

BICATEGORIES OF FRACTIONS REVISITED: TOWARDS SMALL HOMS AND CANONICAL 2-CELLS

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ABSTRACT. This paper addresses two issues in dealing with bicategories of fractions. The first is to introduce a set of conditions on a class of arrows in a bicategory which is weaker than the one given in [5] but still allows a bicalculus of fractions. These conditions allow us to invert a smaller collection of arrows so that in some cases we may obtain a bicategory of fractions with small hom-categories. We adapt the construction of the bicategory of fractions to work with the weaker conditions. The second issue is the difficulty in dealing with 2-cells, which are defined by equivalence classes. We discuss conditions under which there are canonical representatives for 2-cells, and how pasting of 2-cells can be simplified in the presence of certain pseudo pullbacks. We also discuss how both of these improvements apply in the category of orbispaces.

1. Introduction

The purpose of this paper is to study some aspects of the structure of bicategories of fractions in more detail. We focus specifically on two goals. The first is to develop a weaker version of the calculus of fractions conditions of [5] that is still strong enough to allow us to create a bicategory of fractions where arrows are given by spans rather than zig-zags. (We will show that the conditions in [5] are not necessary in order to use fractions, although they make the construction slightly easier.) The second goal is to develop conditions under which we have canonical representatives for 2-cells, thus clarifying the structure of the category and its composition operations. Although this second goal could be considered independently from the first, we will in fact give our proofs in the context of the weaker conditions; since these imply the conditions of [5], our 2-cell results will apply in both contexts. For both of these goals, we will discuss how it applies to the example of orbispaces, defined as the bicategory of fractions of proper étale groupoids of suitable topological spaces with respect to the class of essential equivalences as in described in [4, 2].

For our first goal, we introduce a set of conditions on a class of arrows in a bicategory which is weaker than the one given in [5] but still allows us to form the localization as a bicalculus of fractions. One potential issue with localizations which are constructed as

The first author's research is funded by an NSERC Discovery Grant.

Received by the editors 2019-08-04 and, in final form, 2022-08-02.

Transmitted by Ross Street. Published on 2022-08-18.

2020 Mathematics Subject Classification: 18D05, 18E35.

Key words and phrases: bicategories of fractions, categories of fractions, localizations, orbifolds, small homs.

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categories, or bicategories, of fractions is that the hom-sets, or hom-categories, may not be small, as there is no guarantee in general that the fractions with a given domain and codomain form a set. To ensure that we do get a set in the classical bicategory of fractions construction, we need the class of arrows \mathfrak{W} to be inverted be small over each object; i.e., for any given object C there is only a *set* of arrows in \mathfrak{W} with codomain C . We may try to find a locally small subclass of the arrows to be inverted which generates the larger class in the sense that it induces an equivalent category (or bicategory) of fractions. This subclass may not satisfy all of the conditions for forming a (bi)category of fractions, so we consider whether any of the conditions can be weakened. When an arrow can be factored as a composite of arrows that are to be inverted, this arrow will receive an inverse in any localization that adds inverses for the arrows in the factorization. This observation leads us to consider the second condition of [5], the requirement that the class of arrows to be inverted is closed under composition, as an axiom that could potentially be weakened. We cannot completely omit it: some version of this axiom is needed to be able to define horizontal composition in the bicategory of fractions. However, we can replace it by the following condition:

[WB2] For each pair of composable arrows $B \xrightarrow{v} C \xrightarrow{w} D$ in \mathfrak{W} , there is an arrow $A \xrightarrow{u} B$ such that $A \xrightarrow{wvu} D$ is in \mathfrak{W} .

When a class of arrows satisfies this condition together with the other conditions for a bicalculus of fractions given in [5], it generates (through composition and closure under 2-isomorphisms) a larger class of arrows that satisfies all the bicalculus of fractions conditions. In this paper we will carefully consider all the conditions for the bicalculus of fractions and give more optimal versions of these conditions, and then provide an adjusted construction of the bicategory of fractions. This construction is still given with arrows that are single spans rather than zig-zags. This also provides us with a slightly weaker set of conditions for the classical construction of the category of fractions as given by Gabriel and Zisman in [3], spelled out in Corollary 4.13.

Our motivating example for this is the bicategory of orbispaces [10, 4, 2]. A priori, the hom-categories in this category are not small unless one requires all spaces to be second countable topological manifolds. We can work with a larger class of spaces, however, by observing that the class of essential equivalences has a subclass of essential covering maps that is small over each object, and satisfies the weakened conditions for a bicategory of fractions.

Related results and conditions have been presented in [9]. Roberts shows that for the case where \mathfrak{W} is a singleton pretopology satisfying the *WISC* condition that each object have a *set* of covers that is weakly initial among all covers, the bicategory of fractions will be locally essentially small: each hom-category is equivalent to a small one. By weakening the conditions to obtain a right calculus of fractions we are able to restrict ourselves to only invert the sets of covers when constructing the localization and obtain a locally small bicategory of fractions.

A different construction, of so called *faithful fractions*, was introduced in [1]. The result of this construction has small hom-categories as well. Different additional conditions need to be met to use this construction.

Another issue when working with a (bi)category of fractions is that the homs are defined by equivalence classes. For categories, arrows are given by equivalence classes; for bicategories the same is true for 2-cells. This makes the hom-categories in the bicategory of fractions a priori very large and somewhat mysterious and hard to work with. Horizontal composition of 2-cells for instance is rather cumbersome to describe and calculate. Our second goal in this paper is to address this issue by providing conditions under which there are canonical representatives for 2-cells and under which the horizontal composition operation is significantly simplified. A partial simplification of the presentation of 2-cells was provided in the appendix of [12] under additional hypotheses, but this was not integrated with the operations of horizontal and vertical composition. In our motivating example of orbispaces, essential equivalences have several nice cancellation properties that allow for a simplification of the 2-cell structure and allow us to use canonical representatives for 2-cells when this is convenient. These cancellation properties were identified as being (representably) fully faithful (ff) and co-ff in [1] and [7, 8] and used there to obtain related results about 2-cells in their representations of specific cases of 2-localizations.

In this work, we prove two types of results about the 2-cell structure: about the choice of representatives for 2-cells, and about conditions that allow us to simplify the pasting of 2-cells. Each representative diagram for a 2-cell in the bicategory of fractions, as in diagram (1) in Section 3, is given by two 2-cells in the original bicategory. The ‘left-hand’ 2-cell α is invertible, and we think of this as the cell that allows the ‘right-hand’ 2-cell β to be defined. We focus on the role of the left-hand 2-cell. Tommasini indirectly addresses the question of when a 2-cell can be represented by a diagram with a given left-hand 2-cell in [12]. In general this is not always possible, and moreover, two diagrams with the same left-hand 2-cell but different right-hand 2-cells may still represent the same 2-cell in the bicategory of fractions, so the universal homomorphism mapping a bicategory to its bicategory of fractions is in general neither 2-faithful nor 2-full. However, if the arrows to be inverted satisfy suitable subsets of the fully faithful or co fully faithful conditions, the situation simplifies and for each pair of spans we may choose any left-hand 2-cell and we show that each 2-cell in the bicategory of fractions can then be uniquely represented by a diagram involving the given left-hand 2-cell.

Additionally, for the case when the bicategory has certain pseudo pullbacks, we develop results to simplify the horizontal composition of 2-cells in the bicategory of fractions. Overall, our goal is to make the role of 2-cells in the bicategory of fractions more transparent. In our motivating example of orbispaces these conditions are satisfied; this will be explored further in [6].

Note that in [1] the authors use the ff and co-ff cancellation properties of essential equivalences between internal categories in a regular category to describe the localization with respect to essential equivalences as a faithful bicategory of fractions. One of its notable properties is that 2-cells in the fractions bicategory correspond to suitable 2-cells

in the original bicategory without needing to take equivalence classes. Similar results are obtained by Roberts for the bicategory of fractions of a pretopology consisting of ff and co-ff arrows. He also gives a canonical presentation for the 2-cells that corresponds to taking the strict pullback as the left-hand cell in the 2-cell diagrams of the bicategory of fractions. The work in this paper sheds further light on why this can be done: if the arrows in \mathfrak{W} are ff and co-ff one may choose any class of left-hand 2-cells to obtain canonical representations of 2-cells and avoid the need for equivalence classes.

This paper is structured as follows. In Section 2, we introduce the new, weakened, conditions on a class \mathfrak{W} to give rise to a bicalculus of fractions, and develop some theory on liftings of 2-cells related to the fourth condition on \mathfrak{W} , and on relating squares required by the third condition. In Section 3 we give the new bicategory of fractions construction $\mathcal{B}(\mathfrak{W}^{-1})$, a generalization of the one given in [5], with horizontal composition of arrows and 2-cells adjusted to account for the weaker assumption. In Section 4 we investigate the connection between our new construction and the original construction of [5], and show that if \mathfrak{W} satisfies the weaker conditions of Section 2, then the class of arrows obtained by taking the closure of \mathfrak{W} under composition and 2-isomorphism satisfies the original conditions from [5] and gives a bi-equivalent bicategory of fractions. Additionally, we introduce the notion of weakly initial subclasses of arrows, designed to allow us to pass to an even smaller subclass of arrows to obtain a subclass of a given class of arrows that is small over each object. Sections 5 and 6 develop our results about simplifying 2-cells. In Section 5 we introduce conditions that allow us to simplify the form of the 2-cells in the bicategory of fractions and obtain canonical representatives for the equivalence classes, and in Section 6 we investigate the case when the original bicategory has certain pseudo pullbacks and show how this can be used to simplify horizontal composition of 2-cells in the bicategory of fractions. In Section 7 we indicate how this work applies to orbispaces, to be further explored in [6]. The last sections are appendices containing technical proofs. The first one gives the associativity 2-cells for composition. The second appendix proves associativity coherence. The third appendix proves that horizontal and vertical composition are well-defined on equivalence classes of 2-cell diagrams. And the fourth appendix gives the proof for a result about the horizontal composition of 2-cell diagrams when the left-hand 2-cells are pseudo pullbacks.

Acknowledgements The authors would like to thank Matteo Tommasini for contributing Lemmas 2.7 and 5.5 as a way to strengthen the result in Theorem 5.6, Michael Johnson for his helpful conversations and suggestions related to this work, Martin Szyld for helpful conversations in regard to the universal property of the bicategory of fractions, David Roberts for pointing us to related work by him and others, and the referee for an extremely careful reading of an earlier version of this manuscript, leading us to tighten some of the statements and the proofs.

2. Weaker Conditions for a Bicalculus of Fractions

In the first part of this section we introduce the new conditions on a class of arrows in a bicategory that will give rise to a bicalculus of fractions. These are a weakening of the conditions **BF1–BF5** given in [5]. In the second part of this section we develop general results about the structure of the 2-cells in a bicategory with a class of arrows satisfying our new conditions.

2.1. **THE NEW CONDITIONS.** We list our new conditions on a class of arrows. In Section 3 we will show that these are sufficient for the existence of the bicategory of fractions, although the specific construction of this bicategory needs to be changed.

- **[WB1]** All identities are in \mathfrak{W} .
- **[WB2]** For each pair of composable arrows $B \xrightarrow{v} C \xrightarrow{w} D$ in \mathfrak{W} , there is an arrow $A \xrightarrow{u} B$ such that $A \xrightarrow{wvu} D$ is in \mathfrak{W} .
- **[WB3]** For every pair $w: A \rightarrow B, f: C \rightarrow B$ with $w \in \mathfrak{W}$, there exist maps h, v , where $v \in \mathfrak{W}$, and an invertible 2-cell α as in the following diagram.

$$\begin{array}{ccc} D & \xrightarrow{h} & A \\ v \downarrow & \xleftarrow{\alpha} & \downarrow w \\ C & \xrightarrow{f} & B \end{array}$$

- **[WB4]** For any 2-cell

$$\alpha: w \circ f \Rightarrow w \circ g$$

with $w \in \mathfrak{W}$, there exists an arrow $u \in \mathfrak{W}$ and a 2-cell

$$\beta: f \circ u \Rightarrow g \circ u$$

such that $\alpha \circ u = w \circ \beta$. Furthermore, the collection of such pairs (u, β) has the following property: when (u_1, β_1) and (u_2, β_2) are two such pairs, there exist arrows s, t , such that $u_1 \circ s$ and $u_2 \circ t$ are in \mathfrak{W} , and there is an invertible 2-cell $\varepsilon: u_1 \circ s \Rightarrow u_2 \circ t$ such that the following diagram commutes:

$$\begin{array}{ccc} f \circ u_1 \circ s & \xrightarrow{\beta_1 \circ s} & g \circ u_1 \circ s \\ f \circ \varepsilon \downarrow & & \downarrow g \circ \varepsilon \\ f \circ u_2 \circ t & \xrightarrow{\beta_2 \circ t} & g \circ u_2 \circ t. \end{array}$$

- **[WB5]** When $w \in \mathfrak{W}$ and there is an invertible 2-cell $\alpha: v \Rightarrow w$, then $v \in \mathfrak{W}$.

2.2. REMARKS.

1. The original condition **BF1** stated that all equivalences were in the class \mathfrak{W} . It is well-known that it is sufficient to replace this with the given **[WB1]**; see for instance, [12].
2. Condition **[WB2]** is a significantly weaker version of the original condition **BF2**, which required that \mathfrak{W} be closed under composition.
3. Conditions **[WB3]** and **[WB5]** are the same as the old conditions **BF3** and **BF5** respectively.
4. When α and β are 2-cells as in condition **[WB4]**, we will refer to β as a *lifting of α with respect to w* . In [5], condition **BF4** additionally required that if α is invertible, it has a lifting β that is invertible. We will show in Proposition 2.5 that this assumption is not needed, as it can be derived from the other assumptions.

2.3. PROPERTIES OF LIFTINGS OF 2-CELLS. In this section we prove that our condition **[WB4]**, together with the conditions **[WB1]**–**[WB3]** and **[WB5]**, imply the original condition **BF4**. To do this, we develop some properties of the 2-cell liftings that **[WB4]** requires, and show that they can be chosen to respect composition.

We assume throughout this section that \mathfrak{W} is a class of arrows satisfying conditions **[WB1]**–**[WB5]**. We begin by showing that for fixed $w \in \mathfrak{W}$, the collection of the liftings of cells given by **[WB4]** inherits the vertical composition structure in the sense that the vertical composition of two liftings gives a lifting for the vertical composition of the original cells.

2.4. LEMMA. *Let \mathfrak{W} satisfy **[WB1]**–**[WB5]**. Suppose that we have arrows*

$$B \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \\ \xrightarrow{h} \end{array} C \xrightarrow{w} D$$

with $w \in \mathfrak{W}$, and let $\alpha_1: wf \Rightarrow wg$ and $\alpha_2: wg \Rightarrow wh$ be 2-cells. Then there exists an arrow $u: A \rightarrow B$ in \mathfrak{W} with 2-cells $\beta_1: fu \Rightarrow gu$ and $\beta_2: gu \Rightarrow hu$ such that $w\beta_1 = \alpha_1u$ and $w\beta_2 = \alpha_2u$. It follows that $w(\beta_2 \cdot \beta_1) = (\alpha_2 \cdot \alpha_1)u$.

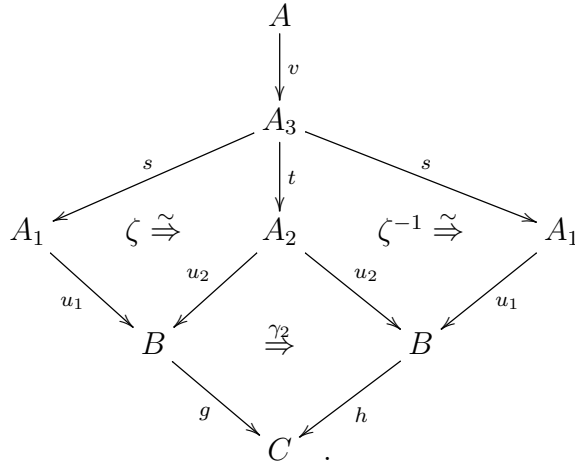
PROOF. We begin by choosing two arbitrary arrows and cells as in condition **[WB4]**: let $u_1: A_1 \rightarrow B$ and $u_2: A_2 \rightarrow B$ be two arrows in \mathfrak{W} with 2-cells $\gamma_1: fu_1 \Rightarrow gu_1$ and $\gamma_2: gu_2 \Rightarrow hu_2$ such that $w\gamma_1 = \alpha_1u_1$ and $w\gamma_2 = \alpha_2u_2$.

Since u_1 and u_2 are in \mathfrak{W} , condition **[WB3]** gives us a square

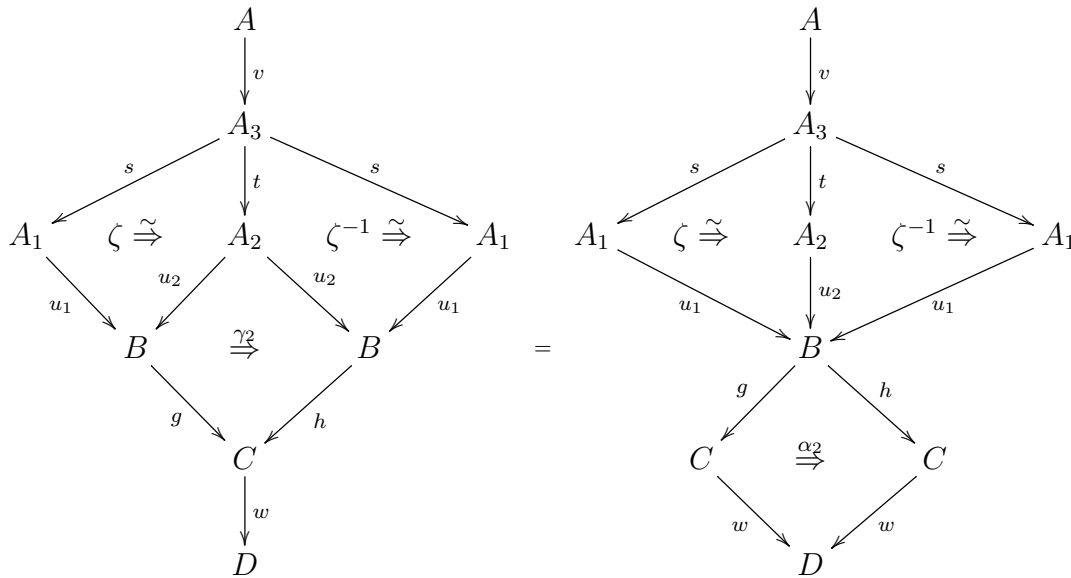
$$\begin{array}{ccc} A_3 & \xrightarrow{s} & A_1 \\ t \downarrow & \begin{array}{c} \xleftarrow{\zeta} \\ \xrightarrow{\zeta} \end{array} & \downarrow u_1 \\ A_2 & \xrightarrow{u_2} & B, \end{array}$$

with $t \in \mathfrak{W}$ and ζ an invertible 2-cell. By Condition **[WB2]**, there is an arrow $v: A \rightarrow A_3$ such that the composition u_2tv is in \mathfrak{W} , and hence by Condition **[WB5]** and the invertibility of ζ , $u_1sv \in \mathfrak{W}$ also.

We claim that the following arrow and 2-cells satisfy the conditions of this lemma: $u = u_1sv$, $\beta_1 = \gamma_1sv$ and $\beta_2 = ((h\zeta^{-1}) \cdot (\gamma_2t) \cdot (g\zeta)) \circ v$, as in the diagram,



To prove this claim, first note that since γ_1 was chosen to satisfy **[WB4]**, $w\beta_1 = w\gamma_1sv = \alpha_1u_1sv = \alpha_1u$. Now using the fact that γ_2 was also chosen so that $w\gamma_2 = \alpha_2u_2$, we calculate $w\beta_2$ in the following diagrams:



and this is clearly equal to $\alpha_2u_1sv = \alpha_2u$, as required. ■

We now use this lemma to prove that whenever the 2-cell $\alpha: wf \Rightarrow wg$ is invertible, there is at least one choice of a pair (u, β) for **[WB4]** such that β is also invertible.

2.5. PROPOSITION. *Let \mathfrak{W} satisfy the conditions [WB1]–[WB5]. If $w \in \mathfrak{W}$ and $\alpha: wf \Rightarrow wg$ is an invertible 2-cell, then there is an arrow $u \in \mathfrak{W}$ with an invertible 2-cell $\beta: fu \Rightarrow gu$ such that $w\beta = \alpha u$.*

PROOF. We begin by applying Lemma 2.4 to the case where $h = f$, $\alpha_1 = \alpha$ and $\alpha_2 = \alpha^{-1}$. This gives us an arrow $v \in \mathfrak{W}$ and 2-cells $\gamma: fv \Rightarrow gv$ and $\gamma': gv \Rightarrow fv$ such that $w\gamma = \alpha v$ and $w\gamma' = \alpha^{-1}v$. So $w(\gamma' \cdot \gamma) = (\alpha^{-1} \cdot \alpha)v = \text{id}_{wf}v$. This does not guarantee that γ and γ' are inverses, but we will show that there is a further lifting v' such that $vv' \in \mathfrak{W}$ and $\gamma v'$ and $\gamma' v'$ are inverses.

We create v' in two stages. First we find u_1 such that $(\gamma' u_1)(\gamma u_1) = \text{id}_{fu_1}$, and then we find w_1 such that $(\gamma u_1 w_1)(\gamma' u_1 w_1) = \text{id}_{fu_1 w_1}$. To find u_1 , we observe that both $w(\gamma' \gamma) = \text{id}_{wf}v$ and $w \circ \text{id}_{fv} = \text{id}_{wf}v$. Thus, $(v, \gamma' \cdot \gamma)$ and (v, id_{fv}) are both pairs of liftings of id_{wf} with respect to w as in [WB4]. The second half of [WB4] gives a relationship between any two such pairs, so applying that here gives two maps, u_1 and u_2 , and an invertible 2-cell,

$$\begin{array}{ccc} & \xrightarrow{u_1} & \\ u_2 \downarrow & \delta \Downarrow \cong & \downarrow v \\ & \xrightarrow{v} & \end{array}$$

with $vu_i \in \mathfrak{W}$ and such that

$$\begin{array}{ccc} \begin{array}{ccc} & u_1 & \\ & \swarrow & \searrow \\ & \delta & \\ & \swarrow & \searrow \\ v & & v \\ & \searrow & \swarrow \\ & \text{id}_{fv} & \\ & \swarrow & \searrow \\ & f & \end{array} & = & \begin{array}{ccc} & u_1 & \\ & \swarrow & \searrow \\ & \gamma^{u_1} & \\ & \swarrow & \searrow \\ v & & v \\ & \searrow & \swarrow \\ & f & \end{array} \end{array}$$

The left-hand side of this equation is equal to the identity 2-cell, id_{fvu_1} , so $\gamma' u_1 \cdot \gamma u_1 = \text{id}_{fvu_1}$.

Now we create w_1 via the same argument applied to the 2-cells $\gamma u_1 \cdot \gamma' u_1$ and id_{gvu_1} . We know that $w(\gamma u_1 \cdot \gamma' u_1) = (\alpha \cdot \alpha^{-1})vu_1 = \text{id}_{wg}vu_1 = \text{id}_{wg}vu_1 = \text{id}_{wg}vu_1$. So both $(vu_1, \gamma u_1 \cdot \gamma' u_1)$ and $(vu_1, \text{id}_{gvu_1})$ are liftings of id_{wg} with respect to w , and applying the second half of [WB4] as above gives us w_1, w_2 and an invertible 2-cell ϵ such that $vu_1 w_i \in \mathfrak{W}$ and $\gamma u_1 w_1 \cdot \gamma' u_1 w_1 = \text{id}_{gvu_1 w_1}$. We conclude that $\gamma' u_1 w_1 = (\gamma u_1 w_1)^{-1}$. Therefore setting $v' = u_1 w_1$, $u = vv' = vu_1 w_1$ and $\beta = \gamma u_1 w_1$ satisfies the requirements of the proposition. ■

2.6. REMARK. Combining the proofs for Proposition 2.5 and Lemma 2.4 shows that if α in Proposition 2.5 is invertible, for any arrow $u \in \mathfrak{W}$ with 2-cell $\beta: fu \Rightarrow gu$ such that $w\beta = \alpha u$, there is an arrow s such that $\beta \circ s$ is invertible.

The following result concerning cancellability of arrows in \mathfrak{W} was communicated to us by Matteo Tommasini [11].

2.7. LEMMA. Let \mathfrak{W} satisfy the conditions [WB1]–[WB4]. For any diagram

$$C \begin{array}{c} \xrightarrow{f} \\ \Downarrow \beta_1 \quad \Downarrow \beta_2 \\ \xrightarrow{g} \end{array} B \xrightarrow{w} A$$

with $w \in \mathfrak{W}$, if $w\beta_1 = w\beta_2$ then there exists an arrow $v: D \rightarrow C$ in \mathfrak{W} such that $\beta_1 v = \beta_2 v$.

PROOF. Apply the second part of [WB4] to $\alpha := w\beta_1 = w\beta_2$, $u_1 = u_2 := \text{id}_C$ and the 2-cells β_1 and β_2 as given (for simplicity we omit the structure cells from the bicategory in this calculation). This gives us the existence of arrows $v, v': D \rightrightarrows C$ such that $\text{id}_C v, \text{id}_C v' \in \mathfrak{W}$, and hence $v, v' \in \mathfrak{W}$ by [WB5], with an invertible 2-cell $\varepsilon: v \rightrightarrows v'$ such that $\beta_1 \circ \varepsilon = \beta_2 \circ \varepsilon$. Composing with $g\varepsilon^{-1}$ gives us that $\beta_1 v = \beta_2 v$ with $v \in \mathfrak{W}$ as required. ■

2.8. SQUARES AS IN CONDITION [WB3]. In this section we address a question related to condition [WB3]: if there are two squares as in [WB3] for the same cospan, how are these squares related to each other? This question was answered in the proof of Lemma A.1.1 in [5] for cospans where both arrows are in \mathfrak{W} . Here, we prove a more general result, for cospans with just one arrow in \mathfrak{W} and assuming only the weaker condition [WB2]. This result will play a crucial role in the constructions of whiskering of 2-cells with arrows in the bicategory of fractions and in the construction of the associativity isomorphisms. It will also be used in the study of the equivalence relation on the 2-cells diagrams.

2.9. PROPOSITION. For $w: A \rightarrow B$ in \mathfrak{W} and $f: C \rightarrow B$ any arrow in \mathcal{B} , and any two squares,

$$\begin{array}{ccc} D_1 & \xrightarrow{g_1} & A \\ v_1 \downarrow & \alpha_1 \lrcorner & \downarrow w \\ C & \xrightarrow{f} & B \\ u \downarrow & & \\ X & & \end{array} \quad \begin{array}{ccc} D_2 & \xrightarrow{g_2} & A \\ v_2 \downarrow & \alpha_2 \lrcorner & \downarrow w \\ C & \xrightarrow{f} & B \\ u \downarrow & & \\ X & & \end{array}$$

where u, v_1 and v_2 are all in \mathfrak{W} , then there are arrows s_1 and s_2 and invertible 2-cells β and γ as in

$$\begin{array}{ccccc} & & D_1 & & \\ & v_1 \swarrow & \uparrow s_1 & \searrow g_1 & \\ C & \lrcorner \beta & E & \lrcorner \gamma & A \\ & v_2 \swarrow & \downarrow s_2 & \searrow g_2 & \\ & & D_2 & & \end{array}$$

such that $uv_1s_1 \in \mathfrak{W}$, and the composites $(f\beta) \cdot (\alpha_1s_1)$ and $(\alpha_2s_2) \cdot (w\gamma)$ are equal:

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & \xrightarrow{s_1} & \xrightarrow{g_1} & & \\
 s_2 \downarrow & \xleftarrow{\beta} & \downarrow v_1 & \xleftarrow{\alpha_1} & \downarrow w \\
 & \xrightarrow{v_2} & \xrightarrow{f} & & \\
 \end{array} & \equiv & \begin{array}{ccccc}
 & \xrightarrow{s_1} & & & \\
 s_2 \downarrow & \xleftarrow{\gamma} & \downarrow g_1 & & \\
 & \xrightarrow{-g_2} & & & \\
 v_2 \downarrow & \xleftarrow{\alpha_2} & \downarrow w & & \\
 & \xrightarrow{f} & & & \\
 \end{array}
 \end{array}$$

PROOF. Since uv_1 is in \mathfrak{W} , condition [WB3] gives us a square

$$\begin{array}{ccc}
 F & \xrightarrow{\bar{v}_2} & D_1 \\
 \bar{v}_1 \downarrow & \xleftarrow{\beta'} & \downarrow uv_1 \\
 D_2 & \xrightarrow{uv_2} & X
 \end{array}$$

with $\bar{v}_1 \in \mathfrak{W}$. Applying Proposition 2.5 to the 2-cell $\beta': uv_1\bar{v}_2 \Rightarrow uv_2\bar{v}_1$, we get an arrow $\tilde{u}: F' \rightarrow F$ in \mathfrak{W} and an invertible 2-cell $\tilde{\beta}': v_1\bar{v}_2\tilde{u} \Rightarrow v_2\bar{v}_1\tilde{u}$.

Then we have the following invertible 2-cell from $wg_1\bar{v}_2\tilde{u}$ to $wg_2\bar{v}_1\tilde{u}$.

$$\begin{array}{ccccc}
 & & D_1 & \xrightarrow{g_1} & A \\
 & \nearrow \bar{v}_2\tilde{u} & \searrow v_1 & \searrow \alpha_1 \Downarrow & \searrow w \\
 F' & & & & C \xrightarrow{f} B \\
 & \searrow \bar{v}_1\tilde{u} & \nearrow v_2 & \nearrow \alpha_2^{-1} \Downarrow & \nearrow w \\
 & & D_2 & \xrightarrow{g_2} & A
 \end{array}$$

By applying Proposition 2.5 with respect to w , there is an arrow $\tilde{w}: F'' \rightarrow F'$ in \mathfrak{W} with an invertible 2-cell $\gamma': g_1\bar{v}_2\tilde{u}\tilde{w} \Rightarrow g_2\bar{v}_1\tilde{u}\tilde{w}$ such that $w\gamma'$ is equal to the pasting of this last diagram composed with \tilde{w} . Finally, by repeatedly applying condition [WB2] to the string of composable \mathfrak{W} arrows $uv_2, \bar{v}_1, \tilde{u}, \tilde{w}$, there is an arrow $t: E \rightarrow F''$ such that $uv_2\bar{v}_1\tilde{u}\tilde{w}t \in \mathfrak{W}$. By condition [WB5] it follows that $uv_1\bar{v}_2\tilde{u}\tilde{w}t \in \mathfrak{W}$ as well. The reader may verify that $s_1 = \bar{v}_2\tilde{u}\tilde{w}t$, $s_2 = \bar{v}_1\tilde{u}\tilde{w}t$, $\beta = \tilde{\beta}'\tilde{w}t$ and $\gamma = \gamma't$ satisfy the conditions of this proposition. ■

2.10. REMARK. An extension of the result of Proposition 2.9, discussing how any two solutions to the problem of this proposition are related, can be found in Appendix A, Proposition A.1.

3. The New Bicategory of Fractions Construction

We will now show that the conditions introduced in Section 2.1 are sufficient to construct a bicategory of fractions $\mathcal{B}(\mathfrak{W}^{-1})$. Given a bicategory \mathcal{B} and a class of arrows \mathfrak{W} which

satisfies the conditions [WB1]–[WB5], we first describe the new bicategory $\mathcal{B}(\mathfrak{W}^{-1})$, and then show that it has the universal property of the bicategory of fractions. The objects, arrows and 2-cells of $\mathcal{B}(\mathfrak{W}^{-1})$ are defined just as in [5], but we will need to adjust the definition of composition and pasting. We begin by reminding the reader of the definition as given in [5].

- *Objects* are the objects of \mathcal{B} .
- *Arrows* are spans of the form $\xleftarrow{w} \xrightarrow{f}$ with $w \in \mathfrak{W}$ and f an arbitrary arrow in \mathcal{B} .
- *2-Cells* are equivalence classes of diagrams of the form

$$\begin{array}{ccccc}
 & & C & & \\
 & w & \nearrow & f & \\
 A & & D & & B \\
 & \Downarrow \alpha \cong & & \Downarrow \beta & \\
 & & C' & & \\
 & w' & \searrow & f' & \\
 & & & &
 \end{array}, \tag{1}$$

where wu is in \mathfrak{W} (and hence $w'u'$ is). Such a diagram (1) is *equivalent* to another such diagram

$$\begin{array}{ccccc}
 & & C & & \\
 & w & \nearrow & f & \\
 A & & E & & B \\
 & \Downarrow \gamma \cong & & \Downarrow \delta & \\
 & & C' & & \\
 & w' & \searrow & f' & \\
 & & & &
 \end{array}$$

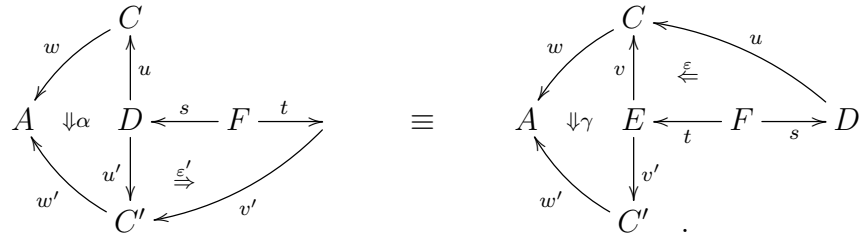
(with wv in \mathfrak{W}) if and only if there exists a diagram of the form

$$\begin{array}{ccccc}
 & & C & & \\
 & u & \nearrow & v & \\
 D & \xleftarrow{s} & F & \xrightarrow{t} & E \\
 & u' & \searrow & v' & \\
 & & C' & &
 \end{array}$$

with $wus \in \mathfrak{W}$, such that

$$\begin{array}{ccccc}
 & & C & & \\
 & u & \nearrow & f & \\
 D & \xleftarrow{s} & F & \xrightarrow{t} & E & \Downarrow \delta & B \\
 & & & & & \Downarrow \beta & \\
 & & & & & & C' \\
 & & & & & & \\
 & & & & & &
 \end{array} \cong \begin{array}{ccccc}
 & & C & & \\
 & & u & \nearrow & f & \\
 E & \xleftarrow{t} & F & \xrightarrow{s} & D & \Downarrow \beta & B \\
 & & & & & \Downarrow \beta & \\
 & & & & & & C' \\
 & & & & & & \\
 & & & & & &
 \end{array}$$

and



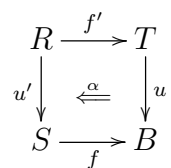
3.1. REMARK. In the description above, we consistently only require half of our arrow compositions to be in \mathfrak{W} . For example, we require only that $wv \in \mathfrak{W}$, and not the corresponding $w'v'$; similarly we only require $wus \in \mathfrak{W}$. However, since the 2-cells are invertible and \mathfrak{W} satisfies [WB5], the other half follows automatically.

The original condition **BF2** was used in [5] in the construction of composition of arrows and horizontal and vertical composition of 2-cells in the bicategory of fractions. In constructing these compositions under our weaker conditions, we need to adjust for the fact that \mathfrak{W} is no longer closed under composition. Instead, we have the condition [WB2] that allows us to pre-compose with an additional arrow to get a composition in \mathfrak{W} . The description of the compositions in [5] relies heavily on the choices of squares as in condition [WB3] and liftings as in condition [WB4] (although, in fact, the construction only depends on the choices of the squares when they are used to compose the spans, as Tommasini [12] has shown that different choices made in the composition of 2-cells give equivalent representatives). In describing the compositions in the new bicategory of fractions, we use a collection of choices for arrows for composites as in [WB2] to augment the choices of squares and liftings to make sure that the necessary arrows are in \mathfrak{W} . We list and label these choices here before beginning the constructions so we can refer back to them.

