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DETECTING MODEL CATEGORIES AMONG QUILLEN CATEGORIES USING HOMOTOPIES

In memory of Professor Aldridge Knight Bousfield

SEUNGHUN LEE

ABSTRACT. A model category has two weak factorizations, a pair of cofibrations and trivial fibrations and a pair of trivial cofibrations and fibrations. Then the class of weak equivalences is the set $\mathcal{W}$ consisting of the morphisms that can be decomposed into trivial cofibrations followed by trivial fibrations. One can build a model category out of such two weak factorizations by defining the class of weak equivalences by $\mathcal{W}$ as long as it satisfies the two out of three property. In this note we show that given a category with two weak factorizations, if every object is fibrant and cofibrant, $\mathcal{W}$ satisfies the two out of three property if and only if $\mathcal{W}$ is closed under the homotopies.

Contents

1. Introduction

The homotopy theory has two layers. The first one is about structures invariant under the homotopies. On the second layer it becomes a classification theory. Here category theory is the proper language. Quillen introduced in [Qui67] the (closed) model category as a categorical framework for homotopy theories. The following is a modern definition. See, for example, [MP12].

1.1. DEFINITION. *Let $\mathcal{M}$ be a category with finite limits and finite colimits. A model structure on $\mathcal{M}$ consists of three classes $\mathcal{W}$, $\mathcal{C}$, and $\mathcal{F}$ of morphisms in $\mathcal{M}$ such that the following two properties hold:*

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1. $\mathcal{W}$ satisfies the two out of three property.

2. $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems.

We do not assume that the weak factorization systems are functorial. A category $\mathcal{M}$ with a model structure $\mathcal{W}, \mathcal{C}, \mathcal{F}$ is called a model category, and we denote it by $(\mathcal{M}; \mathcal{W}, \mathcal{C}, \mathcal{F})$.

We recall the two out of three property for later use.

1.2. Definition. *Let $\mathcal{M}$ be a category. Let $\mathcal{W}$ be a class of morphisms in $\mathcal{M}$. We say that $\mathcal{W}$ satisfies the two out of three property if the following hold: For every pair f, g of composable morphisms in $\mathcal{M}$,*

(M) *$f \in \mathcal{W}$ and $g \in \mathcal{W}$ imply $gf \in \mathcal{W}$.*

(L) *$gf \in \mathcal{W}$ and $g \in \mathcal{W}$ imply $f \in \mathcal{W}$.*

(R) *$gf \in \mathcal{W}$ and $f \in \mathcal{W}$ imply $g \in \mathcal{W}$.*

As a classification theory, the class $\mathcal{W}$ determines the homotopy category. Hence it is the most important one among the three classes of morphisms. In this respect, the two out of three property seems to stand out. It is not shared with either $\mathcal{C}$ or $\mathcal{F}$. It is a reasonable requirement for a set of morphisms that become isomorphisms in its homotopy category. It is crucial in showing that the localization is saturated.

One can always invert a given class of morphisms in a category to obtain its homotopy category. But the hom-sets are hard to manage in general. So, in reality, what makes the hom-sets manageable in a model category are its fibrations and cofibrations. They are the building blocks and the two out of three property is a glue that puts them together. So we introduced in [Lee15] the following definition.

1.3. Definition. *Let $\mathcal{M}$ be a category with finite colimits and finite limits. A Quillen structure on $\mathcal{M}$ is a pair of weak factorization systems*

$$(\mathcal{C}, \mathcal{F}_t) \quad \text{and} \quad (\mathcal{C}_t, \mathcal{F}) \tag{1}$$

such that

$$\mathcal{C}_t \subseteq \mathcal{C} \quad \text{and} \quad \mathcal{F}_t \subseteq \mathcal{F}$$

hold. A category $\mathcal{M}$ with a Quillen structure (1) is called a Quillen category, and we denote it by $(\mathcal{M}; \mathcal{C}, \mathcal{F})$.

Every model category $(\mathcal{M}; \mathcal{W}, \mathcal{C}, \mathcal{F})$ gives a Quillen category $(\mathcal{M}; \mathcal{C}, \mathcal{F})$. And a Quillen category $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ is induced from a model category precisely when $\mathcal{W}$ satisfies the two out of three property where

$$\mathcal{W} = \{p \cdot i \mid p \in \mathcal{F}_t,\ i \in \mathcal{C}_t\}. \tag{2}$$

See Lemma 2.1.1.

In [Qui67], Quillen wrote that he first developed the theory of simplicial model categories on Chapter II. Then because of some examples that had no simplicial structures, he had to write the general theory of model categories on Chapter I. But, curiously, in the set of axioms for the model categories listed on Chapter I.1, the two out of three property that is perhaps the most important for a classification theory is stated as the last item, as if it wasn't really necessary.

In fact, we showed in [Lee15] that if $(\mathcal{M};\mathcal{C},\mathcal{F})$ is a simplicial Quillen category, and if every object is fibrant and cofibrant then $\mathcal{W}$ has the two out of three property. A simplicial Quillen category is a Quillen category enriched in the category of simplicial sets and satisfying the axioms (SM0) and (SM7) just like the simplicial model categories. Every model category is Quillen equivalent to its subcategory of fibrant and cofibrant objects (Theorem 1 on Chapter I.1 in [Qui67]). A large class of model categories are Quillen equivalent to simplicial model categories ([Dug01], [RSS01]). So in this sense, the two out of three property isn't really required in the list of the axioms.

The proofs in [Lee15] rely on the simplicial homotopies. In simplicial model categories the simplicial homotopies coincide with the left and the right homotopies. In model categories the left and the right homotopies can be defined in terms of their Quillen structures. So as we return to the first layer of the homotopy theory, one may ask if the two out of three property can be characterized in terms of homotopies so that model categories can be detected among Quillen categories. The purpose of this note is to answer the question when every object is fibrant and cofibrant. We refer to Definition 3.1.4 for the left and the right homotopies in Quillen categories. Since Quillen categories generalize model categories, we use the same terms and notations for Quillen categories as we do for model categories.

1.4. Definition. *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category. Let $\mathcal{S}$ be a class of morphisms in $\mathcal{M}$. We say that $\mathcal{S}$* ***is closed under the left (resp. right) homotopy*** *if for every pair $f, g : a \to b$ of morphisms in $\mathcal{M}$ with $f \overset{l}{\sim} g$ (resp. $f \overset{r}{\sim} g$),*

$$f \in \mathcal{S} \quad \Leftrightarrow \quad g \in \mathcal{S} \tag{3}$$

holds.

Given a Quillen category $(\mathcal{M};\mathcal{C},\mathcal{F})$, we call an object a of $\mathcal{M}$ **fibrant** if the unique morphism from a to the terminal object of $\mathcal{M}$ is in $\mathcal{F}$. Dually, we call an object a of $\mathcal{M}$ **cofibrant** if the unique morphism from the initial object of $\mathcal{M}$ to a is in $\mathcal{C}$.

1.5. Theorem. *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category and let*

$$\mathcal{W} = \{p \cdot i \mid p \in \mathcal{F}_t,\ i \in \mathcal{C}_t\}. \tag{4}$$

If every object of $\mathcal{M}$ is fibrant and cofibrant then the following are equivalent.

1. *$\mathcal{W}$ satisfies the two out of three property.*
2. *$\mathcal{W}$ is closed under the left homotopy and the right homotopy.*

One may wonder if Theorem 1.5 is true in general. As the proofs on Chapter 1 in [Qui67] demonstrate the left (resp. right) homotopies behave well only with the morphisms whose domains (resp. codomains) are cofibrant (resp. fibrant). So it may be unreasonable to expect such a result.

However there is another closedness property, the property (3) in Theorem 4.2.1, equivalent to the two out of three property when every object is fibrant and cofibrant. It may have a chance for a generalization to arbitrary Quillen categories. This leads us to think about a redundancy of (M) in Definition 1.2 for model categories.

The property (M) is different from (L) and (R) when viewed from Quillen categories. (L) and (R) are about certain fibrations or cofibrations being trivial. However (M) is about the existence of morphisms satisfying certain properties. See Proposition 2.1.4, Remark 2.1.3 and Lemma 2.1.6. So, when one tries to verify these conditions, it is easier to show (L) or (R) than (M). So it would be useful to know if (L) and (R) imply (M) in Quillen categories. Here we have only a partial result.

1.6. Theorem. *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category and let*

$$\mathcal{W} = \{p \cdot i \mid p \in \mathcal{F}_t,\ i \in \mathcal{C}_t\}. \tag{5}$$

We assume that one of the following two conditions holds.

