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COHERENT NERVES FOR HIGHER QUASICATEGORIES

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ABSTRACT. We introduce, for $\mathcal{C}$ a regular Cartesian Reedy category a model category whose fibrant objects are an analogue of quasicategories enriched in simplicial presheaves on $\mathcal{C}$. We then develop a coherent realization and nerve for this model structure and demonstrate that these give a Quillen equivalence, in particular recovering the classical one in the process. We then demonstrate that this equivalence descends to any Cartesian closed left Bousfield localization in a natural way. As an application, we demonstrate a version of Yoneda's lemma for quasicategories enriched in any such Cartesian closed localization.

Contents

Introduction

In his unpublished thesis [Our10], David Oury introduced machinery to give a novel proof that his constructed model structure on Θ_2-sets is Cartesian closed. Around the same time, in [Rez10], Charles Rezk constructed a model structure on Θ_n-spaces, that, in the case $n = 2$, was expected to be Quillen equivalent to a model structure on the category of Θ_2-sets proposed by Joyal and Cisinski (later constructed by Ara in [Ara14] and by the author in an unpublished preprint [Gin12]) that coincides Oury's model structure. However, Rezk's construction allows us to model weak enrichment in a much larger class of model categories, namely Cartesian closed model categories whose underlying categories are simplicial presheaves on a small category $\mathcal{C}$ satisfying some tame restrictions.

Bergner and Rezk, in [BR13] and [BR18], also showed by means of a zig-zag of Quillen equivalences that the category of Θ_n-spaces equipped with Rezk's model structure models the same homotopy theory as the model category of $\mathrm{Psh}_\Delta(\Theta_{n-1})$-enriched categories,

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equipped with the Bergner-Lurie model structure for categories enriched in Θ_{n-1}-spaces equipped with Rezk's model structure. Because the equivalence is indirect, however, many of the ideas from Lurie's work on $(\infty, 1)$-categories cannot be adapted in a straightforward manner, specifically his construction of the Yoneda embedding and his proof of Yoneda's lemma in [Lur09]. In order to rectify this problem, we split the problem up into two parts:

We first introduce using a novel model structure on Θ_n-sets (or more generally $\Theta[\mathcal{C}]$-sets for an appropriate $\mathcal{C}$) that emerges naturally as a hybrid of Oury's model structure on Θ_2-sets and Rezk's model structure on $\Theta[\mathcal{C}]$-spaces. Specifically, we use Oury's machinery to construct a model structure on $\Theta[\mathcal{C}]$-sets that models weak enrichment in simplicial presheaves on $\mathcal{C}$. We then compare this model structure with an intermediate model structure of Rezk, demonstrating they are Quillen equivalent. As a result of this equivalence, we can later use results of Rezk [Rez10] to localize this model structure 'hom-wise' with respect to what Rezk calls a Cartesian presentation on $\mathcal{C}$, which is again equivalent to Rezk's localized model structure by merit of Cisinski's results on simplicial completion (see Appendix or [Cis06]). Like Rezk's model structure, ours is also Cartesian monoidal as a model category. Since we prove many of these theorems using machinery developed by Oury in the unpublished portion of his thesis [Our10], we also provide full proofs of all of his relevant results, but in our more general setting.

We then construct a version of the coherent realization and nerve adjunction between $\Theta[\mathcal{C}]$-sets and categories enriched in simplicial presheaves on $\mathcal{C}$, which reduce to the classical ones in the case where we take $\mathcal{C} = *$ the terminal category. We then demonstrate that this adjunction is a Quillen equivalence between appropriate model structures using an enhanced version of Dugger and Spivak's calculus of necklaces developed in [DS11a] and [DS11b].

Our direct result is strictly stronger than the result of Bergner and Rezk because it allows us to account at the very least for the new case $\Theta = \Theta_\omega$ (which happens to satisfy all of our constraints on $\mathcal{C}$), which the Bergner-Rezk approach could not handle, since one of the categories appearing in the zig-zag (the height-n analogue of Segal categories) only makes sense for $\mathcal{C} = \Theta_n$ for n finite. Their approach goes through rigidification results for homotopy-coherent simplicial models of algebraic theories due to Badzioch (see [BR18, Section 5]). Moreover, all of our Quillen equivalences point in the right direction to generalize Lurie's construction of the Yoneda embedding and his proof of Yoneda's lemma.

The paper is organized into the following chapters:

Formal $\mathcal{C}$-quasicategories. In the first chapter, we apply a general construction to define what we call *labeled simplicial sets* with respect to a monoidal category $\mathcal{V}$. We then specialize to the case where $\mathcal{V}$ is the Cartesian monoidal category of presheaves of sets on a small category $\mathcal{C}$, which we additionally require to be a special kind of Reedy category that axiomatizes a form of the Eilenberg-Zilber shuffle decomposition for products of simplicial sets. We then define $\Theta[\mathcal{C}]$ to be the full subcategory of the labeled simplicial sets whose underlying simplicial sets are simplices and whose edges are all labeled by representable presheaves on $\mathcal{C}$.

We define the category of *$\mathcal{C}$-cellular sets* to be the category of presheaves of sets on this category. We then apply machinery of Cisinski and Oury to construct the *horizontal Joyal model structure* on the category of $\mathcal{C}$-cellular sets that has many of the familiar nice properties of the Joyal model structure. We call the fibrant objects of this category the *formal $\mathcal{C}$-quasicategories*.

We direct the attention of the reader to §1.6, which proves that the model structure is Cartesian monoidal as well as §1.7, where we prove a useful equivalence with an analogous model structure constructed by Rezk.

The chapter culminates with a key technical result that gives a characterization of the fibrant objects by a simple lifting property and the fibrations between them as the *isofibrations*, namely the horizontal inner fibrations that have the right lifting property with respect to the inclusion of a vertex into a freestanding isomorphism, extending an important theorem of Joyal to this setting.

The Coherent Nerve, Horizontal case. In the second chapter, we define an extension of Lurie's coherent realization functor $\mathfrak{C}_\Delta$ to our setting. We leverage the equivalence between $\mathcal{C}$-indexed simplicially-enriched categories with a constant set of objects and $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched categories to induce this functor pointwise from $\mathfrak{C}_\Delta$. We then work to give an explicit calculation of this functor on representables and more generally on $\widehat{\mathcal{C}}$-labeled simplices.

We use the pointwise characterization of this realization to straightforwardly extend the results of Dugger and Spivak [DS11a] on alternative realizations to our setting, while on the other hand, we make use of the explicit characterization to demonstrate directly that the coherent realization and its right adjoint, the coherent nerve, form a Quillen pair

$$\mathfrak{C}\colon \widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}} \rightleftarrows \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}} : \mathfrak{N}.$$

For the next step in this chapter, we introduce cosimplicial resolutions in order to compute mapping objects for formal $\mathcal{C}$-quasicategories. We extend ideas from [DS11b] to demonstrate that the coherent nerve and realization actually specify a Quillen equivalence.