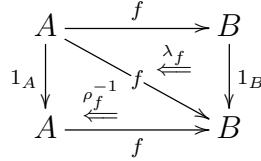
3.2. NOTATION. The following choices of arrows and 2-cells will be used in the construction of the bicategory of fractions $\mathcal{B}(\mathfrak{W}^{-1})$. The first three choices really determine the construction. The last four are just short-cuts for frequently used combinations of the first three.

[C1] For each pair of composable arrows $\xrightarrow{v} \xrightarrow{u}$ in \mathfrak{W} use [WB2] to choose an arrow $w_{u,v}$ such that $uvw_{u,v} \in \mathfrak{W}$. When v is an identity arrow, choose $w_{u,v}$ to be an identity as well.

[C2] For every pair $\xrightarrow{f} \xleftarrow{u}$ with $u \in \mathfrak{W}$ use [WB3] to choose a square



with $u' \in \mathfrak{W}$ and α invertible. When we want to stress the dependence of α on f and u , we denote this cell by $\alpha_{f,u}$. Furthermore, require that when $u = 1_B$, we choose the square,



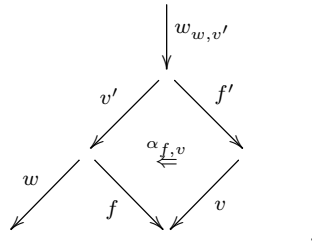
where λ_f and ρ_f are the left and right unitor 2-cell respectively.

[C3] Given $\alpha: w \circ f \Rightarrow w \circ g$, a 2-cell with $w \in \mathfrak{W}$, choose a 1-cell $\tilde{w} \in \mathfrak{W}$ and a 2-cell

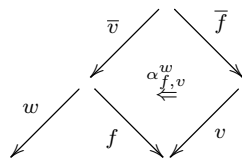
$$\tilde{\alpha}: f \circ u \Rightarrow g \circ u$$

such that $\alpha \circ \tilde{w} = w \circ \tilde{\alpha}$. Using Proposition 2.5, we choose $\tilde{\alpha}$ to be invertible whenever α is.

[C4] For each zig-zag, $\xleftarrow{w} \xrightarrow{f} \xleftarrow{v}$ with v and w in \mathfrak{W} , [C2] determines arrows f' and v' and an invertible 2-cell $\alpha_{f,v}: v f' \Rightarrow f v'$. Compose this with the choice $w_{w,v'}$ from [C1] to get $w v' w_{w,v'} \in \mathfrak{W}$, to obtain the diagram



Defining $\bar{v} = v' w_{w,v'}$, $\bar{f} = f' w_{w,v'}$ and $\alpha_{f,v}^w = \alpha_{f,v} w_{w,v'}$ gives the chosen diagram



with $w\bar{v} \in \mathfrak{W}$. Note that \bar{v} is not guaranteed to be in \mathfrak{W} , but $w\bar{v}$ is always in \mathfrak{W} by construction.

[C5] For each cospan $\xrightarrow{w} \xleftarrow{v}$ with both arrows $w, v \in \mathfrak{W}$, apply [C2] to obtain a square with an invertible 2-cell $\alpha_{w,v}$. Then compose with $w_{w,v'}$ from [C1] to get $v' w_{w,v'} \in \mathfrak{W}$. Define $\hat{v} = v' w_{w,v'}$, $\hat{w} = w' w_{w,v'}$ and $\hat{\alpha}_{w,v} = \alpha_{w,v} w_{w,v'}$ to obtain the chosen square

$$\begin{array}{ccc}
 & \xrightarrow{\hat{w}} & \\
 \hat{v} \downarrow & \hat{\alpha}_{w,v} & \downarrow v \\
 & \xleftarrow{\quad} & \\
 & \xrightarrow{w} &
 \end{array}$$

where $w\hat{v} \in \mathfrak{W}$ and the 2-cell $\hat{\alpha}_{w,v}$ is invertible.

[C6] For each invertible 2-cell $\alpha: w \circ s_1 \Rightarrow w \circ s_2$ with $w, ws_1, ws_2 \in \mathfrak{W}$, apply [C3] to obtain $\tilde{w} \in \mathfrak{W}$ and $\tilde{\alpha}: s_1\tilde{w} \Rightarrow s_2\tilde{w}$, with $\tilde{\alpha}$ invertible. Then ws_1 and \tilde{w} are in \mathfrak{W} , so apply [C1] to obtain an arrow u such that $ws_1\tilde{w}u \in \mathfrak{W}$. Since $\tilde{\alpha}$ in [C3] is invertible, we conclude that $ws_2\tilde{w}u$ is also in \mathfrak{W} . Setting $\bar{w} = \tilde{w}u$, we get the chosen lifting

$$\bar{\alpha}: s_1\bar{w} \Rightarrow s_2\bar{w}$$

such that $ws_1\bar{w} \in \mathfrak{W}$ and $\bar{\alpha}$ is invertible.

[C7] For each configuration,

$$\begin{array}{ccc}
 & \begin{array}{c} \uparrow \\ v \end{array} & \\
 u \swarrow & & \searrow wf \\
 & \downarrow v' & \beta \Downarrow \\
 & \begin{array}{c} \downarrow \\ v' \end{array} & \searrow wf'
 \end{array}$$

with uv and w in \mathfrak{W} , [C3] determines $\tilde{w} \in \mathfrak{W}$ and $\tilde{\beta}: fv\tilde{w} \Rightarrow f'v'\tilde{w}$, and [C1] determines an arrow $w_{\tilde{w},uv}$ with $uv\tilde{w}w_{\tilde{w},uv} \in \mathfrak{W}$. Now write $\bar{w} := \tilde{w}w_{\tilde{w},uv}$ and precomposing $\tilde{\beta}$ by $w_{\tilde{w},uv}$ gives the chosen 2-cell $\bar{\beta}_u$ with $uv\bar{w} \in \mathfrak{W}$.

$$\begin{array}{ccc}
 & \begin{array}{c} \uparrow \\ v\bar{w} \end{array} & \\
 u \swarrow & & \searrow f \\
 & \downarrow v'\bar{w} & \bar{\beta}_u \Downarrow \\
 & \begin{array}{c} \downarrow \\ v'\bar{w} \end{array} & \searrow f'
 \end{array}$$

With these choices determined, we will now define the bicategory of fractions.

Composition of 1-Cells We define the composition of spans

$$A \xleftarrow{u_1} S \xrightarrow{f_1} B \text{ and } B \xleftarrow{u_2} T \xrightarrow{f_2} C$$

in $\mathcal{B}(\mathfrak{W}^{-1})$ using the chosen square in [C4] of Notation 3.2,

$$\begin{array}{ccc}
 & \begin{array}{c} \bar{u}_2 \\ \swarrow \end{array} & \begin{array}{c} \bar{f}_1 \\ \searrow \end{array} \\
 & \alpha_{f_1, u_2}^{u_1} & \\
 u_1 \swarrow & & \searrow u_2 \\
 & \begin{array}{c} \leftarrow \\ f_1 \end{array} & \\
 & \begin{array}{c} \leftarrow \\ f_2 \end{array} &
 \end{array}$$

so that $u_1\bar{u}_2 \in \mathfrak{W}$. Then the composition of spans is given by

$$A \xleftarrow{u_1\bar{u}_2} \xrightarrow{f_2\bar{f}_1} C.$$

3.3. REMARKS.

1. Proposition 2.9 implies that any other choice of a square to define the composition results in an isomorphic arrow in $\mathcal{B}(\mathfrak{W}^{-1})$: Proposition 2.9 gives a 2-cell between the two arrows in $\mathcal{B}(\mathfrak{W}^{-1})$ that is observed to be invertible in Remark 3.4 Part 2. Proposition A.1 below further shows that the isomorphism is unique when certain properties with respect to the defining squares are required. So given the squares used to define the two ways to compose, there is a canonical invertible 2-cell between the two resulting compositions.
2. Horizontal composition of 1-cells is clearly not associative in general. In Appendix A, Proposition A.4 we introduce the family of associativity 2-cells and in Appendix B, Proposition B.4, we show that this family satisfies the associativity coherence conditions. The definition of the associativity cells is a direct generalization of the ones given in [5], but the proof of coherence is a bit more involved. The appendices highlight the technical results that lead to coherence in separate propositions.

Vertical Composition of 2-Cells We define the vertical composition of 2-cell diagrams,



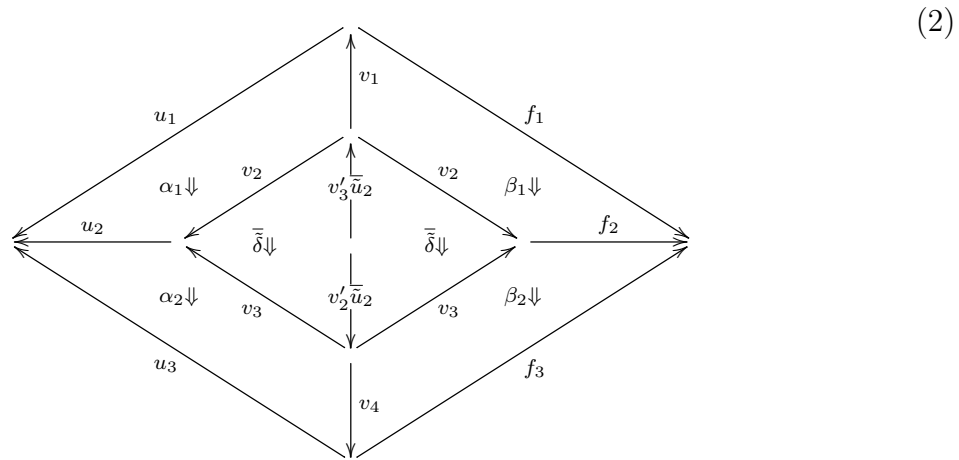
First, since u_2v_3 and u_2v_2 are both in \mathfrak{W} , let

$$\begin{array}{ccc}
 & v'_3 & \\
 v'_2 \downarrow & \xrightarrow{\quad} & u_2v_2 \\
 & \delta \Leftarrow & \\
 & \xrightarrow{\quad} & \\
 & u_2v_3 &
 \end{array}$$

be the chosen square in [C5] of Notation 3.2: $\delta = \hat{\alpha}_{u_2v_3, u_2v_2}$ and $u_2v_3v'_2 \in \mathfrak{W}$. Since δ is invertible, $u_2v_2v'_3 \in \mathfrak{W}$ also.

Next, apply [C6] to $\delta: u_2v_2v'_3 \Rightarrow u_2v_3v'_2$ and obtain an arrow $\bar{u}_2 \in \mathfrak{W}$ and an invertible 2-cell $\bar{\delta}: v_2v'_3\bar{u}_2 \Rightarrow v_3v'_2\bar{u}_2$. Note that $u_2v_2v'_3\bar{u}_2 \in \mathfrak{W}$, as indicated in [C6].

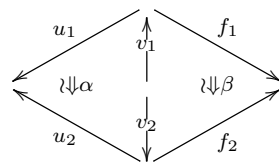
This gives us the following representative for the vertical composition,



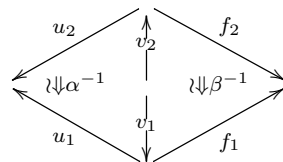
Observe that $u_2 v_2 v_3' \bar{u}_2 \in \mathfrak{W}$ by construction, and $u_1 v_1 v_3' \bar{u}_2$ and $u_3 v_4 v_2' \bar{u}_2$ are in \mathfrak{W} since they are isomorphic to $u_2 v_2 v_3' \bar{u}_2$. So this diagram represents a 2-cell from $\left\langle \begin{array}{c} \xleftarrow{u_1} \xrightarrow{f_1} \\ \xleftarrow{u_3} \xrightarrow{f_3} \end{array} \right\rangle$.

3.4. REMARKS.

1. In Appendix C, Proposition C.1 we show that vertical composition is well-defined on equivalence classes of 2-cell diagrams and in Appendix A, Proposition A.5 we show that it is strictly associative on equivalence classes of 2-cell diagrams.
2. It is straightforward to check that when both the left- and the right-hand 2-cells in a 2-cell diagram

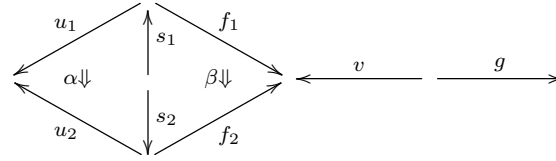


are vertically invertible in the original bicategory \mathcal{B} then the 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ represented by this diagram is vertically invertible with inverse represented by

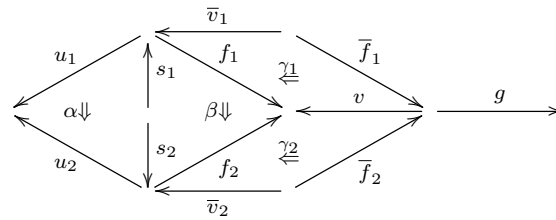


Horizontal Composition of 2-Cells The construction for horizontal composition in [5] is given in terms of whiskering on the left and the right. We will address the two cases in the following two subsections.

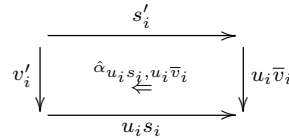
3.5. LEFT WHISKERING. Suppose we have



with $u_i s_i \in \mathfrak{W}$ and α invertible, so that the left side represents a 2-cell. We begin by constructing the composites of the arrows involved. This gives us the cells in the following diagram,

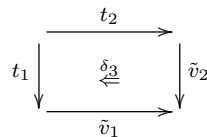


where $\gamma_1 = \alpha_{f_1, v}^{u_1}$ and $\gamma_2 = \alpha_{f_2, v}^{u_2}$ are the chosen squares of [C4] of Notation 3.2. (Note that this is not a pasting diagram.) The next step is to construct squares that complete the cospans $\xrightarrow{s_1} \xleftarrow{\bar{v}_1}$ and $\xrightarrow{s_2} \xleftarrow{\bar{v}_2}$. Neither s_i nor \bar{v}_i (where $i = 1, 2$) are necessarily in \mathfrak{W} , but the $u_i s_i$ are by assumption, and the $u_i \bar{v}_i$ are by [C4]. Now take the squares chosen in [C5] for $i = 1, 2$,



where the composites $u_i s_i v'_i$ are in \mathfrak{W} and the 2-cells $\hat{\alpha}_{u_i s_i, u_i \bar{v}_i}$ are invertible. Now we have $\hat{\alpha}_{u_i s_i, u_i \bar{v}_i}: u_i \bar{v}_i s'_i \Rightarrow u_i s_i v'_i$ where $u_i \in \mathfrak{W}$, and hence [C6] determines arrows \tilde{u}_i and 2-cells $\delta_i: \bar{v}_i s'_i \tilde{u}_i \Rightarrow s_i v'_i \tilde{u}_i$. If we write $v'_i \tilde{u}_i = \tilde{v}_i$ then we have $u_i s_i \tilde{v}_i \in \mathfrak{W}$ for $i = 1, 2$.

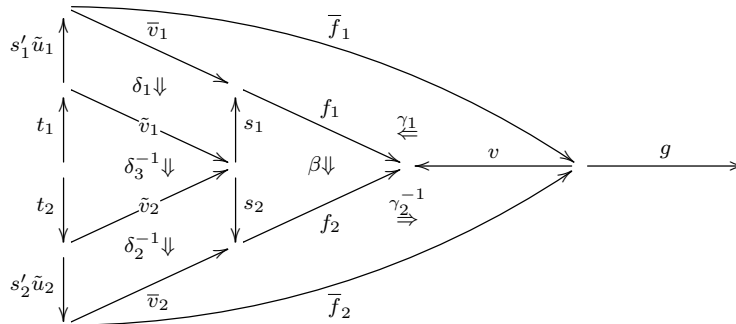
Finally, we want to construct a square to complete the cospan $\xrightarrow{\tilde{v}_1} \xleftarrow{\tilde{v}_2}$. Neither of the \tilde{v}_i is necessarily in \mathfrak{W} , but the $u_i s_i \tilde{v}_i$ are. Also, since $\alpha: u_1 s_1 \Rightarrow u_2 s_2$ is invertible, it follows that $u_1 s_1 \tilde{v}_2 \in \mathfrak{W}$. Using a sequence of chosen squares and lifts as above, we construct a square



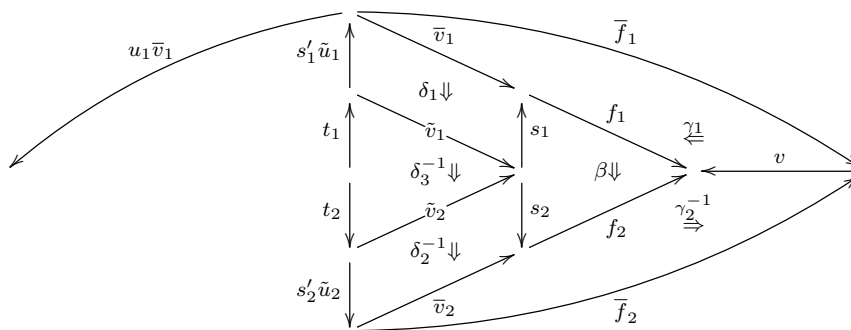
with δ_3 invertible and $u_1 s_1 \tilde{v}_1 t_1 \in \mathfrak{W}$.

To find the right-hand 2-cell in the diagram representing the left whiskering, we want

to apply a choice of lifting as in condition **[WB4]** to the following diagram,

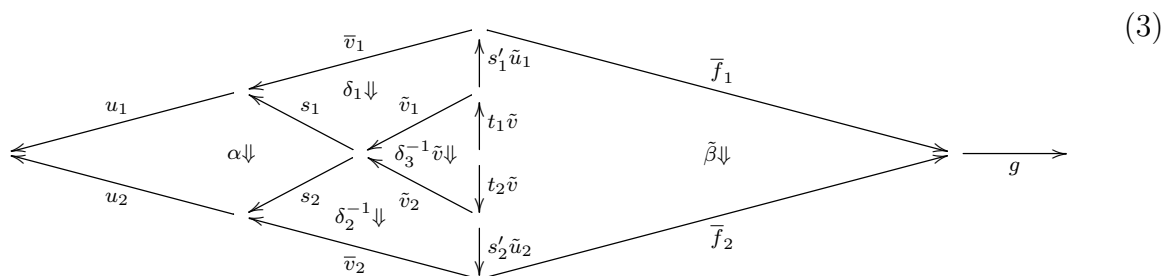


and lift with respect to v . However, we need to do this in such a way that we obtain a valid 2-cell diagram. By construction, $\tilde{v}_1 = v'_1 \tilde{u}_1$, and hence the 2-isomorphism $v_1 \delta_1^{-1} t_1 : u_1 s_1 \tilde{v}_1 t_2 \Rightarrow u_1 \bar{v}_1 s'_1 \tilde{u}_1 t_1 \in \mathfrak{W}$. This allows us to apply **[C7]** to get a lifting for the diagram

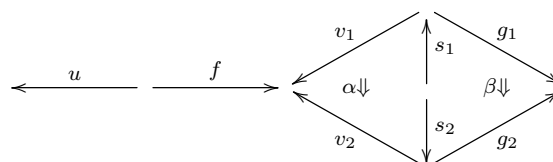


This gives us an arrow \tilde{v} and a 2-cell $\tilde{\beta} : \bar{f}_1 s'_1 \tilde{u}_1 t_1 \tilde{v} \Rightarrow \bar{f}_2 s_2 \tilde{u}_2 t_2 \tilde{v}$ such that $v \tilde{\beta}$ is equal to the pasting of the previous diagram composed with \tilde{v} , and $u_1 \bar{v}_1 s'_1 \tilde{u}_1 t_1 \tilde{v} \in \mathfrak{W}$.

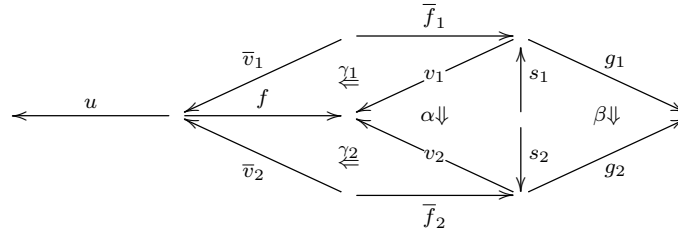
The resulting representative for the horizontal composition can be described by



3.6. RIGHT WHISKERING. Consider a diagram



with v_1s_1 and v_2s_2 in \mathfrak{W} , and α invertible, so the right side represents a 2-cell. Again, we begin by constructing the horizontal compositions of the arrows involved using the squares of [C4] in Notation 3.2 as in the following diagram,

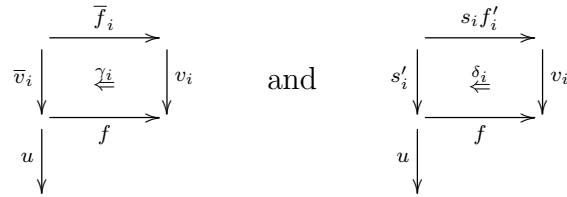


where $\gamma_i = \alpha_{f,v_i}^u$ and $u\bar{v}_i \in \mathfrak{W}$ for $i = 1, 2$. (Note that this is not a pasting diagram.)

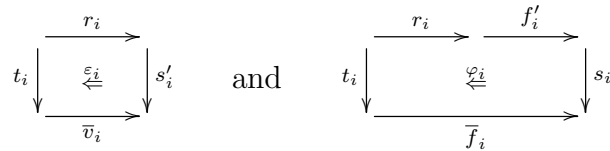
Since $v_i s_i \in \mathfrak{W}$ for $i = 1, 2$ and $u \in \mathfrak{W}$, we have chosen squares from [C4] giving

$$\begin{array}{ccc}
 & \xrightarrow{f'_i} & \\
 s'_i \downarrow & \delta_i \leftarrow & \downarrow v_i s_i \\
 & \xrightarrow{f} &
 \end{array} \tag{4}$$

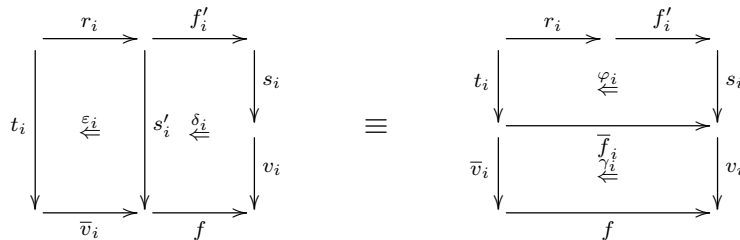
with $us'_i \in \mathfrak{W}$. Now apply Proposition 2.9 to the pairs of squares for $i = 1, 2$,



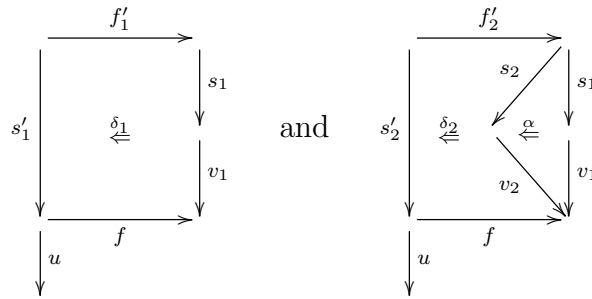
We obtain arrows and invertible 2-cells,



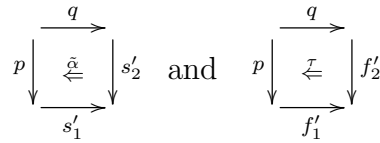
such that $u\bar{v}_i t_i \in \mathfrak{W}$ for $i = 1, 2$ and the composites of the following two pasting diagrams are equal:



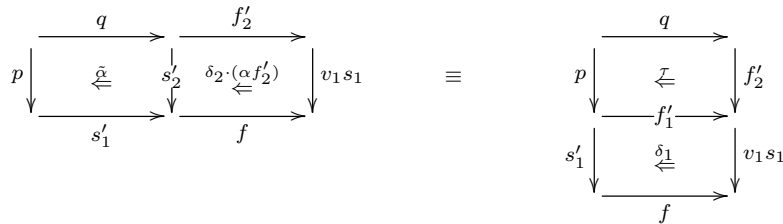
Now apply Proposition 2.9 to the following two squares, where $v_1 s_1, u, u s'_1, u s'_2 \in \mathfrak{W}$:



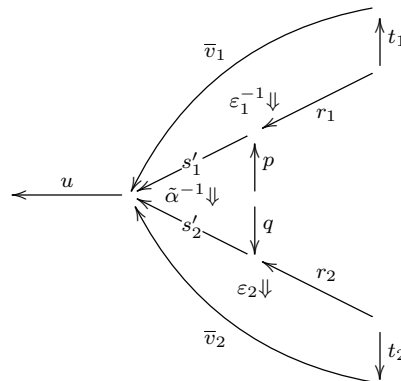
This gives us arrows and invertible 2-cells



such that $u s'_1 p \in \mathfrak{W}$ and the following two pasting diagrams give the same composite:



Thus far we have constructed the following part of the left-hand cell of the whiskered 2-cell diagram,



We fill in the gap in the middle by chosen liftings of chosen squares according to conditions [WB3] and [WB4]. First note that the $u \bar{v}_i t_i$ are in \mathfrak{W} for $i = 1, 2$, and hence since ε_i is

invertible, $us'_i r_i \in \mathfrak{W}$. So we have squares from [C2],

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \xrightarrow{p'} & \\
 r'_1 \downarrow & \rho'_1 \Downarrow & \downarrow us'_1 r_1 \\
 & \xrightarrow{us'_1 p} &
 \end{array}
 & \text{and} &
 \begin{array}{ccc}
 & \xrightarrow{q'} & \\
 r'_2 \downarrow & \rho'_2 \Downarrow & \downarrow us'_2 r_2 \\
 & \xrightarrow{us'_2 q} &
 \end{array}
 \end{array}$$

and we lift with respect to us'_1 and us'_2 respectively (as in [C3]) and add additional arrows w_1 and w_2 to obtain arrows $\bar{r}_1 = r'_1 \tilde{u}_1 w_1$ and $\bar{r}_2 = r'_2 \tilde{u}_2 w_2$ both in \mathfrak{W} . If we denote $\bar{p} = p' \tilde{u}_1 w_1$ and $\bar{q} = q' \tilde{u}_2 w_2$, we obtain invertible 2-cells

$$\begin{array}{ccc}
 & \xrightarrow{\bar{p}} & \\
 \bar{r}_1 \downarrow & \rho_1 \Downarrow & \downarrow r_1 \\
 & \xrightarrow{p} &
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 & \xrightarrow{\bar{q}} & \\
 \bar{r}_2 \downarrow & \rho_2 \Downarrow & \downarrow r_2 \\
 & \xrightarrow{q} &
 \end{array}$$

Finally, we take a chosen square according to [C2],

$$\begin{array}{ccc}
 & \xrightarrow{\tilde{r}_1} & \\
 \tilde{r}_2 \downarrow & \rho_3 \Downarrow & \downarrow \tilde{r}_2 \\
 & \xrightarrow{\tilde{r}_1} &
 \end{array}$$

with $\tilde{r}_2 \in \mathfrak{W}$. Since $us'_1 p, \bar{r}_1, \tilde{r}_2 \in \mathfrak{W}$, let x be a chosen arrow such that $us'_1 p \bar{r}_1 \tilde{r}_2 x \in \mathfrak{W}$. Then the result of the whiskering becomes:

(5)

3.7. REMARKS.

1. When the class \mathfrak{W} of arrows to be inverted satisfies the traditional **BF1–BF5** conditions from [5], this construction reduces to the construction given in that paper when one takes the identity arrow whenever a choice of an arrow based on condition

[WB2] is needed. The definition of horizontal whiskering here is not exactly the same as the one given in [5], but the 2-cell diagrams obtained are equivalent. This is shown in [12], where it is proved that various choices to fill the 2-cell diagrams for whiskering all result in equivalent 2-cell diagrams.

2. The fact that the horizontal whiskering operations described here are well-defined on equivalence classes of 2-cell diagrams is established in Appendix C, Propositions C.2 and C.3.

With these definitions, we get the following:

3.8. THEOREM. *For any bicategory \mathcal{B} with a class \mathfrak{W} of arrows that satisfies conditions [WB1]–[WB5], there is a bicategory of fractions $\mathcal{B}(\mathfrak{W}^{-1})$ with a homomorphism*

$$J_{\mathfrak{W}}: \mathcal{B} \rightarrow \mathcal{B}(\mathfrak{W}^{-1})$$

which sends arrows in \mathfrak{W} to internal equivalences. Moreover, this bicategory satisfies the following universal property: for any bicategory \mathcal{D} , composition with $J_{\mathfrak{W}}$ induces an equivalence of categories

$$\mathrm{Hom}(\mathcal{B}(\mathfrak{W}^{-1}), \mathcal{D}) \simeq \mathrm{Hom}_{\mathfrak{W}}(\mathcal{B}, \mathcal{D}),$$

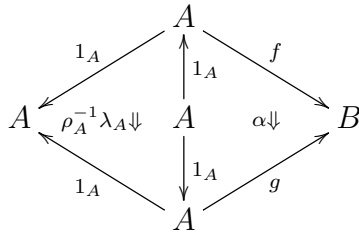
where $\mathrm{Hom}(\mathcal{B}(\mathfrak{W}^{-1}), \mathcal{D})$ denotes the category of homomorphisms and pseudo, resp. lax, resp. oplax, transformations and $\mathrm{Hom}_{\mathfrak{W}}(\mathcal{B}, \mathcal{D})$ denotes the subcategory of homomorphisms and pseudo, resp. lax, resp. oplax, transformations that send arrows in \mathfrak{W} to internal equivalences.

3.9. REMARKS.

1. We can speak of transformations sending arrows to internal equivalences by representing them through a pseudo functor into an appropriate bicategory of cylinders on \mathcal{D} (depending on the type of transformations). For pseudo transformations, the calculus of mates shows that $\mathrm{Hom}_{\mathfrak{W}}(\mathcal{B}, \mathcal{D})$ is a full subcategory of $\mathrm{Hom}(\mathcal{B}, \mathcal{D})$, but for lax and oplax transformations this is not the case in general.
2. The universal property phrased in terms of the pseudo transformations determines the bicategory of fractions up to equivalence of bicategories. The other two universal properties are invariant under equivalence of bicategories. Hence we may view this result as saying that whenever a class of arrows admits a calculus of fractions, its bicategorical localization will also have these other two universal properties.
3. The description of the bicategory of fractions given here depends on the choices made for arrows, squares and liftings used in composition. However, the universal property implies that any other choice would give a biequivalent bicategory of fractions. We actually have a stronger result here: we can give explicit pseudofunctors going back and forth that are the identity in all dimensions (objects, arrows and 2-cells), but

don't preserve horizontal composition strictly: composition in the domain bicategory may have been defined using a different square from the one used in the codomain bicategory. In Proposition A.1 we show that there is a canonical 2-cell between these two compositions. The property established in Proposition A.1 implies that they satisfy the coherence conditions to form the structure cells of a pseudo functor. Furthermore, these functors form commutative triangles with the $J_{\mathfrak{W}}$ functors from \mathcal{B} into the bicategories of fractions. We will also see in the next section that these bicategories of fractions are biequivalent to a bicategory of fractions as defined in [5].

PROOF. Analogous to the situation in [5], we define $J_{\mathfrak{W}}$ as follows: on objects $J_{\mathfrak{W}}(A) = A$; on arrows $J_{\mathfrak{W}}$ sends $A \xrightarrow{f} B$ to $A \xleftarrow{1_A} A \xrightarrow{f} B$; on 2-cells, $J_{\mathfrak{W}}$ sends $A \begin{matrix} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{matrix} B$ to



where ρ_A and λ_A are the right and left unitors respectively for 1_A . By the way we chose squares involving identity arrows, this gives a pseudo functor $\mathcal{B} \rightarrow \mathcal{B}(\mathfrak{W}^{-1})$ with structure cells as defined in [5]. The remainder of the proof goes as in [5]. We have given definitions for all of the composition operations in $\mathcal{B}(\mathfrak{W}^{-1})$ and shown them to be well-defined and suitably associative, sending arrows in \mathfrak{W} to internal equivalences. There are no coherence requirements on the choices of squares or liftings, so this gives a valid construction of a bicategory with all necessary properties.

The resulting homomorphism of bicategories has the same universal properties as the one for the original bicategory of fractions, since the proof of [5, Theorem 21] does not depend on any specific properties of the choices made. ■

A different way to derive this result will be given in Theorem 4.11.

4. Equivalences of Bicategories of Fractions

The first goal of this paper was to provide conditions under which we can take smaller classes of arrows to invert, while still obtaining an equivalent bicategory of fractions. In this section we develop a condition to allow us to restrict to a smaller subclass of arrows, namely when a subclass is weakly initial in the original class of arrows. This is related to the condition *WISC*, where we have weakly initial subsets of the class of arrows to be inverted. This was considered in [7] to obtain a locally essentially small bicategory of fractions.

We show that if we start with a class of arrows satisfying **[WB1]**–**[WB5]**, and we have a weakly initial subclass which satisfies **[WB1]** and **[WB5]**, then in fact the subclass will satisfy all the conditions **[WB1]**–**[WB5]** and the bicategory of fractions for the subclass is equivalent to the one for the original class of arrows. We will then apply this result to a class \mathfrak{W} of arrows satisfying **[WB1]**–**[WB5]**, and consider its closure under composition and invertible 2-cells, $\widehat{\mathfrak{W}}$. We show that $\widehat{\mathfrak{W}}$ satisfies the conditions **BF1**–**BF5** of [5], and that \mathfrak{W} is weakly initial in $\widehat{\mathfrak{W}}$. This gives an equivalence of bicategories

$$\mathcal{B}(\mathfrak{W}^{-1}) \simeq \mathcal{B}(\widehat{\mathfrak{W}}^{-1}),$$

giving another proof that the newly constructed bicategories of fractions of Section 3 are indeed equivalent to the ones introduced in [5].

4.1. WEAKLY INITIAL SUBCLASSES. We begin by reminding the reader of the notion of a weakly initial subclass of arrows. We will show that the new calculus of fractions conditions descend from a class to a weakly initial subclass.

4.2. DEFINITION. Let $\mathfrak{W} \subseteq \mathfrak{V}$ be two classes of arrows in a bicategory \mathcal{B} . Then \mathfrak{W} is *weakly initial* in \mathfrak{V} if for each arrow $v \in \mathfrak{V}$, there is an arrow u such that $vu \in \mathfrak{W}$.

4.3. PROPOSITION. *Let \mathcal{B} be a bicategory with a class of arrows \mathfrak{V} satisfying all the conditions **[WB1]**–**[WB5]**, and a subclass $\mathfrak{W} \subseteq \mathfrak{V}$ which is weakly initial in \mathfrak{V} and satisfies conditions **[WB1]** and **[WB5]**. Then \mathfrak{W} also satisfies conditions **[WB2]**–**[WB4]**.*

PROOF. [WB2] Let $A \xrightarrow{w_1} B$ and $B \xrightarrow{w_2} C$ be a pair of composable arrows in \mathfrak{W} . Since $\mathfrak{W} \subseteq \mathfrak{V}$ and \mathfrak{V} satisfies condition **[WB2]**, there is an arrow u_1 such that $w_2 w_1 u_1 \in \mathfrak{V}$. Since \mathfrak{W} is weakly initial in \mathfrak{V} , there is an arrow u_2 such that $w_2 w_1 u_1 u_2 \in \mathfrak{W}$. So \mathfrak{W} satisfies condition **[WB2]**.

