Theory and Applications of Categories, Vol. 39, No. 1, 2023, pp. 1–20.

JOHNSTONE-GLEASON COVERS FOR PARTIALLY ORDERED SETS

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ABSTRACT. In 1958, Andrew Gleason proved that for every compact Hausdorff space X there exists an extremally disconnected compact Hausdorff space $\tilde{X}$ and a continuous surjection $p : \tilde{X} \to X$ with the property that every other continuous surjection from an extremally disconnected compact Hausdorff space onto X factors via surjection through p. Later, several authors have extended this construction to wider contexts, including the Gleason cover for an elementary topos introduced by Johnstone in 1980.

We investigate properties of the Gleason cover for not necessarily sober T_0 Alexandroff spaces, i. e. spaces determined by partially ordered sets. First, we introduce the notion of co-local homeomorphism for such spaces, and prove that for every finite T_0 topological space X there exists a unique irreducible co-local homeomorphism $p : \tilde{X} \to X$ from finite extremally disconnected space $\tilde{X}$ onto X. Next, we extend this approach to arbitrary Alexandroff topological spaces. We finish with several characterizations of Alexandroff spaces with Alexandroff Gleason covers.

1. Introduction

The classical Gleason cover $\tilde{X}$ of a compact Hausdorff space X is the Stone dual of the complete Boolean algebra of its regular closed sets (regular open sets may be used, equivalently) [Gleason, 1958]. It is equipped with an irreducible surjective continuous map $\tilde{X} \twoheadrightarrow X$ and is defined uniquely up to homeomorphism.

Many authors have introduced various generalizations of this construction to many classes of spaces — see, among others, [Flachmeyer, 1963, Iliadis, 1969, Mioduszewski, 1969, Banaschewski, 1971, Błaszczyk, 1974, Ul'janov, 1975, Šapiro, 1976, Porter & Woods, 1988], — mostly under the name of *absolute*.

We are investigating properties of the Gleason cover in case of not necessarily sober T_0 Alexandroff spaces, i. e. arbitrary partially ordered sets with the Alexandroff topology.

In 1980, Johnstone introduced a construction of the Gleason cover for an arbitrary elementary topos [Johnstone, 1980, Johnstone, 1981] which in particular gives certain version of absolute for any topological space and, more generally, for *locales* in point-free topology. Moreover, the Johnstone-Gleason cover $\tilde{\mathscr{E}}$ of a topos $\mathscr{E}$ can be uniquely characterized as the topos of sheaves over a minimal compact regular extremally disconnected locale in $\mathscr{E}$.

Received by the editors 2021-03-12 and, in final form, 2023-01-05.
Transmitted by Dirk Hofmann. Published on 2023-01-09.
2020 Mathematics Subject Classification: 06F30, 18F60, 18F70, 54B30, 54G05, 54D80.
Key words and phrases: Gleason cover, Alexandroff space, co-local homeomorphism.

Johnstone has demonstrated that his construction indeed gives what is commonly known as the Iliadis absolute for regular topological spaces. More precisely, he has proved that for the topos $\mathrm{Sh}(X)$ of sheaves on a regular space X his construction gives the topos equivalent to $\mathrm{Sh}(\tilde{X})$, where $\tilde{X}$ is the Iliadis absolute of X. However extending such equivalence to not necessarily regular spaces seems to be more subtle and, to our best knowledge, has not been investigated in the existing literature. This issue becomes especially problematic for non-sober spaces, given that the sheaf topos construction (even already the passage from a space to the lattice of its open sets) does not distinguish between a space and its soberification. Although in [Johnstone, 1981] several results deal with non-sober spaces, it is not clear how is his construction related to any possible absolute constructions in the literature cited above.

In [Johnstone, 1980] also the explicit construction of the Gleason cover is provided for the topos of sheaves on a finite category. In the particular case when this category corresponds to a finite partially ordered set, we show that this construction is isomorphic to one obtained by us in [Abashidze, 2016]. However, in our work we encountered an interesting phenomenon not addressed to by Johnstone. When trying to prove the uniqueness of the Gleason cover of a finite T_0 space, we could not use the key projectivity property of extremally disconnected spaces, which does not hold in the category of topological spaces and arbitrary continuous maps. For general spaces uniqueness of the Gleason cover is usually established in a different way — namely, by restricting the class of maps between spaces. In the papers cited above, either the class of *proper* maps or that of *θ-continuous* maps is used.

In [Abashidze, 2016], unaware of this fact, we acted differently to circumvent non-projectivity of extremally disconnected spaces with respect to arbitrary continuous maps; We observed that the Gleason cover map in this case is a *co-local homeomorphism*, which, by definition, means that it becomes a local homeomorphism if we replace spaces corresponding to finite posets with the ones corresponding to the same posets but with the opposite ordering. We proved that the co-local homeomorphism property can be used to prove uniqueness of the Gleason cover.

We address the possibility of extending this approach to Alexandroff topologies of arbitrary partially ordered sets. When this topology is sober, the generalization is relatively straightforward. As it is well known, the Alexandroff topology of a poset P is sober if and only if P is *Noetherian* [Picado & Pultr, 2011], i. e. does not possess infinite ascending chains. As it happens, in this case construction of the Gleason cover merely repeats that of the finite case: we take disjoint unions of downsets of all maximal elements of P

$$\tilde{P} = \bigoplus_{m \in \max(P)} \downarrow m.$$

Note that the resulting map is easily seen to be a co-local homeomorphism, so we can generalize our approach to the proof of uniqueness to this case. However, for non-sober P, the Johnstone-Gleason cover will not necessarily produce a spatial locale, let alone an Alexandroff space of some other poset.

We finish with a characterization of those posets P for which the Gleason cover of the corresponding Alexandroff space is itself Alexandroff. Namely, we prove

Theorem. *The Gleason cover of an Alexandroff topological space X is an Alexandroff space iff for any $x \in X$ and for any infinite antichain $S \subseteq \uparrow x$ above x there exist $y_1, y_2 \in S$ such that $\uparrow y_1 \cap \uparrow y_2 \neq \varnothing$.*

2. Preliminaries

2.1. Definition. *For a family of topological spaces X_i, $i \in I$, the (topological) sum of this family, i. e. the coproduct in the category of topological spaces and continuous maps, is denoted by*

$$\bigoplus_{i \in I} X_i = \bigcup_{i \in I} X_i \times \{i\}.$$

For each $i_0 \in I$ the corresponding coproduct inclusion will be denoted by $j_{i_0} : X_{i_0} \longrightarrow \bigoplus_{i \in I} X_i$.

We will need the following well know fact.

2.2. Definition. [Willard, 1970]*A map $f : X \longrightarrow Y$ is a quotient map if it is surjective, and a subset U of Y is open if and only if $f^{-1}(U)$ is open.*

Closed and open surjections are quotient maps [Willard, 1970].

A surjective map $f : X \longrightarrow Y$ between Alexandroff spaces is a quotient map if and only if for any $V \subseteq Y$, $f^{-1}(V)$ is an upset if and only if V is an upset in Y.

2.3. Definition. [Porter & Woods, 1988] *Let X and Y be topological spaces and let f be a closed surjection from X onto Y. Then f is called irreducible if, whenever A is a proper closed subset of X, $f(A) \neq Y$.*

2.4. Definition. *Let $(A, \leq)$ be an ordered set. We will denote by $(A, \leq)^o = (A, \geq)$ this set with the opposite order on it, hereafter $A^o = (A, \leq)^o$.*

We will use this notation for Alexandroff spaces as well. We will need to introduce the following notion which is dual to local homeomorphism and has the corresponding properties dual to that of a local homeomorphism (it is closed for example).

2.5. Definition. *A map $f : X \longrightarrow Y$ between Alexandroff topological spaces will be called a* co-local homeomorphism *if $f : X^o \longrightarrow Y^o$ is a local homeomorphism.*

2.6. Proposition. [Erné, 1991]*Any nonempty finite ordered set has a maximal element.*

2.7. Definition. *For a topological space X we define the specialization order $\leq_\tau$ on X: for any $x, y \in X$ we will write $x \leq_\tau y$ if and only if for all $O \in \tau$, $x \in O$ implies $y \in O$.*

For any subset S of a partially ordered set we will denote the set of maximal elements of S by $\max(S)$.