The Coherent Nerve, Local case. In the third and final chapter, we give a way to perform a left-Bousfield localization of the horizontal Joyal model structure with respect to Cartesian presentations of the form $(\mathcal{C}, \mathscr{S})$ (though still under the hypothesis that $\mathcal{C}$ is regular Cartesian Reedy). The local objects are exactly the formal $\mathcal{C}$-quasicategories whose mapping objects are $\mathscr{S}$-local. Using our comparison theorem with Rezk's model structure, we can apply his result to show that this model structure is again Cartesian monoidal.

To prove the main result of the paper, we use the compatibility of the coherent realization and nerve with the formation of mapping objects to demonstrate that they remain Quillen equivalences after simultaneous localization.

As a corollary of the main result, we apply a theorem of Lurie to construct a Yoneda embedding. We then demonstrate that it is fully faithful and also prove Yoneda's lemma, which we then leverage to define representability.

Appendix: Recollections on Cisinski Theory. Throughout this paper, we will make extensive use of the extremely elegant theory of Cisinski from [Cis06], which allows for the construction and description of model structures on presheaf categories in which the cofibrations are exactly the monomorphisms. As such, we will recall several key results:

In the first section of the appendix, we will need to recall how to generate Cisinski model structures by anodyne closure with respect to a cellular model, a separating cylinder functor, and a small set of injective maps of presheaves. We will also demonstrate how this plays into the theory of Cartesian monoidal Cisinski model categories. A theorem of Cisinski demonstrates that taking an empty set of generating anodynes together with an injective separating interval object generates the minimal Cisinski model structure on a presheaf category. In particular, this will always exist by taking this object to be the subobject classifier.

We will then recall how the existence of a minimal Cisinski model structure gives rise to the theory of localizers by applying left Bousfield-localization. This theory generalizes the theory of presentation by generating anodynes. In particular, given any small set of maps in a presheaf category, there is a closure of this set such that it generates a minimal Cisinski model structure in which those maps are weak equivalences. Since localizers are defined by a closure operation and determine Cisinski model structures up to identity, it will be clear that Cisinski model structures arrange themselves into the structure of a poset ordered by inclusion of their localizers.

In the first appendix, we will recall Cisinski's theory of simplicial completion and discrete localizers. In particular, it is a theorem of [Cis06] that there is a Galois connection called the simplicial completion between localizers on $\widehat{\mathcal{A}}$ and localizers on $\widehat{\mathcal{A}\times\Delta}$, which restricts to a bijection above the simplicial completion of the minimal localizer on $\widehat{\mathcal{A}}$. Localizers belonging to the image of the simplicial completion are called *discrete*.

We will then describe the tricky relationship between discrete localizers on $\widehat{\mathcal{A}\times\Delta}$ and Dugger presentations on $\mathcal{A}$, which are the localizations of the injective model structure on simplicial presheaves on $\mathcal{A}$. In particular, we will recall the theorem of Cisinski that the simplicial completion of a localizer is also a Dugger presentation if and only if the localizer is *regular*, which is an important property that ensures that every presheaf is canonically the *homotopy colimit* of the functor given by projection from its category of elements. An important fact is that every localizer admits a *regular completion*, which is canonically generated by the regular completion of the minimal localizer together with any localizer. It follows from this fact that the Galois connection also restricts to a bijection between discrete localizers admitting a Dugger presentation on $\widehat{\mathcal{A}\times\Delta}$ and regular localizers on $\widehat{\mathcal{A}}$.

The first appendix concludes with a short digression into the topic of chapter 8 of [Cis06], the theory of skeletal categories. These are generalized Reedy categories with canonical cellular models, which, under certain combinatorial hypotheses, have a minimal localizer that is already regular. These categories will be important in the rest of the paper, as they greatly simplify the generation of the model structures in which we are interested.

In the second much shorter appendix, we prove some useful results about parametrized category theory. In particular, we give a criterion for the total category of the Kan extension of a Grothendieck fibration to be complete and cocomplete. This criterion is used to make sense of the category of labeled simplicial sets.

Questions. We suspect that the arguments here can be generalized to more general small categories $\mathcal{C}$ by replacing the boundary inclusions of $\mathcal{C}$ with a more general cellular model and by replacing the horizontal Joyal model structure with its regular completion (see A.3.3). All of our motivating examples satisfy the requirement that $\mathcal{C}$ is regular Cartesian Reedy, so we haven't attempted to work in this generality.

Looking forward. A major challenge in the theory of higher categories is the problem of coherence, that is to say, defining functors and appropriately-natural transformations valued in a higher category of higher categories. It was observed early as the 1970 that a powerful way to deal with coherence problems even for functors from a 1-category to the 2-category of categories was to perform a rectification of that theory to the theory of Cartesian fibrations.

Lurie extended this point of view to the theory of $(\infty, 1)$-categories for two reasons: Less crucially, one can use the theory of Cartesian fibrations to work with $(\infty, 2)$-categorical notions without ever actually giving a definition of $(\infty, 2)$-category. Much more important than this shortcut, however, is the fact that Cartesian fibrations greatly simplify coherence problems.

Unfortunately, our paper does not even begin to scratch the surface of the fibrational point of view, and as a consequence, it is much more difficult to work in our setting in light of the consequent coherence problems. We expect that to understand the fibrational point of view, attempts will have to be made to understand higher-categorical lax structure. Lax structure is better-understood in the strict setting due to recent work of Ara, Maltsiniotis, and Steiner, but all attempts thusfar to extend these highly combinatorial results to the theory of weak higher categories have produced no tangible results. We suspect that this might change in the future when an equivalence theorem between the Complicial model of Verity and the Θ-style model studied here is proven.

We hope also that new approaches to dealing with coherence problems might be discovered, and if they can be made to work, we expect that the results of this paper will be even more useful.

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We would like to give a special thanks Alexander Campbell for his invaluable help in correcting the proof of Theorem 1.8.7, which previously cited an incorrect result as well as his advice on strictifying the construction of labeled simplicial sets using both the Lack-Paoli resolution and the resolution based on the category of spans construction. We would also like to acknowledge Andrea Gagna for spending his time discussing these ideas with us over several long conversations as well as Eric Peterson for his sage advice and friendship over the years.

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1. Formally enriched quasicategories

1.1. THE WREATH PRODUCT WITH Δ. Segal observed long ago that a monoidal category is classified precisely by a pseudofunctor $M_\bullet \colon \Delta^{\mathrm{op}} \to \mathbf{Cat}$ such that $M_0 = *$ is the terminal category and the maps $M_n \to (M_1)^n$ induced by the inclusion of the spine $Sp[n] \hookrightarrow \Delta^n$ are all equivalences of categories.