[WB3] Consider a cospan of arrows $A \xrightarrow{f} C \xleftarrow{w} B$ with $w \in \mathfrak{W}$. Since \mathfrak{V} satisfies **[WB3]**, there is a square with an invertible 2-cell α ,

$$\begin{array}{ccc} D & \xrightarrow{g} & B \\ v \downarrow & \xleftarrow{\alpha} & \downarrow w \\ A & \xrightarrow{f} & C \end{array}$$

with $v \in \mathfrak{V}$. Since \mathfrak{W} is weakly initial in \mathfrak{V} , there is an arrow $(E \xrightarrow{u} D)$ such that $vu \in \mathfrak{W}$. Then the square

$$\begin{array}{ccc} E & \xrightarrow{gu} & B \\ vu \downarrow & \xleftarrow{\alpha u} & \downarrow w \\ A & \xrightarrow{f} & C \end{array}$$

shows that \mathfrak{W} satisfies condition **[WB3]**.

[WB4] Let $\alpha: wf \Rightarrow wg$ be a 2-cell with $w \in \mathfrak{W}$. Since $w \in \mathfrak{V}$ and \mathfrak{V} satisfies [WB4], there is an arrow $v \in \mathfrak{V}$ with a 2-cell $\beta: fv \Rightarrow gv$ such that $\alpha v = w\beta$. And since \mathfrak{W} is weakly initial in \mathfrak{V} , there is an arrow u such that $vu \in \mathfrak{W}$. Now take $w' = vu \in \mathfrak{W}$ and $\beta' = \beta u$. Then $w\beta' = \alpha w'$.

To check that \mathfrak{W} also satisfies the second part of [WB4], let (w'_1, β_1) and (w'_2, β_2) be pairs such that $w'_1, w'_2 \in \mathfrak{W}$, and $\beta_1: w'_1 f \Rightarrow w'_1 g$, $\beta_2: w'_2 f \Rightarrow w'_2 g$ such that $\alpha w'_1 = w\beta_1$ and $\alpha w'_2 = w\beta_2$. Since $w, w'_1, w'_2 \in \mathfrak{V}$ and we assume that \mathfrak{V} satisfies [WB4], there are arrows s, t such that $w'_1 s, w'_2 t \in \mathfrak{V}$, and an invertible 2-cell $\varepsilon: w'_1 s \Rightarrow w'_2 t$ such that

$$\begin{array}{ccc} fw'_1 s & \xrightarrow{\beta_1 s} & gw'_1 s \\ f\varepsilon \downarrow & & \downarrow g\varepsilon \\ fw'_2 t & \xrightarrow{\beta_2 t} & gw'_2 t \end{array}$$

commutes. Since $w'_1 s \in \mathfrak{V}$, there is an arrow u such that $w'_1 s u \in \mathfrak{W}$. Then $w'_2 t u \in \mathfrak{W}$ as well, since $\varepsilon u: w'_1 s u \Rightarrow w'_2 t u$ is an invertible 2-cell and \mathfrak{W} is closed under invertible 2-cells by condition [WB5]. So define $s' = su$, $t' = tu$, and $\varepsilon' = \varepsilon u: w'_1 s' \xrightarrow{\sim} w'_2 t'$ to obtain a commutative diagram

$$\begin{array}{ccc} fw'_1 s' & \xrightarrow{\beta_1 s'} & gw'_1 s' \\ f\varepsilon \downarrow & & \downarrow g\varepsilon' \\ fw'_2 t' & \xrightarrow{\beta_2 t'} & gw'_2 t' \end{array}$$

as required. ■

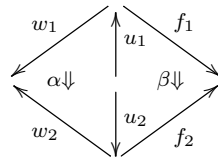
4.4. THEOREM. Let \mathcal{B} be a bicategory with a class of arrows \mathfrak{V} satisfying the conditions [WB1]-[WB5] and a class $\mathfrak{W} \subseteq \mathfrak{V}$ which is initial in \mathfrak{V} and satisfies [WB1] and [WB5]. Then there is an equivalence of bicategories $J: \mathcal{B}(\mathfrak{W}^{-1}) \rightarrow \mathcal{B}(\mathfrak{V}^{-1})$ that makes the following diagram commutative,

$$\begin{array}{ccc} & \mathcal{B}(\mathfrak{W}^{-1}) & \\ J_{\mathfrak{W}} \nearrow & & \downarrow J \\ \mathcal{B} & & \mathcal{B}(\mathfrak{V}^{-1}). \\ J_{\mathfrak{V}} \searrow & & \simeq \end{array}$$

PROOF. By the universal property of $\mathcal{B}(\mathfrak{V}^{-1})$ there is a canonical pseudo functor

$$J: \mathcal{B}(\mathfrak{W}^{-1}) \rightarrow \mathcal{B}(\mathfrak{V}^{-1}),$$

which is the identity on objects, sends the span (w, f) in $\mathcal{B}(\mathfrak{W}^{-1})$ to the span (w, f) in $\mathcal{B}(\mathfrak{V}^{-1})$ and maps the 2-cell represented by the diagram



in $\mathcal{B}(\mathfrak{W}^{-1})$ to the 2-cell represented by this same diagram in $\mathcal{B}(\mathfrak{V}^{-1})$. Note that J sends identity arrows to identity arrows and the comparison cells for compositions of arrows are the canonical 2-cells related to the choices of squares for composition in $\mathcal{B}(\mathfrak{W}^{-1})$ and $\mathcal{B}(\mathfrak{V}^{-1})$, as described in Remark 3.3.(1). It is clear that $J \circ J_{\mathfrak{W}} = J_{\mathfrak{V}}$ as required.

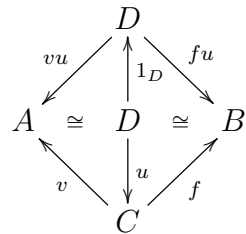
It is obvious that J is an isomorphism on objects. To show that it is essentially surjective on arrows, let

$$A \xleftarrow{v} C \xrightarrow{f} B$$

be an arrow in $\mathcal{B}(\mathfrak{V}^{-1})$. Since \mathfrak{W} is weakly initial in \mathfrak{V} , there is an arrow $(D \xrightarrow{u} C)$ such that $vu \in \mathfrak{W}$. So the span

$$A \xleftarrow{vu} D \xrightarrow{fu} B$$

is in the image of J . Furthermore, there is an invertible 2-cell



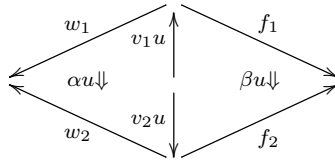
showing that J is essentially surjective on arrows.

It remains to show that J is fully faithful on 2-cells. To show that it is full on 2-cells, consider the 2-cell represented by the diagram,

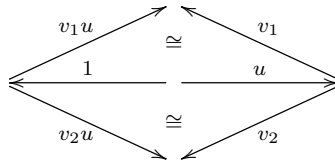


with $w_1, w_2 \in \mathfrak{W}$ and $w_1v_1, w_2v_2 \in \mathfrak{V}$. Since \mathfrak{W} is weakly initial in \mathfrak{V} , there is an arrow u

such that $w_1v_1u \in \mathfrak{W}$. Hence, the 2-cell represented by

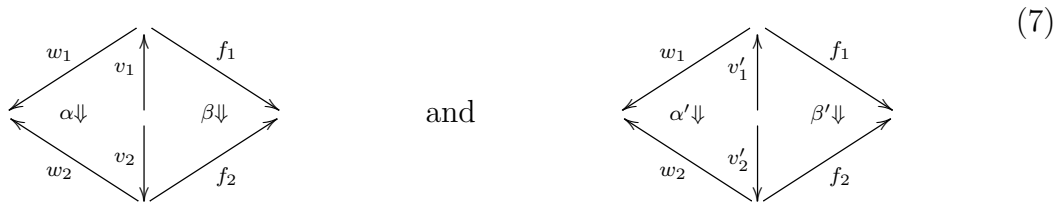


is in the image of J . This diagram represents the same 2-cell as (6), since the following diagram with unitor 2-cells gives an equivalence between them:

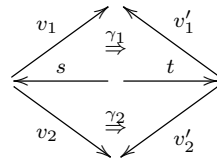


Hence (6) is in the image of J and we conclude that J is full on 2-cells.

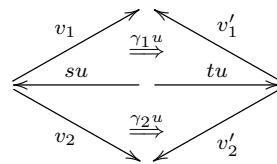
To verify that J is faithful on 2-cells, consider two 2-cells between the same spans of arrows



and suppose that these diagrams represent the same 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$. This means that there is an equivalence given by arrows s and t with 2-cells γ_1 and γ_2 as in



such that the appropriate diagrams of 2-cells commute and $w_1v_1s \in \mathfrak{W}$. Since \mathfrak{W} is weakly initial in \mathfrak{W} , there is an arrow u such that $w_1v_1su \in \mathfrak{W}$. So the diagram



represents an equivalence of the diagrams in (7) in $\mathcal{B}(\mathfrak{W}^{-1})$. We conclude that J is fully faithful on 2-cells, and hence is a biequivalence of bicategories. ■

4.5. **REMARK.** This theorem implies that the choices made in constructing the bicategory of fractions in Section 3 do not matter, since \mathfrak{W} is weakly initial in itself, and Theorem 4.4 provides an equivalence of bicategories created with different choices.

This result can be combined with the condition *WISC* given in [9] to obtain the following.

4.6. **COROLLARY.** *If \mathfrak{V} has a weakly initial subset \mathfrak{S}_X over each object X , and these subsets contain identities and are closed under 2-isomorphism (conditions [WB1] and [WB5]), then the arrows in the weakly initial subsets define a locally small bicategory of fractions $\mathcal{B}(\mathfrak{S}^{-1})$, equivalent to $\mathcal{B}(\mathfrak{V}^{-1})$.*

This strengthens the result in [9] where one would only get a locally essentially small bicategory of fractions.

4.7. **REMARK.** Our notion of a weakly initial class of arrows is a dual notion to that of the right saturation of a class of arrows defined in [13]. The right saturation enlarges the class of arrows to be inverted, rather than restricting to a smaller subclass.

The right saturation of a class \mathcal{W} of arrows consists of those arrows $f: C \rightarrow D$ for which there exist arrows $g: B \rightarrow C$ and $h: A \rightarrow B$ such that gh and fg are both in \mathcal{W} . If \mathcal{W} satisfies the conditions **BF1-BF5**, then so does its saturation, and the saturation gives rise to an equivalent bicategory of fractions. It is not difficult to use [WB3] to show that if $\mathcal{W} \subseteq \mathcal{V}$ is weakly initial in \mathcal{V} , then \mathcal{V} is a subset of the saturation of \mathcal{W} . This does not immediately imply the equivalence of the induced bicategories of fractions, because \mathcal{W} may not satisfy **BF2**. However, Theorem 4.4 implies that the equivalences of bicategories of fractions in [13] apply when we replace **BF2** with [WB2].

4.8. **REMARK.** In the case where one is only interested in obtaining a smaller version of $\mathcal{B}(\mathfrak{V}^{-1})(X, Y)$ for a particular object X (or for a particular class of objects) in the bicategory \mathcal{B} , there is a local version of Theorem 4.4. Given an object X in \mathcal{B} and a class of arrows \mathfrak{V} in \mathcal{B} , we say that a subclass $\mathfrak{A} \subseteq \mathfrak{V}$ is *weakly initial in \mathfrak{V} at X* when the class \mathfrak{A}/X of arrows in \mathfrak{A} with codomain X is weakly initial in the class \mathfrak{V}/X of arrows in \mathfrak{V} with codomain X . We write $\mathcal{B}_{\mathfrak{A}}(X, Y)$ for the category for spans from X to Y with reverse arrows in \mathfrak{A} and 2-cells defined using 2-cell diagrams with the appropriate composites in \mathfrak{A} . Now, if \mathfrak{V} satisfies conditions [WB1]–[WB5], $\mathfrak{A} \subseteq \mathfrak{V}$ satisfies condition [WB1] and [WB5], and \mathfrak{A} is weakly initial in \mathfrak{V} at X , there is an equivalence of categories

$$\mathcal{B}_{\mathfrak{A}}(X, Y) \xrightarrow{\sim} \mathcal{B}(\mathfrak{V}^{-1})(X, Y),$$

for any object Y in \mathcal{B} .

4.9. **CLOSURE UNDER COMPOSITION.** Given a class of arrows \mathfrak{W} in a bicategory \mathcal{B} , let $\widehat{\mathfrak{W}}$ denote the class obtained from \mathfrak{W} by closure under composition and invertible 2-cells. So $\widehat{\mathfrak{W}}$ is the smallest class of arrows in \mathcal{B} such that

- $\mathfrak{W} \subseteq \widehat{\mathfrak{W}}$;

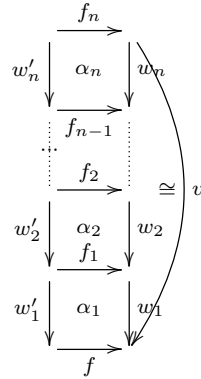
- If $f_1, f_2 \in \widehat{\mathfrak{W}}$, and $f_2 \circ f_1$ is defined, then $f_2 \circ f_1 \in \widehat{\mathfrak{W}}$;
- If $f \in \widehat{\mathfrak{W}}$ and $\alpha: f \xrightarrow{\sim} g$ is an invertible 2-cell in \mathcal{B} , then $g \in \widehat{\mathfrak{W}}$.

Then $\widehat{\mathfrak{W}}$ has the stronger property **BF2**, and each arrow $w \in \widehat{\mathfrak{W}}$ will have an invertible 2-cell $\alpha: w \xrightarrow{\sim} w_n \circ \cdots \circ w_1$ with codomain a finite composite of arrows $w_1, \dots, w_n \in \mathfrak{W}$.

4.10. LEMMA. *If \mathfrak{W} satisfies the conditions [WB1]–[WB5], then $\widehat{\mathfrak{W}}$ defines a wide subcategory which satisfies the conditions from [5] for constructing a bicategory of fractions.*

PROOF. Since \mathfrak{W} contains all identities, so does $\widehat{\mathfrak{W}}$, so $\widehat{\mathfrak{W}}$ satisfies condition **BF1** from [5]. And $\widehat{\mathfrak{W}}$ has been created to be closed under composition, verifying **BF2**. Conditions **BF3–BF5** are equivalent to conditions [WB3]–[WB5] (and **BF3** and **BF5** are identical to their weaker versions); see Remark 2.2. So it suffices to check conditions [WB3]–[WB5] for $\widehat{\mathfrak{W}}$.

Since every arrow v in $\widehat{\mathfrak{W}}$ is isomorphic to a composition $w_1 \circ \dots \circ w_n$ of finitely many arrows in \mathfrak{W} , repeated application of [WB3] for \mathfrak{W} gives us [WB3] for $\widehat{\mathfrak{W}}$:



Note that $w'_n \circ \cdots \circ w'_1 \in \widehat{\mathfrak{W}}$ by definition.

To verify condition [WB4], suppose that $\alpha: wf \Rightarrow wg$ and $\gamma: w_n \cdots w_1 \xrightarrow{\sim} w$ with $w_1, \dots, w_n \in \mathfrak{W}$. Repeatedly applying [WB4] for \mathfrak{W} gives us arrows w'_{n-k} and 2-cells $\beta_{n-k}: w_{n-k-1} \cdots w_1 f w'_n \cdots w'_{n-k} \Rightarrow w_{n-k-1} \cdots w_1 g w'_n \cdots w'_{n-k}$ for $k = 0, \dots, n - 1$ such that $w_{n-k} \cdots w_{n-1} w_n \beta_{n-k} = ((\gamma^{-1}g) \cdot \alpha \cdot (\gamma f)) w'_n w'_{n-1} \cdots w'_{n-k}$. So β_1 with $w'_n w'_{n-1} \cdots w'_1$ is the required lifting.

To check the compatibility condition in [WB4], consider $\alpha: wf \Rightarrow wg$ with liftings $\alpha': fw' \Rightarrow gw'$ and $\alpha'': fw'' \Rightarrow gw''$. Since $w', w'' \in \widehat{\mathfrak{W}}$, there are arrows w'_1, \dots, w'_k and w''_1, \dots, w''_ℓ in \mathfrak{W} with invertible 2-cells, $\delta: w'_k \cdots w'_1 \Rightarrow w'$ and $\gamma: w''_\ell \cdots w''_1 \Rightarrow w''$. By repeatedly applying condition [WB2] for \mathfrak{W} there are arrows u', u'' such that $w'u' \in \mathfrak{W}$ and $w''u'' \in \mathfrak{W}$. Hence we can apply [WB4] for \mathfrak{W} to the liftings $\alpha'u': fw'u' \Rightarrow gw'u'$ and $\alpha''u'': fw''u'' \Rightarrow gw''u''$ and obtain arrows s, t and a 2-cell $\varepsilon: w'u's \Rightarrow w''u''t$ showing compatibility of these liftings. This then gives us also the required arrows $u's$ and $u''t$ with the cell ε to establish compatibility for the original liftings.

Finally, $\widehat{\mathfrak{W}}$ satisfies condition **BF5** by construction. ■

4.11. THEOREM. *If \mathfrak{W} satisfies the conditions [WB1]–[WB5], then there is an equivalence of bicategories $J: \mathcal{B}(\mathfrak{W}^{-1}) \xrightarrow{\sim} \mathcal{B}(\widehat{\mathfrak{W}}^{-1})$, making the following triangle commute,*

$$\begin{array}{ccc}
 & \mathcal{B}(\mathfrak{W}^{-1}) & \\
 J_{\mathfrak{W}} \nearrow & & \downarrow J \\
 \mathcal{B} & & \mathcal{B}(\widehat{\mathfrak{W}}^{-1}) \\
 J_{\widehat{\mathfrak{W}}} \searrow & &
 \end{array}$$

where $\mathcal{B}(\widehat{\mathfrak{W}}^{-1})$ is the bicategory of fractions from [5] and $\mathcal{B}(\mathfrak{W}^{-1})$ is the bicategory of fractions defined in Section 3.

PROOF. We have shown that whenever a class of arrows \mathfrak{W} satisfies the stronger conditions **BF1–BF5**, the resulting bicategory of fractions is equivalent to the traditional one from [5]; see Remarks 3.7(1) and 4.5. So $\mathcal{B}(\widehat{\mathfrak{W}}^{-1})$ may be taken to be the classical bicategory of fractions and Theorem 4.4 now gives us the equivalence of the resulting bicategories of fractions. ■

4.12. COROLLARY. *When \mathfrak{W} satisfies the conditions [WB1]–[WB5], the pseudo functor $J_{\mathfrak{W}}: \mathcal{B} \rightarrow \mathcal{B}(\mathfrak{W}^{-1})$ satisfies the universal property for the bicategory of fractions.*

PROOF. A pseudo functor $\mathcal{B} \rightarrow \mathcal{C}$ sends the arrows in \mathfrak{W} to equivalences if and only if it sends the arrows in $\widehat{\mathfrak{W}}$ to equivalences. ■

This result also applies to results for categories of fractions in the 1-category case given in [3].

4.13. COROLLARY. *A class of arrows W in a category \mathcal{C} allows for the construction of a category of right fractions $\mathcal{C}[W^{-1}]$ if it satisfies the following conditions:*

1. W contains all identities;
2. For any pair of composable arrows $B \xrightarrow{v} C \xrightarrow{w} D$ in W there is an arrow $A \xrightarrow{u} B$ such that $A \xrightarrow{wv} D$ is in W ;
3. For any arrow $w \in W$ and any arrow f which shares its codomain with w , there is an arrow $w' \in W$ and an arrow f' such that the following square is defined and commutes:

$$\begin{array}{ccc}
 & \xrightarrow{f'} & \\
 w' \downarrow & & \downarrow w \\
 & \xrightarrow{f} &
 \end{array}$$

4. Given $w \in W$ and parallel arrows f_1, f_2 such that $wf_1 = wf_2$, then there is an arrow $w' \in W$ such that $f_1w' = f_2w'$,

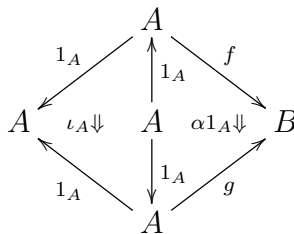
$$\begin{array}{c} w' \longrightarrow \xrightarrow{f_1} w \\ \xrightarrow{f_2} \end{array}$$

4.14. EXAMPLES.

1. When one wants to add the inverse for an arrow w in a monoid, the class W in the traditional Gabriel-Zisman construction of [3] would be required to contain all powers of w . In our case W only needs to contain a cofinal set of powers of w .
2. Consider the category of atlases and atlas maps for manifolds. In order to obtain the category containing all smooth maps between manifolds using the original conditions, one needs to take the category of fractions with respect to all atlas refinements. With the new theory we may restrict ourselves to refinements in which no charts are repeated, or any other family of refinements that is weakly initial among all refinements.

5. Simplifying 2-Cell Representatives

As we have seen, the universal homomorphism $J_{\mathfrak{W}}: \mathcal{B} \rightarrow \mathcal{B}(\mathfrak{W}^{-1})$ is defined to be the identity on objects, and takes an arrow $f: A \rightarrow B$ to the generalized arrow $A \xleftarrow{1_A} A \xrightarrow{f} B$ and a 2-cell $\alpha: f \Rightarrow g$ to a 2-cell diagram of the form below.



As Tommasini observed in Remark 3.5 of [12], this homomorphism is neither 2-full nor 2-faithful in general. The map $J_{\mathfrak{W}}$ fails to be 2-full because not every 2-cell between $J_{\mathfrak{W}}(f)$ and $J_{\mathfrak{W}}(g)$ needs to have a representative of this particular form. The map $J_{\mathfrak{W}}$ fails to be 2-faithful because two 2-cell diagrams of this form, say with distinct right cells β and γ , represent the same 2-cell in the bicategory of fractions when there is an arrow $t \in \mathfrak{W}$ such that $\beta t = \gamma t$. This leads us to consider the more general issue of the equivalence relation on the 2-cell diagrams.

In this section we discuss some variations of [WB4] and consider when a 2-cell in the bicategory of fractions can be represented by a 2-cell diagram with a given left-hand side. In the following section, we will look at choosing these left-hand sides to have nice additional properties that will simplify some of the composition constructions. In some

cases representatives with a given left-hand side will even be unique. We will prove in [6] that some of these properties hold for the case of essential equivalences between orbifold étale groupoids. In fact they apply more generally to any fully faithful maps between étale topological groupoids.

Following the notation of [1] and [7] we say that an arrow $f: A \rightarrow B$ in a bicategory \mathcal{B} has a property \mathcal{P} when the induced functor $f_*: \mathcal{B}(X, A) \rightarrow \mathcal{B}(X, B)$ has this property. Note that for full and faithful, these properties are closely related to Condition [WB4]. In this section we will see that if the arrows in \mathfrak{W} have these properties and/or their duals, we are able to simplify our description of the 2-cells in the bicategory of fractions: each 2-cell will have a representative with a given left-hand side and we won't need equivalence classes if we have chosen representatives.

5.1. DEFINITION. An arrow w in a bicategory \mathcal{B} is

- *full* if for any 2-cell $\alpha: wf \Rightarrow wg$ there is a 2-cell $\tilde{\alpha}: f \Rightarrow g$ such that $w\tilde{\alpha} = \alpha$.
- *fully faithful* or *ff* if for any 2-cell $\alpha: wf \Rightarrow wg$ there is a *unique* 2-cell $\tilde{\alpha}: f \Rightarrow g$ such that $w\tilde{\alpha} = \alpha$.
- *co-full* if for any 2-cell $\alpha: fw \Rightarrow gw$ there is a 2-cell $\alpha': f \Rightarrow g$ such that $\alpha'w = \alpha$.
- *co-fully-faithful* or *co-ff* if for any 2-cell $\alpha: fw \Rightarrow gw$ there is a *unique* 2-cell $\alpha': f \Rightarrow g$ such that $\alpha'w = \alpha$.

Fractions condition [WB4] connects some of these properties as follows:

5.2. LEMMA. *If a class of arrows \mathfrak{W} satisfies condition [WB4] and is co-fully-faithful it is also full.*

PROOF. Consider a 2-cell $\alpha: wf \Rightarrow wg$ with $w \in \mathfrak{W}$. Since \mathfrak{W} satisfies [WB4] there is an arrow $v \in \mathfrak{W}$ with a 2-cell $\beta: fv \Rightarrow gv$ such that $w\beta = \alpha v$. Since \mathfrak{W} is co-full, there is a 2-cell $\tilde{\alpha}: f \Rightarrow g$ such that $\beta = \tilde{\alpha}v$. Hence, $w\tilde{\alpha}v = w\beta = \alpha v$. Since \mathfrak{W} is co-fully-faithful this implies that $w\tilde{\alpha} = \alpha$. ■

5.3. LEMMA. *Let \mathfrak{W} be a class of co-full arrows in \mathcal{B} satisfying the conditions [WB1]–[WB5]. Given any 2-cell diagram*

$$\begin{array}{ccc}
 & & f_1 \\
 & u_1 \nearrow & \uparrow v_1 \\
 & \alpha \Downarrow & \beta \Downarrow \\
 & u_2 \searrow & \downarrow v_2 \\
 & & f_2
 \end{array}
 \tag{8}$$

in $\mathcal{B}(\mathfrak{W}^{-1})$ and any square

$$\begin{array}{ccc}
 & t_1 \rightarrow & \\
 t_2 \downarrow & \gamma & \downarrow u_1 \\
 & \leftarrow & \\
 & u_2 \rightarrow &
 \end{array}$$

in \mathcal{B} with $u_1 t_1 \in \mathfrak{W}$, there is a 2-cell δ such that the diagram

$$\begin{array}{ccc}
 & & f_1 \\
 & \nearrow & \uparrow t_1 \\
 u_1 & & \\
 & \searrow & \downarrow \delta \\
 & & f_2 \\
 & \nwarrow & \uparrow t_2 \\
 & & u_2
 \end{array}
 \tag{9}$$

represents the same 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ as (8).

PROOF. By [WB3] there is a square

$$\begin{array}{ccc}
 & \xrightarrow{\bar{t}_1} & \\
 \bar{v}_1 \downarrow & & \downarrow v_1 \\
 & \theta \Downarrow & \\
 & \xrightarrow{t_1} & \xrightarrow{u_1} \\
 & & \downarrow u_1
 \end{array}$$

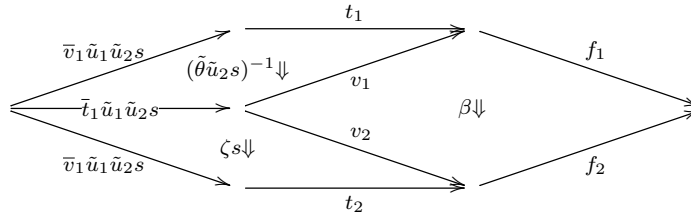
with $\bar{v}_1 \in \mathfrak{W}$ and θ invertible. By [WB4] there is an arrow $\tilde{u}_1 \in \mathfrak{W}$ and an invertible 2-cell $\tilde{\theta}: (v_1 \bar{t}_1) \tilde{u}_1 \Rightarrow (t_1 \bar{v}_1) \tilde{u}_1$. Now consider the pasting of the diagram

$$\begin{array}{ccccc}
 & & v_2 & & \\
 & \nearrow & \rightarrow & \searrow & \\
 \bar{t}_1 \tilde{u}_1 & & & \alpha^{-1} \Downarrow & u_2 \\
 & \searrow & v_1 & & \downarrow \gamma \\
 & & & t_1 & \rightarrow u_1 \\
 \bar{v}_1 \tilde{u}_1 & & \nearrow & \downarrow \gamma & \\
 & & t_2 & & \searrow u_2
 \end{array}
 \tag{10}$$

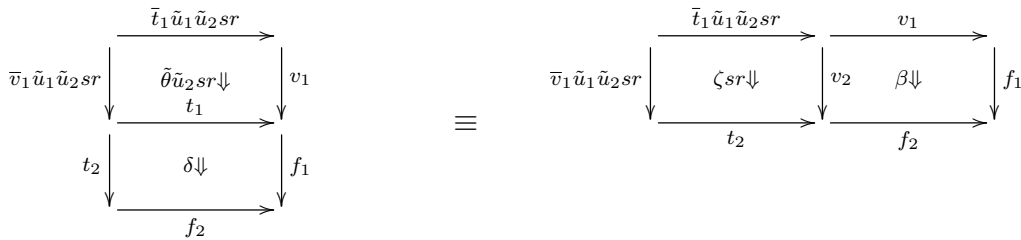
By [WB4] there is an arrow $\tilde{u}_2 \in \mathfrak{W}$ with an invertible 2-cell $\zeta: (v_2(\bar{t}_1 \tilde{u}_1)) \tilde{u}_2 \Rightarrow (t_2(\bar{v}_1 \tilde{u}_1)) \tilde{u}_2$ such that $u_2 \zeta$ is equal to the pasting of the cells in (10) composed with \tilde{u}_2 . Finally, we need to ensure that certain compositions of arrows are in \mathfrak{W} . First consider the composition of arrows $\bar{v}_1 \tilde{u}_1 \tilde{u}_2$. Each of the three arrows in this composition is in \mathfrak{W} , so by [WB2] there is an arrow s such that $\bar{v}_1 \tilde{u}_1 \tilde{u}_2 s \in \mathfrak{W}$. Furthermore, $u_2 t_2 \in \mathfrak{W}$ as well, so there is an arrow r such that $(u_2 t_2)(\bar{v}_1 \tilde{u}_1 \tilde{u}_2 s) r \in \mathfrak{W}$. Then we have the following equality of pastings of 2-cells:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \xrightarrow{\bar{t}_1 \tilde{u}_1 \tilde{u}_2 sr} & \\
 \bar{v}_1 \tilde{u}_1 \tilde{u}_2 sr \downarrow & & \downarrow v_1 \\
 & \tilde{\theta} \tilde{u}_2 sr \Downarrow & \\
 & t_1 & \\
 & \xrightarrow{\quad} & \\
 t_2 \downarrow & & \downarrow u_1 \\
 & \gamma \Downarrow & \\
 & \xrightarrow{\quad} & \\
 & u_2 &
 \end{array}
 & \equiv &
 \begin{array}{ccc}
 & \xrightarrow{\bar{t}_1 \tilde{u}_1 \tilde{u}_2 sr} & \xrightarrow{v_1} \\
 \bar{v}_1 \tilde{u}_1 \tilde{u}_2 sr \downarrow & & \downarrow v_2 \\
 & \zeta sr \Downarrow & \alpha \Downarrow \\
 & \xrightarrow{\quad} & \xrightarrow{\quad} \\
 & t_2 & u_2 \\
 & & \downarrow u_1
 \end{array}
 \end{array}$$

We want to construct a cell δ such that β and δ fit into a similar equality of 2-cell pastings. So consider the following pasting diagram,



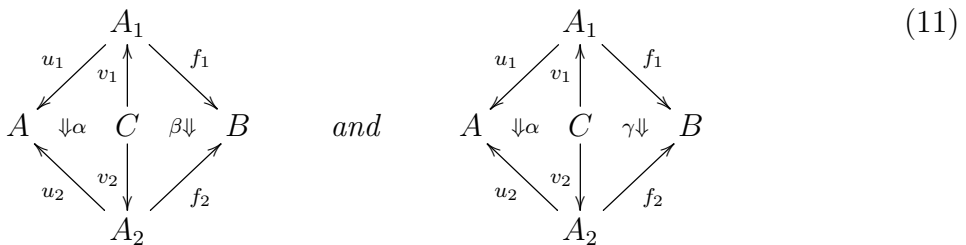
Since the arrows in \mathfrak{W} are co-full, there is a 2-cell $\delta: f_1 t_1 \Rightarrow f_2 t_2$ such that $\delta \bar{v}_1 \tilde{u}_1 \tilde{u}_2 s$ is equal to the pasting of this diagram. Then we get that



and hence we conclude that with δ thus defined, (9) is equivalent to (8). ■

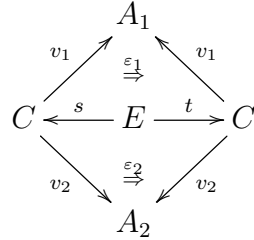
We now want to address the question about uniqueness of 2-cell representatives with a given left-hand side. The following is the closest we can get to uniqueness for 2-cell diagrams with a given left-hand side without adding any further conditions on the class \mathfrak{W} . This result is due to Matteo Tommasini [11], who first pointed it out to us and proved it. We include it here with his permission, with a different proof.

5.4. PROPOSITION. *Let \mathfrak{W} be a class of arrows satisfying conditions [WB1]–[WB5]. Let*



be two equivalent 2-cell diagrams. Then there exists an arrow $w: D \rightarrow C$ such that $u_1 v_1 w \in \mathfrak{W}$ and $\beta w = \gamma w$.

