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THE OVER-TOPOS AT A MODEL

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Abstract. With a model of a geometric theory in an arbitrary topos, we associate a site obtained by endowing a category of generalized elements of the model with a Grothendieck topology, which we call the antecedent topology. Then we show that the associated sheaf topos, which we call the over-topos at the given model, admits a canonical totally connected morphism to the given base topos and satisfies a universal property generalizing that of the colocalization of a topos at a point. We first treat the case of the base topos of sets, where global elements are sufficient to describe our site of definition; in this context, we also introduce a geometric theory classified by the over-topos, whose models can be identified with the model homomorphisms towards the (internalizations of the) model. Then we formulate and prove the general statement over an arbitrary topos, which involves the stack of generalized elements of the model.

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Introduction

The goal of this paper is the construction of a site of definition for the topos classifying model homomorphisms towards (the internalizations of) a fixed model of a geometric theory in a given Grothendieck topos. More precisely, the desired universal property can be formulated as follows. Let $\mathbb{T}$ a geometric theory over a signature $\mathcal{L}$, with $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ its geometric syntactic site, $\mathcal{E}$ a Grothendieck topos and M in $\mathbb{T}[\mathcal{E}]$ a $\mathbb{T}$-model in $\mathcal{E}$. We want to construct the $\mathbb{T}$-*over-topos* associated with M, that is, a geometric morphism $u_M : \mathcal{E}[M] \to \mathcal{E}$ satisfying the universal property that for any $\mathcal{E}$-topos $g : \mathcal{G} \to \mathcal{E}$ one has an equivalence of categories

$$\mathbf{Geom}_{\mathcal{E}}(g, u_M) \simeq \mathbb{T}[\mathcal{G}]/g^*(M)$$

In words, we want $\mathcal{E}[M]$ to classify the theory of $\mathcal{L}$-structures homomorphisms from a $\mathbb{T}$-model to (an interpretation of) M. In some sense, this is a way of forcing M to become

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terminal amongst $\mathbb{T}$-models. In this paper, we explicitly construct a site of definition for the over-topos from model-theoretic data, and describe both the logical and fibrational aspects of this construction.

In section 1 we recall the well-known notion of totally connected topos and the construction of the over-topos as a finite bilimit of Grothendieck toposes. The rest of the paper will be devoted to obtaining a site-theoretic description of this construction.

In section 2, we focus on the case of a set-based model, where the construction of our site of definition for the over-topos simplifies thanks to specific properties of the terminal object of the topos **Set** of sets. In particular, we introduce the *antecedent topology* on a category of elements associated to our model, for obtaining a site of definition of the over-topos.

In section 3, in preparation for the generalization of our construction to an arbitrary topos, we introduce a number of stacks that are used in the sequel. Of particular relevance for our purposes is the notion of *lifted topology* on a category of the form $(1_{\mathcal{F}} \downarrow f^*)$, where $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism, as the smallest topology which makes both projection functors to $\mathcal{E}$ and $\mathcal{F}$ comorphisms to the canonical sites on $\mathcal{E}$ and $\mathcal{F}$; we provide a fully explicit description of this topology by providing a basis for it.

In section 4, we generalize the construction of the over-topos to a model in an arbitrary Grothendieck topos. For this, we construct a canonical stack associated with this model and apply Giraud's general construction of the classifying topos of a stack to prove the desired universal property. In particular, we provide an explicit generalization of the antecedents topology and recover it as a restriction of the lifted topology. We should mention that a construction of this topology also appears in section 4.1 of [1], where site-theoretic descriptions of general comma toposes are provided; however, their construction of this topology only works in the case of small sites with finite limits and under the hypothesis that the relevant geometric morphisms are induced by morphisms of sites, whilst ours is completely general; also, they define this topology by specifying a family of generators (lacking stability under composition), while we actually provide a pretopology, which is moreover obtained from more general considerations on stacks.

Notation

The notations employed in the paper will be standard; in particular,

- We shall denote by **Set** the category of sets (within a fixed model of set theory).
- Given a geometric theory $\mathbb{T}$, we shall denote by $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ its geometric syntactic site and by $\mathbf{Set}[\mathbb{T}]$ its classifying topos, which, as it is well-known, can be represented as $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ (for background on classifying toposes, the reader may refer to [4]).

- For any geometric theory $\mathbb{T}$, we denote by $\mathbb{T}[\mathcal{E}]$ the category of $\mathbb{T}$-models in a geometric category $\mathcal{E}$ and model homomorphisms between them. For any $\mathcal{E}$, we have an equivalence $\mathbb{T}[\mathcal{E}] \simeq \mathbf{Cart}_{J_{\mathbb{T}}}(\mathcal{C}_{\mathbb{T}}, \mathcal{E})$ between $\mathbb{T}[\mathcal{E}]$ and the category of cartesian (that is, finite-limit-preserving) functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{E}$ which are $J_{\mathbb{T}}$-continuous (that is, which send $J_{\mathbb{T}}$-covering families to covering families in $\mathcal{E}$). The functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{E}$ corresponding to a $\mathbb{T}$-model M in $\mathcal{E}$ will be denoted by F_M; it sends any geometric formula $\{\vec{x}.\,\phi\}$ over the signature of $\mathbb{T}$ to its interpretation $[[\vec{x}.\,\phi]]_M$ in M, and acts accordingly on arrows (for more details, see, for instance, Chapter 1 of [4]).

- The 2-category of Grothendieck toposes, geometric morphisms and geometric transformations will be denoted by $\mathbf{GTop}$, and, for any Grothendieck toposes $\mathcal{E}$ and $\mathcal{F}$ over a Grothendieck topos $\mathcal{S}$, the category of geometric morphisms over $\mathcal{S}$ from $\mathcal{F}$ to $\mathcal{E}$ will be denoted by $\mathbf{Geom}_{\mathcal{S}}(\mathcal{F}, \mathcal{E})$, or simply by $\mathbf{Geom}(\mathcal{F}, \mathcal{E})$ if $\mathcal{S}$ is the topos of sets.

1. The concept of over-topos

The central object of study of this work is the over-topos construction, which is known to admit several different abstract descriptions, as a bicomma object or as a bipullback of toposes, as well as an instance of a construction known as *Artin glueing*. In this first section, we recall these abstract points of view on this concept, and also provide a few topological remarks hinting at a link with the notion of *complete spread*.

1.1. Totally connected toposes and comma objects. In this subsection we recall the well-known construction of the over-topos at a geometric morphism through bilimits in the bicategory of Grothendieck toposes, as well as its relation with the notion of *totally connected geometric morphism*. Most of the content of this subsection is standard, and can be found in section C3.6 of [9].

In topology, given a topological space X and a point x, one may look at the *up-set* $\uparrow_{\sqsubseteq} \{x\}$ and *down-set* $\downarrow_{\sqsubseteq} \{x\}$ of x for the specialization order $\sqsubseteq$: $\uparrow_{\sqsubseteq} \{x\}$ contains all the points above x, that is, the points which are contained in any neighborhood of x, while $\downarrow_{\sqsubseteq} \{x\}$ contains all the points below x, that is, whose neighborhoods contain x. One can define, more generally, up-sets and down-sets for arbitrary subsets of X: for any subset A of X, one can consider the up and down closures $\uparrow A = \bigcup_{x\in A} \uparrow \{x\}$ and $\downarrow A = \bigcup_{x\in A} \downarrow \{x\}$ of A.

These topological notions have natural topos-theoretic counterparts. Recall that a Grothendieck topos $\mathcal{F}$ has a *category* of points

$$\mathbf{pt}(\mathcal{F}) \simeq \mathbf{Geom}(\mathbf{Set}, \mathcal{F})$$

For $\mathcal{F}$ a Grothendieck topos and $p : \mathbf{Set} \to \mathcal{F}$ a point of $\mathcal{F}$, one can look at the corresponding *under-category* $(p \downarrow \mathbf{pt}(\mathcal{F}))$ and *over-category* $(\mathbf{pt}(\mathcal{F}) \downarrow p)$, which are respectively the analogues of the up-set and down-set of a point. This generalizes to arbitrary geometric morphisms: indeed, given a Grothendieck topos $\mathcal{F}$ and a geometric morphism $f : \mathcal{E} \to \mathcal{F}$, one can consider respectively the under-category $(f \downarrow \mathbf{Geom}(\mathcal{E}, \mathcal{F}))$ and the over-category $(\mathbf{Geom}(\mathcal{E}, \mathcal{F}) \downarrow f)$ of the hom-category $\mathbf{Geom}(\mathcal{E}, \mathcal{F})$ at f.

In the case where $\mathcal{F}$ is the classifying topos $\mathbf{Set}[\mathbb{T}]$ of a geometric theory $\mathbb{T}$, the hom-category $\mathbf{Geom}(\mathcal{E}, \mathbf{Set}[\mathbb{T}])$ is equivalent to the category $\mathbb{T}[\mathcal{E}]$ of $\mathbb{T}$-models in $\mathcal{E}$, and for a $\mathbb{T}$-model M in $\mathcal{E}$ corresponding to a geometric morphism $f_M : \mathcal{E} \to \mathbf{Set}[\mathbb{T}]$, the under-category $(M \downarrow \mathbb{T}[\mathcal{E}])$ and the over-category $(\mathbb{T}[\mathcal{E}] \downarrow M)$ can be seen as "localizations" forcing M to become respectively initial and terminal.

In this paper we shall be concerned, for a given geometric theory $\mathbb{T}$ and a model M of $\mathbb{T}$ in a Grothendieck topos $\mathcal{E}$, with a certain Grothendieck topos *over* $\mathcal{E}$ whose models in any Grothendieck topos are exactly the homomorphisms of $\mathbb{T}$-models into inverse images of M, in other words, in a topos where M becomes "universally terminal". Such a topos over $\mathcal{E}$ is called the *over-topos* (or *colocalization*) at M. This is dual to the well-known *localization* at M, where one forces M to become universally initial.

The kind of geometric morphism into $\mathcal{E}$ that one gets through the over-topos construction is axiomatized by the following notion:

1.2. DEFINITION. [Theorem C3.6.16 [9]] *A geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ *is said to be* totally connected *if the following equivalent conditions are fulfilled:*

- f^* *has a* $\mathcal{E}$*-indexed cartesian left adjoint* $f_!$*;*
- f *has a right adjoint in* $\mathbf{GTop}/\mathcal{E}$*;*
- f *is connected and has a right adjoint in* $\mathbf{GTop}$*;*
- f *has a universal terminal section, that is, there exists* $s : \mathcal{E} \to \mathcal{F}$ *with* $fs = 1_{\mathcal{E}}$ *and for any* $g : \mathcal{G} \to \mathcal{E}$*, the composite* $sg : \mathcal{G} \to \mathcal{F}$ *is the terminal object of* $\mathbf{Geom}_{\mathcal{E}}(g, f)$

In particular, a Grothendieck topos $\mathcal{E}$ *is* totally connected *if its terminal geometric morphism* $!_{\mathcal{E}} : \mathcal{E} \to \mathbf{Set}$ *is totally connected.*

1.3. REMARK. The condition for a topos $\mathcal{E}$ to be totally connected amounts to requiring $\mathcal{E}$ to have a "universally terminal" point given by the terminal section of its terminal geometric morphism

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{!_{\mathcal{E}}} & \mathbf{Set} \\ & \nwarrow^{t_{\mathcal{E}}} \ \simeq & \| \\ & & \mathbf{Set} \end{array}$$

In particular, if $\mathcal{E}$ is the classifying topos of some geometric theory $\mathbf{Set}[\mathbb{T}]$, then being totally connected means that the category of $\mathbb{T}$-models $\mathbb{T}[\mathbf{Set}]$ has a "universally terminal" object.

In the case where $\mathcal{E}$ is **Set**, that is, where M is a set-valued model of $\mathbb{T}$, the over-topos construction yields a totally connected geometric morphism over the terminal topos **Set**, that is, a totally connected topos. In fact, there is a canonical, abstract way to construct totally connected toposes by means of finite bilimits in the bicategory of Grothendieck toposes. Recall that Grothendieck toposes have all finite bilimits, hence in particular *bipowers with 2* and *bipullbacks*. Explicit descriptions of such bilimits can be found, for instance, in section B1.1 of [9]. In particular, the *bipower* of a topos $\mathcal{F}$ is equipped with its universal 2-cell

$$\mathcal{F}^2 \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \mathcal{F} \qquad \Downarrow \mu_{\mathcal{F}}$$

where $\partial_1 : \mathcal{F}^2 \to \mathcal{F}$ is the *universal codomain* of $\mathcal{F}$.

1.4. Proposition. [C.3.6.19 [9]] *For a Grothendieck topos $\mathcal{F}$, the universal codomain $\partial_1 : \mathcal{F}^2 \to \mathcal{F}$ is a totally connected geometric morphism.*

This generic totally connected geometric morphism can be used to construct other ones thanks to the following stability property for totally connected morphisms (cf. Lemma C3.6.18 of [9]):

1.5. Proposition. [C3.6.18(iii) [9]] *Totally connected geometric morphisms are stable under bipullbacks.*

1.6. Definition. *For $\mathcal{E}$ a Grothendieck topos and $f : \mathcal{E} \to \mathcal{F}$ a geometric morphism, we define the* over-topos of $\mathcal{E}$ at f *as the $\mathcal{E}$-topos $u_f : \mathcal{E}[f] \to \mathcal{E}$ given by the left projection of the following bicomma topos:*

$$\begin{array}{ccc} \mathcal{E}[f] & \xrightarrow{\xi_f} & \mathcal{F} \\ {\scriptstyle u_f}\downarrow & \swarrow_{\lambda_f} & \| \\ \mathcal{E} & \xrightarrow[f]{} & \mathcal{F} \end{array}$$

In other words, there is a morphism $\xi_f : \mathcal{E}[f] \to \mathcal{F}$ such that for any $\mathcal{E}$-topos $g : \mathcal{G} \to \mathcal{E}$, there is an equivalence

$$\mathbf{Geom}_{\mathcal{E}}(g, u_f) \simeq \mathbf{Geom}(\mathcal{E}, \mathcal{F})/(f \circ g)$$

natural in g, induced by composition with ξ_f.

1.7. Remark. Equivalently, we can define $\mathcal{E}[f] \to \mathcal{E}$ as a bipullback:

$$\begin{array}{ccc} \mathcal{E}[f] & \longrightarrow & \mathcal{F}^2 \\ {\scriptstyle u_f}\downarrow & \lrcorner & \downarrow{\scriptstyle \partial_1} \\ \mathcal{E} & \xrightarrow{f} & \mathcal{F} \end{array}$$

1.8. Proposition. *Let be $f : \mathcal{E} \to \mathcal{F}$ and $g : \mathcal{G} \to \mathcal{E}$ geometric morphisms. Then the following square is a bipullback:*

$$\begin{array}{ccc} \mathcal{G}[f \circ g] & \longrightarrow & \mathcal{E}[f] \\ \downarrow & \lrcorner & \downarrow u_f \\ \mathcal{G} & \xrightarrow{g} & \mathcal{E} \end{array}$$

Proof. By cancellation of bipullbacks. ∎

More generally, we can define, for a $\mathcal{S}$-toposes $p : \mathcal{F} \to \mathcal{S}$ and a morphism of $\mathcal{S}$-topos $(f, \alpha) : (\mathcal{E}, q) \to (\mathcal{F}, p)$ the *over-topos* $u_{(f,\alpha)}$ *of* q *at* (f, α) as the $\mathcal{S}$-topos given by the left outer projection of the following bipullback

$$\begin{array}{ccc} q[(f,\alpha)] & \longrightarrow & \mathbf{Inv}(p * \mu) \\ \downarrow & \lrcorner & \downarrow q_\mu \\ \mathcal{E}[f] & \longrightarrow & \mathcal{F}^2 \\ \downarrow u_f & \lrcorner & \downarrow \partial_1 \\ \mathcal{E} & \xrightarrow{f} & \mathcal{F} \\ & {}_q\searrow \overset{\alpha}{\simeq} \swarrow_p & \\ & \mathcal{S} & \end{array} \qquad u_{(f,\alpha)} : q[(f,\alpha)] \to \mathcal{E}$$

which exhibits its domain $q[(f, \alpha)]$ as a subtopos of $\mathcal{E}[f]$.

Beware that the over-topos construction *depends* on the base topos with respect to which it is calculated.

In the sequel $\mathcal{F}$ is replaced with the classifying topos $\mathbf{Set}[\mathbb{T}]$ of a geometric theory $\mathbb{T}$, and f with the geometric morphism f_M corresponding to a $\mathbb{T}$-model M in $\mathcal{E}$ via the universal property of the classifying topos of $\mathbb{T}$: in this case, we denote by $u_M : \mathcal{E}[M] \to \mathcal{E}$ the over-topos at M.

By Proposition 1.6, the $\mathbb{T}$-over-topos $u_M : \mathcal{E}[M] \to \mathcal{E}$ is totally connected; indeed, its terminal point is the identity morphism 1_M. In particular, all the fibers of this morphisms are also totally connected, while a section

$$\begin{array}{ccc} & & \mathcal{E}[M] \\ & {}^{s}\nearrow & \downarrow u_M \\ \mathcal{E} & = & \mathcal{E} \end{array}$$

just is the name of some homomorphism $f_s : N \to M$ in $\mathbb{T}[\mathcal{E}]$.

In particular, the fact that the identity geometric morphism classifies the universal $\mathbb{T}$-model ensures the following:

1.9. Proposition. *The universal codomain* $\partial_1 : \mathbf{Set}[\mathbb{T}]^2 \to \mathbf{Set}[\mathbb{T}]$ *is the over-topos of* $\mathbf{Set}[\mathbb{T}]$ *at the universal model* $U_{\mathbb{T}}$*, that is, we have a geometric equivalence*

$$\mathbf{Set}[\mathbb{T}]^2 \simeq \mathbf{Set}[\mathbb{T}][U_{\mathbb{T}}]$$

and an invertible 2-cell:

$$\begin{array}{ccc} \mathbf{Set}[\mathbb{T}]^2 & \xrightarrow{\quad \simeq \quad} & \mathbf{Set}[\mathbb{T}][U_{\mathbb{T}}] \\ & {\scriptstyle \partial_1} \searrow \quad \simeq \quad \swarrow {\scriptstyle u_{U_{\mathbb{T}}}} & \\ & \mathbf{Set}[\mathbb{T}] & \end{array}$$

1.10. Remark. Before going further, let us give a few remarks on the $\mathbb{T}$-over-topos construction in the particular case $\mathbb{T} = \mathbb{O}$, the (one-sorted) theory of objects, and compare it with the more standard notion of the *slice topos* at an object. For any object E in a topos $\mathcal{E}$, the "totally connected component" of $\mathcal{E}$ in E

$$\begin{array}{ccc} \mathcal{E}[E] & \longrightarrow & \mathbf{Set}[\mathbb{O}]^2 \\ {\scriptstyle u_E}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \partial_1} \\ \mathcal{E} & \xrightarrow{\quad E \quad} & \mathbf{Set}[\mathbb{O}] \end{array}$$

classifies generalized elements $F \to E$ of the object E, as opposed to the usual étale topos

$$\begin{array}{ccc} \mathcal{E}/E & \longrightarrow & \mathbf{Set}[\mathbb{O}_\bullet] \\ {\scriptstyle \widehat{E}}\big\downarrow & \lrcorner & \big\downarrow \\ \mathcal{E} & \xrightarrow{\quad E \quad} & \mathbf{Set}[\mathbb{O}] \end{array}$$

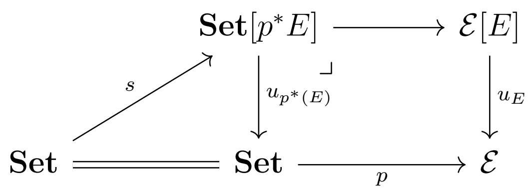
which classifies *global elements* of E (which are those $a : 1 \to E$), where $\mathbf{Set}[\mathbb{O}_\bullet]$ is the classifier of the theory of *pointed objects* - this is equivalently the slice topos $\mathbf{Set}[\mathbb{O}]/\mathbb{O}$ at the object $\mathbb{O}$ corresponding to the universal model $U_{\mathbb{O}} : \mathbf{Set}[\mathbb{O}] \to \mathbf{Set}[\mathbb{O}]$ of the theory of objects in $\mathbf{Set}[\mathbb{O}]$ itself.

Whilst the *objects* of the slice topos $\mathcal{E}/E$ are generalized elements of E, that is, morphisms $F \to E$, the corresponding geometric morphism $\widehat{E} : \mathcal{E}/E \to \mathcal{E}$ actually classifies *global* elements of E in the sense that its sections

$$\begin{array}{ccc} & & \mathcal{E}/E \\ & {\scriptstyle s} \nearrow & \big\downarrow{\scriptstyle \widehat{E}} \\ \mathcal{E} & = & \mathcal{E} \end{array}$$

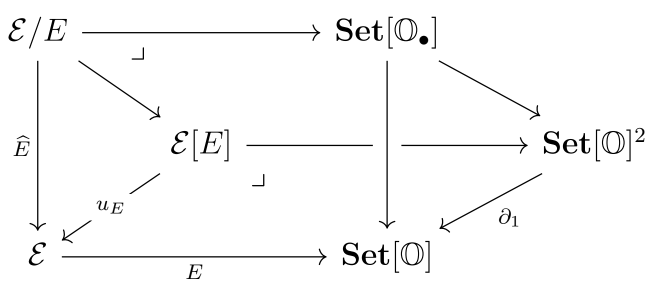
correspond to global elements $1 \to E$. In particular, its fibers at points are discrete (as E only has a *set* of global elements).

