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VARIATION ON A COMPREHENSIVE THEME

Dedicated to the memory of R.F.C. (Bob) Walters.

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Abstract. The main result concerns a bicategorical factorization system on the bicategory Cat of categories and functors. Each functor $A \xrightarrow{f} B$ factors up to isomorphism as $A \xrightarrow{j} E \xrightarrow{p} B$ where j is what we call an ultimate functor and p is what we call a groupoid fibration. Every right adjoint functor is ultimate. Functors whose ultimate factor is a right adjoint are shown to have bearing on the theory of polynomial functors.

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Introduction

As an undergraduate I came across Russell [14] and was quite disturbed by the state of foundations for mathematics. The comprehension schema seemed central as a connection between mathematics and language. Then I was happy with the breakthrough I saw in the papers [10, 11, 12] of Lawvere.

The factorization described here is an old idea I have been meaning to check thoroughly and write up but only now have found a reason to do so. The reason relates to Cat as an example of a polynomic bicategory in the sense of my recent paper [18]. We want to define a property of a functor in terms of one of its factors being special in some way.

The idea for the present paper is a variant of the comprehensive factorization of a functor $A \xrightarrow{f} B$ as a composite $A \xrightarrow{j} E \xrightarrow{p} B$ where j is a final functor (in the sense

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of [13] and used by Walters and the author in [20] but sometimes called cofinal) and p is a discrete fibration. The name for the factorization system was chosen because of its relationship to the comprehension scheme for sets. This is an orthogonal factorization system in the usual sense on Cat as an ordinary category and in the enriched sense on Cat as a (strict) 2-category. Here "discrete" means, of course, that the fibres of p are sets.

Now we wish to think of Cat as a bicategory and consider whether we obtain a factorization system in a bicategorical sense when we behave totally bicategorically and close our fibrations up under composition with equivalences and ask that the pseudofibres be groupoids.

This works. Our proof models the proof of the usual comprehensive factorization as described by Verity and the author in [19]. The final functors are replaced by what we call *ultimate* functors and the discrete fibrations by what we call *groupoid fibrations*. In our application, we are concerned with functors whose ultimate factor is a right adjoint.

I am grateful to Alexander Campbell for pointing to the significantly related work of Joyal where n-final, n-fibration and homotopy factorization system are defined in the context of quasicategories; see page 170 of [8] and Sections A.6-8 of [9].

1. Groupoid fibrations

The following concept is called "strongly cartesian" by Grothendieck. These morphisms are always closed under composition (unlike those he called "cartesian").

1.1. Definition. *Let $p : E \to B$ be a functor. A morphism $\chi : e' \to e$ in E is called* cartesian *for p when the square (1.1) is a pullback for all $k \in E$.*

$$\begin{tikzcd} E(k,e') \arrow[r, "E(k,\chi)"] \arrow[d, "p"'] & E(k,e) \arrow[d, "p"] \\ B(pk,pe') \arrow[r, "B(pk,p\chi)"'] & B(pk,pe) \end{tikzcd} \tag{1.1}$$

Since any commutative square with a pair of opposite sides invertible is a pullback, we see that all invertible morphisms in E are cartesian, and, if p is fully faithful, then all morphisms of E are cartesian.

1.2. Definition. *The functor $p : E \to B$ is a* groupoid fibration *when*

(i) for all $e \in E$ and $\beta : b \to pe$ in B, there exist $\chi : e' \to e$ in E and invertible $b \cong pe'$ such that $\beta = (b \cong pe' \xrightarrow{p\chi} pe)$, and

(ii) every morphism of E is cartesian for p.

Our groupoid fibrations include all equivalences of categories and so are not necessarily fibrations in the sense of Grothendieck.

From the pullback (1.1) it follows that groupoid fibrations are conservative (that is, reflect invertibility). So their pseudofibres E_b are groupoids.

For functors $A \xrightarrow{f} C \xleftarrow{g} B$, we write f/g for the comma category (or slice) of f and g; it is the top left vertex of a universal square

$$\begin{tikzcd} f/g \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\xRightarrow{\lambda}"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{1.2}$$

in the bicategory Cat. In particular, the arrow category of E is $E^{\mathbf{2}} = 1_E/1_E = E/E$. For a functor $E \xrightarrow{p} B$ and writing $B/p = 1_B/p$, there is a canonical functor $E^{\mathbf{2}} \xrightarrow{r} B/p$ defined as follows.

$$\begin{tikzcd} E^{\mathbf{2}} \arrow[dr, dotted, "r"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "ps"'] & & \\ & B/p \arrow[r, "h"] \arrow[d, "u"'] \arrow[dr, phantom, "\xRightarrow{\lambda}"] & E \arrow[d, "p"] \\ & B \arrow[r, "1_B"'] & B \end{tikzcd} = \begin{tikzcd} E^{\mathbf{2}} \arrow[r, "t"] \arrow[d, "ps"'] \arrow[dr, phantom, "\xRightarrow{p\lambda}"] & B \arrow[d, "p"] \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$

We write $f/_{\mathrm{ps}}g$ for the full subcategory of the comma category f/g of (1.2) consisting of those objects at which the component of λ is invertible. It is called the *pseudopullback* or *isocomma category* of the cospan $A \xrightarrow{f} C \xleftarrow{g} B$; it is the top left vertex of a universal square

$$\begin{tikzcd} f/_{\mathrm{ps}}g \arrow[r, "t'"] \arrow[d, "s'"'] \arrow[dr, phantom, "\overset{\lambda'}{\underset{\cong}{\Longrightarrow}}"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{1.3}$$

in the bicategory Cat.