1. *Every object of $\mathcal{M}$ is fibrant.*
2. *Every object of $\mathcal{M}$ is cofibrant.*

If the properties (L) and (R) in Definition 1.2 hold for $\mathcal{W}$ then the property (M) also holds for $\mathcal{W}$.

The model categories as the second layer of the homotopy theory are at the intersections of classification theory, the weak equivalences, and homotopy theory, the Quillen structures. In [Rie11], Riehl introduced the beautiful theory of algebraic model categories. Algebraic model categories are the model categories whose Quillen structures are induced from algebraic data, two pairs of a monad and a comonad, called algebraic weak factorization systems. They were introduced in [GT06] by Grandis and Tholen in the name of natural weak factorization systems. Many cofibrantly generated model categories of interest become algebraic model categories when applying a variant of Quillen's small object argument developed in [Gar07] and [Gar09] by Garner. So it seemed natural to ask if the pair of algebraic weak factorization systems in an algebraic model category determines the weak equivalences. This has been the motivation for [Lee15] and the current paper.

There is at least one more reason to be interested in Quillen structures. In [Bar19], Barton studies the totality of model categories to answer the question raised by Hovey in [Hov99]: Does the 2-category of model categories have a model 2-category structure with the Quillen equivalences as the weak equivalences? Because model categories lack limits and colimits, Barton considers the premodel categories instead. A premodel category is a complete and cocomplete category with a Quillen structure. He shows that the 2-category of combinatorial premodel categories admits all limits and colimits. Next, to explain what it means for a left Quillen functor between premodel categories to be a weak equivalence, he imposes an additional hypothesis, the relaxedness. One consequence of the relaxedness is the existence of a cylinder object for each cofibrant object and a path object for each fibrant object satisfying the conclusions in Lemma 3.1.3. This property is enough for the existence of the localization $\gamma : \mathcal{M}_{cf} \to \pi\mathcal{M}_{cf}$ of the subcategory $\mathcal{M}_{cf}$ of fibrant and cofibrant objects of a given premodel category $\mathcal{M}$. However because this construction is not functorial with respect to the left Quillen functors, Barton uses the subcategory $\mathcal{M}^{cof}$ of cofibrant objects instead. He shows that there exists a well-behaved class of morphisms, the left weak equivalences, making a cofibration category $\mathcal{M}^{cof}$ whenever $\mathcal{M}$ is relaxed. Using the homotopy theory of cofibration category $\mathcal{M}^{cof}$ he moves on to show that such a model 2-category exists by restricting to combinatorial premodel categories enriched over a tractable symmetric monoidal model category. When $\mathcal{M} = \mathcal{M}_{cf}$, our hypothesis, the compatibility with fibrations and cofibrations, shares the same property, Lemma 3.1.3, with the relaxedness. But it seems different from the relaxedness in that the localization γ is saturated under the compatibility.

This note is organized as follows. In section 2, we review the Quillen categories. In section 3, we verify that, as in model categories, one can do a basic homotopy theory in Quillen categories if every object is fibrant and cofibrant and if a condition, which will be proved in Theorem 4.2.1 to be equivalent to the two out of three property, holds. We prove Theorem 1.5 in section 4. Section 5 is about a speculation on the redundancy of property (M) in Definition 1.2 for model categories. Here we prove Theorem 1.6.

We basically use the same notations as in [Qui67]. We use the same notations and terms for Quillen categories as we do for model categories. For example, if $(\mathcal{M};\mathcal{C},\mathcal{F})$ is a Quillen category then $\mathcal{M}_c$ is the full subcategory of cofibrant objects of $\mathcal{M}$. We denote by $\bullet \twoheadrightarrow \bullet$, $\bullet \overset{\sim}{\twoheadrightarrow} \bullet$, $\bullet \rightarrowtail \bullet$ and $\bullet \overset{\sim}{\rightarrowtail} \bullet$, fibrations, trivial fibrations, cofibrations and trivial cofibrations in Quillen categories respectively. We denote by $\bullet \xrightarrow{\sim} \bullet$ the morphisms in $\mathcal{W}$.

Finally, I would like to thank the referee for correcting errors and inaccuracies and suggesting better names for key notions. I am also grateful for informing me of the thesis [Bar19] of Reid William Barton.

2. Review on Quillen categories

Here we collect some properties of Quillen categories used later in the proofs. Recall that given a Quillen category $(\mathcal{M};\mathcal{C},\mathcal{F})$, we use the notation

$$\mathcal{W} = \{p \cdot i \mid p \in \mathcal{F}_t,\, i \in \mathcal{C}_t\}. \tag{6}$$

2.1. Five necessary conditions. Here we list five special but important cases of the two out of three property, which we will use quite often later in the proofs, and collect some of their properties.

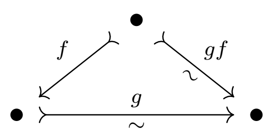
2.1.1. Lemma. [cf. lemma 2.4 in [Lee15]] *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category. Then the following hold.*

1. $\mathcal{C}\cap\mathcal{W}=\mathcal{C}_t$.
2. $\mathcal{F}\cap\mathcal{W}=\mathcal{F}_t$.

In Definition 2.1.2 below, the first two are related with the property (L). Next two are related with the property (R). The last one is equivalent to (M). The first four names (CtC), (FtC), (FFt) and (FCt) were meant to help readers to remember the hypothesis. For example, the name (CtC) indicates that $g\in\mathcal{C}_t$ and $f\in\mathcal{C}$. In other words $g\cdot f$ is an element of $\mathcal{C}_t\cdot\mathcal{C}$. Similarly (FFt) indicates that $g\in\mathcal{F}$ and $f\in\mathcal{F}_t$. The last one is about the existence of a (C_t,F_t)-factorization. Hence the name (Fact).

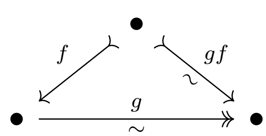
2.1.2. Definition. *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category.*

1. *We say that* ***(CtC)*** *holds if for every pair f,g of composable morphisms in $\mathcal{M}$, $g\in\mathcal{C}_t$, $gf\in\mathcal{C}_t$ and $f\in\mathcal{C}$ imply $f\in\mathcal{C}_t$.*



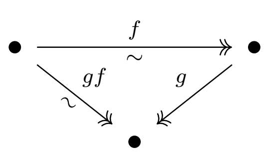
$$\Rightarrow f\in\mathcal{C}_t \tag{7}$$

2. *We say that* ***(FtC)*** *holds if for every pair f,g of composable morphisms in $\mathcal{M}$, $g\in\mathcal{F}_t$, $gf\in\mathcal{C}_t$ and $f\in\mathcal{C}$ imply $f\in\mathcal{C}_t$.*



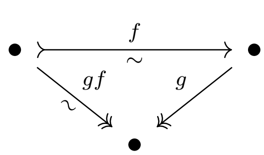
$$\Rightarrow f\in\mathcal{C}_t \tag{8}$$

3. *We say that* ***(FFt)*** *holds if for every pair f,g of composable morphisms in $\mathcal{M}$, $f\in\mathcal{F}_t$, $gf\in\mathcal{F}_t$ and $g\in\mathcal{F}$ imply $g\in\mathcal{F}_t$.*



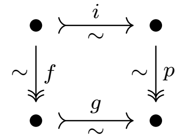
$$\Rightarrow g\in\mathcal{F}_t \tag{9}$$

4. *We say that* ***(FCt)*** *holds if for every pair f,g of composable morphisms in $\mathcal{M}$, $f\in\mathcal{C}_t$, $gf\in\mathcal{F}_t$ and $g\in\mathcal{F}$ imply $g\in\mathcal{F}_t$.*



$$\Rightarrow g\in\mathcal{F}_t \tag{10}$$

5. *We say that* ***(Fact)*** *holds if for every pair f, g of composable morphisms in $\mathcal{M}$, $f \in \mathcal{F}_t$ and $g \in \mathcal{C}_t$ imply that there exist $p \in \mathcal{F}_t$ and $i \in \mathcal{C}_t$ such that $gf = pi$ holds.*


(11)

2.1.3. REMARK. In a Quillen category $(\mathcal{M}; \mathcal{C}, \mathcal{F})$, $\mathcal{W}$ satisfies the property (M) in Definition 1.2 if and only if (Fact) holds.