Note that any point of $\mathcal{E}[E]$ defines by composition with u_E a point p of $\mathcal{E}$ over which it lies. By the universal property of the pullback, it thus yields a section s of the totally connected geometric morphism at the corresponding fiber:



Note that s defines a generalized element $X \to p^*(E)$ of the stalk of E at p, equivalently a X-indexed family of global elements of $p^*(E)$; this should be compared with the étale case, where points of the étale topos are mere global elements of the stalk.

Finally, we observe that the two constructions are related as in the following diagram, where the upper square is a pullback as the bottom and front square are:



1.11. Relation with colocalizations. Let us discuss the relation between the over-topos construction and that of colocalization of a topos at a section of it, studied in Theorem C3.6.19 of [9].

If M is a model of $\mathbb{T}$ in **Set**, hence a point $f_M : \mathbf{Set} \to \mathbf{Set}[\mathbb{T}]$ of $\mathbf{Set}[\mathbb{T}]$ over **Set**, the over-topos $u_M : \mathbf{Set}[M] \to \mathbf{Set}$ is a totally connected topos (over **Set**) which coincides with the colocalization of $\mathbf{Set}[\mathbb{T}]$ at M; indeed, in this case the over-topos $u_M : \mathbf{Set}[M] \to \mathbf{Set}$ admits a **Set**-point $s_M : \mathbf{Set} \to \mathbf{Set}[M]$ providing a factorization of $f_M : \mathbf{Set} \to \mathbf{Set}[\mathbb{T}]$ as $s_M : \mathbf{Set} \to \mathbf{Set}[M]$ followed by the geometric morphism $f_M \circ u_M : \mathbf{Set}[M] \to \mathbf{Set}[\mathbb{T}]$, which satisfies the universal property of the colocalization of $\mathbf{Set}[\mathbb{T}]$ at M.

Still, for an arbitrary base topos $\mathcal{S}$, given a section $s : \mathcal{S} \to \mathcal{E}$ of a $\mathcal{S}$-topos $p : \mathcal{E} \to \mathcal{S}$, the colocalization of p at s differs in general from the over-topos of $\mathcal{S}$ at s, since the universal property of the former provides an equivalence between $\mathbf{Geom}_{\mathcal{S}}(\mathcal{F}, u_s)$ and $\mathbf{Geom}_{\mathcal{S}}(\mathcal{F}, \mathcal{E})/(s \circ g)$ (where $\mathcal{E}$ is regarded as a $\mathcal{S}$-topos via p), while the universal property of the latter provides an equivalence between $\mathbf{Geom}_{\mathcal{S}}(\mathcal{F}, u_s)$ and $\mathbf{Geom}(\mathcal{F}, \mathcal{E})/(s \circ g)$. The condition $p \circ s = 1$ is not sufficient in general to ensure that ξ_s is a morphism over $\mathcal{S}$ (i.e., that $p\alpha_s : p \circ \xi_s \to p \circ s \circ u_s \cong u_s$, where $\alpha_s : \xi_s \to s \circ u_s$ is the canonical geometric transformation, is an isomorphism), which explains why the colocalization of $\mathcal{E}$ at s is not in general equivalent to the over-topos of $\mathcal{S}$ at s. On the other hand, the colocalization of p at s coincides with the over-topos of p at s, regarded as a morphism of $\mathcal{S}$-toposes from the identical morphism on $\mathcal{S}$ to p.

1.12. Over-topos as an Artin glueing. In this subsection we discuss some links between our work and the notion of *Artin glueing*. This construction, as introduced in Theorem 9.5.6 of *Exposé IV* of [2], is a way of glueing two toposes along a cartesian functor. Remarkably, it is a bilimit or a bicolimit depending on the 2-category where it is performed:

– the original Artin glueing, performed for toposes and inverse image parts of geometric morphisms, yields a *bicocomma* topos;

– on the other hand, the Artin glueing of cartesian categories is simultaneously an instance of a bicomma and a bicocomma. When applied to a morphism of cartesian sites, the obtained bicocomma notably provides a site of definition for the *bicomma* in the category of Grothendieck toposes (as explained in Example C2.3.15 (b) of [9]), encompassing in particular the over-topos construction.

The Artin glueing for toposes, as described in Theorem 9.5.6 of *Exposé IV* of [2], assigns to a cartesian functor $f^* : \mathcal{F} \to \mathcal{E}$ - typically the inverse image part of a geometric morphism - the comma object in **Cat**

$$\begin{array}{ccc} (1_{\mathcal{E}} \downarrow f^*) & \xrightarrow{p_1} & \mathcal{E} \\ {\scriptstyle p_2}\downarrow & \lambda_{f^*} \swarrow & \| \\ \mathcal{F} & \xrightarrow[f^*]{} & \mathcal{E} \end{array}$$

whose objects are the triples $(E, F, u : E \to f^*F)$ and whose morphisms are the squares between them. This is actually a comma object in the 2-category **Cart** of (large) cartesian categories and cartesian functors, and, as in general bicolimits of Grothendieck toposes are computed as pseudolimits of the corresponding diagrams made of the underlying categories and inverse image functors, this object is the underlying category of the *bicocomma object* $(\mathcal{E} \uparrow f)$ in **GTop**, the projections p_1 and p_2 being the inverse images of the canonical inclusions into it.

As mentioned above, the Artin glueing construction can also be used to construct a site of definition of a bicomma topos through a computation of a bicocomma in the category **Cart**. In the 2-category of (either large or small) cartesian categories, several kinds of pseudolimits happen to be also bicolimits for a dual diagram consisting of some adjoints; see, for example, [6] and [7] for some instances of this phenomenon. In our case, the comma object $(1_{\mathcal{E}} \downarrow f^*)$ can be equipped with the structure of a bicocomma object in the 2-category **Cart** of cartesian categories, through the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f^*} & \mathcal{E} \\ \| & \overset{\mu_f}{\nearrow} & \downarrow{\scriptstyle q_1} \\ \mathcal{F} & \xrightarrow[q_2]{} & (1_{\mathcal{E}} \downarrow f^*) \end{array}$$

where q_1 and q_2 are right adjoints respectively to p_1, p_2, sending an object E of $\mathcal{E}$ to the unique map $!_E : E \to f^*1 = 1$ and an object F of $\mathcal{F}$ to the identity $1_{f^*(F)}$, and μ_f is the mate of λ_{f^*}. This construction behaves naturally with respect to sites. Indeed, given a span consisting of *small* cartesian sites and cartesian functors

$$\begin{array}{ccc} (\mathcal{C}, J) & \xrightarrow{f_2} & (\mathcal{C}_2, J_2) \\ {\scriptstyle f_1}\downarrow & & \\ (\mathcal{C}_1, J_1) & & \end{array}$$

one can construct the bicomma topos $(\mathbf{Sh}(f_1) \downarrow \mathbf{Sh}(f_2))$ by equipping the relevant *bicocomma* object in **Cart** with the topology induced by the cocomma inclusions. More specifically, given a morphism of (small) cartesian sites $\overline{f} : (\mathcal{D}, K) \to (\mathcal{C}, J)$, the Artin glueing defines a bicocomma object in **Cart**

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\overline{f}} & \mathcal{C} \\ \| & \overset{\mu_{\overline{f}}}{\nearrow} & \downarrow \\ \mathcal{D} & \longrightarrow & (1_{\mathcal{C}} \downarrow \overline{f}) \end{array}$$

which we can equip with the smallest topology $J_{\overline{f}}$ making the bicocomma inclusions morphisms of sites respectively from $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, thus obtaining a site of definition for $(1_{\mathbf{Sh}(\mathcal{D},K)} \downarrow \mathbf{Sh}(\overline{f}))$.

1.13. Remark. In 3.6, we shall consider the same comma category and equip it with a *lifted topology*, defined as the smallest topology making the canonical comma projections *comorphisms of sites*. This generalizes the above construction to the non-cartesian setting, where the right adjoints to the canonical projections do not necessarily exist. In the cartesian setting, the two topologies coincide. Indeed, given a pair of adjoint functors, the right adjoint is a morphism of sites if and only if the left adjoint is a comorphism of sites (cf. Proposition 3.14(iii) [5]); so, since in our case each of the bicomma projections is left adjoint to the corresponding inclusion into the Artin glueing, the smallest topology making the Artin inclusions morphisms of sites is also the smallest topology making the bicomma projections comorphisms of sites.

The above considerations directly yield a site of definition for the over-topos construction, as shown by the following result:

1.14. Proposition. [Example C2.3.15 (b) [9]] *Let f be a geometric morphism $\mathcal{E} \to \mathcal{F}$ induced by a morphism of sites $\overline{f} : (\mathcal{D}, K) \to (\mathcal{C}, J)$ where $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ are small cartesian sites of definition respectively for $\mathcal{E}$ and $\mathcal{F}$. Then the over-topos at f can be constructed as the sheaf topos over the Artin glueing equipped with the induced topology:*

$$\mathcal{E}[f] \simeq \mathbf{Sh}((1_{\mathcal{C}} \downarrow \overline{f}), J_{\overline{f}})$$

We should emphasize that this result requires a choice of small sites of definition for both the domain and codomain topos, such that the model at which we compute the over-topos is induced by a morphism between these sites. In practice in model theory, this is rarely the case: one generally has both a fixed geometric theory $\mathbb{T}$ (determining the corresponding syntactic site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$) classified by the codomain topos $\mathcal{F} = \mathbf{Set}[\mathbb{T}]$, and a fixed small (cartesian) site $(\mathcal{C}, J)$ for the domain topos $\mathcal{E}$, *but the model M is only seldom induced by a morphism of sites between those two fixed sites*, as it corresponds to a J-continuous cartesian functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{E}$. One could argue that this still defines a morphism of sites into $(\mathcal{E}, J^{\mathrm{can}}_{\mathcal{E}})$ for the canonical topology on $\mathcal{E}$, knowing that $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{E}, J^{\mathrm{can}}_{\mathcal{E}})$, even though this only begets *large sites*. Still, the point of this work is to generalize this idea without the need to consider larges sites, but strictly using the data contained in an arbitrarily chosen small site for the domain topos $\mathcal{E}$, even when the model *is not* induced from a morphism of sites as above. In the case where $\mathcal{E}$ is $\mathbf{Set}$, we moreover show that we can restrict to a simpler subcategory of the Artin glueing, where we only take morphisms whose domain is the terminal object.

1.15. The over-topos as a colocalization. We close this introductory section with a remark on the possible dual of a certain limit formula for the localization: for a point $p : \mathcal{S} \to \mathcal{E}$, one can prove that the localization $\mathcal{E}_p$ of $\mathcal{E}$ at p (which can be defined as the pullback $\mathcal{S} \times^{\partial_0}_{\mathcal{E}} \mathcal{E}$ of the universal *domain*) is given by the cofiltered bilimit

$$\mathcal{E}_p \simeq \lim_{(X,a) \in \int p^*} \mathcal{E}/X$$

(where one can restrict to inverse images of representable sheaves, for a small site of definition, in order to have a small indexing category). This is a categorification of the formula exhibiting the upset of a point as the intersection of its open neighborhoods

$$\uparrow_{\sqsubseteq} \{x\} = \bigcap \{U \text{ open} \mid x \in U\}$$

open subsets being replaced by étale geometric morphisms over $\mathcal{E}$.

Is is natural to ask for a categorification of the dual formula exhibiting the downset of a point as the intersection of all the closed sets it belongs to:

$$\downarrow_{\sqsubseteq} \{x\} = \bigcap \{F \text{ closed} \mid x \in F\}$$

While open subsets are generalized to sheaves and the corresponding étale geometric morphisms, closed sets can be generalized to *cosheaves*, which correspond to the so-called *complete spreads* as defined in [3].

It is known that any geometric morphisms admits a *(pure geometric morphism, complete spread)* factorization, and moreover, Example 2.4.11 of [3] tells us that the middle topos in the (pure, complete spread) factorization of a point (corresponding to a set-based model M)

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\quad M \quad} & \mathbf{Set}[\mathbb{T}] \\ & {\scriptstyle p_M} \searrow \quad \nearrow {\scriptstyle s_M} & \\ & \mathbf{Set}[M] & \end{array}$$

is actually the over-topos at M. It would be interesting to give a limit decomposition of this complete spread s_M; we defer this to a future work investigating the role of cosheaves and complete spreads in topos-theoretic model theory.

2. Site for the over-topos at a set-based $\mathbb{T}$-model

We turn to the construction of a canonical site of the over-topos in the case of a set-based model. We suppose in this section that $\mathcal{E}$ is equal to the topos **Set** of sets (within a fixed model of set theory). We first list some specific properties of the terminal object 1 of **Set**, we are going to make use of in this section, and which may fail in arbitrary toposes, as we are going to see in section 3 which address the general case. First, **Set** is generated under coproducts by 1, that is, for any set X one has

$$X \simeq \coprod_{\mathrm{Hom}_{\mathbf{Set}}(1,X)} 1$$

Moreover, 1 is *projective*, that is, any epimorphism $X \twoheadrightarrow 1$ admits a section $1 \to X$, and *indecomposable*, which means that for any arrow from 1 to a coproduct $\coprod_{i \in I} X_i$, there is a section for at least one $i \in I$:

$$\begin{array}{ccc} X_i & \rightarrowtail & \coprod_{i \in I} X_i \\ \uparrow & & \uparrow \\ 1 & = & 1 \end{array}$$

Those properties simplifies substantially the description of the site for the over-topos, which will be provided by a category of *global elements* together with a certain *antecedents topology* we are going to describe.

2.1. The antecedents topology for global elements. We are going to use these properties in order to define a cartesian, subcanonical site for the $\mathbb{T}$-over-topos associated with M. This involves the *category of elements* of M, and a certain topology related to the syntactic topology $J_{\mathbb{T}}$ of $\mathbb{T}$. Let $\mathcal{C}_{\mathbb{T}}$ denote the geometric syntactic of $\mathbb{T}$, and, for any object $\{\vec{x}^{\vec{A}}.\, \phi\}$ of $\mathcal{C}_{\mathbb{T}}$, that is, a geometric formula ϕ in the signature $\mathcal{L}$ in the context $\vec{x}$ of sort $\vec{A}$, $[\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$ the interpretation of $\{\vec{x}^{\vec{A}}.\, \phi\}$ in M. In particular the category $\int M$ of elements of M - seen as a geometric functor $F_M : \mathcal{C}_{\mathbb{T}} \to \mathbf{Set}$ - has as objects the pairs

$(\{\vec{x}^{\vec{A}}.\,\phi\},\vec{a})$, where $\vec{a}\in[\![\vec{x}^{\vec{A}}.\,\phi]\!]_M$ is a global element of $M\vec{A}$ satisfying the formula ϕ, and as arrows $(\{\vec{x}_1^{\vec{A_1}}.\,\phi_1\},\vec{a_1})\to(\{\vec{x}_2^{\vec{A_2}}.\,\phi_2\},\vec{a_2})$ the arrows $[\theta]:\{\vec{x}_1^{\vec{A_1}}.\,\phi_1\}\to\{\vec{x}_2^{\vec{A_2}}.\,\phi_2\}$ in $\mathcal{C}_{\mathbb{T}}$ such that $[\![\theta]\!]_M(\vec{a_1})=\vec{a_2}$, that is diagramatically in **Set**:

$$\begin{array}{ccc} & 1 & \\ {}^{\vec{a_1}}\swarrow & & \searrow{}^{\vec{a_2}} \\ [\![\vec{x}_1^{A_1}.\,\phi_1]\!]_M & \xrightarrow{[\![\theta]\!]_M} & [\![\vec{x}_2^{A_2}.\,\phi_2]\!]_M \end{array}$$

We shall find it convenient to present our Grothendieck topologies in terms of bases generating them. Recall that a basis $\mathcal{B}$ for a Grothendieck topology on a category $\mathcal{C}$ is a collection of *presieves* on objects of $\mathcal{C}$ (by a presieve we simply mean a small family of arrows with common codomain) satisfying the following properties (where we denote by $\mathcal{B}(c)$, for an object c of $\mathcal{C}$, the collection of presieves in $\mathcal{B}$ on the object c):

(a) If f is the identity then $\{f\}$ lies in $\mathcal{B}(\mathrm{cod}(f))$.

(b) If $R\in\mathcal{B}(c)$ then for any arrow $g:d\to c$ in $\mathcal{C}$ there exists a presieve T in $\mathcal{B}(d)$ such that for each $t\in T$, $g\circ t$ factors through some arrow in R.

(c) $\mathcal{B}$ is closed under "multicomposition" of families; that is, given a presieve $\{f_i:c_i\to c\mid i\in I\}$ in $\mathcal{B}(c)$ and for each $i\in I$ a presieve $\{g_{ij}:d_{ij}\to c_i\mid j\in I_i\}$ in $\mathcal{B}(c_i)$, the "multicomposite" presieve $\{\,f_i\circ g_{ij}:d_{ij}\to c\mid i\in I, j\in I_i\}$ belongs to $\mathcal{B}(c)$.

The Grothendieck topology generated by a basis has as covering sieves precisely those which contain a presieve in the basis.

It is well known that the Grothendieck topology $J_{\mathbb{T}}$ on $\mathcal{C}_{\mathbb{T}}$ has as a basis the collection $\mathcal{B}_{\mathbb{T}}$ of small families $\{[\theta_i]:\{\vec{x_i}^{\vec{A_i}}.\,\phi_i\}\to\{\vec{x}^{\vec{A}}.\,\phi\}\mid i\in I\}$ such that the sequent

$$(\phi\vdash_{\vec{x}}\bigvee_{i\in I}(\exists\vec{x_i})\theta_i(\vec{x_i},\vec{x}))$$

is provable in $\mathbb{T}$.

The fact that the following definition is well-posed in ensured by the subsequent Lemma.

2.2. Definition. *Let M be a set-based model of a geometric theory $\mathbb{T}$. We define the* antecedents topology *at M as the Grothendieck topology J_M^{ant} on $\int M$ generated by the basis $\mathcal{B}_{\mathbb{T}}^{\mathrm{ant}}$ consisting of the families*

$$\big((\vec{b},\{\vec{x}_i^{\vec{A_i}}.\,\phi_i\})\xrightarrow{[\theta_i]}(\vec{a},\{\vec{x}^{\vec{A}}.\,\phi\})\big)_{i\in I,\,\vec{b}\,\mid\,[\![\theta_i]\!]_M(\vec{b})=\vec{a}}$$

(indexed by the objects $(\vec{a},\{\vec{x}.\,\phi\})$ of $\int M$ and the families $([\theta_i])_{i\in I}$ in $\mathcal{B}_{\mathbb{T}}$ on $\{\vec{x}.\,\phi\}$) consisting of all the "antecedents" $\vec{b}$ of a given $\vec{a}\in[\![\vec{x}^{\vec{A}}.\,\phi]\!]_M$ with respect to some $[\theta_i]$.

2.3. Lemma. *The collection $\mathcal{B}_{\mathbb{T}}^{\text{ant}}$ of sieves in $\int M$ is a basis for a Grothendieck topology.*

Proof.

- Condition (a) is trivially satisfied.
- Condition (b): given an arrow $[\theta] : (\vec{c}, \{y^{\vec{B}}.\, \psi\}) \to (\vec{a}, \{\vec{x}^{\vec{A}}.\, \phi\})$ in $\int M$ and a family $[\theta_i]_{\mathbb{T}} : \{\vec{x}_i^{\vec{A}_i}.\, \phi_i\} \to \{\vec{x}^{\vec{A}}.\, \phi\})_{i\in I}$ in $\mathcal{B}_{\mathbb{T}}$ on $\{\vec{x}^{\vec{A}}.\, \phi\}$, we have the following pullback squares in $\int M$ for each antecedent $\vec{b}$ of $\vec{a}$:

$$\begin{array}{ccc} ((\vec{c},\vec{b}), \{\vec{y}^{\vec{B}}, \vec{x}_i^{\vec{A}_i}.\, \theta(\vec{y}) = \theta_i(\vec{x}_i)\}) & \longrightarrow & (\vec{b}, \{\vec{x}_i^{\vec{A}_i}.\, \phi_i\})) \\ {\scriptstyle [\theta^*\theta_i]}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle [\theta_i]} \\ (\vec{c}, \{\vec{y}^{\vec{B}}.\, \psi\}) & \xrightarrow{\quad [\theta] \quad} & (\vec{a}, \{\vec{x}^{\vec{A}}.\, \phi\}) \end{array}$$

  Note that the family $\{[\theta^*\theta_i] \mid i \in I\}$ lies in $\mathcal{B}_{\mathbb{T}}(\{\vec{y}^{\vec{B}}.\, \psi\})$ (as $\mathcal{B}_{\mathbb{T}}$ is stable under pullback). So the family

$$\big([\theta^*\theta_i] : ((\vec{c},\vec{b}), \{\vec{y}^{\vec{B}}, \vec{x}_i^{\vec{A}_i}.\, \theta(\vec{y}) = \theta_i(\vec{x}_i)\}) \to (\vec{c}, \{\vec{y}^{\vec{B}}.\, \psi\})\big)_{i\in I}$$

  is the family of antecedents of $\vec{c}$ indexed by it, whence it lies in $\mathcal{B}_{\mathbb{T}}^{\text{ant}}$, as desired; indeed, $[\![\theta^*\theta_i]\!]_M = [\![\theta]\!]_M^*[\![\theta_i]\!]_M$ since F_M is cartesian.
- Condition (c) follows immediately from the fact that $\mathcal{B}_{\mathbb{T}}$ is a basis for $J_{\mathbb{T}}$.