For functors $E \xrightarrow{p} B$ and $X \xrightarrow{b} B$, we sometimes (non-symmetrically) write E_b for the pseudopullback $p/_{\mathrm{ps}}b$ and call it the *pseudofibre* of p over b.

Here are four fairly easy observations; indeed (d) is Proposition 5.1 of [18].

1.3. Proposition.

(a) A functor $E \xrightarrow{p} B$ is a groupoid fibration if and only if the canonical $E^{\mathbf{2}} \xrightarrow{r} B/p$ is an equivalence.

(b) Suppose $E \xrightarrow{p} B$ is a groupoid fibration. A functor $F \xrightarrow{q} E$ is a groupoid fibration if and only if the composite $F \xrightarrow{q} E \xrightarrow{p} B$ is. In particular, in the case E is a groupoid, $F \xrightarrow{q} E$ is a groupoid fibration if if and only if F is a groupoid.

(c) *The pseudopullback of a groupoid fibration along any functor is a groupoid fibration. That is, if in* (1.3) *the functor g is a groupoid fibration, so too is s'.*

(d) *A functor $E \xrightarrow{p} B$ is a groupoid fibration if and only if the square*

$$\begin{tikzcd} E^{\mathbf{2}} \arrow[r, "\mathrm{cod}"] \arrow[d, "p^{\mathbf{2}}"'] & E \arrow[d, "p"] \\ B^{\mathbf{2}} \arrow[r, "\mathrm{cod}"] & B \end{tikzcd}$$

is a pseudopullback.

There is a 2-category $\mathrm{GFib}B$ of groupoid fibrations over B defined as follows: The objects are groupoid fibrations $E \xrightarrow{p} B$ over B. The hom categories are given by the pseudopullbacks:

$$\begin{tikzcd} \mathrm{GFib}B(p,q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\cong}{\Longleftarrow}"] & {[E,F]} \arrow[d, "{[E,q]}"] \\ \mathbf{1} \arrow[r, "\ulcorner p \urcorner"'] & {[E,B]} \end{tikzcd}$$

So the morphisms are triangles with a natural isomorphism therein.

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & \arrow[d, phantom, "\overset{\phi}{\underset{\cong}{\Longrightarrow}}"] & F \arrow[dl, "q"] \\ & B & \end{tikzcd} \tag{1.4}$$

We call the morphism *strict* when ϕ is an identity.

We also consider Cat/B with the same convention on its morphisms.

2. Some fully faithful right adjoint functors

$$\begin{tikzcd} & & \mathrm{Ord} \arrow[dr, "\mathrm{incl}"] & & \\ \mathrm{Set} \arrow[r, "\sim"] & \mathrm{EqR} \arrow[ur, "\mathrm{incl}"] \arrow[dr, "\mathrm{incl}"'] & & \mathrm{Cat} \arrow[r, "\mathrm{nerve}"] & {[\Delta^{\mathrm{op}}, \mathrm{Set}]} \\ & & \mathrm{Gpd} \arrow[ur, "\mathrm{incl}"'] & & \end{tikzcd} \tag{2.5}$$

All the categories in the diagram (2.5) are cartesian closed. All the functors are "closed under exponentiation". The left adjoints all preserve finite products (by Day Reflection Theorem). Our focus here is on the inclusion $\mathrm{Gpd} \xrightarrow{\mathrm{incl}} \mathrm{Cat}$ with left 2-adjoint π_1 and right adjoint Inv. The subcategory $\mathrm{Inv}A$ of the category A contains all and only the invertible morphisms of A. Note that Inv preserves cotensoring with the free living isomorphism category $\mathbf{I}$ but cotensoring with the free living morphism category $\mathbf{2}$.

2.1. Lemma. *A functor $E \xrightarrow{p} B$ is an equivalence if and only if both* $\mathrm{Inv}E \xrightarrow{\mathrm{Inv}p} \mathrm{Inv}B$ *and* $\mathrm{Inv}(E^{\mathbf{2}}) \xrightarrow{\mathrm{Inv}(p^{\mathbf{2}})} \mathrm{Inv}(B^{\mathbf{2}})$ *are equivalences.*

Proof. Only if is clear since Inv is a 2-functor when restricted to invertible natural isomorphisms. For the converse first note that surjectivity on objects up to isomorphism for p is the same as for $\mathrm{Inv}p$.