2.1.4. PROPOSITION. [cf. Lemma 2.6 and Lemma 2.8.(1) in [Lee15]] *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. Then $\mathcal{W}$ has the two out of three property if and only if (CtC), (FtC), (FFt), (FCt) and (Fact) hold.*

2.1.5. REMARK. The proof of Proposition 2.1.4 relies on the existence of pullbacks and pushouts in $\mathcal{M}$.

2.1.6. LEMMA. *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. Then the following hold.*

1. *(L) implies (CtC) and (FtC).*
2. *(R) implies (FFt) and (FCt).*

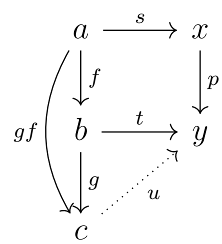
PROOF. It follows from Lemma 2.1.1. ∎

2.1.7. LEMMA. *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category.*

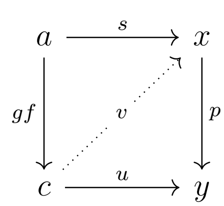
1. *If $\mathcal{M} = \mathcal{M}_f$ then (CtC) holds.*
2. *If $\mathcal{M} = \mathcal{M}_c$ then (FFt) holds.*

PROOF. We will only prove (1). (2) is dual to (1).

Consider a commutative diagram


(12)

of solid arrows where $g, gf \in \mathcal{C}_t$ and $p \in \mathcal{F}$. Because of $g \in \mathcal{C}_t$ and $y \in \mathcal{M}_f$, there is a lifting u such that $u \cdot g = t$. Then because of $g \cdot f \in \mathcal{C}_t$, we have a lifting $v : c \to x$ in the square


(13)

induced from (12). So the diagram

$$\begin{array}{ccc} a & \xrightarrow{s} & x \\ {\scriptstyle f}\downarrow & \overset{vg}{\nearrow} & \downarrow{\scriptstyle p} \\ c & \xrightarrow{t} & y \end{array} \tag{14}$$

commutes. Hence $f \in \mathcal{C}_t$. ■

2.2. Compatibility. Here we introduce a property related with the two out of three property for $\mathcal{W}$.

2.2.1. Definition. *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. Let $\mathcal{S}$ be a subset of $\mathcal{W}$.*

1. *We say that $\mathcal{S}$ **is compatible with** $\mathcal{C}$ if for every morphism h in $\mathcal{S}$ and every $(\mathcal{C}, \mathcal{F}_t)$-factorization $h = gf$*

$$\bullet \overset{f}{\rightarrowtail} \bullet \underset{\sim}{\overset{g}{\twoheadrightarrow}} \bullet \tag{15}$$

   *of h, $f \in \mathcal{C}_t$ holds. If $\mathcal{S}$ consists of a single morphism s, then we also say that s **is compatible with** $\mathcal{C}$.*

2. *We say that $\mathcal{S}$ **is compatible with** $\mathcal{F}$ if for every morphism h in $\mathcal{S}$ and every $(\mathcal{C}_t, \mathcal{F})$-factorization $h = gf$*

$$\bullet \underset{\sim}{\overset{f}{\rightarrowtail}} \bullet \overset{g}{\twoheadrightarrow} \bullet \tag{16}$$

   *of h, $g \in \mathcal{F}_t$ holds. If $\mathcal{S}$ consists of a single morphism s, then we also say that s **is compatible with** $\mathcal{F}$.*

2.2.2. Remark. Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category.

1. (FtC) holds if and only if $\mathcal{C}_t$ is compatible with $\mathcal{C}$.
2. (FCt) holds if and only if $\mathcal{F}_t$ is compatible with $\mathcal{F}$.

2.2.3. Lemma. [cf. Lemma 2.7.(2) in [Lee15]] *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. Then the following hold.*

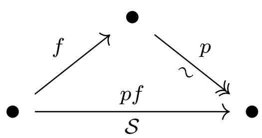
1. *(FtC) holds if and only if $\mathcal{W}$ is compatible with $\mathcal{C}$.*
2. *(FCt) holds if and only if $\mathcal{W}$ is compatible with $\mathcal{F}$.*

2.2.4. Remark. The proof of Lemma 2.2.3 relies on the existence of pullbacks and pushouts in $\mathcal{M}$.

We will often use the following two simple lemmas.

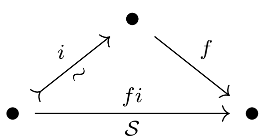
2.2.5. Lemma. *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category. Let $\mathcal{S}$ be a subset of $\mathcal{W}$.*

1. *We assume that $\mathcal{S}$ is compatible with $\mathcal{C}$. Then for every pair f, p of composable morphisms in $\mathcal{M}$, $p \cdot f \in \mathcal{S}$ and $p \in \mathcal{F}_t$ imply $f \in \mathcal{W}$.*



$\Rightarrow f \in \mathcal{W}$ (17)

2. *We assume that $\mathcal{S}$ is compatible with $\mathcal{F}$. Then for every pair i, f of composable morphisms in $\mathcal{M}$, $i \in \mathcal{C}_t$ and $f \cdot i \in \mathcal{S}$ imply $f \in \mathcal{W}$.*



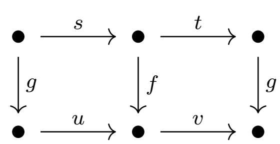
$\Rightarrow f \in \mathcal{W}$ (18)

Proof. We will only prove (2). (1) is dual to (2).

We decompose f into $j \in \mathcal{C}_t$ followed by $p \in \mathcal{F}$. Then since $j \cdot i \in \mathcal{C}_t$ and $p \cdot (j \cdot i) = f \cdot i$ is compatible with $\mathcal{F}$, $p \in \mathcal{F}_t$. Hence $f \in \mathcal{W}$. ∎

2.3. Retract lemma. The following lemma is proved in [JT07] in the context of model structures. The same proof also works for Quillen structures. For the convenience of readers, we reproduce their proof. It plays a critical role in the proof of Lemma 3.2.3, an analogue of Lemma 1 on Chapter 1.5 in [Qui67].

2.3.1. Lemma. [cf. Proposition 7.8 in [JT07]] *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category. Let*



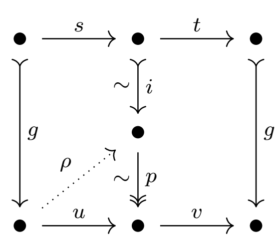
(19)

be a retract diagram.

1. *If $f \in \mathcal{W}$ and $g \in \mathcal{C}$ then $g \in \mathcal{C}_t$ holds.*
2. *If $f \in \mathcal{W}$ and $g \in \mathcal{F}$ then $g \in \mathcal{F}_t$ holds.*

Proof. We will only prove (1). (2) is dual to (1).

We decompose f into $i \in \mathcal{C}_t$ followed by $p \in \mathcal{F}_t$. The we have a lifting ρ in the diagram (19).



(20)

Then

$$\begin{array}{ccccc} \bullet & \xrightarrow{s} & \bullet & \xrightarrow{t} & \bullet \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle \sim}\big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle g} \\ \bullet & \xrightarrow{\rho} & \bullet & \xrightarrow{v\cdot p} & \bullet \end{array} \tag{21}$$

is a retract diagram. Hence $g \in \mathcal{C}_t$ because of $i \in \mathcal{C}_t$. ∎

3. Homotopies in Quillen categories

The purpose of this section is to show that if $\mathcal{M} = \mathcal{M}_{cf}$ and all the identities in $\mathcal{M}$ are compatible with $\mathcal{C}$ and $\mathcal{F}$, one can do a basic homotopy theory with Quillen categories. The proofs are the same as the ones on Chapter 1.1 and 1.5 in [Qui67] except a part of that of Lemma 3.2.3. So, what we do in this section is merely to verify that the proofs in [Qui67] also work for Quillen categories under the assumptions.

3.1. Cylinder and path. Here we define the cylinder objects and the path objects for objects of Quillen categories, and the left homotopies and the right homotopies for morphisms in Quillen categories. We also prove some properties of them.

3.1.1. Definition. [cf. Definition 4 on Chapter I.1 in [Qui67]] *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category.*

1. *A cylinder object for an object a of $\mathcal{M}$ is a factorization*

$$\nabla : a \vee a \xrightarrow{\partial_0 + \partial_1} a \times I \xrightarrow[\sim]{\sigma} a \tag{22}$$

of codiagonal map $\nabla : a \vee a \to a$ into $\partial_0 + \partial_1 \in \mathcal{C}$ followed by $\sigma \in \mathcal{W}$.

2. *A path object for an object b of $\mathcal{M}$ is a factorization*

$$\Delta : b \xrightarrow[\sim]{s} b^I \xrightarrow{(d_0, d_1)} b \times b \tag{23}$$

of diagonal map $\Delta : b \to b \times b$ into $s \in \mathcal{W}$ followed by $(d_0, d_1) \in \mathcal{F}$.

3.1.2. Remark. Every object in a Quillen category has a cylinder object and a path object by the factorization properties of the two weak factorization systems.

