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THE MOORE COMPLEX OF A SIMPLICIAL COCOMMUTATIVE HOPF ALGEBRA

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ABSTRACT. We study the Moore complex of a simplicial cocommutative Hopf algebra through Hopf kernels. The most striking result to emerge from this construction is the coherent definition of 2-crossed modules of cocommutative Hopf algebras. This unifies the 2-crossed module theory of groups and of Lie algebras when we take the group-like and primitive functors into consideration.

Contents

1. Introduction

A simplicial group $\mathcal{G} = (G_n, d_i^n, s_j^{n+1})$ is a simplicial object [May, 1967] in the category of groups. It is given by a collection of groups G_n, together with group homomorphisms $d_i^n\colon G_n \longrightarrow G_{n-1}$ and $s_j^{n+1}\colon G_n \longrightarrow G_{n+1}$ for $i, j = 0, \ldots, n$ called faces and degeneracies, respectively, satisfying the well known simplicial identities. The Moore complex [Moore, 1955] of a simplicial group is a chain complex

$$N(\mathcal{G}) = \Big(\ldots \xrightarrow{\partial_{(n+1)}} N(G)_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_3} N(G)_2 \xrightarrow{\partial_2} N(G)_1 \xrightarrow{\partial_1} G_0\Big)$$

of groups, where

$$N(G)_n = \bigcap_{i=0}^{n-1} \ker(d_i^n)$$

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at level n, and the boundary morphisms $\partial_n\colon N(G)_n \longrightarrow N(G)_{n-1}$ are the restrictions of the $d_n^n\colon G_n \longrightarrow G_{n-1}$. Moreover, $N(\mathcal{G})$ defines a normal chain complex of groups, namely $\partial_n(N(G)_n) \trianglelefteq N(G)_{n-1}$, for all $n \geq 1$. Thus, the Moore complex can be considered as the normalized chain complex of a simplicial group. This construction was first considered as a Moore complex functor from the category of simplicial abelian groups to the category of chain complexes of abelian groups - that yields an equivalence between these two categories (called Dold-Kan correspondence) which was independently proven by [Dold, 1958; Kan, 1958]. For more details, see [Goerss and Jardine, 1999]. Afterwards, the Dold-Kan correspondence was studied for various cases, and consequently generalized to abelian categories in [Dold and Puppe, 1961], to semi-abelian categories in [Bourn, 2007], and to more general source categories and settings rather than simplicial objects in [Lack and Street, 2015]. Above all, we focus our attention on [Bourn, 2007] which considers the Moore complex structure in semi-abelian categories, of which the category of cocommutative Hopf algebras are an instance – see [Gran et al., 2016, 2018], also [Vespa and Wambst, 2018] and [Gran et al., 2019]. Bourn's work builds on [Everaert and Van der Linden, 2004], and semi-abelian categories were introduced in [Janelidze et al., 2002]. In the interests of brevity, a semi-abelian category is a category with binary coproducts which is pointed, Barr exact and Bourn protomodular. And the category of cocommutative Hopf algebras over any field is semi-abelian, as well as the category of groups and Lie algebras.

Not only in the category-theoretic sense, but the Moore complex also has many roles in the context of algebraic topology. For instance, it is a well-known property that the n^{th} homotopy group of a simplicial group $\mathcal{G}$ is the n^{th} homology group of the Moore complex $N(\mathcal{G})$. Furthermore, as an extension of the Dold-Kan correspondence to the category of arbitrary groups (i.e. not only abelian groups), it is shown in [Carrasco and Cegarra, 1991] that the Moore complex construction yields a functor from the category of simplicial groups to the category of hypercrossed complexes of groups. A recent study from the same authors also deals with the Lie algebraic case of the same problem [Carrasco and Cegarra, 2017]. A hypercrossed complex is a chain complex of groups that comes equipped with a specific type of binary operations satisfying certain axioms. Hypercrossed complexes have the following essential property, which is strongly related to this study: they capture crossed modules and 2-crossed modules for dimensions one and two, respectively.

A crossed module is a group homomorphism $\partial\colon E \longrightarrow G$, together with an action $\triangleright$ of G on E by automorphisms satisfying $\partial(g \triangleright e) = g\,\partial(e)\,g^{-1}$ (equivariance) and $\partial(e) \triangleright f = efe^{-1}$ (Peiffer condition), for all $e, f \in E$ and $g \in G$. The notion is introduced in [Whitehead, 1949] as an algebraic model of connected homotopy 2-types. From another point of view, a crossed module can be considered as an encoded strict 2-group [Brown and Spencer, 1976]. Categorically, crossed modules are equivalent to cat^1-groups, which can be shown to be equivalent to internal categories in the category of groups [Loday, 1982]. For more details, see [Porter, 1982; Brown, 1987, 1999; Baez and Lauda, 2004; Faria Martins and Picken, 2010; Morton and Picken, 2015]. Also a thorough discussion of crossed modules from the topological and algebraic point of view is given in [Brown, 2018].

For a given simplicial group $\mathcal{G}$, we say that the Moore complex $N(\mathcal{G})$ is of length n, if $N(G)_i$ is trivial for all $i > n$. In the case that $n = 1$ it yields a crossed module $N(G)_1 \xrightarrow{\partial_1} G_0$, where the action of G_0 on $N(G)_1$ is defined using the conjugate action via s_0. Inspired by this functorial relationship between simplicial groups and crossed modules, Conduché introduced 2-crossed modules of groups in [Conduché, 1984]. Namely, for a given simplicial group $\mathcal{G}$ with Moore complex of length two, it is shown that the first three levels of the Moore complex $N(G)_2 \xrightarrow{\partial_2} N(G)_1 \xrightarrow{\partial_1} G_0$ leads to the 2-crossed module definition. A 2-crossed module of groups is a complex $L \longrightarrow E \longrightarrow G$ of groups together with left actions $\triangleright$ of G on L, E, and on itself by conjugation; and a G-equivariant function $\{\,,\}\colon E \times E \longrightarrow L$ called Peiffer lifting, satisfying certain properties. An alternative way to obtain a 2-crossed module from a simplicial group without considering the length of the Moore complex is given in [Mutlu and Porter, 1998]. Another analogy from crossed modules is that 2-crossed modules are also algebraic models for connected homotopy 3-types, that is, pointed CW-complexes X such that $\pi_i(X) = 0$ for $i \geq 3$ [Carrasco and Cegarra, 1991]. Additionally, there are some other algebraic models of homotopy 3-types such as braided crossed modules [Brown and Gilbert, 1989], neat crossed squares [Faria Martins, 2011] and Gray 3-groups [Kamps and Porter, 2002] (these three are equivalent to the 2-crossed modules); crossed squares [Ellis, 1993a] and quadratic modules [Baues, 1991] (being the particular case of 2-crossed modules). Furthermore, as a generalization, 2-quasi crossed modules are introduced [Carrasco and Porter, 2016] in which some conditions are relaxed.

Regarding to all group theoretic terminology given above, and in the light of the close analogy between general algebraic properties of groups and Lie algebras; the Lie algebraic case of the whole 2-crossed module theory is given in [Ellis, 1993b], based on [Kassel and Loday, 1982] in which the crossed modules are introduced in the context of Lie algebras. However, the analogy between groups and Lie algebras becomes more powerful in the category of Hopf algebras that allows us to unify both of these group and Lie algebraic theories in a functorial way, which was the main motivation of this study.

Hopf algebras [Sweedler, 1969] can be thought of as a unification of groups and Lie algebras as being the group algebra of a group via the functor $\mathcal{G}^\sharp$, and the universal enveloping algebra of a Lie algebra via the functor $\mathcal{F}^\sharp$.[1] Conversely, we have the functors *Gl* and *Prim* from the category of Hopf algebras to the category of groups and of Lie algebras which assigned group-like and primitive elements, respectively (for full details, we refer to [Gran et al., 2016][§2.2]). It is crucial that both the group algebra and the universal enveloping algebra turns into a specific type of Hopf algebras called *cocommutative*. Following from the cocommutativity property, we also have a coreflection functor $R\colon \{\text{Hopf}\} \longrightarrow \{\text{Hopf}^{\text{cc}}\}$ [Porst, 2008] from the category of *all* Hopf algebras to the category of *cocommutative* Hopf algebras, the latter being a full and replete subcategory which is also semi-abelian. There exists a category $\{\text{Grp} \ltimes \text{Lie}\}$ whose objects are triples

[1]The functors denoted by $\mathcal{F}^\sharp$ and $\mathcal{G}^\sharp$ are usually written as U and $\kappa[\cdot]$, respectively, where κ is the field on which the Hopf algebras are built.

$(G, L, \triangleright)$, where G is a group, L is a Lie algebra and $\triangleright$ is a representation of G on L by Lie algebra maps. We have a functor $\mathcal{G}^\sharp \ltimes \mathcal{F}^\sharp \colon \{\mathrm{Grp} \ltimes \mathrm{Lie}\} \longrightarrow \{\mathrm{Hopf}^{\mathrm{cc}}\}$ sending $(G, L, \triangleright)$ to $\mathcal{F}^\sharp(L) \otimes_\rho \mathcal{G}^\sharp(G)$ (for the notation, see Example 2.8). Conversely, we naturally have $Gl \ltimes Prim \colon \{\mathrm{Hopf}^{\mathrm{cc}}\} \longrightarrow \{\mathrm{Grp} \ltimes \mathrm{Lie}\}$ noting that the group of group-like elements acts on the set of primitive elements by conjugation. In particular, if the base field of a cocommutative Hopf algebra is algebraically closed and of characteristic zero, then the functors $\mathcal{G}^\sharp \ltimes \mathcal{F}^\sharp$ and $Gl \ltimes Prim$ define an equivalence of categories $\{\mathrm{Hopf}^{\mathrm{cc}}\} \cong \{\mathrm{Grp} \ltimes \mathrm{Lie}\}$. This is one of the ways to state the Cartier-Gabriel-Kostant-Milnor-Moore theorem. The full diagram of these categories and functors is:

$$\begin{tikzcd}
\{\mathrm{Hopf}\} \arrow[r, shift left, "R"] \arrow[r, phantom, "\top"] & \{\mathrm{Hopf}^{\mathrm{cc}}\} \arrow[l, shift left, "\iota"] \arrow[rr, shift right, bend right=10, "Gl \ltimes Prim"'] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left, "Gl"] \arrow[dr, phantom, "\top"] \arrow[rrrd, shift left, bend left=40, "Prim"] \arrow[rrrd, phantom, bend left=40, "\perp"] & & \{\mathrm{Grp} \ltimes \mathrm{Lie}\} \arrow[ll, shift right, bend right=20, "\mathcal{G}^\sharp \ltimes \mathcal{F}^\sharp"'] \arrow[dl, shift left, "U_1"] \arrow[dl, phantom, "\perp"] \arrow[dr, shift right, "U_2"'] \arrow[dr, phantom, "\perp"] & \\
& & \{\mathrm{Grp}\} \arrow[ul, shift left, bend left, "\mathcal{G}^\sharp"] \arrow[ur, shift left, bend left, "\iota_1"] \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \{\mathrm{Set}\} \arrow[l, shift right, "\mathcal{F}"'] \arrow[r, shift right, "\mathcal{F}"'] \arrow[r, phantom, "\top"] & \{\mathrm{Lie}\} \arrow[l, shift right, "U"'] \arrow[ul, shift right, bend right, "\iota_2"'] \arrow[lllu, shift right, bend right=40, "\mathcal{F}^\sharp"']
\end{tikzcd} \tag{1}$$

where $\mathcal{F}$ is the free functor, U is the forgetful functor, ι is the inclusion, and the forgetful functors $U_1 \colon \{\mathrm{Grp} \ltimes \mathrm{Lie}\} \longrightarrow \{\mathrm{Grp}\}$ and $U_2 \colon \{\mathrm{Grp} \ltimes \mathrm{Lie}\} \longrightarrow \{\mathrm{Lie}\}$ selecting the first and second component of $(G, L, \triangleright)$. In addition, ι_1 and ι_2 being $\iota_1(G) = G \mapsto (G, \{0\}, \triangleright)$ and $\iota_2(L) = (\{1\}, L, \triangleright)$. These two actions are the trivial ones.

Crossed modules of Hopf algebras are introduced in [Majid, 2012] as encoding strict 2-quantum groups. Independently, there exists another definition of crossed modules from the perspective of symmetric monoidal categories given in [Fernández Vilaboa et al., 2007] which is different from the definition of Majid, but coincides with it in the case of cocommutative Hopf algebras. Furthermore, a more general notion is given in [Frégier and Wagemann, 2011] and there is no agreement as to the unique crossed module definition for Hopf algebras; see [Alonso Alvarez et al., 2018] for the discussion. In this paper, we follow [Majid, 2012] because of the following additional facts: it is clear from [Gran et al., 2019] that Majid's definition is not only coherent with [Fernández Vilaboa et al., 2007], but also from the point of view of semi-abelian categories. Moreover, it is proven in [Faria Martins, 2016] that this crossed module structure is also preserved under the functors Gl and $Prim$. Therefore, following Majid's definition, crossed modules of Hopf algebras can be seen as a unification of crossed modules of groups and of Lie algebras.