So it remains to deduce from the groupoid equivalences that p is fully faithful. Take $e, e' \in E$ and $pe \xrightarrow{\beta} pe'$ in B. Since $\mathrm{Inv}(p^{\mathbf{2}})$ is essentially surjective, there exists $e_1 \xrightarrow{\xi} e_1'$ in E and a commutative square

$$\begin{tikzcd}
pe_1 \arrow[r, "\sigma", "\cong"'] \arrow[d, "p\xi"'] & pe \arrow[d, "\beta"] \\
pe_1' \arrow[r, "\cong", "\sigma'"'] & pe' .
\end{tikzcd}$$

Since $\mathrm{Inv}p$ is full, there exist invertible $e_1 \xrightarrow{\chi} e$ and $e_1' \xrightarrow{\chi'} e'$ in E such that $p\chi = \sigma$ and $p\chi' = \sigma'$. Consequently, $\beta = p(\chi'\xi\chi^{-1})$ proving that p is full.

Since $\mathrm{Inv}p$ is faithful, the only automorphisms in E taken to identities by p are identities. We will use this special case in our proof now that p is faithful. Take $\xi, \xi' : e \to e_1$ in E with $p\xi = p\xi'$. Think of these two morphisms as objects of $E^{\mathbf{2}}$ which are taken to two equal objects $p\xi, p\xi' : pe \to pe_1$ of $B^{\mathbf{2}}$. Since $\mathrm{Inv}(p^{\mathbf{2}})$ is full, the two objects ξ and ξ' are isomorphic by an isomorphism in $E^{\mathbf{2}}$ made up of automorphisms of e and e_1 which are taken to identities by p. Since those automorphisms must be identities, we deduce that $\xi = \xi'$, as required. ∎

2.2. Lemma. *If $E \xrightarrow{p} B$ is a groupoid fibration and* $\mathrm{Inv}E \xrightarrow{\mathrm{Inv}p} \mathrm{Inv}B$ *is an equivalence then $E \xrightarrow{p} B$ is an equivalence.*

Proof. Since Inv is a right adjoint, it preserves the pseudopullback of Proposition 1.3 (d), so that both $\mathrm{Inv}p$ and $\mathrm{Inv}(p^{\mathbf{2}})$ are equivalences. The result follows by Lemma 2.1. ∎

2.3. Proposition. *The usual "Grothendieck construction" 2-functor*

$$\wr : \mathrm{Hom}(B^{\mathrm{op}}, \mathrm{Gpd}) \longrightarrow \mathrm{GFib}B$$

is a biequivalence. If $\wr(T) \simeq (E \xrightarrow{p} B)$ then Tb is equivalent to the pseudofibre E_b of p over $b \in B$.

Proof. The 2-functor $\wr$ takes pseudofunctors H to actual Grothendieck fibrations with groupoidal fibres. Every groupoid fibration is the composite of an equivalence and a Grothendieck fibration. So $\wr$ is surjective on objects up to equivalence. The 2-functor $\wr$ takes pseudonatural transformations $\Phi : H \to K$ to strict morphisms over B. Each morphism of $\mathrm{GFib}B$ with codomain a Grothendieck fibration is isomorphic (using path lifting) to a strict morphism. It follows that $\wr$ is locally essentially surjective. It is straightforward to see that 2-cells $\wr\Phi \Rightarrow \wr\Psi$ in $\mathrm{GFib}B$ between commutative-triangle morphisms in the image of $\wr$ are in the image of $\wr$ for a unique 2-cell in $\mathrm{Hom}(B^{\mathrm{op}}, \mathrm{Gpd})$. So $\wr$ is locally fully faithful. ∎

The result of applying $\mathrm{Hom}(B^{\mathrm{op}}, -)$ to the 2-adjunction

$$\mathrm{Cat} \underset{\mathrm{incl}}{\overset{\pi_1}{\rightleftarrows}} \mathrm{Gpd}$$

transports to a biadjunction

$$\mathrm{Fib}B \underset{\mathrm{incl}}{\overset{\pi_{1B}}{\rightleftarrows}} \mathrm{GFib}B$$

via the biequivalences

$$\mathrm{Hom}(B^{\mathrm{op}}, \mathrm{Gpd}) \xrightarrow{\sim} \mathrm{GFib}B \quad \text{and} \quad \mathrm{Hom}(B^{\mathrm{op}}, \mathrm{Cat}) \xrightarrow{\sim} \mathrm{Fib}B\ .$$

2.4. Remark. The inclusion 2-functor $\mathrm{GFib}B \hookrightarrow \mathrm{Cat}/B$ is fully faithful with a left biadjoint whose value at the object $A \xrightarrow{f} B$ is the groupoid fibration $\pi_{1B}(B/f \xrightarrow{\mathrm{dom}} B)$ which corresponds to the pseudofunctor $B^{\mathrm{op}} \to \mathrm{Gpd}$ taking $b \in B$ to $\pi_1(b/f)$.