3.1.3. Lemma. [cf. Lemma 2 on Chapter I.1 in [Qui67]] *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category.*

1. *We assume that (CtC) holds. If $a \in \mathcal{M}_c$ and 1_a is compatible with $\mathcal{C}$ then for any cylinder object (22) for a*

$$\partial_i \in \mathcal{C}_t \tag{24}$$

holds where $i = 0, 1$.

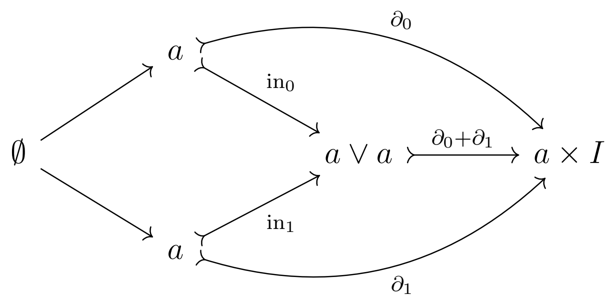
2. *We assume that (FFt) holds. If $b \in \mathcal{M}_f$ and 1_b is compatible with $\mathcal{F}$ then for any path object (23) for b*

$$d_i \in \mathcal{F}_t \tag{25}$$

holds where $i = 0, 1$.

Proof. We will only prove (1). (2) is dual to (1).

Since a is cofibrant, the canonical maps $\mathrm{in}_i : a \to a \vee a$ are cofibrations for $i = 0, 1$. Hence $\partial_i \in \mathcal{C}$ for $i = 0, 1$.



(26)

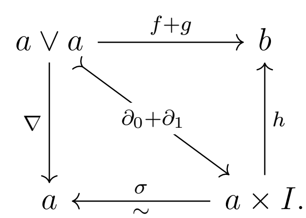
We factor σ into $j \in \mathcal{C}_t$ followed by $p \in \mathcal{F}_t$.

$$1_a : a \xrightarrow{\partial_i} a \times I \xrightarrow[\sim]{j} \bullet \xrightarrow[\sim]{p} a \qquad (\text{with } \sigma = p \cdot j) \tag{27}$$

We have $j \cdot \partial_i \in \mathcal{C}_t$ for $i = 0, 1$ because 1_a is compatible with $\mathcal{C}$. Then $\partial_i \in \mathcal{C}_t$ by (CtC). ■

3.1.4. Definition. [cf. Definition 4 on Chapter I.1 in [Qui67]] *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. Let $f, g : a \to b$ be morphisms in $\mathcal{M}$.*

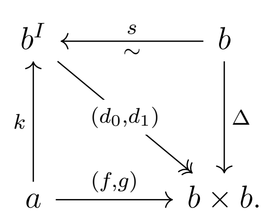
1. *A left homotopy from f to g on a cylinder object (22) is a commutative diagram*



(28)

We denote such a left homotopy by $h : f \overset{l}{\sim} g$. We write $f \overset{l}{\sim} g$ if a left homotopy from f to g exists.

2. *A right homotopy from f to g on a path object (23) is a commutative diagram*



(29)

We denote such a right homotopy by $k : f \overset{r}{\sim} g$. We write $f \overset{r}{\sim} g$ if a right homotopy from f to g exists.

3.1.5. Lemma. [cf. Lemma 5.(i) on Chapter I.1 in [Qui67]] *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category. Let $f,g : a \to b$ be morphisms in $\mathcal{M}$.*

1. *We assume that (CtC) holds. If $a \in \mathcal{M}_c$ and 1_a is compatible with $\mathcal{C}$ then*
$$f \overset{l}{\sim} g \Rightarrow f \overset{r}{\sim} g \tag{30}$$
holds.
2. *We assume that (FFt) holds. If $b \in \mathcal{M}_f$ and 1_b is compatible with $\mathcal{F}$ then*
$$f \overset{r}{\sim} g \Rightarrow f \overset{l}{\sim} g \tag{31}$$
holds.

Proof. We will only prove (1). (2) is dual to (1).

Suppose we are given a left homotopy (28). By Lemma 3.1.3.(1), $\partial_1 \in \mathcal{C}_t$. So if (23) is a path object for b, the following commutative diagram of solid arrows has a lifting ρ.

$$\begin{array}{ccc} a & \xrightarrow{sg} & b^I \\ \sim\big\downarrow \partial_1 & \overset{\rho}{\nearrow} & \big\downarrow (d_0,d_1) \\ a \times I & \xrightarrow{(h,g\sigma)} & b \times b \end{array} \tag{32}$$

Then $\rho \cdot \partial_0$ is a right homotopy from f to g on (23). ∎

3.1.6. Lemma. [cf. Lemma 3 and Lemma 4 on Chapter I.1 in [Qui67]] *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category. Let $a,b \in \mathcal{M}$.*

1. *We assume that (CtC) holds. If $a \in \mathcal{M}_c$ and 1_a is compatible with $\mathcal{C}$ and $\mathcal{F}$ then $\overset{l}{\sim}$ is an equivalence relation on $\mathcal{M}(a,b)$.*
2. *We assume that (FFt) holds. If $b \in \mathcal{M}_f$ and 1_b is compatible with $\mathcal{C}$ and $\mathcal{F}$ then $\overset{r}{\sim}$ is an equivalence relation on $\mathcal{M}(a,b)$.*

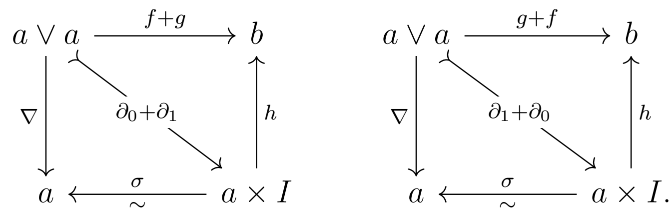
Proof. We will only prove (1). (2) is dual to (1).

We use the hypothesis only for the proof of transitivity.

For any morphism $f : a \to b$ in $\mathcal{M}$, $f \overset{l}{\sim} f$ holds by the following commutative diagram.

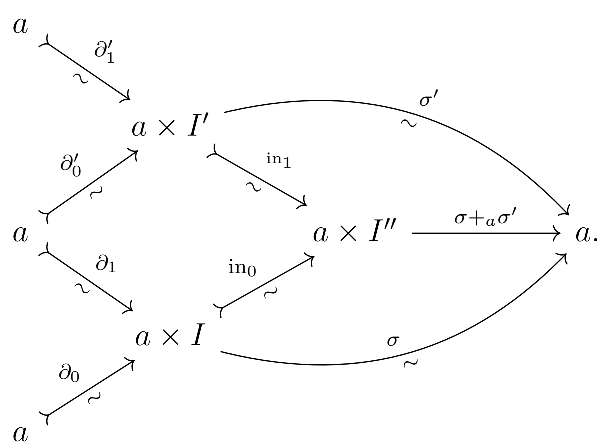
$$\begin{array}{ccc} a \vee a & \xrightarrow{f+f} & b \\ \nabla\big\downarrow & \overset{1_a+1_a}{\searrow} & \big\uparrow f \\ a & \underset{\sim}{\xleftarrow{1_a}} & a \end{array} \tag{33}$$

Let $f, g : a \to b$ be morphisms in $\mathcal{M}$. If the following diagram on the left is a left homotopy from f to g then the diagram on the right is a right homotopy from g to f.



(34)

For the transitivity, we consider morphisms $f_1, f_2, f_3 : a \to b$ in $\mathcal{M}$ and left homotopies $h : f_1 \overset{l}{\sim} f_2$ and $h' : f_2 \overset{l}{\sim} f_3$. Then we have a commutative diagram



(35)

where the square is the pushout and $\sigma +_a \sigma'$ is the induced morphism. By Lemma 3.1.3.(1), $\partial_0', \partial_1 \in \mathcal{C}_t$. Hence $\text{in}_0, \text{in}_1 \in \mathcal{C}_t$ by the pushout diagram, and $\text{in}_0 \cdot \partial_0, \text{in}_1 \cdot \partial_1' \in \mathcal{C}_t$. Since 1_a is compatible with $\mathcal{F}$,

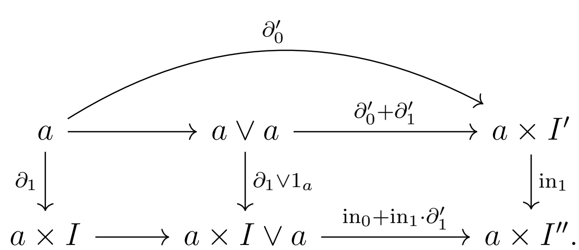
$$\sigma +_a \sigma' \in \mathcal{W} \tag{36}$$

by Lemma 2.2.5.(2).