1.1.1. NOTE. For the sake of readability of this section, we will consider all limits taken in $\mathbf{Cat}$ to be the appropriate 2-categorical pseudo-limits. A $\mathbf{Cat}$-valued pseudofunctor with 1-categorical domain will consequently be called continuous if it sends limits to pseudo-limits. With this out of the way, we proceed to our first definition

1.1.2. DEFINITION. Suppose $\mathcal{V}$ is a monoidal category. Then we construct a fibration

$$\Delta \int \mathcal{V} \to \Delta$$

by applying the Grothendieck construction to the pseudofunctor

$$\mathcal{V}_\bullet \colon \Delta^{\mathrm{op}} \to \mathbf{Cat}$$

classifying $\mathcal{V}$. We call the total space of this fibration the *wreath product* of Δ with $\mathcal{V}$. The objects of $\Delta \int \mathcal{V}$ can be identified with pairs $([n], (v_1, \ldots, v_n))$, where $(v_1, \ldots, v_n)$ is a tuple of objects of $\mathcal{V}$. We will write such an object as $[n](v_1, \ldots, v_n)$.

We will also make use of a more elaborate construction from [Our10] that extends the wreath product to arbitrary simplicial sets:

1.1.3. DEFINITION. Notice that since $\mathbf{Cat}$ is conically complete, the pseudofunctor $\mathcal{V}_\bullet$ extends essentially uniquely along the co-Yoneda embedding to a continuous pseudofunctor

$$\mathcal{V}_\bullet^\dagger \colon \widehat{\Delta}^{\mathrm{op}} \to \mathbf{Cat},$$

which is exactly the pseudo-right Kan extension of $\mathcal{V}_\bullet$ along the co-Yoneda embedding $\Delta^{\mathrm{op}} \hookrightarrow \widehat{\Delta}^{\mathrm{op}}$. Applying the Grothendieck construction to the functor $\mathcal{V}_\bullet^\dagger$, we define the Grothendieck fibration

$$\widehat{\Delta} \int \mathcal{V} \to \widehat{\Delta}.$$

The total space of this fibration is called the category of $\mathcal{V}$-*labeled simplicial sets*.

1.1.4. NOTE. It will be useful to explicitly compute the value of $\mathcal{V}_\bullet^\dagger(S)$ for a simplicial set S in somewhat simpler terms. First, consider Δ^n to be a discrete simplicial object in $\mathbf{Cat}$, we can naturally identify $\mathcal{V}_n$ with the category $\mathbf{Cat}^{\Delta^{\mathrm{op}}}(\Delta^n, \mathcal{V}_\bullet)$ whose objects are pseudonatural transformations of simplicial objects and whose morphisms are modifications. We can then compute

$$\begin{aligned}\mathcal{V}_\bullet^\dagger(S) &= \lim_{\Delta^n \in (\Delta \downarrow S)} \mathcal{V}_n \\ &\simeq \lim_{\Delta^n \in (\Delta \downarrow S)} \mathbf{Cat}^{\Delta^{\mathrm{op}}}(\Delta^n, \mathcal{V}_\bullet) \\ &\simeq \mathbf{Cat}^{\Delta^{\mathrm{op}}}(\operatorname{colim}_{\Delta^n \in (\Delta \downarrow S)} \Delta^n, \mathcal{V}_\bullet)\end{aligned}$$

since the cosimplicial object $\Delta^\bullet$ in $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ is Reedy-cofibrant, and therefore

$$\simeq \mathbf{Cat}^{\Delta^{\mathrm{op}}}(S, \mathcal{V}_\bullet).$$

In particular, we can identify the category $\mathcal{V}_\bullet^\dagger(S)$ with the category whose objects are pseudonatural transformations $\Omega\colon S \to \mathcal{V}_\bullet$ and whose morphisms are modifications, viewing S as a simplicial object in $\mathbf{Cat}$.

1.1.5. NOTE. It is possible, by careful application of coherence results, to rectify everything in sight. First, notice that if $\mathcal{V}_\bullet$ is Reedy-fibrant with respect to the canonical model structure on $\mathbf{Cat}$, we can compute the pseudolimit as a strict limit in the 1-category $\mathbf{Cat}$ while also replacing pseudonatural transformations with strict ones. In particular if $\mathcal{V}_\bullet$ is Reedy-fibrant, we have a natural *isomorphism* of categories

$$\operatorname{Nat}(\Delta^n, \mathcal{V}_\bullet) \cong \mathcal{V}_n,$$

where Nat denotes the category of strict natural transformations and modifications between them. Then for a general simplicial set S using the Reedy-cofibrancy of $\Delta^\bullet$, we also have an isomorphism of categories

$$\operatorname{Nat}(S, \mathcal{V}_\bullet) \cong \mathcal{V}_\bullet^\dagger(S).$$

This raises the question of how to obtain a Reedy-fibrant $\mathcal{V}_\bullet$ from a monoidal category $\mathcal{V}$. For this, consider the monoidal category $\mathcal{V}$ as a one-object bicategory and apply the 2-nerve of Lack and Paoli [LP08]. This produces a simplicial category whose object in degree 0 is the terminal category and whose object in degree 1 is in fact isomorphic to $\mathcal{V}$. This simplicial object is also Reedy-fibrant and satisfies the Segal condition. We can unwind $\mathcal{V}_\bullet$ as follows:

- The objects of $\mathcal{V}_n$ are the normal pseudofunctors $[n] \to \mathbf{B}\mathcal{V}$, where $\mathbf{B}\mathcal{V}$ denotes the associated single-object bicategory.
- The morphisms are given by *icons* between pseudofunctors. These are oplax natural transformations whose object components are identities.

If $\mathcal{V}$ is a category with finite products, we can avoid the Lack-Paoli resolution and directly construct a Reedy-fibrant simplicial category using a trick taught to us by Alexander Campbell:

Let A be a small category. Then the Grothendieck construction of the functor $\mathrm{Hom}\colon A\times A^{\mathrm{op}} \to \mathbf{Set}$ is called the *twisted arrow category of* A and is denoted by $\mathrm{Tw}(A)$. Of particular interest are the opposites of the twisted arrow categories of the objects of $\Delta \subset \mathbf{Cat}$. In particular, $\mathrm{Tw}[n]^{\mathrm{op}}$ is the full subcategory of $[n] \times [n]^{\mathrm{op}}$ spanned by the pairs (i, j) such that $i \leq j$. The opposite twisted arrow construction can be readily seen to define a cosimplicial object $\mathrm{Tw}[\bullet]^{\mathrm{op}}\colon \Delta \to \mathbf{Cat}$.