The major outcome of this paper is to define 2-crossed modules of cocommutative Hopf algebras, which extend crossed modules to one level further. From a categorical point of view, this notion will unify the theory of 2-crossed modules of groups and of Lie algebras when we take the functors Gl and $Prim$ into consideration. As for the group and Lie algebra case, we find out the functorial relationship between simplicial objects and 2-crossed modules in the category of cocommutative Hopf algebras. For this aim, we first give the explicit definition of a Moore complex of a simplicial cocommutative Hopf algebra, which will be constructed via Hopf kernels. No doubt this definition again unifies the Moore complex of groups and Lie algebras in the sense of the same functors. Then we obtain a 2-crossed module structure from a simplicial cocommutative Hopf algebra with Moore complex of length 2 with the aid of iterated Peiffer pairings. Consequently, we obtain the functor $\{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq 2}\} \longrightarrow \{\mathrm{X_2Hopf^{cc}}\}$. On the other hand, we already have the functor $\{\mathrm{SimpGrp}_{\leq 2}\} \longrightarrow \{\mathrm{X_2Grp}\}$ from the category of simplicial groups with Moore complex of length two, to the category of 2-crossed modules of groups [Mutlu and Porter, 1998]; and similarly we have $\{\mathrm{SimpLie}_{\leq 2}\} \longrightarrow \{\mathrm{X_2Lie}\}$ for the category of Lie algebras [Ellis, 1993b]. Finally, these two functors meet in the same diagram that proves the coherence of our 2-crossed module definition as being:

$$\begin{tikzcd}
\{\mathrm{SimpGrp}_{\leq 2}\} \arrow[d] & \{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq 2}\} \arrow[l] \arrow[r] \arrow[d] & \{\mathrm{SimpLie}_{\leq 2}\} \arrow[d] \\
\{\mathrm{X_2Grp}\} & \{\mathrm{X_2Hopf^{cc}}\} \arrow[l] \arrow[r] & \{\mathrm{X_2Lie}\}
\end{tikzcd}$$

in which the horizontal arrows are extended from Gl and $Prim$, respectively. Recall that, the outer vertical arrows in this diagram are known to be equivalences of categories (essentially by construction). We add to this not only the existence of a middle vertical arrows that makes everything commute, but also the result that the middle arrow is an equivalence as well.

This study is the first step towards enhancing our understanding of Hopf algebras in terms of category theory and algebraic topology for higher dimensions, where the crossed modules can be considered as one dimensional categorical objects. As it stands, there already exist many higher dimensional categorical objects defined in the categories of groups and of Lie algebras. We are confident that the results of the present paper will serve as a base from which to unify these other higher dimensional structures and their properties in the category of (cocommutative) Hopf algebras.

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2. Quick Review of Hopf Algebras

All Hopf algebras will be defined over an arbitrary field κ. Towards the end of the paper, we will mainly work in the cocommutative setting.

2.1. Hopf algebraic conventions. Roughly speaking, a "Hopf algebra" H is a bialgebra with an antipode [Majid, 1995]. In full, it is a sextuple $H = (H, \mu, \eta, \delta, \epsilon, S)$ where H is a κ-vector space together with the following data:

- (H, μ, η) is a unital associative algebra. Thus
  - $\mu\colon H \otimes H \longrightarrow H$ is an associative product. In short the product in H induces a map $\mu\colon H \otimes H \longrightarrow H$, where $x \otimes y \mapsto xy$.
  - $\eta\colon \kappa \longrightarrow H$ is an algebra map endowing H with a unit. In short $\eta\colon \lambda \in \kappa \mapsto \lambda 1_H \in H$ (Here 1_H is the identity element of H).
- (H, δ, ϵ) is a counital coassociative coalgebra. Thus
  - $\delta\colon H \longrightarrow H \otimes H$ is a coassociative coproduct. We use Sweedler's sigma notation as in [Kassel, 1995] to denote the coproduct. Explicitly, if $x \in H$, then the element $\delta(x) \in H \otimes H$ will be written in the following form[2]:

$$\delta(x) = \sum_{(x)} x' \otimes x''.$$

  - $\epsilon\colon H \longrightarrow \kappa$ is the counit. So, for all $x \in H$, we have

$$x = \sum_{(x)} \epsilon(x')x'' = \sum_{(x)} x'\epsilon(x'').$$

- Additionally, the following hold:
  - η and μ are coalgebra morphisms,
  - ϵ and δ are algebra morphisms.

  In fact, these two statements are equivalent; and we call the quintuple $H = (H, \mu, \eta, \delta, \epsilon)$ a bialgebra.
- There exists an antipode, namely an (inverse-like) anti-homomorphism $S\colon H \longrightarrow H$ at the level of algebra and coalgebra, satisfying

$$\sum_{(x)} S(x')x'' = \sum_{(x)} x'S(x'') = \epsilon(x)1_H.$$

[2] In fact, the comultiplication of a coalgebra is denoted by $\Delta(x)$ in Kassel [1995]. However, to avoid the potential confusion with the simplicial Δ, we rather use the notation $\delta(x)$ throughout the text.

Moreover:

- A Hopf algebra H is said to be "cocommutative" if, for all $x \in H$, we have

$$\sum_{(x)} x' \otimes x'' = \sum_{(x)} x'' \otimes x'.$$

- A Hopf algebra morphism (map) is exactly a bialgebra morphism, since bialgebra morphisms automatically preserve antipodes.

- Let H be *any* Hopf algebra. An element $x \in H$ is said to be:
  - primitive, if $\delta(x) = x \otimes 1_H + 1_H \otimes x$,
  - group-like, if $\delta(x) = x \otimes x$ and $\epsilon(x) = 1_H$.

  It can be easily proven that $\epsilon(x) = 0$, if x primitive. Then we have the set of primitive elements $Prim(H)$ that gives rise to a Lie algebra with the usual commutator bracket $[x, y] = xy - yx$. Similarly, the set of group-like elements $Gl(H)$ gives rise to a group with the product of H, where the inverse of any group-like element is $x^{-1} = S(x)$.

- Conversely:
  - Let L be a Lie algebra. The universal enveloping algebra $U(L)$ turns into a Hopf algebra where $\delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$ and $S(x) = -x$, $\forall x \in L$.
  - Let G be a group. We have the group algebra $\kappa[G]$ given by the free vector space on G together with the multiplication induced by the group operation. In addition, $\kappa[G]$ forms a Hopf algebra structure together with $\delta(g) = g \otimes g$, $\epsilon(g) = 1$ and $S(g) = g^{-1}$, on the base elements $g \in G$.

  Moreover, (considered as Hopf algebras) both the group Hopf algebra and the universal enveloping algebra come equipped with the cocommutativity property.

- Thus, we have the functors

$$\{\mathrm{Grp}\} \underset{\mathcal{G}^\sharp}{\overset{Gl}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \{\mathrm{Hopf}^{\mathrm{cc}}\} \underset{Prim}{\overset{\mathcal{F}^\sharp}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \{\mathrm{Lie}\} \tag{2}$$

  between the categories of groups, cocommutative Hopf algebras, and Lie algebras, respectively. For an explicit details on the constructions, see [Gran et al., 2016][§2.2].

- Let H be a (cocommutative) Hopf algebra. A sub-Hopf algebra $A \subset H$ is a subvector space A, such that $\mu(A \otimes A) \subset A$, $\delta(A) \subset A \otimes A$ and $\eta(\kappa) \subset A$. Clearly a sub-Hopf algebra inherits a (cocommutative) Hopf algebra structure from H.

- A (cocommutative) sub-Hopf algebra A of H is called normal [Vespa and Wambst, 2018], if $x \triangleright_{ad} a \in A$ for all $x \in H$ and $a \in A$. Here we put
$$x \triangleright_{ad} a = \sum_{(x)} x' a S(x'') \,,$$
which is called the "adjoint action".

2.2. Hopf algebra modules and smash products. Let H be a Hopf algebra and I be a vector space. I is said to be an H-module with an action $\triangleright_\rho \colon H \otimes I \longrightarrow I$, explicitly $x \otimes v \mapsto x \triangleright_\rho v$, satisfying:

- $1_H \triangleright_\rho v = v$, for all $v \in I$,
- $(xy) \triangleright_\rho v = x \triangleright_\rho (y \triangleright_\rho v)$, for all $x, y \in H$ and $v \in I$.

In addition to this, let I be a bialgebra. Then I is said to be an H-module algebra if:

- $x \triangleright_\rho 1_I = \epsilon(x) 1_I$, for all $x \in H$,
- $x \triangleright_\rho (uv) = \sum\limits_{(x)} (x' \triangleright_\rho u)(x'' \triangleright_\rho v)$, for all $x \in H$ and $u, v \in I$,

and similarly, an H-module coalgebra if:

- $\delta(x \triangleright_\rho v) = \sum\limits_{(x)(v)} (x' \triangleright_\rho v') \otimes (x'' \triangleright_\rho v'')$, for all $x \in H$, $v \in I$,
- $\epsilon(x \triangleright_\rho v) = \epsilon(x)\epsilon(v)$, for all $x \in H$, $v \in I$.

The following is well known with a proof from [Majid, 1995].

2.3. Definition. *Let I, H be cocommutative Hopf algebras, where I is an H-module algebra and coalgebra (one can call it H-module bialgebra) under the action $\triangleright_\rho \colon H \otimes I \longrightarrow I$. Then we have a cocommutative Hopf algebra $I \otimes_\rho H$ called the "smash product" with the underlying vector space $I \otimes H$ such that:*

- $(u \otimes x)(v \otimes y) = \sum\limits_{(x)} \Big(u \,(x' \triangleright_\rho v) \Big) \otimes x'' y,$
- $\delta(u \otimes x) = \sum\limits_{(u)(x)} (u' \otimes x') \otimes (u'' \otimes x''),$
- $S(u \otimes x) = \Big(1_I \otimes S(x)\Big)\Big(S(u) \otimes 1_H\Big),$

with the identity $1_I \otimes 1_H$ and the co-identity $\epsilon(u \otimes x) = \epsilon(u)\epsilon(x)$.

2.4. Example. If H is a *cocommutative* Hopf algebra, then H itself has a natural H-module algebra and coalgebra structure given by the adjoint action
$$(x \otimes y) \in H \otimes H \mapsto x \triangleright_{ad} y = \sum_{(x)} x' y S(x'') \in H.$$

2.5. Proposition. *If H is cocommutative, the antipode $S\colon H \longrightarrow H$ becomes an idempotent. We therefore obtain $S(x \triangleright_{ad} y) = x \triangleright_{ad} S(y)$. Furthermore, this property is still true for any action – which is a direct consequence of the action conditions.*

2.6. Remark. Example 2.4 is not true in non-cocommutative case, since it does not form an H-module coalgebra:

$$\begin{aligned}\delta(x \triangleright_{ad} y) = \sum_{(x \triangleright_{ad} y)} (x \triangleright_{ad} y)' \otimes (x \triangleright_{ad} y)'' &= \sum_{(x)(y)} x'y'S(x'''') \otimes x''y''S(x''') \\ &\neq \sum_{(x)(y)} (x' \triangleright_{ad} y') \otimes (x'' \triangleright_{ad} y'').\end{aligned}$$

2.7. Example. For *any* Hopf algebra H, the regular action

$$(x \otimes y) \in H \otimes H \mapsto x \triangleright_{\rho} y = xy \in H$$

turns H into an H-module coalgebra but not an H-module algebra. In other words, this example means that the multiplication is a coalgebra morphism but not an algebra morphism.

We also have a trivial action which turns H into an H-module algebra and coalgebra which is given by $x \triangleright_{\rho} y = \epsilon(x)y$.

2.8. Example. Recall the functors given in (2). Let L be a Lie algebra and $i\colon L \longrightarrow U(L)$ be the inclusion. If there exists an action $\triangleright$ of a group G on L by Lie algebra maps [Agore and Militaru, 2018], then $\mathcal{F}^{\sharp}(L)$ becomes a $\mathcal{G}^{\sharp}(G)$-module bialgebra where the action is defined on the base elements by $g \triangleright_{\rho} v$, for each $v \in L$ and $g \in G$. Consequently, we can form $\mathcal{F}^{\sharp}(L) \otimes_{\rho} \mathcal{G}^{\sharp}(G)$. If the base field is algebraically closed with zero characteristic, then we have an isomorphism $H \cong \mathcal{F}^{\sharp}(Prim(H)) \otimes_{\rho} \mathcal{G}^{\sharp}(Gl(H))$ given in [Cartier, 2007].

2.9. Hopf kernels. Some categorical properties of the cocommutative Hopf algebras we need in this paper are given below. For more details we refer to [Andruskiewitsch and Devoto, 1995; Agore, 2011; Vespa and Wambst, 2018; Porst, 2011].

- The category $\{\mathrm{Hopf}^{\mathrm{cc}}\}$ of cocommutative Hopf algebras over an arbitrary field is semi-abelian. Moreover, it is a full subcategory being complete and cocomplete. For more details, see [Gran et al., 2019].
- The zero object is κ, considered as a cocommutative Hopf algebra with the obvious structure maps. Given Hopf algebras A and B, the zero map $z_{A,B}\colon A \longrightarrow B$ is $\eta_B \epsilon_A$.
- The categorical product of cocommutative Hopf algebras A and B is $A \otimes B$ with underlying vector space $A \otimes B = A \otimes_{\kappa} B$. We have two projections $a \otimes b \in A \otimes B \mapsto a\epsilon(b) \in A$ and $a \otimes b \in A \otimes B \mapsto \epsilon(a)b \in B$. The extension to a finite number of components is the obvious one.