Now we set $\partial_0'' = \text{in}_0 \cdot \partial_0$ and $\partial_1'' = \text{in}_1 \cdot \partial_1'$. We decompose $\partial_0'' + \partial_1''$ into $\partial_0 \vee 1_a$ followed by $\text{in}_0 + \text{in}_1 \partial_1'$.

$$\partial_0'' + \partial_1'' : a \vee a \xrightarrow{\partial_0 \vee 1_a} a \times I \vee a \xrightarrow{\text{in}_0 + \text{in}_1 \partial_1'} a \times I'' \tag{37}$$

$\partial_0 \vee 1_a$ is a pushout of ∂_0. Thus $\partial_0 \vee 1_a \in \mathcal{C}_t$. The pushout square in (35) is decomposed into the following two pushout squares.



(38)

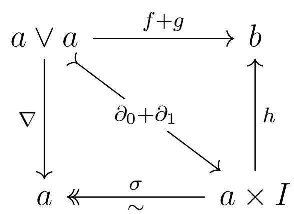
The second morphism $\mathrm{in}_0 + \mathrm{in}_1 \cdot \partial'_1$ in (37) is a pushout of $\partial'_0 + \partial'_1$. Thus $\mathrm{in}_0 + \mathrm{in}_1 \cdot \partial'_1 \in \mathcal{C}$. Then

$$\nabla : a \vee a \overset{\partial''_0 + \partial''_1}{\rightarrowtail} a \times I'' \underset{\sim}{\overset{\sigma +_a \sigma'}{\longrightarrow}} a \tag{39}$$

is a cylinder object for a, and $h +_a h' : a \times I'' \to b$ is a left homotopy from f_1 to f_3 on the cylinder object (39). ∎

3.1.7. LEMMA. [cf. Lemma 5.(ii) on Chapter I.1 in [Qui67]] *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. Let $u : a \to b$, $f, g : b \to x$ and $v : x \to y$ be morphisms in $\mathcal{M}$.*

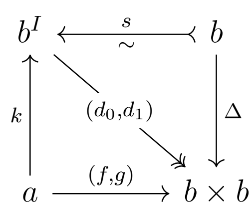
1. *If $f \overset{l}{\sim} g$ and $x \in \mathcal{M}_f$ hold, then there exists a left homotopy from f to g*



(40)

   such that $\sigma \in \mathcal{F}_t$ holds.

2. *If $f \overset{r}{\sim} g$ and $b \in \mathcal{M}_c$ hold, then there exists a right homotopy from f to g*

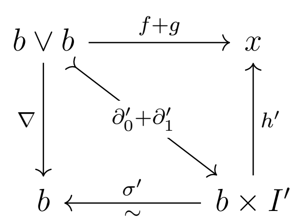


(41)

   such that $s \in \mathcal{C}_t$ holds.

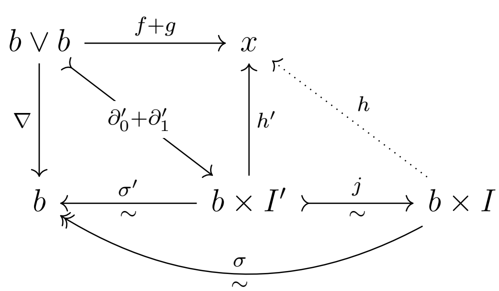
PROOF. We will only prove (1). (2) is dual to (1).

Let



(42)

be a left homotopy diagram. Since $x \in \mathcal{M}_f$, we have a lifting h in the diagram



(43)

where $\sigma' = \sigma \cdot j$. If we set $\partial_0 + \partial_1 = j \cdot (\partial'_0 + \partial'_1)$ then we have the left homotopy (40). ■

3.1.8. Lemma. [cf. Lemma 5.(iii) on Chapter I.1 in [Qui67]] *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. Let $u : a \to b$, $f, g : b \to x$ and $v : x \to y$ be morphisms in $\mathcal{M}$.*

1. *If $f \overset{l}{\sim} g$ and $x \in \mathcal{M}_f$ hold then $fu \overset{l}{\sim} gu$ holds.*
2. *If $f \overset{l}{\sim} g$ holds then $vf \overset{l}{\sim} vg$ holds.*
3. *If $f \overset{r}{\sim} g$ holds then $fu \overset{r}{\sim} gu$ holds.*
4. *If $f \overset{r}{\sim} g$ and $b \in \mathcal{M}_c$ hold then $vf \overset{r}{\sim} vg$ holds.*

Proof. We will only prove (1) and (2). (3) and (4) are dual to (2) and (1) respectively.

If $f \overset{l}{\sim} g$ then $vf \overset{l}{\sim} vg$ by definition. Assume that $f \overset{l}{\sim} g$ and let

$$\begin{array}{ccc} b \vee b & \xrightarrow{f+g} & x \\ {\scriptstyle \nabla}\downarrow & \overset{\partial_0+\partial_1}{\searrow} & \uparrow{\scriptstyle h} \\ b & \underset{\sim}{\overset{\sigma}{\twoheadleftarrow}} & b \times I \end{array} \tag{44}$$

be a homotopy diagram. Since $x \in \mathcal{M}_f$, we may assume that $\sigma \in \mathcal{F}_t$ by Lemma 3.1.7.(1). Since $\sigma \in \mathcal{F}_t$, given a cylinder object

$$a \vee a \xrightarrow{\partial'_0+\partial'_1} a \times I \xrightarrow[\sim]{\sigma'} a \tag{45}$$

for a, we can make a lifting ρ in the following commutative diagram.

$$\begin{array}{ccc} a \vee a & \xrightarrow{u \vee u} & b \vee b \\ \downarrow{\scriptstyle \partial'_0+\partial'_1} & & \downarrow{\scriptstyle \partial_0+\partial_1} \\ a \times I & \overset{\rho}{\dashrightarrow} & b \times I \\ \sim\downarrow{\scriptstyle \sigma'} & & \sim\downarrow{\scriptstyle \sigma} \\ a & \xrightarrow{u} & b \end{array} \tag{46}$$

Then

$$\begin{array}{ccc} a \vee a & \xrightarrow{fu+gu} & x \\ {\scriptstyle \nabla}\downarrow & \overset{\partial'_0+\partial'_1}{\searrow} & \uparrow{\scriptstyle h\cdot\rho} \\ a & \underset{\sim}{\overset{\sigma'}{\longleftarrow}} & a \times I \end{array} \tag{47}$$

is a left homotopy from fu to gu. ■

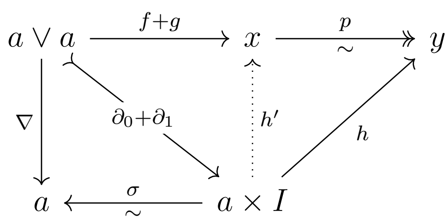
Unlike the previous lemmas, the following lemma has no additional assumption.

3.1.9. Lemma. [cf. Lemma 7 on Chapter I.1 in [Qui67]] *Let* $(\mathcal{M};\mathcal{C},\mathcal{F})$ *be a Quillen category.*

1. *Let* $f, g : a \to x$ *and* $p : x \to y$ *be morphisms in* $\mathcal{M}$. *If* $p \in \mathcal{F}_t$ *then* $pf \overset{l}{\sim} pg$ *implies* $f \overset{l}{\sim} g$.
2. *Let* $f, g : x \to y$ *and* $i : a \to x$ *be morphisms in* $\mathcal{M}$. *If* $i \in \mathcal{C}_t$ *then* $fi \overset{r}{\sim} gi$ *implies* $f \overset{r}{\sim} g$.

Proof. We will only prove (1). (2) is dual to (1).

Given a left homotopy of solid arrows from pf to pg,



(48)

we have a lifting h' because of $p \in \mathcal{F}_t$ and $\partial_0 + \partial_1 \in \mathcal{C}$. Then h' is a left homotopy from f to g. ∎

3.2. Localization. The results in this section are for Quillen categories $(\mathcal{M};\mathcal{C},\mathcal{F})$ satisfying $\mathcal{M} = \mathcal{M}_{cf}$.