■

2.4. Remark. For a family $([\theta_i])_{i\in I}$ in $\mathcal{B}_{\mathbb{T}}(\{\vec{x}.\, \phi\})$ and a global element $\vec{a}$ of the interpretation $[\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$ of its codomain, the fiber at $\vec{a}$ of some $[\![\theta_i]\!]_M$ can be categorically characterized as the following pullback:

$$\begin{array}{ccc} [\![\theta_i]\!]_M^{-1}(\vec{a}) & \longrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \vec{a}} \\ [\![\vec{x}_i^{\vec{A}_i}.\, \phi_i]\!]_M & \xrightarrow{[\![\theta_i]\!]_M} & [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M \end{array}$$

As F_M is a $J_{\mathbb{T}}$-continuous cartesian functor, it sends $J_{\mathbb{T}}$-covering families to jointly surjective families in **Set**, so by the stability of epimorphisms under pullback the global fiber of the cover at $\vec{a}$ is also an epimorphism:

$$\begin{array}{ccc} \langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a}) & \twoheadrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \vec{a}} \\ \coprod_{i\in I} [\![\vec{x}_i^{\vec{A}_i}.\, \phi_i]\!]_M & \xrightarrow{\langle [\![\theta_i]\!]_M \rangle_{i\in I}} \!\!\!\!\twoheadrightarrow & [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M \end{array}$$

Note that, by the stability of colimits under pullback, the global fiber of $\vec{a}$ decomposes as the coproduct

$$\langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a}) \simeq \coprod_{i\in I} [\![\theta_i]\!]_M^{-1}(\vec{a})$$

One can then identify the set of antecedents of $\vec{a}$ with the set of global elements $\vec{b} : 1 \to \langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a})$, which decomposes as the disjoint union of the sets of global elements of the fibers $[\![\theta_i]\!]_M^{-1}(\vec{a})$ (since 1 is indecomposable and coproducts are disjoint and stable under pullback).

Note however that the projection $\int M \to \mathcal{C}_{\mathbb{T}}$ does *not* send J_M^{ant}-covers to $J_{\mathbb{T}}$-covers as $\vec{a} \in [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$ may have antecedents in only some of the $[\![\vec{x}_i^{\vec{A}_i}.\, \phi_i]\!]_M$.

2.5. Proposition. *The category $\int M$ of elements of M is geometric.*

Proof. Actually all the properties we have to check are inherited from the geometricity of $\mathcal{C}_{\mathbb{T}}$ and the fact that M is a model of $\mathbb{T}$:

- $\int M$ is cartesian: for any finite diagram $D \to \int M$, the underlying diagram in $\mathcal{C}_{\mathbb{T}}$ has a limit

$$\lim_{d\in D} \{\vec{x}_d^{\vec{A}_d}.\, \phi_d\} = \{(\vec{x}_d^{\vec{A}_d})_{d\in D}, \bigwedge_{\delta : d \to d'} \theta_\delta(\vec{a}_d) = \vec{a}_{d'}\}$$

  which is sent to a limit in **Set** by the cartesian functor F_M; note that an element of $\lim\limits_{d\in D} [\![\vec{x}_d^{\vec{A}_d}.\, \phi_d]\!]_M$ is a family $(\vec{a}_d)_{d\in D}$ with $\vec{a}_d \in [\![\vec{x}_d^{\vec{A}_d}.\, \phi_d]\!]_M$ and $[\![\theta_\delta]\!]_M(\vec{a}_d) = \vec{a}_{d'}$ for each transition morphism $\delta : d \to d'$ in D. This exactly says that

$$((\vec{a}_d)_{d\in D}, \lim_{d\in D} \{\vec{x}_d^{\vec{A}_d}.\, \phi_d\}) = \lim_{d\in D} (a_d, \{\vec{x}_d^{\vec{A}_d}.\, \phi_d\}).$$

- The image factorization in $\int M$ of an arrow $[\theta] : (\vec{b}, \{\vec{y}^{\vec{B}}.\, \psi\}) \to (\vec{a}, \{\vec{x}^{\vec{A}}.\, \phi\})$ is given by

$$\begin{array}{ccc} (\vec{b}, \{\vec{y}^{\vec{B}}.\, \psi\}) & \xrightarrow{\quad [\theta] \quad} & (\vec{a}, \{\vec{x}^{\vec{A}}.\, \phi\}) \\ & \searrow\!\!\!\!\searrow \qquad \nearrow & \\ & (\vec{a}, \{\exists \vec{y}\theta(\vec{y}, \vec{x})\}) & \end{array}$$

  and is easily seen to be pullback stable.

- Subobjects in $\int M$ are arrows of the form $(\vec{a}, \{\vec{x}^{\vec{A}}.\, \phi\}) \hookrightarrow (\vec{a}, \{\vec{x}^{\vec{A}}.\, \psi\})$ with $(\phi \vdash_{\mathbb{T}} \psi)$. It thus follows at once that lattices of subobjects are frames, as their finite meets (resp. arbitrary joins) are given by

$$(\vec{a}, \{\vec{x}^{\vec{A}}.\, \bigwedge_{i\in I} \phi_i\}) \text{ (resp. } (\vec{a}, \{\vec{x}^{\vec{A}}.\, \bigvee_{i\in I} \phi_i\}))$$

  for any finite family (resp. arbitrary family) $\{(\vec{a}, \{\vec{x}^{\vec{A}}.\, \phi_i\}) \hookrightarrow (\vec{a}, \{\vec{x}^{\vec{A}}.\, \phi\}) \mid i \in I\}$ of subobjects of a given $(\vec{a}, \{\vec{x}^{\vec{A}}.\, \phi\})$.

∎

2.6. DEFINITION. *We define the* $\mathbb{T}$-over-topos at M *as the sheaf topos* $\mathbf{Sh}(\int M, J_M^{\mathrm{ant}})$, *and denote it by* $\mathbf{Set}[M]$.

2.7. SYNTACTIC PRESENTATION AND SET-BASED VERSION OF THE MAIN THEOREM. In this section we define a geometric theory $\mathbb{T}_M$ whose models in any Grothendieck topos $\mathcal{G}$ coincide with homomorphisms of $\mathbb{T}$-models $g : N \to \gamma^* M$, where γ is the unique geometric morphism $\mathcal{G} \to \mathbf{Set}$; this theory is classified by the $\mathbb{T}$-over-topos at M.

2.8. DEFINITION. *Let* $\mathbf{Set}_M$ *be the language with a sort* $S_{(\vec{a},\{\vec{x}^{\vec{A}}.\,\phi\})}$ *for each object* $(\vec{a},\{\vec{x}^{\vec{A}}.\,\phi\})$ *of the category of elements of* M *and a function symbol*

$$S_{(\vec{a_1},\{\vec{x_1}^{\vec{A}_1}.\,\phi_1\})} \xrightarrow{f_\theta^{\vec{a_1},\vec{a_2}}} S_{(\vec{a_2},\{\vec{x_2}^{\vec{A}_2}.\,\phi_2\})}$$

for each $[\theta(\vec{x}_1^{\vec{a_1}}, \vec{x}_2^{\vec{a_2}})]_{\mathbb{T}} : \{\vec{x}_1^{\vec{a_1}}.\,\phi_1\} \longrightarrow \{\vec{x}_2^{\vec{a_2}}.\,\phi_2\}$ *such that* $[\![\theta]\!]_M(\vec{a_1}) = \vec{a_2}$.

Let $\mathcal{L}_M^c$ *be the extension of the language* $\mathcal{L}$ *with a tuple of constant symbols* $c_{(\vec{a},\{x^{\vec{A}}.\,\phi\})}$ *for each* $(\vec{a},\{x^{\vec{A}}.\,\phi\}) \in \int M$. *There is a canonical* $\mathcal{L}_M^c$*-structure* M^c *extending* M*, obtained by interpreting each constant by the corresponding element of* M.

We can naturally interpret $\mathbf{Set}_M$ *in* $\mathcal{L}_M^c$ *by replacing, in the obvious way, each variable of sort* $S_{(\vec{a},\{\vec{x}^{\vec{A}}.\,\phi\})}$ *appearing freely in a formula with the corresponding tuple of constants* $c_{(\vec{a},\{x^A.\,\phi\})}$ *and each function symbol with the corresponding* $\mathbb{T}$*-provably functional formula. This yields, for each formula-in-context* $\{\vec{z}^{\vec{S}}.\,\psi\}$ *over* $\mathbf{Set}_M$ *a closed formula* $\psi^\sharp$.

Let $\mathbb{T}_M$ *be the theory over* $\mathbf{Set}_M$ *having as axioms all the geometric sequents*

$$(\phi \vdash_{\vec{x}^{\vec{S}}{}_{(\vec{a},\{\vec{x}^{\vec{A}}.\,\phi\})}} \psi)$$

such that the corresponding sequent

$$(\phi^\sharp \vdash \psi^\sharp)$$

is valid in M^c.

We are now endowed with a site presentation of the over-topos, as well as a geometric theory classified by it.

2.9. REMARK. Before stating the theorem, let us give the following brief remark on the set of interpretations of formulas and how they navigates along inverse images, for it will guide our intuition in the main proof. Given a Grothendieck topos $\mathcal{G}$ with global section functor γ, the model M is sent in $\mathcal{G}$ to a $\mathbb{T}$-model $\gamma^* M$ and for each $\{\vec{x}^A.\phi\}$, one has

$$[\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^* M} = \gamma^*([\![\vec{x}^{\vec{A}}.\,\phi]\!]_M) = \coprod_{\vec{a}:1\to[\![\vec{x}^{\vec{A}}.\,\phi]\!]_M} 1$$

so any global element $\vec{a} : 1 \to [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M$ is sent into a global element $\gamma^*(\vec{a}) : 1 \to [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^* M}$ in $\mathcal{G}$. Note that each element of $[\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^* M}$ is counted exactly once in this coproduct (since coproducts are disjoint in a topos).

2.10. THEOREM. *Let $\mathbb{T}$ be a geometric theory and M a model of $\mathbb{T}$ in* **Set**.

(i) The theory $\mathbb{T}_M$ axiomatizes the $\mathbb{T}$-model homomorphisms to (internalizations of) M; that is, for any Grothendieck topos $\mathcal{G}$ with global section functor $\gamma : \mathcal{G} \to \mathbf{Set}$, we have an equivalence of categories

$$\mathbb{T}_M[\mathcal{G}] \simeq \mathbb{T}[\mathcal{G}]/\gamma^* M.$$

(ii) There is a geometric functor

$$F_U : \mathcal{C}_{\mathbb{T}_M} \to \int M$$

classifying a $\mathbb{T}_M$-model U internal to the category $\int M$:

$$\begin{array}{ccc} S_{(\vec{a}, \{\vec{x}^{\vec{A}}.\,\phi\})} & \mapsto & (\vec{a}, \{\vec{x}^{\vec{A}}.\,\phi\}) \\ f_\theta^{\vec{a_1}, \vec{a_2}} & \mapsto & [\theta] : (\vec{a_1}, \{\vec{x_1}^{\vec{A_1}}.\,\phi_1\}) \to (\vec{a_2}, \{\vec{x_2}^{\vec{A_2}}.\,\phi_2\}) \end{array}$$

(iii) The sheaf topos $\mathbf{Set}[M] = \mathbf{Sh}(\int M, J_M^{\text{ant}})$ is the classifying topos of $\mathbb{T}_M$; that is, for any Grothendieck topos $\mathcal{G}$ with global section functor $\gamma : \mathcal{G} \to \mathbf{Set}$, we have an equivalence of categories

$$\mathbf{Geom}[\mathcal{G}, \mathbf{Set}[M]] \simeq \mathbb{T}[\mathcal{G}]/\gamma^* M.$$

(iv) The $\mathbb{T}_M$-model U in $\int M$ as in (ii) is sent by the canonical functor $\int M \to \mathbf{Sh}(\int M, J_M^{\text{ant}})$ to 'the' universal model of $\mathbb{T}_M$ inside its classifying topos.

(v) There is a full and faithful canonical functor

$$\begin{array}{ccc} \int M & \xrightarrow{V} & \mathcal{C}_{\mathbb{T}_M} \\ (\vec{a}, \{\vec{x}^{\vec{A}}.\,\phi\}) & \mapsto & \{x^{S_{(\vec{a}, \{\vec{x}^{\vec{A}}.\,\phi\})}}.\,\top\} \\ [\theta] & \mapsto & f_\theta^{\vec{a_1}, \vec{a_2}} \end{array}$$

which is a dense (cartesian but not geometric) morphism of sites $(\int M, J_M^{\text{ant}}) \to (\mathcal{C}_{\mathbb{T}_M}, J_{\mathbb{T}_M})$ such that $F_U \circ V = 1_{\int M}$; in particular, J_M^{ant} is the topology induced by $J_{\mathbb{T}_M}$ via V and, V being full and faithful, it is subcanonical.

PROOF. The proof proceeds as follows. We shall first establish, for any Grothendieck topos $\mathcal{G}$, an equivalence, natural in $\mathcal{G}$, between the category $\mathbb{T}[\mathcal{G}]/\gamma^* M$ of $\mathbb{T}$-model homomorphisms to $\gamma^* M$ and the category $\mathbf{Flat}_{J_M^{\text{ant}}}(\int M, \mathcal{G})$ of J_M^{ant}-continuous flat functors $\int M \to \mathcal{G}$. Next, we shall establish an equivalence between $\mathbb{T}[\mathcal{G}]/\gamma^* M$ and $\mathbb{T}_M[\mathcal{G}]$ (natural in $\mathcal{G}$), obtaining in the process an explicit axiomatization for the theory $\mathbb{T}_M$; moreover, we show that the resulting equivalence

$$\mathbf{Flat}_{J_M^{\text{ant}}}(\int M, \mathcal{G}) \simeq \mathbb{T}_M[\mathcal{G}] \simeq \mathbf{Cart}_{J_{\mathbb{T}_M}}(\mathcal{C}_{\mathbb{T}_M}, \mathcal{G})$$

is induced, on the one hand, by composition with the cartesian cover-preserving functor V, and on the other hand, by composition with the geometric functor F_U. From this we shall deduce that we have an equivalence of toposes

$$\mathbf{Sh}(\textstyle\int M, J_M^{\mathrm{ant}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}_M}, J_{\mathbb{T}_M})$$

whose two functors are induced by the morphisms of sites $V : (\int M, J_M^{\mathrm{ant}}) \to (\mathcal{C}_{\mathbb{T}_M}, J_{\mathbb{T}_M})$ and $F_U : (\mathcal{C}_{\mathbb{T}_M}, J_{\mathbb{T}_M}) \to (\int M, J_M^{\mathrm{ant}})$. This in turn implies (by Proposition 5.3 [5]) that the morphism of sites V is dense and that J_M^{ant} is the Grothendieck topology on $\int M$ induced by the syntactic topology $J_{\mathbb{T}_M}$. Since, as it is easily seen, $F_U \circ V \cong 1_{\int M}$, the functor V is full and faithful and therefore the subcanonicity of $J_{\mathbb{T}_M}$ entails that of J_M^{ant}. The above equivalence of toposes also implies, by the syntactic construction of 'the' universal model of a geometric theory inside its classifying topos, that the $\mathbb{T}_M$-model U is sent by the canonical functor $\int M \to \mathbf{Sh}(\int M, J_M^{\mathrm{ant}})$ to 'the' universal model for $\mathbb{T}_M$ inside this topos, thus completing the proof of the theorem.

For the first half of the theorem, let N be in $\mathbb{T}[\mathcal{G}]$ and

$$N \xrightarrow{g} \gamma^* M$$

be a $\mathcal{L}$-structure homomorphism in $\mathcal{G}$. By the categorical equivalence between models of a geometric theory and cartesian cover-preserving functors on its syntactic site, g is the same thing as a natural transformation

$$[\![\vec{x}^{\vec{A}}.\,\phi]\!]_N \xrightarrow{g_{\{\vec{x}^{\vec{A}}.\,\phi\}}} [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^*(M)}$$

(for $\{\vec{x}^{\vec{A}}.\,\phi\} \in \mathcal{C}_{\mathbb{T}}$).

Note that $[\![\vec{x}^{\vec{A}}.\,\phi]\!]_N$ is the disjoint union

$$[\![\vec{x}^{\vec{A}}.\,\phi]\!]_N = \coprod_{\vec{a} \in [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}$$

of the fibers:

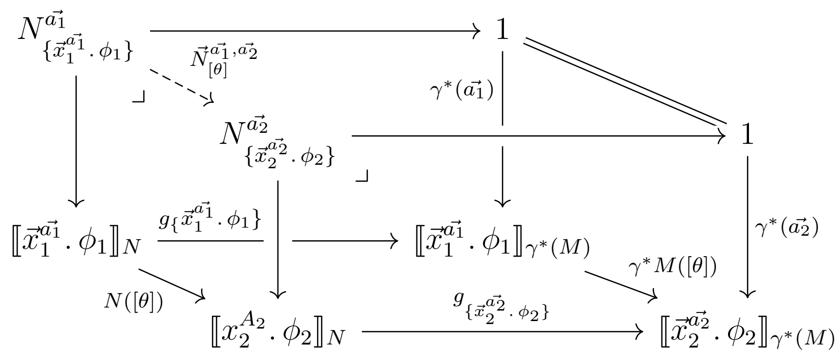
$$\begin{array}{ccc} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}} & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \gamma^*(\vec{a})} \\ [\![\vec{x}^{\vec{A}}.\,\phi]\!]_N & \xrightarrow{g_{\{\vec{x}^{\vec{A}}.\,\phi\}}} & [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^*(M)} \end{array}$$

Thus $g_{\{\vec{x}^{\vec{A}}.\,\phi\}}$ yields a family of objects $(N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}})_{(\{\vec{x}^{\vec{A}}.\,\phi\},\vec{a}) \in \int M}$ indexed by the category of elements of M.

The naturality of g implies that for any morphism in $\int M$ corresponding to a $[\theta]$ in $\mathcal{C}_{\mathbb{T}}$, one has a unique arrow

$$N^{\vec{a_1}}_{\{\vec{x}_1^{\vec{a_1}}.\,\phi_1\}} \overset{N^{\vec{a_1},\vec{a_2}}_{[\theta]}}{\to} N^{\vec{a_2}}_{\{\vec{x}_2^{\vec{a_2}}.\,\phi_2\}}$$

making the following diagram commute:



Let us show that g defines a J_M^{ant}-continuous flat functor:

$$\begin{array}{ccc} \int M & \xrightarrow{\overline{g}} & \mathcal{G} \\ (\vec{a}, \{\vec{x}^{\vec{A}}.\,\phi\}) & \mapsto & N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}} \\ [\theta] & \mapsto & N^{\vec{a_1},\vec{a_2}}_{[\theta]} \end{array}$$

We have to check that $\overline{g}$ preserves the terminal object and pullbacks, and that it sends J_M^{ant}-covering families to jointly epimorphic families. These are actually consequences of N being a $\mathbb{T}$-model:

– For the terminal object, observe that both $[\![[].\top]\!]_N$ and $[\![[].\top]\!]_{\gamma^* M}$ are equal to the terminal object 1, whence $N^*_{\{[].\top\}} = 1$ too.