2.8. PROPOSITION. [Erné, 1991]*Suppose* $(X, \leq)$ *is a finite poset. For any* $A \subset X$ *we have* $\downarrow A = \downarrow \max(A)$.

2.9. PROPOSITION. [Erné, 1991]*Let* X *be an Alexandroff topological space with the specialization order* $\leq_\tau$. *Then for any* $S \subset X$ *its interior is given by* $int(S) = \{x \in S \mid \uparrow x \subseteq S\}$.

2.10. PROPOSITION. [Erné, 1991]*Let* X *be an Alexandroff topological space with specialization* $\leq_\tau$. *Then for* $S \subset X$ *one has* $\overline{S} = \bigcup\{\downarrow x \mid x \in S\}$.

Recall that a set $A \subset X$ is called regular closed (resp. open) if $A = \overline{\mathrm{int}(A)}$ (resp. $A = \mathrm{int}(\overline{A})$). The set of regular closed (resp. open) sets of the topological space X will be denoted by $R(X)$ (resp. $RO(X)$).

2.11. PROPOSITION. *Let* X *be a finite* T_0 *topological space with specialization* $\leq_\tau$, *then for any* $F \in R(X)$, *where* $R(X)$ *is the set of regular closed sets of* X, *we have* $\max(F) = F \cap \max(X)$.

PROOF. $F \cap \max(X) \subset \max(F)$ because $F \cap \max(X) = \max(F) \cap \max(X) \subset \max(F)$. Let us show that also conversely $F \cap \max(X) \supset \max(F)$. Suppose not, and there is an $x \in \max(F)$ such that $x \notin \max(X)$, so that there is an $x' \notin F$ with $x \leq x'$. From this by definition of interior $x \notin \mathrm{int}(F)$ and because $x \in \max(F)$, there does not exist $x_0 \in \mathrm{int}(F)$ with $x \leq x_0$. Hence by definition of closure $x \notin \overline{\mathrm{int}(F)}$, and from this we conclude, that $F \notin R(X)$. ∎

Recall that Alexandroff spaces are locally connected, and locally connected spaces are topological sums of connected clopens, so that we have

2.12. PROPOSITION. [Erné, 1991]*Every Alexandroff topological space is the sum of its connected clopen subspaces.*

2.13. PROPOSITION. [Erné, 1991] *Any open subset of an extremally disconnected space is extremally disconnected in the induced topology.*

2.14. PROPOSITION. [Erné, 1991] *A topological space* X *is extremally disconnected and connected if and only if every nonempty open set of* X *is dense in* X.

2.15. COROLLARY. *For an Alexandroff topological space* X, *the following two assertions are equivalent:*

1. X *is extremally disconnected and connected.*
2. *the space* X *is a directed set with respect to* $\leq_\tau$.

PROOF. $1 \Rightarrow 2$ Let us consider any $x, y \in X$. As X is extremally disconnected and connected, since $\uparrow x$ is an open set, it follows that $\downarrow\uparrow x = X$, thus $y \in \downarrow\uparrow x$. Therefore, for any two different points $x, y \in X$ there exists a third point $z \in X$ such that $x, y \leq_\tau z$.

$2 \Rightarrow 1$ Suppose X is directed with respect to $\leq_\tau$. Consider any nonempty open set U. Let us show that $x \in \overline{U}$ for any $x \in X$. As X is directed, for any $x_0 \in U$ and any $x \in X$ there exists $z \in X$ such that $x, x_0 \leq z$. Since every open set is an upset, U is an upset, so $z \in U$. It thus follows that $x \in \overline{U}$. ∎

2.16. Corollary. *Let X be an extremally disconnected Alexandroff space, and let*

$$X = \bigoplus_{I \in I} A_i$$

be its decomposition into topological sum of connected clopen Alexandroff spaces as in Proposition 2.12. Then each A_i is a directed subset of X.

Proof. By virtue of Proposition 2.12 X is a topological sum of connected spaces. Each of them will be clopen in X. So according to Proposition 2.13 they are extremally disconnected, hence by Corollary 2.15 each of them is a directed set. It is easy to see that union of these directed sets will be X. ∎

3. Finite Case

In case X is compact Hausdorff, the total space $\tilde{X}$ of its Gleason cover $p : \tilde{X} \longrightarrow X$ can be taken to be the Stone space of the Boolean algebra $R(X)$ of its regular closed sets [Porter & Woods, 1988].

It is known for such spaces that for any ultrafilter S of $R(X)$, the intersection of all those regular closed sets which are elements of S is a singleton and the map p sends S to the unique element of this singleton intersection.

We want to find a similar fact for arbitrary T_0 spaces. In this section we consider only finite spaces.

Because $R(X)$ also will be finite, its ultrafilters will be in one-to-one correspondence with its atoms.

So it is interesting to study what are atoms of $R(X)$.

3.1. Proposition. *If a topological space X is a finite T_0 space, then the atoms $A \in R(X)$ are precisely the sets of the form $\downarrow a$ for $a \in \max(X)$.*

Proof. Let us show, that $\max(A)$ is a singleton. Indeed let us show that if $a, a' \in \max(A)$ and $a \neq a'$ then A is not an atom of $R(X)$.

As we know that A is regular closed, by Proposition 2.11 we have $\max(A) \subseteq \max(X)$. Thus $a, a' \in \max(X)$, so that $\uparrow a = \{a\}$, $\uparrow a' = \{a'\}$, i. e. each singleton $\{a\}$, $\{a'\}$ is open, so that (using Proposition 2.6) $\downarrow a, \downarrow a' \in R(X)$. But $a' \notin \downarrow a$, since a' is maximal. hence $\downarrow (a) \subsetneqq A$. So A contains a proper regular closed subset, hence is not an atom. ∎

Let $A_1, ..., A_n$ be all atoms of $R(X)$, and denote $N = \{1, 2, ..., n\}$. By Proposition 3.1 each A_i, $i = 1, ..., n$ is a downset, which has the greatest element, which also is in the maximum of the space X. Consider the atoms A_i, $i \leq n$ as spaces with the induced topology, form their topological sum and denote it by

$$\tilde{X} := \bigoplus_{i \in N} A_i.$$

We also can consider this sum as

$$\bigoplus_{i\in N} \tilde{A}_i = \{(x,a) \mid x \leq a, a \in \max(X)\},$$

where $\tilde{A}_i$ denotes the copy of A_i inside $\tilde{X}$. The specialization order in these terms will be:

$$(x,a_1) \leq (y,a_2) \ \Leftrightarrow \ x \leq y \ \&\ a_1 = a_2.$$

Also note, that

$$(a,a) \leq (x,a) \ \Rightarrow \ a \leq x \ \Rightarrow \ a = x.$$

Also, because always $(x,a) \leq (a,a)$, we obtain that (x,a) is a maximal element of $\tilde{X}$ if and only if $x = a$. This fact will be convenient for us afterwards; the number of maximal elements of this sum will be the same as in the maximum of the space X, because downset of a maximal element in X is a regular closed set, which will be an atom of $R(X)$. Obviously from this sum we have a continuous and surjective map $p : \bigoplus_{i\in N} A_i \to X$, namely

$$p(x,a) = x.$$

Moreover p is a closed map. Indeed it is the induced map from the topological sum of closed subsets A_i of X to X. Since this sum is finite, this induced map is also closed. So the space $\tilde{X}$ which is constructed by us is a candidate for the Gleason cover of X.