- The equalizer (in the category of *all* Hopf algebras) of $f, g\colon A \longrightarrow B$ is given by

$$\{x \in A\colon \sum_{(x)} x' \otimes f(x'') \otimes x''' = \sum_{(x)} x' \otimes g(x'') \otimes x'''\}\,,$$

  which turns out to be, in the *cocommutative* case:

$$\{x \in A\colon \sum_{(x)} x' \otimes f(x'') = \sum_{(x)} x' \otimes g(x'')\} = \{x \in A\colon \sum_{(x)} f(x') \otimes x'' = \sum_{(x)} g(x') \otimes x''\}\,.$$

- From this description of the equalizer, we can obtain the kernel (in the category of *all* Hopf algebras) as the equalizer of f and $z_{A,B}$ as being

$$\{x \in A\colon \sum_{(x)} x' \otimes f(x'') \otimes x''' = \sum_{(x)} x' \otimes 1_B \otimes x''\}\,.$$

  However, in the *cocommutative* case, this can be simplified to:[3]

$$\mathrm{HKer}(f) = \{x \in A\colon \sum_{(x)} x' \otimes f(x'') = x \otimes 1_B\} = \{x \in A\colon \sum_{(x)} f(x') \otimes x'' = 1_B \otimes x\}\,. \tag{3}$$

- Consider the Hopf kernel $\mathrm{HKer}(f\colon A \longrightarrow B)$. If we apply $\mu(\epsilon \otimes \mathrm{id})$ in (3), we obtain

$$x \in \mathrm{HKer}(f) \implies f(x) = \epsilon(x)1_B\,,$$

  that means Hopf kernels are specific cases of linear kernels.

- The functors Gl and $Prim$ preserve kernels. In other words, we have

$$Gl(\mathrm{HKer}(f)) = \ker(Gl(f)) \quad \text{and} \quad Prim(\mathrm{HKer}(f)) = \ker(Prim(f))\,, \tag{4}$$

  for the corresponding categories.

- For any Hopf algebra map $f\colon A \longrightarrow B$, Hopf kernel $\mathrm{HKer}(f)$ defines a normal sub-Hopf algebra.

- $\mathrm{HKer}(\mathrm{id}) = \kappa$. More generally, if a Hopf algebra map f is injective (therefore monic), then we have $\mathrm{HKer}(f) = \kappa$.

[3] In general case, these sets only define subalgebras of H which are denoted by $\mathrm{RKer}(f)$ and $\mathrm{LKer}(f)$, respectively [Andruskiewitsch and Devoto, 1995]. Regarding the cocommutative setting: since $\mathrm{HKer}(f)$ is the kernel in the category of cocommutative Hopf algebras, one can also denote it by $\ker(f)$. To avoid the confusion with the usual linear kernel, we prefer using the notation $\mathrm{HKer}(f)$.

2.10. Crossed modules of cocommutative Hopf algebras. The following definition is well established as a crossed module of cocommutative Hopf algebras that independently introduced in [Majid, 2012] and [Fernández Vilaboa et al., 2007] which are coherent in the cocommutative setting.

2.11. Definition. *A "crossed module" of cocommutative Hopf algebras is given by a Hopf algebra map $\partial\colon I \longrightarrow H$ together with an action of H of I denoted by $\triangleright_\rho\colon H \otimes I \longrightarrow I$ which turns I into an H-module bialgebra, satisfying:*

- $\partial(x \triangleright_\rho v) = x \triangleright_{ad} \partial(v)$,
- $\partial(u) \triangleright_\rho v = u \triangleright_{ad} v$,

for all $x \in H$ and $u, v \in I$.[4] *Without the second condition, we call it a precrossed module.*

2.12. Remark. Group-like and primitive elements preserve crossed module structures. Consequently, we can write down the crossed module conditions of groups and Lie algebras in the sense of the functors Gl and $Prim$ as follows:

- A crossed module of groups is given by a group homomorphism $\partial\colon E \longrightarrow G$ together with an action $\triangleright$ of G on E by automorphisms, such that:

  $$\partial(g \triangleright e) = g\,\partial(e)\,g^{-1}, \quad \partial(e) \triangleright f = e\,f\,e^{-1},$$

  for all $e, f \in E$ and $g \in G$.

- A crossed module of Lie algebras (i.e. differential crossed module) is given by a Lie algebra homomorphism $\partial\colon \mathfrak{e} \longrightarrow \mathfrak{g}$ together with an action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{e}$ by derivations, such that:

  $$\partial(g \triangleright e) = [g, \partial(e)], \quad \partial(e) \triangleright f = [e, f],$$

  for all $e, f \in \mathfrak{e}$ and $g \in \mathfrak{g}$.

Therefore, we have the functors:

$$\{\mathrm{XLie}\} \xleftarrow{Prim} \{\mathrm{XHopf}^{\mathrm{cc}}\} \xrightarrow{Gl} \{\mathrm{XGrp}\} \tag{5}$$

between the categories of crossed modules of Lie algebras, cocommutative Hopf algebras, and groups, respectively. See [Faria Martins, 2016] for more details.

3. The Moore Complex

From now on, all Hopf algebras will be considered cocommutative and we just use the term "Hopf algebra" for the sake of simplicity.

[4] In general, there is an extra crossed module condition called "compatibility" that automatically holds in the cocommutative setting.

3.1. Definition. *A chain complex $(A_\bullet, \partial_\bullet)$ of Hopf algebras is given by a sequence of Hopf algebra maps*

$$\longrightarrow A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \longrightarrow \dots \longrightarrow A_1 \xrightarrow{\partial_1} A_0,$$

such that: $\forall n \geq 1$, we have $\partial_n \partial_{n+1} = \eta_{A_{n-1}} \epsilon_{A_{n+1}}$, and the latter is the zero morphism $x \in A_{n+1} \mapsto \epsilon(x) 1_{A_{n-1}} \in A_{n-1}$. Given a chain complex of Hopf algebras, then $\partial_n(A_n)$ is, a priori, only a sub-Hopf algebra of A_{n-1}. We say that a chain complex of Hopf algebras is normal if $\partial_n(A_n)$ is a normal sub-Hopf algebra of A_{n-1}, for each positive integer n.

3.2. Simplicial Hopf algebras.

3.3. Remark. Recall that the category Δ is the category whose objects are the non-negative integers $n \in \mathbb{N}$ and whose morphisms $n \longrightarrow m$ are the order preserving maps from $\{0, \dots, n\}$ to $\{0, \dots, m\}$. This category is generated by certain maps $D_i^n \colon \{0, \dots, n-1\} \longrightarrow \{0, \dots, n\}$ and $S_i^{n+1} \colon \{0, \dots, n+1\} \longrightarrow \{0, \dots, n\}$, where $n > 0$ and $0 \leq i < n$. In short $D_i^n \colon \{0, \dots, n-1\} \longrightarrow \{0, \dots, n\}$ is injective and its image does not include i, $S_i^{n+1} \colon \{0, \dots, n+1\} \longrightarrow \{0, \dots, n\}$ is surjective, and i is the only element with a double pre-image. These morphisms satisfy the well known (co)simplicial relations, which we will write, dually, below in 3.4. A simplicial object [May, 1967] in a category $\mathcal{C}$ is a covariant functor $F \colon \Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$. And normally we put $d_i^n = F(D_i^n)$ and $s_i^{n+1} = F(S_i^{n+1})$.

We have a category $\mathcal{C}^{\Delta^{op}}$, whose objects are the functors $\Delta^{op} \longrightarrow \mathcal{C}$ (i.e. the simplicial objects in $\mathcal{C}$) and whose morphisms are the natural transformations between functors. If $\mathcal{C}$ is complete and cocomplete (as will always be the case in this paper), then $\mathcal{C}^{\Delta^{op}}$ is complete and cocomplete and all limits and colimits can be computed pointwise.

3.4. Definition. *A simplicial Hopf algebra $\mathcal{H}$ is therefore a simplicial object in the category of Hopf algebras. In other words, it is given by a collection of Hopf algebras H_n $(n \in \mathbb{N})$ together with Hopf algebra maps called faces and degeneracies, respectively*

$$\begin{array}{llll} d_i^n \colon & H_n \longrightarrow H_{n-1} & , & 0 \leq i \leq n \\ s_j^{n+1} \colon & H_n \longrightarrow H_{n+1} & , & 0 \leq j \leq n \end{array}$$

which are to satisfy the following simplicial identities:[5]

$$\begin{array}{llll} \textit{(i)} & d_i d_j = d_{j-1} d_i & \textit{if} & i < j \\ \textit{(ii)} & s_i s_j = s_{j+1} s_i & \textit{if} & i \leq j \\ \textit{(iii)} & d_i s_j = s_{j-1} d_i & \textit{if} & i < j \\ & d_j s_j = d_{j+1} s_j = \mathrm{id} & & \\ & d_i s_j = s_j d_{i-1} & \textit{if} & i > j+1 \end{array}$$

A simplicial Hopf algebra can be pictured as:

$$\begin{tikzcd} \mathcal{H} \; = \; \cdots\cdots\cdots \; H_3 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & H_2 \arrow[l, bend left] \arrow[l, bend left=40] \arrow[l, bend left=60] \arrow[r, shift left=2, "d_2"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_0"] & H_1 \arrow[l, bend left, "s_0"] \arrow[l, bend left=50, "s_1"] \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"] & H_0 \arrow[l, bend left, "s_0"] \end{tikzcd} \tag{6}$$

and we denote the category of simplicial Hopf algebras by $\{\mathrm{SimpHopf}^{\mathrm{cc}}\}$.

[5]To avoid overloaded notation, we will not use superscripts for faces and degeneracies.

3.5. Remark. An n-truncated simplicial Hopf algebra is defined like a simplicial Hopf algebra except that we forget what happens in degree greater than n. More precisely it is a contravariant functor from Δ^n (the full category of Δ whose objects are the naturals $\leq n$) into the category of Hopf algebras. We denote the category of n-truncated simplicial Hopf algebras by $\{\mathrm{SimpHopf}^{\mathrm{cc}}_{|\mathrm{n}}\}$.

The forgetful functor $\{\mathrm{SimpHopf}^{\mathrm{cc}}\} \longrightarrow \{\mathrm{SimpHopf}^{\mathrm{cc}}_{|\mathrm{n}}\}$ has a left adjoint given by left Kan extension, and a right adjoint given by right Kan extension. These are called the skeleton and coskeleton functors, respectively, and are contructed using colimits and limits (as first defined in [Duskin, 1975]). Let us quickly revise the construction of the coskeleton functor for Hopf algebraic case: For a given n-truncated simplicial Hopf algebra $\mathcal{H}^{|n}$, consider the Hopf algebra H_n^{n+1}. Then obtain for each pair $0 \leq i < j \leq n+1$ the equalizer $M_{i,j}$ of $d_i\, p_j$ and $d_{j-1}\, p_i$ where p_i, p_j denote the projections. And then consider the intersection H_{n+1} of all of the $M_{i,j}$. Naturally, the face maps $H_{n+1} \longrightarrow H_n$ are given by the projections p_i where $(0 \leq i \leq n+1)$ and degeneracies $H_n \longrightarrow H_{n+1}$ are uniquely defined by a generator $\alpha_{i,j}$ such that $p_i s_j = \alpha_{i,j}$, for all $0 \leq j \leq n$. This yields an $(n+1)$-truncated simplicial Hopf algebra that is so-called the simplicial kernel. And proceding inductively, we thus define the coskeleton functor.

3.6. Lemma. *Given a simplicial Hopf algebra* (6)*, we have the chain complex of Hopf algebras* $(NH_\bullet, \partial_\bullet)$ *given by:*

- $NH_0 = H_0$,
- $NH_n = \bigcap\limits_{i=0}^{n-1} \mathrm{HKer}(d_i)$, *for* $n \geq 1$,
- $\partial_n \colon NH_n \longrightarrow NH_{n-1}$ *as being the restriction of* d_n *to* NH_n.

Proof. The proof is evident. We refer to the respective results given in [Everaert and Van der Linden, 2004] for semi-abelian categories. ∎

3.7. Definition. [Moore Complex] *For a given simplicial Hopf algebra* $\mathcal{H}$*, the chain complex* $(NH_\bullet, \partial_\bullet)$ *will be called the "Moore complex" of* $\mathcal{H}$.

The Moore complex of groups is also known as a normalized chain complex of simplicial groups in the literature [Mutlu and Porter, 1998]. The following lemma proves that the Moore complex of a simplicial Hopf algebra has the same property.

3.8. Lemma. *The Moore complex* $(NH_\bullet, \partial_\bullet)$ *of a simplicial Hopf algebra* $\mathcal{H}$ *is a normal chain complex.*

Proof. As in the proof of Lemma (3.6), we refer to [Everaert and Van der Linden, 2004]. ∎

3.9. Definition. *We say that a simplicial Hopf algebra has Moore complex of length n, if NH_i is zero object, for all $i > n$.*

We denote the corresponding category by $\{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq \mathrm{n}}\}$.

3.10. Proposition. *Following the property (4), one can say that the functors Gl and $Prim$ preserve the Moore complex definition, as well as the length of it. We therefore have the functors*

$$\{\mathrm{SimpLie}_{\leq \mathrm{n}}\} \xleftarrow{Prim} \{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq \mathrm{n}}\} \xrightarrow{Gl} \{\mathrm{SimpGrp}_{\leq \mathrm{n}}\}$$

with referring to [Conduché, 1984; Ellis, 1993b] for the corresponding categories.

3.11. Remark. Let $\partial\colon I \longrightarrow H$ and $i\colon H \longrightarrow I$ be Hopf algebra maps such that $\partial\, i = \mathrm{id}_H$. Such a couple of maps is usually called a "point" and we denote it by $(\partial\colon I \longrightarrow H, i)$.

From the categorical point of view, it is equal to say that ∂ is a split epimorphism with a chosen splitting i. Together with the (Hopf) kernel of ∂, it is a split extension:

$$\kappa \longrightarrow \mathrm{HKer}(\partial) \hookrightarrow I \underset{i}{\overset{\partial}{\rightleftarrows}} H \longrightarrow \kappa$$

The following theorem is due to [Majid, 1994; Radford, 1985].

