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GYSIN FUNCTORS, CORRESPONDENCES, AND THE GROTHENDIECK-WITT CATEGORY

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ABSTRACT. We introduce some general categorical machinery for studying Gysin functors (certain kinds of functors with transfers) and their associated categories of correspondences. These correspondence categories are closed, symmetric monoidal categories where all objects are self-dual. We also prove a limited reconstruction theorem: given such a closed, symmetric monoidal category (and some extra information) it is isomorphic to the correspondence category associated to a Gysin functor. Finally, if k is a field we use this technology to define and explore a new construction called the Grothendieck-Witt category of k.

1. Introduction

Fix a ground field k. In this paper we describe a category $\mathrm{GWC}(k)$ whose objects are the finite separable field extensions of k. The morphisms are a Grothendieck group of certain kinds of "correspondences" built up from bilinear forms, and there is an intrinsic notion of composition. We call this the **Grothendieck-Witt category** of k. In order to study this category, we generalize to the theory of what we call *Gysin functors* and their associated *categories of correspondences*. These notions simultaneously generalize the classical theory of Green functors and Burnside categories for finite groups. The majority of the paper is devoted to building up the requisite categorical machinery for studying this structure.

The motivation for this paper comes from motivic homotopy theory. Consider the motivic stable homotopy category over k, and restrict to the full subcategory whose objects are the suspension spectra of separable field extensions of k. This should coincide with the Grothendieck-Witt category $\mathrm{GWC}(k)$ defined here, and the results in this paper are tools for producing a careful proof of this folklore result.

To further explain the ideas and motivation of this paper we take a brief detour into equivariant homotopy theory. Let G be a finite group, and let $G\mathcal{T}op$ be the category of G-spaces and equivariant maps. We regard $G\mathcal{S}et$, the category of G-sets, as the full

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subcategory of $G\mathcal{T}op$ consisting of the discrete G-spaces. The **orbit category** $Or(G)$ of G is the full subcategory of $G\mathcal{S}et$ consisting of the G-sets on which G acts transitively. Every object in $Or(G)$ is isomorphic to a quotient G/H, for some subgroup H.

Next consider the stabilization functor $\Sigma^\infty(-)_+ \colon \mathcal{H}o(G\mathcal{T}op) \to \mathcal{H}o(G\mathcal{S}pectra)$ from the homotopy category of G-spaces to that of genuine G-spectra (the version of G-spectra where representation spheres are invertible). When restricted to $G\mathcal{S}et$ this map is an embedding, but it is not full. The full subcategory of $\mathcal{H}o(G\mathcal{S}pectra)$ whose objects are $\Sigma^\infty\mathcal{O}_+$ for $\mathcal{O}$ a G-set is called the **stable category** of G-sets, and denoted $G\mathcal{S}et^{st}$. We will actually focus on $G\mathcal{S}et^{st}_{\mathrm{fin}}$, where we restrict $\mathcal{O}$ to be a finite G-set. The full subcategory of $G\mathcal{S}et^{st}_{\mathrm{fin}}$ consisting of the objects $\Sigma^\infty(G/H)_+$ is called the **stable orbit category**.

There are two common ways of describing $G\mathcal{S}et^{st}_{\mathrm{fin}}$:

(1) Given two finite G-sets $\mathcal{O}_1$ and $\mathcal{O}_2$, define a **span** from $\mathcal{O}_1$ to $\mathcal{O}_2$ to be a diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ \mathcal{O}_2 & & \mathcal{O}_1 \end{tikzcd}$$

in the category of finite G-sets. A map between spans is a map of diagrams that is the identity on $\mathcal{O}_1$ and $\mathcal{O}_2$. This category has a monoidal structure given by disjoint union in the "P"-variable. Define $\boldsymbol{\mathcal{B}urn(\mathcal{O}_1,\mathcal{O}_2)}$ to be the Grothendieck group of isomorphisms classes of spans from $\mathcal{O}_1$ to $\mathcal{O}_2$, with respect to this disjoint union operation.

Note that we will sometimes refer to spans as "correspondences", as that terminology is often used in geometric settings. Also note that the right-to-left convention we use in drawing spans is explained further in Section 1.6 below.

If we have three finite G-sets $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ then we can define a composition of spans via the pullback operation shown in the following diagram:

$$\begin{tikzcd} & & Q \times_{\mathcal{O}_2} P \arrow[dl, dotted] \arrow[dr, dotted] & & \\ & Q \arrow[dl] \arrow[dr] & & P \arrow[dl] \arrow[dr] & \\ \mathcal{O}_3 & & \mathcal{O}_2 & & \mathcal{O}_1. \end{tikzcd}$$

This operation induces a map

$$\mathcal{B}urn(\mathcal{O}_2,\mathcal{O}_3) \times \mathcal{B}urn(\mathcal{O}_1,\mathcal{O}_2) \to \mathcal{B}urn(\mathcal{O}_1,\mathcal{O}_3)$$

which is readily checked to be unital and associative. So we have defined a category $\mathcal{B}urn$ whose objects are the finite G-sets. This is usually called the **Burnside category** of G-sets.

Here are some things to take note of:

(a) There is a functor $R\colon G\mathcal{S}et_{\text{fin}} \to \mathcal{B}urn$ that is the identity on objects and sends a map $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ to the span $[\mathcal{O}_2 \xleftarrow{f} \mathcal{O}_1 \xrightarrow{id} \mathcal{O}_1]$.

(b) The category $\mathcal{B}urn$ has a duality anti-automorphism $(-)^*$ which is the identity on objects, and on morphisms sends a span $[\mathcal{O}_2 \leftarrow P \to \mathcal{O}_1]$ to the similar span $[\mathcal{O}_1 \leftarrow P \to \mathcal{O}_2]$ obtained by reversing the order of the maps. The duality functor is an isomorphism

$$(-)^*\colon \mathcal{B}urn^{op} \to \mathcal{B}urn\,.$$

(c) In particular, setting $I = (-)^* \circ R$ gives a functor $I\colon G\mathcal{S}et_{\text{fin}}^{op} \to \mathcal{B}urn$. If $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ then $I(f)$ is the span $[\mathcal{O}_1 \xleftarrow{id} \mathcal{O}_1 \xrightarrow{f} \mathcal{O}_2]$.

If $\mathcal{A}$ is an additive category then additive functors $\mathcal{B}urn^{op} \to \mathcal{A}$ are the same as what are usually called *Mackey functors*. (One could also identify Mackey functors with additive functors $\mathcal{B}urn \to \mathcal{A}$, since $\mathcal{B}urn$ is self-dual; however, our notation for the R and I maps fits better with the contravariant option).

It is a classical theorem [LMS, Proposition V.9.6] that $\mathcal{B}urn$ is isomorphic to the stable category of finite G-sets. See also [Li] for the initial introduction of this category and its relationship to Mackey functors.

(2) The stable orbit category $Or(G)^{st}$ can also be described in terms of generators and relations. This is the additive category whose objects are the transitive G-sets and whose morphisms are freely generated by the maps $R_f\colon \mathcal{O}_1 \to \mathcal{O}_2$ and $I_f\colon \mathcal{O}_2 \to \mathcal{O}_1$ for every map of G-sets $f\colon \mathcal{O}_1 \to \mathcal{O}_2$, subject to the morphisms satisfying the relations

(i) $R_{gf} = R_g \circ R_f$;

(ii) $I_{gf} = I_f \circ I_g$;

(iii) Given a pullback diagram of G-sets

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "p"'] & \mathcal{O}_3 \arrow[d, "q"] \\ \mathcal{O}_1 \arrow[r, "g"] & \mathcal{O}_2 \end{tikzcd}$$

where the actions on $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ are transitive, write $P = \coprod_i X_i$ where each X_i is a transitive G-set. Then

$$I_g \circ R_q = \sum_i R_{p_i} \circ I_{f_i}$$

where f_i and p_i are the restrictions of f and p to X_i.

It is again a classical theorem that this category, defined in terms of generators and relations, is isomorphic to the stable orbit category.

Now let us return to our original setting, where k is a fixed ground field. Keeping the above discussion in mind, we would like to examine the full subcategory of the motivic stable homotopy category over k whose objects are the suspension spectra of fields. This is vaguely analogous to the stable orbit category, although in the case of G-spectra the orbits generate the category whereas field spectra do not generate the category in the motivic setting. Our goal (though not fully realized in the present paper) is to give descriptions of this category that are analogs of (1) and (2). To give a sense of this in the first case, the Grothendieck-Witt category of k is defined to be the category $\mathrm{GWC}(k)$ whose objects are $\operatorname{Spec} E$ for E a finite, separable field extension of k. The morphisms from $\operatorname{Spec} E$ to $\operatorname{Spec} F$ are the Grothendieck group $GW(F \otimes_k E)$ of quadratic spaces over $F \otimes_k E$ (see Section 2 for details). The definition of composition is a little too cumbersome to be included in this introduction, but it mimics the composition we saw in (1) above.

Morel [Mo] proved that if k is perfect and F/k is a separable field extension then

$$[\Sigma^\infty(\operatorname{Spec} F)_+, S] \cong \mathrm{GW}(F)$$

where S is the motivic sphere spectrum and $[-,-]$ denotes maps in the motivic stable homotopy category of smooth k-schemes. If J/k is another separable extension one can then argue that

$$\begin{aligned}
[\Sigma^\infty(\operatorname{Spec} F)_+, \Sigma^\infty(\operatorname{Spec} J)_+] &\cong [\Sigma^\infty(\operatorname{Spec} F)_+ \wedge \Sigma^\infty(\operatorname{Spec} J)_+, S] \\
&\cong [\Sigma^\infty(\operatorname{Spec}(F \otimes_k J))_+, S] \\
&\cong \mathrm{GW}(F \otimes_k J)
\end{aligned}$$

where the first isomorphism uses a self-duality $\Sigma^\infty(\operatorname{Spec} J)_+ \cong \mathcal{F}(\Sigma^\infty(\operatorname{Spec} J)_+, S)$ and the last isomorphism is the aforementioned one of Morel (using that $F \otimes_k J$ decomposes as a product of separable field extensions of k). The self-duality is dealt with in the appendix to [H], and in the equivariant context it is in modern times usually couched in the machinery of the Wirthmüller isomorphism (cf. [Ma2], for example).

Accepting the above computation, it remains to compute the composition in the motivic stable homotopy category and relate it to the appropriate pairing of Grothendieck-Witt groups. The present paper exists partly because attempting to do this by ad hoc methods proved unwieldy.

In the narrative we provide here, everything comes down to the existence of transfer maps. Transfer maps coupled with diagonal maps give rise to duality structures, and quite general categorical computations show that any reasonable category with this kind of structure may be described by a "correspondence-like" description of composition.

Let us now explain the results in a bit more detail. Let $\mathcal{C}$ be a finitary lextensive category (see Section 3.1, but understand that this is basically just a category where coproducts behave nicely with respect to pullbacks). We define a **Gysin functor** on $\mathcal{C}$ to be an assignment $X \mapsto E(X)$ from $\mathrm{ob}(\mathcal{C})$ to commutative rings, together with pullback and pushforward maps f^* and $f_!$ satisfying certain compatibility properties. Given this situation, one can construct a **category of correspondences** $\mathcal{C}_E$ where the object set is $\mathrm{ob}(\mathcal{C})$, maps from X to Y are the abelian group $E(Y \times X)$, and composition is obtained by a familiar formula using the pullback and pushforward maps. The category $\mathcal{C}_E$ is enriched over abelian groups, is closed symmetric monoidal, and has the property that all objects are self-dual.

Now suppose $\mathcal{H}$ is a closed tensor category (additive category with compatible symmetric monoidal structure) with tensor $\otimes$ and unit S. Suppose given functors $R\colon \mathcal{C} \to \mathcal{H}$ and $I\colon \mathcal{C}^{op} \to \mathcal{H}$ satisfying some reasonable hypotheses (see Section 4). For $f\colon X \to Y$ in $\mathcal{C}$ we think of Rf as the "regular" map associated to f in $\mathcal{H}$, whereas If is an associated transfer map. The prototype for this situation is where $\mathcal{H}$ is the genuine G-equivariant stable homotopy category, $\mathcal{C}$ is the category of finite G-sets, $R(X) = I(X) = \Sigma^\infty(X_+)$, Rf is the usual map induced by $f\colon X \to Y$, and If is the corresponding transfer map.

Write π^0 for the functor $\mathcal{C}^{op} \to \mathcal{A}b$ given by $\pi^0(X) = \mathcal{H}(RX, S)$. This inherits the structure of a Gysin functor, and we prove the following:

1.1. Theorem. *Under mild hypotheses, the category of correspondences $\mathcal{C}_{(\pi^0)}$ is equivalent to the full subcategory of $\mathcal{H}$ whose objects lie in the image of R.*

That is, we prove that one can reconstruct the appropriate subcategory of $\mathcal{H}$ as the category of correspondences associated to the Gysin functor π^0. See Theorem 4.16 for a precise version of the above theorem.

The second result of this paper concerns the structure of the category of correspondences $\mathcal{C}_E$ for a general Gysin functor E. In the Burnside category of a finite group, there are special collections of maps Rf and Ig and every map in the category may be written as a composite $Rf \circ Ig$. There are also rules for rewriting compositions $If \circ Rg$ in the above form. In the case of a general Gysin functor, there are *three* collections of special maps, elements of which are written Rf, Ig, and Da where f and g are maps in $\mathcal{C}$ and $a \in E(X)$ for some object X in $\mathcal{C}$. We prove the following:

1.2. Theorem. *Every map in $\mathcal{C}_E$ can be written as a sum of maps $Rf \circ Da \circ Ig$. Other composites of the R-D-I maps can be rewritten in this form using the rules*

(a) $Da \circ Rf = Rf \circ D(f^*a)$,

(b) $If \circ Da = D(f^*a) \circ If$,

(c) $If \circ Rg = Rp \circ Iq$ where p and q are the maps in the pullback diagram

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & A \arrow[d, "g"] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

inside the category $\mathcal{C}$.

Moreover, for a map $f\colon X \to Y$ in $\mathcal{C}$ and $a \in E(X)$ one has $Rf \circ Da \circ If = D(f_!(a))$.

The following corollary is really just a reformulation of the theorem:

1.3. Corollary. *Maps in $\mathcal{C}_E(X,Y)$ can be represented by a pair consisting of a span $[Y \xleftarrow{f} Z \xrightarrow{g} X]$ and an element $a \in E(Z)$: this pair represents $Rf \circ Da \circ Ig$. If a map in $\mathcal{C}_E(U,X)$ is represented by $[X \xleftarrow{f'} Z' \xrightarrow{g'} U,\ a' \in E(Z')]$ then the composite is represented by the pullback span*

$$\begin{tikzcd}[column sep=small] & & P \arrow[dl, "s"'] \arrow[dr, "t"] & & \\ & Z \arrow[dl, "f"'] \arrow[dr, "g"] & & Z' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ Y & & X & & U \end{tikzcd}$$

and the element $(s^*a)(t^*a') \in E(P)$, where the product is the multiplication in $E(P)$. That is to say,

$$(Rf \circ Da \circ Ig) \circ (Rf' \circ Da' \circ Ig') = R(fs) \circ D((s^*a)(t^*a')) \circ I(g't).$$

Moreover, we have the extra relation

$$\Big[Y \xleftarrow{f} Z \xrightarrow{f} Y,\ a \in E(Z)\Big] = \Big[Y \xleftarrow{id} Y \xrightarrow{id} Y,\ f_!(a) \in E(Y)\Big].$$

If one assumes the category $\mathcal{C}$ to have some basic Galois-type properties (which model the behavior of the category of G-sets) then explicit computations become easier. For example, one can prove the following:

1.4. Proposition. *Assume $\mathcal{C}$ is a Galoisien category (see Section 5.13), and let X be an object in $\mathcal{C}$ that is Galois. Then in $\mathcal{C}_E$ one has*

$$\mathrm{End}_{\mathcal{C}_E}(X) = [\widetilde{\mathrm{Aut}_{\mathcal{C}}(X)}]E(X)$$

where on the right we have the twisted group ring whose elements are finite sums $\sum_i [g_i]a_i$ with $g_i \in \mathrm{Aut}_{\mathcal{C}}(X)$ and $a_i \in E(X)$, and the multiplication is determined by the formula

$$[g]a \cdot [h]b = [gh](h^*a \cdot b).$$

(Here $[g]a$ corresponds to the element $Rg \circ Da$).

The above proposition describes the full subcategory of $\mathcal{C}_E$ consisting of a single Galois object. In a similar vein, one can explicitly describe the full subcategories generated by multiple Galois objects. See Section 5.

Although the motivation for this paper comes from a concrete question concerning motivic homotopy theory, here we only develop the categorical backdrop. In the future we hope to explain how this backdrop applies to both the G-equivariant and motivic settings.

1.5. Organization of the paper. In Section 2 we write down a complete definition of the Grothendieck-Witt category. In Section 3 we generalize this, by introducing the notions of a Gysin functor and its associated category of correspondences (a Gysin functor is the same thing as what is called a commutative Green functor in the group theory literature). Section 4 continues the development of this machinery and proves the main "reconstruction theorem", which in this generality is a simple exercise in category theory.

Section 5 gives a deeper investigation into the structure of correspondence categories, and serves as a prelude to Section 6 where we work out some basic computations inside Grothendieck-Witt categories over a field.

1.6. Notation and terminology. The common notation "$f(x)$" establishes a right-to-left trend in symbology: one starts with x and then applies f to it. The common notation $\mathrm{Hom}(A,B)$ is based on the opposite left-to-right trend. The opposing nature of these two notations is one of the most common annoyances in modern mathematics. Our general philosophy in this paper is that we will always use the right-to-left convention, except when we write $\mathrm{Hom}(A,B)$. This has already appeared in our treatment of the Burnside category, where spans from $\mathcal{O}_1$ to $\mathcal{O}_2$ were drawn with the $\mathcal{O}_1$ term on the right. That particular convention will have various incarnations throughout the paper.

The projection map $X\times Y\times Z\to X\times Z$ will be written π^{XYZ}_{XZ}, and similarly for other projection maps. If $f\colon A\to X$ and $g\colon A\to Y$, then it is sometimes useful to denote the induced map $A\to X\times Y$ as $f\times g$. Unfortunately, $f\times g$ also denotes the map $A\times A\to X\times Y$. Usually it is clear from context which one is meant, but when necessary we will write $(f\times g)^A_{XY}$ and $(f\times g)^{AA}_{XY}$ to distinguish them. In all these conventions, the superscript is the domain and the subscript is the range.

2. Background on Grothendieck-Witt groups and composition

In this section we recall the definition and basic properties of the Grothendieck-Witt group of a ring. Then we explain how these groups can be assembled to give the hom-sets in a certain category $\mathrm{GWC}(R)$. This sets the stage for the rest of the paper, which develops tools that help analyze generalizations of this construction.

2.1. Grothendieck-Witt groups. Let R be a commutative ring. A **quadratic space** over R is a pair (P,b) consisting of a finitely-generated, projective R-module P together with a map $b\colon P\otimes_R P\to R$ that is symmetric in the sense that $b(x,y)=b(y,x)$ for all $x,y\in P$. One says that (P,b) is **nondegenerate** if the adjoint map $P\to\mathrm{Hom}_R(P,R)$ associated to b is an isomorphism of R-modules.

Given any maximal ideal m of R there is an induced map

$$\begin{tikzcd} P\otimes_R P \arrow[r, "b"] \arrow[d] & R \arrow[d] \\ (P/mP)\otimes_{R/m}(P/mP) \arrow[r, dotted, "b_m"] & R/m \end{tikzcd}$$

giving a symmetric bilinear form b_m on the R/m-vector space P/mP. One readily checks that (P,b) is nondegenerate if and only if $(P/mP, b_m)$ is nondegenerate for every maximal ideal m of R. In many cases nondegeneracy is most easily checked using this criterion.

It will be useful for us to sometimes think geometrically. A quadratic space is an algebraic vector bundle on $\operatorname{Spec} R$ equipped with a fibrewise symmetric bilinear form, and it is nondegenerate if the bilinear forms on the closed fibers are all nondegenerate.

Note that there is an evident direct sum operation on quadratic spaces. There is also a tensor product: if (P,b) and (Q,c) are quadratic spaces then $(P\otimes_R Q, b\otimes_R c)$ denotes the projective module $P\otimes_R Q$ equipped with the bilinear form

$$(P\otimes_R Q)\otimes_R (P\otimes_R Q) \xrightarrow{id\otimes t\otimes id} P\otimes_R P\otimes_R Q\otimes_R Q \xrightarrow{b\otimes c} R\otimes_R R \xrightarrow{\mu} R.$$

It is easy to check that the direct sum and tensor product of nondegenerate quadratic spaces are again nondegenerate.

The **Grothendieck-Witt group** of R, denoted $\mathrm{GW}(R)$, is the Grothendieck group of nondegenerate quadratic spaces with respect to direct sum. It has a ring structure induced by tensor product. If $f\colon R\to S$ is a map of commutative rings then there is an induced map of rings $f_*\colon\ \mathrm{GW}(R)\to\mathrm{GW}(S)$ given by $(P,b)\mapsto(P\otimes_R S, b\otimes_R id_S)$.

It turns out that $\mathrm{GW}(-)$ is also a contravariant functor, but only with respect to certain kinds of maps. This will take a while to explain.

2.2. Definition. *A map of commutative rings $R\to S$ is* ***sheer*** *if S is a finitely-generated, projective R-module.*

2.3. Remark. The following conditions on $R\to S$ are equivalent:

(1) $R\to S$ is sheer;

(2) S is finitely-presented and projective as an R-module;

(3) S is finitely-presented and flat as an R-module.

The equivalence (1) $\Longleftrightarrow$ (2) is elementary, and (2) $\Longleftrightarrow$ (3) can be found as [E, Corollary 6.6] or [R, Theorem 3.56].

When $f\colon R\to S$ is sheer there is a trace map $\mathrm{tr}_{S/R}\colon S\to R$ defined in the evident way: $\mathrm{tr}_{S/R}(s)$ is the trace of the multiplication-by-s map $x\mapsto xs$ on S. The map $\mathrm{tr}_{S/R}$ is R-linear. We can then use this trace map to take quadratic spaces over S and induce quadratic spaces over R: if (Q,c) is a quadratic space over S then we regard Q as an R-module (via restriction of scalars along f) equipped with the bilinear pairing

$$Q\otimes_R Q \longrightarrow Q\otimes_S Q \xrightarrow{c} S \xrightarrow{\mathrm{tr}_{S/R}} R.$$

This does not quite give a map $\mathrm{GW}(S)\to\mathrm{GW}(R)$ because the nondegeneracy condition on the form need not be preserved. For example, if $F=\mathbb{F}_p(x)$ and $E=F[u]/(u^p-x)$ then $F\hookrightarrow E$ is sheer but the trace map $\mathrm{tr}_{E/F}$ is zero. In order to get around this issue, we need to assume a little more about the map $R\to S$. This is our next subject.