– For pullbacks, one knows that

$$[\![\vec{x}_1^{\vec{a_1}}, \vec{x}_2^{\vec{a_2}}.\,\theta_1(\vec{x}_1) = \theta_2(\vec{x}_2)]\!]_N \simeq [\![\vec{x}_1^{\vec{a_1}}.\,\phi_1]\!]_N \times_{[\![\vec{x}^{\vec{A}}.\,\phi]\!]_N} [\![\vec{x}_2^{\vec{a_2}}.\,\phi_2]\!]_N$$

as N is a model; then for $\vec{a_1} \in [\![\vec{x}_1^{\vec{a_1}}.\,\phi_1]\!]_M$, $\vec{a_2} \in [\![\vec{x}_2^{\vec{a_2}}.\,\phi_2]\!]_M$ such that $[\![\theta_1]\!]_M(\vec{a_1}) = \vec{c} = [\![\theta_2]\!]_M(\vec{a_2})$, one has

$$\begin{aligned} & N^{\vec{a_1},\vec{a_2}}_{\{\vec{x}_1^{\vec{a_1}},\vec{x}_2^{\vec{a_2}}.\,\theta_1(\vec{x}_1)=\theta_2(\vec{x}_2)\}} = 1 \times^{\gamma^*(\vec{a_1}),\gamma^*(\vec{a_2})}_{[\![\vec{x}_1^{\vec{a_1}},\vec{x}_2^{\vec{a_2}}.\,\theta_1(\vec{x}_1)=\theta_2(\vec{x}_2)]\!]_{\gamma^* M}} [\![\vec{x}_1^{\vec{a_1}}, \vec{x}_2^{\vec{a_2}}.\,\theta_1(\vec{x}_1) = \theta_2(\vec{x}_2)]\!]_N \\ & \simeq 1 \times^{\gamma^*(\vec{a_1}),\gamma^*(\vec{a_2})}_{[\![\vec{x}_1^{\vec{a_1}},\vec{x}_2^{\vec{a_2}}.\,\theta_1(\vec{x}_1)=\theta_2(\vec{x}_2)]\!]_{\gamma^* M}} ([\![\vec{x}_1^{\vec{a_1}}.\,\phi_1]\!]_N \times_{[\![\vec{x}^{\vec{A}}.\,\phi]\!]_N} [\![\vec{x}_2^{\vec{a_2}}.\,\phi_2]\!]_N) \\ & \simeq (1 \times^{\gamma^*(\vec{a_1})}_{[\![\vec{x}_1^{\vec{a_1}}.\,\phi_1]\!]_{\gamma^* M}} [\![\vec{x}_1^{\vec{a_1}}.\,\phi_1]\!]_N) \times_{1 \times^{\gamma^*(\vec{c})}_{[\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^* M}} [\![\vec{x}^{\vec{A}}.\,\phi]\!]_N} (1 \times^{\gamma^*(\vec{a_2})}_{[\![\vec{x}_2^{\vec{a_2}}.\,\phi_2]\!]_{\gamma^* M}} [\![\vec{x}_2^{\vec{a_2}}.\,\phi_2]\!]_N) \\ & \simeq N^{\vec{a_1}}_{\{\vec{x}_1^{\vec{a_1}}.\,\phi_1\}} \times_{N^{\vec{c}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}} N^{\vec{a_2}}_{\{\vec{x}_2^{\vec{a_2}}.\,\phi_2\}}. \end{aligned}$$

– For J_M^{ant}-continuity: let

$$([\theta_i] : (\vec{b}, \{\vec{x}_i^{A_i}.\,\phi_i\}) \to (\vec{a}, \{x^{\vec{A}}.\,\phi\}))_{\vec{b} \in \langle [\![\theta_i]\!]_M \rangle^{-1}_{i \in I}(\vec{a})}$$

be a family in $\mathcal{B}_{\mathbb{T}}^{\mathrm{ant}}$. As $\gamma^* M$ and N are $\mathbb{T}$-models, they both send $([\theta_i])_{i\in I}$ to epimorphic families in $\mathcal{G}$. On the other hand, one can express the fiber of a along the coproduct map $\langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}$ as

$$\coprod_{\langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a})} 1 \simeq \coprod_{i\in I} [\![\theta_i]\!]_M^{-1}(\vec{a}) \to \coprod_{i\in I} [\![\vec{x}_i^{\vec{A}_i}.\, \phi_i]\!]_M.$$

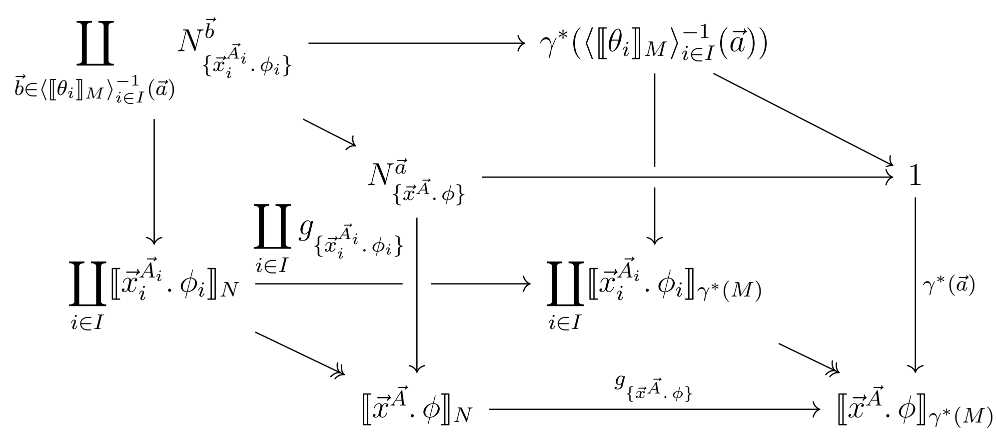
Moreover, this pullback is preserved by γ^*, which sends it to

$$\gamma^*(\langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a})) \simeq \coprod_{i\in I} \gamma^*([\![\theta_i]\!]_M^{-1}(\vec{a})) \simeq \coprod_{\langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a})} \gamma^*(1) \simeq \coprod_{\langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a})} 1$$

which is the $\langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a})$-indexed coproduct of the terminal object of $\mathcal{G}$. Then by the stability of coproducts under pullbacks one has

$$\begin{aligned}\gamma^*(\langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a})) \times_{[\![\vec{x}^{\vec{A}}.\, \phi]\!]_{\gamma^* M}} [\![\vec{x}^{\vec{A}}.\, \phi]\!]_N &\simeq \Big(\coprod_{\vec{b}\in \langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a})} 1\Big) \times_{[\![\vec{x}^{\vec{A}}.\, \phi]\!]_{\gamma^* M}} [\![\vec{x}^{\vec{A}}.\, \phi]\!]_N \\ &= \coprod_{\vec{b}\in \langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a})} N^{\vec{b}}_{\{\vec{x}_i^{\vec{A}_i}.\, \phi_i\}}.\end{aligned}$$

So in the diagram



the front, right and back squares are pullbacks, whence the left square is a pullback too: but this forces the upper left arrow

$$\coprod_{\vec{b}\in \langle [\![\theta_i]\!]_M \rangle_{i\in I}^{-1}(\vec{a})} N^{\vec{b}}_{\{\vec{x}_i^{\vec{A}_i}.\, \phi_i\}} \to N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\, \phi\}}$$

to be an epimorphism by the stability of epimorphisms under pullback in $\mathcal{G}$.

The data of the $N^{a}_{\{\vec{x}^{A}.\phi\}}$'s with their transitions morphisms define a $\mathbf{Set}_M$-structure S_g:

$$\begin{array}{lcc} S_{(\vec{a},\{\vec{x}^{\vec{A}}.\,\phi\})} & \mapsto & N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}} \\ f_{\theta}^{\vec{a_1},\vec{a_2}} & \mapsto & N^{\vec{a_1},\vec{a_2}}_{[\theta]} : N^{\vec{a_1}}_{\{\vec{x}_1^{\vec{a_1}}.\,\phi_1\}} \to N^{\vec{a_2}}_{\{\vec{x}_2^{\vec{a_2}}.\,\phi_2\}} \end{array}$$

This structure is actually a $\mathbb{T}_M$-model. Indeed, this follows at once from the fact that the interpretation in S_g of any geometric formula ψ over the signature S_M can be expressed as a pullback of the interpretation of the corresponding formula $\psi^\sharp$ in the $\mathcal{L}^c_M$-structure M^c, as in the following diagram:

$$\begin{array}{ccc}
[[z_1^{S_{(\vec{a_1},\{\vec{x_1}^{\vec{A_1}}.\,\phi_1\})}},\ldots,z_n^{S_{(\vec{a_n},\{\vec{x_n}^{\vec{A_n}}.\,\phi_n\})}}.\,\psi]]_{S_g} & \longrightarrow & [[[].\psi^\sharp]]_{M^c} \\
\downarrow & \lrcorner & \downarrow \\
[[z_1^{S_{(\vec{a_1},\{\vec{x_1}^{\vec{A_1}}.\,\phi_1\})}},\ldots,z_n^{S_{(\vec{a_n},\{\vec{x_n}^{\vec{A_n}}.\,\phi_n\})}}.\,\top]]_{S_g} & \longrightarrow & 1 \\
\downarrow & \lrcorner & \downarrow{\scriptstyle <\vec{a_1},\ldots,\vec{a_n}>} \\
[[\vec{x_1}^{\vec{A_1}},\ldots,\vec{x_n}^{\vec{A_n}}.\,\top]]_N & \xrightarrow{g\vec{A_1}\times\ldots\times g\vec{A_n}} & [[\vec{x_1}^{\vec{A_1}},\ldots,\vec{x_n}^{\vec{A_n}}.\,\top]]_{\gamma^*(M)}
\end{array}$$

This analysis shows that a simple axiomatization for the theory $\mathbb{T}_M$ may be obtained by phrasing in logical terms the property that the functor $V : \int M \to \mathcal{C}_{\mathbb{T}_M}$ in the statement of the theorem be cartesian and cover-preserving. Recalling the well-known characterizations of pullbacks and terminal objects in the internal language of a topos, this leads to the following axioms for $\mathbb{T}_M$:

$$\big(\top \vdash_{[]} (\exists x^{S_{(\vec{a},\{[].\top\})}})\top\big);$$

$$\big(\top \vdash_{x^{S_{(\vec{a},\{[].\top\})}},x'^{S_{(\vec{a},\{[].\top\})}}} (x = x')\big);$$

$$\big(\top \vdash_{z^{S_{(\vec{a_1},\vec{a_2},\{\vec{x_1'}^{\vec{a_1}},\vec{x_2'}^{\vec{a_2}}.\,\theta_1(\vec{x_1'})=\theta_2(\vec{x_2'})\})}}} f_{\theta_1}^{\vec{a_1},\vec{c}}(f^{(\vec{a_1},\vec{a_2}),\vec{a_1}}_{\vec{x_1}=\vec{x_1'}}(z)) = f_{\theta_1}^{\vec{a_2},\vec{c}}(f^{(\vec{a_1},\vec{a_2}),\vec{a_2}}_{\vec{x_2}=\vec{x_2'}}(z))\big),$$

$$\big(f^{(\vec{a_1},\vec{a_2}),\vec{a_1}}_{\vec{x_1}=\vec{x_1'}}(z) = f^{(\vec{a_1},\vec{a_2}),\vec{a_1}}_{\vec{x_1}=\vec{x_1'}}(z')) \wedge (f^{(\vec{a_1},\vec{a_2}),\vec{a_2}}_{\vec{x_2}=\vec{x_2'}}(z) = f^{(\vec{a_1},\vec{a_2}),\vec{a_2}}_{\vec{x_2}=\vec{x_2'}}(z')) \vdash_{z,z'} z = z'\big),$$

$$\big(f_{\theta_1}^{\vec{a_1},\vec{c}}(y_1) = f_{\theta_2}^{\vec{a_2},\vec{c}}(y_2) \vdash_{y_1^{S_{(\vec{a_1},\{\vec{x_1}^{\vec{A_1}}.\,\phi_1\})}},y_2^{S_{(\vec{a_2},\{\vec{x_2}^{\vec{A_2}}.\,\phi_2\})}}} (\exists z)((f^{(\vec{a_1},\vec{a_2}),\vec{a_1}}_{\vec{x_1}=\vec{x_1'}}(z) = y_1) \wedge (f^{(\vec{a_1},\vec{a_2}),\vec{a_2}}_{\vec{x_2}=\vec{x_2'}}(z) = y_2))\big)$$

for any $\vec{a_1} \in [\![\vec{x}_1^{\vec{a_1}}.\,\phi_1]\!]_M$, $\vec{a_2} \in [\![\vec{x}_2^{\vec{a_2}}.\,\phi_2]\!]_M$ such that $[\![\theta_1]\!]_M(\vec{a_1}) = \vec{c} = [\![\theta_2]\!]_M(\vec{a_2})$;

$$\big(\top \vdash_{y^{S_{(\vec{a},\{\vec{x}^{\vec{A}}.\,\phi\})}}} \bigvee_{i\in I,\vec{b_i}\in[\![\theta_i]\!]_M^{-1}(\vec{a})} (\exists z^{S_{(\vec{b_i},\{\vec{x}_i^{\vec{A}_i}.\,\phi_i\})}})(f_{\theta_i}^{\vec{b_i},\vec{a}}(z) = y)\big)$$

for any family $\{[\theta_i] : \{\vec{x}_i^{\vec{A}_i}.\,\phi_i\} \to \{\vec{x}^{\vec{A}}.\,\phi\} \mid i \in I\} \in \mathcal{B}_{\mathbb{T}}(\{\vec{x}^{\vec{A}}.\,\phi\})$ and any $\vec{a} \in [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M$.

The first two axioms express the fact that $N^{*}_{\{[].\top\}} = 1$, the following three express the fact that for any $\vec{a_1} \in [\![\vec{x}_1^{\vec{a_1}}.\,\phi_1]\!]_M$, $\vec{a_2} \in [\![\vec{x}_2^{\vec{a_2}}.\,\phi_2]\!]_M$ such that $[\![\theta_1]\!]_M(\vec{a_1}) = \vec{c} = [\![\theta_2]\!]_M(\vec{a_2})$, we have a pullback square

$$\begin{array}{ccc} N^{\vec{a_1},\vec{a_2}}_{\{\vec{x_1'}^{\vec{a_1}},\vec{x_2'}^{\vec{a_2}}.\,\theta_1(\vec{x_1'})=\theta_2(\vec{x_2'})\}} & \xrightarrow{N^{(\vec{a_1},\vec{a_2}),\vec{a_1}}_{\vec{x_1}=\vec{x_1'}}} & N^{\vec{a_1}}_{\{\vec{x}_1^{\vec{a_1}}.\,\phi_1\}} \\ \downarrow{\scriptstyle N^{(\vec{a_1},\vec{a_2}),\vec{a_2}}_{\vec{x_2}=\vec{x_2'}}} & & \downarrow{\scriptstyle N^{\vec{a_1},\vec{c}}_{\theta_1}} \\ N^{\vec{a_2}}_{\{\vec{x}_2^{\vec{a_2}}.\,\phi_2\}} & \xrightarrow{N^{\vec{a_2},\vec{c}}_{\theta_2}} & N^{\vec{c}}_{\{\vec{x}^{\vec{A}}.\,\phi\}} \end{array}$$

and the last one corresponds to the property of V being cover-preserving.

It is immediate to see that the $J_{\mathbb{T}_M}$-continuous cartesian functor $F_{S_g} : \mathcal{C}_{\mathbb{T}_M} \to \mathcal{G}$ corresponding to the $\mathbb{T}_M$-model S_g is given by $\overline{g} \circ F_U$ and that, conversely, the flat functor $\overline{g}$ can be recovered from the $\mathbb{T}_M$-model S_g as the composite $F_{S_g} \circ V$.

We now prove the second half of the theorem: given families

$$\big(N^{\vec{a}}_{\{\vec{x}_1^{\vec{A}}.\,\phi\}}\big)_{(\vec{a},\{\vec{x}^{\vec{A}}.\,\phi\})\in\int M}$$

and

$$\big(f^{\vec{a_1},\vec{a_2}}_{\theta} : N^{\vec{a_1}}_{\{\vec{x}_1^{\vec{a_1}}.\,\phi_1\}} \to N^{\vec{a_2}}_{\{\vec{x}_2^{\vec{a_2}}.\,\phi_2\}}\big)_{[\theta]:\{\vec{x_1}.\,\phi_1\}\to\{\vec{x_1}.\,\phi_1\}\in\mathcal{C}_{\mathbb{T}}\text{ with }[[\theta]]_M(\vec{a_1})=\vec{a_2}}$$

respectively of objects and arrows in $\mathcal{G}$ defining a J_M^{ant}-continuous flat (equivalently, cartesian) functor $G : \int M \to \mathcal{G}$, or, equivalently a $\mathbb{T}_M$-model in $\mathcal{G}$, we can associate with it a $\mathbb{T}$-model N in $\mathcal{G}$ and a homomorphism of $\mathbb{T}$-models $g : N \to \gamma^*(M)$.

First, for each each $\{\vec{x}^A.\phi\}$ and $\vec{a} \in [\![\vec{x}^A.\,\phi]\!]_M$, we set $g^{\vec{a}}_{\{\vec{x}^A.\phi\}}$ equal to the composite arrow

$$\begin{array}{ccc} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}} & \xrightarrow{\;!\;} & 1 \\ & {\scriptstyle g^{\vec{a}}_{\{\vec{x}^A.\phi\}}}\searrow & \downarrow{\scriptstyle \gamma^*(\vec{a})} \\ & & [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^*(M)} \end{array}$$

Then we define, for each object $\{\vec{x}^A.\phi\}$ of $\mathcal{C}_{\mathbb{T}}$, $g_{\{\vec{x}^{\vec{A}}.\,\phi\}}$ as the arrow determined by the universal property of the coproduct as in the following diagrams (where the vertical arrows are the canonical coproduct inclusions):

$$\begin{array}{ccc} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}} & & \\ \downarrow & \searrow{\scriptstyle g^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}} & \\ \coprod_{\vec{a}\in[\![\vec{x}^A.\phi]\!]_M} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}} & \overset{}{\dashrightarrow}_{g_{\{\vec{x}^{\vec{A}}.\,\phi\}}} & [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^*(M)} \end{array}$$

We need to ensure that:

– we have a $J_{\mathbb{T}}$-continuous cartesian functor:

$$\begin{array}{rcl} \mathcal{C}_{\mathbb{T}} & \xrightarrow{F_N} & \mathcal{G} \\ \{\vec{x}^{\vec{A}}.\,\phi\} & \mapsto & N_{\{\vec{x}^{\vec{A}}.\,\phi\}} = \coprod\limits_{a\in[\![x^{\vec{A}}.\,\phi]\!]_M} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}} \\ \theta : \{\vec{x_1}^{\vec{A_1}}.\,\phi_1\} \to \{\vec{x_2}^{\vec{A_2}}.\,\phi_2\} & \mapsto & \coprod\limits_{\vec{a_1}\in[\![x_1^{\vec{A_1}}.\,\phi]\!]_M} f^{\vec{a_1},[[\theta]]_M(\vec{a_1})} : \coprod\limits_{\vec{a_1}\in[\![x_1^{\vec{A_1}}.\,\phi]\!]_M} N^{\vec{a_1}}_{\{\vec{x_1}^{\vec{A_1}}.\,\phi_1\}} \to \coprod\limits_{\vec{a_2}\in[\![x_2^{\vec{A_1}}.\,\phi]\!]_M} N^{\vec{a_2}}_{\{\vec{x_2}^{\vec{A_2}}.\,\phi_2\}} \end{array}$$

and hence a $\mathbb{T}$-model N in $\mathcal{G}$;

– we have a natural transformation

$$g = (g_{\{\vec{x}^{\vec{A}}.\,\phi\}})_{\{\vec{x}^{\vec{A}}.\,\phi\}\in\mathcal{C}_{\mathbb{T}}} : N \to \gamma^* M.$$

This amounts to the following conditions:

– If $\{[].\top\}$ is the terminal object of the syntactic site, then its interpretation in M has exactly one global element $1 \to [\![[],\top]\!]_M = 1$ so requiring F_N to preserve the terminal object is equivalent to demanding $N^*_{\{[].\top\}} = 1$, which is ensured by the fact that the functor G preserves the terminal object by our hypotheses.