The construction of $\pi_1 A$ by generators and relations is awkward to work with; instead we use the following universal property of the coinverter construction. Write $[A, X]_{\cong}$ for the full subcategory of $[A, X]$ consisting of those functors $f : A \to X$ which invert all the morphisms of A. The adjunction unit $A \to \pi_1 A$ induces an isomorphism

$$[\pi_1 A, X] \cong [A, X]_{\cong}$$

for all categories X (not just groupoids).

3. Ultimate functors

3.1. Definition. *A functor $j : A \to B$ is called* ultimate *when, for all objects $b \in B$, the fundamental groupoid $\pi_1(b/j)$ of the comma category b/j is equivalent to the terminal groupoid:*

$$\pi_1(b/j) \simeq \mathbf{1}\ .$$

3.2. Proposition. *Every right adjoint functor is ultimate.*

Proof. If $k \dashv j : A \to B$ then $b/j \simeq kb/A \to \mathbf{1}$ has a left adjoint owing to the initial object 1_{kb} of kb/A. Applying the 2-functor π_1 to the adjunction yields an adjunction between groupoids. ∎

3.3. Proposition. *Ultimate functors are taken by π_1 to equivalences.*

Proof. Let $j : A \to B$ be ultimate. We must prove $\pi_1 A \xrightarrow{\pi_1 j} \pi_1 B$ is an equivalence. What we prove is that, for any category X, if each diagonal functor $X \xrightarrow{\delta_b} [b/j, X]_{\cong}$ is an equivalence then $[B, X]_{\cong} \xrightarrow{[j,1]_{\cong}} [A, X]_{\cong}$ is an equivalence. Since δ_b is 2-natural in $b \in B$, any choice γ_b of adjoint equivalence is pseudonatural: choose also counit $\varepsilon_b : \gamma_b \delta_b \overset{\cong}{\Longrightarrow} 1_X$

and unit $\eta_b : 1_A \overset{\cong}{\Longrightarrow} \delta_b\gamma_b$. We will show that we have an inverse equivalence θ for $[j,1]_{\cong}$ defined by

$$\begin{array}{ccc} \theta(f)b & = & \gamma_b(b/j \xrightarrow{\text{cod}} A \xrightarrow{f} X) \\ {\scriptstyle \theta(f)\beta}\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma_{\beta,f\text{cod}}} \\ \theta(f)b' & = & \gamma_b(b'/j \xrightarrow{\beta/j} b/j \xrightarrow{\text{cod}} A \xrightarrow{f} X)\ . \end{array}$$

For $g \in [B,X]_{\cong}$, we have isomorphisms

$$\begin{array}{rcl} (\theta[j,1]_{\cong}g)b & = & \gamma_b(b/j \xrightarrow{\text{cod}} A \xrightarrow{j} B \xrightarrow{g} X) \\ & \cong & \gamma_b(b/j \xrightarrow{!} 1 \xrightarrow{b} B \xrightarrow{g} X) \\ & \cong & \gamma_b\delta_b(gb) \\ & \xRightarrow{\varepsilon_b\ \cong} & gb \end{array}$$

naturally in g and b, while, for $f \in [A,X]_{\cong}$, we have isomorphisms

$$\begin{array}{rcl} ([j,1]_{\cong}\theta)(f)a & = & \theta(f)ja \\ & \cong & \gamma_{ja}(ja/j \xrightarrow{\text{cod}} A \xrightarrow{f} X) \\ & \xRightarrow{\eta^{-1}\ \cong} & (f\text{cod})(ja \xrightarrow{1_{ja}} ja, a) \\ & = & fa \end{array}$$

naturally in f and a. ∎

3.4. Proposition. *A functor is ultimate if and only if its pseudopullback along any (groupoid) opfibration is taken by π_1 to an equivalence.*

Proof. The pseudopullback $P \xrightarrow{\bar{j}} X$ of $A \xrightarrow{j} B$ along an opfibration $F \xrightarrow{q} B$ has $x/\bar{j} \simeq qx/j$; so $\bar{j}$ is ultimate if j is. So π_1 takes $\bar{j}$ to an equivalence by Proposition 3.3. For the rest, in the pseudopullback

$$\begin{array}{ccc} b/j & \longrightarrow & b/B \\ {\scriptstyle \text{cod}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{cod}} \\ A & \xrightarrow{\quad j \quad} & B\ , \end{array}$$

note that b/B has an initial object and cod is a groupoid opfibration. ∎

3.5. Proposition. *Every coinverter (localization) is ultimate.*

Proof. Pullback along an opfibration has a right adjoint so coinverters are taken to coinverters. Also, π_1 takes coinverters to isomorphisms since it is a left adjoint and all 2-cells in Gpd are already invertible. Proposition 3.4 applies. ∎

3.6. Proposition. *Suppose $A \xrightarrow{j} B$ is ultimate. A functor $B \xrightarrow{k} C$ is ultimate if and only if the composite $A \xrightarrow{j} B \xrightarrow{k} C$ is ultimate.*