2.4. Separable algebras. The following material is classical, but perhaps not as readily accessible in the literature as it could be. See [J], [DI], and [L], though.

2.5. Definition. *Let $A \to B$ be a map of commutative rings. We say that B is a **separable** A-algebra if any of the following equivalent conditions is satisfied:*

(1) B is projective as a $B \otimes_A B$-module,

(2) The multiplication map $\mu\colon B \otimes_A B \to B$ is split in the category of $B \otimes_A B$-modules,

(3) There exists an element $\omega \in B \otimes_A B$ such that $\mu(\omega) = 1$ and $(b \otimes 1)\omega = \omega(1 \otimes b)$ for all $b \in B$.

(4) The ring map $B \otimes_A B \to B$ is sheer.

The equivalence of the conditions in the above definition is straightforward: clearly (1)⇔(2), and (2)⇔(3) by letting ω be the image of 1 under the splitting. Note that $B \otimes_A B \to B$ is necessarily surjective, and so B is always cyclic as a $B \otimes_A B$-module (and in particular, finitely-generated). This explains why (1) is equivalent to (4).

If ω is a class as in (3) of the above definition, then for any $z \in B \otimes_A B$ one has

$$z.\omega = (\mu(z) \otimes 1).\omega = \omega.(1 \otimes \mu(z)).$$

To see this, write $z = \sum a_i \otimes b_i$ and then just compute that

$$\begin{aligned} z.\omega = \sum(a_i \otimes b_i).\omega = \sum(a_i \otimes 1)(1 \otimes b_i).\omega &= \sum(a_i \otimes 1)(b_i \otimes 1).\omega \\ &= \Big(\big(\sum a_i b_i\big) \otimes 1\Big).w = (\mu(z) \otimes 1).\omega. \end{aligned}$$

In particular, notice that the class ω from (3) will be unique: if ω' is another such class then we would have

$$\omega.\omega' = (\mu(\omega) \otimes 1).\omega' = (1 \otimes 1).\omega' = \omega'$$

and likewise $\omega.\omega' = \omega$. Also notice that ω is idempotent. Consequently, we have the isomorphism of rings

$$B \otimes_A B \cong (B \otimes_A B)/w \times (B \otimes_A B)/(1 - w)$$

given in each component by projection. The second component can be identified with B. Indeed, certainly $1-\omega$ belongs to $\ker \mu$. Conversely, if $s \in \ker \mu$ then $s.\omega = (\mu(s) \otimes 1).\omega = 0$, and so $s = s - s.\omega = s(1-\omega) \in (1-\omega)$. It follows that μ induces an isomorphism of rings $(B \otimes_A B)/(1-\omega) \cong B$.

2.6. Remark. It helps to have some geometric intuition here. When $E \to B$ is a topological covering space, the diagonal $\Delta\colon E \to E \times_B E$ gives a homeomorphism from E onto an open-and-closed subspace of $E \times_B E$. Similarly, when $A \to B$ is separable then $\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B$ splits off the diagonal copy of $\operatorname{Spec} B$ as an open-and-closed summand. The idempotent $\omega \in B \otimes_A B$ is the algebraic culprit for this splitting.

2.7. Remark. There is another description of ω that is sometimes useful. Since B is a finitely-generated, projective $B\otimes_A B$-module there is a trace map $\mathrm{tr}_{B/(B\otimes_A B)}\colon \mathrm{End}(B)\to B\otimes_A B$. The element ω is simply $\mathrm{tr}_{B/(B\otimes_A B)}(id_B)$.

We will shortly restrict ourselves to studying maps $R\to S$ which are both sheer and separable. Such maps are commonly referred to using different language:

2.8. Proposition. *For a map of commutative rings $R\to S$ the following are equivalent:*

(1) $R\to S$ is both sheer and separable (we will refer to such maps as "sheerly separable");

(2) $R\to S$ is sheer and S is flat as an $S\otimes_R S$-module;

(3) $R\to S$ is étale and S is finitely-presented as an R-module.

2.9. Remark. Algebraic geometers tend to use the phrase "finite étale" to describe the maps $R\to S$ satisfying the condition of Proposition 2.8. But this terminology is a bit confusing in the non-Noetherian case, since "finite étale" does not mean "both finite and étale". Because of this confusion (which seems entirely unnecessary) we avoid the geometers' phrasing in what follows.

Proof of Proposition 2.8. The implication (1)⇒(2) is trivial. For (2)⇒(1), consider the exact sequence

$$0\longrightarrow I\longrightarrow S\otimes_R S\xrightarrow{\mu} S\longrightarrow 0. \tag{2.10}$$

The map $s\mapsto s\otimes 1$ is a splitting for μ as a map of left S-modules, and so there is an associated splitting $S\otimes_R S\to I$ of left S-modules. If $R\to S$ is sheer then S is finitely-presented as an R-module, and so $S\otimes_R S$ is finitely-generated as a left S-module. Consequently, I is finitely-generated as a left S-module. Thus I is also finitely-generated as an $S\otimes_R S$-module, and therefore S is finitely-presented as $S\otimes_R S$-module. Flatness of S as an $S\otimes_R S$-module then implies $S\otimes_R S\to S$ is sheer by Remark 2.3, and so we have proven (2)⇒(1).

Now we turn to (1)⇒(3), so suppose $R\to S$ is sheerly separable. Separability implies that (2.10) is split as a sequence of $S\otimes_R S$-modules. So there is an $S\otimes_R S$-linear map $\chi\colon S\otimes_R S\to I$ splitting the inclusion. Linearity implies that this map sends I into I^2, and so surjectivity gives us $I=I^2$. So $\Omega_{S/R}=I/I^2=0$. Since S is flat and finite-type over R, and $\Omega_{S/R}=0$, it follows that $R\to S$ is étale by [Mi, Proposition I.3.5].

Now suppose that $R\to S$ is étale and S is finitely-presented as an R-module. Since $R\to S$ is flat it follows from Remark 2.3 that $R\to S$ is sheer. The map $f\colon S\to S\otimes_R S$ given by $f(s)=s\otimes 1$ is also étale (geometrically, étale maps are closed under pullback). If $\mu\colon S\otimes_R S\to S$ is the multiplication, then $\mu\circ f=id$. Since f and id are étale, so is μ by [Mi, Corollary I.3.6]. Therefore S is flat over $S\otimes_R S$, and so we have shown (3)⇒(2). ∎

2.11. Corollary. *Let k be a field. A map of commutative rings $k \to E$ is sheer and separable if and only if there is an isomorphism of k-algebras $E \cong E_1 \times E_2 \times \cdots \times E_n$ where each E_i is a separable (in the classical sense) field extension of k.*

Proof. By Proposition 2.8 we can replace "sheerly separable" by "finite étale", and then the result is standard (for example, see [Mi, Proposition I.3.1]). ∎

2.12. Remark. Suppose we are working in a category that has finite limits. Let $\mathcal{P}$ be a property of morphisms that is closed under composition and pullback. Say that a morphism $X \to Y$ has property $\mathcal{PP}$ if $X \to Y$ has $\mathcal{P}$ and $\Delta\colon X \to X \times_Y X$ also has $\mathcal{P}$. Then it follows by general category theory that property $\mathcal{PP}$ is closed under composition and pullback, and has the feature that if composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ are given such that both g and gf have $\mathcal{PP}$ then so does f. For the proof of the latter, the main ideas can be found in any standard reference dealing with the case where $\mathcal{P}$ is "étale" (e.g. [Mi, Corollary I.3.6]). In the present context, we can apply this principle to the opposite category of commutative rings, where $\mathcal{P}$ is "sheer" and $\mathcal{PP}$ is therefore "sheerly separable". So the sheerly separable maps are closed under pullbacks and composition, and have the indicated two-out-of-three property.

2.13. Example. Here are three examples to keep in mind when dealing with these concepts:

(a) If R and S are commutative rings then the projection $R \times S \to R$ is sheerly separable, but not an injection.

(b) If R is a commutative ring then the map $R[x] \to R$ sending $x \mapsto 0$ is separable but not sheer.

(c) Given any non-separable, finite field extension $k \hookrightarrow E$, this map is sheer but not separable.

2.14. Remark. The maps we are calling "sheerly separable" are called "strongly separable" in [J], and "projective separable" in [L]. The following two conditions on a map of commutative rings $R \to S$ are also equivalent to being sheerly separable:

(1) S is separable over R and S is projective as an R-module (but not required to be finitely-generated);

(2) S is a finitely-generated projective module over R and the trace form $S \otimes_R S \to R$ (given by $x \otimes y \mapsto \operatorname{tr}_{S/R}(xy)$) is nondegenerate.

The proof of these equivalences, or at least a sketch of such, is available in [L, Proposition 6.11].

2.15. Corollary. *Suppose that $R \to S$ is sheerly separable. Then for every finitely-generated projective S-module Q, the map $\operatorname{Hom}_S(Q, S) \to \operatorname{Hom}_R(Q, R)$ given by $g \mapsto \operatorname{tr}_{S/R} \circ g$ is an isomorphism of R-modules.*

PROOF. Part (2) of Remark 2.14 shows that the given map is an isomorphism when $Q = S$. It then follows immediately that it is an isomorphism for $Q = S^n$, and therefore whenever Q is a retract of S^n. ∎

2.16. SCHARLAU TRANSFER MAPS FOR GROTHENDIECK-WITT GROUPS.

Suppose $f\colon R \to S$ is sheer and separable. If (Q, c) is a nondegenerate quadratic space over S, consider Q as an R-module via restriction of scalars and equip it with the bilinear form b given as the composite

$$Q \otimes_R Q \to Q \otimes_S Q \xrightarrow{c} S \xrightarrow{\mathrm{tr}_{S/R}} R.$$

Note that as an S-module Q is a direct summand of some S^n, and S is projective as an R-module, so Q is also projective as an R-module. We claim that (Q, b) is nondegenerate. To see this, consider the commutative diagram

$$\begin{tikzcd} Q \arrow[r, "\tilde{c}"] \arrow[dr, "\tilde{b}"'] & \mathrm{Hom}_S(Q, S) \arrow[d, "T"] \\ & \mathrm{Hom}_R(Q, R) \end{tikzcd}$$

where the top map is the adjoint of c, the diagonal map is the adjoint of b, and the vertical map T sends $g\colon Q \to S$ to the composition $\mathrm{tr}_{S/R} \circ g$. Then $\tilde{c}$ is an isomorphism by nondegeneracy of (Q, c), T is an isomorphism by Corollary 2.15, and so $\tilde{b}$ is also an isomorphism.

It is now clear that we obtain a well-defined map of groups $f^!\colon \mathrm{GW}(S) \to \mathrm{GW}(R)$. This is sometimes called the **Scharlau transfer**; one can find a version of it (not done in full generality) in [S, Chapter 2.5]. Note that $f^!$ is only additive, not a map of rings.

When B is a separable A-algebra there is a canonical quadratic space over the ring $B \otimes_A B$: it is B itself (with the usual structure of $B \otimes_A B$-module), equipped with the following bilinear form:

$$B \otimes_{(B \otimes_A B)} B \to B \otimes_A B, \qquad x \otimes y \to (xy \otimes 1).\omega.$$

A moment's check shows that this is indeed $B \otimes_A B$-bilinear, as required. It is also nondegenerate: this is an easy exercise using the properties of ω. We will denote this quadratic space as $(B, \mu \cdot \omega)$. We can also describe this construction in another way. Observe that $\mu\colon B \otimes_A B \to B$ is sheerly separable (it is sheer because $A \to B$ is separable, and it is separable because $B \otimes_{B \otimes_A B} B \to B$ is actually an isomorphism). The quadratic space $(B, \mu \cdot \omega)$ is simply the element $\mu^!(1)$, where $\mu^!\colon \mathrm{GW}(B) \to \mathrm{GW}(B \otimes_A B)$ is the Scharlau transfer.

More generally, for any quadratic space (P, b) over B we obtain a quadratic space $(P, b \cdot \omega)$ over $B \otimes_A B$. The underlying module is P (regarded as a $B \otimes_A B$-module, where it is necessarily projective) equipped with the bilinear form

$$P \otimes_{(B \otimes_A B)} P \to B \otimes_A B, \qquad x \otimes y \mapsto (b(x, y) \otimes 1).\omega.$$

This construction induces a map of groups (not rings)

$$\mathrm{GW}(B) \to \mathrm{GW}(B \otimes_A B), \qquad [P, b] \mapsto [P, b \cdot \omega]$$

and this is precisely the map $\mu^!$.

2.17. Remark. The significance of the quadratic space $(B, \mu \cdot \omega)$ will become clear in Section 2.18 below. It plays the role of the identity morphism in the Grothendieck-Witt category.

2.18. The Grothendieck-Witt category of a commutative ring. We next restrict to a somewhat specialized setting. Assume that S, T, and U are R-algebras, but also assume that $R \to T$ is sheer and separable.

Now suppose given a quadratic space (Q, c) over $U \otimes_R T$ and another quadratic space (P, b) over $T \otimes_R S$. In the following diagram, it is readily checked that the "across-the-top-then-down" composite satisfies the appropriate T-invariance condition to induce the dotted map:

$$\begin{tikzcd}
Q \otimes_R P \otimes_R Q \otimes_R P \arrow[r, "1\otimes t\otimes 1"] \arrow[ddd] & Q \otimes_R Q \otimes_R P \otimes_R P \arrow[r, "c\otimes b"] & (U \otimes_R T) \otimes_R (T \otimes_R S) \arrow[d, "1\otimes\mu\otimes 1"] \\
& & U \otimes_R T \otimes_R S \arrow[d, "1\otimes \mathrm{tr}_{T/R}\otimes 1"] \\
& & U \otimes_R R \otimes_R S \arrow[d, "\cong"] \\
(Q \otimes_T P) \otimes_R (Q \otimes_T P) \arrow[rr, dotted, "c\hat{\otimes}_T b"] & & U \otimes_R S.
\end{tikzcd}$$

This produces a quadratic space $(Q \otimes_T P, c\hat{\otimes}_T b)$ over the ring $U \otimes_R S$. It is easy to check that this is nondegenerate if (P, b) and (Q, c) were, and the construction is evidently compatible with direct sums. So we obtain a pairing

$$\mathrm{GW}(U \otimes_R T) \otimes \mathrm{GW}(T \otimes_R S) \xrightarrow{\hat{\otimes}_T} \mathrm{GW}(U \otimes_R S). \tag{2.19}$$

If we denote the evident maps as

$$j_{12} \colon U \otimes_R T \to U \otimes_R T \otimes_R S, \quad j_{23} \colon T \otimes_R S \to U \otimes_R T \otimes_R S,$$
$$j_{13} \colon U \otimes_R S \to U \otimes_R T \otimes_R S$$

then the pairing of (2.19) can also be expressed as

$$\alpha \hat{\otimes}_T \beta = j_{13}^!\big((j_{12*}\alpha) \cdot (j_{23*}\beta)\big)$$

(and this observation avoids the need to check nondegeneracy). Note that j_{13} is a pushout of the map $R \to T$, and so is sheerly separable by Remark 2.12; this is why the transfer map $j_{13}^!$ is defined.

It is easy to check that these pairings satisfy associativity. They are also unital, with the unit being the canonical element $(T, \mu \cdot \omega)$ in $\mathrm{GW}(T \otimes_R T)$.

2.20. Definition. *Let R be a commutative ring. The* ***Grothendieck-Witt category*** *of R is the category enriched over abelian groups defined as follows:*

(1) The objects are $\operatorname{Spec} T$ *for T a sheerly separable R-algebra,*

(2) The set of morphisms from $\operatorname{Spec} T$ *to* $\operatorname{Spec} U$ *is the additive group* $\mathrm{GW}(U \otimes_R T)$*;*

(3) Composition of morphisms is defined by (2.19).

This category will be denoted $\mathrm{GWC}(R)$.

3. The general theory of Gysin functors

When studying the Grothendieck-Witt categories $\mathrm{GWC}(R)$, it turns out to be advantageous to investigate the story in greater generality. We do this in the present section. The "Gysin functors" that we introduce here are simply functors with pullback and push-forward maps which are compatible in familiar ways. Certainly such functors have been encountered time and again in the literature, and so it is unlikely that anything in this section is actually "new". A very early reference is [G], whereas a more recent reference is [B]. In the setting of finite group theory, our Gysin functors are precisely the *commutative Green functors*.

Being unaware of a reference that serves as a perfect source for what we need, we take some time here to develop the theory from first principles. In doing so, we have tried to provide a unity of discussion that justifies this. We stress, though, that much of the material from this section is in [B].

The main things we do here are:

- Give the definition of a Gysin functor and develop the basic properties;
- Observe the existence of a "universal" Gysin functor, called the Burnside functor;
- Observe that any Gysin functor E on a category $\mathcal{C}$ gives rise to an associated closed, symmetric monoidal category, denoted $\mathcal{C}_E$, of "E-correspondences" between the objects of $\mathcal{C}$. These symmetric monoidal categories have the properties that all objects are dualizable, and moreover every object is self-dual.

3.1. Gysin functors. Let $\mathcal{C}$ be a category with finite limits and finite coproducts, with the property that pullbacks distribute over coproducts: that is, given any maps $A \to X$, $P_1 \to X$, and $P_2 \to X$ the natural map

$$(A \times_X P_1) \amalg (A \times_X P_2) \to A \times_X (P_1 \amalg P_2)$$

is an isomorphism. We also assume that for any objects A and B in $\mathcal{C}$ the following diagrams are pullbacks, where $\emptyset$ is the initial object:

$$\begin{array}{ccc} A \xrightarrow{id} A & B \xrightarrow{id} B & \emptyset \longrightarrow B \\ {\scriptstyle id}\downarrow \quad \downarrow{\scriptstyle i_0} & {\scriptstyle id}\downarrow \quad \downarrow{\scriptstyle i_1} & \downarrow \quad \downarrow{\scriptstyle i_1} \\ A \xrightarrow{i_0} A \amalg B & B \xrightarrow{i_1} A \amalg B & A \xrightarrow{i_0} A \amalg B. \end{array} \tag{3.2}$$

Such categories are called **finitary lextensive** [CLW, Corollary 4.9]. The strange term "lextensive" comes from combining "extensive" with "Limits". Standard examples to keep in mind are the categories $\mathcal{S}et$ and $G\mathcal{S}et$, and more generally any topos. Note that the category of pointed sets $\mathcal{S}et_*$ is *not* finitary lextensive, as the distributive axiom fails. Here is another important example:

3.3. PROPOSITION. *Fix a commutative ring R. Let $\mathcal{A}\mathrm{ff}_R$ be the category of affine schemes over R (the opposite category of commutative R-algebras) and let $\mathcal{A}\mathrm{ff}_{R,ss}$ be the full subcategory consisting of all objects $\operatorname{Spec} T$ where $R \to T$ is sheerly separable. Then $\mathcal{A}\mathrm{ff}_{R,ss}$ is finitary lextensive.*

PROOF. The two-out-of-three property from Remark 2.12 shows that every map in $\mathcal{A}\mathrm{ff}_{R,ss}$ is sheerly separable. We claim that finite coproducts exist in $\mathcal{A}\mathrm{ff}_{R,ss}$ and are the same as those in $\mathcal{A}\mathrm{ff}_R$. To see this one needs to check that if $R \to T$ and $R \to U$ are sheerly separable then so is $R \to T \times U$. It is certainly sheer. For separability one uses that

$$(T \times U) \otimes_R (T \times U) \cong (T \otimes_R T) \times (T \otimes_R U) \times (U \otimes_R T) \times (U \otimes_R U).$$

The multiplication map on $T \times U$ makes it a module over the four-fold Cartesian product on the right by having the middle two factors act as 0, whereas the first and last factors act via $T \otimes_R T \to T$ and $U \otimes_R U \to U$. Since T (resp. U) is projective as $T \otimes_R T$ (resp. $U \otimes_R U$)-module, we get projectivity of $T \times U$ over the four-fold Cartesian product.

We next claim that pullbacks (and in particular, finite products) exist in $\mathcal{A}\mathrm{ff}_{R,ss}$ and are the same as those in $\mathcal{A}\mathrm{ff}_R$. This follows immediately from the properties of sheerly separable maps given in Remark 2.12.

At this point we have shown that $\mathcal{A}\mathrm{ff}_{R,ss}$ has finite coproducts and finite limits, and that these agree with those in $\mathcal{A}\mathrm{ff}_R$. Verification of the distributivity axiom and the necessary pullback squares are then trivial exercises. ∎

3.4. REMARK. The analog of Proposition 3.3 is not true for the subcategory $\mathcal{A}\mathrm{ff}_{R,sh}$ consisting of objects $\operatorname{Spec} T$ where $R \to T$ is sheer, and where the maps in the category are also required to be sheer. The problem lies with finite limits. If $R \to T$ is sheer but not separable and $X = \operatorname{Spec} T$, $Y = \operatorname{Spec} R$, then the diagonal map $X \to X \times_Y X$ is not a map in the category. So either the pullback of $X \to Y \leftarrow X$ does not exist in $\mathcal{A}\mathrm{ff}_{R,sh}$ or it is different from the pullback in $\mathcal{A}\mathrm{ff}_R$.

We record one useful lemma about finitary lextensive categories (note here again that $\emptyset$ always denotes the initial object of $\mathcal{C}$):

3.5. LEMMA. *Let $\mathcal{C}$ be finitary lextensive. Then any map $Z \to \emptyset$ is an isomorphism, and the pullback of any diagram $\emptyset \to Y \leftarrow X$ is $\emptyset$.*

PROOF. The first part is [CLW, Proposition 2.8], but we recount the proof here. If $f\colon Z \to \emptyset$ is a map then we can write

$$Z = Z \times_\emptyset \emptyset = Z \times_\emptyset (\emptyset \amalg \emptyset) = (Z \times_\emptyset \emptyset) \amalg (Z \times_\emptyset \emptyset) = Z \amalg Z.$$

So the fold map $\nabla\colon Z \amalg Z \to Z$ is an isomorphism, and hence any two maps from Z to another object must be identical. In particular, the composite $Z \to \emptyset \to Z$ must be the identity, from which it follows that f is an isomorphism.