3.2. Lemma. *The map $p : \tilde{X} \longrightarrow X$ where X is a finite T_0 topological space and $p(x,a_i) = x, a_i \in \max(\tilde{X})$, is irreducible.*

Proof. Suppose to the contrary that there is a closed set $F \subsetneqq \tilde{X}$ such that $p(F) = X$. Then in the maximum of $\tilde{X}$ there exists an element (x,x) which does not belong to F, since otherwise F would be equal to $\tilde{X}$. Since

$$x = p(x,x) \in p(F) = X,$$

there exists $(y',x') \in F$ such that $p(y',x') = x$. From the facts $y' = x$, $x' \leq x$ and x' is maximal, it follows that $x' = x$, so we conclude that $y = x$ and $(x,x) \in F$. We obtained contradiction. Lemma is proved. ∎

3.3. Lemma. *For a finite T_0 space X, the map $p : \tilde{X} \longrightarrow X$ is closed.*

Proof. Indeed, let us consider any point $(x,a) \in \tilde{X}$; its closure is $\downarrow(x,a) = \downarrow(x) \times \{a\}$, and

$$p(\downarrow(x,a)) = p(\downarrow(x) \times a) = \downarrow(x).$$

Now consider a closed set $F \subset \tilde{X}$. Clearly $F = \bigcup_{(x,a)\in F} \downarrow(x,a)$. From this we have

$$p(F) = p(\bigcup_{(x,a)\in F} \downarrow(x,a)) = \bigcup_{(x,a)\in F} p(\downarrow(x,a)) = \bigcup_{(x,a)\in F} \downarrow(x).$$

As we know X is a finite space, hence this union is closed too. ∎

3.4. Lemma. *Let X be a finite T_0 space. Then for the map $p : \tilde{X} \longrightarrow X$ the restriction $p|_{\tilde{A}} : \tilde{A} \longrightarrow A$ is a homeomorphism for every atom A of $R(X)$.*

Proof. This is a particular case of the following obvious fact: if we have any family of homeomorphisms $\tilde{A}_i \approx A_i$ for some subspaces $A_i \subseteq X$, then under the induced map $\bigoplus_i \tilde{A}_i \longrightarrow X$, each subspace $\tilde{A}_i \subseteq \bigoplus_i \tilde{A}_i$ maps homeomorphically onto its image in X. ∎

3.5. Corollary. *The map $p : \tilde{X} \longrightarrow X$ is a* co-local homeomorphism*.*

Proof. The map P has the following property:

$$\forall\ x \in \tilde{X}\ \exists\ x \in F \subset \tilde{X}\ p|_F : F \xrightarrow{\approx} p(F);$$

that is, for any $x \in X$ there exists a closed set $F \ni x$ such that $p|_F$ is a homeomorphism onto its image. ∎

3.6. Theorem. *Let X be a finite T_0 topological space and consider the map $p : \tilde{X} \longrightarrow X$ constructed above. If we have a map $f : Y \longrightarrow X$ from some finite extremally disconnected Y to X, then there exists a continuous map $\pi : Y \longrightarrow \tilde{X}$ such that $p \circ \pi = f$.*

Proof. By Corollary 2.16 we may assume that Y is a topological sum of finite spaces, each of them having the greatest element, so

$$Y = \bigoplus_{i \in N} \downarrow (y_i).$$

Hence by universal property of topological sum we can define the map π for every $\downarrow (y_i)$ separately, as follows: for every $i \in I$, for $f(y_i)$ there exists an atom of regularly closed sets $A_k \subset X$ such that $f(y_i) \in A_k$ (obviously $f(\downarrow y_i) \subset A_k$). For $y \in (\downarrow y_i)$ we have $\pi(y) = j_k(f(y))$ where j_k is defined as in Definition 2.1. From this we conclude that $p \circ \pi = f$ for every $\downarrow (y_i)$. Indeed π equals the composite

$$\downarrow (y_i) \subset Y \xrightarrow{f} A_k \subset \tilde{X}.$$

At the same time the composite

$$A_k \subset \tilde{X} \xrightarrow{p} X$$

coincides with the inclusion of A_k into X. Hence the composite

$$\downarrow (y_i) \subset Y \xrightarrow{f} A_k \subset \tilde{X} \xrightarrow{p} X$$

coincides with the restriction of f to $\downarrow (y_i)$. ∎

3.7. Theorem. *For every irreducible surjective co-local homeomorphism $f : Y \longrightarrow X$ from a finite extremally disconnected space Y to a finite T_0-space X, the space Y is homeomorphic to $\tilde{X}$.*

Moreover, there exists a homeomorphism $h : Y \longrightarrow \tilde{X}$ such that $p \circ h = f$.

Proof. By virtue of Theorem 3.6 there exists $h : Y \longrightarrow \tilde{X}$ and $h' : \tilde{X} \longrightarrow Y$ such that $p \circ h = f$ and $f \circ h' = p$.

As we know [Johnstone, 2002] h' and h are co-local homeomorphisms, in particular $h(Y)$ is a closed subset of $\tilde{X}$. Since p is an irreducible map and $p \circ h(Y) = X$, we conclude that $h(Y) = \tilde{X}$. Similarly $h'(\tilde{X}) = Y$ since $\tilde{X}$ and Y are finite spaces, $\tilde{X}$ and Y have the same cardinalities, so any surjective map between them is bijective. Dually to the well known fact that any bijective local homeomorphism is a homeomorphism, also clearly every bijective co-local homeomorphism is a homeomorphism. ∎

We have thus obtained that for every finite T_0 topological space X there exists an up to homeomorphism unique irreducible co-local homeomorphism $p : \tilde{X} \longrightarrow X$ from a finite extremally disconnected space $\tilde{X}$ onto X.

4. General Case

In this section we present a general construction of the Gleason cover for an arbitrary topological space as given for example in [Šapiro, 1976].

Let X be topological space and let K be the set of all open ultrafilters on X. In what follows, by "ultrafilter" we will always mean open ultrafilter, i. e. maximal filter of the lattice of open sets, if not otherwise stated.

Denote

$$\tilde{X} = \{(\xi, x) \in K \times X : x \in \bigcap_{U \in \xi} \overline{U}\}.$$

The point $(\xi, x) \in \tilde{X}$ will be denoted by ξ_x. Let us define the natural projection by $\pi_X : \tilde{X} \to X$, $\pi_X(\xi_x) = x$.

For each open set $U \in \tau(X)$, let $O_U = \{\xi_x \in \tilde{X} \mid U \in \xi\}$. We note the following main properties of the sets O_U.

a) $O_{U \cup V} = O_U \cup O_V$

b) $O_{U \cap V} = O_U \cap O_V$

c) $\overline{U} = \overline{V} \Rightarrow O_U = O_V$

The sets $O_U \cap \pi_X^{-1}(H)$ for all open sets U, H of X form a base for a topology. By definition, this is the topology on $\tilde{X}$ that we will consider.

Denote: $x \tilde{\in} F$ iff x is convergent point of F, i.e. $(F, x) \in \tilde{X}$.

4.1. Theorem. *Let $(X_i)_{i \in I}$ be a family of topological spaces and for each i let $p : \tilde{X}_i \to X_i$ be its Gleason cover. Then the Gleason cover of the topological sum $\bigoplus_{i \in I} X_i$ is homeo-*

morphic to the topological sum $\bigoplus_{i\in I}\tilde{X}_i$.

$$\widetilde{(\bigoplus_{i\in I} X_i)} \cong \bigoplus_{i\in I}\tilde{X}_i.$$

Proof. This follows from the following three facts: that a topological sum of extremally disconnected spaces is extremally disconnected, that a topological sum of closed maps is closed, and that a topological sum of irreducible maps is irreducible. ■

4.2. Lemma. *Let X be an Alexandroff space. A point $x \in X$ is a convergent point of the ultrafilter F, i. e. $x\tilde{\in}F$, if and only if $\uparrow x \in F$.*

Proof. Necessity : Suppose $\uparrow x \in F$. Then for all $U \in F$ we have $\uparrow x \cap U \neq \varnothing$ which means that $x\tilde{\in}F$.

Sufficiency: Suppose to the contrary that there exists an $x\tilde{\in}F$ with $\uparrow x \notin F$. Then $\mathrm{int}(X\setminus \uparrow x) \in F$ since $\uparrow x \cap \mathrm{int}(X\setminus \uparrow x) = \varnothing$ and F is an ultrafilter. But $x\tilde{\in}F$ means that $\uparrow x \cap U \neq \varnothing$ for all $U \in F$, contradiction. ■

4.3. Lemma. *Let X be an Alexandroff space. A point $x \in X$ belongs to an atom $A \in R(X)$ of regular closed sets if and only if $A \subseteq\downarrow\uparrow x$.*

Proof. Necessity : Suppose $A \subseteq\downarrow\uparrow x$. Then $\mathrm{int}(A) \subseteq\downarrow\uparrow x$. Since $\downarrow\uparrow x$ is the closure of $\uparrow x$ in the Alexandroff topology, by definition of closure we conclude that any neighborhood O_y of any $y \in \mathrm{int}(A)$ has nonempty intersection with $\uparrow x$; in particular because $\mathrm{int}(A)$ is an open neighborhood of y, there exists an $y' \in (\mathrm{int}(A)\cap \uparrow x)$. Since $x \leq y' \in \mathrm{int}(A)$ we have $x \in\downarrow (\mathrm{int}(A)) = A$.