– Pullbacks in $\mathcal{C}_{\mathbb{T}}$

$$\begin{array}{ccc} \{\vec{x_1}^{\vec{A_1}}, \vec{x_2}^{\vec{A_2}}.\,\theta_1(\vec{x_1}) = \theta_2(\vec{x_2})\} & \longrightarrow & \{\vec{x_1}^{\vec{A_1}}.\,\phi_1\} \\ \big\downarrow & \lrcorner & \big\downarrow [\theta_1] \\ \{\vec{x_2}^{\vec{A_2}}.\,\phi_2\} & \xrightarrow{\quad [\theta_2] \quad} & \{\vec{x}^{\vec{A}}.\,\phi\} \end{array}$$

are sent by F_M to pullbacks in **Set**

$$\begin{array}{ccc} [\![\vec{x_1}^{\vec{A_1}}, \vec{x_2}^{\vec{A_2}}.\,\theta_1(\vec{x_1}) = \theta_2(\vec{x_2})]\!]_M & \longrightarrow & [\![\vec{x_1}^{\vec{A_1}}.\,\phi_1]\!]_M \\ \big\downarrow & \lrcorner & \big\downarrow [\![\theta_1]\!]_M \\ [\![\vec{x_2}^{\vec{A_2}}.\,\phi_2]\!]_M & \xrightarrow{\quad [\![\theta_2]\!]_M \quad} & [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M, \end{array}$$

where elements of $[\![\vec{x_1}^{\vec{A_1}}, \vec{x_2}^{\vec{A_2}}.\,\theta_1(\vec{x_1}) = \theta_2(\vec{x_2})]\!]_M$ are pairs $(\vec{a_1}, \vec{a_2}) : 1 \to [\![\vec{x_1}^{\vec{A_1}}.\,\phi_1]\!]_M \times [\![\vec{x_2}^{\vec{A_2}}.\,\phi_2]\!]_M$ such that $[\![\theta_1]\!]_M(\vec{a_1}) = \vec{a} = [\![\theta_2]\!]_M(\vec{a_2})$. Now, if all the squares of the form

$$\begin{array}{ccc} N^{\vec{a_1},\vec{a_2}}_{\{\vec{x_1}^{\vec{A_1}}, \vec{x_2}^{\vec{A_2}}.\,\theta_1(\vec{x_1})=\theta_2(\vec{x_2})\}} & \longrightarrow & N^{\vec{a_1}}_{\{\vec{x_1}^{\vec{A_1}}.\,\phi_1\}} \\ \big\downarrow & & \big\downarrow N^{\vec{a_1},\vec{a}}_{[\theta_1]} \\ N^{\vec{a_2}}_{\{\vec{x_2}^{\vec{A_2}}.\,\phi_2\}} & \xrightarrow{\quad N^{\vec{a_2},\vec{a}}_{[\theta_2]} \quad} & N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}} \end{array}$$

are pullbacks, which is the case since the functor G preserves pullbacks by our hypotheses, then, by the stability of coproducts along pullbacks, we have

$$\begin{aligned}
N_{\{\vec{x_1}^{\vec{A_1}},\vec{x_2}^{\vec{A_2}}.\,\theta_1(\vec{x_1})=\theta_2(\vec{x_2})\}} &= \coprod_{[\![\vec{x_1}^{\vec{A_1}},\vec{x_2}^{\vec{A_2}}.\,\theta_1(\vec{x_1})=\theta_2(\vec{x_2})]\!]_M} N^{\vec{a_1},\vec{a_2}}_{\{\vec{x_1}^{\vec{A_1}},\vec{x_2}^{\vec{A_2}}.\,\theta_1(\vec{x_1})=\theta_2(\vec{x_2})\}} \\
&= \coprod_{[\![\vec{x_1}^{\vec{A_1}},\vec{x_2}^{\vec{A_2}}.\,\theta_1(\vec{x_1})=\theta_2(\vec{x_2})]\!]_M} N^{\vec{a_1}}_{\{\vec{x_1}^{\vec{A_1}}.\,\phi_1\}} \times_{N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}} N^{\vec{a_2}}_{\{\vec{x_2}^{\vec{A_2}}.\,\phi_2\}} \\
&= \coprod_{[\![\vec{x_1}^{\vec{A_1}}.\,\phi_1]\!]_M} N^{\vec{a_1}}_{\{\vec{x_1}^{\vec{A_1}}.\,\phi_1\}} \times_{\coprod_{\vec{a}\in[\![\vec{x}^{\vec{A}}.\,\phi]\!]_M} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}} \coprod_{[\![\vec{x_2}^{\vec{A_2}}.\,\phi_2]\!]_M} N^{\vec{a_2}}_{\{\vec{x_2}^{\vec{A_2}}.\,\phi_2\}} \\
&= N_{\{\vec{x_1}^{\vec{A_1}}.\,\phi_1\}} \times_{N_{\{\vec{x}^{\vec{A}}.\,\phi\}}} N_{\{\vec{x_2}^{\vec{A_2}}.\,\phi_2\}}.
\end{aligned}$$

– $J_{\mathbb{T}}$-continuity: given a small $J_{\mathbb{T}}$-cover $([\theta_i] : \{\vec{x_i}^{\vec{A_i}}.\,\phi_i\} \to \{\vec{x}^{\vec{A}}.\,\phi\})_{i\in I}$, since M is a $\mathbb{T}$-model, the corresponding functor $F_M : \mathcal{C}_{\mathbb{T}} \to \mathcal{G}$ sends it to a jointly epimorphic family $([\![\theta_i]\!]_M : [\![\vec{x_i}^{\vec{A_i}}.\,\phi_i]\!]_M \to [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M)_{i\in I}$; so each $\vec{a} \in [\![x^A.\,\phi]\!]_M$ has an antecedent $\vec{b} \in [\![\vec{x_i}^{\vec{A_i}}.\,\phi_i]\!]_M$ for some $i \in I$. Therefore for each $\vec{a} \in [\![\vec{x}.\,\phi]\!]$ the cocone of fibers

$$\big(N^{\vec{b}}_{\{\vec{x_i}^{\vec{A_i}}.\,\phi_i\}} \xrightarrow{N^{\vec{b},\vec{a}}_{[\theta_i]}} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}\big)_{i\in I,\ [\![\theta_i]\!]_M(\vec{b})=\vec{a}}$$

in each of its antecedents is jointly epimorphic in $\mathcal{G}$ if and only if the coproduct of fibers

$$\big(N_{\{\vec{x_i}^{\vec{A_i}}.\,\phi_i\}} \xrightarrow{N_{[\theta_i]}} N_{\{\vec{x}^{\vec{A}}.\,\phi\}}\big)_{i\in I}$$

is. But this exactly amounts to requiring the following functor to be J^{ant}_M-continuous:

$$\begin{array}{ccc}
\int M & \xrightarrow{G} & \mathcal{G} \\
(\vec{a}, \{\vec{x}^{\vec{A}}.\,\phi\}) & \mapsto & N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}} \\
[\theta] & \mapsto & N^{\vec{a_1},\vec{a_2}}_{[\theta]}
\end{array}$$

The naturality of g follows immediately from the definition of the functor F_N on the arrows of $\mathcal{C}_{\mathbb{T}}$.

To conclude our proof of the categorical equivalence between flat J^{ant}_M-continuous functors on $\int M$ and $\mathbb{T}$-model homomorphisms to $\gamma^*(M)$, we have to check that the two functors defined above are mutually quasi-inverse. For the construction starting from a homomorphism of $\mathcal{L}$-structures between $\mathbb{T}$-models $g : N \to \gamma^* M$, observe that as the codomain of g decomposes as the coproduct of all elements $[\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^* M} = \coprod_{\vec{a}:1\to[\![\vec{x}^{\vec{A}}.\,\phi]\!]_M} 1$, by the stability of coproducts under pullbacks one has

$$N_{\{\vec{x}^{\vec{A}}.\,\phi\}} \simeq \coprod_{\vec{a}\in[\![\vec{x}^{A}.\phi]\!]_M} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}.$$

For the converse process, if one starts with a J_M^{ant}-continuous flat functor $N_{(-)} : \int M \to \mathcal{G}$ and defines, for each $\{\vec{x}^{\vec{A}}.\,\phi\}$, $N_{\{\vec{x}^{\vec{A}}.\,\phi\}}$ as the above coproduct, then pulling it back along $\gamma^*(\vec{a}) : 1 \to [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^* M}$, one has (again, by the stability of coproducts under pullback)

$$\gamma^*(\vec{a})^*\Big(\coprod_{\vec{b}\in[\![\vec{x}^{A}.\phi]\!]_M} N^{\vec{b}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}\Big) \simeq \coprod_{\vec{b}\in[\![\vec{x}^{A}.\phi]\!]_M} \gamma^*(\vec{a})^*(N^{\vec{b}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}).$$

For any element $\vec{b}$ of $[\![\vec{x}^{\vec{A}}.\,\phi]\!]_M$, we have the following pullback squares:

$$\begin{array}{ccccc} \gamma^*(\vec{a})^*(N^{\vec{b}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}) & \longrightarrow & 1 \times^{\gamma^*(\vec{a}),\gamma^*(\vec{b})}_{[\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^* M}} 1 & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle \gamma^*(\vec{a})} \\ N^{\vec{b}}_{\{\vec{x}^{\vec{A}}.\,\phi\}} & \longrightarrow & 1 & \xrightarrow{\gamma^*(\vec{b})} & [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^* M} \end{array}$$

Now, there are two possible values for the pullback on the right-hand side:

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & \lrcorner \ \text{whenever } \vec{a_1}\neq\vec{a_2} & \downarrow{\scriptstyle \gamma^*(\vec{a_2})} \\ 1 & \xrightarrow{\gamma^*(\vec{a_1})} & [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^* M} \end{array} \qquad\qquad \begin{array}{ccc} 1 & = & 1 \\ \| & \lrcorner \ \text{whenever } \vec{a_1}=\vec{a_2} & \downarrow{\scriptstyle \gamma^*(\vec{a_2})} \\ 1 & \xrightarrow{\gamma^*(\vec{a_1})} & [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{\gamma^* M} \end{array}$$

So the middle pullback is the initial object whenever $\vec{a} \neq \vec{b}$, whence $\gamma^*(\vec{a})^*(N^{\vec{b}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}) \cong 0$; on the other hand, $\gamma^*(\vec{a})^*(N^{\vec{b}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}) \cong N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\,\phi\}}$ whenever $\vec{a} = \vec{b}$. This clearly implies our thesis. ■

2.11. REMARK. We have made use of specific properties of **Set** at several steps of the above constructions and proofs:

- In the definition of the antecedent topology, we only had to consider global elements of the sets $[\![\vec{x}^{\vec{A}}.\,\phi]\!]_M$. This is because 1 is a generator of **Set**, so that generalized elements would just be coproducts of global elements.
- As a consequence, the very notion of antecedent element is simplified. As we shall see in section 4, considering antecedents of generalized elements gives rise to complications when considering jointly epimorphic families, as antecedents may be indexed by objects of the topos other than the domain of the generalized elements whose antecedents we seek. In this case, we just had to consider the global elements of the fiber of a global element; in other words, the antecedent topology exists already in the comma category $(1 \downarrow F_M)$; in the general case, it is scattered on the fibers of a comma $(\mathrm{y} \downarrow F_M)$, for y the Yoneda embedding of a small, cartesian subcanonical site for the given topos.
- We also used that 1 is indecomposable to retrieve global elements of the fibers of a jointly surjective family. This is not anymore a valid argument in an arbitrary Grothendieck topos.

3. An interlude on stacks

We turn in the next section to the construction of the over-topos at a model in the general case, where the valuation topos in which the given model lives is an arbitrary Grothendieck topos. We chose to treat this general case separately as it requires Giraud's theory of the classifying topos of a stack, while the set-valuated case requires more conventional tools. In this section we present a number of results on stacks that we shall need in our analysis.

3.1. A canonical comma construction for geometric morphisms. While in the set-valued case, it was sufficient to see the over-topos $\mathbf{Set}[M]$ as a mere topos as its underlying morphism to $\mathbf{Set}$ is trivial, in the general case, we have to describe the over-topos as a geometric morphism $u_M : \mathcal{E}[M] \to \mathcal{E}$, which is induced from a certain *comorphism of site*. It is worth giving some prerequisites on this notion:

3.2. Definition. *Let $p : \mathcal{M} \to \mathcal{C}$ be a functor, and K a subcanonical Grothendieck topology on $\mathcal{M}$, J a subcanonical Grothendieck topology on $\mathcal{C}$. Then p is said to be a* comorphism of site *if for any object a of $\mathcal{M}$ and any J-cover $(u_i : c_i \to p(a))_{i\in I}$, there is a K-cover $(f_j : a_j \to a)_{j\in J}$ in $\mathcal{M}$ such that $(p(f_j))_{j\in J}$ refines $(u_i)_{i\in I}$ (that is any $p(f_j)$ factorizes through some u_i).*

3.3. Proposition. *Any comorphism of site $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$, with K, J subcanonical, induces a geometric morphism $(p^*, p_*) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$, where p^* is precomposition with p while $p_* = Ran_{\mathrm{y}}\mathrm{y}\, p$:*

$$\begin{array}{ccc} (\mathcal{D}, K) & \xrightarrow{\;p\;} & (\mathcal{C}, J) \\ \big\downarrow & & \big\downarrow \\ \mathbf{Sh}(\mathcal{D}, K) & \xrightarrow{\;p_*\;} & \mathbf{Sh}(\mathcal{C}, J) \end{array}$$

Given a pseudofunctor $\mathbb{I} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CAT}$, we denote by $\pi_{\mathbb{I}}$ the canonical projection functor $\mathcal{G}(\mathbb{I}) \to \mathcal{C}$, where $\mathcal{G}(\mathbb{I})$ is the category obtained from $\mathbb{I}$ by applying the Grothendieck construction. Given a Grothendieck topology J on $\mathcal{C}$, there is a smallest topology $J_{\mathbb{I}}^{\mathrm{Gir}}$ on $\mathcal{G}(\mathbb{I})$ which makes $\pi_{\mathbb{I}}$ a comorphism of sites to $(\mathcal{C}, J)$; this topology, which we call the *Giraud topology*, has as covering sieves those which contain cartesian lifts of J-covering families in $\mathcal{C}$.

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. There are three pseudofunctors naturally associated with it:

– We define
$$t_f : \mathcal{F}^{\mathrm{op}} \to \mathrm{CAT}$$
as the functor sending any object F of $\mathcal{F}$ to the category $(F \downarrow f^*)$ and any arrow $v : F \to F'$ to the functor $(F' \downarrow f^*) \to (F \downarrow f^*)$ induced by composition with v.

– We define

$$r_f : \mathcal{E}^{\mathrm{op}} \to \mathrm{CAT}$$

as the pseudofunctor sending an object E of $\mathcal{E}$ to $\mathcal{F}/f^*(E)$ and an arrow $u : E \to E'$ to the pullback functor $(f^*(u))^* : \mathcal{F}/f^*(E') \to \mathcal{F}/f^*(E)$.

– We define

$$s_f : \mathcal{E} \to \mathrm{CAT}$$

as the functor sending an object E of $\mathcal{E}$ to $\mathcal{F}/f^*(E)$ and an arrow $u : E \to E'$ to the functor $\Sigma_{f^*(u)} : \mathcal{F}/f^*(E) \to \mathcal{F}/f^*(E')$ induced by composition with $f^*(u)$ (which is left adjoint to the pullback functor $(f^*(u))^* : \mathcal{F}/f^*(E') \to \mathcal{F}/f^*(E)$).

By the adjunction between $\Sigma_{f^*(u)}$ and $(f^*(u))^*$ (for any arrow u in $\mathcal{E}$), the fibration to $\mathcal{E}$ associated with r_f coincides with the opfibration associated with s_f; in particular, this functor is both a fibration and an opfibration. This fibration is a stack for the canonical topology on $\mathcal{E}$ by the results in [8]. Note that the domain of this fibration also admits a canonical functor to $\mathcal{F}$, which is precisely the fibration associated with t_f.

3.4. Proposition. *t_f is a (split) stack for the canonical topology on $\mathcal{F}$.*

Proof. Let $\{f_i : F_i \to F \mid i \in I\}$ be an epimorphic family in $\mathcal{F}$ and

$$\{A_i = (F_i, E_i, \alpha_i : F_i \to f^*(E_i)) \mid i \in I\}, \{f_{ij} : {\pi_i}^*(A_i) \xrightarrow{\sim} {\pi'_j}^*(A_j) \mid (i,j) \in I \times I\},$$

where π_i and π'_j are defined, for each $(i,j) \in I \times I$, by the pullback square

$$\begin{array}{ccc} F_i \times_F F_j & \xrightarrow{\pi_i} & F_i \\ {\scriptstyle \pi'_j}\downarrow & & \downarrow{\scriptstyle f_i} \\ F_j & \xrightarrow[f_j]{} & F, \end{array}$$

be a collection of descent data indexed by it. For any $(i,j) \in I$,

$$\pi_i^*(A_i) = (F_i \times_F F_j, E_i, \alpha_i \circ \pi_i : F_i \times_F F_j \to f^*(E_i)),$$

$${\pi'_j}^*(A_j) = (F_i \times_F F_j, E_j, \alpha_j \circ \pi'_j : F_i \times_F F_j \to f^*(E_j)).$$

So the isomorphism f_{ij} actually identifies with an isomorphism

$$f_{ij} : E_i \xrightarrow{\sim} E_j$$

in $\mathcal{E}$ such that the following triangle commutes:

$$\begin{array}{ccc} F_i \times_F F_j & \xrightarrow{\alpha_i \circ \pi_i} & f^*(E_i) \\ & {\scriptstyle \alpha_j \circ \pi'_j}\searrow & \downarrow{\scriptstyle f^*(f_{ij})} \\ & & f^*(E_j) \end{array}$$

Note that the fact that t_f is split easily implies that

– for any $i \in I$, $f_{ii} = 1_{A_i}$,

– for any $i, j, k \in I$, $f_{jk} \circ f_{ij} = f_{ik}$.

Let E be the colimit of the diagram in $\mathcal{E}$ having the E_i's as vertices and the f_{ij}'s as edges between them (with the above relations); the above identities actually imply that, for each $i \in I$, $E \cong E_i$. The commutativity of the above triangles ensures that we have a cocone from the F_i's to $f^*(E)$ whose legs are given by the composites of the arrows α_i with the image under f^* of the corresponding canonical colimit arrow $E_i \to E$. Since the representable $\mathrm{Hom}_{\mathcal{F}}(-, f^*(E))$ is a sheaf for the canonical topology on $\mathcal{F}$, it follows that there is a unique arrow $F \to f^*(E)$ which restricts on the F_i's to the legs of this cocone, and which therefore provides the required 'amalgamation' for our descent data. The uniqueness of the amalgamation (up to isomorphism) also follows at once from the sheaf property. ∎

3.5. Corollary. *Let M be a model of a geometric theory $\mathbb{T}$ in a Grothendieck topos $\mathcal{E}$. Then the functor*

$$t_{f_M} : \mathcal{E}^{\mathrm{op}} \to \mathrm{CAT}$$

associated with the geometric morphism $f_M : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ corresponding to M via the universal property of the classifying topos for $\mathbb{T}$ is a stack for the canonical topology on $\mathcal{E}$.

In particular, if $(\mathcal{C}, J)$ is a site of definition for $\mathcal{E}$ then the functor

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{op}} & \xrightarrow{\mathbb{M}} & \mathrm{CAT} \\ c & \longmapsto & (c \downarrow F_M), \\ c_1 \xrightarrow{u} c_2 & \longmapsto & (c_2 \downarrow F_M) \xrightarrow{u^*} (c_1 \downarrow F_M) \end{array}$$

where $u^* : (c_2 \downarrow F_M) \to (c_1 \downarrow F_M)$ is the pre-composition functor sending any generalized element $a : c_2 \to [\![\vec{x}^{\vec{A}}. \phi]\!]_M$ to $a \circ u : c_1 \to [\![\vec{x}^{\vec{A}}. \phi]\!]_M$ is a stack for the topology J, where F_M is the functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{E}$ taking the interpretations of formulae in the model M.

3.6. The lifted topology. The category $(1_{\mathcal{F}} \downarrow f^*) = \mathcal{G}(r_f) = \mathcal{G}(t_f)$ has as objects the triplets $(F, E, \alpha : F \to f^*(E))$ (where $E \in \mathcal{E}$, $F \in \mathcal{F}$ and α is an arrow in $\mathcal{F}$) and as arrows $(F, E, \alpha : F \to f^*(E)) \to (F', E', \alpha' : F' \to f^*(E'))$ the pairs of arrows $(v : F \to F', u : E \to E')$ in $\mathcal{F}$ and $\mathcal{E}$ such that $f^*(u) \circ \alpha = \alpha' \circ v$; the functors π_{r_f} and π_{t_f} are respectively the canonical projection functors from this category to $\mathcal{E}$ and $\mathcal{F}$.

Since via these functors the category $\mathcal{G}(r_f) = \mathcal{G}(t_f)$ is fibered both over $\mathcal{E}$ and over $\mathcal{F}$, it is natural to consider the smallest Grothendieck topology on it which makes π_{r_f} and π_{t_f} comorphisms of sites when $\mathcal{E}$ and $\mathcal{F}$ are endowed with their canonical topologies, in other words the join of the Giraud topologies on it induced by the canonical topologies on $\mathcal{E}$ and $\mathcal{F}$. As we shall see, this topology plays a crucial role in connection with our construction of the over-topos.

To this end, we more generally describe, for any category $\mathcal{C}$ and basis $\mathcal{B}$ for a Grothendieck topology on $\mathcal{C}$ such that $(\mathcal{C}, J_{\mathcal{B}})$ is a site of definition for $\mathcal{E}$, a basis for the Grothendieck topology on $\mathcal{G}(r'_f)$, where r'_f is the 'restriction' of r_f to $\mathcal{C}$ (that is, the composite of r_f

with the opposite of the canonical functor $\mathcal{C} \to \mathcal{E}$), which is generated by the Giraud topology induced by the canonical topology on $\mathcal{F}$ and by the Giraud topology induced by $J_{\mathcal{B}}$. This result is to be applied subsequently to the syntactic site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of definition of a geometric theory $\mathbb{T}$ inside its classifying topos.