The second statement in the lemma is an immediate consequence of the first. ∎

3.6. Definition. *A* ***Gysin functor*** *on a finitary lextensive category $\mathcal{C}$ is a contravariant functor E from $\mathcal{C}$ to $\mathfrak{CommRing}$ together with a covariant functor $\tilde{E}\colon \mathcal{C} \to \mathcal{A}b$ such that $E(X) = \tilde{E}(X)$ for every object X. If $f\colon X \to Y$ is a map we write $f^* = E(f)$ and $f_! = \tilde{E}(f)$. The maps $f_!$ will be called Gysin maps. For $a \in E(X)$ and $b \in E(Y)$ we write*

$$a \otimes b = (\pi_X^{XY})^*(a) \cdot (\pi_Y^{XY})^*(b) \in E(X \times Y).$$

We require the following axioms:

(1) [Zero axiom] $E(\emptyset) = 0$.

(2) [Behavior on sums] *For any objects X and Y, the natural map*

$$i_X^* \times i_Y^*\colon E(X \amalg Y) \to E(X) \times E(Y)$$

is an isomorphism of rings. Here $i_X\colon X \to X \amalg Y$ and $i_Y\colon Y \to X \amalg Y$ are the canonical maps.

(3) [Push-product axiom] *For any maps $f\colon X \to X'$, $g\colon Y \to Y'$ and $a \in E(X)$, $b \in E(Y)$ one has*

$$(f \times g)_!(a \otimes b) = f_!(a) \otimes g_!(b).$$

(4) [Push-Pull axiom] *For every pullback diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

one has $f_! p^* = q^* g_!$.

A natural transformation between Gysin functors is a natural transformation of contravariant functors that is also a natural transformation of the covariant piece.

3.7. Remark.

(a) The above definition starts with the "internal" multiplications on the abelian groups $E(X)$ and derives the external pairings $E(X) \otimes E(Y) \to E(X \times Y)$. As usual, the opposite approach can also be taken: we could have written the above definition in terms of external pairings, and then constructed the internal pairings using the diagonal maps. The two approaches are equivalent.

(b) When $\mathcal{C}$ is the category of finite G-sets, what we have called Gysin functors are more commonly called *commutative Green functors*; see [B, Chapter 2]. We adopted the term "Gysin functor" due to its brevity.

The following lemmas are useful to record:

3.8. Lemma. *If f is an isomorphism in $\mathcal{C}$ then $f_! = (f^*)^{-1}$ in any Gysin functor.*

Proof. This follows immediately from the push-pull axiom, using the pullback diagram

$$\begin{tikzcd} A \arrow[r, "id"] \arrow[d, "id"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & A. \end{tikzcd}$$

■

3.9. Lemma. *For any objects A and B, the composition*

$$E(A) \oplus E(B) \xrightarrow{(i_0)_! \oplus (i_1)_!} E(A \amalg B) \xrightarrow{(i_0)^* \times (i_1)^*} E(A) \times E(B)$$

sends a pair (x, y) to (x, y) (we refrain from calling this the identity only because the domain and target are perhaps not "equal"). Consequently, the pushforward map $(i_0)_! \oplus (i_1)_!\colon E(A) \oplus E(B) \to E(A \amalg B)$ is an isomorphism of abelian groups.

Proof. For the first statement use the push-pull axiom applied to the three pullback squares listed in the original introduction of $\mathcal{C}$, together with $E(\emptyset) = 0$. The second statement of the lemma then follows directly from Axiom (2) in Definition 3.6. ■

3.10. Lemma. *Let $f\colon A \to X$ and $g\colon B \to X$. Then $(f \times_X g)_!(1) = f_!(1) \cdot g_!(1)$.*

Proof. Use push-pull for the square

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d, "f \times_X g"'] & A \times B \arrow[d, "f \times g"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

Start with $1 = 1 \otimes 1 \in E(A \times B)$, and use the push-product axiom. ■

3.11. Proposition. [Projection formula] *Let E be a Gysin functor. Then given a map $f\colon X \to Y$, $\alpha \in E(X)$, and $\beta \in E(Y)$ one has*

$$f_!(\alpha \cdot f^*(\beta)) = f_!(\alpha).\beta.$$

Proof. Using push-pull applied to $\alpha \otimes \beta \in E(X \times Y)$, the pullback diagram

$$\begin{tikzcd} X \arrow[r, "id \times f"] \arrow[d, "f"'] & X \times Y \arrow[d, "f \times id"] \\ Y \arrow[r, "\Delta"] & Y \times Y \end{tikzcd}$$

implies that $f_!(\alpha \cdot f^*\beta) = \Delta^*(f \times id)_!(\alpha \otimes \beta)$. The push-product axiom finishes the proof. ■

3.12. Example. We give various examples of Gysin functors.

(a) Let $\mathcal{C}$ be the category of sets but with morphisms the maps where all fibers are finite (called **quasi-finite maps** from now on). Let $E(S) = \operatorname{Hom}(S, \mathbb{Z})$, with the ring operations given by pointwise addition and multiplication. If $f\colon S \to T$ then f^* is the evident map and $f_!\colon E(S) \to E(T)$ sends a map $\alpha\colon S \to \mathbb{Z}$ to the assignment $t \mapsto \sum_{s \in f^{-1}(t)} \alpha(s)$. This satisfies the axioms of Definition 3.6, except $\mathcal{C}$ is not finitary lextensive (there is no terminal object). Taking $\mathcal{C}$ to instead be the category of finite sets repairs this.

(b) Let G be a finite group, and let $\mathcal{C}$ be the category of finite G-sets. For S in $\mathcal{C}$ define $\mathcal{A}(S)$ to be the Grothendieck group of maps $X \to S$ (where X is a finite G-set) formed with respect to the direct sum operation $\amalg$, made into a ring via $[X \to S]\cdot[Y \to S] = [X \times_S Y \to S]$. Given $f\colon S \to T$ one gets maps $f^*\colon \mathcal{A}(T) \to \mathcal{A}(S)$ by pulling back along f, and $f_!\colon \mathcal{A}(S) \to \mathcal{A}(T)$ by composing with f.

(c) Fix a commutative ring k and consider the category $\mathcal{A}\mathrm{ff}_{k,ss}$ from Proposition 3.3. For R a k-algebra let $K^0(\operatorname{Spec} R)$ be the Grothendieck group of finitely-generated R-projectives. For $f\colon \operatorname{Spec} R \to \operatorname{Spec} S$ we have $f^*\colon K^0(\operatorname{Spec} S) \to K^0(\operatorname{Spec} R)$ given by $[P] \mapsto [P \otimes_S R]$, and $f_!\colon K^0(\operatorname{Spec} R) \to K^0(\operatorname{Spec} S)$ given by restriction of scalars (so $[P]_R \mapsto [P]_S$).

(d) Again considering $\mathcal{A}\mathrm{ff}_{k,ss}$ as in the previous example, the assignment $\operatorname{Spec} S \mapsto \mathrm{GW}(S)$ has the structure of a Gysin functor, as detailed in Section 2.

(e) Let $\mathcal{C}$ be the category of topological spaces, with morphisms the quasi-finite fibrations. Define $E(X) = \operatorname{Hom}(\pi_0(X), \mathbb{Z}) = H^0(X)$. The pullback maps are as expected. For $f\colon X \to Y$ and $\alpha \in E(X)$ define $f_!(\alpha)$ to be the assignment $[y] \mapsto \sum_{x \in f^{-1}(y)} \alpha([x])$ where $[x]$ and $[y]$ denote the path-components containing x and y. This satisfies the axioms of a Gysin functor (the fibration condition is needed only to show that $f_!$ is well-defined), but the category $\mathcal{C}$ is not finitary lextensive (it does not have a terminal object). This example has a strong relation to that in (a) above.

(f) The following is not an example of a Gysin functor, but is nevertheless instructive. Let $\mathcal{C}$ be the category of finite sets, and let $\mathcal{P}(X)$ be the powerset of the set X; this is not quite a ring, but it does have the intersection operation $\cap$ which we will regard as a multiplication. Given $f\colon X \to Y$ one has the inverse-image map $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ (which preserves the multiplication) and the image map $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ (which does not). The axioms of Definition 3.6 are all satisfied, when suitably interpreted. The powerset functor is something like a "non-additive Gysin functor".

3.13. Remark. Let $\mathcal{C}$ be the category of compact, oriented topological manifolds, and let $E(X) = H^*(X)$. With the usual pullbacks and Gysin morphisms, this is *almost* (but not quite) a Gysin functor as we defined above. In addition to the category $\mathcal{C}$ not being finitary lextensive (finite limits do not always exist), the difficulty is that the push-pull axiom only holds for pullback squares satisfying a suitable transversality condition. This same problem arises if one uses smooth algebraic varieties and the Chow ring, or if one uses oriented manifolds and complex cobordism. But all of these settings represent appearances in the literature of structure similar to what we consider in the present paper. Especially in the case of cobordism, see the axiomatic treatment in [Q, Section 1]. Prior to [Q, Proposition 1.12] Quillen refers to, but does not give, an axiomatic treatment related to the multiplicative structure; the axioms for a Gysin functor are essentially this.

3.14. The universal Gysin functor. Let $\mathcal{C}$ be finitary lextensive. Given an object X in $\mathcal{C}$, define $\mathcal{A}_{\mathcal{C}}(X)$ to be the Grothendieck group of (isomorphism classes of) maps $S \to X$ where

$$[(S \amalg T) \to X] = [S \to X] + [T \to X].$$

The multiplication $[S \to X] \cdot [T \to X] = [S \times_X T \to X]$ is well-defined and makes $\mathcal{A}_{\mathcal{C}}(X)$ into a commutative ring with identity $[id\colon X \to X]$. We call $\mathcal{A}_{\mathcal{C}}(X)$ the **Burnside ring** of X. Note that $\mathcal{A}_{\mathcal{C}}$ has the evident structure of a contravariant functor to rings, as well as that of a covariant structure to abelian groups, generalizing the situation in Example 3.12(b). When the category $\mathcal{C}$ is understood we abbreviate $\mathcal{A}_{\mathcal{C}}$ to just $\mathcal{A}$.

3.15. Proposition. *The Burnside functor $\mathcal{A}_{\mathcal{C}}$ is a Gysin functor.*

Proof. Axioms (3) and (4) are immediate. Axiom (1) follows from Lemma 3.5, which says that every map $Z \to \emptyset$ is an isomorphism. For axiom (2) one checks directly that the composition

$$E(X) \oplus E(Y) \xrightarrow{(i_0)_! \oplus (i_1)_!} E(X \amalg Y) \xrightarrow{(i_0)^* \times (i_1)^*} E(X) \times E(Y)$$

is the identity; this follows from the pullback squares in (3.2) together with the pullback property from Lemma 3.5. The check that the composite $[(i_0)_! \oplus (i_1)_!] \circ [(i_0)^* \times (i_1)^*]$ equals the identity is similar, since for $S \to X \amalg Y$ one can write the distributivity formula

$$S = (X \amalg Y) \times_{X \amalg Y} S = (X \times_{X \amalg Y} S) \amalg (Y \times_{X \amalg Y} S).$$

∎

3.16. Example. When $\mathcal{C}$ is the category of finite sets, note that there is a natural isomorphism $\mathcal{A}(S) \cong \mathrm{Hom}(S, \mathbb{Z})$, sending the element $[f\colon M \to S]$ to the assignment $s \mapsto \# f^{-1}(s)$. The Gysin functor given in Example 3.12(a) is the Burnside functor for $\mathcal{C}$.

The Burnside functor has the following universal property:

3.17. Proposition. *If E is a Gysin functor on the category $\mathcal{C}$ then there is a unique map of Gysin functors $\mathcal{A}_{\mathcal{C}} \to E$. It sends $[f\colon A \to X]$ in $\mathcal{A}_{\mathcal{C}}(X)$ to $f_!(1) \in E(X)$.*

PROOF. For existence, use the given formula. The fact that $\mathcal{A}_{\mathcal{C}}(X) \to E(X)$ is well-defined follows using Lemma 3.9, which implies that $(f \amalg g)_!(1) = f_!(1) + g_!(1)$. The fact that it is a ring map follows from Lemma 3.10. Compatibility with pullbacks and pushforwards is trivial. Uniqueness follows from the fact that $[f\colon A \to X] \in \mathcal{A}(X)$ equals $f_!^{\mathcal{A}}(1)$, the pushforward in the Gysin functor $\mathcal{A}$. ∎

3.18. CATEGORIES DERIVED FROM GYSIN FUNCTORS. Given a Gysin functor E on $\mathcal{C}$ we can define an additive category $\mathcal{C}_E$ as follows. First, the objects of $\mathcal{C}_E$ are the same as the objects of $\mathcal{C}$. Second, for any objects A and B define

$$\mathcal{C}_E(A, B) = E(B \times A).$$

Really what we mean here is that $\mathcal{C}_E(A, B)$ is the underlying abelian group of $E(B \times A)$. Third, define the composition law

$$\mu_{C,B,A}\colon \mathcal{C}_E(B, C) \otimes \mathcal{C}_E(A, B) \to \mathcal{C}_E(A, C)$$

by

$$\mu_{C,B,A}(\alpha \otimes \beta) = (\pi_{13})_!\big((\pi_{12})^*(\alpha) \cdot {\pi_{23}}^*(\beta)\big)$$

where the π_{rs} maps are the evident ones

$$\pi_{12}\colon A \times B \times C \to A \times B, \quad \pi_{23}\colon A \times B \times C \to B \times C,$$

$$\pi_{13}\colon A \times B \times C \to A \times C.$$

We will use the notation

$$\alpha \circ \beta = \mu_{C,B,A}(\alpha \otimes \beta).$$

Finally, for any object A define i_A to be ${\Delta^A}_!(1)$; that is, consider the map

$$E(A) \xrightarrow{{\Delta^A}_!} E(A \times A)$$

and take the image of the unit element of the ring $E(A)$. Note that $E(A \times A)$ is a commutative ring and so has a unit element 1, but this is not necessarily equal to i_A. One may check (see Proposition 3.21(a) below) that this structure makes $\mathcal{C}_E$ into a category with identity maps i_A.

We refer to elements of $E(B \times A)$ as "E-correspondences" from A to B. The category $\mathcal{C}_E$ itself will be referred to as the **category of *E*-correspondences**.

3.19. REMARK. The construction of the category $\mathcal{C}_E$ is one that appears countless times in the algebraic geometry literature, ultimately going back to Grothendieck. For the category of algebraic varieties over some field k, forming the category of correspondences with respect to the Chow ring functor is the first step in Grothendieck's attempts to define a category of motives. See for example [M, Section 2].

3.20. Example.

(a) Let G be a finite group, let $\mathcal{C}$ be the category of finite G-sets, and let $\mathcal{A}$ be the Burnside functor from Example 3.12(b). The category $\mathcal{C}_{\mathcal{A}}$ is precisely the category $\mathcal{B}urn$ mentioned in Section 1.

(b) Fix a commutative ring R, and let $\mathcal{A}\mathrm{ff}_{R,ss}$ be the category from Proposition 3.3. Then $\mathcal{C}_{GW}$ is the Grothendieck-Witt category over R, defined in Section 2.

(c) Let $\mathcal{C}$ be the category of finite sets, and let E be the Gysin functor from Example 3.12(a). Then we obtain the category of correspondences $\mathcal{C}_E$. It turns out this category has a familiar model: it is equivalent to the category of finitely-generated, free abelian groups. Proving this is not hard, but it will also fall out of our general "reconstruction theorem" (Theorem 4.16). See Example 4.17.

The following proposition details many (and perhaps too many) useful facts about the category $\mathcal{C}_E$. Recall one piece of notation: maps into products can be unlabelled if there is a self-evident candidate for how the map projects onto each of the factors. For example, if $f\colon A \to B$ then $A \to A \times B$ denotes the evident map that is the identity on the first factor and f on the second.

3.21. Proposition. *Let E be a Gysin functor on a finitary lextensive category $\mathcal{C}$.*

(a) The structure described above defines a category $\mathcal{C}_E$ that is enriched over abelian groups, where $i_A \in \mathcal{C}_E(A, A)$ is the identity map on A.

(b) A natural transformation of Gysin functors $E \to E'$ induces a functor $\mathcal{C}_E \to \mathcal{C}_{E'}$.

(c) There is a functor $R\colon \mathcal{C} \to \mathcal{C}_E$ that is the identity on objects and has the property that for $f\colon A \to B$ in $\mathcal{C}$ we have

$$R_f = (id_B \times f)^*(i_B) = (B \times A \to B \times B)^*(i_B) \in E(B \times A) = \mathcal{C}_E(A, B).$$

One also has $R_f = (A \to B \times A)_!(1)$.

(d) The category $\mathcal{C}_E$ has an anti-automorphism $(-)^*$ that is the identity on objects, and for $\alpha \in \mathcal{C}_E(A, B)$ is given by

$$\alpha^* = t^*(\alpha)$$

where $t\colon A \times B \to B \times A$ is the evident isomorphism. We define $I\colon \mathcal{C}^{op} \to \mathcal{C}_E$ to be the identity on objects, and to be given on maps by $I(f) = (R_f)^*$. We often write $I_f = I(f)$. If $f\colon A \to B$ then $I_f = (f \times id_B)^*(i_B) = (A \to A \times B)_!(1)$.

(e) Suppose given $\alpha \in \mathcal{C}_E(W, Z)$, $f\colon Y \to W$, $g\colon Z \to U$, $f'\colon W \to Y$, and $g'\colon U \to Z$. Then

(i) $\alpha \circ R_f = (id_Z \times f)^*(\alpha)$;

(ii) $R_g \circ \alpha = (g \times id_W)_!(\alpha)$;

(iii) $\alpha \circ I_{f'} = (id_Z \times f')_!(\alpha)$;

(iv) $I_{g'} \circ \alpha = (g' \times id_W)^*(\alpha)$.

(f) *Given* $A \xrightarrow{f} B \xleftarrow{q} C$ *in* $\mathcal{C}$ *one has* $I_f \circ R_q = (f \times q)^*(i_B) = (\pi_{AC}^{A\times_B C})_!(1)$ *in* $\mathcal{C}_E$.

(g) *Given* $A \xleftarrow{p} D \xrightarrow{g} C$ *in* $\mathcal{C}$ *one has* $R_p \circ I_g = (p \times g)_!(i_D) = \big((p \times g)^D_{AC}\big)_!(1)$ *in* $\mathcal{C}_E$.

(h) *Given a pullback diagram in* $\mathcal{C}$

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "p"'] & W \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

one has $R_p \circ I_g = I_f \circ R_q$ *in* $\mathcal{C}_E$.

(i) *If* f *is an isomorphism in* $\mathcal{C}$ *then* $R_f = I_f^{-1} = I_{f^{-1}}$.

PROOF. This proof is tedious, but completely formal. See Appendix B for details. ∎

Our next goal is to observe that the Gysin functor E, which is both co- and contravariant, extends to a single functor defined on all of $\mathcal{C}_E$. Before embarking on the explanation of this, here is some useful notation. If B is an object of $\mathcal{C}$, note that the abelian group $E(B)$ may be identified with both $\mathcal{C}_E(*, B)$ and $\mathcal{C}_E(B, *)$, where $*$ is the terminal object of $\mathcal{C}$. If $x \in E(B)$ we write x_* for x regarded as an element of $\mathcal{C}_E(*, B) = E(B \times *)$ and ${}_*x$ for x regarded as an element of $\mathcal{C}_E(B, *)$. This notation makes sense if one remembers our general "right-to-left" convention; e.g., x_* is x regarded as a map *from* the object $*$.

Define a functor $E' \colon \mathcal{C}_E^{op} \to \mathcal{A}b$ as follows. On objects it is the same as E: $E'(A) = E(A)$. For $g \in \mathcal{C}_E(A, B)$ define $E'(g) \colon E(B) \to E(A)$ by

$$E'(g)(x) = {}_*x \circ g = \big(\pi_A^{BA}\big)_!\Big[\big(\pi_B^{BA}\big)^*(x) \cdot g\Big].$$

The fact that this is a functor is immediate from the associativity and unital properties of the circle product $\circ$ (the composition product in $\mathcal{C}_E$).

3.22. PROPOSITION. *The functor* $E' \colon \mathcal{C}_E^{op} \to \mathcal{A}b$ *has the property that* $E'(R_f) = f^*$ *and* $E'(I_f) = f_!$ *for any map* f *in* $\mathcal{C}$.

PROOF. Use Proposition 3.21(e), parts (i) and (iii) to write

$$E'(R_f)(x) = {}_*x \circ R_f = (id_* \times f)^*(x) = f^*(x), \quad E'(I_f)(y) = {}_*y \circ I_f = (id_* \times f)_!(y) = f_!(y).$$

(Note that id_* denotes the identity map $* \to *$). ∎

3.23. Remark. Note that E' is not a functor from $\mathcal{C}_E^{op}$ into $\mathcal{C}omm\mathcal{R}ing$. This would of course be too much to ask, since the transfer maps $f_!$ do not respect the multiplicative products.

3.24. Further properties of $\mathcal{C}_E$. The category $\mathcal{C}_E$ has some extra structure that we have not yet accounted for. The categorical product in $\mathcal{C}$ induces a symmetric monoidal product on $\mathcal{C}_E$: that is, for objects X and Y we define

$$X \otimes Y = X \times_{\mathcal{C}} Y.$$

We must define $f \otimes g$ for $f \in \mathcal{C}_E(X, X')$ and $g \in \mathcal{C}_E(Y, Y')$. We do this by

$$f \otimes g = \big(t_{X'XY'Y}^{X'Y'XY}\big)^*(f \otimes g).$$

This formula appears self-referential, but the two tensor symbols mean something different: in the second case, we have $f \in E(X' \times X)$ and $g \in E(Y' \times Y)$ and $f \otimes g$ is the element in $E(X' \times X \times Y' \times Y)$ that was introduced in Definition 3.6. Note that t is the twist map that interchanges the middle two factors.

It takes a little work to verify bi-functoriality, but this is routine. The unit object is $S = *$, the terminal object of $\mathcal{C}$ (note that this is not a terminal object of $\mathcal{C}_E$). The symmetry isomorphism $\tau_{XY} \in \mathcal{C}_E(X \otimes Y, Y \otimes X)$ is defined to be

$$\tau_{XY} = R(t_{XY})$$

where $t_{XY} \colon X \times Y \to Y \times X$ is the canonical isomorphism in $\mathcal{C}$. One must verify that the structure we have defined satisfies the basic commutative diagrams for a symmetric monoidal structure, and we again leave this with simply the remark that it is tedious but not challenging.