Sufficiency: Suppose $x \in A$, where A is atom of $R(X)$. Since $x \in A =\downarrow \mathrm{int}(A)$, there exists a $z \in\uparrow x \cap \mathrm{int}(A)$. Hence $\uparrow z \subset \mathrm{int}(A)$ and $\downarrow\uparrow z \subset\downarrow (\mathrm{int}(A)) = A$. Since $\downarrow\uparrow z$ is a regular closed set and A is an atom of regular closed sets, $\downarrow\uparrow z = A$. Thus, since $\downarrow\uparrow z \subseteq\downarrow\uparrow x$, we obtain that $A \subseteq\downarrow\uparrow x$. ■

4.4. Lemma. *In the space $\tilde{X}$, $\eta_x \leq \psi_y$ holds in the specialization order of its topology iff both $(\eta = \psi)$ and $(x \leq y)$ in the specialization order of X.*

Proof. Necessity: If $\eta = \psi$ and $x \leq y$, we have to show that $\eta_x \leq \psi_y$. It is the same as to show that $\eta_x \in V \Rightarrow \psi_y \in V$, for any open set V of $\tilde{X}$. It suffices to show this when V is a basic open set, $V = O_{V'} \cap \pi_X^{-1}(H)$. Then $\eta_x \in V$ implies that $(x \in H)$ and $(V' \in \eta)$.

Since $x \leq y$, we have that $y \in H$ and because $\eta = \psi$, $V' \in \psi$. Thus $\psi_y \in V$, and we obtain that $\eta_x \leq \psi_y$.

Sufficiency: Let us show that, if $\eta_x \in V \Rightarrow \psi_y \in V$, for any open V, then $\eta = \psi \wedge x \leq y$. In particular the assumption is true for $V = O_U \cap \pi_X^{-1}(H)$. Thus, if $\eta_x \in O_U \cap \pi_X^{-1}(H)$ then $\psi_y \in O_U \cap \pi_X^{-1}(H)$.

Assume $H = X$, then $U \in \eta$ implies $U \in \psi$ for any U, thus $\eta \subset \psi$. Since η and ψ are ultrafilters, we conclude that $\eta = \psi$. Hence, for any H we have

$$(U \in \eta) \wedge (x \in H) \Rightarrow (U \in \eta) \wedge (y \in H) \Rightarrow x \leq y.$$

■

4.5. LEMMA. *In an Alexandroff topological space X, a point $x \in X$ is convergent point of no more than finite number of ultrafilters if and only if for any family of points $(y_j)_{j\in J}$ satisfying $y_j \geq x$ and $\uparrow(y_j)\cap\uparrow(y_k) = \varnothing$ for any $j \neq k$, $j, k \in J$, the set J is finite.*

PROOF. Necessity: Let us suppose to the contrary that $x\tilde{\in}F_i$, $i \in I$, where I is infinite. Let us show that there is a set J of any finite cardinality such that for any $i, j \in J$ $y_j > x$ and $\uparrow y_i \cap \uparrow y_j = \varnothing$ for $i \neq j$.

For F_1, F_2, there exist $U_1 \in F_1, U_2 \in F_2$ such that $U_1 \cap U_2 = \varnothing$. For F_1, F_2, F_3 there exist $U_1 \in F_1, U_2 \in F_2, U_3 \in F_3$ such that $U_1 \cap U_2 = \varnothing, U_1 \cap U_3 = \varnothing, U_2 \cap U_3 = \varnothing$... For $F_1, ..., F_j$ there exist $U_1 \in F_1, ..., U_j \in F_j$ such that $U_k \cap U_j = \varnothing$, $\forall k, j \in J$, $k \neq j$, for $J = \{1, ..., j\}$. For each U_j, since $x \in \overline{U}_j$, there exists $y_j \in U_j$ with $y_j > x$; let us fix such $y_j \in U_j$, so that $y_j > x, j \in J$ and $\uparrow y_i \cap \uparrow y_j = \varnothing$ for all $j, i \in J \subset I$, $j \neq i$. Thus for each natural number j we found pairwise non-intersecting $\uparrow y_1$, ..., $\uparrow y_j$ with $y_1, ..., y_j > x$.

Sufficiency: Let us show that for any point $x \in X$, if there is a family y_j indexed by an infinite set J such that $y_j > x, j \in J$ and $\uparrow y_i \cap \uparrow y_j = \varnothing$ for all $j, i \in J$ with $i \neq j$, then there is an infinite set of ultrafilters for which x is a convergent point.

Indeed: for each $y_j > x, j \in J$ there exists an ultrafilter F_j such that $\uparrow y_j \in F_j$. For all $U \in F_j$, $\uparrow y_j \cap U \neq \varnothing$, thus $y_j \in\downarrow U$. This implies that $x \in\downarrow U$ for all $U \in F_j$, so that $x\tilde{\in}F_j$. ■

4.6. LEMMA. *Let an Alexandroff topological space X be given with an infinite set of ultrafilters Σ, such that the point $x \in X$ is convergent point of each ultrafilter from Σ. Then X possesses a free (non-principal) ultrafilter with x as a convergent point.*

PROOF. Suppose $F_k \in \Sigma$, $k \in K$, where K is infinite. On the set K we have a free ultrafilter Φ. Consider the filter $\tilde{F} = \{U \mid \{k \mid U \in F_k\} \in \Phi\}$. Let us show that $\tilde{F}$ is an ultrafilter. This means to show that for any open U, $\tilde{F}$ contains U or $\mathrm{int}(X \setminus U)$. Suppose $U \notin \tilde{F}$, which means that $S_U := \{k \mid U \in F_k\} \notin \Phi$. Since F_k are ultrafilters, $K \setminus S_U = \{k \mid \mathrm{int}(X \setminus U) \in F_k\} \in \Phi$. This implies $\mathrm{int}(X \setminus U) \in \tilde{F}$.

Now let us show that $\tilde{F}$ is a free ultrafilter. If any of the F_k is non-principal, we are done. Otherwise for each F_k by virtue of Lemma 3.1 from [Johnstone, 1980] we have a corresponding ultrafilter on $RO(X)$ which is generated by an atom $A_k \in RO(X)$. So we may assume that each F_k is principal, generated by an atom $A_k \in RO(X)$, thus $F_k = \{U \mid \mathrm{int}(\downarrow U) \supseteq A_k\}$. Suppose $\tilde{F}$ is principal, then there exists an atom $A \in RO(X)$ such that $\tilde{F} = \{U \mid \mathrm{int}(\downarrow U) \supseteq A\}$. Since $A \in \tilde{F}$, we have that $I := \{i \mid A \in F_i\} \subset \Phi$, and for any $i \in I$, $\mathrm{int}(\downarrow A) \supseteq A_i$. But A is an atom of $RO(X)$, so there exists at most one $A_i \in RO(X)$ such that $\mathrm{int}(\downarrow A) \supseteq A_i$. ■

4.7. LEMMA. *Let a and b be two elements of Boolean algebra B. If $a \in \eta$ implies $b \in \eta$ for any ultrafilter η of B, then $a \leq b$.*

PROOF. Let us show equivalently, that if $a \not\leq b$ then there exists an ultrafilter η such that $a \in \eta$ and $b \notin \eta$. Indeed, in any Boolean algebra $a \not\leq b$ if and only if $a \wedge \neg b \neq 0$. If the latter holds, then there exists an ultrafilter η such that $a \wedge \neg b \in \eta$. Thus $a \in \eta$ and $\neg b \in \eta$, therefore $b \notin \eta$. ■

4.8. THEOREM. *The Gleason cover of an Alexandroff topological space X is an Alexandroff space iff for any point $x \in X$ and any family $(y_k)_{k\in K}$ with $x < y_k$, $k \in K$ such that $\uparrow y_i \cap \uparrow y_j = \varnothing$, $i, j \in K$, $i \neq j$, the set K is finite.*

PROOF. Necessity: Suppose to the contrary that the Gleason cover of an Alexandroff topological space X is Alexandroff itself and there exist $x \in X$, $x < y_k$, $k \in K$ with $\uparrow y_k \cap \uparrow y_j = \varnothing$, for $k, j \in K$, $k \neq j$, where K is an infinite set. By virtue of Lemma 4.5 x is a convergent point of an infinite number of ultrafilters. For each y_k, $k \in K$ there is an ultrafilter $F_k \ni \uparrow y_k$, and by Lemma 4.2 $x \tilde{\in} F_k$. By Lemma 4.6 there is a free ultrafilter $\tilde{F}$ with $x \tilde{\in} \tilde{F}$.