3.7. Definition. *Given a Grothendieck topology J on $\mathcal{C}$ (resp. a basis $\mathcal{B}$ for a Grothendieck topology on $\mathcal{C}$) such that $(\mathcal{C}, J)$ (resp. $(\mathcal{C}, J_{\mathcal{B}})$) is a site of definition for the topos $\mathcal{E}$, we shall call the Grothendieck topology on $\mathcal{G}(r'_f)$ generated by the Giraud topology induced by the canonical topology on $\mathcal{F}$ and by the Giraud topology induced by J (resp. by $J_{\mathcal{B}}$) the* (f, J)-lifted topology *(resp. the* $(f, \mathcal{B})$-lifted topology*) and shall denote it by $L_{(f,J)}$ (resp. $L_{(f,\mathcal{B})}$).*

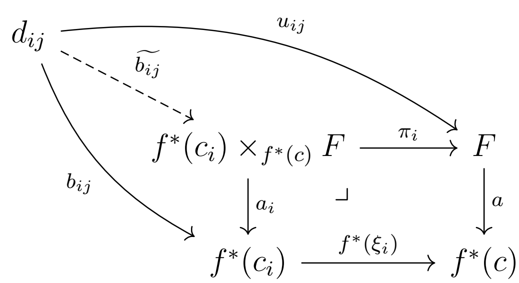
3.8. Theorem. *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism and $\mathcal{B}$ be a basis for a Grothendieck topology on $\mathcal{C}$ such that $(\mathcal{C}, J_{\mathcal{B}})$ is a site of definition for $\mathcal{E}$. Then, with the above notation, the $(f, \mathcal{B})$-lifted Grothendieck topology on $\mathcal{G}(r'_f)$ has as a basis the collection of multicomposites of a family of cartesian lifts (with respect to r'_f) of arrows in a family of $\mathcal{B}$ with $J_{\mathcal{F}}^{\mathrm{Gir}}$-covering families, that is, the collection of families*

$$((d_{ij}, (c_i, b_{ij})) \xrightarrow{(u_{ij}, (\xi_i, \widetilde{b_{ij}}))} (F, (c, a)))_{i \in I, j \in J_i}$$

where $(\xi_i : c_i \to c)_{i \in I}$ is a family in $\mathcal{B}(c)$ and the families

$$(\widetilde{b_{ij}} : d_{ij} \to f^*(c_i) \times_{f^*(c)} F)_{j \in J_i}$$

are epimorphic in $\mathcal{E}$ for each $i \in I$:



Proof. Note that the collection of arrows

$$\big((d_{ij}, (c_i, b_{ij})) \xrightarrow{(u_{ij}, (\xi_i, \widetilde{b_{ij}}))} (F, (c, a))\big)_{i \in I, j \in J_i}$$

is the multicomposite of the family

$$\big((\pi_i, (\xi_i, 1)) : (f^*(c_i) \times_{f^*(c)} F, (c_i, a_i)) \to (F, (c, a))\big)_{i \in I},$$

each of whose arrows is cartesian with respect to the fibration r_f to $\mathcal{E}$, with the families

$$\big((\widetilde{b_{ij}}, (1_{c_i}, \widetilde{b_{ij}})) : (d_{ij}, (c_i, b_{ij})) \to (f^*(c_i) \times_{f^*(c)} F, (c_i, a_i))\big)_{j \in J_i}$$

(for $i \in I$), each of whose arrows is cartesian with respect to the fibration t_f to $\mathcal{F}$ (and vertical with respect to the fibration r_f).

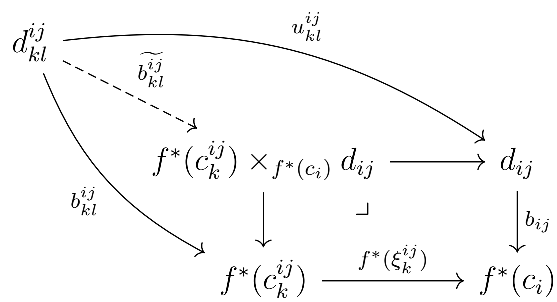
The first condition in the definition of basis is clearly satisfied, since $\mathcal{B}$ is a basis. The second condition follows from the stability under pullback of epimorphic families in $\mathcal{F}$ as well as of families in $\mathcal{B}$, in light of the compatibility of multicomposition with respect to pullback. It remains to show that the collection of families specified in the theorem is closed with respect to multicomposition.

Let

$$\big((d_{ij}, (c_i, b_{ij})) \xrightarrow{(u_{ij}, (\xi_i, \widetilde{b_{ij}}))} (F, (c, a))\big)_{i \in I,\, j \in J_i}$$

and, for each $i \in I, j \in J_i$,

$$\big((d^{ij}_{kl}, (c^{ij}_k, b^{ij}_{kl})) \xrightarrow{(u^{ij}_{kl}, (\xi^{ij}_k, \widetilde{b^{ij}_{kl}}))} ((d_{ij}, (c_i, b_{ij}))\big)_{k \in K^{ij},\, l \in L^{ij}_k}$$



be families satisfying the conditions in the theorem.

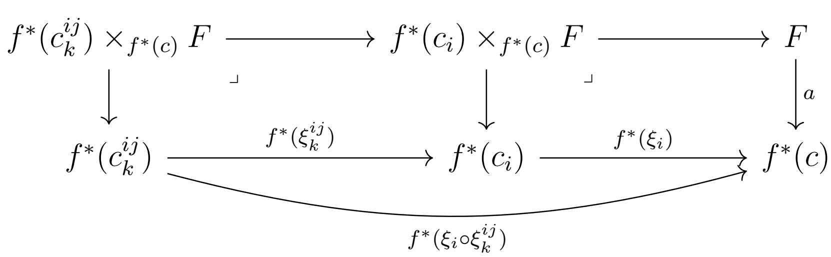
We want to prove that their multicomposite also satisfies these conditions.

As $\mathcal{B}$ is a basis for a Grothendieck topology, the family

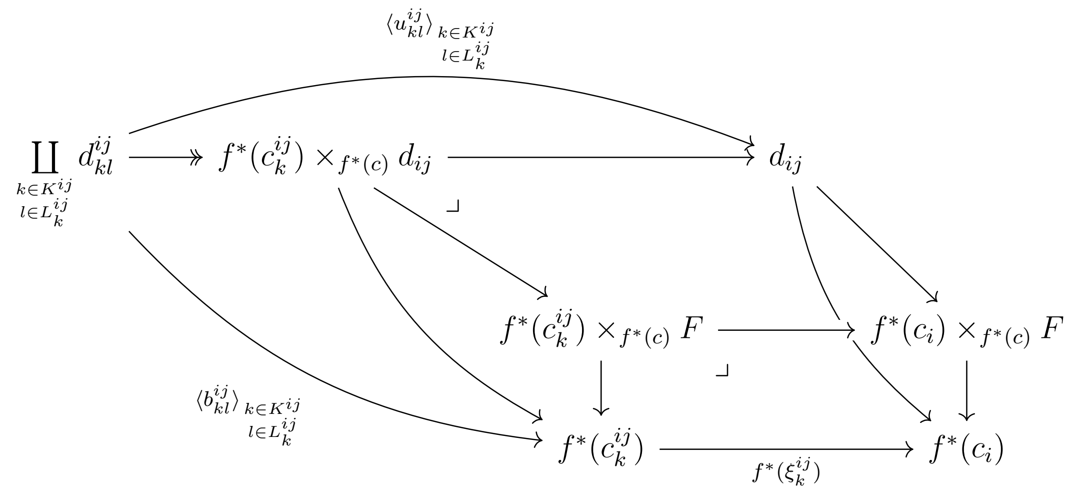
$$\big(c^{ij}_k \xrightarrow{\xi_i \circ \xi^{ij}_k} c\big)_{i \in I, j \in J_i, k \in K^{ij}}$$

is in $\mathcal{B}$.

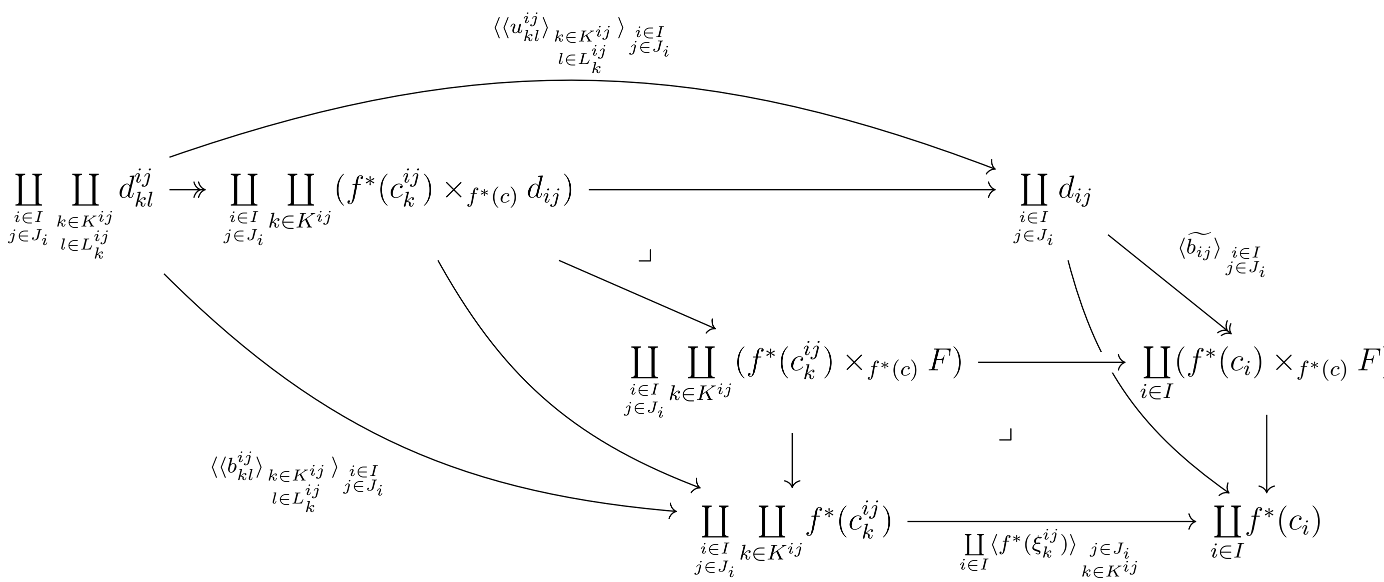
Consider the following pullback diagrams:



For each $i \in I$ and $j \in J_i$, in the diagram



both the front lower and back squares are pullbacks, hence so is the top square. Then in the diagram



the upper square is a pullback, whence the left-hand arrow in it is an epimorphism (as it is the pullback of the epimorphism $\widetilde{\langle b_{ij} \rangle}_{\substack{i\in I \\ j\in J_i}}$). Therefore the arrow

$$\langle\langle \widetilde{b^{ij}_{kl}} \rangle_{\substack{k\in K^{ij} \\ l\in L^{ij}_k}} \rangle_{\substack{i\in I \\ j\in J_i}} : \coprod_{\substack{i\in I \\ j\in J_i}} \coprod_{\substack{k\in K^{ij} \\ l\in L^{ij}_k}} d^{ij}_{kl} \to \coprod_{\substack{i\in I \\ j\in J_i}} \coprod_{k\in K^{ij}} (f^*(c^{ij}_k) \times_{f^*(c)} F)$$

is also an epimorphism, as it is the composite of two epimorphisms. But this is precisely the arrow corresponding to our multicomposite family, whence we can conclude that the latter satisfies the conditions of the theorem, as required (as coproducts are disjoint in a topos, a coproduct of arrows is an epimorphism if and only if each of the arrows are). ∎

We now apply the theorem to the geometric morphism to the classifying topos of a geometric theory $\mathbb{T}$, represented as the topos of sheaves on its geometric syntactic site, induced by a model of $\mathbb{T}$:

3.9. Corollary. *Let M be a model of a geometric theory $\mathbb{T}$ in a Grothendieck topos $\mathcal{E}$ and $f_M : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}_\mathbb{T}, J_\mathbb{T})$ the geometric morphism corresponding to it via the universal property of the classifying topos for $\mathbb{T}$. Then we have:*

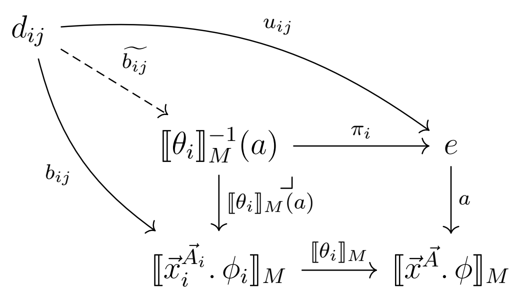
(i) The $(f_M, J_\mathbb{T})$-lifted topology $L_{(f_M,J_\mathbb{T})}$ on $(1_\mathcal{E} \downarrow F_M)$ has as a basis the collection of families

$$\Big((d_{ij}\,(\{\vec{x}_i^{\vec{A}_i}.\,\phi_i\}\,, [\![\theta_i]\!]_M(a))) \xrightarrow{(u_{ij},\,([\theta_i]_\mathbb{T},\widetilde{b_{ij}}))} (e, (\{\vec{x}^{\vec{A}}.\,\phi\}\,, a)\Big)_{i\in I, j\in J_i}$$

where $e \in \mathcal{E}$, $([\theta_i]_\mathbb{T} : \{\vec{x}_i^{\vec{A}_i}.\,\phi_i\} \to \{\vec{x}^{\vec{A}}.\,\phi\})_{i\in I}$ is a family in $\mathcal{B}_\mathbb{T}(\{\vec{x}^{\vec{A}}.\,\phi\})$ and the families

$$\big(\widetilde{b_{ij}} : d_{ij} \to [\![\theta_i]\!]_M^{-1}(a)\big)_{j\in J_i}$$

are epimorphic in $\mathcal{E}$ for each $i \in I$:



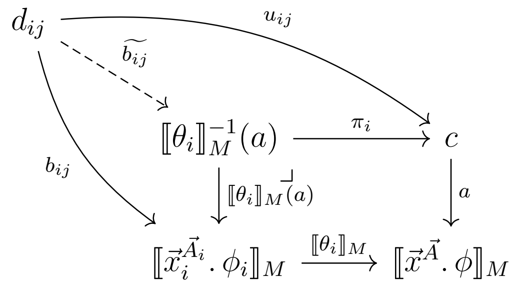
(ii) For any separating set $\mathcal{C}$ for $\mathcal{E}$, the canonical functor $\mathcal{C} \to \mathcal{E}$ induces a $L_{(f_M,J_\mathbb{T})}$-dense functor $(i_\mathcal{C} \downarrow F_M) \to (1_\mathcal{E} \downarrow F_M)$, where $i_\mathcal{C}$ is the canonical embedding of $\mathcal{C}$ in $\mathcal{E}$, and the Grothendieck topology induced by $L_{(f_M,J_\mathbb{T})}$ on the category $(i_\mathcal{C} \downarrow F_M)$ has as a basis the collection of families

$$\Big((d_{ij}, (\{\vec{x}_i^{\vec{A}_i}.\,\phi_i\}\,, [\![\theta_i]\!]_M(a))) \xrightarrow{(u_{ij},\,([\theta_i]_\mathbb{T},\widetilde{b_{ij}}))} (c, (\{\vec{x}^{\vec{A}}.\,\phi\}\,, a)\Big)_{i\in I, j\in J_i}$$

where $c, d_{ij} \in \mathcal{C}$ for each i and j, $([\theta_i]_\mathbb{T} : \{\vec{x}_i^{\vec{A}_i}.\,\phi_i\} \to \{\vec{x}^{\vec{A}}.\,\phi\})_{i\in I}$ is a family in $\mathcal{B}_\mathbb{T}(\{\vec{x}^{\vec{A}}.\,\phi\})$ and the families

$$\big(\widetilde{b_{ij}} : d_{ij} \to [\![\theta_i]\!]_M^{-1}(a)\big)_{j\in J_i}$$

are epimorphic in $\mathcal{E}$ for each $i \in I$:



(iii) If $\mathcal{E}$ is the topos **Set** *of sets and $1 : \{*\} \to \mathcal{E}$ is the functor from the one-object and one-arrow category $\{*\}$ to $\mathcal{E}$ picking the terminal object of* **Set**, *the embedding $(\int M) = (1_{\{*\}} \downarrow F_M) \to (1_{\mathcal{E}} \downarrow F_M)$ induced by the functor 1 is $L_{(f_M, J_{\mathbb{T}})}$-dense and the topology induced by $L_{(f_M, J_{\mathbb{T}})}$ on $(\int M)$ is the antecedent topology of Definition 2.2.*

Proof. (i) This is the particular case of Theorem 3.8 by taking $\mathcal{E}$ to be the classifying topos of $\mathbb{T}$ and $(\mathcal{C}, J)$ the geometric syntactic site of $\mathbb{T}$.

(ii) This easily follows from the definition of induced topology on a full dense subcategory.

(iii) This is the particular case of (ii) when $\mathcal{E}$ is **Set** and $\mathcal{C}$ is the category $\{*\}$, regarded as a separating set for $\mathcal{E}$ through the embedding $1 : \{*\} \to \mathcal{E}$. ∎

3.10. Giraud classifier of a cartesian stack. We now fix a Grothendieck topos $\mathcal{E}$ and a small subcanonical cartesian site $(\mathcal{C}, J)$ of definition for $\mathcal{E}$. Recall that by a *cartesian category* we mean a category with finite limits.

3.11. Definition. *A* cartesian *stack is a stack $\mathbb{M}$ on $(\mathcal{C}, J)$ such that any $\mathbb{M}(c)$ is a cartesian category, and such that any transition functor $\mathbb{M}(u)$, for u a morphism in $\mathcal{C}$, is cartesian. A morphism of cartesian stacks is a morphism of stacks $\alpha : \mathbb{M}_1 \to \mathbb{M}_2$ such that in any c, $\alpha_c : \mathbb{M}_1(c) \to \mathbb{M}_2(c)$ is cartesian.*

We shall denote by $\mathbf{St}_{\mathrm{cart}}(\mathcal{C}, J)$ the category of cartesian stacks on a site $(\mathcal{C}, J)$ and morphisms of cartesian stacks between them.

3.12. Remark. A typical example of cartesian stacks can be obtained from the pseudofunctor r_g associated with a geometric morphism $g : \mathcal{G} \to \mathcal{E}$. Recall that we defined $r_g : \mathcal{E}^{\mathrm{op}} \to \mathbf{Cat}$ as sending each object E to the slice category $\mathcal{G}/g^*E$, which, as a Grothendieck topos, is in particular is cartesian, and any $u : E_1 \to E_2$ in $\mathcal{E}$ to the pullback functor $u^* : \mathcal{G}/g^*E_2 \to \mathcal{G}/g^*E_1$, which is cartesian. In particular, if $\mathcal{E}$ has $(\mathcal{C}, J)$ as a standard site of definition, then one can restricts to objects in $\mathcal{E}$ that come from $\mathcal{C}$ through the Yoneda embedding. Since r_g is a stack for the canonical topology as recalled above, its restriction to $(\mathcal{C}, J)$ is a cartesian stack.