We can also define function objects in $\mathcal{C}_E$. For objects X and Y define

$$F(X, Y) = X^* \otimes Y$$

where $(-)^*$ is the anti-automorphism from Proposition 3.21(d). Of course the object X^* is exactly equal to X, but we wrote X^* because this is more compatible with the way the maps work: for $g \colon Y \to Y'$ define $F(X, g)$ to be the map $i_{X^*} \otimes g$, and for $f \colon X \to X'$ define $F(f, Y)$ to be the map $I_f \otimes i_Y$.

At this point it is useful to recall the notion of dualizability in symmetric monoidal categories. See Appendix A for this. We also need the notion of "tensor category". The meaning of this phrase varies somewhat in the literature, but in this paper we will adopt the following:

3.25. Definition. *A* **tensor category** *is an additive category equipped with a symmetric monoidal product $\otimes$ (called the tensor) for which the tensor product of morphisms is bilinear. It follows as a consequence that the tensor product preserves finite coproducts. A tensor category is closed if it is equipped with function objects related to the tensor by the usual adjunction formula, which is required to be linear. We will typically denote the unit for $\otimes$ as S.*

With the above notions in place, we leave the reader to check the following:

3.26. Proposition. *The above structure makes $\mathcal{C}_E$ into a closed tensor category where the unit is $S = *$, and in which every object is dualizable. Moreover, every object is isomorphic to its own dual.*

Proof. Tedious, but routine. Perhaps the only thing that needs remark is that the evaluation and co-evaluation morphisms for an object X are

$$cev_X = i_X \in E(X \times X) = E(X \times X \times *) = \mathcal{C}_E(*, X \times X) = \mathcal{C}_E(S, X \otimes X)$$

and

$$ev_X = i_X \in E(X \times X) = E(* \times X \times X) = \mathcal{C}_E(X \times X, *) = \mathcal{C}_E(X \otimes X, S).$$

■

The following proposition is easy but important. It will be used implicitly in several later calculations.

3.27. Proposition. *Let $f\colon A \to B$ and $g\colon X \to Y$ be maps in $\mathcal{C}$. Then $R(f \times g) = Rf \otimes Rg$ and $I(f \times g) = If \otimes Ig$.*

Proof. Using Proposition 3.21(d) and the definition of tensor product, we have

$$\begin{aligned} If \otimes Ig &= (t^{AXBY}_{ABXY})^* \Big[(A \to AB)_!(1) \otimes (X \to XY)_!(1)\Big] \\ &= (t^{AXBY}_{ABXY})^* \Big[(A \times X \to A \times B \times X \times Y)_!(1)\Big] \\ &= (A \times X \to A \times X \times B \times Y)_!(1) \\ &= I(f \times g). \end{aligned}$$

The second equality uses the Push-Product Axiom, the third equality uses Push-Pull, and the last equality is Proposition 3.21(d) again.

The proof of the other identity is entirely similar. ■

4. Gysin schema and the reconstruction theorem

We have seen that given a Gysin functor E on a finitary lextensive category $\mathcal{C}$, there is an associated symmetric monoidal category $\mathcal{C}_E$ called the category of E-correspondences. One could try to run this process in reverse: given a symmetric monoidal category $\mathcal{D}$, what do you need to know in order to guarantee that $\mathcal{D}$ is the category of E-correspondences for an appropriately chosen E and $\mathcal{C}$? We might term this the "reconstruction problem": can $\mathcal{D}$ be reconstructed as a category of correspondences? Of course for this to work one must at least require that all objects in $\mathcal{D}$ be self-dual.

Unfortunately, in this form the reconstruction problem is a little awkward. The category $\mathcal{C}_E$ comes equipped with two distinguished subcategories, one consisting of the forward maps Rf and one consisting of the backward maps If. If we are just given a symmetric monoidal category $\mathcal{D}$, there is no clear way to separate out analogs of either of these distinguished subcategories.

The way around this problem is to add these special subcategories into the initial data. Then the reconstruction problem becomes solvable, albeit for almost tautological reasons. See Theorem 4.16 below.

4.1. Gysin schema.

4.2. Definition. *A **Gysin schema** consists of the following data:*

- *A finitary lextensive category $\mathcal{C}$, together with an explicit choice $*$ for terminal object and for each objects X and Y of $\mathcal{C}$ an explicit choice of product $X \times Y$;*
- *A tensor category $(\mathcal{D}, \otimes, S)$ (see Definition 3.25);*
- *A map of sets $\Theta\colon \operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$ and two functors $R\colon \mathcal{C} \to \mathcal{D}$ and $I\colon \mathcal{C}^{op} \to \mathcal{D}$;*
- *Isomorphisms $\theta_*\colon \Theta(*) \to S$ and $\theta_{X,Y}\colon \Theta(X \times Y) \to (\Theta X) \otimes (\Theta Y)$.*

This data is required to satisfy the following axioms:

(1) $R(X) = \Theta(X) = I(X)$ for all objects X of $\mathcal{C}$;

(2) R preserves finite coproducts (including the empty coproduct);

(3) The data (R, θ) makes R into a strong symmetric monoidal functor from $(\mathcal{C}, \times, *)$ to $(\mathcal{D}, \otimes, S)$.

(4) For all maps $f\colon A \to X$ and $g\colon B \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} I(A \times B) \arrow[d, "\theta_{A,B}"', "\cong"] & I(X \times Y) \arrow[l, "I(f\times g)"'] \arrow[d, "\cong"', "\theta_{X,Y}"] \\ I(A) \otimes I(B) & I(X) \otimes I(Y) \arrow[l, "I(f)\otimes I(g)"'] \end{tikzcd}$$

is commutative.

(5) For every pullback diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

in $\mathcal{C}$ one has $Rf \circ Ip = Iq \circ Rg$.

We will write the Gysin schema as $\Theta\colon \mathcal{C} \to \mathcal{D}$, suppressing R, I, and θ from the notation.

4.3. Remark. There are a couple of odd features about the above definition. First, the function Θ is clearly redundant as it can be recovered from either R or I. We include Θ in the definition because it is often useful to have a notation that does not favor either R or I. Secondly, conditions (3) and (4) could have been made more symmetric by replacing (4) with the statement that (I, θ) is strong symmetric monoidal; we leave the equivalence as an exericse. The phrasing from the definition makes applications a little easier, as there is a bit less to verify: in practice one looks for a "nice enough" functor R that admits transfer maps satisfying (4) and (5).

4.4. Example. One readily checks that the following are examples of Gysin schema:

(a) Fix a finite group G, and let $\mathcal{D}$ be the G-equivariant stable homotopy category of genuine G-spectra. Let $\mathcal{C}$ be the category of finite G-sets, and let $R(X) = \Sigma^\infty(X_+)$. The maps $I(f)$ are the usual transfer maps constructed in stable homotopy theory.

(b) Let $\mathcal{D}$ be the category of finitely-generated free abelian groups, equipped with the tensor product. Let $\mathcal{C}$ be the category of finite sets. Let $R(X)$ be the free abelian group on the set X, with its natural functoriality. If $f\colon X \to Y$ then let $I(f)\colon R(Y) \to R(X)$ send the basis element $[y]$ to $\sum_{x\in f^{-1}(y)}[x]$.

When X is an object of $\mathcal{C}$ we let π_X denote the unique map $X \to *$, and Δ_X denote the diagonal $X \to X \times X$. The subscripts will usually be suppressed when understood. Note that $R\pi$ is a map $RX \to R(*)$, and we have a chosen isomorphism $R(*) = \Theta(*) \cong S$; so composing these gives a canonical map $RX \to S$, which we will usually also denote $R\pi$ by abuse. Similarly, $R\Delta$ may be regarded as a map $RX \to RX \otimes RX$. We use these conventions for $I\pi$ and $I\Delta$ as well.

4.5. Transfers and duality.

4.6. Proposition. *Suppose that $\Theta\colon \mathcal{C} \to \mathcal{D}$ is a Gysin schema. Then for every object X in $\mathcal{C}$, ΘX is dualizable in $\mathcal{D}$. In fact, ΘX is self-dual with structure maps η_X and ϵ_X given by*

$$S \xrightarrow{I\pi} \Theta(X) \xrightarrow{R\Delta} \Theta(X \times X) \xrightarrow{\cong} \Theta X \otimes \Theta X$$

and

$$\Theta X \otimes \Theta X \xrightarrow{\cong} \Theta(X \times X) \xrightarrow{I\Delta} \Theta X \xrightarrow{R\pi} S.$$

Proof. The key is the pullback diagram

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \times X \arrow[d, "\Delta\times id"] \\ X \times X \arrow[r, "id\times\Delta"] & X \times X \times X, \end{tikzcd}$$

from which we deduce that $I(id\times\Delta)\circ R(\Delta\times id) = R\Delta\circ I\Delta$. Combining this with axioms (3) and (4) from Definition 4.2 gives the first equality below:

$$(id \otimes I\Delta) \circ (R\Delta \otimes id) = R\Delta \circ I\Delta = (I\Delta \otimes id) \circ (id \otimes R\Delta). \tag{4.7}$$

The second equality comes about in the same way, but starting with the reflection of the above pullback square about its central diagonal.

To prove the proposition we must first check that the composition

$$\begin{tikzcd} \Theta X = \Theta X \otimes S \arrow[r, "1\otimes I\pi"] & \Theta X \otimes \Theta X \arrow[r, "1\otimes R\Delta"] & \Theta X \otimes \Theta X \otimes \Theta X \arrow[r, "I\Delta\otimes 1"] & \Theta X \otimes \Theta X \arrow[d, "R\pi\otimes 1"'] \\ & & & S \otimes \Theta X = \Theta X \end{tikzcd}$$

equals the identity. But using (4.7) this is equal to

$$(R\pi \otimes 1) \circ R\Delta \circ I\Delta \circ I(1 \otimes \pi) = R((\pi \times 1) \circ \Delta) \circ I((1 \times \pi) \circ \Delta) = R(id) \circ I(id) = id.$$

Note that we have again used axioms (3) and (4) of Definition 4.2 to write $R\pi \otimes 1 = R(\pi \times 1)$, and so forth.

The proof that the composite

$$\begin{tikzcd} \Theta X = S \otimes \Theta X \arrow[r, "I\pi\otimes 1"] & \Theta X \otimes \Theta X \arrow[r, "R\Delta\otimes 1"] & \Theta X \otimes \Theta X \otimes \Theta X \arrow[r, "1\otimes I\Delta"] & \Theta X \otimes \Theta X \arrow[d, "1\otimes R\pi"'] \\ & & & S \otimes \Theta X = \Theta X \end{tikzcd}$$

equals the identity is entirely similar. ∎

The following result is also worth recording:

4.8. Proposition. *Let $\Theta\colon \mathcal{C} \to \mathcal{D}$ be a Gysin schema. Then given any map $f\colon X \to Y$ in $\mathcal{C}$, the dual of $Rf\colon \Theta X \to \Theta Y$ (computed using the duality structures provided by Proposition 4.6) is precisely $If\colon \Theta Y \to \Theta X$.*

Proof. The dual of Rf is the following composite:

$$\begin{tikzcd} \Theta Y = \Theta Y \otimes S \arrow[r, "1\otimes\eta_X"] & \Theta Y \otimes \Theta X \otimes \Theta X \arrow[r, "1\otimes Rf\otimes 1"] & \Theta Y \otimes \Theta Y \otimes \Theta X \arrow[d, "\epsilon_Y\otimes 1"] \\ & & S \otimes \Theta X \arrow[r, equal] & \Theta X. \end{tikzcd}$$

One unpacks η and ϵ as $\eta_X = R\Delta_X \circ I\pi_X$ and $\epsilon_Y = R\pi_Y \circ I\Delta_Y$, and then argues precisely as in the proof of Proposition 4.6 but instead using the pullback diagram

$$\begin{tikzcd} X \arrow[r, "f\times 1"] \arrow[d, "f\times 1"'] & Y \times X \arrow[d, "1\times f\times 1"] \\ Y \times X \arrow[r, "\Delta_Y\times 1"] & Y \times Y \times X. \end{tikzcd}$$

The details are encoded in the diagram

$$\begin{tikzcd}
 & & X \arrow[dr, "R(f\times 1)"] \arrow[drr, bend left, "1_X"] & & \\
Y = Y\cdot S \arrow[r, "1\cdot I(\pi_X)"] \arrow[urr, bend left, "If"] & Y\cdot X \arrow[ur, "I(f\times 1)"] \arrow[d, "1\cdot R(\Delta_X)"'] \arrow[dr, "R(1\cdot f\cdot 1)"] & & Y\cdot X \arrow[r, "R(\pi_Y)\cdot 1"] & S\cdot X = X \\
 & Y\cdot X\cdot X \arrow[r, "R(1\cdot f\cdot 1)"'] & Y\cdot Y\cdot X \arrow[ur, "I(\Delta_Y)\cdot 1"] & &
\end{tikzcd}$$

in which for brevity we have omitted the Θ symbols and shortened $\otimes$ to the $\cdot$ symbol. The composite across the bottom edge is the dual of Rf, the middle quadrilateral commutes by the push-pull formula, and all other portions of the diagram commute by simple functoriality. ∎

4.9. The canonical Gysin functor for a Gysin schema. Suppose $\Theta\colon \mathcal{C} \to \mathcal{D}$ is a Gysin schema. Define $\pi_\Theta\colon \mathcal{C}^{op} \to \mathcal{A}b$ to be the functor given by

$$\pi_\Theta(X) = \mathcal{D}(RX, S).$$

Note that the abelian groups $\pi_\Theta(-)$ also inherit the structure of a covariant functor: given $f\colon X \to Y$ in $\mathcal{C}$ define $f_!\colon \pi_\Theta(X) \to \pi_\Theta(Y)$ by the diagram

$$\begin{tikzcd}
\pi_\Theta(X) \arrow[r, "f_!"] \arrow[d, equal] & \pi_\Theta(Y) \arrow[d, equal] \\
\mathcal{D}(\Theta X, S) \arrow[r, "\mathcal{D}(If,S)"] & \mathcal{D}(\Theta Y, S).
\end{tikzcd}$$

Moreover, the abelian groups $\pi_\Theta(X)$ inherit a product: given $a, b \in \pi_\Theta(X)$, define $a \cdot b$ to be the composite

$$\Theta X \xrightarrow{R\Delta} \Theta X \otimes \Theta X \xrightarrow{a\otimes b} S \otimes S \cong S.$$

This gives $\pi_\Theta(X)$ the structure of a commutative ring with identity $R\pi_X$, and if $f\colon X \to Y$ is a map in $\mathcal{C}$ then $f^*\colon \pi_\Theta(Y) \to \pi_\Theta(X)$ is a ring homomorphism.

4.10. Remark. If $a \in \pi_\Theta(X)$ and $b \in \pi_\Theta(Y)$ then we have the element $a\otimes b \in \pi_\Theta(X\times Y)$ defined analogously to as in Definition 3.6. It is easy to check that this is the map

$$\Theta(X\times Y) \xrightarrow{\cong} \Theta X \otimes \Theta Y \xrightarrow{a\otimes b} S \otimes S = S.$$

4.11. Proposition. *If $\Theta\colon \mathcal{C} \to \mathcal{D}$ is a Gysin schema then $\pi_\Theta\colon \mathcal{C}^{op} \to \mathcal{C}omm\mathcal{R}ing$ is a Gysin functor.*

Proof. Axioms (1) and (2) follow from the fact that R preserves finite coproducts (in particular, $R(\emptyset) = 0$). Axioms (3) and (4) are immediate consequences of axioms (4) and (5) in Definition 4.2. ∎

4.12. Preliminaries on the reconstruction problem. Let $(\mathcal{D}, \otimes, S, F(-,-))$ be a closed, symmetric monoidal category in which every object is dualizable. It turns out all such categories have a description that is somewhat reminiscent of the construction of $\mathcal{C}_E$.

For an object X write $X^* = F(X,S)$, and for a map $f\colon X \to Y$ write $f^* = F(f,S)$. Let $ev_X\colon X^* \otimes X \to S$ be the adjoint of the identity map $X^* \to F(X,S)$, and let $cev_X\colon S \to X \otimes X^*$ be the coevaluation map guaranteed by duality (see Appendix A).

Define a new category $\mathcal{D}^{ad}$ as follows. The objects are the same as those in $\mathcal{D}$, and morphisms are given by

$$\mathcal{D}^{ad}(X,Y) = \mathcal{D}(Y^* \otimes X, S).$$

If $\alpha \in \mathcal{D}^{ad}(X,Y)$ and $\beta \in \mathcal{D}^{ad}(Y,Z)$ then $\beta \circ \alpha$ is given as follows:

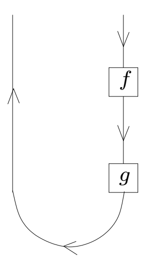
$$Z^* \otimes X \xrightarrow{\cong} Z^* \otimes S \otimes X \xrightarrow{1 \otimes cev_Y \otimes 1} Z^* \otimes Y \otimes Y^* \otimes X \xrightarrow{\beta \otimes \alpha} S \otimes S = S.$$

One readily checks that this composition is associative, and $ev_X \in \mathcal{D}^{ad}(X,X)$ is a two-sided identity. These are very easy categorical arguments, but one can also give proofs using the graphical calculus for compact symmetric monoidal categories (see [BS] for an expository account).

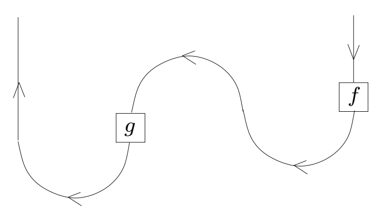
There is a functor $\Gamma\colon \mathcal{D} \to \mathcal{D}^{ad}$ defined as follows. It is the identity on objects, and given $f\colon X \to Y$ we let $\Gamma f \in \mathcal{D}^{ad}(X,Y) = \mathcal{D}(Y^* \otimes X, S)$ be the composite

$$Y^* \otimes X \xrightarrow{id \otimes f} Y^* \otimes Y \xrightarrow{ev_Y} S.$$

The check that this is indeed a functor is best done using the graphical calculus (again, see [BS]). If $f\colon X \to Y$ and $g\colon Y \to Z$ are maps in $\mathcal{D}$, then $\Gamma(gf)$ and $(\Gamma g)(\Gamma f)$ are the composite maps represented by the following diagrams:



$\Gamma(g \circ_{\mathcal{D}} f)$



$\Gamma(g) \circ_{\mathcal{D}^{ad}} \Gamma(f)$

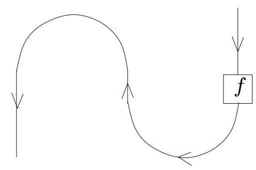
The graphical calculus clearly shows these composites to be identical in $\mathcal{D}$.

4.13. Proposition. *The functor $\Gamma\colon \mathcal{D} \to \mathcal{D}^{ad}$ is an isomorphism of categories.*

Proof. We only need check that the maps $\mathcal{D}(X,Y) \to \mathcal{D}^{ad}(X,Y) = \mathcal{D}(Y^* \otimes X, S)$ are bijections. There is an evident map in the opposite direction that sends a map $h\colon Y^* \otimes X \to S$ to the composite

$$X = S \otimes X \xrightarrow{cev_Y \otimes id_X} Y \otimes Y^* \otimes X \xrightarrow{id_Y \otimes h} Y \otimes S = Y.$$

Proving that these assignments are inverses to each other is another exercise in graphical calculus. For example, the composite in one direction sends the map $f\colon X \to Y$ to the map represented by



and the graphical calculus shows that this is equal to f in $\mathcal{D}$. The other direction is similarly easy. ∎

4.14. Remark. In the above discussion, we never really needed the closed structure on $\mathcal{D}$. We only needed that all objects are dualizable, together with fixed choices of a dual as well as evaluation and coevaluation maps for every object. In fact, by Proposition A.8 below every symmetric monoidal category in which all objects are dualizable can be given a closed structure via $F(X,Y) = X^* \otimes Y$. Thus, Proposition 4.13 applies to these categories and we will use it in that generality.

4.15. The main reconstruction theorem. Recall from Section 3.18 that $\mathcal{C}_{(\pi_\Theta)}$ denotes the category of correspondences associated to the Gysin functor π_Θ.

4.16. Theorem. *Assume given a Gysin schema $\Theta\colon \mathcal{C} \to \mathcal{D}$. Then there is a full and faithful functor of categories $\mathcal{C}_{(\pi_\Theta)} \to \mathcal{D}$ that is equal to Θ on objects and sends a map $f \in \mathcal{C}_{\pi_\Theta}(X,Y) = \mathcal{D}(\Theta Y \otimes \Theta X, S)$ to the composite*

$$\Theta X = S \otimes \Theta X \xrightarrow{\eta_{\Theta Y} \otimes id_{\Theta X}} \Theta Y \otimes \Theta Y \otimes \Theta X \xrightarrow{id_{\Theta Y} \otimes f} \Theta Y \otimes S = \Theta Y.$$

Proof. The proof is easier to understand if we first compare $\mathcal{C}_{(\pi_\Theta)}$ to $\mathcal{D}^{ad}$. Note that for any objects X and Y of $\mathcal{C}$ we have equalities of sets

$$\mathcal{C}_{(\pi_\Theta)}(X,Y) = \pi_\Theta(Y \times X) = \mathcal{D}(\Theta Y \otimes \Theta X, S) = \mathcal{D}^{ad}(\Theta X, \Theta Y).$$

The identity element $i_X \in \mathcal{C}_{(\pi_\Theta)}(X,X) = \pi_\Theta(X \times X)$ is $\Delta_!(1)$, which by unravelling the definitions equals the composite

$$\Theta X \otimes \Theta X \xrightarrow{I\Delta} \Theta X \xrightarrow{R\pi} S,$$

also known as ϵ_X. This is equal to the identity in $\mathcal{D}^{ad}$.