Using the opposite twisted arrow construction, we can assign to any category $\mathcal{V}$ a simplicial category in the obvious way:

$$\widetilde{\mathcal{V}}_\bullet \overset{\mathrm{def}}{=} \mathbf{Cat}(\mathrm{Tw}[\bullet]^{\mathrm{op}}, \mathcal{V}).$$

If $\mathcal{V}$ is a category admitting finite products, we will construct our desired $\mathcal{V}_\bullet^\times$ as a full simplicial subcategory of $\widetilde{\mathcal{V}}_\bullet$ that is levelwise given by the span of those functors $F\colon \mathrm{Tw}[n]^{\mathrm{op}} \to \mathcal{V}$ satisfying the following two properties:

- For all $0 \leq i \leq n$, the object $F(i, i)$ is a terminal object of $\mathcal{V}$.
- For all $0 \leq i \leq i' \leq j' \leq j \leq n$, the commutative square

$$\begin{array}{ccc} F(i,j) & \to & F(i',j) \\ \downarrow & & \downarrow \\ F(i,j') & \to & F(i',j') \end{array}$$

  is Cartesian.

It is straightforward to see that these conditions are compatible with the simplicial structure maps and therefore determine a well-defined full simplicial subcategory $\mathcal{V}_\bullet^\times \subset \widetilde{\mathcal{V}}_\bullet$. In order to see that this construction gives a Reedy-fibrant simplicial category satisfying the Segal condition and classifying the monoidal category $(\mathcal{V}, \times)$, it suffices to show that $\mathcal{V}_n^\times \simeq \mathcal{V}^n$ for each natural number n and that for any simplex $[n]$, the matching map $\mathcal{V}_n^\times \to \mathcal{V}^\times(\partial\Delta^n)$ is an isofibration.

For each natural number n and each $1 \leq i \leq n$, we have an inclusion functor $\iota_i\colon \mathrm{Tw}[1]^{\mathrm{op}} \hookrightarrow \mathrm{Tw}[n]^{\mathrm{op}}$ induced by the inclusion of the interval $[i-1, i] \hookrightarrow [n]$, so by restriction and the universal property of the product, we have a functor $\mathcal{V}_n^\times \to (\mathcal{V}_1^\times)^n$. However, it can be computed immediately that $\mathcal{V}_1^\times \simeq \mathcal{V}$, as the objects of this category are simply spans

$$* \leftarrow v \rightarrow *$$

for $v \in \mathcal{V}$ where $*$ denotes the terminal object of $\mathcal{V}$, from which it follows that we have a natural functor $\mathcal{V}_n^\times \to \mathcal{V}^n$. It is straightforward to see that this functor is essentially surjective, since every family of n objects in $\mathcal{V}$ admits a product and the diagram generated under products and pullback along projections belongs to $\mathcal{V}_n^\times$. To see that it is fully faithful, notice that for any $(i,j) \in \mathrm{Tw}[n]^{\mathrm{op}}$ and any $F \in \mathcal{V}_n^\times$, we have

$$F(i,j) \cong \prod_{i<k\leq j} F(k-1,k),$$

so using naturality and universal properties, we find that the map

$$\mathcal{V}_\bullet^\times(F,G) \to \mathcal{V}^n(\iota^* F, \iota^* G)$$

is bijective.

To prove Reedy fibrancy, notice that the matching map $\mathcal{V}_n^\times \to \mathcal{V}^\times(\partial\Delta^n)$ is an isomorphism of categories for $n \geq 3$ by pasting and an isofibration for $n \leq 1$ as the matching map is the terminal functor. It therefore suffices to prove that it is an isofibration for $n = 2$. By unwinding the definitions, it suffices to show that given isomorphisms $f_A \colon A \to A'$, $f_B \colon B \to B'$, and $f_C \colon C \to A' \times B'$, there exists a span $A \leftarrow C \to B$ exhibiting C as the product of A and B and such that the three isomorphisms above give a natural isomorphism of spans. But we can define $C \to A$ as $f_A^{-1}\pi_0' \circ f_C$ and $C \to B$ similarly, which gives us the desired result by a trivial check.

1.1.6. Proposition. The pullback of the fibration

$$\widehat{\Delta} \int \mathcal{V} \to \widehat{\Delta}$$

along the Yoneda embedding $\Delta \hookrightarrow \widehat{\Delta}$ is exactly the fibration

$$\Delta \int \mathcal{V} \to \Delta,$$

and therefore, the induced map

$$\Delta \int \mathcal{V} \hookrightarrow \widehat{\Delta} \int \mathcal{V}$$

is a fully faithful embedding.

Proof. As the functor $\mathcal{V}_\bullet$ factors as the composite

$$\Delta^{\mathrm{op}} \hookrightarrow \widehat{\Delta}^{\mathrm{op}} \xrightarrow{\mathcal{V}_\bullet^\dagger} \mathbf{Cat},$$

where the first functor is fully faithful, it follows that $\Delta \int \mathcal{V} \to \Delta$ is the pullback of the fibration $\widehat{\Delta} \int \mathcal{V} \to \widehat{\Delta}$ along the fully faithful Yoneda embedding. Ergo, the map in question is fully faithful. ∎

For the purposes of this paper, we do not need this level of generality. We specialize as follows:

1.1.7. Definition. A small regular skeletal Reedy category (also called a regular skeletal category) $\mathcal{C}$ (see Definition A.4.8) is called a *regular Cartesian Reedy category* if it satisfies two conditions:

(CR1) Finite products of representable presheaves on $\mathcal{C}$ are regular (see Definition A.4.8).

(CR2) For any finite set I, an I-indexed family of objects $(c_i)_{i\in I} \in \mathcal{C}^I$, and any object $c \in \mathcal{C}$, the existence of a nondegenerate section (see Definition A.4.7) $(f_i)_{i\in I}\colon h_c \to \prod_{i\in I} h_{c_i}$ implies that $\dim c \leq \sum_{i\in I} \dim c_i$.

(CR3) The category $\mathcal{C}$ admits a terminal object $*_{\mathcal{C}}$ of dimension 0, and $\mathcal{C}(*_{\mathcal{C}}, c) \neq \varnothing$ for all $c \in \mathcal{C}$.

1.1.8. Observation. We will see that the axioms (CR1) and (CR2) imply that a regular Cartesian Reedy category $\mathcal{C}$ is a Reedy multicategory in the sense of [BR11] and admit a weak form of the Eilenberg-Zilber shuffle decomposition. To see this, notice that regularity of finite products of representables tells us that a nondegenerate section

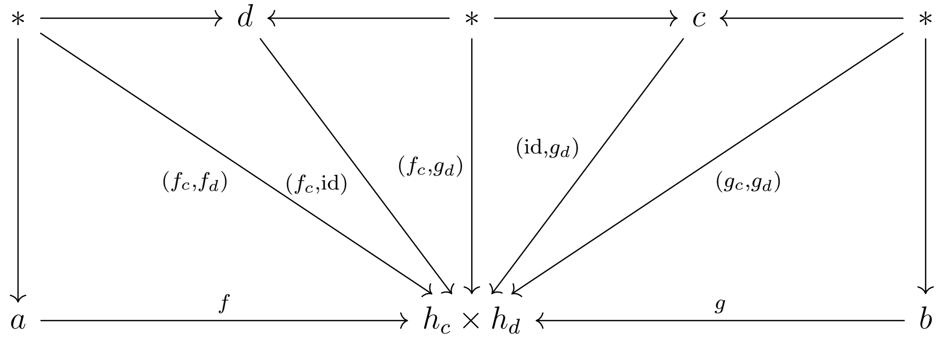
$$(f_i)_{i\in I}\colon h_c \to \prod_{i\in I} h_{c_i}$$

corresponds to an injective map

$$h_c \to \left(\prod_{i\in I} h_{c_i}\right),$$

which in particular has representable image. Then, using that any map from a representable h_c to a presheaf F on a skeletal category admits a unique factorization into a degeneracy followed by a nondegenerate map, we can factor any multimorphism in the sense of Bergner and Rezk as desired. The axiom (CR2) on dimension corresponds to the dimension axiom for Reedy multicategories, as desired.