PROOF. Since the two 2-cell diagrams in (11) are equivalent there is a diagram with invertible 2-cells,



with $u_1 v_1 s \in \mathfrak{W}$ such that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \xrightarrow{s} & & \\
 t \downarrow & \varepsilon_1 \Downarrow & v_1 \downarrow \\
 \xleftarrow{-v_1} & & \downarrow \alpha \\
 v_2 \downarrow & \alpha \Downarrow & u_1 \downarrow \\
 \xrightarrow{u_2} & &
 \end{array} & \equiv & \begin{array}{ccc}
 \xrightarrow{s} & \xrightarrow{v_1} & \\
 t \downarrow & \varepsilon_2 \Downarrow & v_2 \downarrow & \alpha \Downarrow & u_1 \downarrow \\
 \xrightarrow{v_2} & \xrightarrow{u_2} & & &
 \end{array}
 \end{array} \tag{12}$$

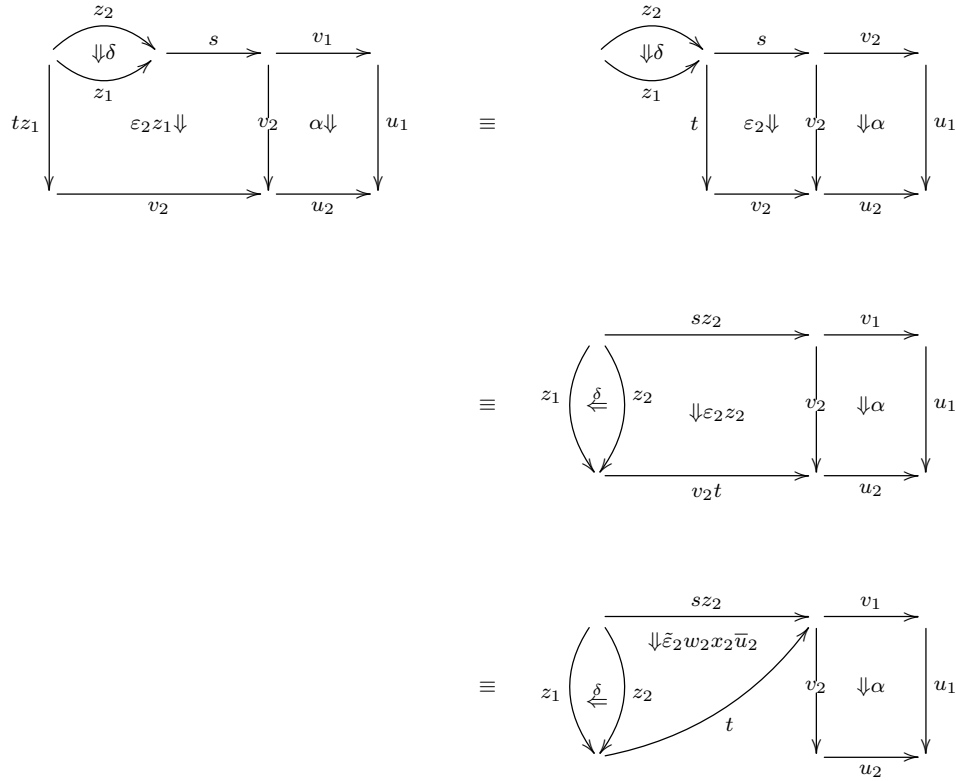
and

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \xrightarrow{s} & & \\
 t \downarrow & \varepsilon_1 \Downarrow & v_1 \downarrow \\
 \xleftarrow{-v_1} & & \downarrow \gamma \\
 v_2 \downarrow & \gamma \Downarrow & f_1 \downarrow \\
 \xrightarrow{f_2} & &
 \end{array} & \equiv & \begin{array}{ccc}
 \xrightarrow{s} & \xrightarrow{v_1} & \\
 t \downarrow & \varepsilon_2 \Downarrow & v_2 \downarrow & \beta \Downarrow & f_1 \downarrow \\
 \xrightarrow{v_2} & \xrightarrow{f_2} & & &
 \end{array}
 \end{array} \tag{13}$$

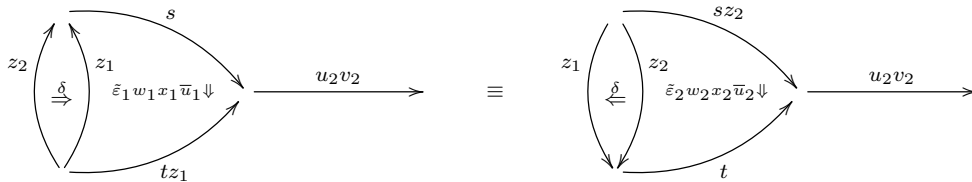
We want to use the first equation to derive a relationship between ε_1 and ε_2 . To make it possible to cancel α we first apply Proposition 2.5 to $u_1 \varepsilon_1: u_1 v_1 s \Rightarrow u_1 v_1 t$ to obtain an arrow $\tilde{u}_1: E_1 \rightarrow E$ in \mathfrak{W} and an invertible 2-cell $\tilde{\varepsilon}_1: s\tilde{u}_1 \rightarrow t\tilde{u}_1$ such that $u_1 v_1 \tilde{\varepsilon}_1 = u_1 \varepsilon_1 \tilde{u}_1$. Furthermore, by Lemma 2.7 there is an arrow $w_1: E'_1 \rightarrow E_1$ in \mathfrak{W} such that $v_1 \tilde{\varepsilon}_1 w_1 = \varepsilon_1 \tilde{u}_1 w_1$. Similarly, applying Proposition 2.5 to $u_2 \varepsilon_2: u_2 v_2 s \Rightarrow u_2 v_2 t$ gives us an arrow $\tilde{u}_2: E_2 \rightarrow E$ in \mathfrak{W} with an invertible 2-cell $\tilde{\varepsilon}_2: s\tilde{u}_2 \rightarrow t\tilde{u}_2$ such that $u_2 v_2 \tilde{\varepsilon}_2 = u_2 \varepsilon_2 \tilde{u}_2$ and there is an arrow $w_2: E'_2 \rightarrow E_2$ in \mathfrak{W} such that $v_2 \tilde{\varepsilon}_2 w_2 = \varepsilon_2 \tilde{u}_2 w_2$. By condition [WB2], let $x_i: E''_i \rightarrow E'_i$ (for $i = 1, 2$) be arrows such that $\tilde{u}_i w_i x_i \in \mathfrak{W}$ for $i = 1, 2$. Now apply condition [WB3] to obtain an invertible 2-cell

$$\begin{array}{ccc}
 F & \xrightarrow{\bar{u}_2} & E_1 \\
 \bar{u}_1 \downarrow & \delta \Downarrow & \downarrow \tilde{u}_2 w_2 x_2 \\
 E_1 & \xrightarrow{\tilde{u}_1 w_1 x_1} & E
 \end{array}$$

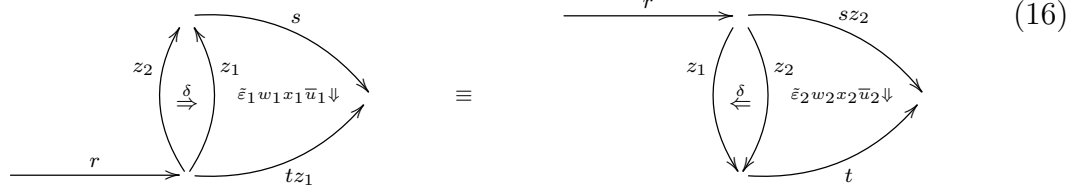
Similarly, we rewrite the right-hand side of (14) as follows:



By composing with $\alpha^{-1}tz_1$ with the rewritten left and right-hand sides of (14) we derive that



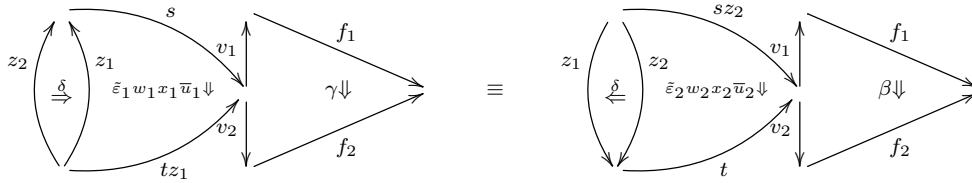
By Lemma 2.7 there is an arrow $(r : G \rightarrow F) \in \mathfrak{W}$ such that



Finally there is an arrow $r' : D \rightarrow G$ such that $u_1v_1sz_1rr' \in \mathfrak{W}$.

We will now combine this result with (15). We first manipulate ε_1 and ε_2 with δ just as we have done above. Note that we did not need the presence of u_1 or u_2 for this, so

the same calculations apply to the compositions with β and γ . This gives us



Now precomposing by rr' and using the result from (16) gives us that β and γ become equal when precomposed by the same invertible cell. So we can conclude that $\beta sz_2 rr' = \gamma sz_2 rr'$ and since $u_1 v_1 sz_2 rr' \cong u_1 v_1 sz_1 rr' \in \mathfrak{W}$, we also have that $u_1 v_1 sz_2 rr' \in \mathfrak{W}$ by [WB5]. So $w = sz_2 rr': D \rightarrow C$ has the required property. ■

We use this result together with the condition that the arrows in \mathfrak{W} be co-fully-faithful to obtain uniqueness of 2-cell representatives with a given left-hand side. The following lemma, proved by Matteo Tommasini [11] and included here with his permission, gives us a key ingredient.

5.5. LEMMA. *Let \mathfrak{W} be a class of arrows satisfying conditions [WB1]–[WB5] and let $a: B \rightarrow A$ and $b: C \rightarrow B$ be arrows such that both a and ab are in \mathfrak{W} . Then there is an arrow $c: D \rightarrow C$ such that $bc \in \mathfrak{W}$.*

PROOF. Since $ab \in \mathfrak{W}$, condition [WB3] gives us the existence of a square with an invertible 2-cell,

$$\begin{array}{ccc} X & \xrightarrow{u} & C \\ v \downarrow & \cong & \downarrow ab \\ B & \xrightarrow{a} & A \end{array}$$

with $v \in \mathfrak{W}$. Since $a \in \mathfrak{W}$, we can apply Proposition 2.5 to $\alpha: a(bu) \xrightarrow{\cong} av$ to obtain an arrow $w: Y \rightarrow X$ in \mathfrak{W} and an invertible 2-cell $\tilde{\alpha}: buw \xrightarrow{\cong} vw$. Since both v and w are in \mathfrak{W} , there is an arrow $z: D \rightarrow Y$ such that $vwz \in \mathfrak{W}$ by condition [WB2]. Now $\tilde{\alpha}z: buwz \xrightarrow{\cong} vwz$, so $buwz \in \mathfrak{W}$ by condition [WB5]. Hence $c = uwz: D \rightarrow C$ has the required property. ■

5.6. THEOREM. *Let \mathfrak{W} be a class of co-ff arrows in a bicategory \mathcal{B} satisfying conditions [WB1]–[WB5]. Then each 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ has at most one representative with a given left-hand 2-cell.*

PROOF. Given two 2-cell diagrams with the same left-hand side as in (11), Proposition 5.4 gives us an arrow w such that $u_1 v_1 w \in \mathfrak{W}$ and $\beta w = \gamma w$. Since $u_1 v_1 \in \mathfrak{W}$ we can apply Lemma 5.5 to obtain an arrow $x: D' \rightarrow D$ such that $wx \in \mathfrak{W}$. Now we have that $\beta wx = \gamma wx$ and since the arrows in \mathfrak{W} are co-ff we conclude that $\beta = \gamma$. ■

5.7. COROLLARY. *Let \mathfrak{W} be a class of co-ff arrows in a bicategory \mathcal{B} satisfying conditions [WB1]–[WB5]. Then each 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ has precisely one representative with a given left-hand 2-cell.*

PROOF. This follows from Lemma 5.3 and Theorem 5.6. ■

5.8. **REMARK.** This provides further understanding in regard to the results provided in [1] and [7] where no equivalence relation is needed for the 2-cells in the localizations: Abbad and Vitale introduce a category of so called *faithful fractions* where the objects are arrows in \mathfrak{W} and hom-categories are hom-categories in the original bicategory between the domains of the objects. Roberts uses these conditions to obtain a description of the 2-cells in his bicategory of fractions that can be viewed as the classical a 2-cell diagram with a strict pullback square as left-hand 2-cell. In the next section we will work out the case where one has pseudo pullbacks for arrows in \mathfrak{W} .

5.9. **COROLLARY.** *Suppose that \mathfrak{W} be a class of co-ff arrows in a bicategory \mathcal{B} satisfying conditions [WB1]–[WB5]. Then the universal homomorphism $J_{\mathfrak{W}}: \mathcal{B} \rightarrow \mathcal{B}(\mathfrak{W}^{-1})$ is 2-full and 2-faithful.*

PROOF. To show that the homomorphism is 2-full, consider an arbitrary 2-cell between $J_{\mathfrak{W}}(f)$ and $J_{\mathfrak{W}}(g)$. This will have a representative of the form

$$\begin{array}{ccccc}
 & & A & & \\
 & \swarrow & \uparrow & \searrow & \\
 & 1_A & s & f & \\
 A & \xrightarrow{\alpha \Downarrow} & C & \xrightarrow{\beta \Downarrow} & B \\
 & \swarrow & \downarrow & \searrow & \\
 & 1_A & t & g & \\
 & & A & &
 \end{array}$$

Now consider the square

$$\begin{array}{ccc}
 A & \xrightarrow{1_A} & A \\
 1_A \downarrow & \iota_A & \downarrow 1_A \\
 A & \xrightarrow{1_A} & A
 \end{array}$$

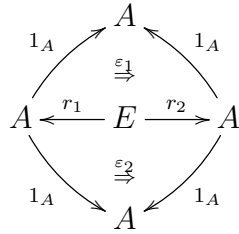
and Lemma 5.3 says that we can represent the 2-cell between $J_{\mathfrak{W}}(f)$ and $J_{\mathfrak{W}}(g)$ using this square on the left side. Thus, the 2-cell is the image of a 2-cell in \mathcal{B} .

To show that the map $J_{\mathfrak{W}}$ is 2-faithful, suppose that we have two 2-cells $J_{\mathfrak{W}}(\alpha)$ and $J_{\mathfrak{W}}(\beta)$, represented by

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & A & & \\
 & \swarrow & \uparrow & \searrow & \\
 & 1_A & 1_A & f & \\
 A & \xrightarrow{\iota_A} & A & \xrightarrow{\alpha 1_A \Downarrow} & B \\
 & \swarrow & \downarrow & \searrow & \\
 & 1_A & 1_A & g & \\
 & & A & &
 \end{array} & \text{and} & \begin{array}{ccccc}
 & & A & & \\
 & \swarrow & \uparrow & \searrow & \\
 & 1_A & 1_A & f & \\
 A & \xrightarrow{\iota_A} & A & \xrightarrow{\beta 1_A \Downarrow} & B \\
 & \swarrow & \downarrow & \searrow & \\
 & 1_A & 1_A & g & \\
 & & A & &
 \end{array} & (17)
 \end{array}$$

which represent the same 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$. Then there must be maps $r_1, r_2: E \rightrightarrows A$ with

2-cells $\varepsilon_1, \varepsilon_2$ as in

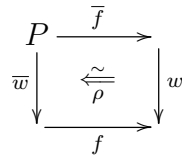


satisfying the equations to make the two diagrams in (17) equivalent and such that $1_A 1_A r_1 \in \mathfrak{W}$. Write $\varepsilon'_1, \varepsilon'_2: r_1 \Rightarrow r_2$ for the induced 2-cells. Since the left-hand squares are just identities, this implies that $\varepsilon'_1 = \varepsilon'_2: r_1 \Rightarrow r_2$. The other equation then implies that $\alpha \circ \varepsilon'_1 = \beta \circ \varepsilon'_1$. Since ε'_1 is invertible, this implies that $\alpha r_1 = \beta r_1$.

Since $1_A 1_A r_1 \in \mathfrak{W}$, we conclude by **[WB5]** that $r_1 \in \mathfrak{W}$. Hence, since the arrows in \mathfrak{W} are co-ff, we get that there is a unique $\gamma: f \Rightarrow g$ such that $\gamma r_1 = \alpha r_1$. Hence, $\alpha = \beta$. ■

6. Bicategories with Pseudo Pullbacks

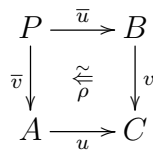
We now apply the ideas of Section 5 to represent generalized 2-cells using pseudo pullbacks. If a bicategory has all pseudo pullbacks of the form



where $w \in \mathfrak{W}$, and the class \mathfrak{W} is stable under these pseudo pullbacks in the sense that $w \in \mathfrak{W}$ implies that $\bar{w} \in \mathfrak{W}$, it is possible to use the pseudo pullbacks as chosen squares as in **[C2]** of Notation 3.2 in the construction of $\mathcal{B}(\mathfrak{W}^{-1})$. This makes the construction of this bicategory more canonical; see **[13]** for instance.

We are interested in a different use of the pseudo pullbacks: as the left-hand sides of the generalized 2-cell diagrams. (The case with strict pullbacks was considered in **[7]**.) This will allow us to simplify the horizontal composition operations. It will require some additional assumptions on \mathcal{B} and \mathfrak{W} , so we will develop conditions under which each 2-cell has a representative diagram where α is a pseudo pullback. The first condition is the following.

6.1. DEFINITION. We say that \mathfrak{W} is *pullback closed* if for any pseudo pullback



with arrows $u, v \in \mathfrak{W}$, the composite $u\bar{v}$ is again in \mathfrak{W} .

Since ρ is invertible, **[WB5]** will imply that $v\bar{u} \in \mathfrak{W}$ as well.

6.2. PROPOSITION. *If \mathcal{B} has all pseudo pullbacks for cospans in \mathfrak{W} , and \mathfrak{W} satisfies conditions [WB1]–[WB5], is pullback closed, and all arrows in \mathfrak{W} are co-full, then each 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ has a representative with the left-hand 2-cell a pseudo pullback.*

PROOF. For any 2-cell diagram,

$$\begin{array}{ccccc}
 & & A' & & \\
 & v \swarrow & \uparrow u & \searrow f & \\
 A & \alpha \Downarrow & C & \beta \Downarrow & B \\
 & v' \swarrow & \downarrow u' & \searrow f' & \\
 & & A'' & &
 \end{array} \tag{18}$$

the pseudo-pullback square

$$\begin{array}{ccc}
 P & \xrightarrow{\bar{v}'} & A' \\
 \bar{v} \downarrow & \cong \rho_{v,v'} & \downarrow v \\
 A'' & \xrightarrow{v'} & A
 \end{array}$$

exists and has the property that $v\bar{v}' \in \mathfrak{W}$. Hence, by Lemma 5.3 there is a representative of (18) with this pseudo-pullback square as left-hand 2-cell. ■

Moreover, the argument from Theorem 5.6 gives the following.

6.3. PROPOSITION. *If \mathfrak{W} satisfies conditions [WB1]–[WB5], is pullback closed, and all arrows in \mathfrak{W} are co-ff, then there is a canonical representation for each 2-cell which is unique up to equivalence of the central object.*

PROOF. The representation using the pseudo pullbacks is canonical and as unique as the choice of pseudo pullbacks. ■

We finally show that if \mathfrak{W} is closed under pseudo pullbacks (rather than pullback closed), we can still use pseudo pullbacks to define the 2-cells:

6.4. PROPOSITION. *If \mathfrak{W} satisfies conditions [WB1]–[WB5], is closed under pseudo pullbacks, and all arrows in \mathfrak{W} are co-ff, then the 2-cells in the bicategory of fractions can be uniquely represented by 2-cell diagrams with a chosen pseudo pullback as left-hand 2-cell.*

PROOF. Let $\widehat{\mathfrak{W}}$ be the class of arrows generated from \mathfrak{W} under composition and closure under 2-isomorphisms. Then $\widehat{\mathfrak{W}}$ satisfies the stronger bicategory of fractions axioms, is pullback-closed and its arrows are still co-ff (this property is preserved by composition and closure under 2-isomorphisms). So the result from Proposition 6.3 applies to $\widehat{\mathfrak{W}}$. Now note that $J: \mathcal{B}(\mathfrak{W}^{-1}) \rightarrow \mathcal{B}(\widehat{\mathfrak{W}}^{-1})$ is an equivalence of bicategories and in particular, it is 2-full and 2-faithful. Hence the 2-cells in $\mathcal{B}(\widehat{\mathfrak{W}}^{-1})$ between arrows in the image of J are in 1-1 correspondence with 2-cells between the original arrows in $\mathcal{B}(\mathfrak{W}^{-1})$. ■

Vertical composition of 2-cells is not simplified by taking representatives with pseudo pullbacks. In fact it is slightly complicated, since we need to calculate the vertical composition of the 2-cell diagrams and then construct an equivalent 2-cell diagram that has the pseudo pullback on the left-hand side, using the lifting as in the proof of Lemma 5.3. However, the horizontal whiskering operations can be significantly simplified by using pseudo pullbacks, as we show in the following two subsections.

6.5. LEFT WHISKERING WITH PSEUDO PULLBACKS. Throughout this subsection, we will assume that \mathcal{B} has all pseudo pullbacks of cospans in \mathfrak{W} and that \mathfrak{W} satisfies all conditions of Proposition 6.2: its arrows are co-full, it satisfies conditions [WB1]–[WB5], and is pullback closed. We will further require \mathfrak{W} to be full. (Note that if \mathfrak{W} is co-fully faithful, this is implied.) We furthermore choose a pseudo pullback

$$\begin{array}{ccc}
 P_{u_1, u_2} & \xrightarrow{\pi_1} & A' \\
 \pi_2 \downarrow & \xleftarrow{\rho_{u_1, u_2}} & \downarrow u_1 \\
 A'' & \xrightarrow{u_2} & A
 \end{array}$$

for each cospan $A' \xrightarrow{u_1} A \xleftarrow{u_2} A''$ in \mathfrak{W} and will now describe the left whiskering operation for 2-cell representatives with these chosen pseudo pullbacks as left-hand 2-cells. So we consider whiskering of the form

$$\begin{array}{ccccccc}
 & & A' & & & & \\
 & u_1 \swarrow & \uparrow \pi_1 & \searrow f_1 & & & \\
 A & \xleftarrow{\rho_{u_1, u_2} \Downarrow} & P_{u_1, u_2} & \xrightarrow{\beta \Downarrow} & B & \xleftarrow{v} & B' \xrightarrow{g} C \\
 & u_2 \swarrow & \downarrow \pi_2 & \searrow f_2 & & & \\
 & & A'' & & & &
 \end{array} \tag{19}$$

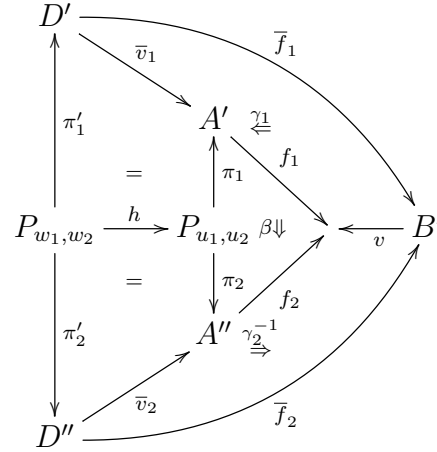
where ρ_{u_1, u_2} is the chosen pseudo pullback. We construct the composition of the 1-cells using chosen squares γ_1 and γ_2 as in Section 3.5,

$$\begin{array}{ccc}
 D' \xrightarrow{\bar{f}_1} B' & & D'' \xrightarrow{\bar{f}_2} B' \\
 \bar{v}_1 \downarrow \cong & \text{and} & \bar{v}_2 \downarrow \cong \\
 A' \xrightarrow{f_1} B & & A'' \xrightarrow{f_2} B \\
 & & \downarrow v
 \end{array}$$

such that $w_1 := u_1 \bar{v}_1$ and $w_2 := u_2 \bar{v}_2$ are in \mathfrak{W} . Let

$$\begin{array}{ccc}
 P_{w_1, w_2} & \xrightarrow{\pi'_1} & D' \\
 \pi'_2 \downarrow & \xleftarrow{\rho_{w_1, w_2}} & \downarrow w_1 = u_1 \bar{v}_1 \\
 D'' & \xrightarrow{w_2 = u_2 \bar{v}_2} & A
 \end{array}$$

be the chosen pseudo pullback. Then there is a unique arrow $h: P_{w_1, w_2} \rightarrow P_{u_1, u_2}$ such that $\pi_1 h = \bar{v}_1 \pi'_1$, $\pi_2 h = \bar{v}_2 \pi'_2$ and $\rho_{u_1, u_2} h = \rho_{w_1, w_2}$. Finally, let $\tilde{\beta}: \bar{f}_1 \pi'_1 \Rightarrow \bar{f}_2 \pi'_2$ be the lifting of the diagram,



with respect to v (this exists because we assume that \mathfrak{W} is full). Then the result of whiskering as in (19) is given by

Diagram (20) is a commutative diagram with nodes $A, A', A'', B', C, D', D''$. Arrows include: $u_1: A \rightarrow A'$, $u_2: A \rightarrow A''$, $\rho_{w_1, w_2}\Downarrow: A \rightarrow P_{w_1, w_2}$, $\rho_{u_1, u_2}\Downarrow: A \rightarrow P_{u_1, u_2}$, $\tilde{\beta}\Downarrow: P_{u_1, u_2} \rightarrow B'$, $\bar{v}_1: D' \rightarrow A'$, $\bar{v}_2: D'' \rightarrow A''$, $\bar{f}_1: D' \rightarrow B'$, $\bar{f}_2: D'' \rightarrow B'$, $\pi'_1: D' \rightarrow P_{w_1, w_2}$, $\pi'_2: D'' \rightarrow P_{w_1, w_2}$, and $g: B' \rightarrow C$.

6.6. LEMMA. *Diagram (20) is equivalent to the diagram (3) obtained for this type of whiskering in Section 3.5.*

PROOF. It was shown in [12] that any pair of choices of the squares and liftings in the composition construction of Section 3.5 give equivalent 2-cell diagrams as long as we use the composition squares from [C2] of Notation 3.2 for the composition of the 1-cells and the squares have the right properties. The only place where the chosen squares are essential is in the composition of the 1-cells, so with the exception of the cells γ_1 and γ_2 we can replace all cells used in the whiskering algorithm from Section 3.5 with cells and squares we have just constructed above. So we will redo the construction from Section 3.5 and use the universal properties of the pseudo pullbacks to adjust the squares to obtain a 2-cell diagram that is clearly equivalent to (20).

Recall that in Section 3.5 we used chosen squares δ_1 , δ_2 and δ_3 to obtain diagrams

$$\begin{array}{c}
 \begin{array}{c}
 \bar{v}_1 \\
 \swarrow \delta_1 \Downarrow \\
 u_1 \nearrow \pi_1 \\
 \rho_{u_1, u_2} \Downarrow P_{u_1, u_2} \\
 \searrow \pi_2 \\
 u_2 \\
 \bar{v}_2
 \end{array}
 \quad
 \begin{array}{c}
 \pi'_1 \tilde{u}_1 \\
 \uparrow t_1 \\
 T \\
 \downarrow t_2 \\
 \pi'_2 \tilde{u}_2
 \end{array}
 \quad
 \text{and} \quad
 \begin{array}{c}
 \bar{v}_1 \\
 \swarrow \delta_1 \Downarrow \\
 \pi'_1 \tilde{u}_1 \nearrow \pi_1 \\
 T \xrightarrow{\tilde{t}} P_{u_1, u_2} \\
 \searrow \delta_3^{-1} \Downarrow \\
 \pi_1 \nearrow f_1 \\
 \bar{v}_1 \searrow \beta \Downarrow \\
 \pi_2 \nearrow f_2 \\
 \delta_2^{-1} \Downarrow \\
 \pi_2 \nearrow f_2 \\
 \bar{v}_2 \searrow \gamma_2^{-1} \Downarrow \\
 \pi'_2 \tilde{u}_2
 \end{array}
 \quad
 \begin{array}{c}
 \bar{f}_1 \\
 \swarrow \underline{\gamma}_1 \\
 P_{u_1, u_2} \xrightarrow{\beta} v \\
 \searrow \underline{\gamma}_2^{-1} \\
 \bar{f}_2
 \end{array}
 \quad
 \begin{array}{c}
 \xrightarrow{g}
 \end{array}
 \end{array}
 \tag{21}$$

By the universal property of the pseudo pullback there is an arrow $\tilde{t}: T \rightarrow P_{u_1, u_2}$ such that the following diagram pastes to the same 2-cell as the first diagram in (21),

$$\begin{array}{c}
 \bar{v}_1 \\
 \swarrow = \\
 u_1 \nearrow \pi_1 \\
 \rho_{u_1, u_2} \Downarrow P_{u_1, u_2} \\
 \searrow \pi_2 \\
 u_2 \\
 \bar{v}_2
 \end{array}
 \quad
 \begin{array}{c}
 \pi'_1 \tilde{u}_1 \\
 \uparrow t_1 \\
 T \\
 \downarrow t_2 \\
 \pi'_2 \tilde{u}_2
 \end{array}
 \quad
 \begin{array}{c}
 \tilde{t} \\
 \leftarrow \\
 \leftarrow \\
 \leftarrow
 \end{array}$$

We now replace the chosen squares δ_1 , δ_2 by the new commuting squares in this diagram and let $\delta_3 = \text{id}_{\tilde{t}}$. We obtain the following diagram,

$$\begin{array}{c}
 \bar{v}_1 \\
 \swarrow = \\
 u_1 \nearrow \pi_1 \\
 \rho_{u_1, u_2} \Downarrow P_{u_1, u_2} \\
 \searrow \pi_2 \\
 u_2 \\
 \bar{v}_2
 \end{array}
 \quad
 \begin{array}{c}
 \bar{\pi}_1 \\
 \uparrow t_1 \\
 T \\
 \downarrow t_2 \\
 \bar{\pi}_2
 \end{array}
 \quad
 \begin{array}{c}
 \bar{v}_1 \\
 \swarrow = \\
 \bar{\pi}_1 \nearrow \pi_1 \\
 P_{u_1, u_2} \xrightarrow{\tilde{t}} P_{u_1, u_2} \\
 \searrow \beta \Downarrow \\
 \bar{\pi}_1 \nearrow f_1 \\
 \bar{v}_1 \searrow \beta \Downarrow \\
 \bar{\pi}_2 \nearrow f_2 \\
 \bar{v}_2 \searrow \gamma_2^{-1} \Downarrow \\
 \bar{\pi}_2
 \end{array}
 \quad
 \begin{array}{c}
 \bar{f}_1 \\
 \swarrow \underline{\gamma}_1 \\
 P_{u_1, u_2} \xrightarrow{\beta} v \\
 \searrow \underline{\gamma}_2^{-1} \\
 \bar{f}_2
 \end{array}
 \quad
 \begin{array}{c}
 \xrightarrow{g}
 \end{array}$$

This is almost a 2-cell diagram: we just need to take a lifting $\tilde{\beta}': \bar{f}_1 \bar{\pi}_1 t_1 \Rightarrow \bar{f}_2 \bar{\pi}_2 t_2$ of the right-hand side with respect to v (which is possible since v is full)

To show that the resulting 2-cell,

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & \bar{v}_1 & & \\
 & \swarrow & \longleftarrow & \searrow & \\
 u_1 & & & & \bar{f}_1 \\
 & \uparrow \pi_1 & & \uparrow \bar{\pi}_1 t_1 & \\
 \rho_{u_1, u_2} \Downarrow & P_{u_1, u_2} & \xleftarrow{\tilde{t}} & T & \xrightarrow{\tilde{\beta}' \Downarrow} \\
 & \downarrow \pi_2 & & \downarrow \bar{\pi}_2 t_2 & \\
 u_2 & & \bar{v}_2 & & \bar{f}_2 \\
 & \swarrow & \longleftarrow & \searrow & \\
 & & & & g
 \end{array}
 \end{array} \tag{22}$$

is equivalent to (20), note that there is a unique arrow $t' : T \rightarrow P_{w_1, w_2}$ such that $\rho_{w_1, w_2} t' = \rho_{u_1, u_2} \tilde{t}$. Now $\tilde{\beta}' t'$ is another lifting of the right-hand side in (19), so the diagrams with $\tilde{\beta}'$ and $\beta' t'$ on the left-hand side are equivalent. Hence, (20) and (22) are equivalent. ■

6.7. RIGHT WHISKERING WITH PULLBACKS. Throughout this section, we will assume all conditions of Proposition 6.2: \mathcal{B} has all pseudo pullbacks of cospans in \mathfrak{W} (and we will use the chosen pseudo pullbacks as in the previous subsection), \mathfrak{W} satisfies conditions [WB1]–[WB5], is pullback closed, and its arrows are co-full. Furthermore, we will require \mathfrak{W} to be full as well. We now consider right whiskering for 2-cell representatives where the left-hand 2-cell is a chosen pseudo pullback. So we start with the composition

$$\begin{array}{ccccc}
 & & & B' & \\
 & & & \uparrow \pi_1 & \\
 & & v_1 & & g_1 \\
 A & \xleftarrow{u} & A' & \xrightarrow{f} & B & \xrightarrow{\rho_{v_1, v_2} \Downarrow} & P_{v_1, v_2} & \xrightarrow{\beta \Downarrow} & C \\
 & & & \downarrow \pi_2 & \\
 & & v_2 & & g_2 \\
 & & & B'' &
 \end{array} \tag{23}$$

where P_{v_1, v_2} is the chosen pseudo pullback of v_1 and v_2 . First we construct the composition of the 1-cells using chosen squares [C2]

$$\begin{array}{ccc}
 D' \xrightarrow{\bar{f}_1} B' & & D'' \xrightarrow{\bar{f}_2} B'' \\
 \bar{v}_1 \downarrow \quad \cong \quad \downarrow v_1 & \text{and} & \bar{v}_2 \downarrow \quad \cong \quad \downarrow v_2 \\
 A' \xrightarrow{f} B & & A' \xrightarrow{f} B
 \end{array}$$

such that $u_1 := u\bar{v}_1$ and $u_2 := u\bar{v}_2$ are in \mathfrak{W} as in Section 3.6. Let

$$\begin{array}{ccc}
 P_{u_1, u_2} \xrightarrow{\bar{\pi}_1} D' & & \\
 \bar{\pi}_2 \downarrow \quad \cong \quad \downarrow u_1 & & \\
 D'' \xrightarrow{u_2} A & &
 \end{array}$$

be the chosen pseudo pullback of u_1 and u_2 . Note that $\rho_{u_1, u_2} : u\bar{v}_1\bar{\pi}_1 \Rightarrow u\bar{v}_2\bar{\pi}_2$. Since u is full, there is a lifting $\tilde{\rho}_{u_1, u_2} : \bar{v}_1\bar{\pi}_1 \Rightarrow \bar{v}_2\bar{\pi}_2$. This cell can be pasted with γ_1 and γ_2^{-1} to

form

$$\begin{array}{ccccc}
 & & D' & \xrightarrow{\bar{f}_1} & B' & & \\
 & \nearrow^{\bar{\pi}_1} & \downarrow \bar{v}_1 & \Downarrow \gamma_1 & \searrow^{v_1} & & \\
 P_{u_1, u_2} & \xrightarrow{\bar{\rho}_{u_1, u_2} \Downarrow} & A' & \xrightarrow{f} & B & & \\
 & \searrow_{\bar{\pi}_2} & \uparrow \bar{v}_2 & \Downarrow \gamma_2^{-1} & \nearrow_{v_2} & & \\
 & & D'' & \xrightarrow{\bar{f}_2} & B'' & &
 \end{array} \tag{24}$$

By the universal property of the pseudo pullback P_{v_1, v_2} , there is a unique arrow

$$h: P_{u_1, u_2} \rightarrow P_{v_1, v_2} \text{ such that } \pi_1 h = \bar{f}_1 \bar{\pi}_1 \text{ and } \pi_2 h = \bar{f}_2 \bar{\pi}_2 \tag{25}$$

and furthermore, $\rho_{v_1, v_2} h$ is equal to the pasting of (24). We claim that the following 2-cell diagram represents the result of whiskering (23):

$$\begin{array}{ccccc}
 & & D' & \xrightarrow{\bar{f}_1} & B' & & \\
 & \nearrow^{u_1} & \uparrow \bar{\pi}_1 & = & \uparrow \pi_1 & \searrow^{g_1} & \\
 A & \xrightarrow{\rho_{u_1, u_2} \Downarrow} & P_{u_1, u_2} & \xrightarrow{h} & P_{v_1, v_2} & \xrightarrow{\beta \Downarrow} & C \\
 & \searrow_{u_2} & \downarrow \bar{\pi}_2 & = & \downarrow \pi_2 & \nearrow_{g_2} & \\
 & & D'' & \xrightarrow{\bar{f}_2} & B'' & &
 \end{array} \tag{26}$$

6.8. LEMMA. *Diagram (26) is equivalent to the diagram (5) obtained for this type of whiskering in Section 3.6.*

PROOF. Again, we use the result from [12] that the equivalence classes of the resulting 2-cell diagrams in the whiskering constructions and vertical composition construction do not depend on the choice of the squares and liftings used as long as we use the chosen composition of 1-cells and the appropriate arrows are in \mathfrak{W} . We will now go through the algorithm of Section 3.6 and substitute the cells above. We will show that the result is precisely (26).