Consider the point $\tilde{F}_x$ of the Gleason cover $\tilde{X}$. Let us show that for all $O \in \tau(\tilde{X})$ containing $\tilde{F}_x$ there exists a proper open subset of O containing $\tilde{F}_x$. Indeed, because of the way the Gleason cover is constructed we may restrict to the case when $O = O_U \cap \pi_X^{-1}(H)$, where $U \in \tilde{F}$ and $H = \uparrow x$, as $\uparrow x$ is smallest neighbourhood of x. $\tilde{F}$ is a free ultrafilter, so according to the construction $I = \{i \mid U \in F_i\} \in \Phi$, hence $\{i \mid U \cap \uparrow y_i \neq \varnothing\} \subseteq I$.

$$\bigcup_{i\in I} \uparrow_U y_i \subseteq U,$$

where $\uparrow_U y_i = \uparrow y \cap U$. Let us denote $U' = \bigcup_{k\in I'} \uparrow_U y_k$, where $I' \subset I$ is such that I' and $I \setminus I'$ are infinite sets, and let U'' be defined similarly with $I'' = I \setminus I'$ in place of I'. Obviously $U', U'' \subseteq \uparrow x$. $U' \cap U'' = \varnothing$ since $\uparrow y_i \cap \uparrow y_j = \varnothing$ for all $i, j \in I$ with $i \neq j$. Since $\tilde{F}$ is an ultrafilter, $U' \in \tilde{F}$ or $U'' \in \tilde{F}$. Without loss of generality let us assume that $U' \notin \tilde{F}$. Take some ultrafilter ψ such that $U' \in \psi$. Then x is convergent point of ψ since $x \leq y_i$ for all $i \in I'$. Therefore $\psi_x \in O$. Because $\tilde{F}$ is an ultrafilter different from $|psi$, there exists $V \in \tilde{F}$ such that $V \notin \psi$. Let us denote $U''' = V \cap U''$. Then $\tilde{F}_x \in O_{U'''} \cap \pi_X^{-1}(H) \subset O_U \cap \pi_X^{-1}(H)$ and $\psi_x \notin O_{U'''} \cap \pi_X^{-1}(H)$, so we get a contradiction.

Sufficiency: Suppose that for any $x \in X$, whenever $y_k > x$ for $k \in K$ and $\uparrow y_i \cap \uparrow y_j = \varnothing$ for all $i, j \in K$ with $i \neq j$, the set K is finite.

By Lemma 4.5 x is convergent point of only a finite number of ultrafilters. Let us show that for any ultrafilter η, where $x \tilde{\in} \eta$, for the point η_x of $\tilde{X}$ there exists a smallest neighborhood. Indeed because x is convergent point of only a finite number of ultrafilters, for each ultrafilter $\psi \neq \eta$, where $x \tilde{\in} \psi$, there exists $U \in \eta$ such that $U \notin \psi$. Consider the intersection of all such open sets U and denote it by U_0. Because the number of such sets U is finite, U_0 is an open set. Moreover $U_0 \in \eta$, and U_0 does not belong to any other ultrafilter ψ with $x \tilde{\in} \psi$. Since by Lemma 4.5 $\uparrow x \in \eta$ and $U_0 \in \eta$, also $\uparrow x \cap U_0 \in \eta$. Consider $O_{\uparrow x \cap U_0} \cap \pi_X^{-1}(\uparrow x)$. Obviously $\eta_x \in O_{\uparrow x \cap U_0} \cap \pi_X^{-1}(\uparrow x)$. Let us show that $O_{\uparrow x \cap U_0} \cap \pi_X^{-1}(\uparrow x)$ is the required smallest neighborhood of η_x. Clearly for this it suffices to show that for any $\xi_y \in O_{\uparrow x \cap U_0} \cap \pi_X^{-1}(\uparrow x)$ we have $\eta_x \leq \xi_y$. Indeed $\xi_y \in \tilde{X}$ means $y \tilde{\in} \xi$, $\xi_y \in O_{\uparrow x \cap U_0}$ implies $\uparrow x \cap U_0 \in \xi$, and $\xi_y \in \pi_X^{-1}(\uparrow x)$ implies $y \geq x$. Now $y \tilde{\in} \xi$ and $y \geq x$ together imply $x \tilde{\in} \xi$, and since U_0 does not belong to any other ultrafilter whose convergent point is x, it follows that $\xi = \eta$. Thus indeed $\eta_x \leq \eta_y = \xi_y$. ∎

4.9. COROLLARY. *If the Gleason cover $\pi_X : \tilde{X} \longrightarrow X$ of an Alexandroff topological space X is an Alexandroff space, then the inverse image of any point $x \in X$ under π_X is finite.*

Proof. X is an Alexandroff topological space and $\tilde{X}$ is Alexandroff too, thus Theorem 4.8 and Lemma 4.5 imply that every $x \in X$ is convergent point of only a finite number ultrafilters. From the construction of $\pi_X : \tilde{X} \to X$, x is the π_X-image of η_x, where η is an ultrafilter of X with $x \tilde{\in} \eta$. The number of such points in $\tilde{X}$ is finite, and the π_X-inverse image of x is exactly this set which is finite. ∎

4.10. Corollary. *If the set of regular open sets $RO(X)$ of an Alexandroff topological space X is finite, then its Gleason cover is Alexandroff.*

Proof. Proof follows from Theorem 4.8. ∎

4.11. Definition. *Call an open ultrafilter F* regularly principal *if there is a regular open $U_F \in F$ such that for any open U, $U \in F$ if and only if $U_F \subseteq int(\overline{U})$.*

4.12. Proposition. *If the Gleason cover of an Alexandroff topological space X is an Alexandroff space then for any point $F_x \in \tilde{X}$, F is a regularly principal ultrafilter.*

Proof. Consider the point $\tilde{F_x}$ of the Gleason cover $\tilde{X}$. Since $\tilde{X}$ is an Alexandroff space, $\tilde{F_x}$ has the smallest neighbourhood. Let O be the smallest neighbourhood of $\tilde{F_x}$. Since each open set is a union of basic open sets, O must be a basic open set. By construction of the base for $\tilde{X}$ we may assume that $O = O_U \cap \pi_X^{-1}(\uparrow x)$. That O is the smallest neighbourhood of $\tilde{F_x}$ means the following: for any $\tilde{F_x} \in O_{U'} \cap \pi_X^{-1}(H)$ we have $O_U \cap \pi_X^{-1}(\uparrow x) \subseteq O_{U'} \cap \pi_X^{-1}(H)$. Recall that $\tilde{F_x} \in O_U \cap \pi_X^{-1}(H)$ means that $U \in F$ and $x \in H$, and take for U the open set U_0 as constructed in the proof of Theorem 4.8 for $\tilde{F}$. Thus, $\uparrow x \subseteq H$ and $\tilde{F_x} \in O_{U_0} \cap \pi_X^{-1}(\uparrow x) \subseteq O_{U_0} \cap \pi_X^{-1}(H)$, hence the smallest neighbourhood of $\tilde{F_x}$ is $O_{U_0} \cap \pi_X^{-1}(\uparrow x)$.

Let us note that x is convergent point of $\tilde{F}$ and by virtue of Lemma 4.2, $\uparrow x \in \tilde{F}$. Since $U_0 \in \tilde{F}$, it follows that $U_0 \cap \uparrow x \in \tilde{F}$. Hence $\tilde{F_x} \in O_{U_0 \cap \uparrow x} \cap \pi_X^{-1}(\uparrow x)$. Since $O_{U_0 \cap \uparrow x} \subseteq O_{U_0}$, in fact the smallest neighbourhood of $\tilde{F_x}$ is $O_{U_0 \cap \uparrow x} \cap \pi_X^{-1}(\uparrow x)$.

