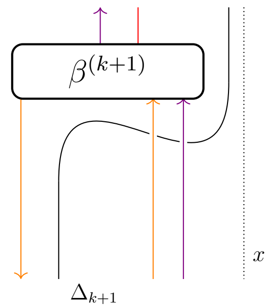


$: F_k(x) \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_k \underset{A_k}{\boxtimes} A_{k+1} \to F_{k+1}Q_{k+1}\Gamma_{k+1}(x)$

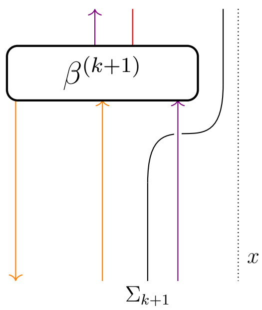
is given as follows:



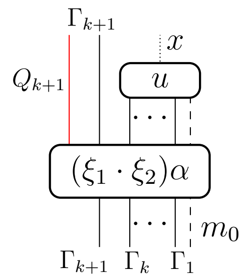
$\left(u \underset{A_k}{\boxtimes} \xi_1 \underset{H_k}{\boxtimes} \xi_2 \underset{A_k}{\boxtimes} \alpha\right) =$



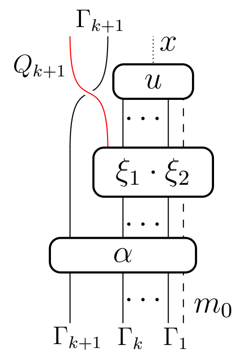
$\left(u \underset{A_k}{\boxtimes} \xi_1 \cdot \xi_2 \underset{H_k}{\boxtimes} \alpha \underset{H_{k+1}}{\boxtimes} 1_{H_{k+1}}\right)$



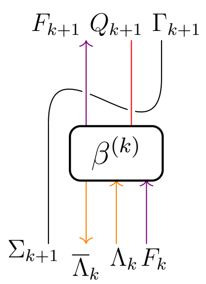
$\left(u \underset{A_k}{\boxtimes} 1_{A_{k+1}} \underset{H_k}{\boxtimes} (\xi_1 \cdot \xi_2)\alpha \underset{H_{k+1}}{\boxtimes} 1_{H_{k+1}}\right) =$



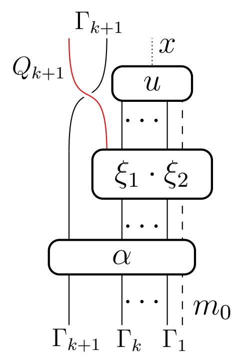
$=$



for every $u \in F_k(x), \xi_1, \xi_2 \in H_k, \alpha \in A_{k+1}$. Also, it will easily follow from the definition of $\beta^{(k)}$'s that for every $u \in F_k(x)$, ξ_1, $\xi_2 \in H_k$, $\alpha \in A_{k+1}$ we have,

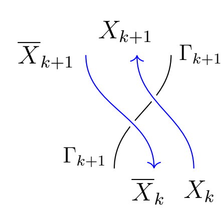


$\left(u \underset{A_k}{\boxtimes} \xi_1 \underset{H_k}{\boxtimes} \xi_2 \underset{A_k}{\boxtimes} \alpha\right) =$



Thus, $\beta^{(k)}$'s satisfy exchange relation for $k \geq l$. ∎

From Remark 2.14, Proposition 4.17, Remark 4.19 and Proposition 4.20 we get the following theorem.

4.21. THEOREM. *$(Q_\bullet, W^Q{}_\bullet)$ is isomorphic to* $\left(\{\overline{X}_k X_k\}_{k\geq 0}, \left\{ \right. \right.$  $\left. \left. \right\}_{k\geq 1} \right)$

as Q-systems in **UC** .

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