In (4), we take for δ_1 and δ_2 respectively,

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \xrightarrow{h} & \\
 \bar{\pi}_1 \downarrow & = & \downarrow \pi_1 \\
 & \xrightarrow{\bar{f}_1} & \\
 \bar{v}_1 \downarrow & \Downarrow \gamma_1 & \downarrow v_1 \\
 & \xrightarrow{f} &
 \end{array} & \text{and} &
 \begin{array}{ccc}
 & \xrightarrow{h} & \\
 \bar{\pi}_2 \downarrow & = & \downarrow \pi_2 \\
 & \xrightarrow{\bar{f}_2} & \\
 \bar{v}_2 \downarrow & \Downarrow \gamma_2 & \downarrow v_2 \\
 & \xrightarrow{f} &
 \end{array}
 \end{array}$$

This allows us to take r_1 and r_2 to be identity arrows and $t_i = \bar{\pi}_i$ for $i = 1, 2$. Furthermore, φ_i is given by

$$\begin{array}{ccc} & \xrightarrow{h} & \\ \bar{\pi}_i \downarrow & = & \downarrow \pi_i \\ & \xrightarrow{\bar{f}_i} & \end{array}$$

and $\varepsilon_i = \text{id}_{\bar{v}_i \bar{\pi}_i}$, for $i = 1, 2$. The next step is then to compare the pastings,

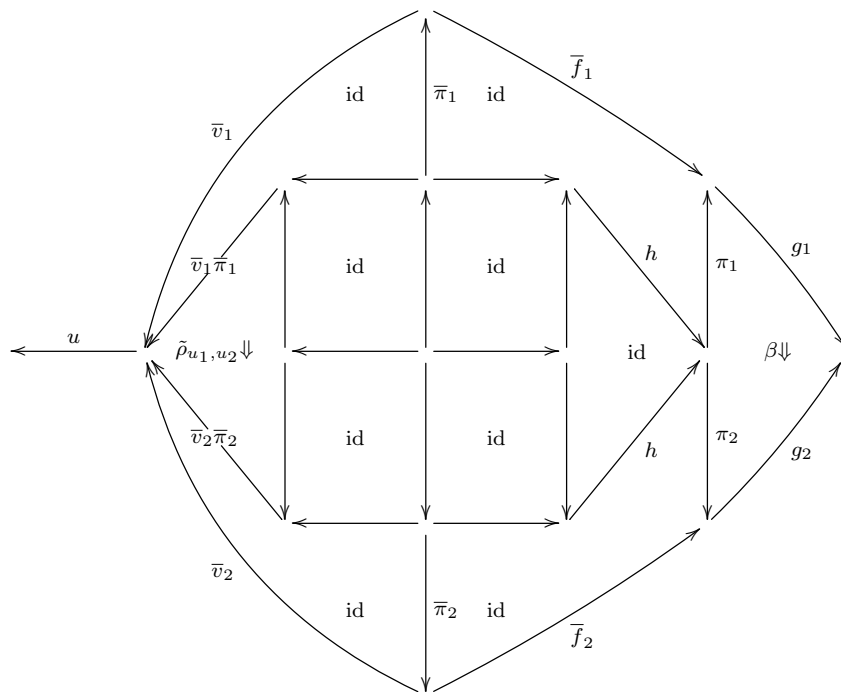
$$\begin{array}{ccc} \begin{array}{ccc} & \xrightarrow{h} & \\ \bar{\pi}_1 \downarrow & = & \downarrow \pi_1 \\ & \xrightarrow{\bar{f}_1} & \\ \bar{v}_1 \downarrow & \xleftarrow{\gamma_1} & v_1 \\ & \xrightarrow{f} & \end{array} & \text{and} & \begin{array}{ccc} & \xrightarrow{h} & \\ \bar{\pi}_2 \swarrow & = & \pi_2 \searrow \\ \bar{f}_2 \rightarrow & \rho_{v_1, v_2} & \downarrow \pi_1 \\ \bar{v}_2 \swarrow & \xleftarrow{\gamma_2} & v_2 \searrow \\ & \xrightarrow{f} & \end{array} \end{array}$$

Here we may choose p and q to be identity arrows, $\tau = \text{id}_h$ and $\tilde{\alpha} = \tilde{\rho}_{u_1, u_2}$, since

$$\begin{array}{ccc} \begin{array}{ccc} & \xrightarrow{h} & \\ \bar{\pi}_2 \swarrow & \bar{\pi}_1 \downarrow = \downarrow \pi_1 & \\ \tilde{\rho}_{u_1, u_2} \xleftarrow{\tilde{\alpha}} & \bar{f}_1 \rightarrow & \\ \bar{v}_2 \swarrow & \bar{v}_1 \downarrow \xleftarrow{\gamma_1} & v_1 \\ & \xrightarrow{f} & \end{array} & = & \begin{array}{ccc} & \xrightarrow{h} & \\ \bar{\pi}_2 \swarrow & = & \pi_2 \searrow \\ \bar{f}_2 \rightarrow & \rho_{v_1, v_2} & \downarrow \pi_1 \\ \bar{v}_2 \swarrow & \xleftarrow{\gamma_2} & v_2 \searrow \\ & \xrightarrow{f} & \end{array} \end{array}$$

by (24) and (25).

Omitting the identity coherence cells, the resulting 2-cell diagram is



(27)

where all unlabeled arrows are identity arrows. Composing the cells in both the left-hand side and the right-hand side of this diagram gives us the 2-cell diagram in (26) as required. ■

6.9. HORIZONTAL COMPOSITION OF 2-CELL DIAGRAMS WITH PSEUDO PULLBACKS.

Suppose that we have two 2-cells that we want to compose:

$$\begin{array}{ccccc}
 & & A' & & B' \\
 & u_1 \swarrow & \uparrow \pi_{A'} & \searrow f_1 & \swarrow v_1 \\
 A & \xrightarrow{\rho_{u_1, u_2} \Downarrow} & P_{u_1, u_2} & \xrightarrow{\beta \Downarrow} & B \\
 & u_2 \searrow & \downarrow \pi_{A''} & \swarrow f_2 & \searrow v_2 \\
 & & A'' & & B'' \\
 & & & & & & B' \\
 & & & & & \swarrow v_1 & \uparrow \pi_{B'} \\
 & & & & & \searrow v_2 & \downarrow \pi_{B''} \\
 & & & & & & B'' \\
 & & & & & \swarrow g_1 & \searrow g_2 \\
 & & & & & & C
 \end{array} \tag{28}$$

The horizontal composition of these two general 2-cell diagrams is rather involved, being a combination of two whiskering operations and a vertical composition. However, for 2-cell diagrams with pseudo pullbacks as left-hand cells, the right-hand side of the horizontal composition can be calculated as a lifting with respect to v_1 of β composed with suitable invertible 2-cells, whiskered with g_1 and then post-composed with γ . If furthermore, β is invertible, the horizontal composition can be calculated by using two universal arrows obtained from the two pseudo-pullback squares in the initial diagram, whiskered with γ . We describe this here.

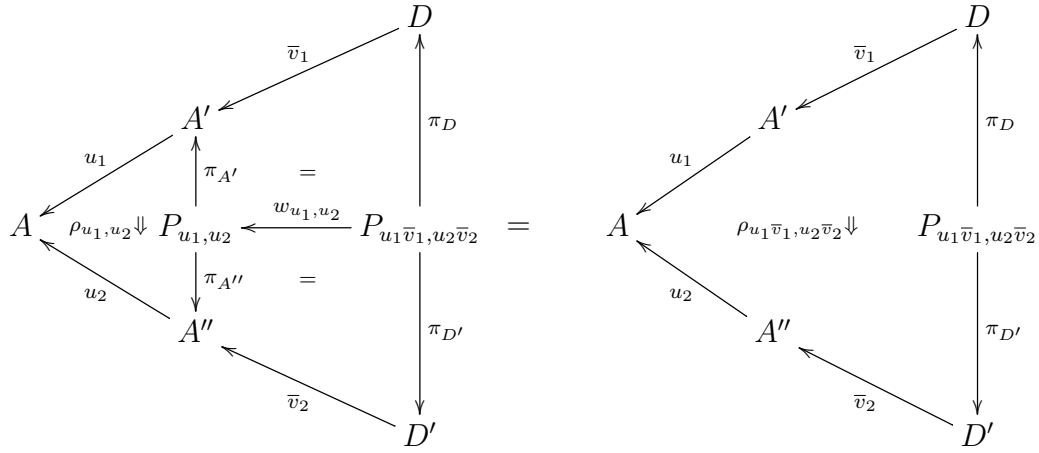
Let δ_1 and δ_2 be chosen squares (as in [C4]) such that $u_1\bar{v}_1$ and $u_2\bar{v}_2$ are in \mathfrak{W} , as in the following diagram.

$$\begin{array}{ccccc}
 & & D & & \\
 & \bar{v}_1 \swarrow & & \searrow \bar{f}_1 & \\
 & & A' & & B' \\
 & u_1 \swarrow & \uparrow \pi_{A'} & \searrow f_1 & \swarrow v_1 \\
 A & \xrightarrow{\rho_{u_1, u_2} \Downarrow} & P_{u_1, u_2} & \xrightarrow{\beta \Downarrow} & B \\
 & u_2 \searrow & \downarrow \pi_{A''} & \swarrow f_2 & \searrow v_2 \\
 & & A'' & & B'' \\
 & & & & & & B' \\
 & & & & & \swarrow v_1 & \uparrow \pi_{B'} \\
 & & & & & \searrow v_2 & \downarrow \pi_{B''} \\
 & & & & & & B'' \\
 & & & & & \swarrow \bar{f}_2 & \searrow g_2 \\
 & & & & & & C \\
 & & & & & \swarrow \bar{v}_2 & \\
 & & & & & & D'
 \end{array}$$

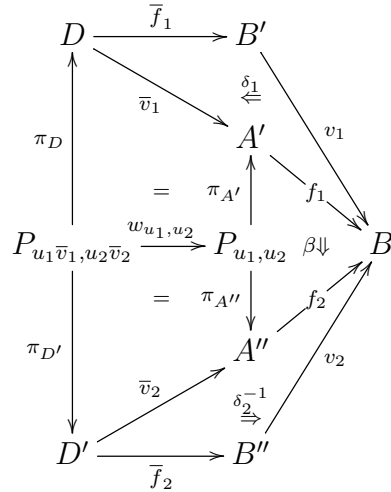
(Note that this diagram is not a pasting diagram.) The left-hand side of the composed 2-cell diagram will be the chosen pseudo pullback $\rho_{u_1\bar{v}_1, u_2\bar{v}_2}$. By the universal property of ρ_{u_1, u_2} , we obtain a unique arrow

$$w_{u_1, u_2} : P_{u_1\bar{v}_1, u_2\bar{v}_2} \rightarrow P_{u_1, u_2}$$

such that $\rho_{u_1, u_2} w_{u_1, u_2} = \rho_{u_1 \bar{v}_1, u_2 \bar{v}_2}$,



The arrow w_{u_1, u_2} can be used to construct the following pasting diagram,



If β is invertible, the universal property of the pseudo pullback P_{v_1, v_2} gives rise to a unique arrow

$$w_{v_1, v_2} : P_{u_1 \bar{v}_1, u_2 \bar{v}_2} \rightarrow P_{v_1, v_2}$$

details, see [2, 4]. We define orbifolds as the bicategory of fractions of orbigroupoids with respect to the class of essential equivalences. Essential equivalences are both ff and co-ff. The class of essential equivalences is also pullback closed as in Definition 6.1, and satisfies the **BF** conditions from [5]. Thus, we can apply the results of Corollary 5.9 and Proposition 6.3 to get the following:

7.1. THEOREM.

1. The universal map from the 2-category of orbigroupoids to its bicategory of fractions with respect to the class \mathfrak{W} of essential equivalences,

$$J_{\mathfrak{W}}: \mathbf{OrbiGroupoids} \longrightarrow \mathbf{OrbiGroupoids}(\mathfrak{W}^{-1})$$

is 2-fully faithful.

2. Each 2-cell in $\mathbf{OrbiGroupoids}(\mathfrak{W}^{-1})$ has a unique representation by a 2-cell diagram with any given left-hand side.
3. Given a choice of pseudo pullbacks for cospans of essential equivalences the 2-cells in $\mathbf{OrbiGroupoids}(\mathfrak{W}^{-1})$ can be uniquely represented by diagrams with these pseudo pullbacks as left-hand 2-cells and horizontal composition can be calculated as in Section 6.

Furthermore, there is a subclass $\mathfrak{C} \subset \mathfrak{W}$ of essential covering maps, defined by,

7.2. DEFINITION. Let \mathcal{G} be an étale groupoid. An essential covering map

$$\varphi^{\mathcal{U}}: \mathcal{G}^*(\mathcal{U}) \rightarrow \mathcal{G}$$

is determined by a (non-repeating) collection of open subsets $\mathcal{U} \subseteq \mathcal{P}(\mathcal{G}_0)$ which meets every orbit of \mathcal{G} (although it may not cover \mathcal{G}_0). Then $\mathcal{G}^*(\mathcal{U})$ is the groupoid defined by $\mathcal{G}^*(\mathcal{U})_0 = \coprod_{U \in \mathcal{U}} U$, with $\varphi_0^{\mathcal{U}}: \mathcal{G}(\mathcal{U})_0 \rightarrow \mathcal{G}_0$ defined by the inclusion maps. Furthermore, the space $\mathcal{G}(\mathcal{U})_1$ and the remaining maps are determined by the pullback diagram

$$\begin{array}{ccc} \mathcal{G}(\mathcal{U})_1 & \xrightarrow{\varphi_1^{\mathcal{U}}} & \mathcal{G}_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ \mathcal{G}(\mathcal{U})_0 \times \mathcal{G}(\mathcal{U})_0 & \xrightarrow{\varphi_0^{\mathcal{U}} \times \varphi_0^{\mathcal{U}}} & \mathcal{G}_0 \times \mathcal{G}_0 \end{array}$$

The class \mathfrak{C} of essential covering maps is locally small and satisfies conditions [WB1]–[WB5]. As essential equivalences they are also ff and co-ff. So we get a bicategory $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ with small hom-categories, where

$$J_{\mathfrak{C}}: \mathbf{OrbiGroupoids} \longrightarrow \mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$$

is 2-fully faithful. Furthermore, the essential covering maps are weakly initial in the essential equivalences in the sense described in Definition 4.2. Hence, there is an equivalence of bicategories, $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1}) \simeq \mathbf{OrbiGroupoids}(\mathfrak{W}^{-1})$.

Now \mathfrak{C} is not pullback-closed. However, because of this equivalence of bicategories we can use the 2-cell diagrams from $\mathbf{OrbiGroupoids}(\mathfrak{W}^{-1})$ as 2-cells between arrows in $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$, and hence represent these by 2-cell diagrams with pseudo pullbacks as left-hand 2-cells; these are not necessarily in the shape required of 2-cell diagrams in $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ because certain composites will not be in \mathfrak{C} , but they can be used as an alternate way to represent the 2-cells in this bicategory. This allows us to use the simplified composition described in Section 6. So we conclude:

7.3. THEOREM.

1. *The bicategory of fractions of orbifoldoids with respect to essential covering maps, $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ has small hom-categories.*
2. *The pseudo functor $J_{\mathfrak{W}}: \mathbf{OrbiGroupoids} \rightarrow \mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ is 2-fully faithful.*
3. *Each 2-cell in $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ has a unique representation by a 2-cell diagram with any given left-hand side.*
4. *Given a choice of pseudo-pullback squares the 2-cells in $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ can be uniquely represented by diagrams with pseudo pullbacks as left-hand 2-cells, and horizontal composition can be calculated as in Section 6.*

For further details, proofs, and applications, see [6].

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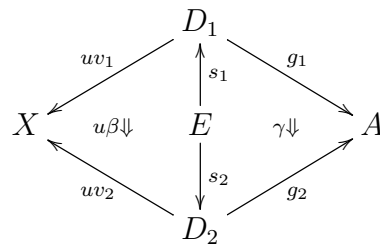
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Appendix A Associativity Part I: Associativity 2-cells

The goal of these appendices is to study associativity coherence and well-definedness for composition in $\mathcal{B}(\mathfrak{W}^{-1})$. In Appendix A we will construct the associativity 2-cells, based on an extension of Proposition 2.9. In Appendix B we will show that these cells satisfy the coherence pentagon condition. In Appendix C we verify that all composition operations are well-defined on equivalence classes. In Appendix D we give a proof for the presentation, given in Section 6.9, of the horizontal composition of two 2-cell diagrams with pull-back squares for left-hand 2-cells and where the left 2-cell diagram is invertible. Throughout the appendices, we assume that \mathcal{B} is a bicategory and \mathfrak{W} is a class of arrows satisfying conditions [WB1]–[WB5].

Consider the 2-cells β and γ in Proposition 2.9. They give rise to a generalized 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$,



We show that this is the unique cell with this property: if β' and γ' also satisfy the conditions of Proposition 2.9, then the 2-cell diagram defined by β' and γ' is equivalent to this one.

A.1 PROPOSITION. For $v: C \rightarrow X$ and $w: A \rightarrow B$ both in \mathfrak{W} and $f: C \rightarrow B$ any arrow in \mathcal{B} , and any two squares,

$$\begin{array}{ccc}
 D_1 & \xrightarrow{f_1} & A \\
 w_1 \downarrow & \alpha_1 \rightrightarrows & \downarrow w \\
 C & \xrightarrow{f} & B \\
 v \downarrow & & \\
 X & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 D_2 & \xrightarrow{f_2} & A \\
 w_2 \downarrow & \alpha_2 \rightrightarrows & \downarrow w \\
 C & \xrightarrow{f} & B \\
 v \downarrow & & \\
 X & &
 \end{array}$$

with $vw_1, vw_2 \in \mathfrak{W}$, there is a unique 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$

$$\begin{array}{ccccc}
 & & D_1 & & \\
 & vw_1 \swarrow & \uparrow s_1 & \searrow f_1 & \\
 X & & E & & A \\
 & v\beta \downarrow & \downarrow s_2 & \uparrow f_2 & \\
 & & D_2 & &
 \end{array}
 \tag{30}$$

such that the composites $(f\beta) \cdot (\alpha_1 s_1)$ and $(\alpha_2 s_2) \cdot (w\gamma)$ are equal.

PROOF. Existence is a consequence of Proposition 2.9, so we need only prove uniqueness. Let

$$\begin{array}{ccccc}
 & & D_1 & & \\
 & vw_1 \swarrow & \uparrow t_1 & \searrow f_1 & \\
 X & & E' & & A \\
 & v\beta' \downarrow & \downarrow t_2 & \uparrow f_2 & \\
 & & D_2 & &
 \end{array}
 \tag{31}$$

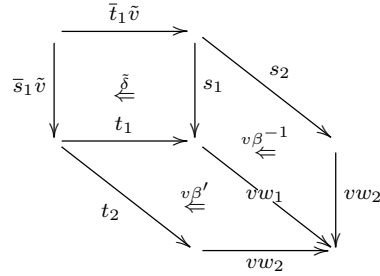
be another 2-cell diagram with the property that the composites $(f\beta') \cdot (\alpha_1 t_1)$ and $(\alpha_2 t_2) \cdot (w\gamma')$ are equal. Let

$$\begin{array}{ccc}
 \xrightarrow{\bar{t}_1} & & \\
 \bar{s}_1 \downarrow & \xleftarrow{\delta} & \downarrow vw_1 s_1 \\
 & \xrightarrow{vw_1 t_1} &
 \end{array}$$

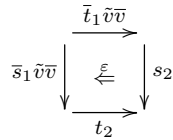
be a square as in condition **[WB3]** and let \tilde{v} with

$$\begin{array}{ccc}
 \xrightarrow{\bar{t}_1 \tilde{v}} & & \\
 \bar{s}_1 \tilde{v} \downarrow & \xleftarrow{\tilde{\delta}} & \downarrow s_1 \\
 & \xrightarrow{t_1} &
 \end{array}$$

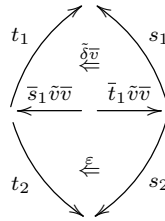
be a lifting as in [WB4] for δ with respect to vw_1 . We use this cell in the following pasting,



and then use condition [WB4] to obtain an arrow \bar{v} and a cell

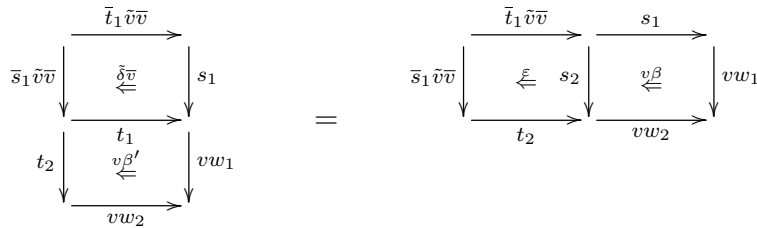


which form a lifting for this pasting with respect to vw_2 . We would like to use the diagram

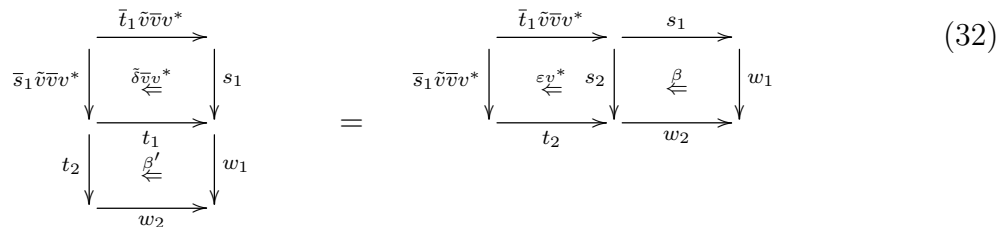


to show that the two 2-cell diagrams are equivalent. However, we still need to make a couple of small adjustments.

By construction we have that the following pastings are equal:



By Lemma 2.7 there is an arrow v^* in \mathfrak{W} such that



(32)

To obtain the corresponding result with γ, γ' instead of β, β' , we need to compose with the arrow w so that the hypothesis of **[WB4]** is satisfied. We will also compose the pasting diagrams we are interested in with the cells α_2 and β'^{-1} . This leads to the following calculation,

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & s_1 & & f_1 \\
 & \nearrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{t}_1 \bar{v} \bar{v} v^* & & \delta \bar{v} v^* \Downarrow & t_1 & \gamma' \Downarrow & f_2 \\
 & \searrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{s}_1 \bar{v} \bar{v} v^* & & t_2 & & \alpha_2 \Downarrow \\
 & & t_1 & & \beta'^{-1} \Downarrow & w_2 \\
 & & & & & w_1 \\
 & & & & & f
 \end{array}
 & = &
 \begin{array}{ccccc}
 & & s_1 & & f_1 \\
 & \nearrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{t}_1 \bar{v} \bar{v} v^* & & \delta \bar{v} v^* \Downarrow & t_1 & & \\
 & \searrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{s}_1 \bar{v} \bar{v} v^* & & t_1 & & \alpha_1 t_1 \Downarrow \\
 & & & & & w_1 \\
 & & & & & f
 \end{array} \\
 \\
 = &
 \begin{array}{ccccc}
 & & s_1 & & f_1 \\
 & \nearrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{t}_1 \bar{v} \bar{v} v^* & & \delta \bar{v} v^* \Downarrow & & \alpha_1 \Downarrow \\
 & \searrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{s}_1 \bar{v} \bar{v} v^* & & t_1 & & w_1 \\
 & & & & & f
 \end{array} \\
 \\
 = &
 \begin{array}{ccccc}
 & & s_1 & & f_1 \\
 & \nearrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{t}_1 \bar{v} \bar{v} v^* & & \delta \bar{v} v^* \Downarrow & s_1 & & \\
 & \searrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{s}_1 \bar{v} \bar{v} v^* & & t_1 & & \alpha_1 s_1 \Downarrow \\
 & & & & & w_1 \\
 & & & & & f
 \end{array} \\
 \\
 = &
 \begin{array}{ccccc}
 & & s_1 & & f_1 \\
 & \nearrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{t}_1 \bar{v} \bar{v} v^* & & \delta \bar{v} v^* \Downarrow & s_1 & \gamma \Downarrow & f_2 \\
 & \searrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{s}_1 \bar{v} \bar{v} v^* & & t_1 & & \beta^{-1} \Downarrow & w_2 \\
 & & & & & w_1 \\
 & & & & & \alpha_2 \Downarrow \\
 & & & & & f
 \end{array} \\
 \\
 = &
 \begin{array}{ccccc}
 & & s_1 & & f_1 \\
 & \nearrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{t}_1 \bar{v} \bar{v} v^* & & \varepsilon v^* \Downarrow & s_2 & \gamma \Downarrow & f_2 \\
 & \searrow & \rightarrow & \rightarrow & \rightarrow \\
 \bar{s}_1 \bar{v} \bar{v} v^* & & t_1 & & \beta'^{-1} \Downarrow & w_2 \\
 & & & & & w_1 \\
 & & & & & \alpha_2 \Downarrow \\
 & & & & & f
 \end{array}
 \end{array}$$

where the last equality follows from (32). Since β' and α_2 are invertible 2-cells, we conclude

that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \bar{s}_1 \bar{v} \bar{v} v^* & \xrightarrow{\bar{t}_1 \bar{v} \bar{v} v^*} & \\
 \downarrow \bar{s}_1 & \downarrow \delta \bar{v} v^* \Downarrow & \downarrow s_1 \\
 t_2 & \xrightarrow{t_1} & \\
 \downarrow t_2 & \downarrow \gamma' \Downarrow & \downarrow f_1 \\
 & \xrightarrow{f_2} & \\
 & & \xrightarrow{w}
 \end{array} & = &
 \begin{array}{ccc}
 \bar{s}_1 \bar{v} \bar{v} v^* & \xrightarrow{\bar{t}_1 \bar{v} \bar{v} v^*} & \xrightarrow{s_1} & \\
 \downarrow \bar{s}_1 & \downarrow \varepsilon v^* \Downarrow & \downarrow s_2 & \downarrow \gamma \Downarrow & \downarrow f_1 \\
 t_2 & \xrightarrow{f_2} & & & \\
 & & \xrightarrow{f_2} & & \xrightarrow{w}
 \end{array}
 \end{array}$$

By Lemma 2.7 there is an arrow $\tilde{w} \in \mathfrak{W}$ such that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \tilde{w} & \xrightarrow{\bar{t}_1 \bar{v} \bar{v} v^*} & \\
 \downarrow \bar{s}_1 \bar{v} \bar{v} v^* & \downarrow \delta \bar{v} v^* \Downarrow & \downarrow s_1 \\
 t_2 & \xrightarrow{t_1} & \\
 \downarrow t_2 & \downarrow \gamma' \Downarrow & \downarrow f_1 \\
 & \xrightarrow{f_2} &
 \end{array} & = &
 \begin{array}{ccc}
 \tilde{w} & \xrightarrow{\bar{t}_1 \bar{v} \bar{v} v^*} & \xrightarrow{s_1} & \\
 \downarrow \bar{s}_1 \bar{v} \bar{v} v^* & \downarrow \varepsilon v^* \Downarrow & \downarrow s_2 & \downarrow \gamma \Downarrow & \downarrow f_1 \\
 & \xrightarrow{f_2} & & & \\
 & & \xrightarrow{f_2} & &
 \end{array}
 \end{array}$$

Finally, let r be an arrow such that the composition $v w_1 s_1 \bar{t}_1 \tilde{v} \bar{v} v^* \tilde{w} r \in \mathfrak{W}$. Then the cells

$$\begin{array}{ccc}
 & \xrightarrow{t_1} & \\
 & \delta \bar{v} v^* \tilde{w} r \Leftarrow & \\
 \bar{s}_1 \bar{v} \bar{v} v^* \tilde{w} r & \xrightarrow{\bar{t}_1 \bar{v} \bar{v} v^* \tilde{w} r} & \\
 & \varepsilon v^* \tilde{w} r \Leftarrow & \\
 & \xrightarrow{t_2} &
 \end{array}$$

satisfy the equations to establish the fact that (30) and (31) are equivalent 2-cell diagrams, as claimed. ■

A.2 NOTATION. We will say that the 2-cell

$$\begin{array}{ccccc}
 & & D_1 & & \\
 & \swarrow v w_1 & \uparrow s_1 & \searrow f_1 & \\
 X & & E & & A \\
 & \swarrow v \beta \Downarrow & \downarrow s_2 & \searrow \gamma \Downarrow & \\
 & & D_2 & & \\
 & \swarrow v w_2 & & \searrow f_2 &
 \end{array}$$

above connects the squares α_1 and α_2 .

A.3 LEMMA. Let $v: C \rightarrow X$ and $w: A \rightarrow B$ both be in \mathfrak{W} and $f: C \rightarrow B$ any arrow in \mathcal{B} , and let

$$\begin{array}{ccc}
 D_i & \xrightarrow{\bar{f}_i} & A \\
 \bar{w}_i \downarrow & \alpha_i \Leftarrow & \downarrow w \\
 C & \xrightarrow{f} & B \\
 v \downarrow & & \\
 X & &
 \end{array}$$

be invertible 2-cells with $v\bar{w}_i \in \mathfrak{W}$ for $i = 1, 2, 3$. For each pair i, j , let $(v\beta_{ij}, \gamma_{ij})$ be the canonical 2-cell connecting the squares α_i and α_j as given in Proposition A.1,

$$\begin{array}{ccccc}
 & & D_i & & \\
 & & \swarrow & & \searrow \\
 & & vw_2 & & f_i \\
 & & \swarrow & & \searrow \\
 X & & v\beta_{ij} \Downarrow & & \gamma_{ij} \Downarrow & A \\
 & & \swarrow & & \searrow \\
 & & D_j & & \\
 & & \swarrow & & \searrow \\
 & & vw_j & & f_j
 \end{array}$$

Then $(v\beta_{ii}, \gamma_{ii})$ is the identity 2-cell on the span $(v\bar{w}_i, f)$ and these two cells are closed under vertical composition: $(v\beta_{jk}, \gamma_{jk}) \cdot (v\beta_{ij}, \gamma_{ij}) = (v\beta_{ik}, \gamma_{ik})$.

PROOF. Straight forward calculation. ■

A.4 PROPOSITION. For any path of composable spans:

$$\begin{array}{ccccccc}
 & & w_1 & & w_2 & & w_3 & & f_3 \\
 & & \swarrow & & \swarrow & & \swarrow & & \searrow \\
 & & & f_1 & & f_2 & & & \\
 & & \searrow & & \searrow & & \searrow & &
 \end{array} \tag{33}$$

there is an associativity 2-cell

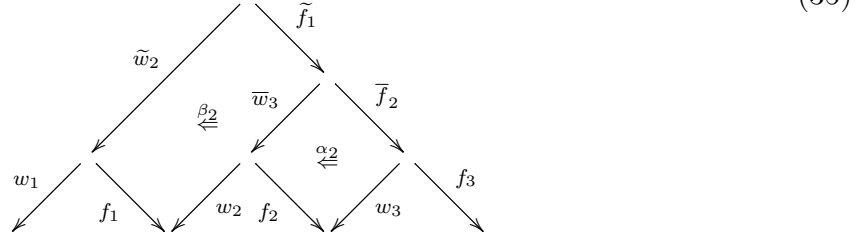
$$\alpha_{(w_3, f_3), (w_2, f_2), (w_1, f_1)}: (w_3, f_3) \circ ((w_2, f_2) \circ (w_1, f_1)) \Rightarrow ((w_3, f_3) \circ (w_2, f_2)) \circ (w_1, f_1)$$

between the composites as constructed in Section 3.

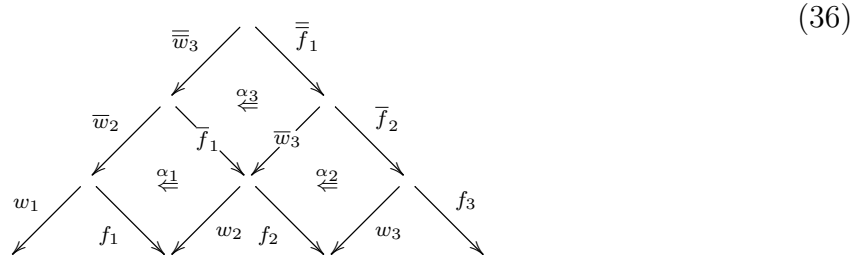
PROOF. If we first compose the left-hand pair and use the choices as described in the construction of $\mathcal{B}(\mathfrak{W}^{-1})$, we obtain $(w_3, f_3) \circ ((w_2, f_2) \circ (w_1, f_1))$ as the following span,

$$\begin{array}{ccccccc}
 & & \tilde{w}_3 & & & & & & \\
 & & \swarrow & & \tilde{f}_2 & & & & \\
 & & \bar{w}_2 & & \bar{f}_1 & & \beta_1 \Leftarrow & & \\
 & & \swarrow & & \swarrow & & & & \\
 w_1 & & & f_1 & & w_2 & f_2 & & w_3 & & f_3 \\
 & & \searrow & & \searrow & & \searrow & & & &
 \end{array} \tag{34}$$

Note that $w_1\bar{w}_2\tilde{w}_3 \in \mathfrak{W}$. If we first compose the right-hand pair we get $((w_3, f_3) \circ (w_2, f_2)) \circ (w_1, f_1)$ as the span,

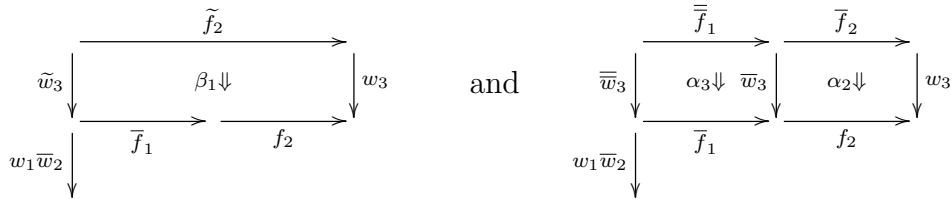


where $w_1\tilde{w}_2 \in \mathfrak{W}$ and $w_2\bar{w}_3 \in \mathfrak{W}$. The associativity 2-cell will be a vertical composite of two 2-cells going through the intermediate:

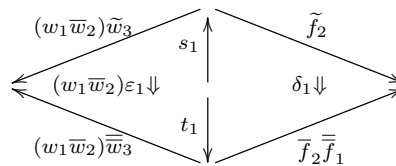


where α_3 is chosen as in [C4] with $w_1\bar{w}_2\bar{w}_3 \in \mathfrak{W}$; also $w_2\bar{w}_3 \in \mathfrak{W}$ by the choice of α_2 as in [C4]. We construct the associativity 2-cell as a vertical composition of two 2-cells: (34) \Rightarrow (36) and (36) \Rightarrow (35). (Note that by Lemma A.3 the resulting associativity 2-cell does not depend on the choice of the square α_3 .)