Let us show that the ultrafilter $\tilde{F}$ is regularly principal. Consider $U_{\tilde{F}} = \mathrm{int}\overline{U_0 \cap \uparrow x}$. Obviously $U_{\tilde{F}}$ is a regular open set. Let us show that $U \in \tilde{F}$ iff $U_{\tilde{F}} \subseteq \overline{U}$ for any open U. Suppose to the contrary that there exists $U' \in \tilde{F}$ such that $U_{\tilde{F}} \not\subseteq \overline{U'}$. Then $\mathrm{int}\overline{U_0 \cap \uparrow x} \not\subseteq \overline{U'}$, which implies $U_0 \cap \uparrow x \not\subseteq \overline{U'}$, hence we can take $x' \in (U_0 \cap \uparrow x) \setminus \overline{U'}$. Obviously $\uparrow x' \subseteq U_0 \cap \uparrow x$ and moreover $\uparrow x' \cap \overline{U'} = \varnothing$ since $\overline{U'}$ is downset. Since $U' \in \tilde{F}$ and $\uparrow x' \cap U' = \varnothing$, it follows that $\uparrow x' \notin \tilde{F}$, so that there exists an ultrafilter $F' \neq \tilde{F}$ such that $\uparrow x' \in F'$ and as $x \leq x'$, $x \tilde{\in} F'$. Then by construction of U_0 we will have $U_0 \notin F'$, so, since $\uparrow x' \in F'$ we must have $\uparrow x' \cap U_0 = \varnothing$, contradicting $x' \in U_0 \cap \uparrow x$.

∎

4.13. Theorem. *The Gleason cover of an Alexandroff topological space X is an Alexandroff space if and only if for any $x \in X$ and for any infinite antichain $S \subseteq \uparrow x$ there exist $y_1, y_2 \in S$ such that $\uparrow y_1 \cap \uparrow y_2 \neq \varnothing$.*

PROOF. <u>*Necessity:*</u> By hypothesis for any $x \in X$ and for any family $(y_k)_{k\in K}$ such that $x \le y_k$ for all $k \in K$ and $\uparrow y_i \cap \uparrow y_j = \varnothing$ for $i, j \in K$ with $i \neq j$ the set K is finite. Therefore the conditions of Theorem 4.8 are satisfied.

<u>*Sufficiency:*</u> Let us suppose that the Gleason cover of X is an Alexandroff space, and for $x \in X$ let $S \subset X$ be an antichain such that $x \le y$ for all $y \in S$. If in S there exists a family $(y_i)_{i\in I}$ with infinite I, such that $\uparrow y_i \cap \uparrow y_j = \varnothing$ for any $i, j \in I$ with $i \neq j$, then we will get contradiction by Theorem 4.8. ∎

4.14. LEMMA. *Let X be an Alexandroff topological space. For any point $x \in X$ and for any atom $A \in R(X)$ of the algebra of regular closed sets of X, $x \in A$ iff for any neighbourhood O_x of x, we have $A \subset \overline{O_x}$.*

PROOF. <u>*Necessity:*</u> If $x \in A$ then for any neighbourhood O_x of x we have $A \cap O_x \neq \varnothing$ and as A is an atom and $\overline{O_x}$ is a regular closed set, $A \cap \overline{O_x} = A$, hence $A \subset \overline{O_x}$.

<u>*Sufficiency:*</u> Given any neighborhood O_x of x, then $A \subseteq \overline{O_x}$ means that any neighborhood of any point of A meets O_x. Since A is regular, $\mathrm{int}(A)$ is nonempty, hence $\mathrm{int}(A) \cap O_x \neq \varnothing$, thus also $A \cap O_x \neq \varnothing$. Since A is a closed set and O_x is any neighbourhood of x, we conclude that $x \in A$. ∎

4.15. THEOREM. *The Gleason cover of an Alexandroff topological space X is an Alexandroff space iff for any $x \in X$ we have $\downarrow\uparrow x = \bigcup_{k\in K} A_k$, where $A_k \in R(X)$ are atoms of the Boolean algebra $R(X)$ and the set K is finite.*

PROOF. <u>*Necessity:*</u> Suppose that for any $x \in X$ we have $\downarrow\uparrow x = \bigcup_{k\in K} A_k$, each A_k an atom of $R(X)$, and $K \neq \varnothing$ is finite.

Let us suppose there exists a family of points $(y_j)_{j\in J}$ such that J is infinite, $y_j \ge x$ for $j \in J$ and $\uparrow y_j \cap \uparrow y_k = \varnothing$ for $j, k \in J$ with $j \neq k$. Let y_1 be one of these y_j. Since $\downarrow\uparrow y_1 \subset \downarrow\uparrow x = \bigcup_{k\in K} A_k$, for this point y_1 choose a subset $K_1 \subset K$ such that $A_k \subset \downarrow\uparrow y_1$ for $k \in K_1$. Then $\bigcup_{k\in K_1} A_k = \downarrow\uparrow y_1$ since otherwise, $\downarrow\uparrow y_1$ also being finite union of atoms of $R(X)$, there will exists an atom of regular closed sets $\tilde{A} \in R(X)$ such that $y_1 \in \tilde{A}$ and $\tilde{A} \not\subset \bigcup_{k\in K_1} A_k \subset \bigcup_{k\in K} A_k = \uparrow\downarrow x$; but as $x \le y$ and $y \in \tilde{A}$ is closed, we would have $x \in \tilde{A}$, contradiction.

Now consider $y_2 \in (y_j)_{j\in J}$, $y_2 \neq y_1$. Then there is a $K_2 \subset K \setminus K_1$ such that $\downarrow\uparrow y_2 = \bigcup_{k\in K_2} A_k$, as $\uparrow y_1 \cap \uparrow y_2 = \varnothing$ clearly implies that $y_2 \notin A_k$ for any $k \in K_1$. In general for integer k, let $y_k \in (y_j)_{j\in J}$ and $K_k \subset K \setminus (K_1 \cup K_2 \cup \cdots \cup K_{k-1})$.

This process will terminate as K is finite, thus there exists n such that

$$K \setminus (K_1 \cup K_2 \cup \cdots \cup K_n) = \varnothing.$$

But since J is infinite, there exists at least one point y_{n+1} such that $x \le y_{n+1}$ but $y_{n+1} \notin \bigcup_{k\in K} A_k$, which leads to contradiction. Therefore, any family of points $(y_j)_{j\in J}$ such that $\forall j \in J$, $y_j \ge x$ and $\uparrow y_j \cap \uparrow y_k = \varnothing$ for all $j, k \in J$ with $j \neq k$ must be necessarily finite.

Therefore by Theorem 4.8, Gleason cover of X is an Alexandroff space.

Sufficiency: By Theorem 4.8, for all $x \in X$, if $x < y_k$, $k \in K$ and $\uparrow y_k \cap \uparrow y_j = \varnothing$ for all $k, j \in K$ with $k \neq j$ then K is finite. Let us take a maximal such K.

Denote $A_k = \downarrow\uparrow y_k$, and let us show that for all $k \in K$ the set A_k is an atom of $R(X)$. Suppose to the contrary that for some $k \in K$ there exist $A', A'' \in R(X)$ such that, $A' \neq A_k$, $A'' \neq A_k$ and $A_k = A' \cup A''$. Thus $\downarrow\uparrow y_k = A' \cup A''$, and there exist points $z_1 \in A' \setminus A''$, $z_2 \in A'' \setminus A'$. Obviously $x \leq z_1, z_2$ and $\uparrow z_1 \cap \uparrow z_2 = \varnothing$, as well as $\uparrow z_1 \cap \uparrow y_i = \uparrow z_2 \cap \uparrow y_i = \varnothing$ where $i \in K \setminus \{k\}$. It follows that we can add to K either z_1 or z_2, or both, contradiction.

Now let us show that $A_k, k \in K$ is the only maximal family which satisfies the required conditions. Suppose to the contrary that there exists $A \subseteq \downarrow\uparrow x$, $A \in R(X)$ and $A \neq A_k$ for all $k \in K$. Then there exists $y \in A$ such that $y \notin A_k$ for all $k \in K$, which gives $\uparrow y \cap \uparrow y_k = \varnothing$. ■

Remark: In Theorem 4.15 each atom A_k is such that $x \in A_k$.