3.2.1. Definition. *Let* $(\mathcal{M};\mathcal{C},\mathcal{F})$ *be a Quillen category. We assume that* $\mathcal{M} = \mathcal{M}_{cf}$ *and every identity in* $\mathcal{M}$ *is compatible with* $\mathcal{C}$ *and* $\mathcal{F}$. *Then the left homotopy and the right homotopy coincide by Lemma 3.1.5, and they are equivalent relations by Lemma 3.1.6. By Lemma 3.1.8, we have the category*

$$\pi\mathcal{M} \tag{49}$$

whose objects are the objects of $\mathcal{M}$ *and whose morphisms are the homotopy equivalent classes of morphisms in* $\mathcal{M}$, *and the functor*

$$\gamma : \mathcal{M} \to \pi\mathcal{M} \tag{50}$$

mapping a morphism in $\mathcal{M}$ *to the homotopy equivalent class associated with it.*

3.2.2. Definition. [cf. Lemma 1 on Chapter 1.5 in [Qui67]] *Let* $(\mathcal{M};\mathcal{C},\mathcal{F})$ *be a Quillen category.*

1. *A morphism $i : a \to b$ in $\mathcal{C}$ is called a strong deformation retract if there are a morphism $r : b \to a$ and a right homotopy $k : i \cdot r \overset{r}{\sim} 1_b$*

$$\begin{array}{ccc} b^I & \xleftarrow[\sim]{s} & b \\ {\scriptstyle k}\uparrow & \searrow^{(d_0,d_1)} & \downarrow{\scriptstyle \Delta} \\ b & \xrightarrow{(ir,1_b)} & b \times b \end{array} \tag{51}$$

such that $r \cdot i = 1_a$ and $k \cdot i = s \cdot i$ hold.

2. *A morphism $p : x \to y$ in $\mathcal{F}$ is called the dual of a strong deformation retract if there are a morphism $t : y \to x$ and a left homotopy $h : t \cdot p \overset{l}{\sim} 1_x$*

$$\begin{array}{ccc} x \vee x & \xrightarrow{tp+1_x} & x \\ {\scriptstyle \nabla}\downarrow & \searrow^{\partial_0+\partial_1} & \uparrow{\scriptstyle h} \\ x & \xleftarrow[\sim]{\sigma} & x \times I \end{array} \tag{52}$$

such that $p \cdot t = 1_x$ and $p \cdot h = p \cdot \sigma$ hold.

The following is Lemma 1 on Chapter 1.5 in [Qui67]. The proof there uses the homotopies of homotopies and does not seem applicable for Quillen categories. Here we use the retract lemma, Lemma 2.3.1, instead.

3.2.3. Lemma. [cf. Lemma 1 on Chapter 1.5 in [Qui67]] *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. We assume that $\mathcal{M} = \mathcal{M}_{cf}$ and all the identities in $\mathcal{M}$ are compatible with $\mathcal{C}$ and $\mathcal{F}$.*

1. *If $i \in \mathcal{C}$ then the following are equivalent.*
   - *(a) $i \in \mathcal{C}_t$.*
   - *(b) i is a strong deformation retract.*
   - *(c) $\gamma(i)$ is an isomorphism in $\pi\mathcal{M}$.*
2. *If $p \in \mathcal{F}$ then the following are equivalent.*
   - *(a) $p \in \mathcal{F}_t$.*
   - *(b) p is the dual of a strong deformation retract.*
   - *(c) $\gamma(p)$ is an isomorphism in $\pi\mathcal{M}$.*

Proof. We will only prove (2). (1) is dual to (2).

We use the compatibility hypothesis only for the proof of (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Let $p : x \to y$. Since $y \in \mathcal{M}_c$ we have a lifting t in the following diagram of solid arrows.

$$\begin{tikzcd} & x \arrow[d, "p"] \\ y \arrow[ur, dotted, "t"] \arrow[r, "1"] & y \end{tikzcd} \tag{53}$$

So, $pt = 1_y$ holds. Then given a cylinder object

$$x \vee x \rightarrowtail^{\partial_0+\partial_1} x \times I \xrightarrow[\sim]{\sigma} x \tag{54}$$

for x, we have a commutative diagram

$$\begin{tikzcd} x \vee x \arrow[r, "tp+1"] \arrow[d, "\nabla"'] \arrow[dr, tail, "\partial_0+\partial_1"] & x \arrow[r, two heads, "p", "\sim"'] & y \\ x & x \times I \arrow[l, "\sigma", "\sim"'] \arrow[u, dotted, "h"] \arrow[ur, "p\sigma"'] & \end{tikzcd} \tag{55}$$

of solid arrows. Since $p \in \mathcal{F}_t$, we have a lifting h, which is a left homotopy from tp to 1_x. Hence p is the dual of a strong deformation retract.

(b) $\Rightarrow$ (c) is clear

(c) $\Rightarrow$ (a) There exists a morphism $t : y \to x$ satisfying $pt \overset{l}{\sim} 1_y$ and $tp \overset{l}{\sim} 1_x$. Suppose that the diagram

$$\begin{tikzcd} y \vee y \arrow[r, "pt+1"] \arrow[d, "\nabla"'] \arrow[dr, tail, "\partial_0+\partial_1"] & y \\ y & y \times I \arrow[l, "\sigma", "\sim"'] \arrow[u, "h"'] \end{tikzcd} \tag{56}$$

is a left homotopy from pt to 1_y. Then the following diagram of solid arrows commutes.

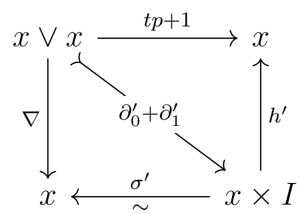
$$\begin{tikzcd} & y \arrow[r, "t"] \arrow[d, tail, "\partial_0", "\sim"'] & x \arrow[d, two heads, "p"] \\ y \arrow[r, tail, "\partial_1", "\sim"'] & y \times I \arrow[r, "h"] \arrow[ur, dotted, "\rho"] & y \end{tikzcd} \tag{57}$$

Since $\partial_0 \in \mathcal{C}_t$ holds by Lemma 2.1.7.(1) and Lemma 3.1.3.(1), $p \in \mathcal{F}$ implies that the diagram has a lifting ρ. Then ρ defines a left homotopy from t to $\rho \cdot \partial_1$. Hence $\rho \cdot \partial_1 \cdot p \overset{l}{\sim}$

$t \cdot p \overset{l}{\sim} 1_x$ by Lemma 3.1.8.(1). So we may assume that $p \cdot t = 1_y$ by replacing t with $\rho \cdot \partial_1$. Then we have the following retract diagram.

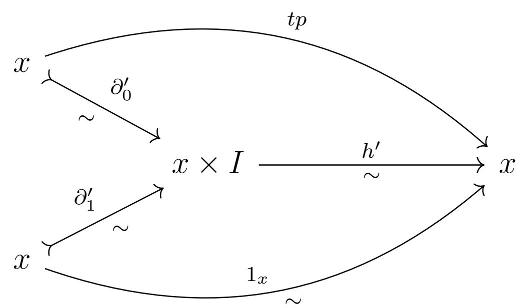
$$\begin{array}{ccccc} x & \xrightarrow{1} & x & \xrightarrow{1} & x \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle tp} & & \downarrow{\scriptstyle p} \\ y & \xrightarrow{t} & x & \xrightarrow{p} & y \end{array} \tag{58}$$

Let



(59)

be a left homotopy from tp to 1_x. We have $\partial'_i \in \mathcal{C}_t$ for $i = 0, 1$ by Lemma 2.1.7.(1) and Lemma 3.1.3.(1). Then $h' \in \mathcal{W}$ by Lemma 2.2.5.(2).



(60)

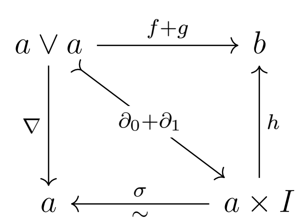
Hence $t \cdot p \in \mathcal{W}$, and $p \in \mathcal{F}_t$ by Lemma 2.3.1.(2). ■

3.2.4. Lemma. [cf. Lemma 8 on Chapter I.1 and Proposition 1 on Chapter I.5 in [Qui67]] *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. We assume that $\mathcal{M} = \mathcal{M}_{cf}$ and all the identities in $\mathcal{M}$ are compatible with $\mathcal{C}$ and $\mathcal{F}$. Then the following hold.*

1. *The functor γ in (50) is the localization of $\mathcal{M}$ with respect to $\mathcal{W}$.*
2. *For any morphism f in $\mathcal{M}$, $f \in \mathcal{W}$ holds if and only if $\gamma(f)$ is an isomorphism in $\pi\mathcal{M}$.*

Proof. (1) First, by Lemma 3.2.3, if $f \in \mathcal{W}$ then $\gamma(f)$ is an isomorphism in $\pi\mathcal{M}$.