The following lemma, whose proof is straightforward, expresses cartesian lifts in cartesian fibrations as pullback squares, and serves for describing one half of the correspondence provided by Giraud's construction of the classifying topos of a stack recalled below:

3.13. Lemma. *Let $\mathbb{M}$ be a cartesian stack, with $1_{\mathbb{M}}(c)$ the terminal object of the fiber of $\mathbb{M}$ at c. Then, in the Grothendieck fibration $\pi_{\mathbb{M}} : \int \mathbb{M} \to \mathcal{C}$, for any arrow $u : c_1 \to c_2$ and any object a in $\mathbb{M}(c_2)$, the following square is a pullback:*

$$\begin{array}{ccc} (c_1, \mathbb{M}(u)(a)) & \xrightarrow{(u, 1_{\mathbb{M}(u)(a)})} & (c_2, a) \\ {\scriptstyle (1_{c_1}, !_{\mathbb{M}(u)(a)})}\downarrow & \lrcorner & \downarrow{\scriptstyle (1_{c_2}, !_a)} \\ (c_1, 1_{\mathbb{M}(c_1)}) & \xrightarrow{(u, 1_{1_{\mathbb{M}(c_1)}})} & (c_2, 1_{\mathbb{M}(c_2)}) \end{array}$$

3.14. Proposition. *For any cartesian stack on a site $\mathbb{M}$ on $(\mathcal{C}, J)$ whose underlying category $\mathcal{C}$ is cartesian, $\int \mathbb{M}$ is a cartesian category.*

Proof. We must prove that $\int \mathbb{M}$ is cartesian. As we shall see, the considered property holds globally on $\int \mathbb{M}$ out of holding in a specific fiber. Let $(c_i, a_i)_{i \in I}$ a finite diagram in $\int \mathbb{M}$; then, as $\mathcal{C}$ is cartesian, we can compute the limit $p_i : \lim_{i \in I} c_i \to c_i$ in $\mathcal{C}$. Moreover, $\mathbb{M}(\lim_{i \in I} c_i)$ is cartesian, so the finite limit $\lim_{i \in I} \mathbb{M}(p_i)(a_i)$ also exists in $\mathbb{M}(\lim_{i \in I} c_i)$, providing a cone

$$((p_i, \pi_i) : (\lim_{i \in I} c_i, \lim_{i \in I} \mathbb{M}(p_i)(a_i)) \to (c_i, a_i))_{i \in I}$$

in $\int \mathbb{M}$, where $\pi_i : \lim_{i \in I} \mathbb{M}(p_i)(a_i) \to \mathbb{M}(p_i)(a_i)$ is the projection in the fiber. Now for any other cone $((v_i, q_i) : (c, a) \to (c_i, a_i))_{i \in I}$, with $q_i : a \to \mathbb{M}(v_i)(a_i)$, there exists a unique arrow $w : c \to \lim_{i \in I} c_i$ by the universal property of the limit in $\mathcal{C}$; but, as the transition functors $\mathbb{M}(p_i)$ are cartesian, we have

$$\mathbb{M}(w)(\lim_{i \in I} \mathbb{M}(p_i)(a_i)) \simeq \lim_{i \in I} \mathbb{M}(w)\mathbb{M}(p_i)(a_i) \simeq \lim_{i \in I} \mathbb{M}(v_i)(a_i)$$

inducing a unique arrow $f : a \to \mathbb{M}(w)(\lim_{i \in I} \mathbb{M}(p_i)(a_i))$, so that there is a unique factorization in $\int \mathbb{M}$

$$\begin{array}{ccc} (c, a) & \overset{(w,f)}{\dashrightarrow} & (\lim_{i \in I} c_i, \lim_{i \in I} \mathbb{M}(p_i)(a_i)) \\ & {\scriptstyle (v_i, q_i)} \searrow \quad \swarrow {\scriptstyle (p_i, \pi_i)} & \\ & (c_i, a_i) & \end{array}$$

as desired. ∎

Let us recall from [8] the following fundamental classification result:

3.15. Theorem. *Let $\mathbb{M}$ be a cartesian stack on a cartesian site $(\mathcal{C}, J)$ of definition for a topos $\mathcal{E}$, and $J^{\mathrm{Gir}}_{\mathbb{M}}$ be the Giraud topology induced by J. Then the sheaf topos $\mathcal{E}[\mathbb{M}] = \mathbf{Sh}(\int \mathbb{M}, J^{\mathrm{Gir}}_{\mathbb{M}})$, with its canonical morphism $p_{\mathbb{M}} : \mathcal{E}[\mathbb{M}] \to \mathcal{E}$ induced by the comorphism of sites $\pi_{\mathbb{M}} : (\int \mathbb{M}, J^{\mathrm{Gir}}_{\mathbb{M}}) \to (\mathcal{C}, J)$, is the classifier of $\mathbb{M}$, in the sense that the following universal property holds: for any geometric morphism $g : \mathcal{G} \to \mathcal{E}$,*

$$\mathbf{Geom}_{\mathcal{E}}(g, p_{\mathbb{M}}) \simeq \mathbf{St}_{\mathrm{cart}}(\mathcal{C}, J)(\mathbb{M}, r_g)$$

where r_g is the cartesian stack sending c on $\mathcal{G}/g^*(c)$ as defined above.

PROOF. Given a triangle

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{f} & \mathcal{E}[\mathbb{M}] \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle p_{\mathbb{M}}} & \\ & \mathcal{E} & \end{array}$$

of geometric morphisms, the cartesian morphism of stacks $\alpha : \mathbb{M} \to \mathcal{G}/g^*$ associated with g as in the theorem can be described as follows.

We have $g^* \cong f^* p_{\mathbb{M}}^*$. The restriction to $\mathcal{C}$ of the functor $p_{\mathbb{M}}^*$ is just the terminal section $1_{\mathbb{M}(-)}$, so we have a commutative (up to isomorphism) triangle

$$\begin{array}{ccc} \int \mathbb{M} & \xrightarrow{f^*} & \mathcal{G} \\ & {\scriptstyle 1_{\mathbb{M}(-)}}\nwarrow \quad \nearrow{\scriptstyle g^*} & \\ & \mathcal{C} & \end{array}$$

Now, for any (c, a) in $\int \mathbb{M}$, there is a unique arrow $!_a : a \to 1_{\mathbb{M}(c)}$ in the fiber $\mathbb{M}(c)$, which is sent by f^* to an arrow $f^*(1_c, !_a) : f^*(c, a) \to f^*(c, 1_{\mathbb{M}(c)}) \simeq g^*(c)$. This yields, for each $c \in \mathcal{C}$, a functor

$$\begin{array}{rcl} \alpha_c : \mathbb{M}(c) & \to & \mathcal{G}/g^*(c) \\ a & \mapsto & f^*(1_c, !_a) : f^*(c, a) \to g^*(c) \end{array}$$

The functors $(\alpha_c : \mathbb{M}(c) \to \mathcal{G}/g^*(c))_{c \in \mathcal{C}}$ actually define a natural transformation, that is, all the squares of the following form commute up to isomorphism:

$$\begin{array}{ccc} \mathbb{M}(c_2) & \xrightarrow{\alpha_{c_2}} & \mathcal{G}/g^*(c_2) \\ {\scriptstyle \mathbb{M}(u)}\downarrow & & \downarrow{\scriptstyle (g^*(u))^*} \\ \mathbb{M}(c_1) & \xrightarrow{\alpha_{c_1}} & \mathcal{G}/g^*(c_1) \end{array}$$

Indeed, by Lemma 3.13 the following square is a pullback in $\int \mathbb{M}$:

$$\begin{array}{ccc} (c_1, \mathbb{M}(u)(a)) & \xrightarrow{(u, 1_{\mathbb{M}(u)(a)})} & (c_2, a) \\ {\scriptstyle (1_{c_1}, !_{\mathbb{M}(u)(a)})}\downarrow & \lrcorner & \downarrow{\scriptstyle (1_{c_2}, !_a)} \\ (c_1, 1_{\mathbb{M}(c_1)}) & \xrightarrow{(u, 1_{1_{\mathbb{M}(c_1)}})} & (c_2, 1_{\mathbb{M}(c_2)}) \end{array}$$

It thus follows that, the functor f^* being cartesian, the following square is also a pullback:

$$\begin{array}{ccc} f^*(c_1, \mathbb{M}(u)(a))) & \xrightarrow{f^*(u, 1_{\mathbb{M}(u)(a)})} & f^*(c_2, a) \\ {\scriptstyle \alpha_{c_1}(\mathbb{M}(u)(a))}\downarrow & \lrcorner & \downarrow{\scriptstyle \alpha_{c_2}(a)} \\ g^*(c_1) & \xrightarrow{g^* u} & g^*(c_2) \end{array}$$

So $\alpha_{c_1}(\mathbb{M}(u)(a)) = (g^*u)^*\alpha_{c_2}(a)$, which is precisely the content of the naturality condition.

The cartesianness of α is actually inherited from that of f^*, as α acts as a restriction of f^* on each fiber in light of Proposition 3.14. ∎

4. Site for the over-topos at a model in an arbitrary topos

We turn to the construction of the over-topos $u_M : \mathcal{E}[M] \to \mathcal{E}$ associated with a $\mathbb{T}$-model M in an arbitrary topos $\mathcal{E}$. Recall that the desired formula is, for any $\mathcal{E}$-topos $g : \mathcal{G} \to \mathcal{E}$,

$$\mathbf{Geom}_{\mathcal{E}}(g, u_M) \simeq \mathbb{T}[\mathcal{G}]/g^*(M).$$

We suppose that we are given a cartesian small subcanonical site of definition $(\mathcal{C}, J)$ for $\mathcal{E}$. Note that the canonical embedding $i_{\mathcal{C}}$ of $\mathcal{C}$ into $\mathcal{E}$ identifies $\mathcal{C}$ with a separating set of objects for $\mathcal{E}$ and J with the Grothendieck topology induced on it by the canonical topology on $\mathcal{E}$.

4.1. The cartesian stack at a model. In the case where $\mathcal{E} = \mathbf{Set}$, as $\mathbf{Set}$ is generated by 1 under coproducts, the global elements $1 \to [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$ are sufficient to generate all the generalized elements $X \simeq \coprod_X 1 \to [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$. In the general case, global elements must be replaced by generalized elements, possibly restricting to those whose domain is an object of $\mathcal{C}$, which we call *basic generalized elements*. Indeed, we shall replace the category of global elements of M by an indexed category of basic generalized elements of interpretations in M of geometric formulas over the language of $\mathbb{T}$, that is, with the comma category $(i_{\mathcal{C}} \downarrow F_M)$, where $F_M : (\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \to \mathcal{E}$ is the $J_{\mathbb{T}}$-continuous cartesian functor sending a geometric formula-in-context $\{\vec{x}^{\vec{A}}.\, \phi\}$ to its interpretation $[\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$ in M. This defines a prestack

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{op}} & \xrightarrow{\mathbb{M}} & \mathrm{CAT} \\ c & \longmapsto & (c \downarrow F_M), \\ c_1 \xrightarrow{u} c_2 & \longmapsto & (c_2 \downarrow F_M) \xrightarrow{u^*} (c_1 \downarrow F_M) \end{array}$$

where $u^* : (c_2 \downarrow F_M) \to (c_1 \downarrow F_M)$ is the pre-composition functor sending some $a : c_2 \to [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$ to $a \circ u : c_1 \to [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$.

4.2. Proposition. *$\mathbb{M}$ is a cartesian stack on $(\mathcal{C}, J)$. That is, for each c in $\mathcal{C}$, $(c \downarrow F_M)$ is cartesian, and for any arrow $u : c' \to c$ in $\mathcal{C}$, the transition functor $(u \downarrow F_M)$ is cartesian.*

Proof. This is a consequence of F_M being cartesian: indeed any finite diagram

$$(a_i : c \to [\![\vec{x}_i^{\vec{A}_i}.\phi_i]\!]_M)_{i \in I}$$

in $(c \downarrow F_M)$ defines a unique arrow $(a_i)_{i \in I} : c \to \lim_{i \in I}[\![\vec{x}_i^{\vec{A}_i}.\phi_i]\!]_M \simeq F_M(\lim_{i \in I}\{\vec{x}_i^{\vec{A}_i}.\phi_i\})$. For any arrow $b : c \to [\![\vec{x}^{\vec{A}}.\phi]\!]_M$ equipped with a cone $([\theta_i] : b \to a_i)_{i \in I}$ in $(c \downarrow F_M)$, there is a canonical arrow $[\theta]_{\mathbb{T}} : \{\vec{x}^{\vec{A}}.\phi\} \to \lim_{i\in I}\{\vec{x}_i^{\vec{A}_i}.\phi_i\}$ in $\mathcal{C}_{\mathbb{T}}$ providing a factorization of the $[\theta_i]_{\mathbb{T}}$'s, whence $[\![\theta]\!]_M$ also factorizes each $[\![\theta_i]\!]_M$, thus providing a universal factorization of the cone in $(c \downarrow F_M)$. For any $u : c' \to c$, it is easy to see that the composite $(a_i)_{i\in I} \circ u : c' \to F_M(\lim_{i\in I}\{\vec{x}_i^{\vec{A}_i}.\phi_i\})$ is the limit of the $a_i \circ u$'s in $(c' \downarrow F_M)$ by the uniqueness of the factorization of a cone through the limit. ∎

As a consequence, the fibred category $\int \mathbb{M} = (i_{\mathcal{C}} \downarrow F_M)$ is cartesian. Its objects are the pairs (c, a), where c is an object of $\mathcal{C}$ and a is a c-indexed basic element $a : c \to [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M$, while an arrow $(c_1, a_1) \to (c_2, a_2)$ is a pair $(u, [\theta]_{\mathbb{T}})$ consisting of an arrow $u : c_1 \to c_2$ in $\mathcal{C}$ and an arrow $[\theta]_{\mathbb{T}}$ in $\mathcal{C}_{\mathbb{T}}$ such that $a_2 \circ u = [\![\theta]\!]_M \circ a_1$:

$$\begin{array}{ccc} c_1 & \xrightarrow{\quad u \quad} & c_2 \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle a_2} \\ [\![\vec{x}_1^{\vec{a_1}}.\,\phi_1]\!]_M & \xrightarrow{[\![\theta]\!]_M} & [\![\vec{x}_2^{\vec{a_2}}.\,\phi_2]\!]_M \end{array}$$

In order to complete our generalization of the construction of section 2, we need to equip the category $\int \mathbb{M}$ with a Grothendieck topology representing the analogue of the antecedents topology. This is not straightforward since, in the general context, there are no vertical covers a priori. Indeed, one would be tempted to define in each fiber $(c \downarrow F_M)$ a topology of c-indexed antecedents, made of families of triangles

$$\begin{array}{ccc} & c & \\ {\scriptstyle b}\swarrow & & \searrow{\scriptstyle [\![\theta_i]\!]_M \circ b = a} \\ [\![\vec{x}_i^{\vec{A}_i}.\,\phi_i]\!]_M & \xrightarrow{\quad [\![\theta_i]\!]_M \quad} & [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M \end{array}$$

for $([\theta_i] : \{\vec{x}_i^{\vec{A}_i}.\,\phi_i\} \to \{\vec{x}^{\vec{A}}.\,\phi\})_{i\in I}$ a family in $\mathcal{B}_{\mathbb{T}}(\{\vec{x}^{\vec{A}}.\,\phi\})$ and b ranging over all the antecedents of a along the arrows $[\![\theta_i]\!]_{i\in I}$. However this would not work because, although F_M sends the family $([\theta_i] : \{\vec{x}_i^{\vec{A}_i}.\,\phi_i\} \to \{\vec{x}^{\vec{A}}.\,\phi\})_{i\in I}$ to an epimorphic family in $\mathcal{E}$, there is no reason for a c-indexed basic element a to have a c-indexed antecedent along any of the $[\![\theta_i]\!]_M$'s. In order to take the horizontal components into account, we need to leave the domain of 'antecedent' generalized elements vary among the objects of $\mathcal{C}$, and require them to cover the fibers $[\![\theta_i]\!]_M^{-1}(a)$ of a along the arrows $[\![\theta_i]\!]_M$:

$$\begin{array}{ccc} \coprod_{i\in I} [\![\theta_i]\!]_M^{-1}(a) & \twoheadrightarrow & c \\ \downarrow & \lrcorner \;\; \coprod_{i\in I} [\![\theta_i]\!]_M & \downarrow{\scriptstyle a} \\ \coprod_{i\in I} [\![\vec{x}_i^{\vec{A}_i}.\,\phi_i]\!]_M & \twoheadrightarrow & [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M \end{array}$$

(Note that, as in the set-based setting, some $[\![\theta_i]\!]_M^{-1}(a)$ may be empty, though they jointly cover c). In light of the characterization of the antecedents topology in the set-based setting in terms of lifted topologies, provided by Corollary 3.9, we are thus led to defining the antecedents topology on $\int \mathbb{M}$ as follows:

4.3. Definition. *The* antecedents topology J_M^{ant} *on* $\int \mathbb{M}$ *is the Grothendieck topology on* $(i_{\mathcal{C}} \downarrow F_M)$ *induced by the* $(f_M, J_{\mathbb{T}})$*-topology on* $(1_{\mathcal{E}} \downarrow F_M)$ *(in the sense of Definition 3.7); that is, it has as a basis the collection of families*

$$\big((d_{ij}, b_{ij}) \xrightarrow{(u_{ij}, [\theta_i]_{\mathbb{T}})} (c, a)\big)_{i \in I,\, j \in J_i}$$

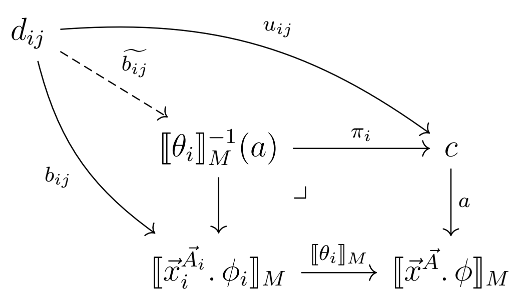
where $a : c \to [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M$ *is a basic generalized element,* $([\theta_i]_{\mathbb{T}} : \{\vec{x}_i^{\vec{A}_i}.\,\phi_i\} \to \{\vec{x}^{\vec{A}}.\,\phi\})_{i \in I}$ *is a family in* $\mathcal{B}_{\mathbb{T}}(\{\vec{x}^{\vec{A}}.\,\phi\})$*,* $(u_{ij} : d_{ij} \to c)_{j \in J_i}$ *is a family of arrows in* $\mathcal{C}$ *(for each* $i \in I$*) and* $(b_{ij} : d_{ij} \to [\![\vec{x}_i^{\vec{A}_i}.\,\phi_i]\!]_M)_{j \in J_i}$ *is a family of arrows (for each* $i \in I$*) making the diagrams*

$$\begin{array}{ccc} d_{ij} & \xrightarrow{\quad u_{ij} \quad} & c \\ {\scriptstyle b_{ij}}\big\downarrow & & \big\downarrow{\scriptstyle a} \\ [\![\vec{x}_i^{\vec{A}_i}.\,\phi_i]\!]_M & \xrightarrow{[\![\theta_i]\!]_M} & [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M \end{array}$$

commutative and such that the family of arrows

$$(\widetilde{b_{ij}} : d_{ij} \to \langle [\![\theta_i]\!]_M \rangle^{-1}(a))_{j \in J_i}$$

as in the following diagram is epimorphic :



4.4. Remark.

(a) *The covering families in the definition of* J_M^{ant} *are indexed by a dependent sum, with first a set indexing a basic cover of* $J_{\mathbb{T}}$ *and for each term of this cover, a basic covering family. To obtain a more conventional presentation, one can equivalently use a single indexing set* J *and require a family of squares* $(\{\theta_i\}, u_i)_{i \in I}$ *to induce an epimorphism* $\langle (b_i, u_i) \rangle_{i \in I}$ *and have* $(\theta_i)_{i \in I}$ *in* $J_{\mathbb{T}}$*. In particular, for a presentation as above, take* $J = \coprod_{i \in I} J_i$ *and impose* $\{\theta_{(i,j)}\} := \{\theta_i\}$*.*

(b) *As in the set-based case, the* $[\![\theta_i]\!]_M$*'s are only jointly epimorphic, so a generalized element* $a : c \to [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M$ *may have no antecedent along a chosen* θ_i*, that is, the corresponding set* J_i *may be empty. However, antecedent families live over* subfamilies *of* $J_{\mathbb{T}}$*-covering families, as shown by Proposition 4.5 below.*

4.5. Proposition. *There is a comorphism of sites* $(\int \mathbb{M}, J^{\text{ant}}_M) \to (\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ *sending a basic generalized element* $a : c \to [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$ *to the underlying sort* $\{\vec{x}^{\vec{A}}.\, \phi\}$.

Proof. This follows immediately from the characterization of J^{ant}_M as the Grothendieck topology on $(i_{\mathcal{C}} \downarrow F_M)$ induced by the $(f_M, J_{\mathbb{T}})$-topology on $(1_{\mathcal{E}} \downarrow F_M)$. ∎

4.6. Remark. The comorphism of sites of Proposition 4.5 is *not* a fibration, unlike its topos-theoretic extension provided by the canonical projection functor $(1_{\mathcal{E}} \downarrow {f_M}^*) \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$. This explains why the antecedent topology is *not* a lifted topology but rather a topology induced on a smaller subcategory by a lifted topology existing at the topos level, and hence that its description is more involved than that for its topos-theoretic extension. This is an illustration of the importance of developing *invariant constructions* at the topos-theoretic level and of investigating only later how such notions can be described at the level of sites.

Since J^{ant}_M is the Grothendieck topology on $(i_{\mathcal{C}} \downarrow F_M)$ induced by the $(f_M, J_{\mathbb{T}})$-lifted topology on $(1_{\mathcal{E}} \downarrow F_M)$, the canonical projection functor $\pi_M : \int \mathbb{M} \to \mathcal{C}$ to $\mathcal{C}$ is a comorphism of sites $(\int \mathbb{M}, J^{\text{ant}}_M) \to (\mathcal{C}, J)$.

4.7. Main theorem in the general case. We are now ready to define the over-topos at M in the general setting:

4.8. Definition. *For a model* M *of* $\mathbb{T}$ *in a Grothendieck topos* $\mathcal{E}$*, we denote as* $\mathcal{E}[M]$ *the sheaf topos* $\mathbf{Sh}(\int \mathbb{M}, J^{\text{ant}}_M)$ *over the category of elements of* $\mathbb{M}$ *together with its antecedents topology, and as* $u_M : \mathcal{E}[M] \to \mathcal{E}$ *the geometric morphism induced by the comorphism of sites* $\pi_M : (\int \mathbb{M}, J^{\text{ant}}_{\mathbb{M}}) \to (\mathcal{C}, J)$.

4.9. Remark.

(a) The inverse image of u_M is the pre-composition sending c to $\mathcal{C}(\pi_M(-), c)$ (since π_M is continuous by Theorem 4.44 [5]), while the direct image is the restriction to sheaves of the right Kan extension along π_M.