Proof. Look at the pasting

$$\begin{tikzcd} Q \arrow[r, "j'"] \arrow[d, "q''"'] & P \arrow[r, "k'"] \arrow[d, "q'"] & F \arrow[d, "q"] \\ A \arrow[r, "j"'] & B \arrow[r, "k"'] & C \end{tikzcd}$$

of two pullbacks with q a groupoid fibration. Since j is ultimate, j' is equivalenced by π_1. So $k'j'$ is equivalenced by π_1 if and only if k' is. ∎

3.7. Lemma. *If $E \xrightarrow{p} B$ is a groupoid fibration and $X \xrightarrow{b} B$ is a functor from a groupoid X then the composite $E_b \to b/p \to \pi_1(b/p)$ is an equivalence.*

Proof. By Proposition 1.3 (b), (c), the pseudofibre E_b is a groupoid and so is invariant under π_1. Also, $E_b \to b/p$ is a left adjoint and so taken to an equivalence by π_1. ∎

3.8. Proposition. *Ultimate groupoid fibrations $E \xrightarrow{p} B$ are equivalences.*

Proof. Let $\mathrm{Inv}B \xrightarrow{b} B$ be the inclusion. Since p is ultimate, the pullback $b/p \xrightarrow{g} b/B$ of p along $b/B \xrightarrow{\mathrm{cod}} B$ is taken to an equivalence by π_1. By Lemma 3.7, $\pi_1(g)$ is equivalent to $\mathrm{Inv}(p)$. By Lemma 2.2, since p is a groupoid fibration with $\mathrm{Inv}(p)$ an equivalence, p is an equivalence. ∎

4. Bicategorical factorization systems

The concept of factorization system in a bicategory is not new; for example, see [2, 3]. Before giving the definition, we revise the bicategorical variant of pullback.

4.1. Definition. *A square*

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\sigma}{\underset{\cong}{\Rightarrow}}"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{4.6}$$

in a bicategory $\mathscr{K}$ is called a bipullback *of the cospan $A \xrightarrow{f} C \xleftarrow{g} B$ when, for all objects $K \in \mathscr{K}$, the functor*

$$\mathscr{K}(K, W) \xrightarrow{(p,\sigma,q)} \mathscr{K}(K, f)/_{\mathrm{ps}}\mathscr{K}(K, g) \ ,$$

obtained from the universal property of the pseudopullback, is an equivalence.

4.2. Remark. In the square (4.6), if g and p are groupoid fibrations, then the square is a bipullback if and only if

$$\mathrm{Inv}(\mathscr{K}(K,W)) \xrightarrow{\mathrm{Inv}(p,\sigma,q)} \mathrm{Inv}(\mathscr{K}(K,f)/_{\mathrm{ps}}\mathscr{K}(K,g))$$

is an equivalence of groupoids. This is because Proposition 1.3 (b) and (c) imply (p,σ,q) is a groupoid fibration so that Lemma 2.2 applies.

A *factorization system on a bicategory* $\mathscr{K}$ consists of a pair $(\mathscr{E},\mathscr{M})$ of sets $\mathscr{E}$ and $\mathscr{M}$ of morphisms of $\mathscr{K}$ satisfying:

FS0. if $f \cong mw$ with $m \in \mathscr{M}$ and w an equivalence then $f \in \mathscr{M}$, while if $f \cong we$ with $e \in \mathscr{E}$ and w an equivalence then $f \in \mathscr{E}$;

FS1. for all $X \xrightarrow{e} Y \in \mathscr{E}$ and $A \xrightarrow{m} B \in \mathscr{M}$, the diagram

$$\begin{tikzcd} \mathscr{K}(Y,A) \arrow[r, "\mathscr{K}(e,A)"] \arrow[d, "\mathscr{K}(Y,m)"'] \arrow[dr, phantom, "\cong"] & \mathscr{K}(X,A) \arrow[d, "\mathscr{K}(X,m)"] \\ \mathscr{K}(Y,B) \arrow[r, "\mathscr{K}(e,B)"] & \mathscr{K}(X,B) \end{tikzcd} \tag{4.7}$$

(in which the isomorphism has components of the associativity constraints for $\mathscr{K}$) is a bipullback;

FS2. every morphism f factorizes $f \cong m \circ e$ with $e \in \mathscr{E}$ and $m \in \mathscr{M}$.

It follows that $\mathscr{E}$ and $\mathscr{M}$ are closed under composition and their intersection consists of precisely the equivalences. Moreover, in the square (4.7), the morphism m is in $\mathscr{M}$ if the square is a bipullback for all $e \in \mathscr{E}$, and dually. Also note that, if all morphisms in $\mathscr{M}$ are groupoid fibrations then Remark 4.2 applies to simplify the bipullback verification for FS1.