Finally, we must compare the composition rules in $\mathcal{C}_{(\pi_\Theta)}$ and $\mathcal{D}^{ad}$. Suppose given $f \in \mathcal{D}(\Theta Y \otimes \Theta X, S)$ and $g \in \mathcal{D}(\Theta Z \otimes \Theta Y, S)$. The composition $g \circ f$ in $\mathcal{C}_{(\pi_\Theta)}$ is given by the composite

$$\begin{tikzcd}
\Theta Z \otimes \Theta X \arrow[r, "\cong", "\theta"'] & \Theta(Z\times X) \arrow[r, "Ip"] & \Theta(Z\times Y\times X) \arrow[d, "R\Delta"] & \\
& & \Theta(Z\times Y\times X)\otimes\Theta(Z\times Y\times X) \arrow[d, "\theta", "\cong"'] & \\
& & \Theta Z\otimes\Theta Y\otimes\Theta X\otimes\Theta Z\otimes\Theta Y\otimes\Theta X \arrow[d, "1\otimes 1\otimes R\pi_X\otimes R\pi_Z\otimes 1\otimes 1"] & \\
& & \Theta Z\otimes\Theta Y\otimes S\otimes S\otimes\Theta Y\otimes\Theta X \arrow[d, "g\otimes 1\otimes 1\otimes f"] & \\
& & S\otimes S\otimes S\otimes S \arrow[r, equal] & S
\end{tikzcd}$$

where $p\colon Z\times Y\times X\to Z\times X$ is the evident projection. The composition $g\circ f$ in $\mathcal{D}^{ad}$ is given by the composite

$$\Theta Z\otimes\Theta X \xrightarrow{\cong} \Theta Z\otimes S\otimes\Theta X \xrightarrow{1\otimes\eta_Y\otimes 1} \Theta Z\otimes\Theta Y\otimes\Theta Y\otimes\Theta X \xrightarrow{g\otimes f} S\otimes S =\!=\!= S.$$

A diagram chase shows these two composites to be equal. This is best left to the reader, but the main idea is to take the first composite and decompose the diagonal on $Z\times Y\times X$ into the three diagonals on the individual components followed by a permutation of the factors. The diagonals on Z and X cancel the π_X and π_Z appearing later, leaving only the diagonal on Y. The map Ip is equal to $1\otimes I\pi_Y\otimes 1$, and the $I\pi_Y$ assembles with the $R\Delta_Y$ to make η_Y. An outline of the relevant diagram is

$$\begin{tikzcd}
ZX \arrow[r, "Ip"] & ZYX \arrow[r, "\Delta"] \arrow[dr, "\Delta\Delta\Delta"'] \arrow[ddrr, bend right, "1\Delta 1"'] & ZYXZYX \arrow[r, "11\pi\pi 11"] & ZYSSYX \arrow[r, "g11f"] & SSSS \\
& & ZZYYXX \arrow[u, "\sigma"] \arrow[r, "1\pi 11\pi 1"] & ZSYYSX \arrow[u, "\sigma"] \arrow[d, equal] & \\
& & & ZYYX \arrow[uur, "gf"'] &
\end{tikzcd}$$

where as usual we have omitted Θ and $\otimes$ symbols, and where the two maps labelled σ are the permutations (composites of twist maps) sending the word $abcdef$ to $acebdf$.

At this point we have constructed a functor $\mathcal{C}_{(\pi_\Theta)}\to\mathcal{D}^{ad}$ which by inspection is a bijection on Hom-sets. Finally, compose this with the isomorphism from Proposition 4.13 (see also Remark 4.14) to get the desired result. ∎

4.17. Example. Let $\mathcal{D}$ be the category of finitely-generated free abelian groups, and let $\mathcal{C}$ be the category of finite sets. Let $\Theta\colon\mathcal{C}\to\mathcal{D}$ be the free abelian group functor, given the structure of a Gysin schema as in Example 4.4. The associated Gysin functor π_Θ is precisely the one of Example 3.12(a). By Theorem 4.16 we conclude that $\mathcal{C}_{(\pi_\Theta)}$ is isomorphic to the category of finitely-generated free abelian groups.

5. The structure of correspondence categories

Suppose $E\colon \mathcal{C} \to \mathcal{C}omm\mathcal{R}ing$ is a Gysin functor. Our goal is to better understand how the category of correspondences $\mathcal{C}_E$ relates to the original category $\mathcal{C}$. Given objects X and Y in $\mathcal{C}$, every element $f \in \mathcal{C}(X,Y)$ gives rise to maps R_f and I_f in $\mathcal{C}_E$. In addition, we will see that every element $a \in E(X)$ gives an endomorphism D_a of X in $\mathcal{C}_E$. We will prove that every map in $\mathcal{C}_E$ may be written in the form $R_f \circ D_a \circ I_g$, and we will explain rules for rewriting the composition of two such expressions into the same form.

These results do not give a simple picture for the structure of $\mathcal{C}_E$, but they do give a reasonable prescription for working with these categories in specific examples. In Section 5.13 we explore this in a general "Galoisien" setting, meaning a setting where the category $\mathcal{C}$ has properties formally similar to the category of G-sets, with G is a finite group.

5.1. The diagonal structure. We will need an extra piece of structure in $\mathcal{C}_E$ coming from the diagonal maps in $\mathcal{C}$. For an object X in $\mathcal{C}$ let $\Delta\colon X \to X \times X$ be the diagonal. This induces a map of abelian groups

$$D = \Delta_!\colon E(X) \to E(X \times X) = \mathcal{C}_E(X,X).$$

The target has two ring structures: it has the generic ring structure that any $E(Z)$ has, and it has the circle product coming from composition in $\mathcal{C}_E$. It is the latter that we consider in the next proposition. This result also uses the $*$-involution on $\mathcal{C}_E(X,X)$ from Proposition 3.21(d).

5.2. Proposition. *$D\colon E(X) \to \mathcal{C}_E(X,X)$ is a ring map, and for any $a \in E(X)$ one has $(Da)^* = Da$.*

Proof. Let $a, b \in E(X)$. We calculate

$$\begin{aligned}
Db \circ Da &= (\pi_{13})_!\big[\pi_{12}^*(\Delta_! b) \cdot \pi_{23}^*(\Delta_! a)\big] \\
&= (\pi_{13})_!\big[(\Delta \times id)_!(\pi_1^* b) \cdot \pi_{23}^*(\Delta_! a)\big] && \text{(push-pull)} \\
&= (\pi_{13})_!(\Delta \times id)_!\big[(\pi_1^* b) \cdot (\Delta \times id)^*\pi_{23}^*(\Delta_! a)\big] && \text{(proj. formula)} \\
&= \pi_1^* b \cdot \Delta_!(a) \\
&= \Delta_!\big((\Delta^*\pi_1^* b) \cdot a\big) \\
&= \Delta_!(b \cdot a) \\
&= D(ba).
\end{aligned}$$

The second statement in the proposition is proven by

$$(Da)^* = t^*(Da) = (t_!)^{-1}(Da) = t_!(Da) = t_!(\Delta_! a) = \Delta_! a = Da,$$

where t is the twist $X \times X \to X \times X$. The second equality is from Proposition 3.21(i), and the third equality is because $t = t^{-1}$. ∎

5.3. Notation. We will usually write Da, or if really necessary $D(a)$, but sometimes we will write D_a for the same thing.

5.4. Proposition. *Suppose given $f\colon X \to Y$ and $a \in E(Y)$. Then $Da \circ R_f = R_f \circ D(f^*a)$ and $I_f \circ Da = D(f^*a) \circ I_f$.*

Proof. We compute

$$\begin{aligned} Da \circ R_f = \Delta_! a \circ R_f = (id_Y \times f)^*(\Delta_! a) &= \big((f \times id)^X_{Y\times X}\big)_!(f^*a) \\ &= (f \times id)_! \Delta_!(f^*a) \\ &= R_f \circ D(f^*a). \end{aligned}$$

The second and fifth equalities are by Proposition 3.21(e), and the third equality is by push-pull.

To conclude, the second statement in the proposition follows by applying $(-)^*$ to the first and using Proposition 5.2. ∎

5.5. Remark. Let $G = \mathrm{Aut}_{\mathcal{C}}(X)$, with the group structure coming from composition. Note that there is a map $G^{op} \to \mathrm{Aut}(E(X))$ given by $f \mapsto f^*$. Let $E(X)[\widetilde{G}]$ be the twisted group ring defined as follows: it is spanned by elements $a[f]$ for $a \in E(X)$ and $f \in G$, and the multiplication is induced by

$$a[f] \cdot b[g] = a\big((f^{-1})^* b\big)[fg].$$

Then Proposition 5.4 shows that there is a map of rings

$$E(X)[\widetilde{G}] \longrightarrow \mathcal{C}_E(X,X), \qquad a[f] \mapsto Da \circ R_f.$$

In good cases this is an isomorphism: see Proposition 5.16 below.

5.6. Initial results on the structure of $\mathcal{C}_E$. If we have maps $Y \xleftarrow{f} Z \xrightarrow{g} X$ and $a \in E(Z)$ then $R_f \circ D_a \circ I_g$ is a morphism from X to Y in $\mathcal{C}_E$. We will refer to such an expression as an RDI formula for the composite morphism. Here are some useful facts that relate these RDI formulas in $\mathcal{C}_E$ to pushforwards in E:

5.7. Proposition. *Suppose given maps $Y \xleftarrow{f} Z \xrightarrow{g} X$ and $a \in E(Z)$. Then:*

(a) $R_f \circ D_a \circ I_g = \big((f \times g)^Z_{YX}\big)_!(a)$.

(b) $R_f \circ D_a \circ I_f = D(f_! a)$.

(c) $R_f \circ I_f = D(f_! 1)$.

Proof. For (a) we use Proposition 3.21(e) to write

$$R_f \circ D_a \circ I_g = (f \times id)_!(id \times g)_!(\Delta_! a) = \big((f \times g)^Z_{YX}\big)_!(a).$$

Part (b) follows from (a) together with $(f \times f)^Z_{YY} = \Delta^Y \circ f$. Finally, (c) is just the special case of (b) where we take $a = 1$ (so $Da = i_X$). ∎

In fact *every* morphism in $\mathcal{C}_E$ can be expressed as an RDI composition. This is actually a triviality, but it is nevertheless important:

5.8. Lemma. *Every element of* $\mathcal{C}_E(X,Y)$ *may be written as* $R_f \circ D_a \circ I_g$ *for some object* Z*, some maps* $f\colon Z \to Y$*,* $g\colon Z \to X$*, and some* $a \in E(Z)$*.*

Proof. Let $a \in \mathcal{C}_E(X,Y) = E(Y \times X)$. Set $Z = Y \times X$. We claim that $a = R_{\pi_1} \circ D_a \circ I_{\pi_2}$ in $\mathcal{C}_E$. This is immediate from Proposition 5.7(a). ∎

Now suppose that we have two maps in RDI form, and that we wish to compose them; that is, consider a composition of the form

$$[R_f \circ D_a \circ I_g] \circ [R_{f'} \circ D_{a'} \circ I_{g'}].$$

There are three rules that allow us to rewrite this in RDI form once again. We indicate these schematically as:

$$\begin{aligned} D \circ R &\rightsquigarrow R \circ D && \text{[Proposition 5.4]} \\ I \circ D &\rightsquigarrow D \circ I && \text{[Proposition 5.4]} \\ I \circ R &\rightsquigarrow R \circ I && \text{[Proposition 3.21(h)].} \end{aligned} \tag{5.9}$$

To use these in our problem, we start by forming the pullback in the following diagram:

$$\begin{tikzcd} & & P \arrow[dl, "s"'] \arrow[dr, "t"] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "g"] & & A' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ Y & & B & & X. \end{tikzcd}$$

Then

$$\begin{aligned} R_f \circ D_a \circ I_g \circ R_{f'} \circ D_{a'} \circ I_{g'} &= R_f \circ D_a \circ R_s \circ I_t \circ D_{a'} \circ I_{g'} \\ &= R_f \circ R_s \circ D_{s^*a} \circ D_{t^*a'} \circ I_t \circ I_{g'} \\ &= R_{fs} \circ D_{(s^*a)\cdot(t^*a')} \circ I_{g't}. \end{aligned} \tag{5.10}$$

The above discussion has proven Corollary 1.3 from the introduction. We also note that we have proven Theorem 1.2 along the way:

Proof of Theorem 1.2. Parts (a) and (b) are Proposition 5.4, whereas (c) is Proposition 3.21(h). The last assertion in the statement of the theorem is Proposition 5.7(b). ∎

5.11. A DETAILED EXAMPLE OF THE BURNSIDE FUNCTOR. Consider the category $\mathcal{C}_\mathcal{A}$, where $\mathcal{A}$ is the Burnside functor for $\mathcal{C}$ from Section 3.14. Given the role of $\mathcal{A}$ as the universal Gysin functor, it is useful to have a particularly good handle on how to work with $\mathcal{C}_\mathcal{A}$. Recall that a map in $\mathcal{C}_\mathcal{A}$ from X to Y is an element of $\mathcal{A}(Y \times X)$, and so is represented by a map $h\colon T \to Y \times X$ in $\mathcal{C}$. It is often useful to represent this data as a span, by writing

$$\begin{tikzcd} & T \arrow[dl, "\pi_1 h"'] \arrow[dr, "\pi_2 h"] & \\ Y & & X. \end{tikzcd}$$

The following list gives a "dictionary" for how certain structures are represented in $\mathcal{C}_\mathcal{A}$.

(1) $$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "id"] & \\ A & & X \end{tikzcd} = Rf, \qquad \begin{tikzcd} & X \arrow[dl, "id"'] \arrow[dr, "g"] & \\ X & & B \end{tikzcd} = Ig$$

(2) $$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd} = Rf \circ Ig$$

(3) $$\begin{tikzcd} & X \arrow[dl, "id"'] \arrow[dr, "id"] & \\ X & & X \end{tikzcd} = i_X \qquad \begin{tikzcd} & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd} = 1_X$$

(4) $$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd} \otimes \begin{tikzcd} & X' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ A' & & B' \end{tikzcd} = \begin{tikzcd} & X \times X' \arrow[dl, "f \times f'"'] \arrow[dr, "g \times g'"] & \\ A \times A' & & B \times B' \end{tikzcd}$$

(5) unit of $\otimes$ is $S = *$

(6) The adjunction $\mathrm{Hom}(X, F(Y,Z)) \to \mathrm{Hom}(X \otimes Y, Z)$ is

$$\begin{tikzcd} & W \arrow[dl, "f \times g"'] \arrow[dr, "h"] & \\ Y \times Z & & X \end{tikzcd} \longleftrightarrow \begin{tikzcd} & W \arrow[dl, "g"'] \arrow[dr, "h \times f"] & \\ Z & & X \times Y \end{tikzcd}$$

(8) The identity $\mathbf{1}_X \colon S \to F(X,X)$ is

$$\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ X \times X & & * \end{tikzcd}$$

(9) The evaluation and coevaluation morphisms are

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "\Delta"] & \\ * & & X \times X \end{tikzcd} = ev_X, \qquad \begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ X \times X & & * \end{tikzcd} = cev_X$$

(10) The transposition $t_{X,Y} \colon X \otimes Y \to Y \otimes X$ is

$$\begin{tikzcd} & X \times Y \arrow[dl, "t"'] \arrow[dr, "id"] & \\ Y \times X & & X \times Y \end{tikzcd}$$

(11) Given $a \colon T \to X$, one has $\begin{tikzcd} & T \arrow[dl, "a"'] \arrow[dr, "a"] & \\ X & & X \end{tikzcd} = Da.$

5.12. Example. Note that in this example of the Burnside functor we have $D_a = R_a \circ I_a$, which means that we can also write

$$R_f \circ D_a \circ I_g = R_f \circ R_a \circ I_a \circ I_g = R_{fa} \circ I_{ga}.$$

So the "D" factors in an RDI-decomposition can always be eliminated, and every map can be written in the form $R_x \circ I_y$. This is, of course, very particular to the Burnside functor.

5.13. Gysin categories in the Galois setting. We now add some extra hypotheses to the category $\mathcal{C}$, all of which are satisfied in the cases of interest. First, say that an object X in $\mathcal{C}$ is **atomic** if $X \neq \emptyset$ and X is not isomorphic to a coproduct $A \amalg B$ where both A and B are different from the initial object. We will assume that

- If X is atomic and Y and Z are any objects, then the natural map $\mathcal{C}(X,Y) \amalg \mathcal{C}(X,Z) \to \mathcal{C}(X, Y \amalg Z)$ is a bijection.
- For every atomic object X in $\mathcal{C}$, the set $\mathrm{Aut}(X)$ is finite.
- If $X \neq \emptyset$ then $\mathcal{C}(X, \emptyset) = \emptyset$.

If $\mathcal{C}$ is finitary lextensive and satisfies the above properties, we will say that $\mathcal{C}$ is a **Galoisien** category.

5.14. Example. Let G be a finite group, and let $\mathcal{C}$ be the category of finite G-sets. Then $\mathcal{C}$ is Galoisien, and the atomic objects are the transitive G-sets.

Let Y be an object of a Galoisien category $\mathcal{C}$, and let $G(Y) = \mathrm{Aut}(Y)$. There is an evident map

$$\coprod_{\sigma\in G(Y)} Y_\sigma \to Y \times Y$$

(where Y_σ denotes a copy of Y labelled by σ), where the map $Y_\sigma \to Y\times Y$ is $id\times\sigma$. We say that Y is **Galois** if the displayed map is an isomorphism. To generalize this somewhat, if $p\colon X \to Y$ is a map then let $G(X/Y) = \{\alpha \in \mathrm{Aut}(X) \,|\, p\alpha = p\}$. Say that $X \to Y$ is Galois if the evident map

$$\coprod_{\sigma\in G(X/Y)} X_\sigma \to X \times_Y X$$

is an isomorphism.

The results in the following lemma can be proven by elementary category theory:

5.15. Lemma. *Suppose that X and Y are atomic.*

(a) If Y is Galois then $\mathcal{C}(X,Y)$ is either empty or else it is a $G(Y)$-torsor.

(b) If Y is Galois then every endomorphism of Y is an isomorphism.

(c) If X and Y are Galois and $f, g\colon X \to Y$, then the evident map

$$\coprod_{\{\sigma\in G(X)\,|\,f=g\sigma\}} X_\sigma \longrightarrow \mathrm{pullback}[X \xrightarrow{f} Y \xleftarrow{g} X]$$

is an isomorphism.

(d) If X and Y are Galois then so is every map $X \to Y$.

(e) Suppose that Y and Z are both Galois, and assume given $f\colon X \to Y$ and $g\colon Z \to Y$. If there exists a map $u\colon X \to Z$ such that $gu = f$, then the evident map

$$\coprod_{\sigma\in G(Z/Y)} X_\sigma \to X \times_Y Z$$

is an isomorphism.

(f) If $f\colon X \to Z$ is a map and Z is Galois, then the map $\coprod_{\sigma\in G(Z)} X_\sigma \to X \times Z$ given by $id \times \sigma f\colon X_\sigma \to X \times Z$ is an isomorphism.

(g) If X and Y are both Galois and $f\colon X \to Y$, then for every $\alpha \in G(X)$ there is a unique $\alpha_f \in G(Y)$ such that $f\alpha = \alpha_f f$. Moreover, the map $G(X) \to G(Y)$ given by $\alpha \to \alpha_f$ is a group homomorphism.

Proof. For (a), suppose that $\mathcal{C}(X,Y) \neq \emptyset$ and let $f\colon X \to Y$ be a map. We need to show that the map $G(Y) \to \mathcal{C}(X,Y)$ given by $\sigma \mapsto \sigma f$ is a bijection. Let $g\colon X \to Y$ be any map, and consider $f \times g\colon X \to Y \times Y$. Composing with the inverse of the isomorphism $\coprod_{G(Y)} Y \to Y \times Y$, the fact that X is atomic shows that the resulting map factors through a map $u\colon X \to Y_\sigma$, for some σ. One then obtains the commutative diagram

$$\begin{tikzcd} X \arrow[d, "u"'] \arrow[dr, "f\times g"] & \\ Y_\sigma \arrow[r, "id\times\sigma"'] & Y \times Y \end{tikzcd}$$

which shows that $u = f$ and $g = \sigma u = \sigma f$. So the action of $G(Y)$ on $\mathcal{C}(X,Y)$ is transitive.

Now suppose that $\alpha, \beta \in G(Y)$ and $\alpha f = \beta f$. Then $f \times \alpha f\colon X \to Y \times Y$ factors through both Y_α and Y_β (under the isomorphism $\coprod_{G(Y)} Y \cong Y \times Y$). Therefore it factors through the pullback $Y_\alpha \to \coprod_{G(Y)} Y \leftarrow Y_\beta$. But if $\alpha \neq \beta$ then this pullback is $\emptyset$, by our standing hypotheses that $\mathcal{C}$ is finitary lextensive. Since the map $f \times \alpha f$ cannot factor through $\emptyset$, this is a contradiction; so we must have $\alpha = \beta$.

Part (b) is an immediate consequence of (a) applied to the case $X = Y$. For (c), the pullback in question is isomorphic to the pullback of

$$Y \xrightarrow{\Delta} Y \times Y \xleftarrow{f\times g} X \times X.$$

Use the decomposition $X \times X \cong \coprod_{\sigma \in G(X)} X_\sigma$ and the fact that pullbacks distribute over finite coproducts to see that our pullback is isomorphic to

$$\coprod_{\sigma \in G(X)} \text{pullback}[Y \xrightarrow{\Delta} Y \times Y \xleftarrow{f\times g\sigma} X].$$

Next use the decomposition $Y \times Y \cong \coprod_{\alpha \in G(Y)} Y_\alpha$, together with the fact that $\Delta\colon Y \to Y \times Y$ factors through the summand Y_{id}. Since X is atomic, we deduce that the pullback inside the above coproduct is either $\emptyset$ (when $f \neq g\sigma$) or X (when $f = g\sigma$). This finishes off part (c).

Part (d) is a direct consequence of (c), applied in the case $f = g$.

For (e), the existence of u implies that $X \times_Y Z$ is isomorphic to the pullback of

$$X \xrightarrow{u} Z \xleftarrow{\pi_1} Z \times_Y Z.$$

Next use that $Z \times_Y Z \cong \coprod_{\sigma \in G(Z/Y)} Z_\sigma$ and the fact that pullbacks distribute over coproducts.

Part (f) is the special case of (e) where $Y = *$. Finally, the proof of (g) uses the same techniques that have been demonstrated in the preceding parts: consider the map $f \times f\alpha\colon X \to Y \times Y$ and factor this through some Y_σ, which must be unique. The fact that $\alpha \mapsto \alpha_f$ is a homomorphism follows from

$$(\alpha\beta)_f f = f(\alpha\beta) = (f\alpha)\beta = \alpha_f f \beta = \alpha_f \beta_f f.$$

■

Before proceeding, let us establish some notation. If R is a ring and S is a set, then $R\langle S\rangle$ denotes the set of all formal finite sums $\sum r_i s_i$ where $r_i \in R$ and $s_i \in S$. This is the free left R-module with basis S. Similarly, let $\langle S\rangle R$ be the set of all formal finite sums $\sum s_i r_i$ with $s_i \in S$ and $r_i \in R$. When R is commutative these are of course isomorphic R-modules, but the difference in notation will be useful to us below.