(34) \Rightarrow (36): the diagrams in (34) and (36) only differ in the following chosen squares:



By Proposition A.1 there is a unique 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ connecting these two squares. Let



be a diagram representing this 2-cell. Composing it with f_3 gives,

$$\begin{array}{ccc}
 & (w_1\bar{w}_2)\tilde{w}_3 & \\
 & \swarrow & \searrow \\
 & (w_1\bar{w}_2)\varepsilon_1\Downarrow & \\
 & \swarrow & \searrow \\
 & (w_1\bar{w}_2)\bar{w}_3 & \\
 \end{array}
 \begin{array}{c}
 \uparrow s_1 \\
 \downarrow t_1
 \end{array}
 \begin{array}{ccc}
 & f_3\tilde{f}_2 & \\
 & \swarrow & \searrow \\
 & f_3\delta_1\Downarrow & \\
 & \swarrow & \searrow \\
 & f_3\bar{f}_2\bar{f}_1 & \\
 \end{array}
 \tag{37}$$

(36) \Rightarrow (35): the diagrams in (36) and (35) only differ by the following two squares:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \bar{f}_1 & \\
 \bar{w}_3 \downarrow & \alpha_3 \Leftarrow & \bar{w}_3 \downarrow \\
 & \bar{f}_1 & \\
 \bar{w}_2 \downarrow & \alpha_1 \Leftarrow & w_2 \downarrow \\
 & f_1 & \\
 w_1 \downarrow & & \\
 \end{array}
 & \text{and} &
 \begin{array}{ccc}
 & \tilde{f}_1 & \\
 \tilde{w}_2 \downarrow & \beta_2 \Leftarrow & \bar{w}_3 \downarrow \\
 & f_1 & \\
 w_1 \downarrow & & \\
 \end{array}
 \end{array}$$

By Proposition A.1 there is a unique 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ connecting these two squares. Let

$$\begin{array}{ccc}
 & w_1\bar{w}_2\bar{w}_3 & \\
 & \swarrow & \searrow \\
 & w_1\bar{w}_2\varepsilon_2\Downarrow & \\
 & \swarrow & \searrow \\
 & w_1\tilde{w}_2 & \\
 \end{array}
 \begin{array}{c}
 \uparrow s_2 \\
 \downarrow t_2
 \end{array}
 \begin{array}{ccc}
 & \bar{f}_1 & \\
 & \swarrow & \searrow \\
 & \delta_2\Downarrow & \\
 & \swarrow & \searrow \\
 & \tilde{f}_1 & \\
 \end{array}$$

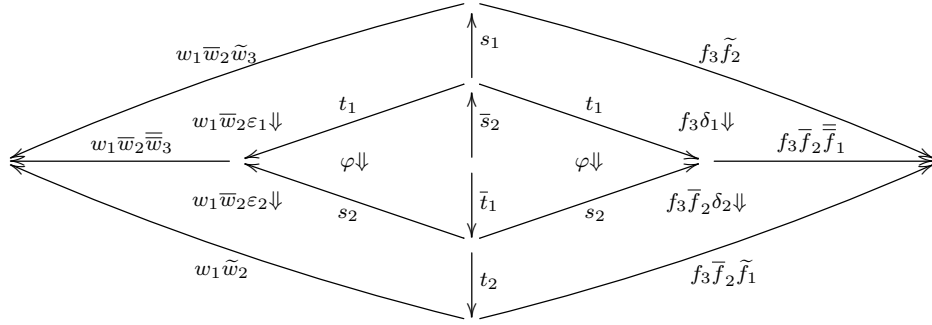
be a diagram representing this 2-cell. Composing with $f_3\bar{f}_2$ gives,

$$\begin{array}{ccc}
 & w_1\bar{w}_2\bar{w}_3 & \\
 & \swarrow & \searrow \\
 & w_1\bar{w}_2\varepsilon_2\Downarrow & \\
 & \swarrow & \searrow \\
 & w_1\tilde{w}_2 & \\
 \end{array}
 \begin{array}{c}
 \uparrow s_2 \\
 \downarrow t_2
 \end{array}
 \begin{array}{ccc}
 & f_3\bar{f}_2\bar{f}_1 & \\
 & \swarrow & \searrow \\
 & f_3\bar{f}_2\delta_2\Downarrow & \\
 & \swarrow & \searrow \\
 & f_3\bar{f}_2\tilde{f}_1 & \\
 \end{array}
 \tag{38}$$

The associativity 2-cell for the composable path given in (33) is the vertical composition of (37) and (38). To calculate this composition (as described in Section 3), we use the choices of [C5] and [C6] of Notation 3.2 to obtain a square

$$\begin{array}{ccc}
 & \bar{s}_2 & \\
 & \swarrow & \searrow \\
 & \mathcal{L} & \\
 & \swarrow & \searrow \\
 & s_2 & \\
 \end{array}
 \begin{array}{c}
 \uparrow \bar{t}_1 \\
 \downarrow t_1
 \end{array}$$

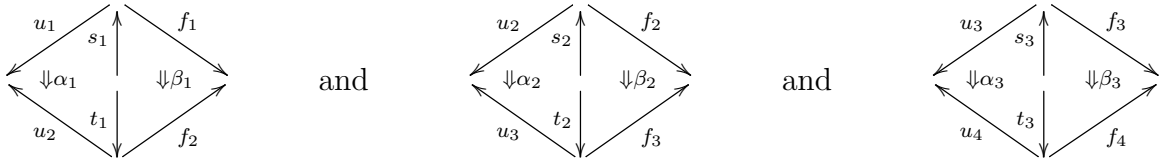
with φ invertible and $w_1\bar{w}_2\bar{w}_3s_2\bar{t}_1 \in \mathfrak{W}$. Then the associativity 2-cell $\alpha_{(w_3,f_3),(w_2,f_2),(w_1,f_1)}$ is represented by



■

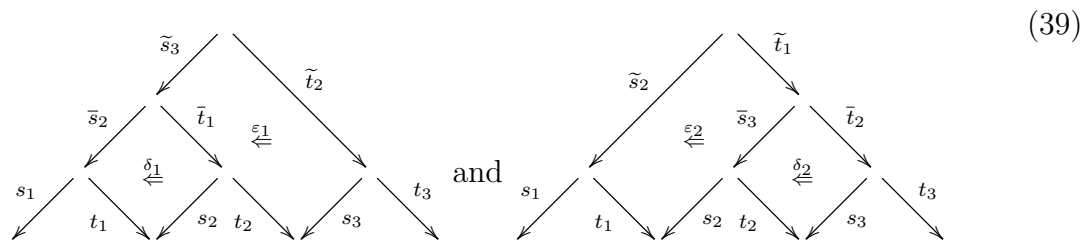
A.5 PROPOSITION. *Vertical composition of 2-cells is strictly associative.*

PROOF. Consider three vertically composable 2-cell diagrams,



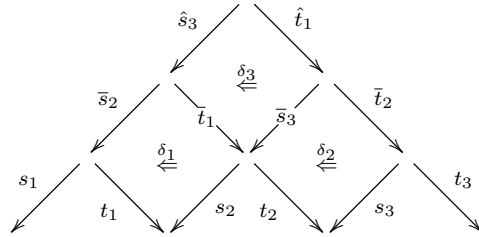
Our proof that the two ways of composing these cells vertically are equivalent will mimic the construction of the associativity isomorphism in the proof of the previous proposition. The constructed cells will in this case become the cells that witness the equivalence. However, since we are only interested in the equivalence rather than the cells witnessing it, we will do this in two steps without composing the cells obtained in the two steps.

The two possible vertical compositions correspond to choices of squares δ_i and ε_i with $i = 1, 2$ as in

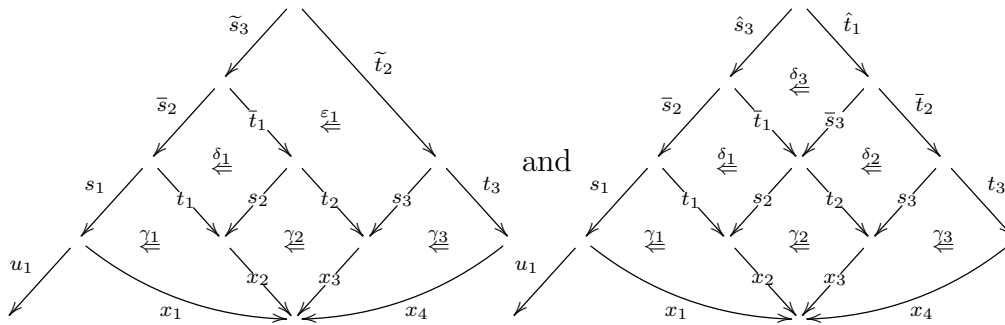


with $u_2 s_2 \bar{t}_1 \tilde{s}_3 \in \mathfrak{W}$ and $u_2 s_2 \bar{s}_3 \hat{t}_1 \in \mathfrak{W}$. We will also consider the following diagram:

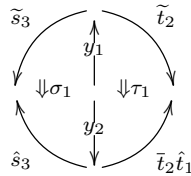
(40)



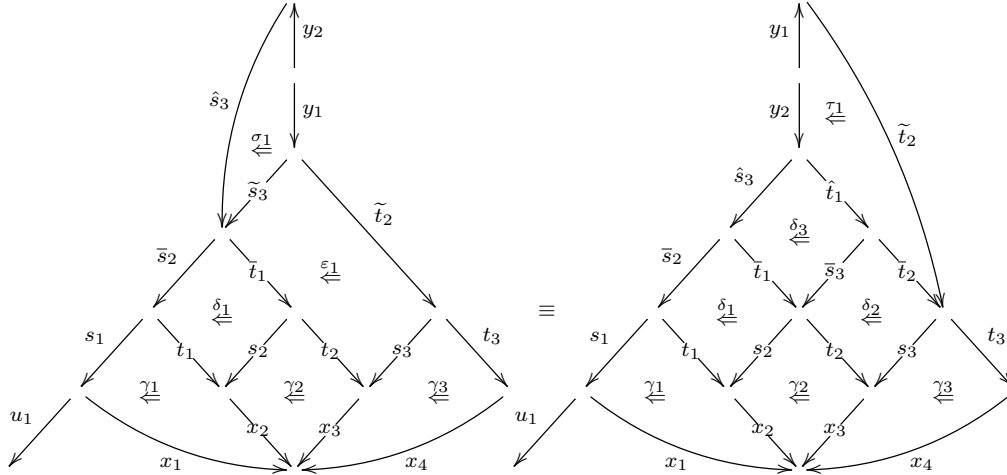
where δ_3 is an invertible 2-cell such that $u_1 s_1 \bar{s}_2 \hat{s}_3$ is in \mathfrak{W} . Note that none of these are pasting diagrams yet, but they can be made into pasting diagrams by adding the cells α_i or the cells β_i as a bottom row to the diagrams. With the α_i cells we obtain the left-hand 2-cells of our composite 2-cell diagrams and with the β_i cells we obtain the right-hand 2-cells of our composite diagrams. As we want to argue about both at the same time, we will give the argument for variable γ_1, γ_2 and γ_3 . We begin by comparing the diagrams



These two diagrams only differ in the rectangle with ε_1 versus the composition of δ_3 and δ_2 . As both $u_1 s_1 \bar{s}_2 \tilde{s}_3$ and $u_1 s_1 \bar{s}_2 \hat{s}_3$ are in \mathfrak{W} , we can apply Proposition 2.9 to these two rectangles and obtain arrows and 2-cells as in the following diagram,



with the property that



By substituting the α_i for the γ_i and by substituting the β_i for the γ_i we see that if the vertical composition had been constructed with the cells δ_1, δ_2 and δ_3 it would have been equivalent to the composition obtained by composing the first two 2-cells first. By a similar argument we see that the new composition is also equivalent to the composition obtained by composing the last two diagrams first. So we conclude that the two compositions considered are equivalent and hence vertical composition is strictly associative. ■

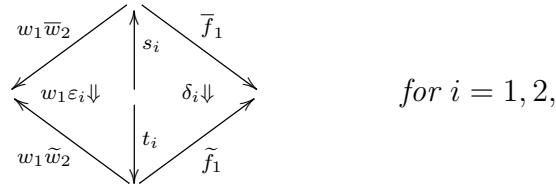
Appendix B Associativity Part II: Coherence

We will only sketch the proof for the associativity pentagon. The other coherence diagrams are straight forward. We will view the diagram (36) as a kind of common subdivision of (34) and (35), and break up the coherence into transitions given by Proposition A.1, and transitions with two layers of cells. There are two versions of this two layer case. They seem dual to each other, but their proofs are not, as the arrows in \mathfrak{W} play very different roles. The two cases are covered in Propositions B.1 and B.2 below.

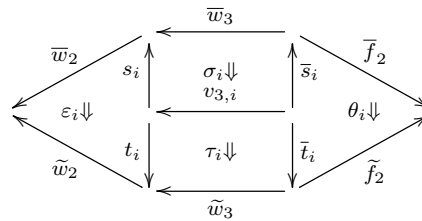
B.1 PROPOSITION. *Suppose we have two diagrams in \mathcal{B} ,*

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \xrightarrow{\bar{f}_2} & \\
 \bar{w}_3 \downarrow & \alpha_2 \Downarrow & \downarrow w_3 \\
 & \xrightarrow{\bar{f}_1} & \\
 \bar{w}_2 \downarrow & \alpha_1 \Downarrow & \downarrow w_2 \\
 & \xrightarrow{f_1} & \\
 w_1 \downarrow & &
 \end{array}
 & \text{and} &
 \begin{array}{ccc}
 & \xrightarrow{\tilde{f}_2} & \\
 \tilde{w}_3 \downarrow & \beta_2 \Downarrow & \downarrow w_3 \\
 & \xrightarrow{\tilde{f}_1} & \\
 \tilde{w}_2 \downarrow & \beta_1 \Downarrow & \downarrow w_2 \\
 & \xrightarrow{f_1} & \\
 w_1 \downarrow & &
 \end{array}
 \end{array}
 \tag{41}$$

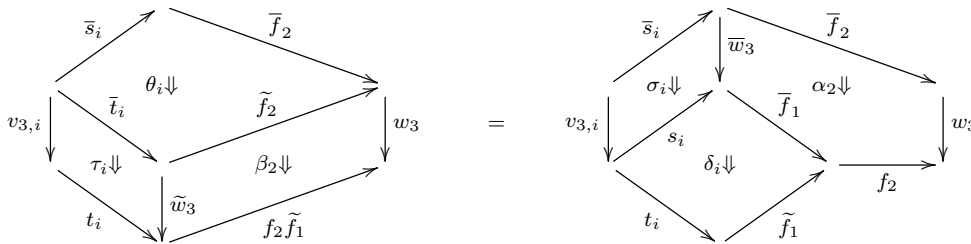
with $\alpha_1, \alpha_2, \beta_1$ and β_2 invertible and all of $w_1, w_1\bar{w}_2, w_1\bar{w}_2\bar{w}_3, w_1\tilde{w}_2$, and $w_1\tilde{w}_2\tilde{w}_3$ in \mathfrak{W} . Furthermore, suppose that we have two 2-cell diagrams



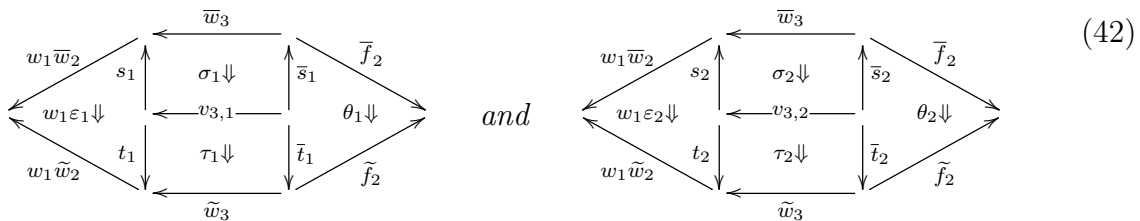
that both connect α_1 and β_1 in the sense of Notation A.2. And, suppose that there are 2-cells σ_i, τ_i and θ_i for $i = 1, 2$ as in



such that $w_1\bar{w}_2\bar{w}_3\bar{s}_i \in \mathfrak{W}$ and

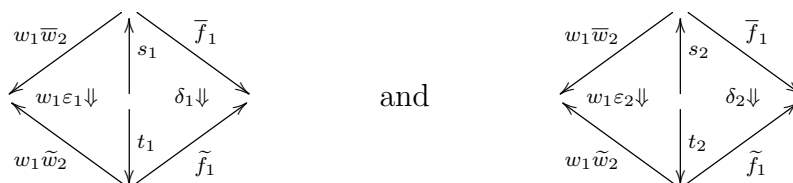


for $i = 1, 2$. Then the 2-cell diagrams,

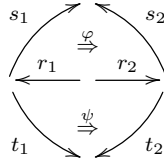


are equivalent.

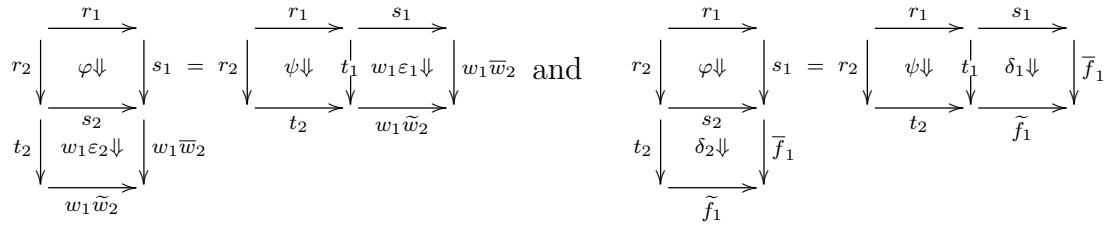
PROOF. By Proposition A.1 we know that



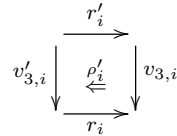
are equivalent 2-cell diagrams as they both connect α_1 and β_1 . So there are 2-cells



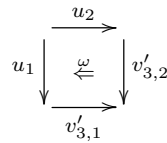
such that $w_1\bar{w}_2s_1r_1 \in \mathfrak{W}$ and



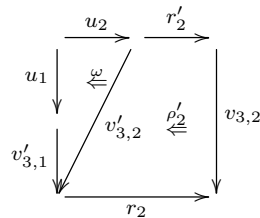
Now consider the cospan $v_{3,i} \xrightarrow{r_i} \leftarrow r'_i$. Since both $w_1\bar{w}_2s_i v_{3,i}$ and $w_1\bar{w}_2s_i r'_i$ are in \mathfrak{W} we can use conditions [WB3], [WB4] and [WB2] to obtain a square with an invertible 2-cell,



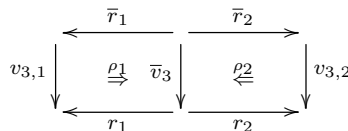
with $w_1\bar{w}_2s_i r'_i v_{3,i} \in \mathfrak{W}$. We apply the same conditions then to $w_1\bar{w}_2s_1r_1v'_{3,1}$ and $w_1\bar{w}_2s_2r_2v'_{3,2}$ to obtain a square with an invertible 2-cell,



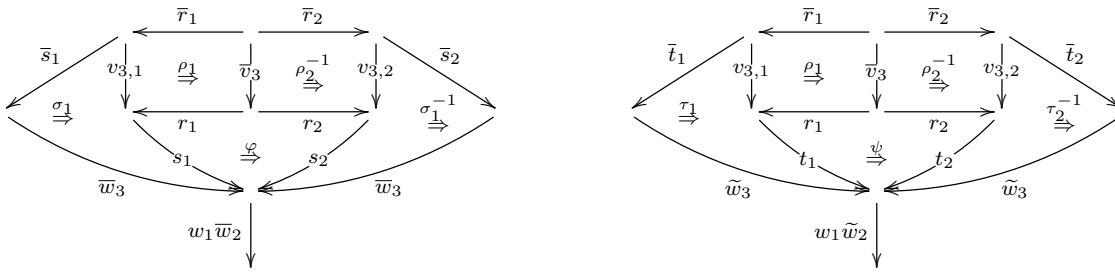
such that $w_1\bar{w}_2s_1r_1v'_{3,1}u_1 \in \mathfrak{W}$. Now write $\rho_1 := \rho'_1u_1$, $\bar{r}_1 := r'_1u_1$, $\bar{v}_3 := v'_{3,1}u_1$, and $\bar{r}_2 := r'_2u_2$. Finally, write ρ_2 for the pasting of



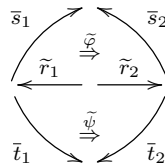
Then we obtain the following diagram,



Now consider the following two pasting diagrams,



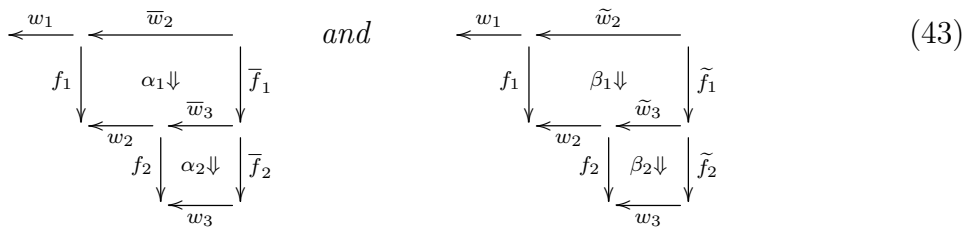
Use condition **[WB4]** to lift the first pasting with respect to $w_1\bar{w}_2\bar{w}_3$ to obtain $\varphi' : \bar{s}_1\bar{r}_1u \Rightarrow \bar{s}_2\bar{r}_2u$; similarly, apply condition **[WB4]** to the pasting of the second diagram composed with u and lift with respect to $w_1\tilde{w}_2\tilde{w}_3$ to obtain $\tilde{\psi} : \bar{t}_1\bar{r}_1uu' \Rightarrow \bar{t}_2\bar{r}_2uu'$. Now write $\tilde{r}_1 = \bar{r}_1uu'$, $\tilde{r}_2 = \bar{r}_2uu'$, and $\tilde{\varphi} = \varphi'u'$. Then the reader may check that the 2-cells



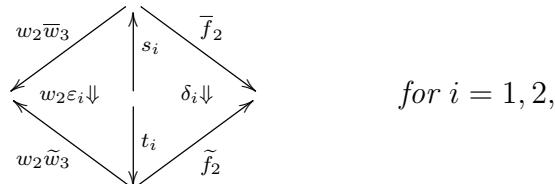
witness to the 2-cell diagrams in (42) being equivalent. ■

The following proposition is the dual to the previous one; however, the proof is quite different, due to the special role played by arrows in \mathfrak{W} .

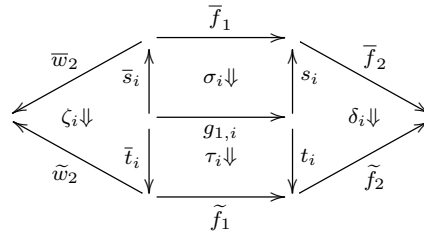
B.2 PROPOSITION. *Suppose we have two diagrams in \mathcal{B} ,*



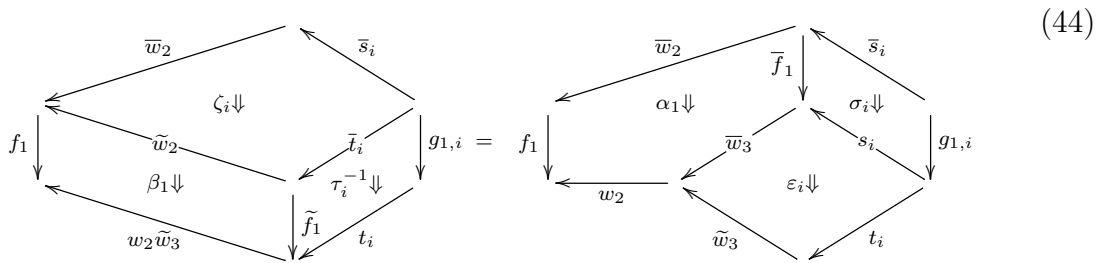
with all 2-cells invertible and all of $w_3, w_2\bar{w}_3, w_2\tilde{w}_3, w_1\bar{w}_2$, and $w_1\tilde{w}_2$ in \mathfrak{W} . Suppose further that we have two 2-cell diagrams



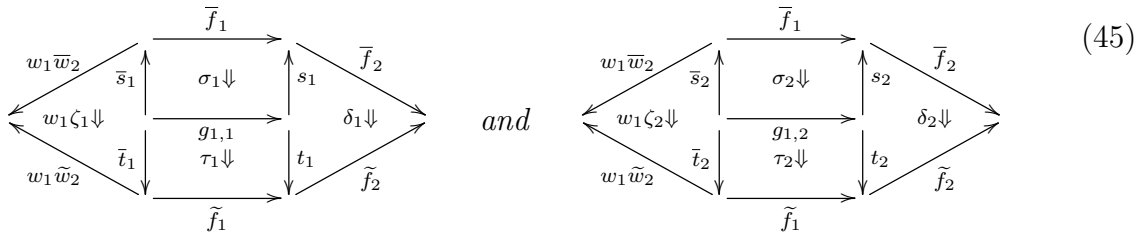
that both connect α_2 and β_2 . Suppose that there are 2-cells σ_i, τ_i and ζ_i for $i = 1, 2$ as in,



such that $w_1\bar{w}_2\bar{s}_i \in \mathfrak{W}$ for $i = 1, 2$, and

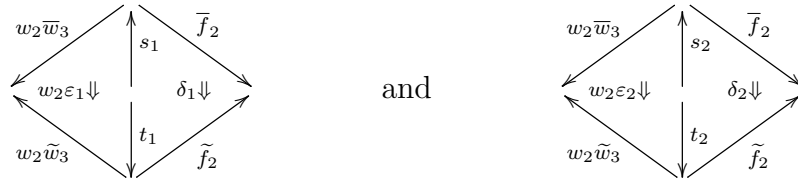


for $i = 1, 2$. Then the 2-cell diagrams,



are equivalent.

PROOF. By Proposition A.1 we know that



are equivalent 2-cell diagrams as they both connect α_2 and β_2 . So there are 2-cells



such that $w_2\bar{w}_3s_1r_1 \in \mathfrak{W}$ and

$$\begin{array}{ccc}
 \begin{array}{c} \xrightarrow{r_1} \\ r_2 \downarrow \varphi \Downarrow \\ \xrightarrow{s_2} \\ t_2 \downarrow w_2\varepsilon_2 \Downarrow \\ \xrightarrow{w_2\bar{w}_3} \end{array} & s_1 = r_2 \begin{array}{c} \xrightarrow{r_1} \xrightarrow{s_1} \\ \psi \Downarrow \downarrow t_1 \\ \xrightarrow{w_2\varepsilon_1 \Downarrow} \\ t_2 \downarrow \\ \xrightarrow{w_2\bar{w}_3} \end{array} & \text{and} & \begin{array}{c} \xrightarrow{r_1} \\ r_2 \downarrow \varphi \Downarrow \\ \xrightarrow{s_2} \\ t_2 \downarrow \delta_2 \Downarrow \\ \xrightarrow{\tilde{f}_2} \end{array} & s_1 = r_2 \begin{array}{c} \xrightarrow{r_1} \xrightarrow{s_1} \\ \psi \Downarrow \downarrow t_1 \\ \xrightarrow{\delta_1 \Downarrow} \\ t_2 \downarrow \\ \xrightarrow{\tilde{f}_2} \end{array} & \begin{array}{c} \xrightarrow{r_1} \\ r_2 \downarrow \varphi \Downarrow \\ \xrightarrow{s_2} \\ t_2 \downarrow \\ \xrightarrow{\tilde{f}_2} \end{array} & \begin{array}{c} \xrightarrow{r_1} \xrightarrow{s_1} \\ \psi \Downarrow \downarrow t_1 \\ \xrightarrow{\delta_1 \Downarrow} \\ t_2 \downarrow \\ \xrightarrow{\tilde{f}_2} \end{array} & (47)
 \end{array}$$

Since the composites $w_1\bar{w}_2\bar{s}_i \in \mathfrak{W}$ for $i = 1, 2$, we can use conditions **[WB3]**, **[WB4]** and **[WB2]** to obtain an invertible 2-cell φ' as in

$$\begin{array}{ccc}
 & \bar{s}_1 & \\
 & \swarrow & \uparrow r'_1 \\
 w_1\bar{w}_2 & \leftarrow \varphi' \Downarrow & \\
 & \searrow & \downarrow r'_2 \\
 & \bar{s}_2 &
 \end{array}$$

with $w_1\bar{w}_2\bar{s}_1r'_1 \in \mathfrak{W}$. We want to define a corresponding cell ψ' . So consider the diagram,

$$\begin{array}{ccc}
 & \bar{t}_1 & \\
 r'_1 \uparrow & \swarrow \bar{s}_1 & \xrightarrow{w_1\tilde{w}_2} \\
 & \varphi' \Downarrow & \xrightarrow{w_1\bar{w}_2} \\
 r'_2 \downarrow & \swarrow \bar{s}_2 & \xrightarrow{w_1\tilde{w}_2} \\
 & \bar{t}_2 &
 \end{array}
 \quad (48)$$

Since $w_1\tilde{w}_2 \in \mathfrak{W}$, we apply conditions **[WB4]** and **[WB2]** to lift the pasting of this diagram with respect to $w_1\tilde{w}_2$ to obtain $\psi' : \bar{t}_1r'_1w' \Rightarrow \bar{t}_2r'_2w'$. Now note that $\tilde{w}_2\psi'$ and the composite of

$$\begin{array}{ccc}
 & \bar{t}_1 & \\
 r'_1w' \uparrow & \swarrow \bar{s}_1 & \xrightarrow{\tilde{w}_2} \\
 & \varphi'w' \Downarrow & \xrightarrow{\bar{w}_2} \\
 r'_2w' \downarrow & \swarrow \bar{s}_2 & \xrightarrow{\tilde{w}_2} \\
 & \bar{t}_2 &
 \end{array}$$

are both liftings of the pasting of (48) with respect to w_1 . So by condition **[WB4]** there is an arrow w'' such that $\psi'w''$ is equal to the composition of this last pasting with w'' . We will need this in our calculations, so we write $\bar{r}_i = r'_iw'w''$, $\tilde{\varphi} = \varphi'w'w''$, and $\tilde{\psi} = \psi'w''$. This gives us the following diagram

$$\begin{array}{ccc}
 & \bar{s}_1 & \\
 & \swarrow & \uparrow \bar{s}_2 \\
 \bar{r}_1 & \leftarrow \tilde{\varphi} \Downarrow & \bar{r}_2 \\
 & \searrow & \downarrow \tilde{\psi} \\
 \bar{t}_1 & & \bar{t}_2
 \end{array}
 \quad (49)$$

These cells satisfy the required equation with the ζ_i by construction:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \xrightarrow{\bar{r}_1} & & \\
 \bar{r}_2 \downarrow & \begin{array}{c} \varphi \Downarrow \\ \bar{s}_2 \end{array} & \downarrow \bar{s}_1 \\
 \xrightarrow{\bar{t}_2} & \begin{array}{c} w_1 \zeta_2 \Downarrow \\ w_1 \tilde{w}_2 \end{array} & \downarrow w_1 \bar{w}_2
 \end{array} & = & \begin{array}{ccc}
 \xrightarrow{\bar{r}_1} & \xrightarrow{\bar{s}_1} & \\
 \bar{r}_2 \downarrow & \begin{array}{c} \tilde{\psi} \Downarrow \\ \bar{t}_1 \end{array} & \begin{array}{c} w_1 \zeta_1 \Downarrow \\ w_1 \bar{w}_2 \end{array} & \downarrow w_1 \bar{w}_2 \\
 \xrightarrow{\bar{t}_2} & \xrightarrow{w_1 \bar{w}_2} &
 \end{array}
 \end{array}$$

We will next see that after precomposing with an appropriate arrow they will also satisfy the equation for the composites of the right-hand sides of (45). Since the cells φ and ψ satisfy the equation with the δ_i as stated in (47), we will focus on the cylinder with the diagram (46) as bottom and (49) as top. The sides of this cylinder are given by

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \xleftarrow{\bar{s}_1} & \xrightarrow{\bar{t}_1} & \\
 \bar{f}_1 \downarrow & \begin{array}{c} \sigma_1 \Downarrow \\ g_{1,1} \end{array} & \begin{array}{c} \tau_1 \Downarrow \\ t_1 \end{array} & \downarrow \tilde{f}_1 \\
 \xleftarrow{s_1} & \xrightarrow{t_1} &
 \end{array} & \text{and} & \begin{array}{ccc}
 \xleftarrow{\bar{s}_2} & \xrightarrow{\bar{t}_2} & \\
 \bar{f}_1 \downarrow & \begin{array}{c} \sigma_2 \Downarrow \\ g_{1,2} \end{array} & \begin{array}{c} \tau_2 \Downarrow \\ t_2 \end{array} & \downarrow \tilde{f}_1 \\
 \xleftarrow{s_2} & \xrightarrow{t_2} &
 \end{array}
 \end{array}$$

Before we can discuss the commutativity of this cylinder, we need to build cells to fill in the following frame,

$$\begin{array}{ccc}
 \xleftarrow{\bar{r}_1} & \xrightarrow{\bar{r}_2} & \\
 g_{1,1} \downarrow & & \downarrow g_{1,2} \\
 \xleftarrow{r_1} & \xrightarrow{r_2} &
 \end{array}$$

Since $w_1 \bar{w}_2 s_1 r_1 \in \mathfrak{W}$, we can use conditions [WB3], [WB4] and [WB2] to construct an invertible 2-cell ρ_1 as in

$$\begin{array}{ccc}
 \xrightarrow{u} & \xrightarrow{\bar{r}_1} & \\
 h_1 \downarrow & \begin{array}{c} \rho_1 \Downarrow \\ r_1 \end{array} & \downarrow g_{1,1} \\
 \xrightarrow{r_1} & &
 \end{array}$$

where $w_1 \bar{w}_2 \bar{s}_1 \bar{r}_1 u \in \mathfrak{W}$. Use this to construct a left-hand square in the frame. To obtain a cell to fill the remaining right-hand square, we consider the following pasting diagram,

$$\begin{array}{ccccccc}
 & & u & \xrightarrow{\quad} & \bar{r}_2 & \xrightarrow{\quad} & \\
 & & \downarrow & & \downarrow \bar{r}_1 & & \downarrow g_{1,2} \\
 h_1 & & \rho_1 \Downarrow & & \tilde{\psi}^{-1} \Downarrow & & \bar{t}_2 \\
 & & \downarrow & & \downarrow \bar{t}_1 & & \downarrow \tau_2 \\
 & & g_{1,1} & \xrightarrow{\quad} & \tau_1^{-1} \Downarrow & & \tilde{f}_1 \\
 & & \downarrow & & \downarrow t_1 & & \downarrow t_2 \\
 & & r_1 & \xrightarrow{\quad} & t_1 & \xrightarrow{\quad} & w_2 \bar{w}_3 \\
 & & \downarrow & & \downarrow \psi & & \\
 & & r_2 & \xrightarrow{\quad} & t_2 & \xrightarrow{\quad} &
 \end{array}$$

Now lift with respect to $w_2\tilde{w}_3t_2$ to obtain $\rho_2: g_{1,2}\bar{r}_2\tilde{u}\tilde{t} \Rightarrow r_2h_1\tilde{t}$. So the middle frame gets filled as follows:

$$\begin{array}{ccc}
 \overleftarrow{\bar{r}_1\tilde{u}\tilde{t}} & \overrightarrow{\bar{r}_2\tilde{u}\tilde{t}} & \\
 \downarrow g_{1,1} & \downarrow h_1\tilde{t} & \downarrow g_{1,2} \\
 \rho_1\tilde{t}\Downarrow & & \rho_2\Downarrow \\
 \overleftarrow{r_1} & \overrightarrow{r_2} &
 \end{array}$$

Furthermore, we have adjusted the top of the cylinder to become

$$\begin{array}{ccc}
 \bar{s}_1 & \overleftarrow{\varphi\tilde{u}\tilde{t}} & \bar{s}_2 \\
 \overleftarrow{\bar{r}_1\tilde{u}\tilde{t}} & \overrightarrow{\bar{r}_2\tilde{u}\tilde{t}} & \\
 \overleftarrow{\bar{t}_1} & \overrightarrow{\tilde{\psi}\tilde{u}\tilde{t}} & \overleftarrow{\bar{t}_2}
 \end{array}$$

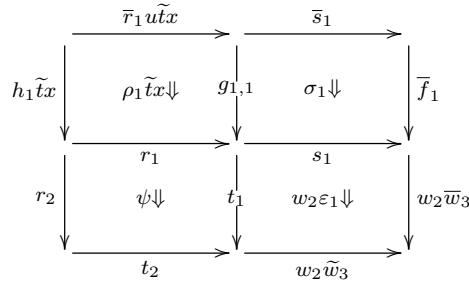
We have defined ρ_2 in such a way that if the half of the cylinder that contains the $\psi, \tilde{\psi}, \tau_1$ and τ_2 gets composed with $w_2\tilde{w}_3$ it commutes. Condition **[WB4]** now gives that there is an arrow x such that if we precompose the top of the cylinder and the middle frame both with x , this half of the cylinder commutes. So now the top and the middle frame are respectively,

$$\begin{array}{ccc}
 \bar{s}_1 & \overleftarrow{\varphi\tilde{u}\tilde{t}x} & \bar{s}_2 \\
 \overleftarrow{\bar{r}_1\tilde{u}\tilde{t}x} & \overrightarrow{\bar{r}_2\tilde{u}\tilde{t}x} & \\
 \overleftarrow{\bar{t}_1} & \overrightarrow{\tilde{\psi}\tilde{u}\tilde{t}x} & \overleftarrow{\bar{t}_2}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \overleftarrow{\bar{r}_1\tilde{u}\tilde{t}x} & \overrightarrow{\bar{r}_2\tilde{u}\tilde{t}x} & \\
 \downarrow g_{1,1} & \downarrow h_1\tilde{t}x & \downarrow g_{1,2} \\
 \rho_1\tilde{t}x\Downarrow & & \rho_2x\Downarrow \\
 \overleftarrow{r_1} & \overrightarrow{r_2} &
 \end{array}$$