3.12. Theorem. [Majid/Radford] *Let $(\partial\colon I \longrightarrow H, i)$ be a point. Then I is an H-module bialgebra where the action $\triangleright_\rho\colon H \otimes I \longrightarrow I$ is the adjoint action via i.*

Moreover, we have an isomorphism of Hopf algebras $I \cong \mathrm{HKer}(\partial) \otimes_\rho H$, where the maps below are mutually inverse:

$$\Psi\colon v \in I \mapsto \sum_{(v)} f(v') \otimes \partial(v'') \in \mathrm{HKer}(\partial) \otimes_\rho H,$$

with $f(x) = \sum_{(x)} x'\, i\partial(S(x''))$, for all $x \in I$; and

$$\Phi\colon a \otimes x \in \mathrm{HKer}(\partial) \otimes_\rho H \mapsto a\, i(x) \in I.$$

3.13. Remark. By using simplicial identities, we can obtain points $(d_i\colon H_{n+1} \longrightarrow H_n, s_i)$ for each n, in a simplicial Hopf algebra (6). Therefore, one can adapt Theorem (3.12) to a simplicial Hopf algebra as follows.

3.14. Proposition. *In a simplicial Hopf algebra $\mathcal{H}$, there exists an action*

$$\begin{array}{rcl} \triangleright_{\rho_i}\colon s_i(H_{n-1}) \otimes \mathrm{HKer}\Big(d_i\Big) & \longrightarrow & \mathrm{HKer}\Big(d_i\Big) \\ (a, x) & \longmapsto & a \triangleright_{\rho_i} x = a \triangleright_{ad} x\,, \end{array}$$

for all $0 \leq i \leq n-1$ and $0 < n$. Consequently, we have the smash product Hopf algebra

$$\mathrm{HKer}(d_i) \otimes_{\rho_i} s_i(H_{n-1})\,,$$

from Definition 2.3.

3.15. THEOREM. *For any simplicial Hopf algebra $\mathcal{H}$, we have an isomorphism*

$$H_n \cong \mathrm{HKer}(d_i) \otimes_{\rho_i} s_i(H_{n-1}),$$

of Hopf algebras, for all $n \in \mathbb{N}$, $i \leq n-1$.

PROOF. The map

$$\begin{array}{rccl} \phi\colon & H_n & \longrightarrow & \mathrm{HKer}(d_i) \otimes_{\rho_i} s_i(H_{n-1}) \\ & x & \longmapsto & \sum\limits_{(x)} f_i(x') \otimes s_i d_i(x'') \end{array}$$

gives the isomorphism, where the "Hopf kernel generator map" $f_i\colon H_n \longrightarrow \mathrm{HKer}(d_i)$ is a linear map defined by $f_i\colon x \mapsto \sum_{(x)} x' s_i d_i(S(x''))$. ∎

3.16. DÉCALAGE CONSTRUCTION. *Consider the simplicial Hopf algebra:*

$$\begin{tikzcd} \mathcal{H} = \cdots\cdots & H_3 \arrow[r, shift left=4.5] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[r, shift right=4.5] & H_2 \arrow[l, bend left=40] \arrow[l, bend left=60] \arrow[l, bend left=80] \arrow[r, shift left=3, "d_2"] \arrow[r, "d_1"] \arrow[r, shift right=3, "d_0"'] & H_1 \arrow[l, bend left=40, "s_0"] \arrow[l, bend left=60, "s_1"] \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & H_0 \arrow[l, bend left=40, "s_0"] \end{tikzcd} \tag{7}$$

By using Hopf kernels in (7), we obtain a new simplicial Hopf algebra where the first three components are:

$$\begin{tikzcd} \mathrm{HKer}(d_0) \subset H_3 \arrow[r, shift left=6, "d_3"] \arrow[r, shift left=3, "d_2"] \arrow[r, "d_1"] & \mathrm{HKer}(d_0) \subset H_2 \arrow[l, shift left=3, "s_1"] \arrow[l, shift left=6, "s_2"] \arrow[r, shift left=3, "d_2"] \arrow[r, "d_1"] & \mathrm{HKer}(d_0) \subset H_1 \arrow[l, shift left=3, "s_1"] \end{tikzcd} \tag{8}$$

Note that, we used the restrictions of the face and degeneracy maps. This idea can be iterated. For instance, when we take Hopf kernels again in (8), we get a new simplicial Hopf algebra where the first three components are:

$$\begin{tikzcd} \mathrm{HKer}(d_0) \cap \mathrm{HKer}(d_1) \subset H_4 \arrow[r, shift left=6, "d_4"] \arrow[r, shift left=3, "d_3"] \arrow[r, "d_2"] & \mathrm{HKer}(d_0) \cap \mathrm{HKer}(d_1) \subset H_3 \arrow[l, shift left=3, "s_2"] \arrow[l, shift left=6, "s_3"] \arrow[r, shift left=3, "d_3"] \arrow[r, "d_2"] & \mathrm{HKer}(d_0) \cap \mathrm{HKer}(d_1) \subset H_2 \arrow[l, shift left=3, "s_2"] \end{tikzcd} \tag{9}$$

3.17. DECOMPOSING. *If we apply Theorem 3.15 to H_1 in (7), we get*

$$\begin{aligned} H_1 &\cong \mathrm{HKer}(d_0) \otimes_{\rho_0} s_0(H_0) \\ &= NH_1 \otimes_{\rho_0} s_0(NH_0). \end{aligned}$$

Similarly, considering H_2, we first get

$$\begin{aligned} H_2 &\cong \mathrm{HKer}(d_0) \otimes_{\rho_0} s_0(H_1) \\ &\cong \mathrm{HKer}(d_0) \otimes_{\rho_0} s_0\Big(NH_1 \otimes_{\rho_0} s_0(NH_0)\Big), \end{aligned}$$

and by applying Theorem (3.15) in (8), we further have

$$
\begin{aligned}
H_2 &\cong \Big(\mathrm{HKer}(d_1)\mid_{\mathrm{HKer}(d_0)} \otimes_{\rho_1} s_1(\mathrm{HKer}(d_0))\Big) \otimes_{\rho_0} s_0\Big(NH_1 \otimes_{\rho_0} s_0(NH_0)\Big)\\
&= \Big((\mathrm{HKer}(d_1) \cap \mathrm{HKer}(d_0)) \otimes_{\rho_1} s_1(\mathrm{HKer}(d_0))\Big) \otimes_{\rho_0} s_0\Big(NH_1 \otimes_{\rho_0} s_0(NH_0)\Big)\\
&= \Big(NH_2 \otimes_{\rho_1} s_1(NH_1)\Big) \otimes_{\rho_0} \Big(s_0(NH_1) \otimes_{\rho_0} s_1 s_0(NH_0)\Big).
\end{aligned}
$$

One level further, by using (9), we obtain

$$
\begin{aligned}
H_3 \cong \Big(&(NH_3 \otimes_{\rho_2} s_2(NH_2)) \otimes_{\rho_1} (s_1(NH_2 \otimes_{\rho_1} s_2 s_1(NH_1)))\Big)\\
&\otimes_{\rho_0} \Big(s_0(NH_2) \otimes_{\rho_1} s_2 s_0(NH_1) \otimes_{\rho_0} (s_1 s_0(NH_1) \otimes_{\rho_0} s_2 s_1 s_0(NH_0))\Big).
\end{aligned}
$$

By iteration, we get the general formula:

3.18. Theorem. [Simplicial Decomposition] *Let $\mathcal{H}$ be a simplicial Hopf algebra. We have the decomposition of H_n, for any $n \geq 0$ as follows:*[6]

$$
\begin{aligned}
H_n \cong \Big(\cdots\Big(NH_n \otimes s_{n-1} NH_{n-1}\Big) \otimes \cdots s_{n-2} \cdots s_1 NH_1\Big) \otimes&\\
\Big(\cdots\Big(s_0 NH_{n-1} \otimes s_1 s_0 NH_{n-2}\Big) \otimes \cdots \otimes s_{n-1} s_{n-2} \cdots s_0 NH_0\Big)&.
\end{aligned}
$$

4. Iterated Peiffer Pairings

The following notation and terminology is derived from [Carrasco, 1995; Carrasco and Cegarra, 1991] where it is used for both simplicial groups and simplicial algebras.

4.1. The poset of surjective maps. For the ordered set $[n] = \{0 < 1 < \cdots < n\}$, let $\alpha_i = \alpha_i^n \colon [n+1] \longrightarrow [n]$ be the increasing surjective map given by

$$
\alpha_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}
$$

Let $S(n, n-l)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-l]$. This can be generated from the various α_i^n by composition. The composition of these generating maps satisfies the property $\alpha_j \alpha_i = \alpha_{i-1} \alpha_j$ with the condition $j < i$. This implies that each element $\alpha \in S(n, n-l)$ has a unique expression as $\alpha = \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_l}$ with $0 \leq i_1 < i_2 < \cdots < i_l \leq n$, where the indices i_k are the elements of $[n]$ at which $\alpha(i) = \alpha(i+1)$. Clearly, $S(n, n-l)$ is canonically isomorphic to the set $\{(i_l, \cdots, i_1) \colon 0 \leq i_i < i_2 < \cdots < i_l \leq n-1\}$. For instance, the single element of $S(n, n)$ defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly the only elements of $S(n, 0)$ is $(n-1, n-2, \cdots, 0)$. For all $n \geq 0$, let:

$$
S(n) = \bigcup_{0 \leq l \leq n} S(n, n-l).
$$

[6] For the sake of simplicity, each smash product $\otimes_{\rho_i}$ is just written by $\otimes$ in the formula.

Any element of $S(n)$ is of the form $(i_l, \cdots, i_1)$ where $0 \leq i_1 < i_2 < \cdots < i_l < n$. If $\alpha = (i_l, \cdots, i_1)$, then we say α has length l and will write $\#\alpha = l$. Consider the lexicographic order on $S(n)$. That is to say, $\alpha = (i_l, \cdots, i_1) < \beta = (j_m, \cdots, j_1)$ in $S(n)$,

$$\text{if } i_1 = j_1, \cdots, i_k = j_k \text{ but } i_{k+1} > j_{k+1} \ (k > 0)$$

or

$$\text{if } i_1 = j_1, \cdots, i_l = j_l \text{ and } l < m \ ,$$

that makes $S(n)$ a totally ordered set. For instance, the orders of $S(2), S(3)$ and $S(4)$ are respectively:

$$\begin{aligned}
S(2) &= \{\emptyset_2 < (1) < (0) < (1,0)\} \\
S(3) &= \{\emptyset_3 < (2) < (1) < (2,1) < (0) < (2,0) < (1,0) < (2,1,0)\} \\
S(4) &= \{\emptyset_4 < (3) < (2) < (3,2) < (1) < (3,1) < (2,1) < (3,2,1) < (0) < (3,0) \\
&\qquad\qquad < (2,0) < (3,2,0) < (1,0) < (3,1,0) < (2,1,0) < (3,2,1,0)\}
\end{aligned}$$

If $\alpha, \beta \in S(n)$, we define $\alpha \cap \beta$ to be the set of indices which belong to both α and β.

4.2. Iterated Peiffer pairings. Let $\mathcal{H}$ be a simplicial Hopf algebra and $(NH_\bullet, \partial_\bullet)$ be its Moore complex. We define the set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$, with respect to lexicographic ordering in $S(n)$ where $\alpha = (i_l, \cdots, i_1)$, $\beta = (j_m, \cdots, j_1) \in S(n)$. The pairings

$$\{F_{\alpha,\beta} : NH_{n-\#\alpha} \otimes NH_{n-\#\beta} \longrightarrow NH_n \mid (\alpha, \beta) \in P(n), n \geq 0\}$$

are defined as composites in the diagram:

$$\begin{tikzcd}
NH_{n-\#\alpha} \otimes NH_{n-\#\beta} \arrow[r, "F_{\alpha,\beta}"] \arrow[d, "s_\alpha \otimes s_\beta"'] & NH_n \\
H_n \otimes H_n \arrow[r, "\triangleright_{ad}"] & H_n \arrow[u, "f_n"']
\end{tikzcd} \tag{10}$$

such that

$$s_\alpha = s_{i_l} \ldots s_{i_1} : NH_{n-\#\alpha} \to H_n, \quad s_\beta = s_{j_m} \ldots s_{j_1} : NH_{n-\#\beta} \to H_n,$$

and $f_n \colon H_n \to NH_n$ is defined by the composition $f_n = f_{n-1} \ldots f_1 f_0$, where f_i is the Hopf kernel generator map defined in the proof of Theorem 3.15.

4.3. Remark. In the prerequisite for the above idea, one can also consider the case $\alpha > \beta$ which creates same type of Peiffer elements, namely the same elements under the antipode. Moreover, lexicographic order guarantees the Peiffer elements to be non-trivial in the sense of simplicial identities.

4.4. Calculating Peiffer pairings. In this subsection, we obtain the Peiffer pairings that needed in the sequel.

4.4.1. $n = 2$ case. We have unique type of element for this case, by taking $\alpha = (0)$, $\beta = (1)$. Then $F_{(0)(1)}(x, y) \in NH_2$ is calculated for all $x, y \in NH_1$ as follows:

$$
\begin{aligned}
F_{(0)(1)}(x,y) &= f_1 f_0 \Big(s_0(x) \triangleright_{ad} s_1(y)\Big) \\
&= f_1 \left(\sum_{(s_0(x) \triangleright_{ad} s_1(x))} \Big(s_0(x) \triangleright_{ad} s_1(y)\Big)' \, s_0 d_0 \Big(S\Big(s_0(x) \triangleright_{ad} s_1(y)\Big)'' \Big) \right) \\
&= f_1 \left(\sum_{(x)(y)} \Big(s_0(x') \triangleright_{ad} s_1(y')\Big) \, S\Big(s_0 d_0 \Big(s_0(x'') \triangleright_{ad} s_1(y'') \Big) \Big) \right) \\
&= f_1 \left(\sum_{(x)(y)} \Big(s_0(x') \triangleright_{ad} s_1(y')\Big) \, S\Big(s_0(x'') \triangleright_{ad} s_0 s_0 d_0(y'') \Big) \right) \\
&= f_1 \left(\sum_{(x)} \Big(s_0(x') \triangleright_{ad} s_1(y)\Big) \, S\Big(s_0(x'') \triangleright_{ad} 1_{H_2} \Big) \right) \\
&= f_1 \Big(s_0(x) \triangleright_{ad} s_1(y) \Big) \\
&= \sum_{(x)(y)} \Big(s_0(x') \triangleright_{ad} s_1(y')\Big) \, s_1 d_1 \Big(S\Big(s_0(x'') \triangleright_{ad} s_1(y'') \Big) \Big) \\
&= \sum_{(x)(y)} \Big(s_0(x') \triangleright_{ad} s_1(y')\Big) \, S\Big(s_1(x'') \triangleright_{ad} s_1(y'') \Big) \qquad (11)
\end{aligned}
$$

Remark that since $y \in NH_1 = \mathrm{HKer}(d_0)$, we write $\sum_{(y)} y' \otimes d_0(y'') = y \otimes 1_{H_0}$ follows from (3), and this implies $\sum_{(y)} s_1(y') \otimes s_0 s_0 d_0(y'') = s_1(y) \otimes 1_{H_2}$.