When X is Galois we can now determine the ring $\mathcal{C}_E(X, X)$ precisely:

5.16. PROPOSITION. *If X is Galois then the map $E(X)[\widetilde{\mathrm{Aut}(X)}] \to \mathcal{C}_E(X, X)$ from Remark 5.5 is an isomorphism of rings.*

PROOF. Since X is Galois, the usual map $\coprod_{\sigma \in \mathrm{Aut}(X)} X \to X \times X$ is an isomorphism. So

$$B\colon \bigoplus_{\sigma \in \mathrm{Aut}(X)} E(X) \to E(X \times X)$$

is an isomorphism of groups by Lemma 3.9, where on component σ the map B equals $(id \times \sigma)_! \Delta_!$. If $a \in E(X)$ then we have a copy of a in the component of the domain indexed by σ. The image of this class in $E(X \times X)$ is precisely

$$(id \times \sigma)_! \Delta_!(a) = D_a \circ I_\sigma = D_a \circ R_{\sigma^{-1}}$$

by Proposition 3.21(e),(iii). Replacing σ^{-1} by β, this says that every element of $E(X \times X)$ can be written uniquely as a sum of terms $D_a \circ R_\beta$ for $a \in E(X)$ and $\beta \in \mathrm{Aut}(X)$. So the map $E(X)\langle \mathrm{Aut}(X)\rangle \to \mathcal{C}_E(X, X)$ given by $a.\beta \mapsto D_a R_\beta$ is an isomorphism of abelian groups. We already saw in Remark 5.5 that it is a ring homomorphism, where we give the domain the appropriate structure of twisted group ring. ∎

5.17. REMARK. In concrete terms, Proposition 5.16 says that every map in $\mathcal{C}_E(X, X)$ may be uniquely written as a finite sum of terms $D_a R_\alpha$ where $a \in E(X)$ and $\alpha \in \mathrm{Aut}_{\mathcal{C}}(X)$. Composition is done according to the rule

$$D_a R_\alpha \circ D_b R_\beta = D_a D_{(\alpha^{-1})^* b} R_\alpha R_\beta = D_{a \cdot (\alpha^{-1})^* b} R_{\alpha\beta}$$

where in the first equality we have used Proposition 5.4 (together with the fact that α is an isomorphism). The awkwardness of this formula stems from our representation of elements of $\mathcal{C}_E(X, X)$ in the form $D_a R_\alpha$. As we have remarked before, it is better to use the RDI system and represent the elements as $R_\alpha \circ D_a$. If we do this, then the composition law is

$$R_\alpha D_a \circ R_\beta D_b = R_{\alpha\beta} D_{(\beta^* a \cdot b)},$$

which is a little simpler. We will always use this formulation from now on.

We next turn to the case of two objects. Assume that $f\colon X \to Y$ is a map in $\mathcal{C}$, where both X and Y are assumed to be atomic and Galois. Our goal is to describe the full subcategory of $\mathcal{C}_E$ containing X and Y. If f is an isomorphism then this problem reduces to the case of one object, which we handled above. So let us further assume that

f is not an isomorphism. Note that this implies that there cannot exist a map in $\mathcal{C}$ from Y to X: if there were such a map, then the post- and pre-composites with f would be isomorphisms by Lemma 5.15(b), and so f would itself be an isomorphism.

Write $\mathrm{Aut}(X) = \{\alpha_1, \ldots, \alpha_r\}$ and $\mathrm{Aut}(Y) = \{\beta_1, \ldots, \beta_s\}$. Note that $\mathcal{C}(X, Y) = \{\beta_1 f, \ldots, \beta_s f\}$ by Lemma 5.15(a), and $X \times Y \cong \coprod_{\sigma \in \mathrm{Aut}(Y)} X$ by Lemma 5.15(f). Then $\mathcal{C}_E(X, Y) = E(Y \times X) \cong \oplus_{\mathrm{Aut}(Y)} E(X)$, and one can check that the isomorphism is the one that represents each map in $\mathcal{C}_E(X, Y)$ as a sum of maps $R_{\beta_i f} D_a$ where $a \in E(X)$. A similar analysis works for $\mathcal{C}_E(Y, X)$, and so the full subcategory of $\mathcal{C}_E$ containing X and Y may be depicted as follows:

$$\begin{tikzcd}
X \arrow[loop left, "\langle R_{\alpha_1},\ldots,R_{\alpha_r}\rangle E(X)"] \arrow[rr, bend left, "\langle R_{\beta_1 f},\ldots,R_{\beta_s f}\rangle E(X)"] & & Y \arrow[loop right, "\langle R_{\beta_1},\ldots,R_{\beta_s}\rangle E(Y)"] \arrow[ll, bend left, "E(X)\langle I_{\beta_1 f},\ldots,I_{\beta_s f}\rangle"]
\end{tikzcd}$$

The labels on the arrows depict the abelian group of maps in $\mathcal{C}_E$; e.g., the label on the arrow from X to Y depicts $\mathcal{C}_E(X, Y)$. The diagram indicates that every map from X to Y may be uniquely written as a sum of terms $R_{\beta_i f} D_{a_i}$ where $a_i \in E(X)$ (and similarly for other choices of domain and range). The endomorphism monoids of X and Y are as described in Remark 5.17.

Compositions of maps are determined via the RDI rules outlined in (5.9) and (5.10). Here are some examples:

(1) [$X \to X \to Y$ compositions.] Here one uses

$$R_{\beta_i f} D_a \circ R_{\alpha_j} D_b = R_{\beta_i f} R_{\alpha_j} D_{\alpha_j^*(a)} D_b = R_{\beta_i f \alpha_j} D_{(\alpha_j^* a) b}.$$

(2) [$Y \to Y \to X$ compositions.] Here one uses that β_i is invertible and so we have $R_{\beta_i} = I_{\beta_i}^{-1}$ by Proposition 3.21(i). Then

$$\begin{aligned} D_a I_{\beta_j f} \circ R_{\beta_i} D_u = D_a I_{\beta_j f} \circ I_{\beta_i}^{-1} D_u = D_a I_{\beta_j f} I_{\beta_i^{-1}} D_u &= D_a I_{\beta_i^{-1} \beta_j f} D_u \\ &= D_{(a \cdot (\beta_i^{-1} \beta_j f)^*(u))} I_{\beta_i^{-1} \beta_j f}. \end{aligned}$$

(3) [$Y \to X \to Y$ compositions.] In this case we consider

$$\begin{aligned} R_{\beta_j f} D_a \circ D_b I_{\beta_i f} = R_{\beta_j} \circ R_f \circ D_{ab} \circ I_f \circ I_{\beta_i} &= R_{\beta_j} \circ D_{f_!(ab)} \circ I_{\beta_i} \\ &= R_{\beta_j} \circ I_{\beta_i} \circ D_{(\beta_i^{-1})^*(f_!(ab))} \\ &= R_{\beta_j} \circ R_{\beta_i^{-1}} \circ D_{(\beta_i^{-1})^*(f_!(ab))} \\ &= R_{\beta_j \beta_i^{-1}} D_{(\beta_i^{-1})^*(f_!(ab))}. \end{aligned}$$

In the second equality we have used Proposition 5.7(b) and in the third equality we have used Proposition 5.4 (which applies because β_i is invertible).

(4) [$X \to Y \to X$ compositions.] Fix i and j and let $T = \{\sigma \in G(X) \,|\, \beta_j f = \beta_i f \sigma\}$. Observe that by Lemma 5.15(c) since X and Y are Galois we have a pullback diagram

$$\begin{tikzcd} \coprod_{\sigma \in T} X_\sigma \arrow[r] \arrow[d] & X \arrow[d, "\beta_i f"] \\ X \arrow[r, "\beta_j f"] & Y \end{tikzcd}$$

where the vertical map $X_\sigma \to X$ is the identity and the horizontal map $X_\sigma \to X$ is σ. We then write

$$\begin{aligned} D_{a'} I_{\beta_j f} \circ R_{\beta_i f} D_a = \sum_\sigma D_{a'} I_\sigma D_a &= \sum_\sigma I_\sigma D_{(\sigma^{-1})^*(a')} D_a \\ &= \sum_\sigma R_{\sigma^{-1}} D_{((\sigma^{-1})^*(a') \cdot a)}. \end{aligned}$$

The second equality is by Proposition 5.4, using that σ is an isomorphism.

(5) [Remaining cases.] The cases that have not been treated so far are all very similar to (1) or (2).

As the reader can see from the above analysis, a complete description of the maps between Galois objects is relatively simple. But the description of compositions becomes unwieldy, although in practice it is a purely mechanical process to work out any given composition.

6. Grothendieck-Witt categories over a field

Let k be a field of characteristic not equal to 2, and recall the Grothendieck-Witt category $\mathrm{GWC}(k)$ over k defined in Section 2.

Let fEt/k be the full subcategory of $\mathcal{A}\mathrm{ff}/\operatorname{Spec} k$ consisting of the objects $\operatorname{Spec} E$ where $k \to E$ is finite étale (in our terminology, sheerly separable—see Remark 2.9 and Corollary 2.11). Let $\mathcal{A}_{\mathrm{fEt}}$ be the Burnside Gysin functor, and let $\chi\colon \mathcal{A}_{\mathrm{fEt}} \to \mathrm{GW}$ be the natural transformation from Proposition 3.17.

The following result is essentially [Dr, Appendix B, Theorem 3.1]. We include the proof for completeness. For the proof, recall that if $a \in E$ then $\langle a \rangle$ denotes the quadratic space (E, b_a) where $b_a(x, y) = axy$, and $\langle a, b \rangle = \langle a \rangle \oplus \langle b \rangle$.

6.1. Proposition. *The map $\chi\colon \mathcal{A}_{\mathrm{fEt}}(E) \to \mathrm{GW}(E)$ is surjective, for any finite separable field extension $k \to E$.*

Proof. Recall that $\mathrm{GW}(E)$ is generated as an abelian group by the classes $\langle a \rangle$ for $a \in E^*$. We will show that each of these classes is in the image of χ.

If a is not a square in E then consider the field extension $E_a = E[x]/(x^2 - a)$. Then E_a is a separable field extension of E, and $\chi(E_a)$ is simply E_a (regarded as an E-vector space) equipped with the trace form. An easy computation shows this form is isomorphic to $\langle 2, 2a\rangle = \langle 2\rangle + \langle 2a\rangle$. So we have $\langle 2\rangle + \langle 2a\rangle = \chi(E_a)$.

We claim that $\langle 2\rangle \in \operatorname{im}\chi$. If 2 is a square in E then this is clear, since $\langle 2\rangle = \langle 1\rangle$. If 2 is not a square in E then we may apply the above analysis with a replaced by 2 to find that $\langle 2\rangle + \langle 4\rangle \in \operatorname{im}\chi$. Since $\langle 4\rangle = \langle 1\rangle \in \operatorname{im}\chi$, we again have $\langle 2\rangle \in \operatorname{im}\chi$.

At this point we know that $\langle 2\rangle + \langle 2a\rangle \in \operatorname{im}\chi$ and $\langle 2\rangle \in \operatorname{im}\chi$, and so $\langle 2a\rangle \in \operatorname{im}\chi$. But then $\langle 4a\rangle = \langle 2\rangle \cdot \langle 2a\rangle \in \operatorname{im}\chi$. Since $\langle 4a\rangle = \langle a\rangle$, we are done. ∎

6.2. Example. The map χ is usually not an isomorphism. To see this in one example, let $k = \mathbb{F}_p$ where p is odd. Then $\mathcal{A}_{\mathrm{fEt}}(k)$ is a free abelian group on a countably-infinite set of generators, whereas $\mathrm{GW}(k) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. In general, it might be interesting to have a set of generators for the kernel of $\mathcal{A}_{\mathrm{fEt}}(k) \to \mathrm{GW}(k)$ together with some kind of geometric source for them. See Example 6.7 below.

6.3. Remark. If $f\colon R \to S$ is a sheerly separable map of rings, we have the induced maps $f_*\colon \mathrm{GW}(R) \to \mathrm{GW}(S)$ and $f^!\colon \mathrm{GW}(S) \to \mathrm{GW}(R)$ from Section 2. However, for most purposes it is more convenient to use the geometric setting of affine schemes: there we would write $f^*\colon \mathrm{GW}(\operatorname{Spec} R) \to \mathrm{GW}(\operatorname{Spec} S)$ and $f_!\colon \mathrm{GW}(\operatorname{Spec} S) \to \mathrm{GW}(\operatorname{Spec} R)$. The disadvantage here is that it becomes tedious to write Spec repeatedly. We will tend to mix the two notations and write $f^*\colon \mathrm{GW}(R) \to \mathrm{GW}(S)$ and $f_!\colon \mathrm{GW}(S) \to \mathrm{GW}(R)$. In effect, this is basically just dropping the “Spec” and letting it be understood. In practice there is never any confusion here.

Our goal is to be able to analyze pieces of the categories $\mathrm{GWC}(k)$ for some explicit choices of k. Galois theory gives an equivalence of categories between sheerly separable extensions of k and continuous $\mathrm{Gal}(k^{sep}/k)$-sets, and this is a useful tool to exploit.

Fix a finite-dimensional Galois extension L/k, and set $G = \mathrm{Gal}(L/k)$. Say that a separable k-algebra A is ***L*-constructible** if it is isomorphic to a product $\prod_i A_i$ where each A_i is an algebraic field extension of k that admits an embedding into L. For each finite G-set S, let $\mathcal{F}(S, L)$ be the set of G-maps from S to L, with ring structure given by pointwise addition and multiplication. Clearly $\mathcal{F}(G/H, L) \cong L^H$ and $\mathcal{F}(S \amalg T, L) \cong \mathcal{F}(S, L) \times \mathcal{F}(T, L)$, hence each $\mathcal{F}(S, L)$ is L-constructible. In the opposite direction, given a sheerly separable k-algebra A the set of k-algebra maps $k\text{-alg}(A, L)$ inherits an action of G. Galois theory says that we have an equivalence of categories

$$\mathrm{fin}G\mathcal{S}et \rightleftarrows \mathrm{fEt}_k^{L-con}$$

where the upper arrow is $S \mapsto \mathcal{F}(S, L)$ and the lower arrow is $\operatorname{Spec} A \mapsto k\text{-alg}(A, L)$.

The Grothendieck-Witt functor on fEt_k restricts, via the above Galois equivalence, to a Gysin functor on finite G-sets. Let us write

$$\mathrm{GW}_L(S) = \mathrm{GW}(\mathcal{F}(S, L))$$

for this restricted Gysin functor. Clearly the correspondence category $\mathrm{fin}G\mathcal{S}et_{(\mathrm{GW}_L)}$ is the full subcategory of $(\mathrm{fEt}_k)_{\mathrm{GW}}$ whose objects are the L-constructible k-algebras.

The universality of the Burnside functor gives a natural transformation $\mathcal{A}_G \to \mathrm{GW}_L$, and therefore a functor between correspondence categories $\mathrm{fin}G\mathcal{S}et_{(\mathcal{A}_G)} \to \mathrm{fin}G\mathcal{S}et_{(\mathrm{GW}_L)}$. Composing with the inclusion into $(\mathrm{fEt}_k)_{\mathrm{GW}}$, we have constructed a functor from the Burnside category of G to the Grothendieck-Witt category of k:

$$\mathrm{fin}G\mathcal{S}et_{(\mathcal{A}_G)} \to \mathrm{fin}G\mathcal{S}et_{(\mathrm{GW}_L)} \to (\mathrm{fEt}_k)_{\mathrm{GW}} = \mathrm{GWC}(k).$$

We now look at some examples:

6.4. Example. The category $\mathrm{GWC}(\mathbb{R})$ has two objects: $\operatorname{Spec}\mathbb{R}$ and $\operatorname{Spec}\mathbb{C}$. Let π denote the unique map $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$, and let $\sigma\colon \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$ be the nontrivial automorphism over $\operatorname{Spec}\mathbb{R}$. Since $\mathrm{GW}(\mathbb{C}) = \mathbb{Z}$ and $\mathrm{GW}(\mathbb{R}) = \mathbb{Z}\langle\langle 1\rangle, \langle -1\rangle\rangle$, the category $\mathrm{GWC}(\mathbb{R})$ is readily computed to be as shown in the right-hand diagram below. One only needs check that $I_\pi \circ R_\pi = 1 + \sigma$ and $R_\pi \circ I_\pi = \langle 1\rangle + \langle -1\rangle$.

Similarly, the Burnside category for $\mathbb{Z}/2$ has two objects: $*$ and $\mathbb{Z}/2$. We write $\pi\colon \mathbb{Z}/2 \to *$ and $\sigma\colon \mathbb{Z}/2 \to \mathbb{Z}/2$ for the evident maps. Then $\mathcal{A}_{\mathbb{Z}/2}(\mathbb{Z}/2) = \mathbb{Z}$ and $\mathcal{A}_{\mathbb{Z}/2}(*) = \mathbb{Z}\langle [*], [\mathbb{Z}/2]\rangle$. Here one computes that $I_\pi \circ R_\pi = 1 + \sigma$ and $R_\pi \circ I_\pi = [\mathbb{Z}/2]$.

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[loop above, "\mathbb{Z}\langle 1,\sigma\rangle"] \arrow[d, bend left, "\mathbb{Z}\langle R\pi\rangle"] \\ * \arrow[u, bend left, "\mathbb{Z}\langle I\pi\rangle"] \arrow[loop below, "\mathbb{Z}\langle [*],[\mathbb{Z}/2]\rangle"] \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}\mathbb{C} \arrow[loop above, "\mathbb{Z}\langle 1,\sigma\rangle"] \arrow[d, bend left, "\mathbb{Z}\langle R\pi\rangle"] \\ \operatorname{Spec}\mathbb{R}. \arrow[u, bend left, "\mathbb{Z}\langle I\pi\rangle"] \arrow[loop below, "\mathbb{Z}\langle \langle 1\rangle,\langle -1\rangle \rangle"] \end{tikzcd}$$

The map from the Burnside category to the Grothendieck-Witt category has the evident behavior (in particular, it sends $[\mathbb{Z}/2]$ to $\langle 1\rangle + \langle -1\rangle$), and by inspection is an isomorphism.

Before considering our next example we need to recall some facts about finite fields. If F is a finite field of odd characteristic then $F^\times$ is cyclic of even order and so $(F^\times)/(F^\times)^2 = \mathbb{Z}/2$. Thus when we partition $F^\times$ into the squares and the non-squares, any two non-squares are equivalent: if a and b are non-squares then $a = \lambda^2 b$ for some λ. A little work shows when $\mathrm{char}(F) \neq 2$ that $\mathrm{GW}(F)$ is generated by $\langle 1\rangle$ and $\langle g\rangle$, where $g \in F^\times$ is any choice of non-square. Moreover, $2\langle g\rangle = 2\langle 1\rangle$ and $\mathrm{GW}(F) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ with corresponding generators $\langle 1\rangle$ and $\langle g\rangle - \langle 1\rangle$. See [S] or [D, Appendix A] for details. It is useful to write $\alpha = \langle g\rangle - \langle 1\rangle$. Note that

$$\alpha^2 = \langle g^2\rangle - 2\langle g\rangle + 1 = 2 - 2\langle g\rangle = -2\alpha = 0.$$

Note that the calculation of $\mathrm{GW}(F)$ gives a classification of non-degenerate quadratic spaces over F: in each dimension there are exactly two, namely $n\langle 1\rangle$ and $(n-1)\langle 1\rangle + \langle g\rangle = n\langle 1\rangle + \alpha$. The discriminant of the form, regarded as an element of $F^\times/(F^\times)^2$, distinguishes the two isomorphism types.

The following lemma calculates the behavior of the Grothendieck-Witt group under a quadratic extension.

6.5. LEMMA. *Let $q = p^e$ where p is an odd prime. Fix a non-square $g \in \mathbb{F}_q$, and fix a non-square $h \in \mathbb{F}_{q^2}$. If $j\colon \mathbb{F}_q \hookrightarrow \mathbb{F}_{q^2}$ is a fixed embedding then the pullback and pushforward maps for $\mathrm{GW}(-)$ are given by the formulas*

$$j^*(\langle 1\rangle) = j^*(\langle g\rangle) = \langle 1\rangle, \qquad j_!(\langle 1\rangle) = \langle 1\rangle + \langle g\rangle, \quad j_!(\langle h\rangle) = 2\langle 1\rangle.$$

Every automorphism of $\mathbb{F}_q$ induces the identity on $\mathrm{GW}(\mathbb{F}_q)$ (both via pullback and pushforward).

PROOF. First note that if λ is an automorphism of $\mathbb{F}_q$ then λ preserves the property of being a square or non-square; consequently, λ^* is the identity since $\lambda^*(\langle g\rangle) = \langle g\rangle$. Since $\lambda_!$ is the inverse of λ^* (Lemma 3.8), this is also the identity. So we have verified the last sentence of the lemma.

Observe that $\mathbb{F}_{q^2}$ may be identified with the extension $\mathbb{F}_q[x]/(x^2 - g)$, and we may assume that j is the evident inclusion of $\mathbb{F}_q$ (using the previous paragraph). Since $g = x^2$ in $\mathbb{F}_{q^2}$ we have $j^*(\langle g\rangle) = \langle 1\rangle$.

To compute $j_!(\langle 1\rangle)$ we must analyze the trace form on $\mathbb{F}_{q^2}$. This is represented by the 2×2 matrix

$$\begin{bmatrix} \mathrm{tr}(1) & \mathrm{tr}(x) \\ \mathrm{tr}(x) & \mathrm{tr}(x^2) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2g \end{bmatrix}.$$

The discriminant is $4g$, which is equivalent to g modulo squares. So $j_!(\langle 1\rangle) = \langle 1\rangle + \langle g\rangle$, as the forms have the same discriminant class.

The above work readily generalizes to compute $j_!(\langle a + bx\rangle)$ for any $a, b \in \mathbb{F}_q$. This form is represented by the matrix

$$\begin{bmatrix} \mathrm{tr}(a + bx) & \mathrm{tr}(ax + bx^2) \\ \mathrm{tr}(ax + bx^2) & \mathrm{tr}(ax^2 + bx^3) \end{bmatrix} = \begin{bmatrix} 2a & 2bg \\ 2bg & 2ag \end{bmatrix}.$$

The discriminant is $4a^2g - 4b^2g^2 = 4g(a^2 - b^2g)$, and so $j_!(\langle a + bx\rangle) = \langle 1\rangle + \langle g(a^2 - b^2g)\rangle$.