5. Main theorem

5.1. Theorem. *Ultimate functors and groupoid fibrations form a bicategorical factorization system on* Cat. *So every functor $f : A \to B$ factors pseudofunctorially as $f \cong (A \xrightarrow{j} E \xrightarrow{p} B)$ with j ultimate and p a groupoid fibration.*

Proof. FS0 is obvious. For FS2 construct the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B/f \arrow[r, "n"] \arrow[d, "\mathrm{dom}"] & E \arrow[d, "p"] \\ B \arrow[r, "1"] & B \arrow[r, "1"] & B \end{tikzcd}$$

where $(E \xrightarrow{p} B) = \pi_{1B}(B/f \xrightarrow{\mathrm{dom}} B)$, the squares commute up to isomorphism, i has a left adjoint cod, and n is a coinverter.

It remains to prove FS1. By Remark 4.2, we must prove that, for any groupoid fibration $E \xrightarrow{p} C$ and any ultimate functor $A \xrightarrow{j} B$, the functor

$$([j, E], [B, p]) : [B, E] \longrightarrow [A, p]/_{\mathrm{ps}}[j, C]$$

is taken to an equivalence of groupoids by Inv. By Remark 2.4, the value of the left biadjoint to $\mathrm{GFib}B \hookrightarrow \mathrm{Cat}/B$ at the ultimate functor $A \xrightarrow{j} B$ is equivalent to $B \xrightarrow{1_B} B$. So every morphism $j \xrightarrow{(f,\phi)} q$ over B with q a groupoid fibration factors up to isomorphism as

$$\begin{tikzcd}
j \arrow[rr, "{(j,1_j)}"] \arrow[dr, "{(f,\phi)}"'] & \arrow[d, phantom, "\overset{\sigma}{\underset{\cong}{\Rightarrow}}"] & 1_B \arrow[dl, "{(w,\psi)}"] \\
& q &
\end{tikzcd} \tag{5.8}$$

uniquely up to a unique isomorphism. In this, we have $f \overset{\sigma}{\Rightarrow} wj$ and $1_B \overset{\psi}{\Rightarrow} qw$ such that $\psi j = (j \overset{\phi}{\Rightarrow} qf \overset{q\sigma}{\Longrightarrow} qwj)$. Take any object (u, γ, v) of $[A, p]/_{\mathrm{ps}}[j, C]$; it consists of functors $A \xrightarrow{u} E, B \xrightarrow{v} C$ and an invertible natural transformation $pu \overset{\gamma}{\Rightarrow} vj$. By the universal property of the pseudopullback $p/_{\mathrm{ps}}v$, the isomorphism γ is equal to the pasted composite

$$\begin{tikzcd}
A \arrow[dr, dotted, "u'"] \arrow[drr, bend left, "j"] \arrow[ddr, bend right, "u"'] & & \\
& p/_{\mathrm{ps}}v \arrow[r, "t'"] \arrow[d, "s'"'] \arrow[dr, phantom, "\overset{\lambda'}{\underset{\cong}{\Rightarrow}}"] & B \arrow[d, "v"] \\
& E \arrow[r, "p"'] & C\,.
\end{tikzcd}$$

By Proposition 1.3, $p/_{\mathrm{ps}}v \xrightarrow{t'} B$ is a groupoid fibration. We can apply (5.8) with $f = u'$, $q = t'$ and ϕ the identity of $j = t'u'$ to obtain $u' \overset{\sigma}{\Rightarrow} wj$ and $1_B \overset{\psi}{\Rightarrow} t'w$ such that $\psi j = (j = t'u' \overset{t'\sigma}{\Longrightarrow} t'wj)$ uniquely up to a unique isomorphism of (w, ψ, σ). This gives us $w' = s'w \in [B, E]$ and an isomorphism $(u, \gamma, v) \cong (w'j, 1_{pw'j}, pw') = ([j, E], [B, p])w'$ determined by the isomorphisms

$$u = s'u' \overset{s'\sigma}{\Longrightarrow} s'wj = w'j \quad \text{and} \quad v \overset{v\psi}{\Longrightarrow} vt'w = vt'w \overset{(\lambda' w)^{-1}}{\Longrightarrow} ps'w = pw'\ .$$

This proves that the functor $\mathrm{Inv}([j, E], [B, p])$ is essentially surjective (that is, surjective on objects up to isomorphism). Now suppose we also have $h \in [B, E]$ and an isomorphism

$$(\xi, \zeta) : ([j, E], [B, p])h \cong (w'j, 1_{pw'j}, pw')$$

which means we have invertible $hj \overset{\xi}{\Rightarrow} s'wj$ and $ph \overset{\zeta}{\Rightarrow} ps'w$ such that $p\xi = \zeta j$. By the universal property of the pseudopullback, there exists a unique $k : B \to p/_{\mathrm{ps}v}$ such that $s'k = h$, $t'k = 1_B$ and $\lambda' k = (ph \overset{\zeta}{\Rightarrow} ps'w \xRightarrow{\lambda' w} vt'w \xRightarrow{v\psi^{-1}} v)$, and there also exists a unique invertible $\tau : kj \Rightarrow wj$ such that $\xi = (hj = s'kj \xRightarrow{s'\tau} s'wj)$ and $\psi j = (t'kj \xRightarrow{t'\tau} t'wj)$. So we have