(b) In the set-based case, choosing as a presentation site the category $\{*\}$ with the trivial topology on it, there is only one fiber, and the antecedents topology on $\int M$ of Definition 2.2 is completely concentrated in it.

4.10. Theorem. *The* $\mathcal{E}$*-topos* $u_M : \mathcal{E}[M] \to \mathcal{E}$ *satisfies the universal property of the* $\mathbb{T}$*-over-topos at* M*: that is, for any* $\mathcal{E}$*-topos* $g : \mathcal{G} \to \mathcal{E}$ *there is an equivalence of categories*

$$\mathbf{Geom}_{\mathcal{E}}(g, \mathcal{E}[\mathbb{M}]) \simeq \mathbb{T}[\mathcal{G}]/g^* M$$

natural in g.

Proof. The proof naturally generalizes that for a set-based model.

In one direction, suppose that f is a geometric morphism over $\mathcal{E}$ from g to u_M:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\quad f \quad} & \mathcal{E}[\mathbb{M}] \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle u_M} & \\ & \mathcal{E} & \end{array}$$

This defines a J_M^{ant}-continuous, cartesian functor

$$\int \mathbb{M} \xrightarrow{f^*} \mathcal{G}.$$

By Theorem 3.15, it also induces a morphism of cartesian stacks over $\mathcal{E}$

$$\mathbb{M} \xrightarrow{f} \mathcal{G}/g^*$$

whose components

$$(c \downarrow F_M) \xrightarrow{f_c} \mathcal{G}/g^*c$$

(for each c in $\mathcal{C}$) are cartesian functors sending a given basic generalized element $a : c \to [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$ to a certain arrow $f_c(a) : N_{(c,a)} \to g^*c$ in $\mathcal{G}$. We want to associate with f a morphism $\tilde{f} : N \to g^*M$ in $\mathbb{T}[\mathcal{G}]$. For each $\{\vec{x}^{\vec{A}}.\, \phi\}$ in $\mathcal{C}_{\mathbb{T}}$, the category $(i_{\mathcal{C}} \downarrow [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M)$ of basic generalized elements is small; we can thus define

$$N_{\{\vec{x}^{\vec{A}}.\, \phi\}} := \operatorname*{colim}_{(i_{\mathcal{C}} \downarrow \{\vec{x}^{\vec{A}}.\, \phi\})} N_{(c,a)} \simeq \operatorname*{colim}_{c \in \mathcal{C}} \coprod_{a \in \mathcal{E}(c, [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M)} N_{(c,a)}$$

and, by using the universal property of the colimit, an arrow $f_{\{\vec{x}^{\vec{A}}.\, \phi\}}$ as in the following diagram (where the j_a's are the legs of the colimit cocone):

$$\begin{array}{ccc} N_{(c,a)} & \xrightarrow{\quad f_c(a) \quad} & g^*c \\ {\scriptstyle j_a}\downarrow & & \downarrow{\scriptstyle g^*(a)} \\ N_{\{\vec{x}^{\vec{A}}.\, \phi\}} & \overset{f_{\{\vec{x}^{\vec{A}}.\, \phi\}}}{\dashrightarrow} & [\![\vec{x}^{\vec{A}}.\, \phi]\!]_{g^*M} \end{array}$$

Let us first check this yields a $J_{\mathbb{T}}$-continuous cartesian functor $N : (\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \to \mathcal{G}$, showing how the above assignment on objects naturally extends to arrows.

Given an arrow $[\theta]_{\mathbb{T}} : \{\vec{x_1}^{\vec{A_1}}.\, \phi_1\} \to \{\vec{x_2}^{\vec{A_2}}.\, \phi_2\}$ in $\mathcal{C}_{\mathbb{T}}$, composing with $[\![\theta]\!]_M$ allows one to associate with any generalized element $a : c \to [\![\vec{x_1}^{\vec{A_1}}.\, \phi]\!]_M$ a generalized element $[\![\theta]\!]_M \circ a : c \to [\![\vec{x_2}^{\vec{A_2}}.\, \phi]\!]_M$. So by the functoriality of f_c we have:

$$\begin{array}{ccc} N_{(c,a)} & \xrightarrow{\quad f_c([\theta]_{\mathbb{T}}) \quad} & N_{(c, [\![\theta]\!]_M \circ a)} \\ & {\scriptstyle f_c(a)}\searrow \quad \swarrow{\scriptstyle f_c([\![\theta]\!]_M \circ a)} & \\ & g^*(c) & \end{array}$$

We can thus define $N_{[\theta]_{\mathbb{T}}} : N_{\{\vec{x_1}^{\vec{A_1}}\,.\,\phi_1\}} \to N_{\{\vec{x_2}^{\vec{A_2}}\,.\,\phi_2\}}$ as the arrow determined by the universal property of the colimit by the requirement that all the diagrams of the form

$$\begin{array}{ccc} N_{(c,a)} & \xrightarrow{f_c([\theta]_{\mathbb{T}})} & N_{(c,[\![\theta]\!]_M \circ a)} \\ \downarrow{\scriptstyle j_a} & & \downarrow{\scriptstyle j_{[\![\theta]\!]_M \circ a}} \\ N_{\{\vec{x_1}^{\vec{A_1}}\,.\,\phi_1\}} & \xrightarrow{N_{[\theta]_{\mathbb{T}}}} & N_{\{\vec{x_2}^{\vec{A_2}}\,.\,\phi_2\}} \end{array}$$

should commute.

The fact that N is cartesian follows at once from the fact that the functor $f^* : \int \mathbb{M} \to \mathcal{G}$ is, in light of the definition of N in terms of colimits and of the stability of these under pullback. So it remains to prove its $J_{\mathbb{T}}$-continuity.

Let $([\theta_i]_{\mathbb{T}} : \{\vec{x}_i^{\vec{A}_i}\,.\,\phi_i\} \to \{\vec{x}^{\vec{A}}\,.\,\phi\})_{i \in I}$ be a family in $\mathcal{B}_{\mathbb{T}}(\{\vec{x}^{\vec{A}}\,.\,\phi\})$; we want to show that the family $\big(N_{[\theta_i]_{\mathbb{T}}} : N_{\{\vec{x}_i^{\vec{A}_i}\,.\,\phi_i\}} \to N_{\{\vec{x}^{\vec{A}}\,.\,\phi\}}\big)_{i \in I}$ is epimorphic. First, we notice that this condition can be conveniently phrased in terms of basic generalized elements, as follows: for any basic generalized element $w : e \to N_{\{\vec{x}^{\vec{A}}\,.\,\phi\}}$ there are an epimorphic family $\{u_k : e_k \to e \mid k \in K\}$ lying in $\mathcal{C}$ and for each $k \in K$ an element $i_k \in I$ and a basic generalized element $w_k : e_k \to N_{\{\vec{x_{i_k}}^{\vec{A_{i_k}}}\,.\,\phi_{i_k}\}}$ such that the diagram

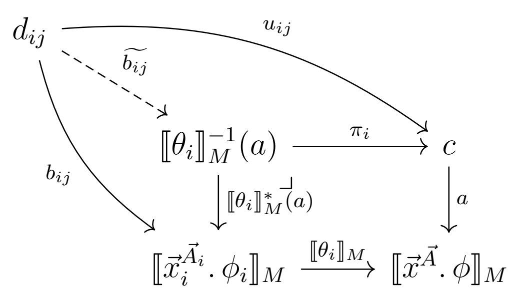
$$\begin{array}{ccc} e_k & \xrightarrow{u_k} & e \\ \downarrow{\scriptstyle w_k} & & \downarrow{\scriptstyle w} \\ N_{\{\vec{x_{i_k}}^{\vec{A_{i_k}}}\,.\,\phi_{i_k}\}} & \xrightarrow{N_{[\theta_{i_k}]_{\mathbb{T}}}} & N_{\{\vec{x}^{\vec{A}}\,.\,\phi\}} \end{array}$$

commutes. Since colimits yield epimorphic families in a topos, in light of the definition of N we can further rewrite this condition as follows: for any basic generalized element $w : e \to N_{(c,a)}$, where $a : c \to [\![\vec{x}^{\vec{A}}\,.\,\phi]\!]_M$, there are an epimorphic family $\{u_k : e_k \to e \mid k \in K\}$ lying in $\mathcal{C}$ and for each $k \in K$ an element $i_k \in I$, an object (d_k, b_k) of the category $(i_{\mathcal{C}} \downarrow [\![\vec{x_{i_k}}^{\vec{A_{i_k}}}\,.\,\phi_{i_k}]\!]_M)$ and an arrow $w_k : e_k \to N_{(d_k,b_k)}$ such that the diagram

$$\begin{array}{ccc} e_k & \xrightarrow{u_k} & e \\ \downarrow{\scriptstyle w_k} & & \downarrow{\scriptstyle w} \\ N_{(d_k,b_k)} & \xrightarrow{N_{[\theta_{i_k}]_{\mathbb{T}}}} & N_{(c,a)} \end{array}$$

commutes.

Now, given a generalized element $a : c \to [\![\vec{x}^{\vec{A}}\,.\,\phi]\!]_M$, we may obtain a covering family for the antecedent topology J_M^{ant} (with respect to our original $J_{\mathbb{T}}$-cover) by covering each of the fibers $[\![\theta_i]\!]_M^{-1}(a)$ by basic generalized elements $\widetilde{b_{ij}} : d_{ij} \to [\![\theta_i]\!]_M^{-1}(a)$:

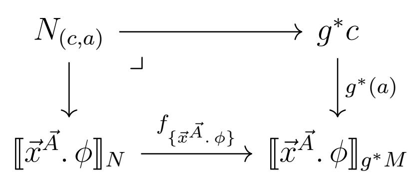


The functor f^* being J_M^{ant}-continuous, f^* sends this family to an epimorphic family

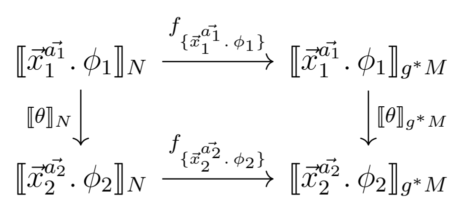
$$\big(f(u_{ij}, [\theta_i]_{\mathbb{T}}) : N_{(d_{ij}, b_{ij})} \to N_{(c,a)}\big)_{i \in I,\, j \in J_i}$$

in $\mathcal{G}$; but this clearly implies our thesis.

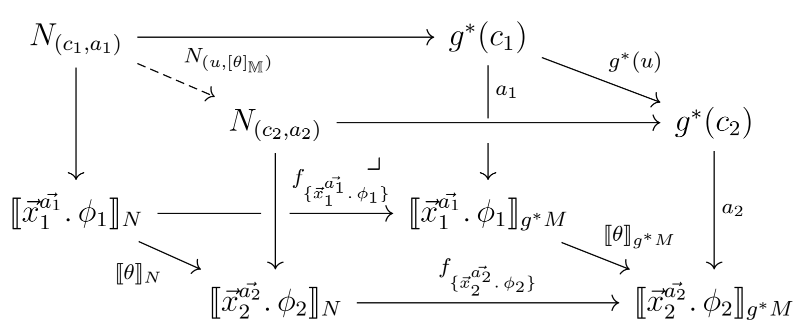
Conversely, let $f : N \to g^*M$ be a morphism of $\mathbb{T}$-models in $\mathcal{G}$. We can regard f as a $\mathcal{C}_{\mathbb{T}}$-indexed family $\{f_{\{\vec{x}^{\vec{A}}.\,\phi\}} : [\![\vec{x}^{\vec{A}}.\,\phi]\!]_N \to [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{g^*M}\}$ of morphisms in $\mathcal{G}$ (subject to the naturality conditions). For each basic generalized element $a : c \to [\![\vec{x}^{\vec{A}}.\,\phi]\!]_M$, we define $N_{(c,a)}$ as the following pullback:



Given a morphism $(u, [\theta]_{\mathbb{T}}) : (c_1, a_1) \to (c_2, a_2)$ in $\int \mathbb{M}$, we have by the naturality of f (seen as a morphism of $J_{\mathbb{T}}$-continuous cartesian functors) the following commutative square:



We can thus define $N_{(u,[\theta]_{\mathbb{M}})} : N_{(c_1,a_1)} \to N_{(c_2,a_2)}$ as the unique arrow making the following diagram (where the front and back faces are pullback) commute:



This yields a functor

$$\textstyle\int \mathbb{M} \xrightarrow{N_{(-)}} \mathcal{G}.$$

We want to show that this functor is cartesian and J_M^{ant}-continuous. The fact that it is cartesian follows easily from the fact that the functor $F_N : \mathcal{C}_{\mathbb{T}} \to \mathcal{G}$ corresponding to the model N is, in light of the construction of finite limits in the category $\int \mathbb{M}$ provided by Proposition 3.14.

Concerning J_M^{ant}-continuity, we shall consider the extension of $N_{(-)}$ to the category $(1_{\mathcal{E}} \downarrow F_M)$ and show its continuity with respect to the extended topology, as the latter admits a more natural characterization as the $(f_M, J_{\mathbb{T}})$-lifted topology; recall that this topology has a basis consisting of multicomposites of covering families of horizontal arrows for the fibration r_{f_M} and of covering families of horizontal arrows for the fibration t_{f_M}, so continuity can be checked separately with respect to each of these families.

Let us start by showing the continuity with respect to the covering families of horizontal arrows for t_{f_M}; this can be checked without any problems in terms of the site of definition $(\mathcal{C}, J)$ for $\mathcal{E}$. For a J-covering family $(u_i : c_i \to c)_{i \in I}$ and a generalized element $a : c \to [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$, consider the following diagram:

$$\begin{array}{ccc}
N_{(c_i, a \circ u_i)} & \longrightarrow & g^* c_i \\
{\scriptstyle N_{(u_i, 1_{\{\vec{x}^{\vec{A}}.\, \phi\}})}} \big\downarrow & & \big\downarrow {\scriptstyle g^* u_i} \\
N_{(c,a)} & \longrightarrow & g^* c \\
\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle g^*(a)} \\
[\![\vec{x}^{\vec{A}}.\, \phi]\!]_N & \overline{}_{f_{\{\vec{x}^{\vec{A}}.\, \phi\}}} & [\![\vec{x}^{\vec{A}}.\, \phi]\!]_{g^* M}
\end{array}$$

The lower square and the outer rectangle are pullbacks, whence by the pullback lemma the upper square is also a pullback. Since $(\mathcal{C}, J)$ is a site of definition for $\mathcal{E}$, g^* sends J-covering families to epimorphic families; so we have a pullback of epimorphisms

$$\begin{array}{ccc}
\coprod_{i \in I} N_{(c_i, a \circ u_i)} & \longrightarrow & \coprod_{i \in I} g^* c_i \\
{\scriptstyle \langle N_{(u_i, 1_{\{\vec{x}^{\vec{A}}.\, \phi\}})} \rangle_{i \in I}} \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle \langle g^* u_i \rangle_{i \in I}} \\
N_{(c,a)} & \longrightarrow & g^* c
\end{array}$$

ensuring that $N_{(-)}$ sends the horizontal covering family

$$\big((u_i, 1_{\{\vec{x}^{\vec{A}}.\, \phi\}}) : (c_i,\, (\{\vec{x}^{\vec{A}}.\, \phi\}, a \circ u_i)) \to (c,\, (\{\vec{x}^{\vec{A}}.\, \phi\}, a))\big)_{i \in I}$$

to an epimorphic family.

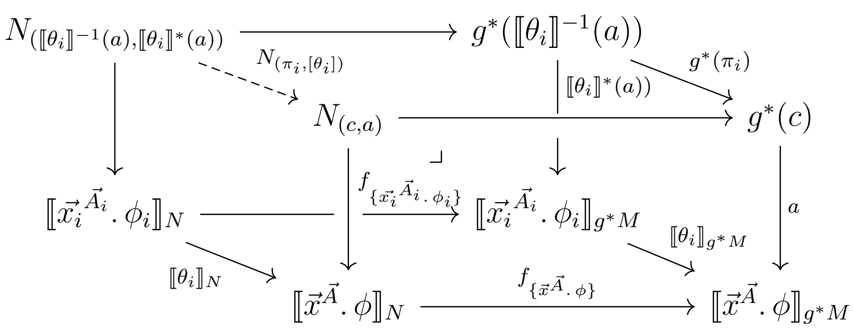
Now we turn to the proof of the continuity of $N_{(-)}$ with respect to the covering families of horizontal arrows for the fibration r_{f_M}.

Given a family $\big([\theta_i] : \{\vec{x_i}.\, \phi_i\} \to \{\vec{x}.\, \phi\}\big)_{i\in I}$ in $\mathcal{B}_{\mathbb{T}}$ and a generalized element $a : c \to [\![\vec{x}^{\vec{A}}.\, \phi]\!]_M$, we want to prove that $N_{(-)}$ sends the family

$$\big((\pi_i, [\theta_i]) : ([\![\theta_i]\!]_M^{-1}(a), (\{\vec{x_i}.\, \phi_i\}, [\![\theta_i]\!]_M^{*}(a))) \to (c, (\{\vec{x}.\, \phi\}, a))\big)_{i\in I}$$

in $\int \mathbb{M}$ to an epimorphic family.

Consider, for each $i \in I$, the following diagram:



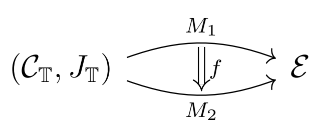
The front square is a pullback by definition of $N_{(c,a)}$, the back square is a pullback by definition of $N_{([\![\theta_i]\!]^{-1}(a),[\![\theta_i]\!]^*(a))}$ and the right-hand lateral one is a pullback since g^* is cartesian. So the composite of the back square with the right-hand lateral square is a pullback, and hence by the pullback lemma the left-hand lateral square is also a pullback (as the front square is a pullback). Our thesis thus follows from the fact that, since N is a $\mathbb{T}$-model by our hypotheses, the family

$$\big([\![\theta_i]\!]_N : [\![\vec{x_i}^{\vec{A}_i}.\, \phi_i]\!]_N \to [\![\vec{x}^{\vec{A}}.\, \phi]\!]_N\big)_{i\in I}$$

is epimorphic.

Finally, proving that the functors defined above are mutually quasi-inverses is a straightforward exercise which we leave to the reader. ∎

To conclude this section, we remark that our construction of the over-topos is functorial; that is, a morphism of $\mathbb{T}$-models naturally induces a canonical morphism between the corresponding $\mathbb{T}$-over-toposes. Let $f : M_1 \to M_2$ be a morphism in $\mathbb{T}[\mathcal{E}]$. This is a same as a natural transformation



whose components

$$[\![\vec{x}^{\vec{A}}.\, \phi]\!]_{M_1} \xrightarrow{f_{\{\vec{x}^{\vec{A}}.\, \phi\}}} [\![\vec{x}^{\vec{A}}.\, \phi]\!]_{M_2}$$

are indexed by the objects of $\mathcal{C}_{\mathbb{T}}$. This induces an indexed functor $\alpha : \mathbb{M}_1 \Rightarrow \mathbb{M}_2$ assigning to each object c of $\mathcal{C}$ the functor

$$(c \downarrow F_{M_1}) \xrightarrow{\alpha_c = (c \downarrow f)} (c \downarrow F_{M_2})$$

sending a basic global element $a : c \to [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{M_1}$ to $f_{\{\vec{x}^{\vec{A}}.\,\phi\}} \circ a : c \to [\![\vec{x}^{\vec{A}}.\,\phi]\!]_{M_2}$. This clearly yields a comorphism of sites

$$((i_{\mathcal{C}} \downarrow F_{M_1}), J^{\text{ant}}_{M_1}) \to ((i_{\mathcal{C}} \downarrow F_{M_2}), J^{\text{ant}}_{M_2})$$

over $\mathcal{C}$, which thus induces a geometric morphism

$$\mathcal{E}[\alpha] : \mathcal{E}[\mathbb{M}_1] \to \mathcal{E}[\mathbb{M}_2]$$

over $\mathcal{E}$ between the associated over-toposes.

4.11. Corollary. *For any geometric theory $\mathbb{T}$, $(\mathcal{C}^2_{\mathbb{T}}, J_{U_{\mathbb{T}}})$ is a small, cartesian site for* $\mathbf{Set}[\mathbb{T}]^2$.

Proof. It suffices to apply the construction of the $\mathbb{T}$-over-topos provided by Theorem 4.10 in the particular case $\mathcal{E} = \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$, $M = U_{\mathbb{T}}$ and $(\mathcal{C}, J) = (\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$. Note that $i_{\mathcal{C}}$ is simply the Yoneda embedding $y_{\mathbb{T}} : \mathcal{C}_{\mathbb{T}} \hookrightarrow \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$, and also F_M coincides with $y_{\mathbb{T}}$, whence the site for $\mathbf{Set}[\mathbb{T}][U_{\mathbb{T}}]$ provided by Definition 4.8 is $(\mathrm{y}_{\mathbb{T}} \downarrow \mathrm{y}_{\mathbb{T}}) \simeq \mathcal{C}^2_{\mathbb{T}}$, that is, the arrow category of $\mathcal{C}_{\mathbb{T}}$. ∎

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