4.4.2. $n = 3$ case. The possible six Peiffer elements belonging to NH_3 are:

$$F_{(1,0)(2)}\,,\ F_{(2,0)(1)}\,,\ F_{(0)(2,1)}\,,\ F_{(0)(1)}\,,\ F_{(0)(2)}\,,\ F_{(1)(2)}\,.$$

If we calculate these elements, we get:

a) for all $x \in NH_1$ and $y \in NH_2$,

- $F_{(1,0)(2)}(x,y) = \sum_{(x)(y)} \Big(s_1 s_0(x') \triangleright_{ad} s_2(y')\Big) \, S\Big(s_2 s_0(x'') \triangleright_{ad} s_2(y'')\Big)$

- $F_{(2,0)(1)}(x,y)$

$$
\begin{aligned}
&= \sum_{(x)(y)} \Big(s_2 s_0(x') \triangleright_{ad} s_1(y')\Big) \, S\Big(s_2 s_1(x'') \triangleright_{ad} s_1(y'')\Big) \\
&\qquad\qquad S\Big(\Big(s_2 s_0(x''') \triangleright_{ad} s_2(y''')\Big) \, S\Big(s_2 s_1(x'''') \triangleright_{ad} s_2(y'''')\Big) \Big)
\end{aligned}
$$

b) for all $x \in NH_2$ and $y \in NH_1$,

- $F_{(0)(2,1)}(x,y)$

$$= \sum_{(x)(y)} \Big(s_0(x') \rhd_{ad} s_2s_1(y')\Big)\, S\Big(s_1(x'') \rhd_{ad} s_2s_1(y'')\Big)$$
$$S\Big(s_2s_1(y''')\, S\Big(s_2(x''') \rhd_{ad} s_2s_1(y'''')\Big)\Big)$$

c) for all $x, y \in NH_2$,

- $F_{(0)(1)}(x,y)$

$$= \sum_{(x)(y)} \Big(s_0(x') \rhd_{ad} s_1(y')\Big)\, S\Big(s_1(x'') \rhd_{ad} s_1(y'')\Big)\, S\Big(s_2(y''')\, S\Big(s_2(x''') \rhd_{ad} s_2(y'''')\Big)\Big)$$

- $F_{(0)(2)}(x,y) = \sum_{(y)} \Big(s_0(x) \rhd_{ad} s_2(y')\Big)\, S\Big(s_2(y'')\Big)$

- $F_{(1)(2)}(x,y) = \sum_{(x)(y)} \Big(s_1(x') \rhd_{ad} s_2(y')\Big)\, S\Big(s_2(x'') \rhd_{ad} s_2(y'')\Big)$

5. More on Crossed Modules

In this section, we deal with the functorial relationship between the categories of crossed modules of Hopf algebras and simplicial Hopf algebras.

5.1. From simplicial Hopf algebras to crossed modules.

5.2. Theorem. *Let $\mathcal{H}$ be a simplicial Hopf algebra with a Moore complex of length one. We have the crossed module*

$$\partial_1 \colon NH_1 \longrightarrow H_0,$$

where the action $\rhd_\rho \colon H_0 \otimes NH_1 \longrightarrow NH_1$ is defined by

$$k \rhd_\rho x = s_0(k) \rhd_{ad} x,$$

for all $k \in H_0$, $x \in NH_1$.

This construction yields a functor $\mathrm{X}_1 \colon \{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq 1}\} \longrightarrow \{\mathrm{XHopf}^{\mathrm{cc}}\}$.

PROOF. The first crossed module condition ($\partial_1(k \rhd_\rho x) = k \rhd_{ad} \partial_1(x)$, $\forall k \in H_0$ and $\forall x \in NH_1$) follows immediately. Thus, let us prove the second one ($\partial_1(x) \rhd_\rho y = x \rhd_{ad} y$, $\forall x, y \in NH_1$). Since the length of the Moore complex of $\mathcal{H}$ is one, we straightforwardly have $F_{(0)(1)}(x,y) \in NH_2 = \kappa$. Therefore, the restriction of $d_2 \colon H_2 \longrightarrow H_1$ to NH_2 (i.e. ∂_2) becomes the zero morphism. Consequently we obtain

$$d_2\Big(F_{(0)(1)}(x,y)\Big) = \epsilon(x)\epsilon(y)1_{H_1} \,.$$

And, following from (11), we can write

$$\sum_{(x)(y)} (d_2 s_0(x') \rhd_{ad} y')\, S(x'' \rhd_{ad} y'') = \epsilon(x)\epsilon(y)1_{H_1} \,, \tag{12}$$

that yields

$$d_2 s_0(x) \rhd_{ad} y = x \rhd_{ad} y \,,$$

through Appendix A.1. Finally, we have

$$\begin{aligned} \partial_1(x) \rhd_\rho y &= s_0 d_1(x) \rhd_{ad} y \\ &= d_2 s_0(x) \rhd_{ad} y \\ &= x \rhd_{ad} y \,, \end{aligned}$$

for all $x, y \in NH_1$, and complete the proof. ■

5.3. FROM CROSSED MODULES TO SIMPLICIAL HOPF ALGEBRAS. The following lemma is an obvious analogue to the group and Lie algebra versions given in [Conduché, 1984; Ellis, 1993b].

5.4. LEMMA. *For a given n-truncated simplicial Hopf algebra $\mathcal{H}_{|n}$, the length of the Moore complex of $\mathrm{cosk_n}(\mathcal{H}_{|n})$ is $n+1$. Moreover, we have:*

- $N\Big(\mathrm{cosk_n}(\mathcal{H}_{|n})\Big)_{n+1} \cong \mathrm{HKer}\Big(\partial_n \colon N(\mathcal{H}_{|n})_n \longrightarrow N(\mathcal{H}_{|n})_{n-1}\Big)$,
- $\partial_{n+1} \colon N\Big(\mathrm{cosk_n}(\mathcal{H}_{|n})\Big)_{n+1} \longrightarrow N\Big(\mathrm{cosk_n}(\mathcal{H}_{|n})\Big)_n$ *is injective.*

5.5. THEOREM. *The category of crossed modules of Hopf algebras is equivalent to the category of simplicial Hopf algebras with Moore complex of length one.*

PROOF. Let $\partial \colon I \longrightarrow H$ be a crossed module. Put $\mathfrak{H}_0 = H$. From Definition 2.3, we can define the smash product Hopf algebra $\mathfrak{H}_1 = I \otimes_\rho H$ with two face $d_0, d_1 \colon \mathfrak{H}_1 \longrightarrow \mathfrak{H}_0$ and one degeneracy $s_0 \colon \mathfrak{H}_0 \longrightarrow \mathfrak{H}_1$ maps:

$$d_0\Big(u \otimes x\Big) = \epsilon(u)x \,, \quad d_1\Big(u \otimes x\Big) = \partial(u)x, \quad s_0(x) = \Big(1_I \otimes x\Big) \,. \tag{13}$$

Also, there exists an action of $I \otimes_\rho H$ on I given via (for all $u, v \in I$ and $x \in H$):

$$(u \otimes x) \rhd_\star v = (\partial(u)\, x) \rhd_\rho v \,,$$

that we define $\mathfrak{H}_2 = I \otimes_\star (I \otimes_\rho H)$. Then we obtain three face maps $d_0, d_1, d_2 \colon \mathfrak{H}_2 \longrightarrow \mathfrak{H}_1$ given by

$$d_0\Big(u \otimes v \otimes x\Big) = \Big(\epsilon(u)v \otimes x\Big)\,,\; d_1\Big(u \otimes v \otimes x\Big) = \Big(uv \otimes x\Big)\,,\; d_2\Big(u \otimes v \otimes x\Big) = \Big(u \otimes \partial(v)x\Big)\,, \tag{14}$$

and also two degeneracies $s_0, s_1 \colon \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ given by

$$s_0(u \otimes x) = (1_I \otimes u \otimes x), \quad s_1(u \otimes x) = (u \otimes 1_I \otimes x)\,.$$

Thus, we have a 2-truncated simplicial Hopf algebra $\mathfrak{H}_{|2}$ as follows:

$$\begin{tikzcd} I \otimes_\star (I \otimes_\rho H) \arrow[r, shift left=4, "d_2"] \arrow[r, shift left=2, "d_1"] \arrow[r, "d_0"] & I \otimes_\rho H \arrow[l, shift left=2, "s_0"] \arrow[l, shift left=4, "s_1"] \arrow[r, shift left=2, "d_1"] \arrow[r, "d_0"] & H \arrow[l, shift left=2, "s_0"] \end{tikzcd}$$

which is associated with the crossed module $\partial \colon I \longrightarrow H$.

Now, consider the simplicial Hopf algebra $\mathcal{H}' = \operatorname{cosk}_2(\mathfrak{H}_{|2})$. The length of the Moore complex of $\mathcal{H}'$ is basically three, from Lemma (5.4). In other words, $N(\mathcal{H}')$ has the form:

$$\cdots \longrightarrow \kappa \longrightarrow \operatorname{HKer}(\partial_2) \longrightarrow NH'_2 \xrightarrow{\partial_2} NH'_1 \xrightarrow{\partial_1} NH'_0$$

It is clear that $NH'_1 = I$ and $NH'_0 = H$ by definition. However, when we calculate NH'_2, we see that it is the zero object, namely κ. This implies that $NH'_3 \cong \operatorname{HKer}(\partial_2)$ is also κ, that means the Moore complex of H' is of length one[7]. Therefore, we obtain a functor $\mathrm{G}_1 \colon \{\mathrm{XHopf}^{\mathrm{cc}}\} \longrightarrow \{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq 1}\}$.

Consequently, it is immediate from the face morphisms given in (13), (14) that, $\mathrm{X}_1\mathrm{G}_1 \cong 1_{\mathrm{XHopf}^{\mathrm{cc}}}$. On the other hand, considering the simplicial decomposition given in Theorem 3.15, we have $\mathrm{G}_1\mathrm{X}_1 \cong 1_{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq 1}}$. ■

5.6. Remark. An alternative construction of the functor G_1 can be given due to [Porter, 2019, §2.3.5 and §3.7.4]. Briefly, for a given crossed module $X := \Big(I \xrightarrow{\partial} H\Big)$, we obtain its nerve $K(X)$ as follows:

- $K(X)_0 = H$
- $K(X)_1 = I \otimes_{\rho_1} H$
- $K(X)_2 = I \otimes_{\rho_2} (I \otimes_{\rho_1} H)$

[7]Here we need to obtain 2-truncated simplicial Hopf algebra. If we stop construction at 1-truncation level and apply cosk_1, we will get a simplicial Hopf algebra with Moore complex length two, and there will be no cancellation.

(with the same face and degeneracy maps (13) and (14)) in low dimensions; and the construction continues with:

$$\begin{aligned} K(X)_n &= I \otimes_{\rho_n} K(X)_{n-1} \\ &= I \otimes_{\rho_n} (\cdots (I \otimes_{\rho_2} (I \otimes_{\rho_1} H)) \cdots) \end{aligned}$$

for all dimensions $n \geq 1$. Here $K(X)_n$ includes n-copies of I that acts on I via the unique composed face map ρ_{n+1}, i.e. $\rho_1 = \rho$ and $\rho_2 = \star$ in the context of Theorem (5.5). Note that $K(X)_n$ is isomorphic to H_n in the simplicial decomposition form when put $NH_n = \kappa$, for all $n \geq 2$ in the decomposition formula. The equivalence of these two alternative proofs comes from the well-known categorical property that, the nerve of a category is 2-coskeletal.

6. 2-Crossed Modules

For completeness, we first recall the 2-crossed modules of groups and of Lie algebras. We note that, a lot of different conventions appear in the literature for 2-crossed modules. We follow Conduché, [Conduché, 1984] in defining group 2-crossed modules, and as derived from that, the definition in [Faria Martins and Picken, 2011] of 2-crossed module of Lie algebras (also called differential 2-crossed modules).

6.1. 2-crossed modules of groups, and Lie algebras.