The axiom (CR3) is interesting and arose from a counterexample brought up by the referee. It implies a useful property, namely that binary products of representable presheaves are connected (see Definition A.4.6). To see this, suppose we have a pair of maps $f\colon h_a \to h_c \times h_d$ and $g\colon h_b \to h_c \times h_d$. Since h_a and h_b admit points, we have maps $(f_c, f_d)\colon * \to h_a \to h_c \times h_d$ and $(g_c, g_d)\colon * \to h_b \to h_c \times h_d$. Then consider the following zig-zag in the category of elements $(\Delta \downarrow (h_c \times h_d))$



This proves that f and g are connected by a zig-zag. Since f and g were arbitrary, it follows that the category of elements $(\mathcal{C} \downarrow (h_c \times h_d))$ is connected.

The connectedness of binary products of representable presheaves implies even more, namely that connected presheaves are closed under finite products. To see this, suppose $A, B \in \widehat{\mathcal{C}}$ are connected. Then we can compute

$$A \times B \cong \left(\operatorname*{colim}_{c \in (\mathcal{C} \downarrow A)} c \right) \times \left(\operatorname*{colim}_{c' \in (\mathcal{C} \downarrow B)} c' \right),$$

and by two applications of universality of colimits

$$\cong \operatorname*{colim}_{c \in (\mathcal{C} \downarrow A)} \operatorname*{colim}_{c' \in (\mathcal{C} \downarrow B)} (c \times c'),$$

but the categories of elements $(\mathcal{C} \downarrow A)$ and $(\mathcal{C} \downarrow B)$ are connected, so this is a connected colimit of products of representables, which are connected by assumption. It follows that $A \times B$ is connected as connected colimits of connected presheaves are connected (see Definition A.4.6). For finite products of higher arity, apply induction.

In the sequel, we assume that $\mathcal{V} = \widehat{\mathcal{C}}$ is the category of presheaves of sets on a small regular Cartesian Reedy category $\mathcal{C}$. We use the construction $\mathcal{V}^{\times}_{\bullet}$ in order to produce a Reedy-fibrant simplicial category satisfying the (monoidal) Segal condition. Then we give the following definition:

1.1.9. Definition. For any regular Cartesian Reedy category $\mathcal{C}$, we define the category of $\mathcal{C}$-cells to be the skeleton $\Theta[\mathcal{C}]$ of the full subcategory

$$\widetilde{\Theta[\mathcal{C}]} \subset \Delta \int \widehat{\mathcal{C}}$$

spanned by those objects $([n], \Omega)$ for $n \geq 0$, where $\Omega \colon \Delta^n \to \widehat{\mathcal{C}}^{\times}_{\bullet}$ is a natural transformation, such that for every inclusion $\iota_i \colon [i-1, i] \hookrightarrow [n]$, the restriction $\iota_i^* \Omega$ classifies a representable presheaf $h_{c_i} \in \widehat{\mathcal{C}}$.

We deduce from the construction of $\widehat{\mathcal{C}}^{\times}_{\bullet}$ that given objects $([n], \Omega)$ and $([n], \Omega')$ of $\widehat{\mathcal{C}}^{\times}_n$ such that $\iota_i^* \Omega \cong \iota_i^* \Omega'$ for all $1 \leq i \leq n$, the family of isomorphisms extends to an isomorphism $([n], \Omega) \cong ([n], \Omega')$.

Therefore, we introduce the notation

$$[n](c_1, \cdots, c_n) \quad \text{for} \quad c_1, \cdots, c_n \in \mathcal{C},$$

to refer to the isomorphism class in the skeleton $\Theta[\mathcal{C}]$ corresponding to all pairs $([n], \Omega)$ such that $\iota_i^* \Omega \cong h_{c_i}$ for $1 \leq i \leq n$.

An arrow $f\colon [m](c_1, \ldots, c_m) \to [n](c'_1, \ldots, c'_n)$ in $\Theta[C]$ is given by the following data:

- A map $f_\Delta\colon [m] \to [n]$ in Δ
- For each pair of natural numbers $i, j \in \mathbb{N}$ such that $1 \leq i \leq n$ and $f_\Delta(i-1)+1 \leq j \leq f(i)$ a map $f_{ij}\colon c_i \to c'_j$.

Given a pair of maps

$$[\ell](c_1, \ldots, c_\ell) \xrightarrow{g} [m](c'_1, \ldots, c'_m) \xrightarrow{f} [n](c''_1, \ldots, c''_n),$$

we can compute the composite h as the map given by the following data:

- The map h_Δ is $f_\Delta \circ g_\Delta$
- For each $0 \leq i \leq \ell$ and each $h_\Delta(i-1)+1 \leq j \leq h_\Delta(i)$, there exists a unique $k \in [m]$ with $g_\Delta(i-1)+1 \leq k \leq g_\Delta(i)$ such that $f_\Delta(k-1)+1 \leq j \leq f_\Delta(k)$, we can define $h_{ij}\colon c_i \to c''_j$ as the composite $f_{kj} \circ g_{ik}$.

These formulas are obtained by a direct computation of the pullback of each of these maps to the fibre over [1] for each interval $[i-1, i] \hookrightarrow [\ell]$ (resp. $[k-1, k] \hookrightarrow [m]$).

1.1.10. Proposition. Let $\mathcal{C}$ be a regular Cartesian Reedy category. Then the category of $\widehat{\mathcal{C}}$-labeled simplicial sets $\widehat{\Delta} \int \widehat{\mathcal{C}}$ is complete and cocomplete.