Let $F : \mathcal{M} \to \mathcal{N}$ be a functor to a category $\mathcal{N}$ such that for every $f \in \mathcal{W}$, $F(f)$ is an isomorphism in $\mathcal{N}$. Let $f, g : a \to b$ and let



(61)

be a left homotopy from f to g. Since $\sigma \in \mathcal{W}$ and $F(\sigma)$ is an isomorphism, $F(\partial_0) = F(\sigma)^{-1} = F(\partial_1)$. Thus $F(f) = F(g)$, and the functor F factors through γ.

(2) Let f be a morphism in $\mathcal{M}$. If f is in $\mathcal{W}$ then $\gamma(f)$ is an isomorphism in $\pi\mathcal{M}$ as in (1). Now if $\gamma(f)$ is an isomorphism in $\pi\mathcal{M}$ we decompose f into $i \in \mathcal{C}_t$ followed by $p \in \mathcal{F}$. $\gamma(i)$ is an isomorphism in $\pi\mathcal{M}$ by Lemma 3.2.3.(1). Then because $\gamma(f)$ is an isomorphism in $\pi\mathcal{M}$, so is $\gamma(p)$. Then $p \in \mathcal{F}_t$ by Lemma 3.2.3.(2), hence $f \in \mathcal{W}$. ∎

3.2.5. Remark. Up to the proof of (1) that the functor γ is a localization, we only need the conclusions of Lemma 3.1.3, the existence of a cylinder object with $\partial_i : a \to a \times I$ in $\mathcal{C}_t$ for each $a \in \mathcal{M}_c$ and of a path object with $d_i : x^I \to x$ in $\mathcal{F}_t$ for each $x \in \mathcal{M}_f$. But for the proof of (2) that the functor γ is saturated, we need the implication (c) ⇒ (a) in Lemma 3.2.3. So all the identities are required to be compatible with $\mathcal{C}$ and $\mathcal{F}$.

4. Proof of Theorem 1.5

Here we prove Theorem 1.5. For this, we need to relate being closed under the left homotopy and the right homotopy with being compatible with $\mathcal{C}$ and $\mathcal{F}$.

4.1. Homotopy and compatibility.

4.1.1. Lemma. *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category. We assume that $\mathcal{W}$ is compatible with $\mathcal{C}$ and $\mathcal{F}$. Let $f, g : a \to b$ be morphisms in $\mathcal{M}$.*

1. *We assume that (CtC) holds. If $a \in \mathcal{M}_c$ and $f \overset{l}{\sim} g$ hold then $f \in \mathcal{W}$ holds if and only if $g \in \mathcal{W}$ holds.*
2. *We assume that (FFt) holds. If $b \in \mathcal{M}_f$ and $f \overset{r}{\sim} g$ hold then $f \in \mathcal{W}$ holds if and only if $g \in \mathcal{W}$ holds.*

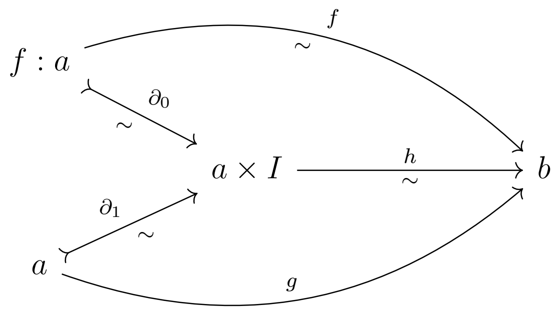
Proof. We will only prove (1). (2) is dual to (1).

Let

$$\begin{array}{ccc} a \vee a & \xrightarrow{f+g} & b \\ {\scriptstyle\nabla}\downarrow & \searrow^{\partial_0+\partial_1} & \uparrow{\scriptstyle h} \\ a & \xleftarrow[\sim]{\sigma} & a \times I \end{array} \tag{62}$$

be a left homotopy from f to g. Then $\partial_i \in \mathcal{C}_t$ holds for $i = 0, 1$ by Lemma 3.1.3.(1)

because of $1_a \in \mathcal{W}$. If $f \in \mathcal{W}$ then $h \in \mathcal{W}$ holds by Lemma 2.2.5.(2).



(63)

Thus $g \in \mathcal{W}$ holds. ■

The following lemma follows from Lemma 4.1.1.

4.1.2. Lemma. *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category. We assume that $\mathcal{W}$ is compatible with $\mathcal{C}$ and $\mathcal{F}$.*

1. *If $\mathcal{M} = \mathcal{M}_c$ and (CtC) hold, then $\mathcal{W}$ is closed under the left homotopy.*
2. *If $\mathcal{M} = \mathcal{M}_f$ and (FFt) hold, then $\mathcal{W}$ is closed under the right homotopy.*

4.1.3. Lemma. *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category.*

1. *If all the left homotopy classes of identities in $\mathcal{M}$ are contained in $\mathcal{W}$, then all the identities in $\mathcal{M}$ are compatible with $\mathcal{C}$.*
2. *If all the right homotopy classes of identities in $\mathcal{M}$ are contained in $\mathcal{W}$, then all the identities in $\mathcal{M}$ are compatible with $\mathcal{F}$.*

Proof. We will only prove (1). (2) is dual to (1).

Let $a \in \mathcal{M}$. Consider a $(\mathcal{C}, \mathcal{F}_t)$-factorization

$$1_a : a \overset{i}{\rightarrowtail} b \underset{\sim}{\overset{p}{\twoheadrightarrow}} a \tag{64}$$

of the identity 1_a. Since $p \cdot i = 1$, we have the following retract diagram.

$$\begin{array}{ccccc} a & \xrightarrow{i} & b & \xrightarrow{p} & a \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle ip} & & \downarrow{\scriptstyle i} \\ b & \xrightarrow{1} & b & \xrightarrow{1} & b \end{array} \tag{65}$$

Since $p = p \cdot i \cdot p$ holds, we have $1_b \overset{l}{\sim} i \cdot p$ by Lemma 3.1.9.(1). Then $1_b \in \mathcal{W}$ implies $i \cdot p \in \mathcal{W}$. So by Lemma 2.3.1.(1), $i \in \mathcal{C}_t$ holds. ■

The following lemma follows from Lemma 4.1.3

4.1.4. Lemma. *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category.*

1. *If $\mathcal{W}$ is closed under the left homotopy then all the identities in $\mathcal{M}$ are compatible with $\mathcal{C}$.*

2. *If $\mathcal{W}$ is closed under the right homotopy then all the identities in $\mathcal{M}$ are compatible with $\mathcal{F}$.*

4.2. Proof of Theorem 1.5. It follows from Theorem 4.2.1 below right away.

4.2.1. Theorem. *Let $(\mathcal{M};\mathcal{C},\mathcal{F})$ be a Quillen category. If $\mathcal{M} = \mathcal{M}_{cf}$ then the following are equivalent.*

1. *$\mathcal{W}$ satisfies the two out of three property.*

2. *$\mathcal{W}$ satisfies the properties (L) and (R) in Definition 1.2.*

3. *$\mathcal{W}$ is compatible with $\mathcal{C}$ and $\mathcal{F}$.*

4. *$\mathcal{W}$ is closed under the left homotopy and the right homotopy.*

5. *All the identities in $\mathcal{M}$ are compatible with $\mathcal{C}$ and $\mathcal{F}$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) follows from Lemma 2.1.1.

(3) $\Rightarrow$ (4) Since $\mathcal{M} = \mathcal{M}_{cf}$, the left homotopy and the right homotopy coincide by Lemma 3.1.5. So it follows from either Lemma 2.1.7.(1) and Lemma 4.1.2.(1) or Lemma 2.1.7.(2) and Lemma 4.1.2.(2).

(4) $\Rightarrow$ (5) follows from Lemma 4.1.4.

(5) $\Rightarrow$ (1) From our assumption we have the homotopy category (Definition 3.2.1)

$$\pi\mathcal{M} \tag{66}$$

and, by Lemma 3.2.4.(1), the localization functor

$$\gamma : \mathcal{M} \to \pi\mathcal{M}. \tag{67}$$

Then the conclusion (1) follows from Lemma 3.2.4.(2) because the set of all isomorphisms in $\pi\mathcal{M}$ satisfies the two out of three property. ∎

4.2.2. Remark. Theorem 4.2.1 implies that the property (M) in Definition 1.2 holds under (L) and (R) if every object is fibrant and cofibrant.

4.3. Simplicial Quillen category. For the rest of this section, we give a new proof of a result in [Lee15] by verifying the property (3) in Theorem 4.2.1.

In [Lee15], we defined a simplicial Quillen category as a Quillen category enriched in the category of simplicial sets and satisfying the axioms (SM0) and (SM7) on Chapter 2.2 in [Qui67], just like the simplicial model categories. So the only difference between them is again the two out of three property for $\mathcal{W}$.