4.16. Corollary. *The Gleason cover of an Alexandroff topological space X is an Alexandroff space iff for any $x \in X$ the set of atoms $A \in R(X)$ such that $x \in A$ is non-empty and finite.*

Proof. The proof depends on the following well known fact:

Let B be Boolean algebra and $at(B)$ be the set of atoms of B. If $b \in B$ is the join of a finite set $A \subseteq at(B)$ of atoms of B, then this set A necessarily coincides with the set $\mathbf{A}(b) = \{a \in at(B) \mid a \leq b\}$ of all atoms below b. ■

4.17. Lemma. *Let X be an Alexandroff space. If the Gleason cover of X is an Alexandroff space then $R(X)$ is atomic.*

Proof. Let us show that for any $R \in R(X)$ there exists an atom A of $R(X)$ such that $A \subseteq R$. Consider any point $x \in \text{int}(R)$ (since R is a regular closed set, its interior is not empty). Thus $\uparrow x \subseteq \text{int}(R)$, hence $\downarrow\uparrow x \subseteq \downarrow \text{int}(R) = R$. By virtue of Theorem 4.15 there exists an atom A of $R(X)$ such that $x \in A$. Lemma 4.3 gives $A \subseteq \downarrow\uparrow x \subseteq R$, therefore $A \subseteq R$. ■

4.18. Lemma. *Suppose X is an Alexandroff extremally disconnected space, then $X = \bigoplus A_i$, where A_i are atoms of $R(X)$.*

Proof. Extremal disconnectedness of X implies that each $A \in R(X)$ is clopen, because $\text{int}(A) = \overline{\text{int}(A)} = A$ and $\overline{\text{int}(A)}$ is an open set. Moreover each connected clopen is an atom of $R(X)$. Hence each connected component of X is an atom of $R(X)$. ■

4.19. Theorem. *If a Gleason cover of an Alexandroff topological space X is Alexandroff itself, then the Gleason cover of X is the topological sum of atoms of the complete Boolean algebra of regular closed sets of X,*

$$\tilde{X} = \bigoplus_{A \in atR(X)} A$$

PROOF. Let us show that for any ultrafilter F the set $\{F_x \in \tilde{X} \mid x\tilde{\in}F\}$ is clopen in $\tilde{X}$. It is equivalent to show that each set $\downarrow\uparrow \{F_x\}$ is clopen in $\tilde{X}$. Thus we have to show that $\downarrow\uparrow F_x$ is open. Indeed if there exists $\eta_{x'} \in \tilde{X}$ such that $\eta_{x'} \geq F_y \in\downarrow\uparrow F_x$ for some y, then by virtue of Lemma 4.4 $\eta = F$ and $x'\tilde{\in}F$, hence $\eta_{x'} = F_{x'} \in\downarrow\uparrow F_x$.

Consider any point $F_x \in \tilde{X}$. By construction of $\tilde{X}$ we have that $x\tilde{\in}F$ iff $x \in \overline{U}$ for all $U \in F$. Since $\tilde{X}$ is Alexandroff, by virtue of Proposition 4.12 F is a regularly principal ultrafilter. Thus there exists a regular open set $U_F \in F$ such that $U_F \subseteq \mathrm{int}\overline{U}$ for all $U \in F$.

Consider $A = \overline{U}_F \ni x$, and let us show that it is a regular closed atom. It is clear that A is regular closed. Suppose that A is not an atom. Then there exists a nonempty regular closed set A' such that $A' \subset A = \overline{U}_F$ and $A' \neq A$. Since U_F is regular open, $\mathrm{int}(A') \subseteq U_F$. Moreover $\mathrm{int}(A') \in F$. Indeed, for any $U \in F$ we have $\mathrm{int}(A') \subseteq U_F \subseteq \mathrm{int}\overline{U}$ hence $\mathrm{int}(A') \subset \overline{U}$, hence $\mathrm{int}(A') \cap U \neq \varnothing$. Since F is an ultrafilter, this implies that $\mathrm{int}(A') \in F$. By regular principality of F we conclude that $U_F \subseteq \mathrm{int}\overline{\mathrm{int}(A')} = \mathrm{int}(A') \subseteq A'$ hence $\overline{U}_F \subseteq \overline{\mathrm{int}(A')} = A'$, which implies $A' = \overline{U}_F = A$.

Therefore, for each ultrafilter F of X there exists unique regular closed atom A_F, such that for all $U \in F$, $A_F \subseteq \overline{U}$. Hence for each $x\tilde{\in}F$ we have $x \in A$ and conversely, if $x \in A_F$ then $x\tilde{\in}F$. Thus $\{F_x \in \tilde{X} \mid x\tilde{\in}F\} = A_F$. By Lemma 4.2 for any $x\tilde{\in}F$ we have $\uparrow x \in F$, hence for any $x'\tilde{\in}F$ $x' \in\downarrow\uparrow x$, thus $\downarrow\uparrow F_x = A_F$.

Thus, we get

$$\tilde{X} = \bigoplus_{F\in\mathbf{F}} \downarrow\uparrow F_x = \bigoplus_{A\in \mathrm{at}R(X)} A$$

where $\mathbf{F}$ is the set of those ultrafilters F on X for which there exists an $x \in X$ with $x\tilde{\in}F$. ∎

4.20. THEOREM. *For the Gleason cover $\pi_X : \tilde{X} \to X$ of an Alexandroff space X which satisfies conditions of Theorem 4.8, the map π_X is a co-local homeomorphism.*

PROOF. Take any point $\eta_x \in \tilde{X}$. By construction of $\tilde{X}$, η is one of the ultrafilters on X and x is one of the convergent points of η. Let us show that $|pi_X$ maps $\downarrow \eta_x$ homeomorphically onto $\downarrow x$.

Indeed, for any $y \leq x$, if $x \in\downarrow U$ then $y \in\downarrow U$, hence $y\tilde{\in}\eta$ and $\eta_y \leq \eta_x$ and vice versa because of Lemma 4.4.

Because of construction of the function π_X, its restriction to $\downarrow \eta_x$ obviously is a map onto $\downarrow x$. For any $\eta_{y_1}, \eta_{y_2} \leq \eta_x$, $\pi_X(\eta_{y_1}) = y_1 \neq y_2 = \pi_X(\eta_{y_2})$, i. e. this restriction is injection and therefore bijection.

From construction of topology on the $\tilde{X}$ and the function π_X for any $\eta_y \leq \eta_x$, $\pi_X(\downarrow \eta_y) =\downarrow y$. Hence each restriction $\pi_X|_{\downarrow\eta_x}$ is a homeomorphism. ∎

4.21. PROPOSITION. *Suppose X is an Alexandroff extremally disconnected space, then each atom A of $R(X)$ is a maximal ideal of X as a poset.*

PROOF. Let us consider any atom $A \in R(X)$, and show that it is a maximal ideal. Obviously $A \neq \varnothing$, A is a closed set, therefore A is a downset. Let us show that for any $x, y \in A$ such that $x \neq y$ there exists $z \in A$ such that $x, y \leq z$. Indeed, if not then $\uparrow x \cap \uparrow y = \varnothing$, hence also $\downarrow\uparrow x \cap \uparrow y = \varnothing$, so that $\downarrow\uparrow x$ would be a regularly closed proper subset of A, thus A would not be an atom.

Let us now show that A is maximal. If not, then there exists an ideal J such that $A \subseteq J$, $x \in J$ and $x \notin A$. For any $x' \in A$ there exists $y \in J$ such that $x', x \leq y$. But since X is extremally disconnected and $A \in R(X)$, A must be clopen, which contradicts $x' \in A$, $y \notin A$ and $x' \leq y$.

■

4.22. THEOREM. *Suppose we have Gleason cover $\pi_X : \tilde{X} \to X$, where X is an Alexandroff topological space which satisfies the conditions of Theorem 4.8. For any continuous map $f : Y \to X$ with Y extremally disconnected Alexandroff there exists a continuous map $\pi : Y \to \tilde{X}$ such that $\pi_X \circ \pi = f$.*

PROOF. According to Proposition 2.12 the space Y is disjoint topological sum of its connected components, and by virtue of Lemma 4.18 each component is an atom in the algebra of regularly closed sets of Y. For any point $y \in Y$ obviously there exists a component $C \subset Y$ with $y \in C$, therefore for any $y_1, y_2 \in C$ such that $y \leq y_1, y_2$ there exists $z \in C$ such that $y_1, y_2 \leq z$.

For a point $y \in Y$ let us show that in X there exists a regularly closed atom $A \in R(X)$ such that $f(\uparrow y) \subseteq A$.