Proof. Since the Grothendieck fibration

$$\widehat{\Delta} \int \widehat{\mathcal{C}} \to \widehat{\Delta}$$

is the right Kan extension of the fibration

$$\Delta \int \widehat{\mathcal{C}} \to \Delta$$

to the complete and cocomplete locally small category of simplicial sets along the Yoneda embedding $\Delta \hookrightarrow \widehat{\Delta}$, it will suffice to show by Proposition B.2.5 that the Grothendieck fibration

$$\Delta \int \widehat{\mathcal{C}} \to \Delta$$

is presentable (see Definition B.2.2). First, notice that its fibres are all equivalent to presheaf categories, so its fibres are presentable. It therefore suffices to show that for any map $\alpha\colon [n] \to [m]$ in Δ, the associated map α^* on fibres admits a left adjoint. By factorization into faces and degeneracies, we have three cases:

1. If $n \geq 0$ and $m = n+1$ and α is the inclusion of an outer face (by symmetry, assume it is the inclusion of the face opposite the vertex $n+1$), the functor α^* sends an $n+1$-tuple to an n-tuple by the rule

$$(A_1, \ldots, A_{n+1}) \mapsto (A_1, \ldots, A_n).$$

In this case, we can compute its left adjoint to be the functor sending an n-tuple to an $n+1$-tuple by the rule

$$(B_1, \ldots, B_n) \mapsto (B_1, \ldots, B_n, \varnothing).$$

2. If $n \geq 0$ and $m = n+1$ and α is the inclusion of the face opposite the kth vertex for $0 < k < n+1$, the functor α^* sends an $n+1$-tuple to an n-tuple by the rule

$$(A_1, \ldots, A_{n+1}) \mapsto (A_1, \ldots, A_k \times A_{k+1}, \ldots, A_n).$$

In this case, we can compute its left adjoint to be the functor sending an n-tuple to an $n+1$-tuple by the rule

$$(B_1, \ldots, B_n) \mapsto (B_1, \ldots, B_k, B_k, \ldots, B_n).$$

3. If $n \geq 1$ and $m = n-1$, and the map α hits the kth vertex twice for $0 \leq k \leq n-1$, the functor α^* sends an $n-1$-tuple to an n-tuple by the rule

$$(A_1, \ldots, A_{n-1}) \mapsto (A_1, \ldots, A_{k-1}, *, A_k, \ldots, A_{n-1}).$$

In this case, we can compute its left adjoint to be the functor sending an n-tuple to an $n-1$-tuple by the rule

$$(B_1, \ldots, B_n) \mapsto (B_1, \ldots, B_{k-1}, B_{k+1}, \ldots, B_n).$$

Therefore, the fibration is presentable, and we are done. ∎

1.1.11. Note. For any small category $\mathcal{C}$, we have a projection functor

$$\pi \colon \Theta[\mathcal{C}] \to \Theta[*] = \Delta$$

sending an object to the associated underlying simplex. It is straightforward to see that the functor π agrees with the restriction of the left adjoint in the $\widehat{\Delta}$-relative adjunction

$$\widehat{\Delta} \int \tau_! \colon \widehat{\Delta} \int \widehat{\mathcal{C}} \rightleftarrows \widehat{\Delta} \int \mathbf{Set} \colon \widehat{\Delta} \int \tau^*,$$

to $\Theta[\mathcal{C}]$, where $\tau \colon \mathcal{C} \to *$ is the terminal functor. Note however that in general, this is not an adjunction in the category of Grothendieck fibrations over $\widehat{\Delta}$. Since the functor $\tau_! \colon \widehat{\mathcal{C}} \to \mathbf{Set}$ need not preserve Cartesian products, the associated morphism of monoidal categories is only oplax monoidal, which only gives an oplax transformation of simplicial categories $\tau_!^\times : \widehat{\mathcal{C}}_\bullet^\times \to \mathbf{Set}_\bullet^\times$. After applying the Grothendieck construction we see that the functor $\widehat{\Delta}\int \tau_!$ need not preserve Cartesian morphisms.

Notice however that this defect disappears if we demand that finite Cartesian products of representable presheaves in $\widehat{\mathcal{C}}$ are connected. Under this assumption, notice that the natural map

$$\tau_!(h_c \times h_{c'}) \to \tau_!(h_c) \times \tau_!(h_{c'})$$

is an isomorphism for any pair of objects $c, c' \in \mathcal{C}$, since the source is the terminal set by merit of the connectedness of $h_c \times h_{c'}$ and the righthand side is the terminal set by merit of the fact that representables are connected and the product of two terminal sets is a terminal set. Then more generally, given $A, B \in \widehat{\mathcal{C}}$,

$$\begin{aligned}
\tau_!(A \times B) &\cong \tau_!(\operatorname*{colim}_{c\in(\mathcal{C}\downarrow A)} h_c \times \operatorname*{colim}_{c'\in(\mathcal{C}\downarrow B)} h_{c'}) \\
&\cong \tau_!(\operatorname*{colim}_{c\in(\mathcal{C}\downarrow A)} \operatorname*{colim}_{c'\in(\mathcal{C}\downarrow B)} (h_c \times h_{c'})) \\
&\cong \operatorname*{colim}_{c\in(\mathcal{C}\downarrow A)} \operatorname*{colim}_{c'\in(\mathcal{C}\downarrow B)} \tau_!(h_c \times h_{c'}) \\
&\cong \operatorname*{colim}_{c\in(\mathcal{C}\downarrow A)} \operatorname*{colim}_{c'\in(\mathcal{C}\downarrow B)} (\tau_!(h_c) \times \tau_!(h_{c'})) \\
&\cong \operatorname*{colim}_{c\in(\mathcal{C}\downarrow A)} \tau_!(h_c) \times \operatorname*{colim}_{c'\in(\mathcal{C}\downarrow B)} \tau_!(h_{c'}) \\
&\cong \tau_!(\operatorname*{colim}_{c\in(\mathcal{C}\downarrow A)} h_c) \times \tau_!(\operatorname*{colim}_{c'\in(\mathcal{C}\downarrow B)} h_{c'}) \\
&\cong \tau_!(A) \times \tau_!(B).
\end{aligned}$$

This demonstrates that the functor $\tau_!$ preserves finite products and is therefore a strong monoidal functor, which ensures that the relative adjunction

$$\widehat{\Delta}\int \tau_! \quad \dashv_{\widehat{\Delta}} \quad \widehat{\Delta}\int \tau^*$$

is actually a fibrational adjunction. In particular, by (CR3), every regular Cartesian Reedy category satisfies this property.

When $\mathcal{C}$ has a terminal object, the terminal object of $\widehat{\mathcal{C}}$ is representable. This ensures that the restriction of the functor $\widehat{\Delta}\int \tau^*$ to $\Delta = \Theta[*] \subset \widehat{\Delta}\int \mathbf{Set}$ factors through $\Theta[\mathcal{C}]$. Since the restriction of an adjunction to full subcategories remains an adjunction, we have a restricted adjunction

$$\pi \colon \Theta[\mathcal{C}] \rightleftarrows \Theta[*] = \Delta \colon \eta,$$

where η is the functor sending a simplex $[n]$ to the object $[n](*, \dots, *)$ in $\Theta[\mathcal{C}]$. Passing to presheaf categories, these functors also extend to a quadruple adjunction by a routine calculation of Kan extensions. However, we will only name and make use of three of the four adjoints.

$$\widehat{\Theta[\mathcal{C}]} \underset{\mathscr{N}}{\overset{\pi}{\underset{\perp}{\overset{\perp}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{\mathscr{H}}}}}}}} \widehat{\Delta},$$

where, by abuse of notation, we denote the *simplicial projection* functor $\pi_!$ simply by π, we denote the *local termination* functor $\pi^* = \eta_!$ by $\mathscr{H}$, and we denote the *underlying simplicial set* functor $\pi_* = \eta^*$ by $\mathscr{N}$.