In [Lee15], we showed that if every object of a simplicial Quillen category is fibrant and cofibrant, then it is already a simplicial model category (Theorem 1.7 in [Lee15]).

The strategy was to use the following two sets and the simplicial homotopies instead of the left and the right homotopies in the underlying Quillen categories.

1. $\mathcal{SC} = \{f \in \mathrm{Mor}(\mathcal{M}) \mid \pi_0\underline{\mathcal{M}}(f, z) \text{ is bijective for all } z \in \mathrm{ob}\,\mathcal{M}_f\}$
2. $\mathcal{SF} = \{g \in \mathrm{Mor}(\mathcal{M}) \mid \pi_0\underline{\mathcal{M}}(a, g) \text{ is bijective for all } a \in \mathrm{ob}\,\mathcal{M}_c\}$

These sets clearly satisfy the two out of three property, and

$$\mathcal{C}_t \subseteq \mathcal{SC} \qquad \mathcal{F}_t \subseteq \mathcal{SF} \tag{68}$$

hold (Lemma 3.8.(1) in [Lee15]).

Given a simplicial Quillen category $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ satisfying $\mathcal{M} = \mathcal{M}_{cf}$, the following two inclusions hold (Corollary 3.15[1] in [Lee15]).

$$\mathcal{C} \cap \mathcal{SC} \subseteq \mathcal{C}_t \qquad\qquad \mathcal{F} \cap \mathcal{SF} \subseteq \mathcal{F}_t \tag{69}$$

They were the key for the proof and follow from Lemma 3.13 in [Lee15], which is a generalization of Lemma 7 on Chapter 2.3 in [Qui67]. Below we use them to derive the aforementioned result from Theorem 4.2.1.

Suppose that we have a decomposition $g \cdot f$ of a morphism $q \cdot j$ in $\mathcal{W}$ such that $f \in \mathcal{C}$ and $g \in \mathcal{F}_t$.

$$\begin{array}{ccc} \bullet & \overset{j}{\underset{\sim}{\rightarrowtail}} & \bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q}\,\sim \\ \bullet & \overset{g}{\underset{\sim}{\twoheadrightarrow}} & \bullet \end{array} \tag{70}$$

If $q, g \in \mathcal{F}_t$ then with an argument similar to that of the proof of Lemma 3.2.3 but using simplicial homotopies instead, one can show that $q, g \in \mathcal{SC}$ (Lemma 3.8.(2) in [Lee15]). Thus by the two out of three property for $\mathcal{SC}$ we have $f \in \mathcal{SC}$. Hence by the inclusion (69), $f \in \mathcal{C}_t$. Dually, using $\mathcal{SF}$ instead, one can show that if $f \in \mathcal{C}_t$ and $g \in \mathcal{F}$ then $g \in \mathcal{F}_t$. So the property (3) in Theorem 4.2.1 holds.

[1] In Corollary 3.5 in [Lee15], $\mathcal{SC} \cap \mathrm{Mor}(\mathcal{M})_f \subseteq \mathcal{C}_t$ should be corrected to $\mathcal{C} \cap \mathcal{SC} \cap \mathrm{Mor}(\mathcal{M})_f \subseteq \mathcal{C}_t$. Similarly, $\mathcal{SF} \cap \mathrm{Mor}(\mathcal{M})^c \subseteq \mathcal{F}_t$ should be corrected to $\mathcal{F} \cap \mathcal{SF} \cap \mathrm{Mor}(\mathcal{M})^c \subseteq \mathcal{F}_t$.

5. Proof of Theorem 1.6

The property (Fact) in Definition 2.1.2 which is equivalent to the property (M) in Definition 1.2 is different from the others because it is about the existence of a $(\mathcal{C}_t, \mathcal{F}_t)$-factorization. Hence it is not something that can be verified as we did at the end of the previous section. However, from Theorem 4.2.1 and Lemma 2.1.7, one may speculate on the possibility of deducing (Fact) from (CtC), (FtC), (FFt) and (FCt), hence (M) from (L) and (R). Here we provide a positive evidence for it.

5.1. A sufficient condition for (M). The essential point of the assumption in the following lemma is that y is fibrant.

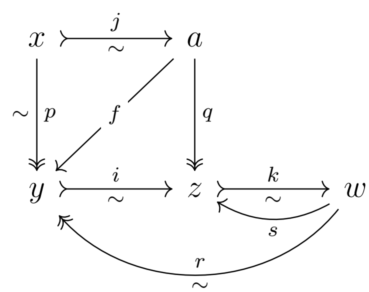
5.1.1. Lemma. *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. We assume that $\mathcal{W}$ is compatible with $\mathcal{C}$ and $\mathcal{F}$, and that (FFt) holds. Let $p : x \to y$ and $i : y \to z$. If $p \in \mathcal{F}_t$, $i \in \mathcal{C}_t$ and $y, z \in \mathcal{M}_f$ hold then there exist morphisms $j \in \mathcal{C}_t$ and $q \in \mathcal{F}_t$ that make the following diagram commute.*

$$\begin{array}{ccc} x & \overset{j}{\underset{\sim}{\rightarrowtail}} & a \\ {\scriptstyle \sim}\downarrow p & & {\scriptstyle \sim}\downarrow q \\ y & \overset{i}{\underset{\sim}{\rightarrowtail}} & z \end{array} \tag{71}$$

Proof. Let

$$\begin{array}{ccc} x & \overset{j}{\underset{\sim}{\rightarrowtail}} & a \\ {\scriptstyle \sim}\downarrow p & & \downarrow q \\ y & \overset{i}{\underset{\sim}{\rightarrowtail}} & z \end{array} \tag{72}$$

be a $(\mathcal{C}_t, \mathcal{F})$-factorization of $i \cdot p$. Since $y \in \mathcal{M}_f$, there is a morphism $f : a \to y$ such that $f \cdot j = p$, and there is a morphism $t : z \to y$ such that $t \cdot i = 1_y$. Because of $1_y \in \mathcal{W}$ and $i \in \mathcal{C}_t$, we can decompose t into $k : z \to w \in \mathcal{C}_t$ followed by $r \in \mathcal{F}_t$ by Lemma 2.2.5.(2). Since $z \in \mathcal{M}_f$, there is a morphism $s : w \to z$ such that $s \cdot k = 1_z$.



(73)

Since $p \in \mathcal{F}_t$ and $j \in \mathcal{C}_t$, we have $f \in \mathcal{W}$ by Lemma 2.2.5.(2). Since $fj = p = rkip = rkqj$ and $j \in \mathcal{C}_t$, we have $f \overset{r}{\sim} rkq$ by Lemma 3.1.9.(2). Then $rkq \in \mathcal{W}$ by Lemma 4.1.1.(2), so $kq \in \mathcal{W}$ by Lemma 2.2.5.(1). Consider the following retract diagram.

$$\begin{array}{ccccc} a & \xrightarrow{1} & a & \xrightarrow{1} & a \\ \downarrow q & & \downarrow kq & & \downarrow q \\ z & \xrightarrow{k} & w & \xrightarrow{s} & z \end{array} \tag{74}$$

Since $q \in \mathcal{F}$ and $kq \in \mathcal{W}$ hold, $q \in \mathcal{F}_t$ by Lemma 2.3.1.(2). ∎

5.2. Proof of Theorem 1.6. It follows immediately from Lemma 2.1.6 and the following Theorem 5.2.1 and Theorem 5.2.2.

5.2.1. Theorem. *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. If $\mathcal{M} = \mathcal{M}_f$ then the following are equivalent.*

1. *$\mathcal{W}$ satisfies the two out of three property.*
2. *(FtC), (FFt) and (FCt) hold.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1) We use Proposition 2.1.4. Since $\mathcal{M} = \mathcal{M}_f$, (CtC) holds by Lemma 2.1.7.(1). $\mathcal{W}$ is compatible with $\mathcal{C}$ and $\mathcal{F}$ by Lemma 2.2.3 because of (FtC) and (FCt). Then the property (Fact) follows from Lemma 5.1.1. ∎

Dually, we have the following theorem.

5.2.2. Theorem. *Let $(\mathcal{M}; \mathcal{C}, \mathcal{F})$ be a Quillen category. If $\mathcal{M} = \mathcal{M}_c$ then the following are equivalent.*

1. *$\mathcal{W}$ satisfies the two out of three property.*
2. *(CtC), (FtC) and (FCt) hold.*

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Department of Mathematics, Konkuk University, Kwangjin-Gu Hwayang-dong 1, Seoul 143-701, Korea
Email: `mbrs@konkuk.ac.kr`

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