In a finite field every element can be written as a sum of two squares [S, Lemma 2.3.7], so we can write $g^{-1} = b^2 + r^2$ for some $b, r \in \mathbb{F}_q$. Neither b nor r is zero, since g is not a square. Then

$$j_!(\langle 1 + bx\rangle) = \langle 1\rangle + \langle g(1 - b^2g)\rangle = \langle 1\rangle + \langle g(r^2g)\rangle = \langle 1\rangle + \langle 1\rangle = 2\langle 1\rangle.$$

Hence $\langle 1 + bx\rangle \neq \langle 1\rangle$ (since their images under $j_!$ are different), and so $1 + bx$ is a non-square class; i.e. $\langle 1 + bx\rangle = \langle h\rangle$ in $\mathrm{GW}(\mathbb{F}_{q^2})$. So we have in fact proven that $j_!(\langle h\rangle) = 2\langle 1\rangle$. ∎

6.6. PROPOSITION. *Let q be a power of an odd prime, and consider a field extension $j\colon \mathbb{F}_q \hookrightarrow \mathbb{F}_{q^e}$. Let g and g' be non-squares in $\mathbb{F}_q$ and $\mathbb{F}_{q^e}$, respectively. Then the induced maps j^* and $j_!$ are given by*

$$j^*(\langle 1\rangle) = \langle 1\rangle, \qquad j^*(\langle g\rangle) = \begin{cases} \langle g'\rangle & \text{if } e \text{ is odd,} \\ \langle 1\rangle & \text{if } e \text{ is even,} \end{cases}$$

$$j_!(\langle 1\rangle) = \begin{cases} e\langle 1\rangle & e \text{ odd,} \\ (e-1)\langle 1\rangle + \langle g\rangle & e \text{ even,} \end{cases} \qquad j_!(\langle g'\rangle) = \begin{cases} (e-1)\langle 1\rangle + \langle g\rangle & e \text{ odd,} \\ e\langle 1\rangle & e \text{ even.} \end{cases}$$

These formulas can also be written as:

$$j^*(\langle 1\rangle) = \langle 1\rangle, \qquad j^*(\alpha) = \begin{cases} \alpha & e \text{ odd,} \\ 0 & e \text{ even,} \end{cases}$$

$$j_!(\langle 1\rangle) = \begin{cases} e\langle 1\rangle & e \text{ odd} \\ e\langle 1\rangle + \alpha & e \text{ even} \end{cases}, \qquad j_!(\alpha) = \alpha.$$

Proof. The statement about j^* is immediate: the extension $\mathbb{F}_{q^e}$ contains a square root of g if and only if it contains $\mathbb{F}_{q^2}$, which happens precisely when e is even.

The form $j_!(\langle 1\rangle)$ is either $e\langle 1\rangle$ or $(e-1)\langle 1\rangle + \langle g\rangle$, and these are distinguished by the discriminant. So it suffices to analyze the discriminant of the trace form on $\mathbb{F}_{q^e}$: the discriminant is a square if and only if the form is $e\langle 1\rangle$. A classical computation says this coincides with the discriminant of the minimal polynomial of any primitive element for the extension $\mathbb{F}_{q^e}/\mathbb{F}_q$. If $r_1, \dots, r_e$ are the roots of this minimal polynomial, then this discriminant is $\Delta = Q^2$ where

$$Q = \prod_{i<j}(r_i - r_j).$$

If the roots are indexed appropriately then the Galois group of $\mathbb{F}_{q^e}/\mathbb{F}_q$ acts by cyclic permutation. It follows that Q is invariant under the Galois action if and only if e is odd. So we see that Δ is a square in $\mathbb{F}_q$ (and equivalently, $j_!(\langle 1\rangle) = e\langle 1\rangle$) if and only if e is odd.

Finally, we analyze $j_!(\langle g'\rangle)$. When e is odd this is easy, as we can write

$$j_!(\langle g'\rangle) = j_!(j^*(\langle g\rangle)\cdot 1) = \langle g\rangle \cdot j_!(\langle 1\rangle) = \langle g\rangle \cdot e\langle 1\rangle = e\langle g\rangle = (e-1)\langle 1\rangle + \langle g\rangle$$

where in the last equality we have used that $\langle g, g\rangle = \langle 1, 1\rangle$. When e is even the pushforward $\mathrm{GW}(\mathbb{F}_{q^e}) \to \mathrm{GW}(\mathbb{F}_{q^{e/2}})$ sends $\langle g'\rangle$ to $2\langle 1\rangle$ by Lemma 6.5. It follows that $j_!(\langle g'\rangle)$ is a multiple of 2, and of course it also has rank e. The only such element of $\mathrm{GW}(\mathbb{F}_q)$ is $e\langle 1\rangle$. ∎

6.7. Example. [The Euler characteristic of a finite field extension] Our goal is to explicitly compute the map $\chi\colon \mathcal{A}_{\mathrm{fEt}}(\mathbb{F}_q) \to \mathrm{GW}(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. Given a finite field extension $j\colon \mathbb{F}_q \hookrightarrow \mathbb{F}_{q^e}$, the Euler characteristic is another name for $j_!(1)$. Using Proposition 6.6, this is equal to $\chi(\mathbb{F}_{q^e}) = e\langle 1\rangle + \epsilon_e\alpha \in \mathrm{GW}(\mathbb{F}_q)$ where

$$\epsilon_e = \begin{cases} 0 & \text{if } e \text{ is odd,} \\ 1 & \text{if } e \text{ is even.} \end{cases}$$

It is an amusing exercise to use the above computation to check the multiplicativity formula

$$\chi(\mathbb{F}_{q^e} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^f}) = \chi(\mathbb{F}_{q^e}) \cdot \chi(\mathbb{F}_{q^f}),$$

which is the analog of the topological formula $\chi(X \times Y) = \chi(X) \times \chi(Y)$.

We can use the above computation to give generators for the kernel of the canonical map $\chi\colon \mathcal{A}_{\mathrm{fEt}}(\mathbb{F}_q) \to \mathrm{GW}(\mathbb{F}_q)$. If we set $E_n = [\mathbb{F}_{q^n}]$ then by inspection a complete set of generators is

$$E_{n+3} - E_{n+2} - E_{n+1} + E_n \quad (n \geq 1), \quad 2E_2 - E_1 - E_3, \quad E_3 - 3E_1.$$

It could be interesting to find an explicit geometric explanation for these relations. For example, one might try to produce a degree 4 étale map $f\colon X \to Y$ of $\mathbb{F}_q$-schemes where Y is $\mathbb{A}^1$-connected and where one fiber of f is $\operatorname{Spec}\mathbb{F}_{q^2} \amalg \operatorname{Spec}\mathbb{F}_{q^2}$ and another fiber is $\operatorname{Spec}\mathbb{F}_q \amalg \operatorname{Spec}\mathbb{F}_{q^3}$.

6.8. Example. We next explore a small piece of $\mathrm{GWC}(\mathbb{F}_p)$, where p is odd. Specifically, consider the full subcategory whose objects are $\operatorname{Spec}\mathbb{F}_q$ for $q = p^{2^i}$ and $0 \leq i \leq 3$. Set $G = \mathrm{Gal}(\mathbb{F}_{p^8}/\mathbb{F}_p) = \mathbb{Z}/8$. Let g_{2^i} denote some specific choice of non-square element in $\mathbb{F}_{p^{2^i}}$, and write $\alpha_{2^i} = \langle g_{2^i}\rangle - \langle 1\rangle$. Also write $J_{2^i} = \mathrm{GW}(\mathbb{F}_{p^{2^i}})$; this is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$ with corresponding generators 1 and α_{2^i}, subject to the multiplicative relation $\alpha_{2^i}^2 = -2\alpha_{2^i} = 0$. Finally, let σ always denote the Frobenius $x \mapsto x^p$ and fix specific embeddings $j_{2^i}\colon \mathbb{F}_{q^{2^i}} \hookrightarrow \mathbb{F}_{q^{2^{i+1}}}$. Let their induced maps be denoted $\pi_{2^i}\colon \operatorname{Spec}\mathbb{F}_{q^{2^{i+1}}} \to \operatorname{Spec}\mathbb{F}_{q^i}$.

The following diagrams show the Burnside category for $\mathbb{Z}/8$ as well as the relevant piece of $\mathrm{GWC}(\mathbb{F}_p)$. Recall that if R is a ring and S is a set then we write $R\langle S\rangle$ and $\langle S\rangle R$ for the sets of finite sums $\sum r_i s_i$ and $\sum s_i r_i$ where $r_i \in R$, $s_i \in S$. We let $A_{2^i} = \mathcal{A}_{\mathbb{Z}/8}(\mathbb{Z}/2^i)$, the Grothendieck ring of $\mathbb{Z}/8$-sets over $\mathbb{Z}/2^i$. So $A_{2^i} = \mathbb{Z}\langle[\mathbb{Z}/2^i], [\mathbb{Z}/2^{i+1}], \ldots, [\mathbb{Z}/8]\rangle$. Here when we write $[\mathbb{Z}/2^j]$ we are suppressing the structure map $\pi\colon \mathbb{Z}/2^j \to \mathbb{Z}/2^i$, which we always take to be the map of $\mathbb{Z}/8$-sets sending 0 to 0. In the $\mathbb{Z}/8$-set context we let σ always denote the map $x \mapsto x+1$.

$$\begin{tikzcd}
\mathbb{Z}/8 \arrow[loop left, "\langle 1,\sigma,\ldots,\sigma^7\rangle A_8"] \arrow[d, shift left, "\langle R\pi_4,\ldots,R\pi_4\sigma^3\rangle A_8"] \\
\mathbb{Z}/4 \arrow[loop left, "\langle 1,\sigma,\sigma^2,\sigma^3\rangle A_4"] \arrow[u, shift left, "A_8\langle I\pi_4,\ldots,I(\sigma^3\pi_4)\rangle"] \arrow[d, shift left, "\langle R\pi_2,R\pi_2\sigma\rangle A_4"] \\
\mathbb{Z}/2 \arrow[loop left, "\langle 1,\sigma\rangle A_2"] \arrow[u, shift left, "A_4\langle I\pi_2,I(\sigma\pi_2)\rangle"] \arrow[d, shift left, "\langle R\pi_1\rangle A_2"] \\
\ast \arrow[loop left, "A_1"] \arrow[u, shift left, "A_2\langle I\pi_1\rangle"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F}_{p^8} \arrow[loop left, "\langle 1,\sigma,\ldots,\sigma^7\rangle J_8"] \arrow[d, shift left, "\langle R\pi_4,\ldots,R(\sigma^3\pi_4)\rangle J_8"] \\
\mathbb{F}_{p^4} \arrow[loop left, "\langle 1,\sigma,\sigma^2,\sigma^3\rangle J_4"] \arrow[u, shift left, "J_8\langle I\pi_4,\ldots,I(\sigma^3\pi_4)\rangle"] \arrow[d, shift left, "\langle R\pi_2,R(\sigma\pi_2)\rangle J_4"] \\
\mathbb{F}_{p^2} \arrow[loop left, "\langle 1,\sigma\rangle J_2"] \arrow[u, shift left, "J_4\langle I\pi_2,I(\sigma\pi_2)\rangle"] \arrow[d, shift left, "\langle R\pi_1\rangle J_2"] \\
\mathbb{F}_p \arrow[loop left, "J_1"] \arrow[u, shift left, "J_2\langle I\pi_1\rangle"]
\end{tikzcd}$$

Notice that we have written σ^i instead of $R\sigma^i$. Also, note that σ acts trivially on each J_n by Lemma 6.5 and so the endomorphism ring of $\mathbb{F}_{p^n}$ is the group ring $J_n[\mathbb{Z}/n]$. The analogous remark holds in the Burnside category. Finally, note that while the two categories clearly have very similar forms, the map between them is not an isomorphism because $A_n \ncong J_n$. The map $A_n \to J_n$ sends $[\mathbb{Z}/n]$ to 1; for $k > n$ it sends $[\mathbb{Z}/k]$ to $j_!(1) = \frac{k}{n} + \alpha$, where $j\colon \mathbb{F}_{p^n} \hookrightarrow \mathbb{F}_{p^k}$ is the inclusion (here we have used Proposition 6.6).

Below we list the main relations in $\mathrm{GWC}(\mathbb{F}_p)$. Recall that $\alpha_n \in J_n$ is the unique element of order 2. We simplify D_a to just a, for $a \in J_n$.

$$\begin{aligned} &R\pi_n \circ I\pi_n = \langle 2\rangle + \alpha_n \in J_n & I\pi_n \circ R\pi_n = 1 + \sigma^n \\ &\alpha_n \circ R\pi_n = 0 & I\pi_n \circ \alpha_n = 0 \\ &R\pi_n \circ \alpha_{n+1} \circ I\pi_n = \alpha_n & \end{aligned}$$

We leave the reader to derive these, as they are simple consequences of using the RDI rules from Theorem 1.2 together with the computations in Proposition 6.6. Coupled with the obvious relations that come from the category of fields, e.g. $R\pi_n \circ \sigma^n = R\pi_n$, the above relations allow one to work out all compositions in $\mathrm{GWC}(\mathbb{F}_p)$.

6.9. Example. We describe one last example, this time concerning non-Galois extensions. Most of the details will be left to the reader. Write $E_2 = \mathbb{Q}(\sqrt[3]{2})$, $E_\mu = \mathbb{Q}(\mu_3)$ (the cyclotomic field), and $E_{2,\mu} = \mathbb{Q}(\sqrt[3]{2}, \mu_3)$. Note that $[E_2 : \mathbb{Q}] = 3$, $[E_3 : \mathbb{Q}] = 2$, and $[E_{2,\mu} : \mathbb{Q}] = 6$. The extensions $E_\mu/\mathbb{Q}$ and $E_{2,\mu}/E_\mu$ are Galois, but $E_2/\mathbb{Q}$ is not. Let π_i, $i \in \{0, 1, 2, 3\}$, be the maps of schemes induced by the evident inclusions of fields:

$$\begin{tikzcd} \operatorname{Spec} E_2 \arrow[d, "\pi_0"'] & \operatorname{Spec} E_{2,\mu} \arrow[l, "\pi_1"'] \arrow[d, "\pi_3"] \\ \operatorname{Spec} \mathbb{Q} & \operatorname{Spec} E_\mu. \arrow[l, "\pi_2"'] \end{tikzcd}$$

Finally, write $\mathrm{GW}_\mu = \mathrm{GW}(E_\mu)$, and so forth.

Computing in the Grothendieck-Witt category $\mathrm{GWC}(\mathbb{Q})$, maps between E_μ and $\mathbb{Q}$, or between $E_{2,\mu}$ and E_μ, are handled exactly as the general case discussed at the end of Section 5. For maps from E_2 to E_μ, as an abelian group this is $\mathrm{GW}_{2,\mu}$ since $E_2 \otimes_{\mathbb{Q}} E_\mu = E_{2,\mu}$. A little thought shows that the maps are all of the form $R\pi_3 \circ Da_{2,\mu} \circ I\pi_1$, where $a_{2,\mu} \in \mathrm{GW}_{2,\mu}$.

To compute maps from E_2 to itself, we start with $E_2 \otimes_{\mathbb{Q}} E_2 \cong E_2 \times E_{2,\mu}$. As an abelian group we then have $\mathrm{GWC}(\mathbb{Q})(E_2, E_2) = \mathrm{GW}_2 \oplus \mathrm{GW}_{2,\mu}$. The two summands correpond to elements Da_2 for $a_2 \in \mathrm{GW}_2$ and $R\pi_1 \circ Da_{2,\mu} \circ I\pi_1$ where $a_{2,\mu} \in \mathrm{GW}_{2,\mu}$. The ring structure is determined by the formulas

$$\begin{aligned} Da_2 \circ Db_2 &= D(a_2 b_2), \\ Da_2 \circ (R\pi_1 \circ D_{a_{2,\mu}} \circ I\pi_1) &= R\pi_1 \circ D(\pi_1^*(a_2) \cdot a_{2,\mu}) \circ I\pi_1 \\ (R\pi_1 \circ Da_{2,\mu} \circ I\pi_1) \circ Da_2 &= R\pi_1 \circ D(a_{2,\mu} \cdot \pi_1^*(a_2)) \circ I\pi_1 \\ (R\pi_1 \circ Da_{2,\mu} \circ I\pi_1) \circ (R\pi_1 \circ Db_{2,\mu} \circ I\pi_1) &= [R\pi_1 \circ D(a_{2,\mu} b_{2,\mu}) \circ I\pi_1] + \\ &\qquad [R\pi_1 \circ D(\sigma^*(a_{2,\mu}) b_{2,\mu}) \circ I\pi_1]. \end{aligned}$$

These equations all follow from the rules in Theorem 1.2.

To get a sense of the above computation, let us generalize things just a bit. Let $f\colon R \to S$ be a homomorphism of commutative rings, and let $\sigma\colon S \to S$ be an automorphism such that $\sigma^2 = id$ and $\sigma f = f$. Define a product on $R \times S$ by

$$(r,s)\cdot(r',s') = (rr', (fr)s' + s(fr') + ss' + \sigma(s)s').$$

Check by brute force that this makes the abelian group $R \times S$ into a ring. Let α be the unique E_2-linear automorphism of $E_{2,\mu}$ that has order 2. Applying the above construction to $\pi_1^*\colon \mathrm{GW}(E_2) \to \mathrm{GW}(E_{2,\mu})$, where $\sigma = \alpha^*$, yields the endomorphism ring of E_2 in the Grothendieck-Witt category $\mathrm{GWC}(\mathbb{Q})$.

A. Symmetric monoidal categories and duality

In this section we review some elements from the theory of closed, symmetric monoidal categories. Then we recall the notion of a dualizable object, as well as some standard properties.

A.1. Basic conventions. Let $(\mathcal{C}, \otimes, S, F(-,-))$ be a closed symmetric monoidal category. This means $\otimes$ is the monoidal structure, S is the unit, and $X, Y \mapsto F(X,Y)$ is the cotensor.

In this setting there are evident evaluation maps

$$F(A,B)\otimes A \to B$$

defined as the adjoint to the identity on $F(A,B)$. Likewise, there are certain canonical maps

$$F(X,S)\otimes Y \to F(X,Y) \qquad \text{and} \qquad F(A,B)\otimes F(X,Y) \to F(A\otimes X, B\otimes Y)$$

defined to be the adjoints of evident compositions involving symmetry isomorphisms and evaluations. In general, we will use ψ to denote any such canonical map that arises in a general closed symmetric monoidal category. It should always be clear from context exactly what map we mean.

There is one special case where it is useful to have a distinguished name, rather than just the generic "ψ". For any object X in a closed symmetric monoidal category, set $X^* = F(X,S)$. Then we let $ev_X\colon X^*\otimes X \to S$ be the adjoint of the identity map $X^* \to F(X,S)$.

A.2. Dualizable objects. The theory of dualizable objects goes back to Dold and Puppe [DP], but in modern times has been used extensively by May and his collaborators (see [LMS] and [Ma1], for example).

A.3. Definition. *An object X in a symmetric monoidal category is called* ***dualizable*** *if there is another object Y together with maps*

$$\eta\colon S \to X \otimes Y, \qquad \epsilon\colon Y \otimes X \to S$$

such that the composite

$$X = S \otimes X \xrightarrow{\eta \otimes id_X} X \otimes Y \otimes X \xrightarrow{id \otimes \epsilon} X \otimes S = X$$

is id_X and the composite

$$Y = Y \otimes S \xrightarrow{id_Y \otimes \eta} Y \otimes X \otimes Y \xrightarrow{\epsilon \otimes id_Y} S \otimes Y = Y$$

is id_Y. We say that Y is a ***dual*** *for X, although it is more precise to say that the dual is (Y, ϵ, η) since all three pieces of structure are needed.*

A.4. Remark. If Y is a dual for X, then there can be several choices for ϵ and η that serve as structure maps. If one fixes Y and ϵ, however, then there is only one corresponding choice for η; similarly, if one fixes Y and η then there is only one choice for ϵ. This follows by the same argument that shows that a functor can have at most one left (or right) adjoint.

The following result can be pulled out of the proof of [LMS, Theorem III.1.6]:

A.5. Proposition. *In a closed symmetric monoidal category suppose that X is dualizable with dual (Y, ϵ, η). Then the map $\tilde{\epsilon}\colon Y \to X^*$, adjoint to ϵ, is an isomorphism. Consequently, X^* is also a dual for X, with structure maps $ev_X\colon X^* \otimes X \to S$ and the composite*

$$S \xrightarrow{\eta} X \otimes Y \xrightarrow{id \otimes \tilde{\epsilon}} X \otimes X^*.$$

Proof. The duality axioms imply that the composite λ_W given by

$$\mathcal{C}(W, Y) \to \mathcal{C}(W \otimes X, Y \otimes X) \to \mathcal{C}(W \otimes X, S) = \mathcal{C}(W, X^*)$$

is a bijection, for all objects W. This composite is natural in W, and for $W = Y$ it sends id_Y to $\tilde{\epsilon}$; it follows that λ_W is post-composition with $\tilde{\epsilon}$ for all W. The Yoneda Lemma then yields that $\tilde{\epsilon}$ is an isomorphism. Finally, one uses that

$$Y \otimes X \xrightarrow{\tilde{\epsilon} \otimes id} X^* \otimes X \xrightarrow{ev_X} S$$

equals ϵ. ∎

If X is dualizable, $ev_X\colon X^*\otimes X\to S$ is the evaluation map from the closed structure, and $cev_X\colon S\to X\otimes X^*$ is any map satisfying the conditions of Definition A.3 then we call cev_X the **coevaluation map** for X (it is uniquely determined, of course). The following two results are standard:

A.6. Proposition. *In a closed symmetric monoidal category, an object X is dualizable if and only if there exists a map c that makes the following diagram commute:*

$$\begin{tikzcd}
S \arrow[r, "c"] \arrow[d, "id_X"'] & X\otimes X^* \arrow[d, "t"] \\
F(X,X) & X^*\otimes X \arrow[l, "\psi"]
\end{tikzcd}$$

Here ψ is an instance of the canonical map $A^*\otimes B\to F(A,B)$. If c exists, it is unique; and moreover, it is precisely the coevaluation map for X.

Proof. See [LMS, Theorem III.1.6]. The uniqueness of c follows from [LMS, Proposition III.1.3], which shows that the horizontal map ψ is an isomorphism. ∎

A.7. Proposition. *If X and Y are dualizable objects in a closed symmetric monoidal category then the following are true:*

(a) $X\otimes Y$ and X^* are dualizable;

(b) $\psi\colon X\to X^{**}$ is an isomorphism;

(c) $\psi\colon X^*\otimes Y^*\to(X\otimes Y)^*$ is an isomorphism.