1.1.12. Definition. We define a special cosimplicial object in $\widehat{\Theta[\mathcal{C}]}$ by the formula

$$E^\bullet = \mathscr{H}(\operatorname{cosk}_0 \Delta^\bullet).$$

This cosimplicial object will be a cosimplicial resolution of a point, once we define our model structures.

1.2. The generalized intertwiner and $\widehat{\Delta} \int \widehat{\mathcal{C}}$. Rezk introduced a functor called the *intertwiner* by means of an explicit construction in [Rez10], but Oury has given an even more powerful version in [Our10], which we recall here:

1.2.1. Definition. Recall that we have a fully-faithful embedding

$$L\colon \Theta[\mathcal{C}] \hookrightarrow \Delta \int \widehat{\mathcal{C}} \hookrightarrow \widehat{\Delta} \int \widehat{\mathcal{C}}.$$

We define the *intertwiner* to be the restricted Yoneda functor

$$\Box\colon \widehat{\Delta} \int \widehat{\mathcal{C}} \to \widehat{\Theta[\mathcal{C}]}$$

by the formula

$$(S,\Omega) \mapsto S\Box\Omega = \operatorname{Hom}_{\widehat{\Delta} \int \widehat{\mathcal{C}}}(L(\cdot),(S,\Omega)).$$

By Proposition 1.1.10, this functor admits a left adjoint

$$L_!\colon \widehat{\Theta[\mathcal{C}]} \to \widehat{\Delta} \int \widehat{\mathcal{C}}$$

defined by cocontinuous extension.

1.2.2. Note. The restriction of the intertwiner to $\Delta \int \widehat{\mathcal{C}}$ is exactly the intertwiner of Rezk. When we apply the intertwiner to an object belonging to the full subcategory $\Delta \int \widehat{\mathcal{C}}$, that is, $(S,\Omega) = [n](A_1,\dots,A_n)$, we will switch to Rezk's notation, namely

$$V[n](A_1,\dots,A_n) \overset{\text{def}}{=} S\Box\Omega$$

.

1.2.3. Definition. An object (S,Ω) of $\widehat{\Delta} \int \widehat{\mathcal{C}}$ is called *normalized* if for every edge $e\colon \Delta^1 \to S$, the presheaf $e^*\Omega$ is connected.

1.2.4. Proposition. Suppose that $\mathcal{C}$ is regular Cartesian Reedy (or more generally, that finite products of connected presheaves on $\mathcal{C}$ are connected). Then a pair $(S,\Omega) \in \widehat{\Delta} \int \widehat{\mathcal{C}}$ is normalized if we have

$$\left(\widehat{\Delta} \int \tau_!\right)(S,\Omega) \cong (S,*).$$

That is to say, after taking connected components of the label, the associated **Set**-labeled simplicial set has the terminal labeling.

Proof. When finite products of connected presheaves on $\mathcal{C}$ are connected, the functor

$$\widehat{\Delta} \int \tau_! \colon \widehat{\Delta} \int \widehat{\mathcal{C}} \to \widehat{\Delta} \int \mathbf{Set}$$

is Cartesian. In particular, we have for any map $f\colon \Delta^1 \to S$ with S a simplicial set and Ω a labeling of S, an isomorphism

$$\left(\widehat{\Delta} \int \tau_!\right)(\Delta^1, f^*\Omega) \cong (\Delta^1, \tau_!(f)^*\tau_!(\Omega)).$$

The pair (S,Ω) is normalized if and only if for every map $f\colon \Delta^1 \to S$, the lefthand side is isomorphic to $(\Delta^1,*)$. On the other hand, the **Set**-labeled simplicial set $\left(\widehat{\Delta} \int \tau_!\right)(S,\Omega)$ has the terminal labeling if and only if the righthand side is isomorphic to $(\Delta^1,*)$. This proves both directions. ■

1.2.5. Proposition. The full subcategory of $\widehat{\Delta} \int \widehat{\mathcal{C}}$ spanned by the normalized objects is closed under colimits.

Proof. Since the functor

$$\widehat{\Delta} \int \tau_! \colon \widehat{\Delta} \int \widehat{\mathcal{C}} \to \widehat{\Delta} \int \mathbf{Set}$$

is a left adjoint, it commutes with colimits, and since a $\widehat{\mathcal{C}}$-labeled simplicial set is normalized if its image under this functor has the terminal labeling, we reduce to the case where $\mathcal{C} = *$ is the terminal category. However, in this case, a labeled simplicial set is normalized if and only if it has the terminal labeling. Therefore, we are reduced to showing that the functor

$$\widehat{\Delta} = \widehat{\Theta[*]} \to \widehat{\Delta} \int \mathbf{Set}$$

preserves colimits. However, this functor is precisely the left adjoint $L_!$ to the intertwiner in the case where $\mathcal{C} = *$, and therefore it preserves colimits. ■

1.2.6. Proposition. Let (S,Ω) be a normalized object in $\widehat{\Delta}\int\widehat{\mathcal{C}}$. Then the counit map $\varepsilon\colon L_!(S\Box\Omega)\to(S,\Omega)$ is an isomorphism.

Proof. Unwinding the definitions, we have

$$L_!(S\Box\Omega)=\operatorname*{colim}_{(\Theta[\mathcal{C}]\downarrow(S,\Omega))}L,$$

where L denotes the inclusion of $\Theta[\mathcal{C}]$, but we have the obvious projection functor

$$\pi^S\colon(\Theta[\mathcal{C}]\downarrow(S,\Omega))\to(\Delta\downarrow S),$$

and we can form the left Kan extension

$$\pi^S_!L\colon(\Delta\downarrow S)\to\widehat{\Delta}\int\widehat{\mathcal{C}},$$

which has the explicit formula for each simplex $g\colon\Delta^n\to S$

$$(\Delta^n,g)\mapsto\operatorname*{colim}_{(\pi^S\downarrow(\Delta^n,g))}L,$$

where the colimit ranges over the category of data comprising an object $[t]\in\Theta[\mathcal{C}]$, a map $\phi\colon L([t])\to(S,\Omega)$, and a map $\psi\colon\pi^S([t])\to\Delta^n$ making the triangle

$$\begin{array}{ccc}\pi^S([t]) & & \\ {\scriptstyle\psi}\downarrow & \searrow^{\pi(\phi)} & \\ \Delta^n & \xrightarrow{\ g\ } & S\end{array}$$

commute.