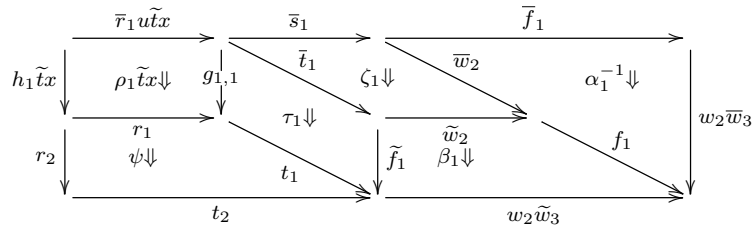
To investigate the commutativity of the other half of the cylinder, we will show that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \overleftarrow{\bar{r}_1\tilde{u}\tilde{t}x} & \overrightarrow{\bar{s}_1} & \\
 \downarrow h_1\tilde{t}x & \downarrow g_{1,1} & \downarrow \bar{f}_1 \\
 \rho_1\tilde{t}x\Downarrow & & \sigma_1\Downarrow \\
 \overleftarrow{r_1} & \overrightarrow{s_1} & \\
 \downarrow r_2 & \downarrow \varphi\Downarrow & \downarrow s_2 \\
 & w_2\varepsilon_2\Downarrow & \\
 & t_2 & w_2\tilde{w}_3
 \end{array}
 & = &
 \begin{array}{ccc}
 \overleftarrow{\bar{r}_1\tilde{u}\tilde{t}x} & \overrightarrow{\bar{s}_1} & \\
 \downarrow h_1\tilde{t}x & \downarrow \bar{f}_1 & \\
 \overleftarrow{\bar{r}_2\tilde{u}\tilde{t}x} & \overrightarrow{\bar{s}_2} & \\
 \downarrow g_{1,2} & \downarrow \sigma_2\Downarrow & \\
 \rho_2x\Downarrow & & \\
 \overleftarrow{r_2} & \overrightarrow{s_2} & \\
 \downarrow t_2 & \downarrow w_2\varepsilon_2\Downarrow & \\
 & w_2\tilde{w}_3 &
 \end{array}
 \end{array}$$

We begin by rewriting the left-hand side. By (47) this pasting is equal to the pasting of



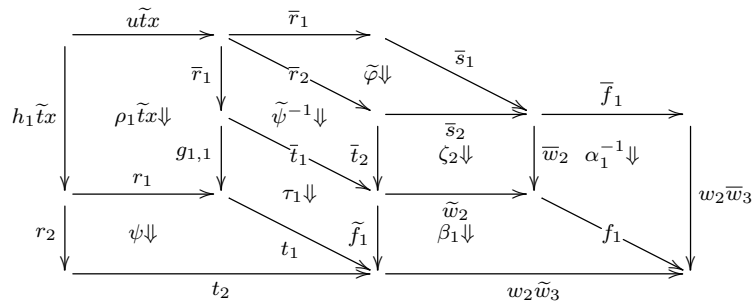
We use (47) to rewrite the right two 2-cells in this diagram to get



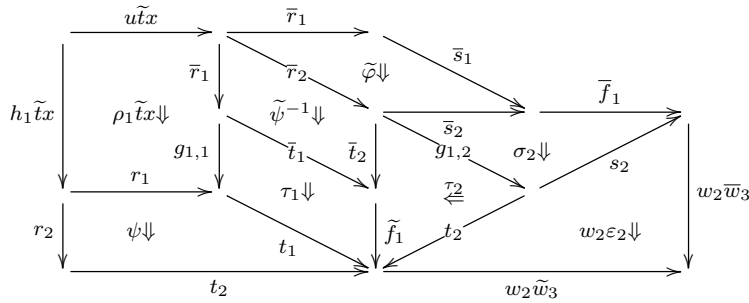
Now note that we have constructed $\tilde{\varphi}$ and $\tilde{\psi}$ such that

$$\begin{array}{ccc} \bar{s}_1 \bar{r}_1 & \xrightarrow{\quad} & \\ \bar{t}_1 \bar{r}_1 \Downarrow & \zeta_1 \bar{r}_1 \Downarrow & \bar{w}_2 \\ & \xrightarrow{\quad} & \bar{w}_2 \end{array} = \begin{array}{ccccc} & \bar{r}_1 & & \bar{s}_1 & \\ \bar{r}_1 \Downarrow & \bar{r}_2 & \tilde{\varphi} \Downarrow & \bar{s}_1 & \\ & \tilde{\psi}^{-1} \Downarrow & & \bar{s}_2 & \\ & \bar{t}_1 & \bar{t}_2 & \zeta_2 \Downarrow & \bar{w}_2 \\ & & & \bar{w}_2 & \\ & & & \tilde{w}_2 & \end{array}$$

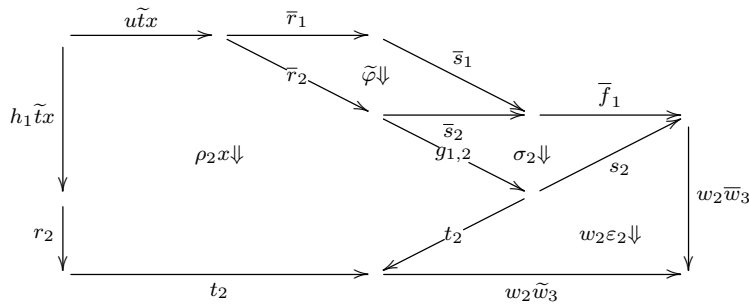
so we make this substitution in the diagram above to obtain,



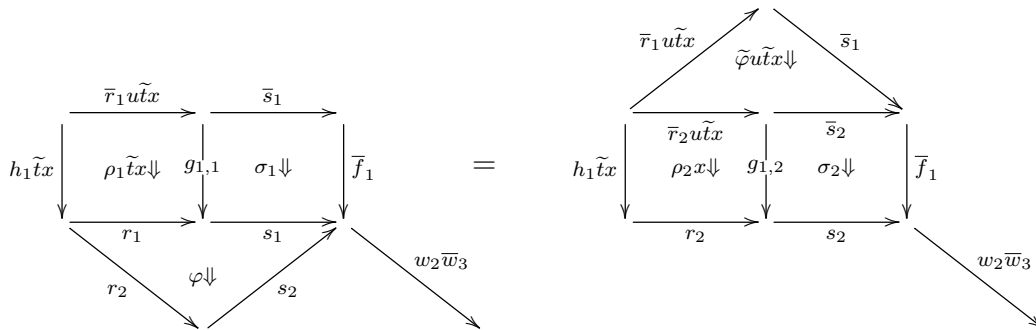
We use (47) again; this time to rewrite the bottom right-hand corner of the diagram:



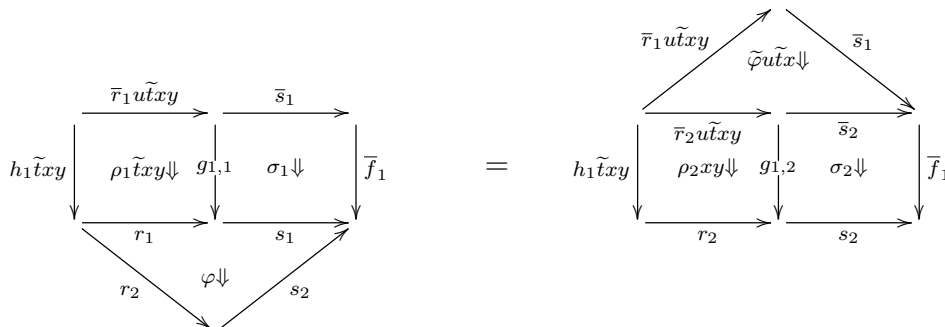
and by the definition of ρ_2 , this is equal to



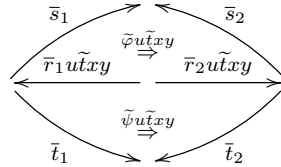
This completes our proof of equation (50). Since ε_2 is invertible we can compose both sides of (50) by $w_2\varepsilon_2^{-1}r_2h_1\tilde{t}x$ and it follows that



It follows from condition [WB4] that there is an arrow y such that



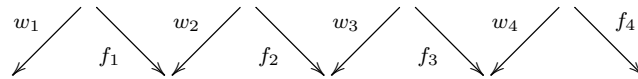
Hence, it follows from the arguments above that the cells



witness to the equivalence of the 2-cell diagrams in (45). ■

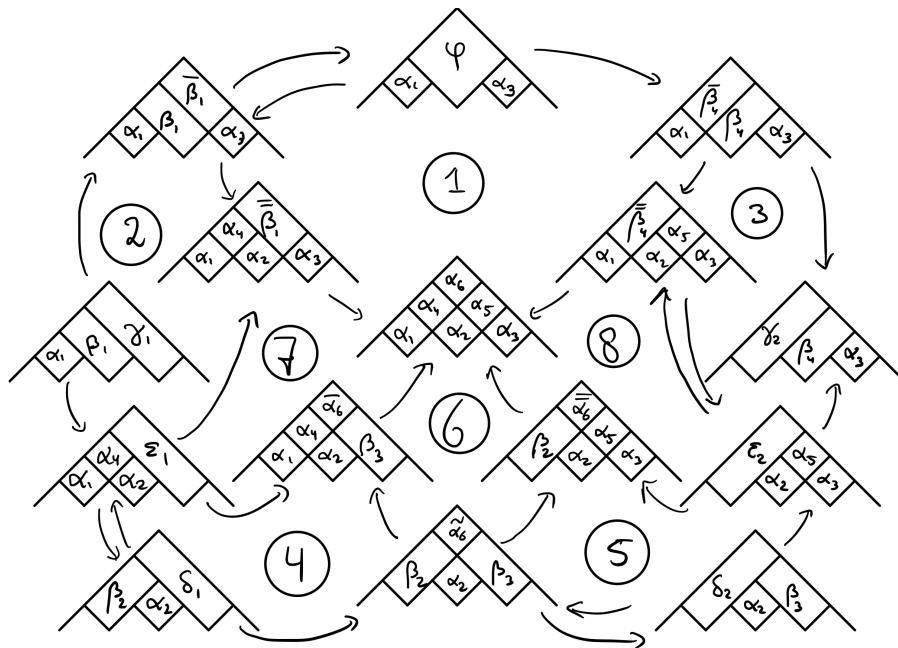
B.3 REMARK. Analogous to the situation in Proposition A.1, we say that the 2-cell diagrams in (42) (respectively in (45)) *connect* the 2-cell configurations in (41) (respectively (43)). Propositions B.1 and B.2 only state uniqueness results, but it is not hard to prove existence as well. Since we will only need uniqueness in the proof of associativity coherence, we will not include the proofs of existence.

B.4 PROPOSITION. *For any composable path of four spans,*



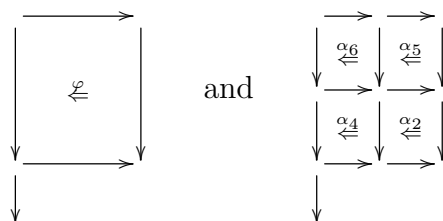
the associativity 2-cells defined in Propostion A.4 make the associativity coherence pentagon commute.

PROOF. The following diagram shows the associativity coherence pentagon.



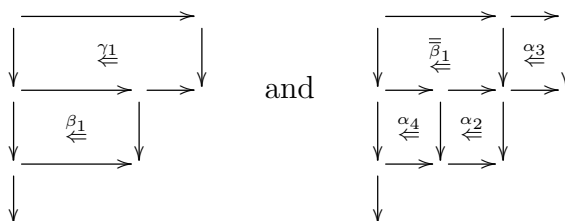
We have divided the pentagon into regions corresponding to various subdivisions, and we will show that each region commutes by one of the three results in Propositions A.1, B.1 and B.2. We sketch the argument for each region, leaving the details for the reader.

For region ① both composites provide a whiskering of a 2-cell that connects the squares



Since there is only one such 2-cell by Proposition A.1, this region commutes.

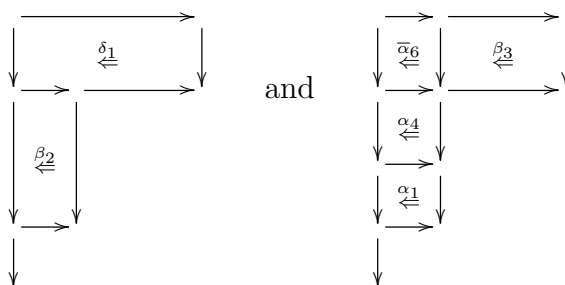
For region ② the two compositions connect the diagrams



as in Proposition B.1.

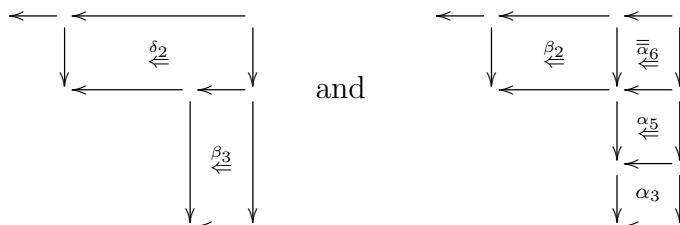
Region ③ is the dual of region ② and follows from Proposition B.2.

For region ④ commutativity is obtained from Proposition B.1 applied to



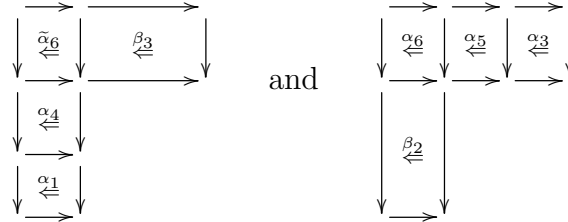
where we view the pasting of α_1 and α_4 as a single cell.

Region ⑤ is the dual of region ④ and commutativity can be obtained by applying Proposition B.2 to



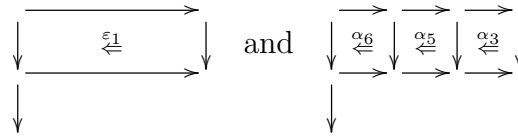
where we view the pasting of α_5 and α_3 as a single cell and the pasting of $\bar{\alpha}_6$ and β_2 as a single cell.

Region ⑥ could be done with an application of either Proposition B.1 or Proposition B.2. If we use Proposition B.1, we focus on the diagrams,



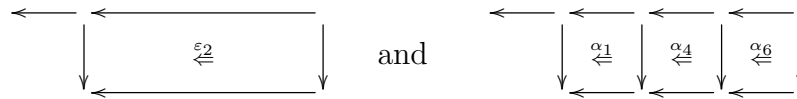
Here we consider the pasting of α_4 and α_1 as a single cell, the pasting of $\tilde{\alpha}_6$ and β_3 as a single cell, and the pasting of α_6 , α_5 and α_3 as a single cell.

For region ⑦ the two ways of composing provide to 2-cells that connect the rectangles,



and there is only one such cell by Proposition A.1, so this region commutes.

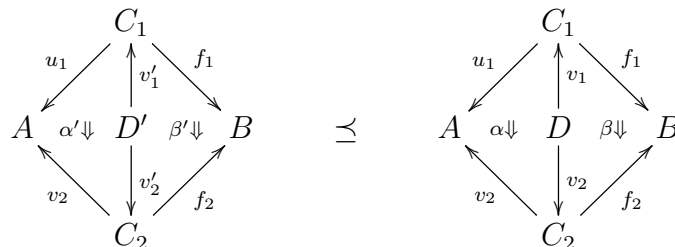
Region ⑧ is the dual of region ⑦ whose two compositions give the 2-cell connecting the rectangles,



■

Appendix C Well-Definedness of Composition

In this appendix we show that vertical composition and horizontal whiskering are well-defined on equivalence classes of 2-cell diagrams. We start by observing that the equivalence relation on 2-cell diagrams is generated by the following non-symmetric relation:



if there are invertible 2-cells γ_1, γ_2 such that

$$\begin{array}{ccc}
 & C_1 & \\
 v'_1 \nearrow & & \searrow u_1 \\
 D' & \xrightarrow[t]{} D & \xrightarrow{\alpha \Downarrow} A \\
 v'_2 \searrow & & \nearrow u_2 \\
 & C_2 & \\
 \end{array}
 \begin{array}{c}
 \xrightarrow{\gamma_1} \\
 \xleftarrow{\gamma_2}
 \end{array}
 =
 \begin{array}{ccc}
 & C_1 & \\
 v'_1 \nearrow & & \searrow u_1 \\
 D' & & \xrightarrow{\alpha' \Downarrow} A \\
 v'_2 \searrow & & \nearrow u_2 \\
 & C_2 & \\
 \end{array}$$

and

$$\begin{array}{ccc}
 & C_1 & \\
 v'_1 \nearrow & & \searrow f_1 \\
 D' & \xrightarrow[t]{} D & \xrightarrow{\beta \Downarrow} B \\
 v'_2 \searrow & & \nearrow f_2 \\
 & C_2 & \\
 \end{array}
 \begin{array}{c}
 \xrightarrow{\gamma_1} \\
 \xleftarrow{\gamma_2}
 \end{array}
 =
 \begin{array}{ccc}
 & C_1 & \\
 v'_1 \nearrow & & \searrow f_1 \\
 D' & & \xrightarrow{\beta' \Downarrow} B \\
 v'_2 \searrow & & \nearrow f_2 \\
 & C_2 & \\
 \end{array}$$

where $u_1 t \in \mathcal{W}$ (equivalently, $u'_1 \in \mathcal{W}$). So it is sufficient to check well-definedness with respect to this relation. The main tool we will use for this is Proposition 2.9. We will repeatedly create squares that can be compared using this proposition and the cells produced that way will show that the 2-cell diagrams resulting from composing or whiskering equivalent 2-cell diagrams are again equivalent.

C.1 PROPOSITION. *Vertical composition of 2-cell diagrams is well-defined on equivalence classes.*

PROOF. Consider two 2-cell diagrams

$$\begin{array}{ccc}
 & & \\
 u_1 \nearrow & & \searrow f_1 \\
 \Downarrow \alpha_1 & & \Downarrow \beta_1 \\
 u_2 \searrow & & \nearrow f_2 \\
 & & \\
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 & & \\
 u_2 \nearrow & & \searrow f_2 \\
 \Downarrow \alpha_2 & & \Downarrow \beta_2 \\
 u_3 \searrow & & \nearrow f_3 \\
 & & \\
 \end{array}
 \tag{51}$$

as in Section 3 and two 2-cell diagrams

$$\begin{array}{ccc}
 & & \\
 u_1 \nearrow & & \searrow f_1 \\
 \Downarrow \alpha'_1 & & \Downarrow \beta'_1 \\
 u_2 \searrow & & \nearrow f_2 \\
 & & \\
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 & & \\
 u_2 \nearrow & & \searrow f_2 \\
 \Downarrow \alpha'_2 & & \Downarrow \beta'_2 \\
 u_3 \searrow & & \nearrow f_3 \\
 & & \\
 \end{array}
 \tag{52}$$

with 2-cells $\gamma_1, \gamma_2, \gamma_3$ and γ_4 such that

$$\begin{array}{ccc}
 & C_1 & \\
 w_1 \nearrow & \uparrow v_1 & \searrow u_1 \\
 D' \xrightarrow{t_1} D & \xrightarrow{\alpha_1 \Downarrow} & A \\
 w_2 \searrow & \downarrow v_2 & \nearrow u_2 \\
 & C_2 &
 \end{array}
 \begin{array}{c}
 \xrightarrow{\gamma_1} \\
 \xleftarrow{\gamma_2}
 \end{array}
 =
 \begin{array}{ccc}
 & C_1 & \\
 w_1 \nearrow & \uparrow & \searrow u_1 \\
 D' \xrightarrow{\quad} D & \xrightarrow{\alpha'_1 \Downarrow} & A \\
 w_2 \searrow & \downarrow & \nearrow u_2 \\
 & C_2 &
 \end{array}
 \tag{53}$$

$$\begin{array}{ccc}
 & C_1 & \\
 w_1 \nearrow & \uparrow v_1 & \searrow f_1 \\
 D' \xrightarrow{t_1} D & \xrightarrow{\beta_1 \Downarrow} & B \\
 w_2 \searrow & \downarrow v_2 & \nearrow f_2 \\
 & C_2 &
 \end{array}
 \begin{array}{c}
 \xrightarrow{\gamma_1} \\
 \xleftarrow{\gamma_2}
 \end{array}
 =
 \begin{array}{ccc}
 & C_1 & \\
 w_1 \nearrow & \uparrow & \searrow f_1 \\
 D' \xrightarrow{\quad} D & \xrightarrow{\beta'_1 \Downarrow} & B \\
 w_2 \searrow & \downarrow & \nearrow f_2 \\
 & C_2 &
 \end{array}
 \tag{54}$$

$$\begin{array}{ccc}
 & C_1 & \\
 w_4 \nearrow & \uparrow v_3 & \searrow u_1 \\
 D' \xrightarrow{t_2} D & \xrightarrow{\alpha_2 \Downarrow} & A \\
 w_3 \searrow & \downarrow v_4 & \nearrow u_2 \\
 & C_2 &
 \end{array}
 \begin{array}{c}
 \xrightarrow{\gamma_3} \\
 \xleftarrow{\gamma_4}
 \end{array}
 =
 \begin{array}{ccc}
 & C_1 & \\
 w_3 \nearrow & \uparrow & \searrow u_1 \\
 D' \xrightarrow{\quad} D & \xrightarrow{\alpha'_2 \Downarrow} & A \\
 w_4 \searrow & \downarrow & \nearrow u_2 \\
 & C_2 &
 \end{array}
 \tag{55}$$

and

$$\begin{array}{ccc}
 & C_1 & \\
 w_3 \nearrow & \uparrow v_3 & \searrow f_1 \\
 D' \xrightarrow{t_2} D & \xrightarrow{\beta_2 \Downarrow} & B \\
 w_4 \searrow & \downarrow v_4 & \nearrow f_2 \\
 & C_2 &
 \end{array}
 \begin{array}{c}
 \xrightarrow{\gamma_3} \\
 \xleftarrow{\gamma_4}
 \end{array}
 =
 \begin{array}{ccc}
 & C_1 & \\
 w_3 \nearrow & \uparrow & \searrow f_1 \\
 D' \xrightarrow{\quad} D & \xrightarrow{\beta'_2 \Downarrow} & B \\
 w_4 \searrow & \downarrow & \nearrow f_2 \\
 & C_2 &
 \end{array}
 \tag{56}$$

Vertical composition of the two 2-cell diagrams in (51) is given by

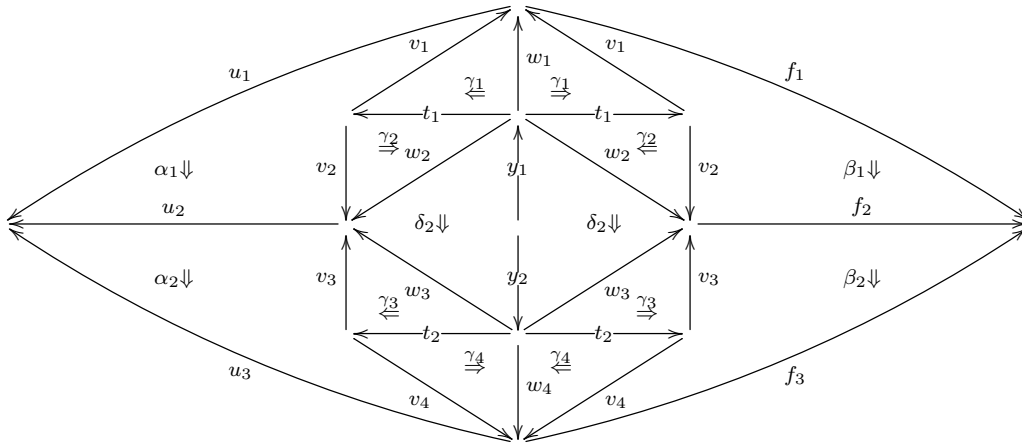
(57)

and vertical composition of the two 2-cell diagrams in (52) is given by:

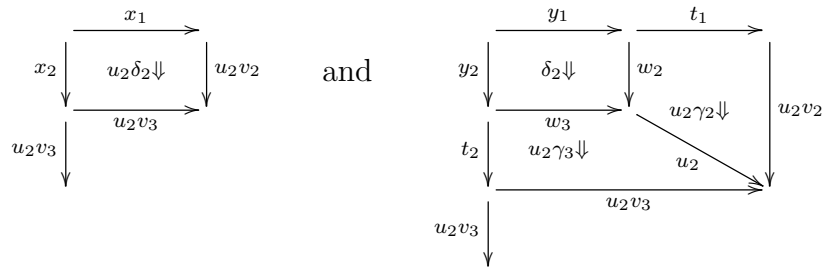
(58)

for suitable arrows x_1, x_2, y_1, y_2 such that $u_1 v_1 x_1$ and $u_1 v_1 y_1$ are in \mathfrak{W} and suitable invertible 2-cells δ_1 and δ_2 . By equations (53)-(56) the 2-cell diagram (58) can be rewritten as:

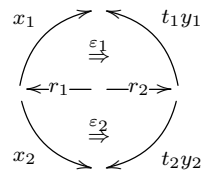
(59)



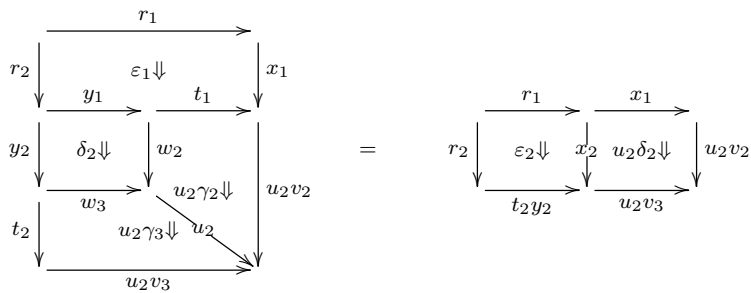
We can now apply Proposition 2.9 to



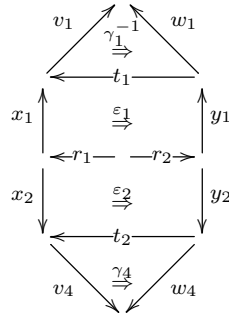
This gives us invertible 2-cells ε_1 and ε_2 as in the following diagram,



where $u_2v_3x_2r_1 \in \mathfrak{W}$ and such that



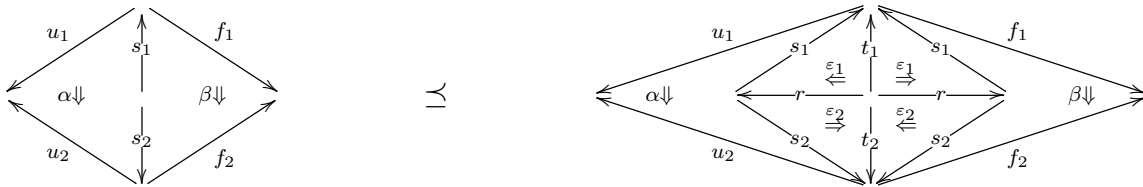
Now the reader may check that the following diagram can be used to show that the 2-cells diagrams (57) and (58) are equivalent:



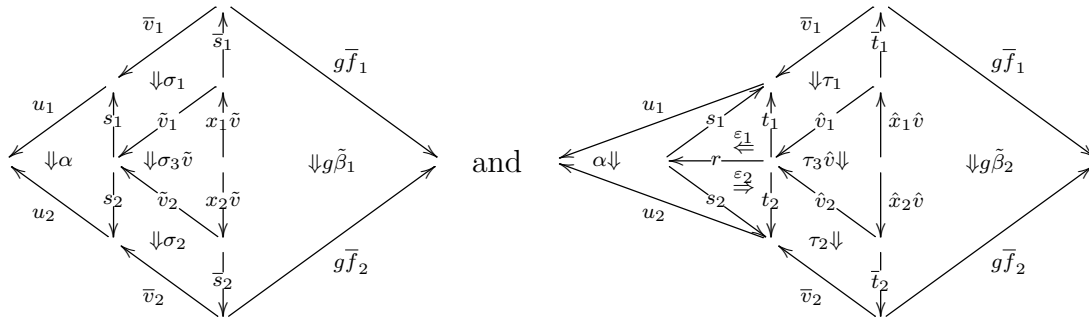
■

C.2 PROPOSITION. *Left whiskering of a 2-cell diagram and an arrow in the bicategory of fractions is well-defined on equivalence classes of 2-cell diagrams.*

PROOF. We will again consider a generator of the equivalence relation:

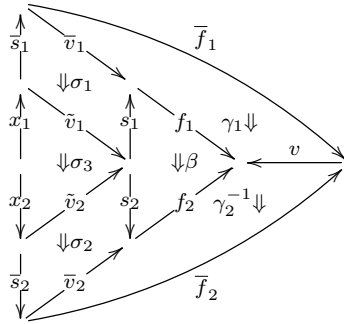


Whiskering these 2-cell diagrams with $\leftarrow \overset{v}{\underset{g}{\longrightarrow}}$ gives us the following 2-cell diagrams:

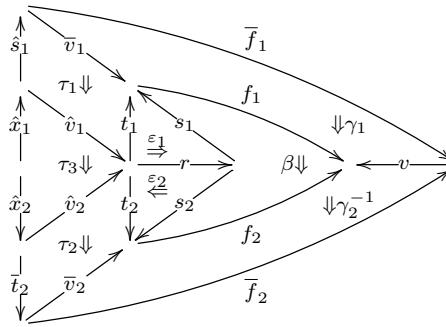


(60)

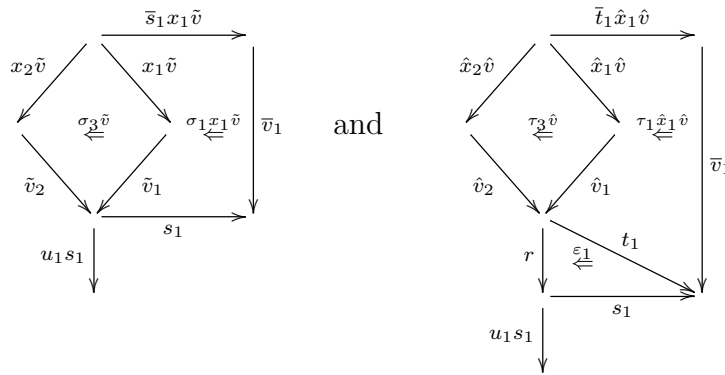
where $\tilde{\beta}_1$ is the lifting with respect to v of



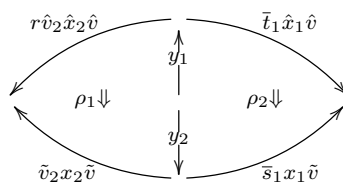
and $\tilde{\beta}_2$ is the lifting with respect to v of:



To show that 2-cell diagrams in (60) are equivalent, we begin by applying Proposition 2.9 to the following two diagrams:



This gives us arrows y_1 and y_2 and cells ρ_1 and ρ_2 as in



with the property that

$$\begin{array}{ccc}
 \begin{array}{c}
 \begin{array}{ccc}
 & y_1 & \\
 \xrightarrow{\quad} & & \xrightarrow{\quad} \\
 \hat{x}_2 \hat{v} & & \hat{x}_1 \hat{v} \\
 \swarrow & \tau_3 \hat{v} & \searrow \\
 \hat{v}_2 & & \hat{v}_1 \\
 \downarrow r & \varepsilon_1 \hat{v} & \downarrow t_1 \\
 \hat{v}_2 x_2 \hat{v} & & s_1
 \end{array} \\
 \begin{array}{c}
 y_2 \\
 \downarrow \rho_1 \\
 \hat{v}_2
 \end{array}
 \end{array}
 \quad \equiv \quad
 \begin{array}{c}
 \begin{array}{ccc}
 & y_1 & \\
 \xrightarrow{\quad} & & \xrightarrow{\quad} \\
 x_2 \hat{v} & & x_1 \hat{v} \\
 \swarrow & \sigma_3 \hat{v} & \searrow \\
 \hat{v}_2 & & \hat{v}_1 \\
 \downarrow r & \varepsilon_1 \hat{v} & \downarrow t_1 \\
 \hat{v}_2 x_2 \hat{v} & & s_1
 \end{array} \\
 \begin{array}{c}
 y_2 \\
 \downarrow \rho_2 \\
 \bar{s}_1 x_1 \hat{v}
 \end{array}
 \end{array}
 \end{array}
 \tag{61}$$

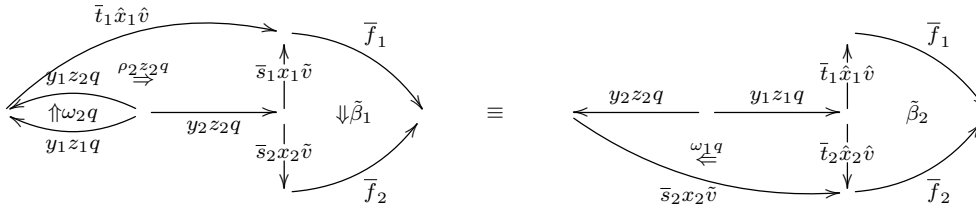
Now we apply Proposition 2.9 to the following two diagrams:

$$\begin{array}{ccc}
 \begin{array}{c}
 \begin{array}{ccc}
 & y_1 & \\
 \xrightarrow{\quad} & & \xrightarrow{\quad} \\
 \hat{v} & & \hat{x}_2 \\
 \downarrow \rho_1 & & \downarrow \\
 \hat{v}_2 & & r \hat{v}_2 \\
 \downarrow \bar{s}_2 & \sigma_2 & \downarrow s_2 \\
 \bar{v}_2 & & s_2 \\
 \downarrow u_2 \bar{v}_2 & & \downarrow
 \end{array}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{c}
 \begin{array}{ccc}
 & y_1 & \\
 \xrightarrow{\quad} & & \xrightarrow{\quad} \\
 \hat{x}_2 & & \hat{v}_2 \\
 \downarrow \tau_2 \hat{x}_2 \hat{v} y_1 & & \downarrow r \\
 \hat{x}_2 & & \hat{v}_2 \\
 \downarrow \bar{t}_2 & t_2 & \downarrow \varepsilon_2 \\
 \bar{v}_2 & & s_2 \\
 \downarrow u_2 \bar{v}_2 & & \downarrow
 \end{array}
 \end{array}
 \end{array}$$

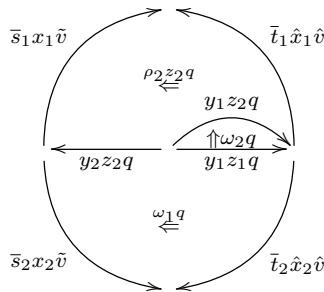
this gives us arrows z_1 and z_2 and cells ω_1 and ω_2 as in the following diagram

$$\begin{array}{ccc}
 \bar{t}_2 \hat{x}_2 \hat{v} y_1 & & y_1 \\
 \swarrow & \uparrow z_1 & \searrow \\
 \omega_1 \Downarrow & & \omega_2 \Downarrow \\
 \bar{s}_2 x_2 \hat{v} y_2 & & y_1 \\
 \swarrow & \downarrow z_2 & \searrow
 \end{array}$$

So by Lemma 2.7 and **WB2** we get an arrow q such that $u_2 s_2 r \hat{v}_2 \hat{x}_2 \hat{v}_1 y_1 z_1 q \in \mathfrak{W}$ and



Applying the calculation above with α instead of δ gives us the remaining result to conclude that the arrows and cells in

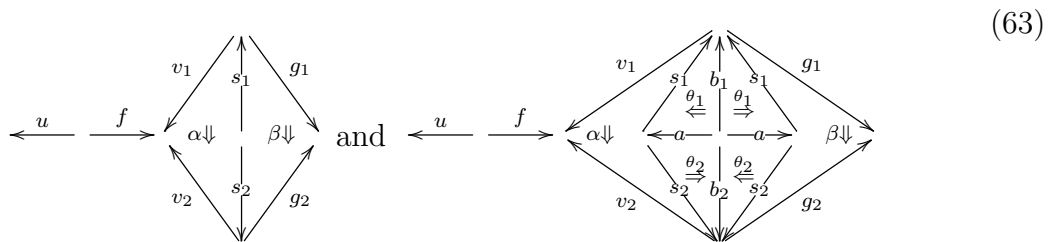


witness to the fact that the two cell diagrams in (60) are equivalent. We conclude that left-whiskering is well-defined on equivalence classes of 2-cell diagrams. ■

C.3 PROPOSITION. *Right whiskering of a 2-cell diagram and an arrow in the bicategory of fractions is well-defined on equivalence classes of 2-cell diagrams.*

PROOF. We will sketch the proof of this result as the details get rather involved and don't necessarily illuminate the idea behind the proof. Any interested reader is welcome to contact the authors for further details.