(d) $cev_X\colon S\to X\otimes X^*$ is the composite

$$\begin{tikzcd}
S \arrow[r, equal] & S^* \arrow[r, "ev_X^*"] & (X^*\otimes X)^* & X^{**}\otimes X^* \arrow[l, "\cong", "\psi"'] & X\otimes X^* \arrow[l, "\psi\otimes id", "\cong"']
\end{tikzcd}$$

Proof. Part (a) is elementary, while parts (b) and (c) are from [LMS, Proposition III.1.3]. For part (d), perhaps the easiest method is to check that ev_X and the given composite satisfy the properties of Definition A.3. To this end, consider the following diagram:

$$\begin{tikzcd}
S^*\otimes X \arrow[r, "ev_X^*\otimes 1_X"] \arrow[dr, "\psi"'] & (X^*\otimes X)^*\otimes X \arrow[d, "\psi"'] & (X^{**}\otimes X^*)\otimes X \arrow[l, "\psi\otimes 1", "\cong"'] & (X\otimes X^*)\otimes X \arrow[l, "\psi\otimes 1\otimes 1", "\cong"'] \arrow[d, "1\otimes ev_X"] \\
& X^{**} & & X\otimes S. \arrow[ll, "\psi"]
\end{tikzcd}$$

The vertical map labelled ψ is the adjoint to the composite

$$(X^*\otimes X)^*\otimes X\otimes X^*\xrightarrow{1\otimes t}(X^*\otimes X)^*\otimes X^*\otimes X\xrightarrow{ev_{X^*\otimes X}}S.$$

We aim to show that the "across-the-top, then down" composition from $S^*\otimes X$ to $X\otimes S$ is the identity (after canonical identifications of the domain and codomain with X). But the triangle and the rectangle commute in any closed symmetric monoidal category, by an easy verification (it suffices to check commutativity in the category of finite-dimensional vector spaces over a field, cf. [HHP]). Since $\psi\colon X \to X^{**}$ is an isomorphism by (b), this completes the verification.

The second condition from Definition A.3 is checked in a similar manner. The relevant diagram is a little easier:

$$\begin{tikzcd}
X^*\otimes S^* \arrow[r, "1\otimes ev^*"] \arrow[drrr, bend right, "id"'] & X^*\otimes(X^*\otimes X)^* \arrow[drr, "\psi"] & X^*\otimes(X^{**}\otimes X^*) \arrow[l, "1_{X^*}\otimes\psi"'] & X^*\otimes(X\otimes X^*) \arrow[l, "1\otimes\psi\otimes 1"'] \arrow[d, "ev_X\otimes 1_{X^*}"] \\
& & & S\otimes X^*
\end{tikzcd}$$

The diagonal map labelled ψ is the adjoint of the composite

$$X^*\otimes(X^*\otimes X)^*\otimes X \xrightarrow{t\otimes 1} (X^*\otimes X)^*\otimes X^*\otimes X \xrightarrow{ev_{X^*\otimes X}} S.$$

The "quadrilateral" and "triangle" in the diagram again commute in any closed symmetric monoidal category, and this completes the verification. ∎

Let $\mathcal{C}$ be a symmetric monoidal category in which every object is dualizable. For each object X, choose a specific dual X^* together with associated maps $\eta_X\colon S \to X\otimes X^*$ and $\epsilon_X\colon X^*\otimes X \to S$. We will show that this data determines a closed structure on $\mathcal{C}$, with

$$F(X,Y) = X^*\otimes Y.$$

This definition has a clear functoriality in Y, but functoriality in X requires a few remarks.

For $f\colon U \to X$, define $D(f)\colon X^* \to U^*$ to be the composite

$$X^* \xrightarrow{1\otimes\eta_U} X^*\otimes U\otimes U^* \xrightarrow{1\otimes f\otimes 1} X^*\otimes X\otimes U^* \xrightarrow{\epsilon_X\otimes 1} U^*.$$

It is easy to see that $D(id)=id$, and we will check below that $D(fg)=D(g)D(f)$. These dual maps then make $F(-,Y)$ into a contravariant functor, via $F(f,Y)=D(f)\otimes id_Y$.

We next describe maps

$$\alpha\colon \mathcal{C}(A, X^*\otimes Y) \rightleftarrows \mathcal{C}(A\otimes X, Y)\colon \beta$$

which will turn out to be an adjunction. Given $h\colon A \to X^*\otimes Y$, let $\alpha(h)$ be the composite

$$A\otimes X \xrightarrow{h\otimes 1} X^*\otimes Y\otimes X \xrightarrow{1\otimes t} X^*\otimes X\otimes Y \xrightarrow{\epsilon_X\otimes 1} Y.$$

And given $f\colon A\otimes X \to Y$, let $\beta(f)$ be the composite

$$A \xrightarrow{1\otimes\eta_X} A\otimes X\otimes X^* \xrightarrow{1\otimes t} A\otimes X^*\otimes X \xrightarrow{t\otimes 1} X^*\otimes A\otimes X \xrightarrow{1\otimes f} X^*\otimes Y.$$

The following proposition seems to be a well-known piece of folklore. For example, see [BS, Discussion after Definition 17].

A.8. Proposition. *Let $\mathcal{C}$ be a symmetric monoidal category in which all objects are dualizable, and where for each object X a triple $(X^*, \eta_X, \epsilon_X)$ has been chosen. Then the above definitions for $F(X,Y)$ and the adjunction make $\mathcal{C}$ into a closed symmetric monoidal category.*

Proof. This just involves checking a bunch of commutative diagrams. We first verify that if $f\colon X \to Y$ and $g\colon Y \to Z$ then $D(gf) = D(f)D(g)$. In the following commutative diagram, these are the two ways of going around the boundary:

$$\begin{tikzcd}
Y^* \arrow[r, "1\eta_X"] & Y^*XX^* \arrow[r, "1f1"] & Y^*YX^* \arrow[r, "\epsilon_Y 1"] & X^* \\
Z^*ZY^* \arrow[u, "\epsilon_Z 1"] \arrow[r, "111\eta_X"] & Z^*ZY^*XX^* \arrow[u, "\epsilon_Z 111"'] \arrow[r, "111f1"] & Z^*ZY^*YX^* \arrow[u, "\epsilon_Z 111"'] \arrow[r, "11\epsilon_Y 1"] & Z^*ZX^* \arrow[u, "\epsilon_Z 1"'] \\
Z^*YY^* \arrow[u, "1g1"] \arrow[r, "111\eta_X"] & Z^*YY^*XX^* \arrow[u, "1g111"] \arrow[r, "111f1"] & Z^*YY^*YX^* \arrow[u, "1g111"] \arrow[r, "11\epsilon_Y 1"] & Z^*YX^* \arrow[u, "1g1"'] \\
Z^* \arrow[u, "1\eta_Y"] \arrow[r, "1\eta_X"'] & Z^*XX^* \arrow[u, "1\eta_Y 11"] \arrow[r, "1f1"'] & Z^*YX^* \arrow[u, "1\eta_Y 11"] \arrow[ur, equal] &
\end{tikzcd}$$

Note that we have left out the tensor symbols, to enhance readability.

We next need a special property of the dual maps $D(f)$. Namely, for $f\colon X \to Y$ the following diagram is commutative:

$$\begin{tikzcd}
Y^* \otimes X \arrow[r, "1\otimes f"] \arrow[d, "D(f)\otimes 1"'] & Y^* \otimes Y \arrow[d, "\epsilon_Y"] \\
X^* \otimes X \arrow[r, "\epsilon_X"'] & S.
\end{tikzcd} \tag{A.9}$$

The proof is immediate upon inspection of the larger commutative diagram below:

$$\begin{tikzcd}
Y^*X \arrow[r, equal] \arrow[dr, "1\eta_X 1"'] & Y^*X \arrow[r, "1f"] & Y^*Y \arrow[r, "\epsilon_Y"] & S \\
& Y^*XX^*X \arrow[u, "11\epsilon_X"'] \arrow[r, "1f11"'] & Y^*YX^*X \arrow[u, "11\epsilon_X"] \arrow[r, "\epsilon_Y 11"'] & X^*X. \arrow[u, "\epsilon_X"']
\end{tikzcd}$$

Now we can check that the adjunction map α is functorial. In the variables A and Y this is obvious, but in the variable X it takes some work. For $f\colon U \to V$ we must check that

$$\begin{tikzcd}
\mathcal{C}(A, U^* \otimes Y) \arrow[r, "\alpha"] & \mathcal{C}(A \otimes U, Y) \\
\mathcal{C}(A, V^* \otimes Y) \arrow[u] \arrow[r, "\alpha"] & \mathcal{C}(A \otimes V, Y) \arrow[u]
\end{tikzcd} \tag{A.10}$$

is commutative. To this end, consider a map $h\colon A \to V^* \otimes Y$ and the following diagram:

$$\begin{tikzcd}
AV \arrow[r, "h1"] & V^*YV \arrow[r, "1t"] & V^*VY \arrow[r, "\epsilon_V 1"] & Y \\
 & & V^*UY \arrow[u, "1f1"'] \arrow[r, "(Df)11"] & U^*UY \arrow[u, "\epsilon_U 1"'] \\
AU \arrow[uu, "1f"] \arrow[r, "h1"'] & V^*YU \arrow[uu, "11f"] \arrow[ur, "1t"] \arrow[r, "(Df)11"'] & U^*YU \arrow[ur, "1t"'] &
\end{tikzcd}$$

The diagram is readily checked to be commutative—the upper left corner by (A.9)—and the two ways of pushing h around (A.10) are the two outer composites.

Finally, we prove that α and β are inverses. Suppose given $f\colon A \otimes X \to Y$ and $h\colon A \to X^* \otimes Y$. The proof that $\alpha(\beta(f)) = f$ is

$$\begin{tikzcd}
AX \arrow[r, "1\eta_X 1"] \arrow[dr, equal] & AXX^*X \arrow[r, "f11"] \arrow[d, "11\epsilon_X"] & YX^*X \arrow[r, "t1"] \arrow[d, "1\epsilon_X"'] \arrow[dr, "t_{Y,X^*X}"] & X^*YX \arrow[d, "1t"] \\
 & AX \arrow[r, "f"'] & Y & X^*XY \arrow[l, "\epsilon_X 1"]
\end{tikzcd}$$

and the proof that $\beta(\alpha(h)) = h$ is

$$\begin{tikzcd}
 & & X^*XYX^* \arrow[r, "\epsilon_X 11"] & YX^* \arrow[r, "t"] & X^*Y \\
AXX^* \arrow[r, "h11"] & X^*YXX^* \arrow[r, "t11"] \arrow[ur, "1t1"] & YX^*XX^* \arrow[ur, "1\epsilon_X 1"] & & \\
A \arrow[u, "1\eta_X"] \arrow[r, "h"] & X^*Y \arrow[u, "11\eta_X"] \arrow[r, "t"] & YX^*. \arrow[u, "11\eta_X"] \arrow[uur, equal] & &
\end{tikzcd}$$

■

B. A leftover proof

Here we give details for the proof of Proposition 3.21. This is mostly routine, but we include details because several steps are a bit hard to remember, and this is the kind of thing one wants to be able to just look up when needed.

Note that in this proof we mostly split with our previous conventions and denote objects by lowercase rather than uppercase letters. This is for typographical reasons, both to save space in subscripts and to reduce the extent to which the long formulas feel like they are shouting at the reader.

Proof of Proposition 3.21. For part (a), here is the check that i_a is a right identity. If $x \in \mathcal{C}_E(a,b) = E(b \times a)$ then

$$\begin{aligned}
x \circ i_a &= (\pi_{13})_!\Big(\pi_{12}^* x \cdot \pi_{23}^*(\Delta_!^a(1))\Big) \\
&= (\pi_{13})_!\Big(\pi_{12}^* x \cdot (id_b \times \Delta^a)_!(1))\Big) && \text{(push-pull)} \\
&= (\pi_{13})_!\Big((id_b \times \Delta^a)_!\big((id_b \times \Delta^a)^*\pi_{12}^* x \cdot 1\big)\Big) && \text{(projection formula)} \\
&= x.
\end{aligned}$$

The last step used that $\pi_{13} \circ (id_b \times \Delta^a) = id_{b \times a}$ and $\pi_{12} \circ (id_b \times \Delta^a) = id_{b \times a}$. The verification that i_a is a left identity is similar.

Write π_{ca}^{cba} for the evident projection map $c \times b \times a \to c \times a$. Let $x \in \mathcal{C}_E(a,b)$, $y \in \mathcal{C}_E(b,c)$, and $z \in \mathcal{C}_E(c,d)$. The proof of associativity proceeds by analyzing the element

$$\Omega = \big(\pi_{da}^{dcba}\big)_!\Big((\pi_{dc}^{dcba})^*(z) \cdot (\pi_{cb}^{dcba})^*(y) \cdot (\pi_{ba}^{dcba})^*(x)\Big)$$

in two different ways. The first proceeds as follows:

$$\begin{aligned}
\Omega &= \big(\pi_{da}^{dca}\big)_!\big(\pi_{dca}^{dcba}\big)_!\Big[(\pi_{dca}^{dcba})^*(\pi_{dc}^{dca})^*(z) \cdot (\pi_{cb}^{dcba})^*(y) \cdot (\pi_{ba}^{dcba})^*(x)\Big] \\
&= \big(\pi_{da}^{dca}\big)_!\Big[(\pi_{dc}^{dca})^*(z) \cdot (\pi_{dca}^{dcba})_!\big[(\pi_{cb}^{dcba})^*(y) \cdot (\pi_{ba}^{dcba})^*(x)\big]\Big] && \text{(proj. form.)} \\
&= \big(\pi_{da}^{dca}\big)_!\Big[(\pi_{dc}^{dca})^*(z) \cdot (\pi_{dca}^{dcba})_!(\pi_{cba}^{dcba})^*\big[(\pi_{cb}^{cba})^*(y) \cdot (\pi_{ba}^{cba})^*(x)\big]\Big] \\
&= \big(\pi_{da}^{dca}\big)_!\Big[(\pi_{dc}^{dca})^*(z) \cdot (\pi_{ca}^{dca})^*(\pi_{ca}^{cba})_!\big[(\pi_{cb}^{cba})^*(y) \cdot (\pi_{ba}^{cba})^*(x)\big]\Big] && \text{(push-pull)} \\
&= z \cdot (y \cdot x).
\end{aligned}$$

The first and third equalities just use functoriality. For example, in the third equality we use that $\pi_{cb}^{dbca} = \pi_{cb}^{cba} \circ \pi_{cba}^{dcba}$ and so forth. We leave the reader to perform a similar series of steps to show that $\Omega = (z \cdot y) \cdot x$. This proves associativity, and so finishes the proof of (a).

Part (b) is obvious.

For (c) we must show that if $f\colon a \to b$ and $g\colon b \to c$ then $R_g \circ R_f = R(gf)$. That is, we must check the formula

$$(\pi_{ca}^{cba})_!\Big[(\pi_{cb}^{cba})^*(id_c \times g)^*(i_c) \cdot (\pi_{ba}^{cba})^*(id_b \times f)^*(i_b)\Big] = (id_c \times gf)^*(i_c).$$

Note that the left side is $(id_c \times g)^*(i_c) \circ (id_b \times f)^*(i_b)$.

The first step is to use the two pullback squares

$$\begin{tikzcd}
c \times a \arrow[r, "\pi_2"] \arrow[d, "\pi_1 \times f \times \pi_2"'] & a \arrow[r, "f"] \arrow[d, "f \times id_a"] & b \arrow[d, "\Delta"] \\
c \times b \times a \arrow[r, "\pi_{ba}^{cba}"] & b \times a \arrow[r, "id_b \times f"] & b \times b
\end{tikzcd}$$

to see that $(\pi^{cba}_{ba})^*(id_b \times f)^*(i_b) = (\pi_1 \times f \times \pi_2)_!(1)$ (here we use that π_2^* and f^* are ring maps and so send 1 to 1). Next we compute that

$$\begin{aligned}
R_g \circ R_f &= (\pi^{cba}_{ca})_!\Big[(\pi^{cba}_{cb})^*(id_c \times g)^*(i_c) \cdot (\pi^{cba}_{ba})^*(id_b \times f)^*(i_b)\Big] \\
&= (\pi^{cba}_{ca})_!\Big[(\pi^{cba}_{cb})^*(id_c \times g)^*(i_c) \cdot (\pi_1 \times f \times \pi_2)_!(1)\Big] \\
&= (\pi^{cba}_{ca})_!(\pi_1 \times f \times \pi_2)_!\Big[(\pi_1 \times f \times \pi_2)^*(\pi^{cba}_{cb})^*(id_c \times g)^*(i_c) \cdot 1\Big] \\
&= (id_c \times gf)^*(i_c) \\
&= R(gf).
\end{aligned}$$

In the second-to-last equality we have used that $\pi^{cba}_{ca} \circ (\pi_1 \times f \times \pi_2) = id_{c\times a}$ and that $(id_c \times g)\pi^{cba}_{cb}(\pi_1 \times f \times \pi_2) = id_c \times gf$.

To prove (d) we must verify that $i_a^* = i_a$ (for every object a) and $(g \circ f)^* = f^* \circ g^*$ for every $f \in \mathcal{C}_E(a,b)$ and $g \in \mathcal{C}_E(b,c)$. For the first of these, consider the twist map $t\colon a \times a \to a \times a$. Since $t^2 = id_{a\times a}$ we have by Lemma 3.8 that $t_! = (t^*)^{-1} = t^*$. So

$$i_a^* = t^*(i_a) = t_!(i_a) = t_!(\Delta^a_!(1)) = (t \circ \Delta^a)_!(1) = \Delta^a_!(1) = i_a.$$

Write t^{ab}_{ba} for the map $t\colon a \times b \to b \times a$, and similarly for other situations. Then

$$\begin{aligned}
(g \circ f)^* &= (t^{ac}_{ca})^*\Big[(\pi^{cba}_{ca})_![(\pi^{cba}_{cb})^*(g) \cdot (\pi^{cba}_{ba})^*(f)]\Big] \\
&= (t^{ca}_{ac})_!\Big[(\pi^{cba}_{ca})_![(\pi^{cba}_{cb})^*(g) \cdot (\pi^{cba}_{ba})^*(f)]\Big] \\
&= (\pi^{cba}_{ac})_!\big[(\pi^{cba}_{cb})^*(g) \cdot (\pi^{cba}_{ba})^*(f)\big] \\
&= (\pi^{cba}_{ac})_!(t^{abc}_{cba})_!(t^{abc}_{cba})^*\big[(\pi^{cba}_{cb})^*(g) \cdot (\pi^{cba}_{ba})^*(f)\big] \\
&= (\pi^{abc}_{ac})_!\big[(\pi^{abc}_{cb})^*(g) \cdot (\pi^{abc}_{ba})^*(f)\big] \\
&= (\pi^{abc}_{ac})_!\big[(\pi^{abc}_{bc})^*(t^{bc}_{cb})^*g \cdot (\pi^{abc}_{ab})^*(t^{ab}_{ba})^*f\big] \\
&= (\pi^{abc}_{ac})_!\big[(\pi^{abc}_{ab})^*(t^{ab}_{ba})^*f \cdot (\pi^{abc}_{bc})^*(t^{bc}_{cb})^*g\big] \\
&= f^* \circ g^*.
\end{aligned}$$

In the second and fourth equalities we have used Lemma 3.8, but all of the other equalities use only simple functoriality.

Continuing with (d), for $f\colon A \to B$ we have

$$I_f = t^*(R_f) = t^*(id \times f)^*(i_B) = (f \times id)^*(i_B)$$

(using part (c) for the second equality) and likewise

$$I_f = t^*(R_f) = t^*(A \to B \times A)_!(1) = t_!(A \to B \times A)_!(1) = (A \to A \times B)_!(1),$$

using (c) for the second equality and Lemma 3.8 to identity t^* and $t_!$.

To prove the first part of (e) we argue as follows:

$$\begin{aligned}
\alpha \circ R_f &= (\pi^{ZWY}_{ZY})_! \Big[(\pi^{ZWY}_{ZW})^*(\alpha) \cdot (\pi^{ZWY}_{WY})^*(id_W \times f)^*(i_W) \Big] \\
&= (\pi^{ZWY}_{ZY})_! \Big[(\pi^{ZWY}_{ZW})^*(\alpha) \cdot (id_Z \times f \times id_Y)_!(1) \Big] \\
&= (\pi^{ZWY}_{ZY})_! (id_Z \times f \times id_Y)_! \Big[(id_Z \times f \times id_Y)^* (\pi^{ZWY}_{ZW})^*(\alpha) \cdot 1 \Big] \\
&= id_! \Big[(id_Z \times f)^* \alpha \cdot 1 \Big] \\
&= (id_Z \times f)^*(\alpha).
\end{aligned}$$

In the second equality we have used the push-pull axiom applied to the pullback diagram

$$\begin{tikzcd}
Z \times W \times Y \arrow[r, "\pi^{ZWY}_{WY}"] & W \times Y \arrow[r, "id_W \times f"] & W \times W \\
Z \times Y \arrow[u, "id_Z \times f \times id_Y"] \arrow[r, "\pi_2"] & Y \arrow[u, "f \times id_Y"'] \arrow[r, "f"] & W, \arrow[u, "\Delta"']
\end{tikzcd}$$

together with $i_W = \Delta_!(1)$. In the third equality we have used the projection formula (Proposition 3.11), and in the fourth equality we have used that $\pi^{ZWY}_{ZY} \circ (id_Z \times f \times id_Y) = id_{ZY}$ and $\pi^{ZWY}_{WY} \circ (id_Z \times f \times id_Y) = id_Z \times f$.

The other parts of (e) are proven by similar arguments. Part (f) follows from (e) using

$$I_f \circ R_q = I_f \circ i_B \circ R_q = (id \times q)^*(I_f \circ i_B) = (id \times q)^*(f \times id)^*(i_B) = (f \times q)^*(i_B).$$

The second part of (f) then follows using push-pull applied to the square

$$\begin{tikzcd}
A \times_B C \arrow[r] \arrow[d] & B \arrow[d, "\Delta"] \\
A \times C \arrow[r, "f \times q"] & B \times B.
\end{tikzcd}$$

Part (g) is similar to (f).

For (h), use that $R_p \circ I_g = \big((p \times g)^Z_{XW}\big)_!(1) = I_f \circ R_q$, by applying (f) and (g) together. Part (i) follows directly from (h) using the pullback diagram

$$\begin{tikzcd}
A \arrow[r, "id"] \arrow[d, "id"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"] & B.
\end{tikzcd}$$

∎

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