6.2. Definition. *A 2-crossed module of groups is given by a chain complex of groups*

$$L \xrightarrow{\partial_2} E \xrightarrow{\partial_1} G$$

together with left actions $\triangleright$ of G on E, L; and with a G-equivariant[8] *bilinear map called Peiffer lifting*

$$\{\ ,\ \} : E \times_G E \longrightarrow L$$

satisfying the following axioms, for all $l, m \in L$ and $e, f, g \in E$:

1) $L \xrightarrow{\partial_2} E \xrightarrow{\partial_1} G$ *is a complex of G-modules where G acts on itself by conjugation,*

2) $\partial_2\{e, f\} = (efe^{-1})\,(\partial_1(e) \triangleright f^{-1})$,

3) $\{\partial_2(l), \partial_2(m)\} = lml^{-1}m^{-1}$,

4) $\Big\{e, fg\Big\} = \{e, f\}\ \Big(\partial_1(e) \triangleright f\Big) \triangleright' \{e, g\}$,

5) $\Big\{ef, g\Big\} = \{e, fgf^{-1}\}\ \partial_1(e) \triangleright \{f, g\}$,

6) $\Big\{\partial_2(l), e\Big\}\Big\{e, \partial_2(l)\Big\} = l\,\Big(\partial_1(e) \triangleright l^{-1}\Big)$.

[8]For groups, G-equivariance means $g \triangleright \{e, f\} = \{g \triangleright e, g \triangleright f\}$, $\forall g \in G$ and $e, f \in E$.

In the fourth condition, we put the action:

$$e \triangleright' l = l\Big\{\partial_2(l^{-1}), e\Big\}, \tag{15}$$

for each $e \in E$, $l \in L$, *that turns* $\Big(\partial_2 \colon L \longrightarrow E, \triangleright'\Big)$ *into a crossed module.*

6.3. Definition. *A 2-crossed module of Lie algebras (i.e. differential 2-crossed module) is given by a chain complex of Lie algebras*

$$\mathfrak{l} \xrightarrow{\partial_2} \mathfrak{e} \xrightarrow{\partial_1} \mathfrak{g}$$

together with left actions $\triangleright$ *of* $\mathfrak{g}$ *on* $\mathfrak{e}, \mathfrak{l}$*; and with a* $\mathfrak{g}$*-equivariant*[9] *bilinear map called Peiffer lifting*

$$\{\ ,\ \} : \mathfrak{e} \times_{\mathfrak{g}} \mathfrak{e} \longrightarrow \mathfrak{l}$$

satisfying the following axioms, for all $x, y \in \mathfrak{l}$ *and* $u, v, w \in \mathfrak{e}$*:*

1) $\mathfrak{l} \xrightarrow{\partial_2} \mathfrak{e} \xrightarrow{\partial_1} \mathfrak{g}$ *is a complex of* $\mathfrak{g}$*-modules where* $\mathfrak{g}$ *acts on itself by adjoint representation,*

2) $\partial_2\{u, v\} = [u, v] - \partial_1(u) \triangleright v,$

3) $\{\partial_2(x), \partial_2(y)\} = [x, y],$

4) $\Big\{u, [v, w]\Big\} = \Big\{\partial_2\{u, v\}, w\Big\} - \Big\{\partial_2\{u, w\}, v\Big\},$

5) $\Big\{[u, v], w\Big\} = \partial_1(u) \triangleright \Big\{v, w\Big\} + \Big\{u, [v, w]\Big\} - \partial_1(v) \triangleright \Big\{u, w\Big\} - \Big\{v, [u, w]\Big\},$

6) $\Big\{\partial_2(x), v\Big\} + \Big\{v, \partial_2(x)\Big\} = -\partial_1(v) \triangleright x.$

When we put:

$$v \triangleright' x = -\Big\{\partial_2(x), v\Big\} \tag{16}$$

for each $x \in \mathfrak{l}$, $v \in \mathfrak{e}$, *that turns* $\Big(\partial_2 \colon \mathfrak{l} \longrightarrow \mathfrak{e}, \triangleright'\Big)$ *into a differential crossed module.*

6.4. 2-crossed modules of Hopf algebras.

6.5. Definition. *A "2-crossed module" of Hopf algebras is given by a chain complex*

$$K \xrightarrow{\partial_2} I \xrightarrow{\partial_1} H$$

of Hopf algebras (i.e. $\partial_1\partial_2$ *is zero morphism) with the actions* $\triangleright_\rho$ *of* H *on* I, K*, and also on itself by the adjoint action; together with an* H*-equivariant*[10] *bilinear map called Peiffer lifting*

$$\{\ ,\ \} : I \times_H I \longrightarrow K$$

satisfying the following axioms, for all $x, y, z \in I$ *and* $k, l \in K$*:*

[9] For Lie algebras, $\mathfrak{g}$-equivariance means $g \triangleright \{u, v\} = \{g \triangleright u, v\} + \{u, g \triangleright v\}$, $\forall g \in \mathfrak{g}$ and $u, v \in \mathfrak{e}$.

[10] Here H-equivariance means $a \triangleright_\rho \{x, y\} = \sum_{(a)}\{a' \triangleright_\rho x, a'' \triangleright_\rho y\}$, $\forall a \in H$ and $x, y \in I$.

1) $K \xrightarrow{\partial_2} I \xrightarrow{\partial_1} H$ *is a complex of* H*-module bialgebras,*[11]

2) $\partial_2\{x,y\} = \sum\limits_{(x)(y)} (x' \triangleright_{ad} y')\ \partial_1(x'') \triangleright_\rho S(y''),$

3) $\{\partial_2(k), \partial_2(l)\} = \sum\limits_{(l)} (k \triangleright_{ad} l')\, S(l'')$[12]$,$

4) $\{x, yz\} = \sum\limits_{(x)(y)} \{x', y'\}\ (\partial_1(x'') \triangleright_\rho y'') \triangleright'_\rho \{x''', z\},$

5) $\{xy, z\} = \sum\limits_{(x)(y)(z)} \{x', y' \triangleright_{ad} z'\}\ \partial_1(x'') \triangleright_\rho \{y'', z''\},$

6) $\sum\limits_{(k)(x)} \{\partial_2(k'), x'\}\{x'', \partial_2(k'')\} = \sum\limits_{(k)} k'\ (\partial_1(x) \triangleright_\rho S(k'')).$

In the fourth condition, the action is:

$$x \triangleright'_\rho k = \sum_{(k)} k' \left\{\partial_2(S(k'')), x\right\}, \tag{17}$$

such that $\left(\partial_2 \colon K \longrightarrow I, \triangleright'_\rho\right)$ *becomes a crossed module. Remark that* ∂_1 *is only a precrossed module in general – just like in the cases of groups and Lie algebras.*

6.6. Definition. *A 2-crossed module morphism of Hopf algebras between* $K \xrightarrow{\partial_2} I \xrightarrow{\partial_1} H$ *and* $K' \xrightarrow{\partial'_2} I' \xrightarrow{\partial'_1} H'$ *is given by a triple* (f_2, f_1, f_0) *that consists of Hopf algebra maps* $f_0 \colon H \longrightarrow H'$, $f_1 \colon I \longrightarrow I'$ *and* $f_2 \colon K \longrightarrow K'$*, making the diagram:*

$$\begin{tikzcd} K \arrow[r, "\partial_2"] \arrow[d, "f_2"] & I \arrow[r, "\partial_1"] \arrow[d, "f_1"] & H \arrow[d, "f_0"] \\ K' \arrow[r, "\partial'_2"'] & I' \arrow[r, "\partial'_1"'] & H' \end{tikzcd}$$

commutative, and also preserving the actions of H *and* H'*, as well as the Peiffer liftings:*

$$\begin{aligned} f_1(a \triangleright_\rho x) &= f_0(a) \triangleright_\rho f_1(x), \text{ for all } a \in H \text{ and } x \in I, \\ f_2(a \triangleright_\rho k) &= f_0(a) \triangleright_\rho f_2(k), \text{ for all } a \in H \text{ and } k \in K, \\ f_2\{x,y\} &= \{f_1(x), f_1(y)\}, \text{ for all } x, y \in I. \end{aligned}$$

Therefore, we have defined the category of 2-crossed modules of Hopf algebras which will be denoted by $\{\mathrm{X_2Hopf^{cc}}\}$.

6.7. Example. For a given crossed module $I \xrightarrow{\partial} H$, we can obtain a natural 2-crossed module $\kappa \longrightarrow I \xrightarrow{\partial} H$ where κ is the zero object as usual. This gives rise to an inclusion functor $\{\mathrm{XHopf^{cc}}\} \longrightarrow \{\mathrm{X_2Hopf^{cc}}\}$.

[11]Namely, $\partial_2(a \triangleright_\rho k) = a \triangleright_\rho \partial_2(k)$, and $\partial_1(a \triangleright_\rho x) = a \triangleright_{ad} \partial_1(x)$, $\forall a \in H$, $x \in I$, $k \in K$.
[12]It is the commutator, i.e. $\sum_{(k)(l)} k'l'S(k')S(l')$.

6.8. Example. Let $I \xrightarrow{\partial_1} H$ be a precrossed module. Then we have a 2-crossed module

$$\mathrm{HKer}(\partial_1) \xrightarrow{\partial_2} I \xrightarrow{\partial_1} H\,,$$

where ∂_2 is the inclusion map, and the Peiffer lifting is necessarily given by ($\forall x, y \in I$):

$$\{x, y\} = \sum_{(x)(y)} (x' \triangleright_{ad} y')\ \partial_1(x'') \triangleright_\rho S(y'')\,.$$

6.9. Coherence with *Prim* and *Gl*.

6.10. Theorem. *The functor Prim preserves the 2-crossed module structure.*

Proof. Suppose that $K \xrightarrow{\partial_2} I \xrightarrow{\partial_1} H$ is a 2-crossed modules of Hopf algebras. We already know that the primitive elements preserve the actions [Faria Martins, 2016]. Therefore, we have the complex of $Prim(H)$-modules of Lie algebras. Since the rest of the proof consists in routine calculations by setting $\delta(a) = 1 \otimes a + a \otimes 1$ and $Prim(a \triangleright_{ad} b) = [a, b]$, we only prove that the derived crossed module action (17) is compatible with (16), namely

$$\begin{aligned} x \triangleright'_\rho k \ &= \sum_{(k)} k' \{\partial_2(S(k'')), x\} \\ &= 1_K \{-\partial_2(k), x\} + k \{\partial_2(1_K), x\} \\ &= -\{\partial_2(k), x\} \\ &= x \triangleright' k\,, \end{aligned}$$

for all $x \in I$ and $k \in K$. Regarding the penultimate step of the calculation: from the fifth condition of the Peiffer lifting, we obtain $\{1_I, x\} = \{x, 1_I\} = \epsilon(x)1_K$. Moreover, since x is primitive, we have $\epsilon(x) = 0$ that yields $\{1_I, x\} = \{x, 1_I\} = 0$. ■

6.11. Theorem. *The functor Gl preserves the 2-crossed module structure.*

Proof. Follows immediately by letting $\delta(x) = x \otimes x$ and $Gl(x \triangleright_{ad} y) = xyx^{-1}$. ■

6.12. Proposition. *Consequently, we have the functors*

$$\{\mathrm{X_2Lie}\} \xleftarrow{Prim} \{\mathrm{X_2Hopf^{cc}}\} \xrightarrow{Gl} \{\mathrm{X_2Grp}\}$$

extending (5) *to the 2-crossed module level.*

6.13. From simplicial Hopf algebras to 2-crossed modules.

6.14. Theorem. *Let $\mathcal{H}$ be a simplicial Hopf algebra with a Moore complex of length two. Consider:*

$$NH_2 \xrightarrow{\partial_2} NH_1 \xrightarrow{\partial_1} H_0\,. \tag{18}$$

Then we have the actions:

- *of H_0 on NH_1, given by $n \triangleright_\rho m = s_0(n) \triangleright_{ad} m$, for all $n \in H_0$ and $m \in NH_1$,*
- *of H_0 on NH_2, given by $n \triangleright_\rho l = s_1 s_0(n) \triangleright_{ad} l$, for all $n \in H_0$ and $l \in NH_2$,*

and the Peiffer lifting $\{\ ,\ \} : NH_1 \otimes_{H_0} NH_1 \longrightarrow NH_2$ as being

$$\{x, y\} = \sum_{(x)(y)} \Big(s_1(x') \triangleright_{ad} s_1(y')\Big)\, S\Big(s_0(x'') \triangleright_{ad} s_1(y'')\Big),$$

for all $x, y \in NH_1$, that makes (18) into a 2-crossed module of Hopf algebras.

Proof. Since the length of the Moore complex of $\mathcal{H}$ is two, we have $\partial_3(x) = \epsilon(x)\, 1_{H_2}$, for all $x \in NH_3$ - which plays the key role in the whole proof. But first, let us let us examine some direct consequences that will be helpful in the calculations:

- We already know that

$$\partial_3\left(F_{(1,0)(2)}(x, y)\right) = \epsilon(x)\epsilon(y)1_{H_2},$$

for all $x \in H$ and $y \in K$. If we calculate the left hand side, we obtain

$$\sum_{(x)(y)} (s_1 s_0 d_1(x') \triangleright_{ad} y')\, S\,(s_0(x'') \triangleright_{ad} y'') = \epsilon(x)\epsilon(y)1_{H_2},$$

which implies

$$(s_1 s_0 d_1(x) \triangleright_{ad} y) = (s_0(x) \triangleright_{ad} y)\,.$$

Moreover, recalling the action of H_0 on NH_2 given above, we explicitly get

$$\partial_1(x) \triangleright_\rho k = (s_1 s_0 d_1(x) \triangleright_{ad} k) = (s_0(x) \triangleright_{ad} k)\,, \tag{19}$$

for all $x \in NH_1$ and $k \in NH_2$.

- The action $x \triangleright'_\rho k$ given in (17) corresponds to $s_1(x) \triangleright_{ad} k$, for all $x \in NH_1$ and $k \in NH_2$; which follows from $\partial_3\left(F_{(0)(2,1)}(k, x)\right) = \epsilon(k)\epsilon(x)1_{H_2}$.

Now, let us use the Peiffer pairings obtained in section 4.4.2 to prove the conditions:

1) Follows from the definition of the Moore complex.