$$\begin{tikzcd} j \arrow[rr, "{(j,1_j)}"] \arrow[dr, "{(u',1_j)}"'] \arrow[rr, phantom, shift right=4, "\overset{\tau^{-1}\sigma}{\underset{\cong}{\Rrightarrow}}"] & & 1_B \arrow[dl, "{(k,1_{1_B})}"] \\ & t' & \end{tikzcd}$$

which allows us to use the uniqueness of (w, ψ, σ) to obtain a unique isomorphism $\kappa : k \Rightarrow w$ such that $\kappa j = \tau$ and $t'\kappa = \psi$. Then $\kappa' = s'\kappa : h \Rightarrow w'$ is such that $\kappa' j = s'\kappa j = s'\tau = \xi$ and $p\kappa' = ps'\kappa = (\lambda' w)^{-1}(v\psi)(\lambda' k) = \zeta$. Hence $\mathrm{Inv}([j, E], [B, p])$ is full and it remains to prove it faithful. So suppose we have an invertible $\delta' : h \Rightarrow w'$ such that $\delta' j = \xi$ and $p\delta' = \zeta = (\lambda' w)^{-1}(v\psi)(\lambda' k)$. The universal property of pseudopullback implies there exists $\delta : k \Rightarrow w$ such that $s'\delta = \delta'$ and $t'\delta = \psi$, and implies we can deduce that $\delta j = \tau$ from the equations $s'\delta j = \xi$ and $t'\delta j = \psi j = t'\tau$. By the uniqueness of κ, we have $\delta = \kappa$ and hence $\delta' = \kappa'$, as required. ∎

6. Other possible variants

It is possible that the factorization carries through for $(\infty, 1)$-categories (also called quasicategories or weak Kan complexes); see [8, 9]. For the case of the tricategory $(2, 1)$-Cat whose objects are bicategories with all 2-cells invertible, a basic ingredient would be the triadjunction

$$\begin{tikzcd} (2,1)\text{-Cat} \arrow[r, shift left=2, "\pi_1"] \arrow[r, phantom, "\perp"] & (2,0)\text{-Cat} \arrow[l, shift left=2, "\mathrm{incl}"] \end{tikzcd}$$

where $(2, 0)$-Cat is the subtricategory of $(2, 1)$-Cat with all morphisms equivalences. There is an obvious core providing a right triadjoint too. This requires the bumping up to factorization systems on tricategories. And, after all, as yet my application only needs the Cat case.

There is presumably also a version of the (ultimate, groupoid fibration) for categories internal to a category $\mathscr{E}$ as done in [19] for the usual comprehensive factorization.

Another direction concerns the laxer hierarchy of comprehension schema proposed by John Gray; see [6, 7]. What kinds of factorization do they provide?

7. Application to polynomials

In this section, we use our factorization to understand the implications of the paper [18] for polynomials in Cat as a bicategory.

A morphism $p : E \to B$ in a bicategory is called a *groupoid fibration* when, for all objects $A \in \mathscr{M}$, the functor $\mathscr{M}(A, p) : \mathscr{M}(A, E) \to \mathscr{M}(A, B)$ is a groupoid fibration as per Definition 1.2.

A morphism $n : Y \to Z$ is called a *right lifter* when, for all $u : K \to Z$, there exists a right lifting of of u through n (in the sense of [21]).

Recall from [18] that a bicategory $\mathscr{M}$ with bipullbacks is always calibrated by the groupoid fibrations as the neat morphisms; that is, such a bicategory is polynomic. This allows for the construction of a bicategory of "polynomials" in $\mathscr{M}$. Indeed, Definition 8.2 of [18] means for this situation that a *polynomial* (m, S, p) from X to Y in $\mathscr{M}$ is a span

$$X \xleftarrow{m} S \xrightarrow{p} Y$$

in $\mathscr{M}$ with m a right lifter and p a groupoid fibration. To have a more explicit description we need to identify the right lifters in the given $\mathscr{M}$.

7.1. Proposition. *A functor is a right lifter in* Cat *if and only if it is a left adjoint.*

Proof. Left adjoints in any bicategory are right lifters since the lifting is given by composing with the right adjoint. Conversely, suppose the functor $Y \xrightarrow{n} Z$ is a right lifter. A right lift $1 \xrightarrow{n_!(z)} Y$ for each object $1 \xrightarrow{z} Z$ of Z gives the components $nn_!(z) \xrightarrow{\epsilon_z} z$ of the counit of an adjunction $n \dashv n_!$; as in any book introducing adjoint functors, we know that the universal property of right lifter allows us to define $n_!$ on morphisms and so on. ∎

In order to distinguish polynomials in the polynomic bicategory Cat from polynomials in Cat, in the sense of Weber [23], as a category with pullbacks, I use the term *abstract polynomial* for the former; that is, it is a span

$$A \xleftarrow{j_*} E \xrightarrow{p} B$$

of functors, where p is a groupoid fibration and $j_* \dashv j$. Recall from Proposition 3.2 that right adjoints are ultimate.