Consider the following whiskering diagrams:



We want to show that the 2-cell diagrams that result after whiskering are equivalent.

consider 2-cell diagrams comparing the following four squares:

$$\begin{array}{ccc}
 \begin{array}{c}
 \begin{array}{ccccc}
 & \tilde{r}_2x & \xrightarrow{p\tilde{r}_1} & f'_1 & \\
 \tilde{r}_2x \downarrow & \rho_1^{-1}\tilde{r}_2x \downarrow & \nearrow r_1 & & \\
 \bar{p} & & \varphi_1 \downarrow & & \\
 t_1 & & \bar{f}_1 & & \\
 \bar{v}_1 & \xrightarrow{\gamma_1 \downarrow} & & \xrightarrow{\alpha^{-1}} & v_2 \\
 u & \xrightarrow{f} & & &
 \end{array}
 & \Rightarrow &
 \begin{array}{ccccc}
 & \tilde{r}_2\hat{x} & \xrightarrow{\hat{p}\tilde{r}_1} & \hat{f}'_1 & \xrightarrow{a} \\
 \tilde{r}_2\hat{x} \downarrow & \hat{\rho}_1^{-1}\tilde{r}_2\hat{x} \downarrow & \nearrow \hat{r}_1 & & \\
 \hat{p} & & \hat{\varphi}_1 \downarrow & & \\
 \hat{t}_1 & & \bar{f}_1 & & \\
 \bar{v}_1 & \xrightarrow{\gamma_1 \downarrow} & & \xrightarrow{\alpha^{-1}} & v_2 \\
 u & \xrightarrow{f} & & &
 \end{array}
 \end{array}
 \tag{66}
 \end{array}$$

$$\begin{array}{ccc}
 \Downarrow & & \Downarrow \\
 \begin{array}{c}
 \begin{array}{ccccc}
 & \tilde{r}_1x & \xrightarrow{q\tilde{r}_2} & f'_2 & \\
 \tilde{r}_1x \downarrow & \rho_2^{-1}\tilde{r}_1x \downarrow & \nearrow r_2 & & \\
 \bar{q} & & \varphi_2 \downarrow & & \\
 t_2 & & \bar{f}_2 & & \\
 \bar{v}_2 & \xrightarrow{\gamma_2 \downarrow} & & \xrightarrow{\alpha^{-1}} & v_2 \\
 u & \xrightarrow{f} & & &
 \end{array}
 & \Rightarrow &
 \begin{array}{ccccc}
 & \tilde{r}_1\hat{x} & \xrightarrow{\hat{q}\tilde{r}_2} & \hat{f}'_2 & \xrightarrow{a} \\
 \tilde{r}_1\hat{x} \downarrow & \hat{\rho}_2^{-1}\tilde{r}_1\hat{x} \downarrow & \nearrow \hat{r}_2 & & \\
 \hat{q} & & \hat{\varphi}_2 \downarrow & & \\
 \hat{t}_2 & & \bar{f}_2 & & \\
 \bar{v}_2 & \xrightarrow{\gamma_2 \downarrow} & & \xrightarrow{\alpha^{-1}} & v_2 \\
 u & \xrightarrow{f} & & &
 \end{array}
 \end{array}$$

By composing these 2-cell diagrams vertically, we obtain two 2-cell diagrams comparing the top left and bottom right square. By Proposition A.1 these 2-cell diagrams are equivalent. This will provide us two additional cells which paste with cells from the 2-cell diagrams to provide us the cells that witness the equivalence of (64) and (65).

We start with the 2-cell diagram comparing the two squares in the top row. However, we will ignore the cells γ_1 and α^{-1} . So by applying Proposition 2.9, we obtain arrows c, \hat{c} and cells ξ_1 and ξ_2 as in

$$\begin{array}{ccc}
 & t_1\bar{p}\tilde{r}_2x & \xrightarrow{f'_1\hat{p}\tilde{r}_1\tilde{r}_2x} & \\
 & \downarrow \xi_1 & & \downarrow \xi_2 \\
 u\bar{v}_1 & \xrightarrow{\hat{c}} & & \xrightarrow{c} \\
 & \hat{t}_1\hat{p}\tilde{r}_2\hat{x} & \xrightarrow{a\hat{f}'_1\hat{p}\tilde{r}_1\tilde{r}_2\hat{x}} &
 \end{array}
 \tag{67}$$

such that

(68)

The 2-cell diagram to compare the two squares on the right-hand side of (66) can be built from cells we have already. The two arrows in the middle can be taken as identity arrows, and we will omit them to avoid adding unitor cells. So the reader may verify that the following 2-cell diagram compares the two squares on the right:

(69)

Composing (67) with (69) gives us:

(70)

Similar to the situation for the right two squares, the 2-cell diagram comparing the two squares on the left of (66) can also be constructed from cells we have constructed already. Again collapsing all identity arrows, the following 2-cell diagram is what is needed to compare the left two squares:

(71)

To compare the bottom two squares in (66), we apply Proposition 2.9 to

and

This gives us cells as in

$$\begin{array}{ccc}
 & \xrightarrow{t_2 \bar{q} \bar{r}_1 x} & \\
 & \omega_1 \Downarrow & \omega_2 \Downarrow \\
 u \bar{v}_2 & \xleftarrow{\quad} & \xrightarrow{f'_2 \hat{q} \bar{r}_2 \bar{r}_1 x} \\
 & \hat{t}_2 \bar{\hat{q}} \bar{\hat{r}}_1 \hat{x} & \xrightarrow{a \hat{f}'_2 \hat{q} \bar{\hat{r}}_2 \bar{\hat{r}}_1 \hat{x}} \\
 & \omega_1 \Downarrow & \omega_2 \Downarrow \\
 & \xrightarrow{\quad} &
 \end{array}
 \begin{array}{c}
 d \\
 \hat{d}
 \end{array}
 \tag{72}$$

such that

$$\begin{array}{ccc}
 \begin{array}{c}
 \xrightarrow{d} \xrightarrow{\bar{r}_1 x} \xrightarrow{\bar{q} \bar{r}_2} \xrightarrow{f'_2} \\
 \hat{d} \downarrow \bar{r}_1 x \downarrow \rho_2 \bar{r}_1 x \downarrow \\
 \hat{r}_1 \hat{x} \downarrow \omega_1 \bar{q} \downarrow \\
 \bar{q} \downarrow t_2 \downarrow \\
 \hat{t}_2 \xrightarrow{\quad} \xrightarrow{\bar{f}_2}
 \end{array}
 & \equiv &
 \begin{array}{c}
 \xrightarrow{d} \xrightarrow{\bar{r}_1 x} \xrightarrow{\bar{r}_2} \xrightarrow{q} \\
 \hat{d} \downarrow \bar{r}_1 \hat{x} \downarrow \hat{q} \bar{r}_2 \downarrow \omega_2 \downarrow \\
 \hat{r}_1 \hat{x} \downarrow \hat{\rho}_2 \bar{\hat{r}}_1 \hat{x} \downarrow \\
 \bar{q} \downarrow \hat{r}_2 \downarrow \\
 \hat{t}_2 \xrightarrow{\quad} \xrightarrow{\bar{f}_2}
 \end{array}
 \end{array}
 \tag{73}$$

Composing (71) with (72) gives us:

$$\begin{array}{ccc}
 & \xleftarrow{\bar{p}} \xleftarrow{\hat{r}_2 x} \xleftarrow{\bar{r}_2 x} & \\
 & \omega_1 \Downarrow & \omega_2 \Downarrow \\
 u & \xleftarrow{\quad} & \xrightarrow{f'_1} \\
 \bar{v}_1 & \xleftarrow{\quad} & \xrightarrow{f'_2} \\
 \bar{v}_2 & \xleftarrow{\quad} & \xrightarrow{f'_2} \\
 & \hat{t}_2 \bar{\hat{q}} \bar{\hat{r}}_1 \hat{x} & \xrightarrow{a \hat{f}'_2 \hat{q} \bar{\hat{r}}_2 \bar{\hat{r}}_1 \hat{x}} \\
 & \omega_1 \Downarrow & \omega_2 \Downarrow \\
 & \xrightarrow{\quad} &
 \end{array}
 \tag{74}$$

As we noted at the beginning, the 2-cell diagrams (70) and (74) are equivalent, so there are arrows and 2-cells as in

$$\begin{array}{ccc}
 c & \xrightarrow{\quad} & d \\
 & \hat{X} & \\
 \leftarrow e_1 & \xrightarrow{\quad} & \rightarrow e_2 \\
 & \hat{X} & \\
 \hat{c} & \xrightarrow{\quad} & \hat{d}
 \end{array}$$

to witness this equivalence; i.e., such that

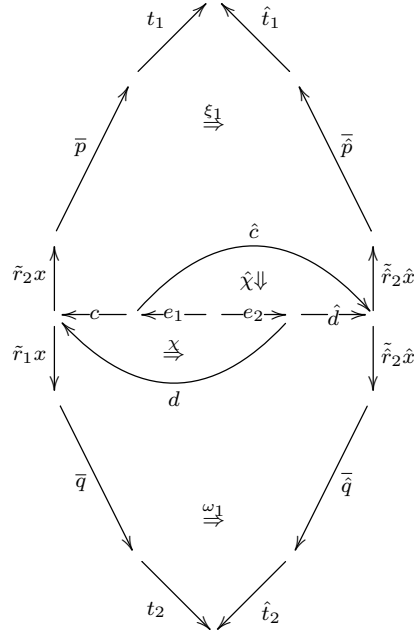
(75)

and

(76)

It can be checked by a long but straightforward calculation using all the equations set up

in this proof that the following cells witness the equivalence of (64) and (65):

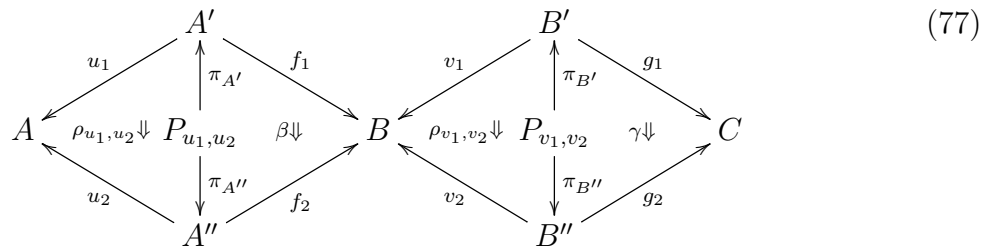


■

Appendix D Horizontal Composition of 2-Cell Diagrams

In this appendix we provide a proof for the following result, described in Section 6.9:

D.1 PROPOSITION. *Let \mathcal{B} be a bicategory and let \mathfrak{W} be a class of arrows in \mathcal{B} that is pullback-closed, satisfies the fractions axioms and is full and co-full. If the cell β in the following diagram of composable 2-cell diagrams is invertible,*



then the horizontal composition of these 2-cells in $\mathcal{B}[\mathfrak{W}^{-1}]$ can be represented by the 2-cell diagram

$$\begin{array}{ccccc}
 & & D & \xrightarrow{\bar{f}_1} & B' \\
 & \swarrow^{u_1 \bar{v}_1} & \uparrow \pi_D & = & \uparrow \pi_{B'} \\
 A & \xleftarrow{\rho_{u_1 \bar{v}_1, u_2 \bar{v}_2} \Downarrow} & P_{u_1 \bar{v}_1, u_2 \bar{v}_2} & \xrightarrow{w_{v_1, v_2}} & P_{v_1, v_2} & \xrightarrow{\gamma \Downarrow} & C \\
 & \searrow_{u_2 \bar{v}_2} & \downarrow \pi_{D'} & = & \downarrow \pi_{B''} \\
 & & D' & \xrightarrow{\bar{f}_2} & B''
 \end{array} \tag{78}$$

as described in Section 6.9.

PROOF. We construct the horizontal composition of the 2-cell diagrams of (77) using whiskering and vertical composition:

$$([\rho_{v_1, v_2}, \gamma](f_2, u_2)) \cdot ((v_1, g_1)[\rho_{u_1, u_2}, \beta])$$

We start by considering the whiskering $(v_1, g_1)[\rho_{u_1, u_2}, \beta]$. To construct this, we need the chosen square:

$$\begin{array}{ccc}
 & v_1^* & \\
 f_2^* \downarrow & \xrightarrow{\quad} & \downarrow f_2 \\
 & \Downarrow \varepsilon_{1,2} & \\
 & v_1 &
 \end{array}$$

This lets us construct the composition of the spans of arrows as in the following diagram (which is not a pasting diagram):

$$\begin{array}{ccccccc}
 & & A' & \xleftarrow{\bar{v}_1} & D & & \\
 & \swarrow^{u_1} & \uparrow \pi_{A'} & \searrow^{f_1} & \downarrow \delta_1 & \searrow^{\bar{f}_1} & \\
 A & \xleftarrow{\rho_{u_1, u_2} \Downarrow} & P_{u_1, u_2} & \xrightarrow{\beta \Downarrow} & B & \xleftarrow{v_1} & B' \xrightarrow{g_1} C \\
 & \searrow_{u_2} & \downarrow \pi_{A''} & \swarrow^{f_2} & \downarrow \varepsilon_{1,2}^{-1} & \swarrow^{f_2^*} & \\
 & & A'' & \xleftarrow{v_1^*} & D^* & &
 \end{array}$$

The left-hand 2-cell for the 2-cell diagram representing the whiskering $(v_1, g_1)[\rho_{u_1, u_2}, \beta]$ is the pseudo-pullback square

$$\begin{array}{ccc}
 P_{u_1 \bar{v}_1, u_2 v_1^*} & \xrightarrow{\pi'_D} & D \\
 \pi'_{D^*} \downarrow & \Downarrow \rho_{u_1 \bar{v}_1, u_2 v_1^*} & \downarrow u_1 \bar{v}_1 \\
 D^* & \xrightarrow{u_2 v_1^*} & A
 \end{array}$$

Let $w_{u_1, u_2}^* : P_{u_1 \bar{v}_1, u_2 v_1^*} \rightarrow P_{u_1, u_2}$ be the unique arrow such that $\rho_{u_1, u_2} w_{u_1, u_2}^* = \rho_{u_1 \bar{v}_1, u_2 v_1^*}$. Then the right-hand 2-cell in the diagram representing the whiskering of $[\rho_{u_1, u_2}, \beta]$ with (v_1, g_1) can be obtained by considering diagram below and then taking a lifting with respect to v_1 for the right-hand pasting diagram (using fullness of \mathfrak{B}):

(79)

We write $\tilde{\beta} : \bar{f}_1 \pi'_D \Rightarrow f_2^* \pi'_{D^*}$ for the lifted cell. We obtain then the following 2-cell diagram in the bicategory of fractions:

Now we consider the other half of the composition, the whiskering $[\rho_{v_1, v_2}, \gamma](u_2, f_2)$. The domain and codomain spans of arrows for the whiskering are constructed in the following diagram (not a pasting diagram):

To find a 2-cell diagram representing this whiskering, we start with the pseudo pullback,

$$\begin{array}{ccc}
 P_{u_2 v_1^*, u_2 \bar{v}_2} & \xrightarrow{\pi''_{D^*}} & D^* \\
 \pi''_{D'} \downarrow & \Downarrow \rho_{u_2 v_1^*, u_2 \bar{v}_2} & \downarrow v_1^* \\
 D' & \xrightarrow{\bar{v}_2} & A'' \xrightarrow{u_2} A
 \end{array}$$

Using fullness of \mathfrak{M} , let

$$\begin{array}{ccc}
 P_{u_2 v_1^*, u_2 \bar{v}_2} & \xrightarrow{\pi''_{D^*}} & D^* \\
 \pi''_{D'} \downarrow & \Downarrow \tilde{\rho}_{u_2 v_1^*, u_2 \bar{v}_2} & \downarrow v_1^* \\
 D' & \xrightarrow{\bar{v}_2} & A''
 \end{array}$$

be the lifting of this diagram with respect to u_2 , and let $x_{v_1, v_2} : P_{u_2 v_1^*, u_2 \bar{v}_2} \rightarrow P_{v_1, v_2}$ be the unique arrow such that the following equality of pasting diagrams holds:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & B' & \\
 f_2^* \pi''_{D^*} \nearrow & \uparrow \pi_{B'} & \searrow v_1 \\
 P_{u_2 v_1^*, u_2 \bar{v}_2} & \xrightarrow{x_{v_1, v_2}} & P_{v_1, v_2} \\
 \parallel & & \parallel \rho_{v_1, v_2} \\
 \bar{f}_2 \pi''_{D'} \searrow & \downarrow \pi_{B''} & \nearrow v_2 \\
 & B'' &
 \end{array}
 & \equiv &
 \begin{array}{ccccc}
 & D^* & \xrightarrow{f_2^*} & B' & \\
 \pi''_{D^*} \nearrow & \downarrow v_1^* & \Downarrow \varepsilon_{1,2} & \searrow v_1 & \\
 P_{u_2 v_1^*, u_2 \bar{v}_2} & \xrightarrow{\tilde{\rho}_{u_2 v_1^*, u_2 \bar{v}_2}} & A'' & \xrightarrow{f_2} & B \\
 \parallel & \nearrow \bar{v}_2 & \Downarrow \delta_2^{-1} & \nearrow v_2 & \\
 \pi''_{D'} \searrow & D' & \xrightarrow{\bar{f}_2} & B'' &
 \end{array}
 \end{array}$$

The whiskering $[\rho_{v_1, v_2}, \gamma](u_2, f_2)$ can now be represented by the diagram

$$\begin{array}{ccccc}
 & D^* & \xrightarrow{f_2^*} & B' & \\
 u_2 v_1^* \nearrow & \uparrow \pi''_{D^*} & = & \uparrow \pi_{B'} & \searrow g_1 \\
 A & \xrightarrow{\Downarrow \rho_{u_2 v_1^*, u_2 \bar{v}_2}} P_{u_2 v_1^*, u_2 \bar{v}_2} & \xrightarrow{x_{v_1, v_2}} & P_{v_1, v_2} & \Downarrow \gamma \\
 u_2 \bar{v}_2 \searrow & \downarrow \pi''_{D'} & = & \downarrow \pi_{B''} & \nearrow g_2 \\
 & D' & \xrightarrow{\bar{f}_2} & B'' &
 \end{array} \tag{80}$$

We now want to construct the vertical composition of the whiskerings $[\rho_{v_1, v_2}, \gamma](f_2, u_2)$ and $(v_1, g_1)[\rho_{u_1, u_2}, \beta]$ as presented in (79) and (80). For this we need the following pseudo

pullback (or any square that commutes up to an invertible 2-cell):

$$\begin{array}{ccc}
 R & \xrightarrow{\pi_1} & P_{u_1\bar{v}_1, \bar{v}_2 v_1^*} \\
 \pi_2 \downarrow & \rho \Downarrow \lambda & \downarrow \pi'_{D^*} \\
 P_{u_2 v_1^*, u_2 \bar{v}_2} & \xrightarrow{\pi''_{D^*}} & D^*
 \end{array}$$

Furthermore, let $r: R \rightarrow P_{u_1\bar{v}_1, u_2\bar{v}_2}$ be the unique arrow such that the following equality of pasting diagrams holds,

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & D & & \\
 & \nearrow^{\pi_D r} & \uparrow \pi_D & \searrow^{u_1 \bar{v}_1} & \\
 R & \xrightarrow{r} & P_{u_1 \bar{v}_1, u_2 \bar{v}_2} & \xrightarrow{\rho_{u_1 \bar{v}_1, u_2 \bar{v}_2}} & A \\
 & \searrow_{\pi_{D'} r} & \downarrow \pi_{D'} & \nearrow_{u_2 \bar{v}_2} & \\
 & & D' & &
 \end{array} & \equiv &
 \begin{array}{ccccc}
 & & D & & \\
 & \nearrow^{\pi_1} & \uparrow \pi'_D & \searrow^{u_1 \bar{v}_1} & \\
 R & \xrightarrow{\pi_1} & P_{u_1 \bar{v}_1, \bar{v}_2 v_1^*} & \xrightarrow{\rho_{u_1 \bar{v}_1, \bar{v}_2 v_1^*}} & D^* \xrightarrow{u_2 v_1^*} & A \\
 & \searrow_{\pi_2} & \downarrow \rho & \nearrow_{\pi''_{D^*}} & \downarrow \pi''_{D^*} & \\
 & & P_{u_2 v_1^*, u_2 \bar{v}_2} & \xrightarrow{\rho_{u_2 v_1^*, u_2, \bar{v}_2}} & D^* & \\
 & & \downarrow \pi''_{D'} & \nearrow_{u_2 \bar{v}_2} & & \\
 & & D' & & &
 \end{array}
 \end{array}$$

Now let

$$\begin{array}{ccccc}
 & & D & & \\
 & \nearrow^{\pi_D} & & \searrow^{g_1 \bar{f}_1} & \\
 P_{u_1 \bar{v}_1, u_2 \bar{v}_2} & & & & C \\
 & \searrow_{\pi_{D'}} & & \nearrow_{g_2 \bar{f}_2} & \\
 & & D' & &
 \end{array}$$

be a lifting with respect to r of the pasting of the following diagram,

(81)

We need to show that the 2-cell diagram (78) that we constructed in Section 6.9,

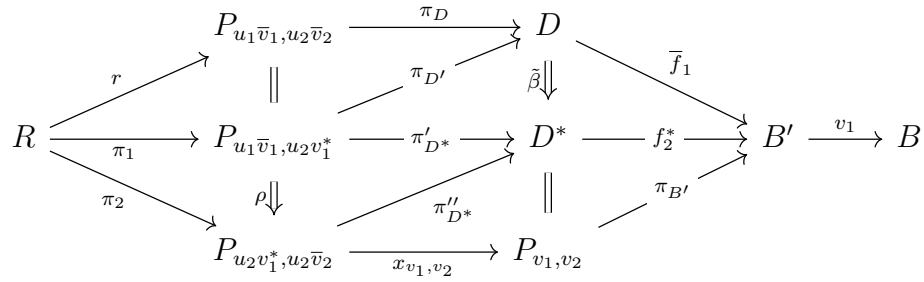
is equivalent to the following diagram, whose right side is the lift of (81) with respect to r :

(82)

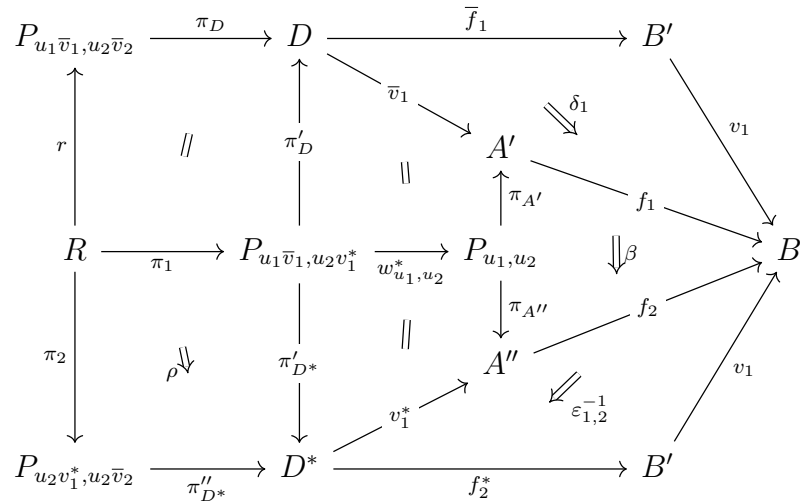
To do this, we precompose $\widetilde{\beta\gamma}$ by r , which allows us to expand $\widetilde{\beta\gamma}$, replacing it with (81). Let diagram **(I)** be the following sub-diagram of the result:

(I)

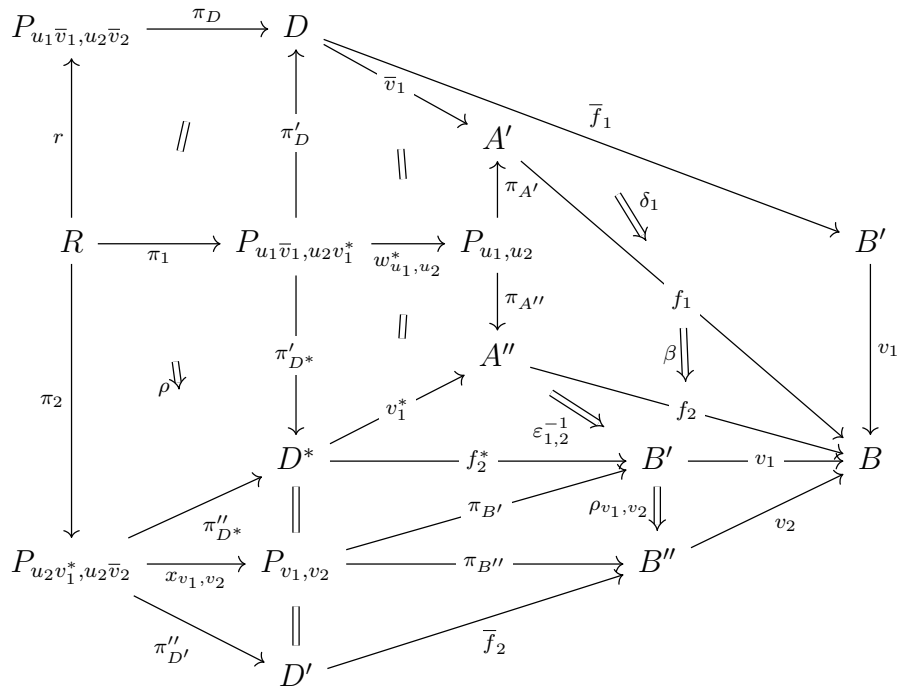
So diagram (81) is obtained from diagram (I) by postcomposing it with g_1 and then with γ . We now take diagram (I) and postcompose with v_1 .



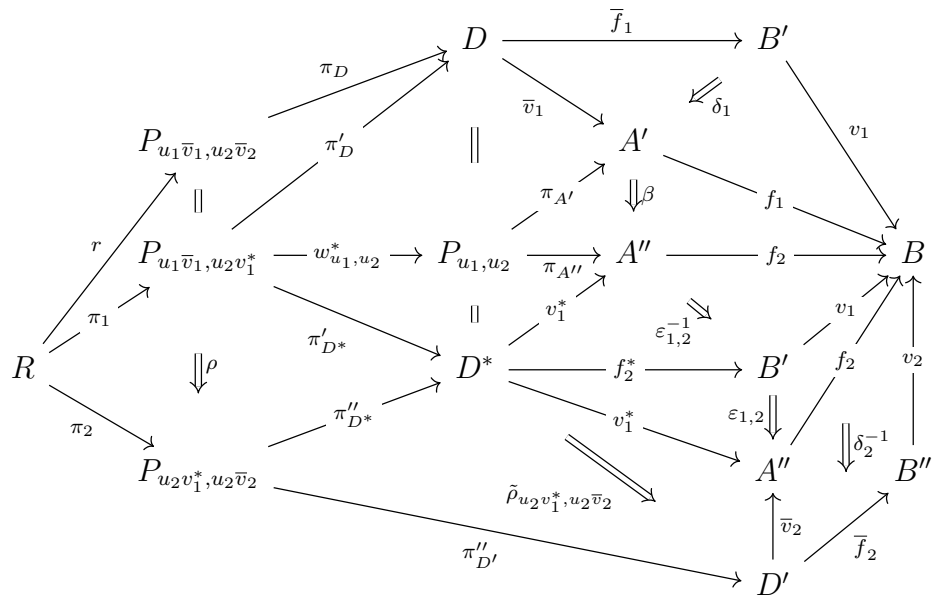
Since $\tilde{\beta}$ was originally defined as a lift with respect to v_1 , this allows us to expand $\tilde{\beta}$:



We now postcompose by ρ_{v_1, v_2} .



By the definition of x_{v_1, v_2} this is equal to



We can now cancel $\varepsilon_{1,2}^{-1}$ and $\varepsilon_{1,2}$:

$$\begin{array}{ccccccc}
 & P_{u_1 \bar{v}_1, u_2 \bar{v}_2} & \xrightarrow{\pi_D} & D & \xrightarrow{\bar{f}_1} & B' & \\
 & \nearrow r & & \nearrow \pi'_D & \parallel & \searrow \bar{v}_1 & \downarrow \delta_1 \\
 R & \xrightarrow{\pi_1} & P_{u_1 \bar{v}_1, u_2 \bar{v}_2} & \xrightarrow{w_{u_1, u_2}^*} & P_{u_1, u_2} & \xrightarrow{-\pi_{A'}} & A' & \xrightarrow{-f_1} & B \\
 & \searrow \pi_2 & \swarrow \rho & \searrow \pi'_{D^*} & \parallel & \searrow \pi_{A''} & \downarrow \beta & \nearrow f_2 & \\
 & P_{u_2 v_1^*, u_2 \bar{v}_2} & \xrightarrow{\pi''_{D^*}} & D^* & \xrightarrow{v_1^*} & A'' & & & \\
 & \searrow \pi''_{D'} & \swarrow \tilde{\rho}_{u_2 v_1^*, u_2 \bar{v}_2} & \swarrow \bar{v}_2 & \downarrow \delta_2^{-1} & & & & \\
 & & & D' & \xrightarrow{\bar{f}_2} & B'' & & & \\
 & & & & & & & & \nearrow v_2
 \end{array} \tag{83}$$

We again decide to focus on just a part of this diagram – we call this part **(II)**.

$$\text{(II)} \quad \equiv \quad \begin{array}{ccccc}
 & P_{u_1 \bar{v}_1, u_2 v_1^*} & \xrightarrow{w_{u_1, u_2}^*} & P_{u_1, u_2} & \xrightarrow{\pi_{A''}} & A'' \\
 & \nearrow \pi_1 & & \nearrow \pi'_{D^*} & \parallel & \searrow v_1^* \\
 R & & \downarrow \rho & & D^* & \xrightarrow{v_1^*} & A'' \\
 & \searrow \pi_2 & & \searrow \pi''_{D^*} & \parallel & \swarrow \tilde{\rho}_{u_2 v_1^*, u_2 \bar{v}_2} & \\
 & P_{u_2 v_1^*, u_2 \bar{v}_2} & \xrightarrow{\pi''_{D'}} & D' & \xrightarrow{\bar{v}_2} & A''
 \end{array}$$

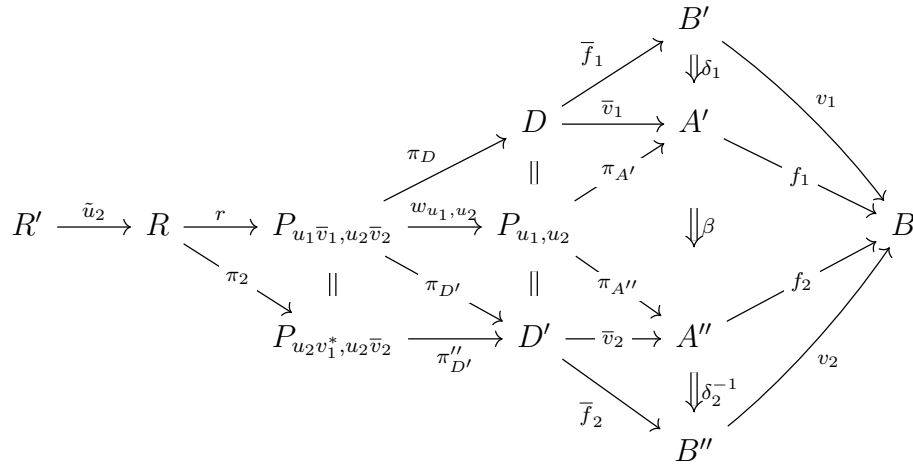
We will now show that if we post-compose diagram **(II)** with u_2 , we get an identity 2-cell. To show this, we post-compose diagram **(II)** with u_2 and then pre-compose with the

invertible 2-cell ρ_{u_1, u_2} :

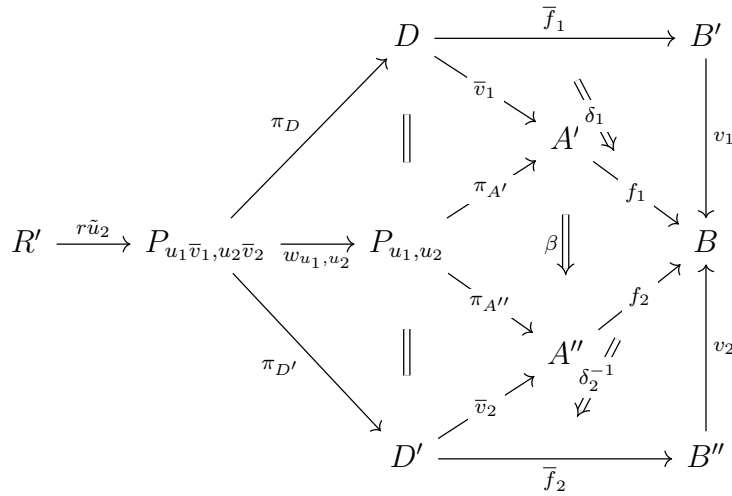
The first equality above follows from the universal property of the arrow w_{u_1, u_2}^* , and the second equality follows from the definition of the arrow r . The definition of w_{u_1, u_2} now implies that this pasting is equal to

Now composing with an appropriate whiskering of ρ_{u_1, u_2}^{-1} gives the promised identity 2-cell. We conclude that there is an arrow $\tilde{u}_2: R' \rightarrow R$ in \mathfrak{W} such that diagram (II) pre-composed with \tilde{u}_2 is an identity 2-cell as claimed. When we substitute this into (83),

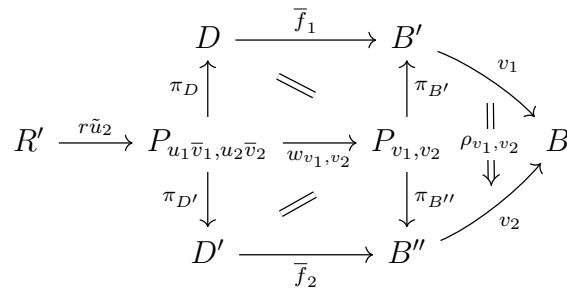
we get the following pasting diagram:



We can rewrite this as



If we assume that β is invertible, this pasting is equal to the following by definition of w_{v_1, v_2} given in Section 6.9:



When we post-compose this with ρ_{v_1, v_2}^{-1} we find that when the pasting of diagram (I) is pre-composed with \tilde{u}_2 and post-composed with v_1 , the result is equal to the identity

2-cell on $v_1\pi_{B'}w_{v_1,v_2}r\tilde{u}_2$. So there is an arrow $\tilde{v}_1: R'' \rightarrow R$ in \mathfrak{W} such that pasting (I) pre-composed by $r\tilde{u}_2\tilde{v}_1$ is the identity 2-cell on $\pi_{B'}w_{v_1,v_2}r\tilde{u}_2\tilde{v}_1$. We finally post-compose with γ to find that $\gamma\beta$ is precomposed with $r\tilde{u}_2\tilde{v}_1$ is equal to

$$\begin{array}{ccccc}
 & & & D & \xrightarrow{\bar{f}_1} & B' & & \\
 & & \nearrow \pi_D & \parallel & \nearrow \pi_{B'} & & \searrow g_1 & \\
 R'' & \xrightarrow{r\tilde{u}_2\tilde{v}_1} & P_{u_1\bar{v}_1,u_2\bar{v}_2} & \xrightarrow{w_{u_1,u_2}} & P_{u_1,u_2} & & \Downarrow \gamma & C \\
 & & \searrow \pi_{D'} & \parallel & \searrow \pi_{B''} & & \nearrow g_2 & \\
 & & & D' & \xrightarrow{\bar{f}_2} & B'' & &
 \end{array}$$

We conclude that diagram (29) given in Section 6.9 and the diagram constructed from the vertical composition of whiskerings are equivalent as claimed. ■

D.2 REMARK. If β is not invertible, the 2-cell diagram (82) within the proof above gives a representation of the horizontal composition. Unfortunately, without further assumptions there is no obvious way to simplify this representation.

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