2) We straightforwardly have (for all $x, y \in NH_1$):

$$\begin{aligned}
\partial_2\{x, y\} &= d_2\left(\sum_{(x)(y)} \Big(s_1(x') \triangleright_{ad} s_1(y')\Big)\, S\Big(s_0(x'') \triangleright_{ad} s_1(y'')\Big)\right) \\
&= \sum_{(x)(y)} \Big(d_2 s_1(x') \triangleright_{ad} d_2 s_1(y')\Big)\, S\Big(d_2 s_0(x'') \triangleright_{ad} d_2 s_1(y'')\Big) \\
&= \sum_{(x)(y)} (x' \triangleright_{ad} y')\, \Big(s_0 d_1(x'') \triangleright_{ad} S(y'')\Big) \\
&= \sum_{(x)(y)} (x' \triangleright_{ad} y')\ \partial_1(x'') \triangleright_\rho S(y'').
\end{aligned}$$

3) We get (for all $k, l \in NH_2$):

$$\begin{aligned}\{\partial_2(k), \partial_2(l)\} &= \sum_{(k)(l)} \Big(s_1d_2(k') \triangleright_{ad} s_1d_2(l')\Big)\, S\Big(s_0d_2(k'') \triangleright_{ad} s_1d_2(l'')\Big) \\ &\because \quad \text{the fact that:} \quad \partial_3\left(F_{(0)(1)}(k,l)\right) = \epsilon(k)\epsilon(l)1_{H_2} \\ &= \sum_{(l)} (k \triangleright_{ad} l')\, S(l'').\end{aligned}$$

4) We get (for all $x, y, z \in NH_1$):

$$\begin{aligned}\{x, yz\} &= \sum_{(x)(yz)} \Big(s_1(x)' \triangleright_{ad} s_1(yz)'\Big)\, S\Big(s_0(x)'' \triangleright_{ad} s_1(yz)''\Big) \\ &\because \quad \text{for the calculations, see Appendix A.2} \\ &= \sum_{(x)(y)} \{x', y'\}\ (\partial_1(x'') \triangleright_\rho y'')\triangleright'_\rho \{x''', z\}.\end{aligned}$$

5) Again for all $x, y, z \in NH_1$, we get:

$$\begin{aligned}\{xy, z\} &= \sum_{(xy)(z)} \Big(s_1(xy)' \triangleright_{ad} s_1(z)'\Big)\, S\Big(s_0(xy)'' \triangleright_{ad} s_1(z)''\Big) \\ &\because \quad \text{for the calculations, see Appendix A.3} \\ &= \sum_{(x)(y)(z)} \{x', y' \triangleright_{ad} z'\}\ d_1(x'') \triangleright_\rho \{y'', z''\}.\end{aligned}$$

6) We first have (for all $k \in NH_2$ and $x \in NH_1$):

$$\begin{aligned}\{\partial_2(k), x\} &= \sum_{(k)(x)} \Big(s_1d_2(k') \triangleright_{ad} s_1(x')\Big)\, S\Big(s_0d_2(k'') \triangleright_{ad} s_1(x'')\Big) \\ &\because \quad \text{the fact that:} \quad \partial_3\left(F_{(0)(2,1)}(k,x)\right) = \epsilon(k)\epsilon(x)1_{H_2} \\ &= \sum_{(x)} \Big(k \triangleright_{ad} s_1(x')\Big)\, S\Big(s_1(x'')\Big) \\ &= \sum_{(k)} k' \ \Big(s_1(x) \triangleright_{ad} S(k'')\Big),\end{aligned}$$

and

$$\begin{aligned}\{x, \partial_2(k)\} &= \sum_{(k)(x)} \Big(s_1(x') \triangleright_{ad} s_1d_2(k')\Big)\, S\Big(s_0(x'') \triangleright_{ad} s_1d_2(k'')\Big) \\ &\because \quad \text{the fact that:} \quad \partial_3\left(F_{(2,0)(1)}(x,k)\right) = \epsilon(x)\epsilon(k)1_{H_2} \\ &= \sum_{(k)(x)} \Big(s_1(x') \triangleright_{ad} k'\Big)\, S\Big(s_0(x'') \triangleright_{ad} k''\Big),\end{aligned}$$

that implies

$$
\begin{aligned}
&\sum_{(k)(x)} \{\partial_2(k'), x'\}\{x'', \partial_2(k'')\} \\
&\qquad\qquad = \sum_{(k)(x)} k' \Big(s_1(x') \rhd_{ad} S(k'')\Big) \Big(s_1(x'') \rhd_{ad} k'''\Big) S\Big(s_0(x''') \rhd_{ad} k''''\Big) \\
&\qquad\qquad \because \quad \text{by using (19)} \\
&\qquad\qquad = \sum_{(k)} k' \left(\partial_1(x) \rhd_\rho S(k'')\right).
\end{aligned}
$$

∎

6.15. Corollary. *We therefore obtain a functor* $\mathrm{X}_2\colon \{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq 2}\} \longrightarrow \{\mathrm{X}_2\mathrm{Hopf}^{\mathrm{cc}}\}$.

6.16. From 2-crossed modules to simplicial Hopf algebras.

6.17. Theorem. *The category of 2-crossed modules of Hopf algebras is equivalent to the category of simplicial Hopf algebras with Moore complex of length two.*

Proof. Let us fix an arbitrary 2-crossed module $K \xrightarrow{\partial_2} I \xrightarrow{\partial_1} H$. We already have $\mathfrak{H}_0 = H$ and $\mathfrak{H}_1 = I \otimes_\rho H$ with the face and degeneracy maps as given in the crossed module case.

To improve the readability of the proof, we also fix the variables $k, l, m, k_2, l_2, m_2 \in K$, $x, y, z, x_2, y_2, z_2 \in I$ and $a \in H$ from now on.

First of all, ∂_2 part of the 2-crossed module yields the smash product Hopf algebra $K \otimes_{\rho'} I$ considering the action $\rhd'_\rho$ of I on K defined in (17). There exists an action of $I \otimes_\rho H$ on $K \otimes_{\rho'} I$ given by

$$
(y \otimes a) \rhd_* (k_2 \otimes x_2) = \sum_{(y)(a)(x_2)} \Big(\Big(\partial_1(y')a' \rhd_\rho k_2\Big) S\Big(\{y'', a'' \rhd_\rho x_2'\}\Big) \otimes y''' \rhd_{ad} (a''' \rhd_\rho x_2'')\Big)
$$

that yields the smash product Hopf algebra

$$
\mathfrak{H}_2 = (K \otimes_{\rho'} I) \otimes_* (I \otimes_\rho H),
$$

with the face maps $d_0, d_1, d_2\colon \mathfrak{H}_2 \longrightarrow \mathfrak{H}_1$:

$$
\begin{aligned}
d_0(k \otimes x \otimes y \otimes a) &= (\epsilon(k)\epsilon(x)y \otimes a) \\
d_1(k \otimes x \otimes y \otimes a) &= (\epsilon(k)xy \otimes a) \\
d_2(k \otimes x \otimes y \otimes a) &= (\partial_2(k)x \otimes \partial_1(y)a)
\end{aligned}
$$

and also with two natural degeneracy maps $s_0, s_1\colon \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$.

Naturally, $K \otimes_{\rho'} I$ acts on K with

$$
(k \otimes x) \rhd_\star l = k \rhd_{ad} (x \rhd'_\rho l),
$$

hence we can construct $K \otimes_{\star} (K \otimes_{\rho'} I)$.

Then I and H acts on $K \otimes_{\star} (K \otimes_{\rho'} I)$ respectively with:

$$\begin{aligned} & y \triangleright_{\dagger_1} (l_2 \otimes m_2 \otimes z_2) \\ &\qquad = \sum_{(y)(z_2)} \Big(\partial_1(y') \triangleright_{\rho} l_2 \, \otimes \, (\partial_1(y'') \triangleright_{\rho} m_2) \, S\Big(\{y''', z_2{}'\}\Big) \, \otimes \, y'''' \triangleright_{ad} z_2{}'' \Big), \end{aligned}$$

and

$$a \triangleright_{\dagger_2} (l_2 \otimes m_2 \otimes z_2) = \sum_{(a)} \Big(a' \triangleright_{\rho} l_2 \otimes a'' \triangleright_{\rho} m_2 \otimes a''' \triangleright_{\rho} z_2 \Big).$$

Therefore, $I \otimes_{\rho} H$ acts on $K \otimes_{\star} (K \otimes_{\rho'} I)$ as follows:

$$(y \otimes a) \triangleright_{\dagger} (l_2 \otimes m_2 \otimes z_2) = y \triangleright_{\dagger_1} \Big(a \triangleright_{\dagger_2} \Big(l_2 \otimes m_2 \otimes z_2 \Big) \Big). \tag{20}$$

On the other hand, we have an action of K on $K \otimes_{\star} (K \otimes_{\rho'} I)$ given by

$$\begin{aligned} & k \triangleright_{\ddagger_1} (l_2 \otimes m_2 \otimes z_2) \\ &\qquad = \sum_{(k)(m_2)(z_2)} \Big(l_2 \, S\Big(\Big\{\partial_2(k'), \partial_2(m_2') \, z_2'\Big\}\Big) \, \otimes \, (k'' \triangleright_{ad} m_2'') \, \{\partial_2(k'''), z_2''\} \otimes z_2''' \Big). \end{aligned}$$

Moreover, we have another nontrivial action of I on $K \otimes_{\star} (K \otimes_{\rho'} I)$ which is

$$\begin{aligned} & x \triangleright_{\ddagger_2} (l_2 \otimes m_2 \otimes z_2) \\ &\qquad = \sum_{(x)(m_2)(z_2)} \Big(\Big(\partial_1(x') \triangleright_{\rho} l_2 \Big) \, S\Big(\Big\{x'', \partial_2(m_2') z_2'\Big\}\Big) \otimes x''' \triangleright_{\rho}' m_2'' \otimes x'''' \triangleright_{ad} z_2'' \Big). \end{aligned}$$

Therefore, $K \otimes_{\rho'} I$ acts on $K \otimes_{\star} (K \otimes_{\rho'} I)$ as follows:

$$(k \otimes x) \triangleright_{\ddagger} (l_2 \otimes m_2 \otimes z_2) = k \triangleright_{\ddagger_1} \Big(x \triangleright_{\ddagger_2} \Big(l_2 \otimes m_2 \otimes z_2 \Big) \Big). \tag{21}$$

Consequently, we have an action of $\mathfrak{H}_2 = (K \otimes_{\rho'} I) \otimes_{*} (I \otimes_{\rho} H)$ on $K \otimes_{\star} (K \otimes_{\rho'} I)$ which is given by[13]

$$(k \otimes x \otimes y \otimes a) \triangleright_{\bullet} (l_2 \otimes m_2 \otimes z_2) = (k \otimes x) \triangleright_{\ddagger} \Big(\Big(y \otimes a \Big) \triangleright_{\dagger} \Big(l_2 \otimes m_2 \otimes z_2 \Big) \Big), \tag{22}$$

that yields the smash product Hopf algebra

$$\mathfrak{H}_3 = \Big(K \otimes_{\star} (K \otimes_{\rho'} I) \Big) \otimes_{\bullet} \Big((K \otimes_{\rho'} I) \otimes_{*} (I \otimes_{\rho} H) \Big),$$

[13] Subscripts on the right-hand side are useful when we calculate the product of two elements of $\mathfrak{H}_3$.

with the face maps $d_0, d_1, d_2, d_3 \colon \mathfrak{H}_3 \longrightarrow \mathfrak{H}_2$:

$$d_0\Big(l \otimes m \otimes z \otimes k \otimes x \otimes y \otimes a\Big) = \Big(\epsilon(lm)k \otimes \epsilon(z)x \otimes y \otimes a\Big)$$

$$d_1\Big(l \otimes m \otimes z \otimes k \otimes x \otimes y \otimes a\Big) = \sum_{(z)}\Big(\epsilon(l)\,\Big(m\,(z' \rhd k)\Big) \otimes z''x \otimes y \otimes a\Big)$$

$$d_2\Big(l \otimes m \otimes z \otimes k \otimes x \otimes y \otimes a\Big) = \Big(\epsilon(k)lm \otimes z \otimes xy \otimes a\Big)$$

$$d_3\Big(l \otimes m \otimes z \otimes k \otimes x \otimes y \otimes a\Big) = \Big(l \otimes \partial_2(m)z \otimes \partial_2(k)x \otimes \partial_1(y)a\Big)$$

and also with three natural degeneracies $s_0, s_1, s_2 \colon \mathfrak{H}_2 \longrightarrow \mathfrak{H}_3$.

We therefore obtain a 3-truncated simplicial Hopf algebra. Analogous to the crossed module case, considering its image under the cosk_3 functor, we get a simplicial Hopf algebra with Moore complex of length two[14]. This yields a functor between the categories $\mathrm{G}_2 \colon \{\mathrm{X_2Hopf^{cc}}\} \longrightarrow \{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq 2}\}$. The functorial constructions given above yield the required equivalence. ■

6.18. Conclusion. Recalling the functors given in 3.10, 6.12, 6.15 and 6.17, all fitting into the diagram:

$$\begin{tikzcd}
\{\mathrm{SimpGrp}_{\leq 2}\} \arrow[d, shift left] & \{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq 2}\} \arrow[l, "Gl"'] \arrow[r, "Prim"] \arrow[d, shift left] & \{\mathrm{SimpLie}_{\leq 2}\} \arrow[d, shift left] \\
\{\mathrm{X_2Grp}\} \arrow[u, shift left] & \{\mathrm{X_2Hopf^{cc}}\} \arrow[l, "Gl"'] \arrow[r, "Prim"] \arrow[u, shift left] & \{\mathrm{X_2Lie}\} \arrow[u, shift left]
\end{tikzcd} \tag{23}$$

where we refer to [Conduché, 1984; Mutlu and Porter, 1998; Ellis, 1993b] for the equivalences between $\{\mathrm{SimpGrp}_{\leq 2}\} \longrightarrow \{\mathrm{X_2Grp}\}$, and $\{\mathrm{SimpLie}_{\leq 2}\} \longrightarrow \{\mathrm{X_2Lie}\}$.