A functor $f : A \to B$ is an *abstract polynomial functor* when, in its factorization

$$f \cong (A \xrightarrow{j} E \xrightarrow{p} B)$$

as per Theorem 5.1, the ultimate functor j is a right adjoint.

The next result follows from the work in [18]; for convenience, we will include a direct proof.

7.2. Proposition. *Abstract polynomial functors compose.*

Proof. Take $A \xrightarrow{j} E \xrightarrow{p} B \xrightarrow{k} F \xrightarrow{q} C$ with $j_* \dashv j$, $k_* \dashv k$ and with p, q groupoid fibrations. Form the pseudopullback

$$\begin{tikzcd} P \arrow[r, "p'"] \arrow[d, "k'_*"'] \arrow[dr, phantom, "\overset{\theta}{\underset{\cong}{\Leftarrow}}"] & F \arrow[d, "k_*"] \\ E \arrow[r, "p"'] & B \end{tikzcd} \tag{7.9}$$

to obtain the required "distributive law". One easily verifies there exists $k'_* \dashv k'$, p' is a groupoid fibration and the Chevalley-Beck condition (as recalled on page 150 of [15])

$$p' \circ k' \cong k \circ p$$

holds. So $q \circ k \circ p \circ j \cong q \circ p' \circ k' \circ j$ where $q \circ p'$ is a groupoid fibration and $k' \circ j$ is a right adjoint. ∎

Write $\mathrm{Cat}_{\mathrm{apf}}$ for the subcategory of Cat obtained by restricting the morphisms to abstract polynomial functors.

The next result is essentially Proposition 8.6 of [18].

7.3. PROPOSITION. *If the bicategory $\mathscr{M}$ is calibrated then, for each $K \in \mathscr{M}$, there is a pseudofunctor $\mathbb{H}_K : \mathrm{Poly}\mathscr{M} \longrightarrow \mathrm{Cat}_{\mathrm{apf}}$ taking the polynomial $X \xleftarrow{m} S \xrightarrow{p} Y$ to the abstract polynomial functor which is the composite*

$$\mathscr{M}(K, X) \xrightarrow{\mathrm{rif}(m,-)} \mathscr{M}(K, S) \xrightarrow{\mathscr{M}(K,p)} \mathscr{M}(K, Y)$$

in Cat.

7.4. COROLLARY. *The pseudofunctor $\mathbb{H}_{\mathbf{1}} : \mathrm{PolyCat} \longrightarrow \mathrm{Cat}_{\mathrm{apf}}$, taking each abstract polynomial $A \xleftarrow{j_*} E \xrightarrow{p} B$ its associated abstract polynomial functor $A \xrightarrow{j} E \xrightarrow{p} B$ with $j_* \dashv j$, is a biequivalence.*

7.5. REMARK. After my talk on this topic in the Workshop on Polynomial Functors `https://topos.site/p-func-2021-workshop/`, Paul Taylor kindly pointed out his 1988 preprint [22] in which he distinguished parametric (or local) right adjoint functors with motivation from proof theory and consequently calling them *stable* functors. His *trace* factorization for such a functor is a right adjoint functor followed by a groupoid fibration. I am grateful to Clemens Berger for observing that the groupoid fibrations so arising are a resticted class: their pseudofibres are coproducts of codiscrete (chaotic) categories. However, it does show that every parametric right adjoint functor provides an example of an abstract polynomial functor. We shall conclude by explaining this.

There are two notions closely related to abstract polynomial functors (apf).

- apf *Parametric right adjoints* in the sense of [17] are functors $A \xrightarrow{f} B$ whose comprehensive factorization $(A \xrightarrow{f} B) = (A \xrightarrow{j} E \xrightarrow{p} B)$, where j is final and p is a discrete fibration, is such that E has a terminal object and j is a right adjoint.
- lra *Local right adjoints* are functors $A \xrightarrow{f} B$ such that, for each $a \in A$, the functor $A/a \xrightarrow{f_a} B/fa$, taking data over a to their value under f, is a right adjoint.

The following lemma is easy.

7.6. LEMMA. *Groupoid fibrations are local equivalences and so local right adjoints. Right adjoints are local right adjoints.*

7.7. Proposition. *Every parametric right adjoint is an abstract polynomial functor. Every abstract polynomial functor is a local right adjoint. For functors whose domain category admits a terminal object, the three properties coincide.*

Proof. If we have $A \xrightarrow{j} E \xrightarrow{p} B$ with j a right adjoint and p a discrete fibration then this is also the (ultimate, groupoid fibration)-factorization. This proves the first sentence. The second sentence follows from Lemma 7.6 and the obvious fact that local right adjoints compose. If $A \xrightarrow{f} B$ is a local right adjoint and A has a terminal object 1 then $A/1 \xrightarrow{f_1} B/f1$ is a right adjoint, so $f = (A \cong A/1 \xrightarrow{f_1} B/f1 \xrightarrow{\mathrm{dom}} B)$ is a parametric right adjoint. ∎

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