Indeed, suppose to the contrary that $f(\uparrow y) \subset \bigcup_{i \in I} A_i$ and for any $i \in I$, $f(\uparrow y) \not\subset A_i$, $f(\uparrow y) \cap A_i \neq \varnothing$ and each A_i is an atom of $R(X)$. Let us consider any two of them A_j, A_k. There exist $y_j, y_k \geq y$ such that $f(y_j) \in A_j$, $f(y_k) \in A_k$ and $f(y_j) \notin \bigcup_{i \in I \setminus j} A_i$, $f(y_k) \notin \bigcup_{i \in I \setminus k} A_i$. In the component of y there exists $z \in Y$ such that $y_j, y_k \leq z$, hence because of continuity of f, $f(y_j), f(y_k) \leq f(z)$, so $f(z) \notin \bigcup_{i \in I \setminus k} A_i$ and $f(z) \notin \bigcup_{i \in I \setminus j} A_i$, i. e. $f(z) \notin \bigcup_{i \in I} A_i$, which contradicts $f(\uparrow y) \subset \bigcup_{i \in I} A_i$.

Thus there indeed exists an atom $A \in R(X)$ such that $f(\uparrow y) \in A$. Since f is continuous and A is closed, we conclude $f(\downarrow\uparrow y) \subseteq \downarrow f(\uparrow y) \subseteq A$.

Any atom A of $R(X)$ generates ultrafilter on $R(X)$, and by virtue of lemma 3.1 from [Johnstone, 1980] to this ultrafilter corresponds an open ultrafilter $\eta_A = \{U \mid A \subseteq \operatorname{int}(\downarrow U)\}$. Let us denote $\tilde{A} = \{(\eta_A)_x \mid x \tilde{\in} \eta_A\}$. Let us define function π as follows: $\pi(y)$ is the unique element of $\pi_X^{-1}(f(y)) \cap \tilde{A}$. Hence for each $y \in Y$ we have $\pi(y) = (\eta_A)_{f(y)}$.

To show monotonicity of the function π, note that for any two points $y_1, y_2 \in Y$ such that $y_1 \leq y_2$ we have $f(y_1) \leq f(y_2)$, hence $\pi(y_1) = (\eta_A)_{f(y_1)} \leq (\eta_A)_{f(y_2)} = \pi(y_2)$.

Obviously for any point $y \in Y$ we have $\pi_X(\pi(y)) = \pi_X((\eta_A)_{f(y)}) = f(y)$. ■

5. Examples

5.1. EXAMPLE. Any finite topological space X will satisfy conditions of Theorem 4.8, hence its Gleason cover will be Alexandroff space too.

5.2. Example. Condition of Corollary 4.10 (that there are only finitely many regular opens) is sufficient but not necessary for the Gleason cover of an Alexandroff space to be Alexandroff, as the following example shows. Let $P = \mathbb{N}$, the set of natural numbers, be ordered as follows: $1 \leq 3 \leq 5 \leq 7 \leq ...$ and $1 \leq 2 \leq 4 \leq 6 \leq ....$ The topology corresponding to this order defines the Alexandroff topological space $\tau(P)$. Let us consider the topological sum $\bigoplus_{i \in I} \tau(P_i)$, where I is an infinite set of indices and each $\tau(P_i)$ is a copy of the Alexandroff space which we defined above. This topological space obviously is an Alexandroff topological space and will satisfy Theorem 4.8, therefore its Gleason cover will also be Alexandroff.

If an Alexandroff space has finitely many regularly closed sets, then by Corollary 26.2 its Gleason cover is Alexandroff. Moreover if an Alexandroff space is a topological sum of Alexandroff spaces with finitely many regularly closed sets, then its Gleason cover will be Alexandroff by combining Theorem 4.1 and Corollary 4.10. The following example shows that there also are Alexandroff spaces which do not satisfy the above conditions but their Gleason covers are still Alexandroff.

5.3. Example. Let $P = \mathbb{N}$, the set of natural numbers, be ordered as follows: $\forall n \in \mathbb{N}$, $2n > 2n-1, 2n+1$. This topological space is an Alexandroff topological space and satisfies the conditions of Theorem 4.8, therefore its Gleason cover will also be Alexandroff.

5.4. Example. There are Alexandroff spaces of finite height whose Gleason covers are not Alexandroff. Let $P = \mathbb{N} \cup \{*\}$ be the set of natural numbers with one more point added. Let us define a topology on P with the following open sets: $\tau(P) = \{U \mid U \subseteq \mathbb{N} \vee U = P\}$. Let us note that (P, τ) is an Alexandroff topological space. The topological space (P, τ) does not satisfy conditions of Theorem 4.8, hence its Gleason cover will not be Alexandroff.

5.5. Example. The previous example had infinite branching, i. e. there was a point with infinitely many immediate successors. The following is an example of an Alexandroff space with finite branching whose Gleason cover is not Alexandroff.

Let $P = \mathbb{N}$, the set of natural numbers, be ordered as follows: each even number is a maximal point, thus for any $x \in \mathbb{N}$ if $x = 2k$ then $\forall y \in \mathbb{N}$ if $x \leq y$ then $x = y$. Moreover for odd numbers there let us have natural ordering. Thus $1 \leq 2, 3, 4, 5, ...$, $3 \leq 4, 5, 6, 7...$, $5 \leq 6, 7, 8, 9, 10...$ and so on. The topology given from this order is an Alexandroff topology. The resulting topological space (P, τ) will not satisfy conditions of Theorem 4.8, therefore its Gleason cover will not be Alexandroff.

In the illustrations below for these examples, red arrows indicate maps, and blue lines describe ordering.

Acknowledgments. The author benefited a lot from communication with the members of the seminar at the Department of Mathematical Logic of the TSU Razmadze Mathematical Institute. In particular, his supervisor M. Jibladze, supervisor's questions, suggestions, and advice were crucial for creating this article, as well as the seminar participants D. Gabelaia, L. Uridia, and B. Tkemaladze provided several essential arguments

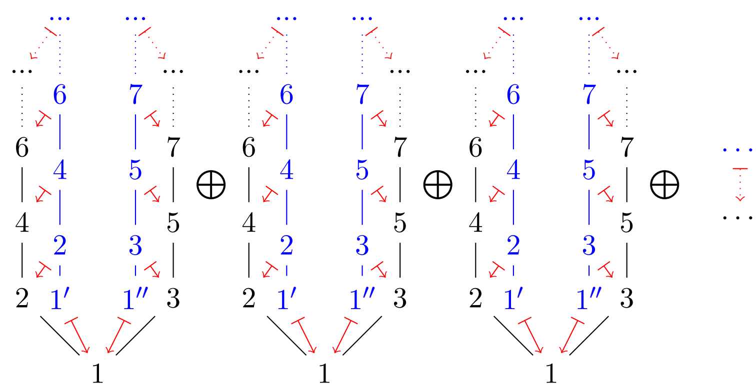

Figure 1: Example 2

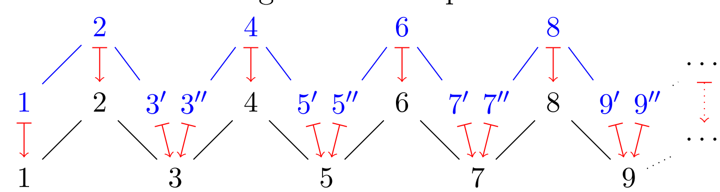

Figure 2: Example 3

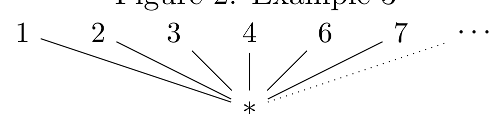

Figure 3: Example 4

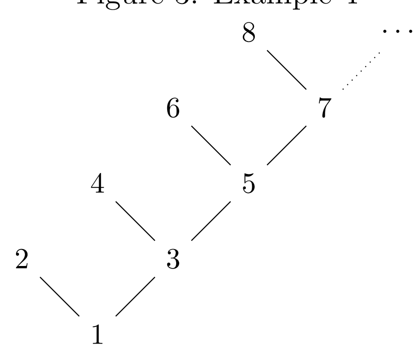

Figure 4: Example 5

and observations. The author also received important detailed comments regarding the paper from Prof. Marcel Erné and an anonymous referee.

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