Moreover, it is a natural question (suggested by the referee) as to whether the horizontal functors of (23) are part of an adjunction as an induced version of the functors given in (1). Related to the upper part: first of all, considering the following diagram:

$$\begin{tikzcd}
\Delta^{\mathrm{op}} \arrow[r, equal] \arrow[d] & \Delta^{\mathrm{op}} \arrow[r, equal] \arrow[d] & \Delta^{\mathrm{op}} \arrow[d] \\
\{\mathrm{Grp}\} \arrow[r, shift right, "\mathcal{G}^\sharp"'] \arrow[r, phantom, "\top"] & \{\mathrm{Hopf^{cc}}\} \arrow[l, shift right, "Gl"'] \arrow[r, shift right, "Prim"'] \arrow[r, phantom, "\perp"] & \{\mathrm{Lie}\} \arrow[l, shift right, "\mathcal{F}^\sharp"']
\end{tikzcd} \tag{24}$$

we get the adjunctions

$$\begin{tikzcd}
\{\mathrm{SimpGrp}\} \arrow[r, shift right, "\mathcal{G}^\sharp"'] \arrow[r, phantom, "\top"] & \{\mathrm{SimpHopf^{cc}}\} \arrow[l, shift right, "Gl"'] \arrow[r, shift right, "Prim"'] \arrow[r, phantom, "\perp"] & \{\mathrm{SimpLie}\} \arrow[l, shift right, "\mathcal{F}^\sharp"']
\end{tikzcd}.$$

[14]Instead of four, since the specific definitions of face maps – as in the crossed module case. Moreover, this simplicial structure can be considered as the 2-nerve of a given 2-crossed module; and it is 3-coskeletal.

However, as a consequence of the well-known adjunctions given in (2), we can immediately say that the left adjoint functors $\mathcal{G}^\sharp$ and $\mathcal{F}^\sharp$ preserve colimits (therefore cokernels). In this context, the behaviors of these functors at the (length of) Moore complex level is not automatically predictable - since the Moore complex construction is based on the kernels. On the other hand, related to the lower part: we do not know whether the functor $\mathcal{G}^\sharp$ or $\mathcal{F}^\sharp$ preserves 2-crossed module structures so far. These will be subject of another study, together with the ones we mentioned below.

Additionally, not only from the viewpoint of Lie algebras and groups, but also considering the 2-crossed module version of the diagram (10) in [Faria Martins, 2016], it could be possible to have further functorial relationships between different algebraic structures. For instance, the categories of bare algebras, commutative algebras, Lie groups, etc. can be examined (in fact, 2-crossed modules of bare algebras or Lie groups have not been defined yet).

One level fewer of diagram (23), we obviously obtain the crossed module version of the same diagram that yields new possible connections to the functors given in [Casas et al., 2014, 2017] where the authors examine the functorial relationships between the category of crossed modules of groups, of Lie algebras, of Leibniz algebras and of associative algebras. Another generalization of crossed modules and their relationships with simplicial objects is recently studied in [Böhm, 2021, 2019] for the category of monoids in which the author proves that the category of crossed modules of monoids is equivalent to the category of simplicial monoids with Moore complex of length one.

All in all, we have unified the 2-crossed module notions of groups and of Lie algebras as well as their relationships with the Moore complex, in the category of cocommutative Hopf algebras. On the other hand, as we mentioned in the introduction, there are some other algebraic models for homotopy 3-types such as crossed squares, cat^2-structures, quadratic modules, etc with some relationships (and also some equivalences) between them. In this context, the notion of crossed squares and their relationship between cat^2-Hopf algebras is an ongoing work due to [Sterck, 2019]. Once its completed, one can examine the connection between these two studies, which will lead to a functorial relationship between crossed squares and 2-crossed modules that will again capture the group and Lie algebra cases.

A. Appendix

A.1. Inverting through the antipode. Let $f, g\colon H' \longrightarrow H$ be two Hopf algebra maps such that

$$\sum_{(x)(y)} f(x' \otimes y')\, S\Big(g(x'' \otimes y'')\Big) = \epsilon(x)\epsilon(y)1_H\,.$$

Then

$$\begin{aligned} f(x\otimes y) &= \sum_{(x)(y)} f(x'\otimes y')\Big(S\big(g(x''\otimes y'')\big)\, g(x'''\otimes y''')\Big) \\ &= \sum_{(x)(y)} \Big(f(x'\otimes y')\, S\big(g(x''\otimes y'')\big)\Big)\, g(x'''\otimes y''') \\ &= \sum_{(x)(y)} \epsilon(x')\epsilon(y')1_H\, g(x''\otimes y'') \\ &= g(x\otimes y). \end{aligned}$$

Note that, this property is not only used in the proof of Theorem 5.2, but also frequently in the proof of Theorem 6.14.

A.2. Proof of (4). First of all, we have (for all $x, y \in NH_1$ and $k \in NH_2$):

$$\begin{aligned} \Big(\partial_1(x) \rhd_\rho y\Big) \rhd'_\rho k &= \Big(s_0d_1(x) \rhd_{ad} y\Big) \rhd'_\rho k \\ &= \sum_{(x)} \Big(s_0d_1(x')\, y\, S(s_0d_1(x''))\Big) \rhd'_\rho k \\ &= \sum_{(x)} s_0d_1(x') \rhd'_\rho \Big(y \rhd'_\rho \Big(s_0d_1(S(x'')) \rhd'_\rho k\Big)\Big) \\ &= \sum_{(x)} s_1s_0d_1(x') \rhd_{ad} \Big(s_1(y) \rhd_{ad} \Big(s_1s_0d_1(S(x'')) \rhd_{ad} k\Big)\Big) \\ &= \sum_{(x)} s_0(x') \rhd_{ad} \Big(s_1(y) \rhd_{ad} \Big(s_0(S(x'')) \rhd_{ad} k\Big)\Big) \\ &= \Big(s_0(x) \rhd_{ad} s_1(y)\Big) \rhd_{ad} k\,. \end{aligned}$$

Then the proof continues as follows (for all $x, y, z \in NH_1$):

$$\begin{aligned} \{x, yz\} &= \sum_{(xy)(z)} \Big(s_1(xy)' \rhd_{ad} s_1(z)'\Big)\, S\Big(s_0(xy)'' \rhd_{ad} s_1(z)''\Big) \\ &= \sum_{(x)(y)(z)} \Big(s_1(x') \rhd_{ad} s_1(y'z')\Big)\, S\Big(s_0(x'') \rhd_{ad} s_1(y''z'')\Big) \\ &= \sum_{(x)(y)(z)} \Big(s_1(x') \rhd_{ad} s_1(y')\Big)\Big(s_1(x'') \rhd_{ad} s_1(z')\Big) \\ &\qquad S\Big(\Big(s_0(x''') \rhd_{ad} s_1(y'')\Big)\Big(s_0(x'''') \rhd_{ad} s_1(z'')\Big)\Big) \end{aligned}$$

$$= \sum_{(x)(y)(z)} \Big(s_1(x') \triangleright_{ad} s_1(y')\Big) S\Big(s_0(x'') \triangleright_{ad} s_1(y'')\Big)$$
$$\Big(s_0(x''') \triangleright_{ad} s_1(y''')\Big) \Big(s_1(x'''') \triangleright_{ad} s_1(z')\Big)$$
$$S\Big(\Big(s_0(x''''') \triangleright_{ad} s_1(y'''')\Big)\Big(s_0(x'''''') \triangleright_{ad} s_1(z'')\Big)\Big)$$

$$= \sum_{(x)(y)(z)} \{x', y'\} \Big(s_0(x'') \triangleright_{ad} s_1(y'')\Big) \Big(s_1(x''') \triangleright_{ad} s_1(z')\Big)$$
$$S\Big(\Big(s_0(x'''') \triangleright_{ad} s_1(y''')\Big) \Big(s_0(x''''') \triangleright_{ad} s_1(z'')\Big)\Big)$$

$$= \sum_{(x)(y)(z)} \{x', y'\} \Big(s_0(x'') \triangleright_{ad} s_1(y'')\Big) \triangleright_{ad}$$
$$\Big(\Big(s_1(x''') \triangleright_{ad} s_1(z')\Big) S\Big(s_0(x'''') \triangleright_{ad} s_1(z'')\Big)\Big)$$

$$= \sum_{(x)(y)} \{x', y'\} \Big(s_0(x'') \triangleright_{ad} s_1(y'')\Big) \triangleright_{ad} \{x''', z\}$$

$$= \sum_{(x)(y)} \{x', y'\} \left(\partial_1(x'') \triangleright_{\rho} y''\right) \triangleright'_{\rho} \{x''', z\} \quad \because \text{ by using (19).}$$

A.3. Proof of (5). For all $x, y, z \in NH_1$, we get:

$$\{xy, z\} = \sum_{(xy)(z)} \Big(s_1(xy)' \triangleright_{ad} s_1(z)'\Big) S\Big(s_0(xy)'' \triangleright_{ad} s_1(z)''\Big)$$

$$= \sum_{(x)(y)(z)} \Big(s_1(x'y') \triangleright_{ad} s_1(z')\Big) S\Big(s_0(x''y'') \triangleright_{ad} s_1(z'')\Big)$$

$$= \sum_{(x)(y)(z)} \Big(s_1(x'y') \triangleright_{ad} s_1(z')\Big) \Big(\epsilon(x'')\epsilon(y'')\epsilon(z'')1_{H_2}\Big) S\Big(s_0(x'''y''') \triangleright_{ad} s_1(z''')\Big)$$

$$= \sum_{(x)(y)(z)} \Big(s_1(x'y') \triangleright_{ad} s_1(z')\Big) \Big(s_0(x'') \triangleright_{ad} \Big(\epsilon(y'')\epsilon(z'')1_{H_2}\Big)\Big) S\Big(s_0(x'''y''') \triangleright_{ad} s_1(z''')\Big)$$

$$= \sum_{(x)(y)(z)} \Big(s_1(x'y') \triangleright_{ad} s_1(z')\Big)$$
$$s_0(x'') \triangleright_{ad} \Big(S\Big(s_1(y'') \triangleright_{ad} s_1(z'')\Big) \Big(s_1(y''') \triangleright_{ad} s_1(z''')\Big)\Big)$$
$$S\Big(s_0(x''''y'''') \triangleright_{ad} s_1(z'''')\Big)$$

$$= \sum_{(x)(y)(z)} \Big(s_1(x'y') \rhd_{ad} s_1(z')\Big)$$
$$s_0(x'') \rhd_{ad} \left(S\Big(s_1(y'') \rhd_{ad} s_1(z'')\Big)\right) s_0(x''') \rhd_{ad} \left(\Big(s_1(y''') \rhd_{ad} s_1(z''')\Big)\right)$$
$$S\Big(s_0(x''''y'''') \rhd_{ad} s_1(z'''')\Big)$$
$$= \sum_{(x)(y)(z)} s_1(x') \rhd_{ad} \left(s_1(y') \rhd_{ad} s_1(z')\right) s_0(x'') \rhd_{ad} \left(S\Big(s_1(y'') \rhd_{ad} s_1(z'')\Big)\right)$$
$$s_0(x''') \rhd_{ad} \left(s_1(y''') \rhd_{ad} s_1(z''')\right) S\Big(s_0(x''''y'''') \rhd_{ad} s_1(z'''')\Big)$$

$$= \sum_{(x)(y)(z)} \left(s_1(x') \rhd_{ad} s_1\Big(y' \rhd_{ad} z'\Big)\right) S\left(s_0(x'') \rhd_{ad} s_1\Big(y'' \rhd_{ad} z''\Big)\right)$$
$$\left(s_0(x''') \rhd_{ad} \Big(s_1(y''') \rhd_{ad} s_1(z''')\Big)\right) S\Big(s_0(x''''y'''') \rhd_{ad} s_1(z'''')\Big)$$

$$= \sum_{(x)(y)(z)} \{x', y' \rhd_{ad} z'\}\ s_0(x'') \rhd_{ad} \Big(s_1(y'') \rhd_{ad} s_1(z'')\Big) S\Big(s_0(x'''y''') \rhd_{ad} s_1(z''')\Big)$$
$$= \sum_{(x)(y)(z)} \{x', y' \rhd_{ad} z'\}\ s_0(x'') \rhd_{ad} \Big(s_1(y'') \rhd_{ad} s_1(z'')\Big)$$
$$s_0(x''') \rhd_{ad} \left(S\Big(s_0(y''') \rhd_{ad} s_1(z''')\Big)\right)$$
$$= \sum_{(x)(y)(z)} \{x', y' \rhd_{ad} z'\}\ s_0(x'') \rhd_{ad} \left(\Big(s_1(y'') \rhd_{ad} s_1(z'')\Big) S\Big(s_0(y''') \rhd_{ad} s_1(z''')\Big)\right)$$
$$= \sum_{(x)(y)(z)} \{x', y' \rhd_{ad} z'\}\ s_0(x'') \rhd_{ad} \{y'', z''\}$$
$$= \sum_{(x)(y)(z)} \{x', y' \rhd_{ad} z'\}\ s_1 s_0 d_1(x'') \rhd_{ad} \{y'', z''\}$$
$$= \sum_{(x)(y)(z)} \{x', y' \rhd_{ad} z'\}\ d_1(x'') \rhd_{\rho} \{y'', z''\} \quad \because \text{ by using (19).}$$

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