By the definition of the colimit as the left Kan extension along the terminal functor and the functoriality of left Kan extensions, we see that $\operatorname{colim}\pi^S_!L\cong L_!(S\Box\Omega)$. However, by choosing a Cartesian lift $\widetilde{g}\colon(\Delta^n,g^*\Omega)\to(S,\Omega)$, we obtain using the unique factorization a natural equivalence of categories between the category of data indexing the colimit noted above and the category $(\Theta[\mathcal{C}]\downarrow(\Delta^n,g^*\Omega))$. Therefore, the formula for the Kan extension $\pi^S_!L$ simplifies to

$$(\Delta^n,g)\mapsto\operatorname*{colim}_{(\Theta[\mathcal{C}]\downarrow(\Delta^n,g^*\Omega))}L,$$

but for each $g\colon\Delta^n\to S$, this colimit is the definition of the $\widehat{\mathcal{C}}$-labeled simplicial set

$$L_!(\Delta^n\Box g^*\Omega).$$

Let $F\colon(\Delta\downarrow S)\to\widehat{\Delta}\int\widehat{\mathcal{C}}$ be the functor defined by the rule

$$(\Delta^n,g)\mapsto L_!(\Delta^n\Box g^*\Omega)$$

Therefore, we have a natural isomorphism

$$L_!(S\square\Omega) \cong \operatorname{colim} F.$$

From a different angle now, notice that by merit of the fact that the fibration

$$\widehat{\Delta} \int \widehat{\mathcal{C}} \to \widehat{\Delta}$$

is presentable and $\widehat{\Delta}$ is cocomplete, together with the fact that the fibre over S can be computed as

$$\left(\widehat{\Delta} \int \widehat{\mathcal{C}}\right)_S \simeq \lim_{g\colon \Delta^n \to S} \left(\widehat{\Delta} \int \widehat{\mathcal{C}}\right)_{\Delta^n},$$

we see that the pair (S, Ω) is isomorphic to the colimit

$$\operatorname*{colim}_{(\Delta^n, g) \in (\Delta \downarrow S)} (\Delta^n, g^*\Omega)$$

in the category of $\widehat{\mathcal{C}}$-labeled simplicial sets. Let $G\colon (\Delta \downarrow S) \to \widehat{\Delta} \int \widehat{\mathcal{C}}$ be the functor defined by the rule

$$(\Delta^n, g\colon \Delta^n \to S) \mapsto (\Delta^n, g^*\Omega).$$

By our observations above, we have a natural transformation $\varepsilon\colon F \to G$ such that the induced map on colimits is precisely the counit map $L_! S\square\Omega \to (S, \Omega)$. Therefore, in order to show that the counit map is an isomorphism when (S, Ω) is normalized, it suffices to show that the natural transformation ε is an isomorphism.

Since the pair (S, Ω) is normalized if and only if for all $g\colon \Delta^n \to S$ the pair $(\Delta^n, g^*\Omega)$ is normalized, and since the component of ε at (Δ^n, g) is precisely the counit map

$$\varepsilon_{(\Delta^n, g^*\Omega)}\colon L_!(\Delta^n \square g^*\Omega) \to (\Delta^n, g^*\Omega),$$

it will therefore suffice to prove the statement in the case where $S = \Delta^n$. We therefore suggestively write the pair (S, Ω) as $[n](A_1, \ldots, A_n)$. Let $f\colon [m](c_1, \ldots, c_m) \to [n](A_1, \ldots, A_n)$ be a map. Since the full subcategory of such maps with $f_\Delta\colon [m] \to [n]$ nondegenerate is cofinal, we may assume that the underlying map of simplices is injective. To deal with the case where $m < n$, we reduce to the case where $m = n-1$. Then if f_Δ is the inclusion of an inner face, we have a map

$$[n-1](c_1, \ldots, c_{n-1}) \to [n-1](A_1, \ldots, A_k \times A_{k+1}, \ldots, A_n).$$

But in this case, the map $f_{\Delta!}$ sends representables to representables, so we have a factorization through the adjunct map

$$[n](c_1, \ldots, c_k, c_k, \ldots, c_{n-1}) \to [n](A_1, \ldots, A_n).$$

In the case where f_Δ is the inclusion of an outer face (say the face opposite the vertex n), it suffices to produce some map $c_n \to A_n$ where c_n is an arbitrary representable, which exists by merit of A_n being connected (and in particular nonempty).

Therefore, we have the case $m = n$ and f_Δ is the identity. Since all of the presheaves in the family $(A_1, \dots, A_n)$ are connected, it follows that the subcategory consisting of those maps

$$[n](c_1, \dots, c_n) \to [n](A_1, \dots, A_n)$$

with f_Δ the identity is none other than the product of the overcategories

$$\prod_{i=1}^{n} (\mathcal{C} \downarrow A_i),$$

which is connected, as it is a finite product of connected categories. Therefore, the full subcategory of the category of elements

$$(\Theta[\mathcal{C}] \downarrow V[n](A_1, \dots, A_n))$$

spanned by those maps $[m](c_1, \dots, c_n) \to V[n](A_1, \dots, A_n)$ where the map $[m] \to [n]$ is the identity forms a connected cofinal subcategory. It follows that the colimit of the functor

$$\pi \circ L\colon (\Theta[\mathcal{C}] \downarrow V[n](A_1, \dots, A_n)) \to \widehat{\Delta}$$

is none other than Δ^n. It follows therefore that the colimit of the functor

$$L\colon (\Theta[\mathcal{C}] \downarrow V[n](A_1, \dots, A_n)) \to \widehat{\Delta} \int \widehat{\mathcal{C}}$$

can be computed directly in the fibre

$$\left(\Delta \int \widehat{\mathcal{C}}\right)_{\Delta^n}.$$

Since the fibre is equivalent to the product $\prod_{i=1}^{n} \widehat{\mathcal{C}}$, we may compute the colimit pointwise. We can compute the composite of the diagram with the projection onto the factor $1 \leq k \leq n$

$$\prod_{i=1}^{n} (\mathcal{C} \downarrow A_i) \to \prod_{i=1}^{n} \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$$

to be the functor sending a tuple $(c_1, \dots, c_n) \to (A_1, \dots, A_n)$ to the representable object c_k, and by manipulation of left Kan extensions, we can compute this colimit as

$$\operatorname*{colim}_{c \in (\mathcal{C} \downarrow A_k)} \operatorname*{colim}_{(p_k \downarrow c)} c,$$

where $p_k : \prod_{i=1}^{n} (\mathcal{C} \downarrow A_i) \to (C \downarrow A_k)$ is the projection. But by direct computation, we see that

$$(p_k \downarrow c) = \prod_{i \neq k} (\mathcal{C} \downarrow A_i) \times (\mathcal{C} \downarrow c),$$

which is connected, so we have

$$\operatorname*{colim}_{c\in(\mathcal{C}\downarrow A_k)} \operatorname*{colim}_{(p_k\downarrow c)} c \cong \operatorname*{colim}_{c\in(\mathcal{C}\downarrow A_k)} c \cong A_k.$$

Therefore, the colimit in the fibre is none other than the tuple $(A_1,\dots,A_n)$, and since this colimit computes $L_!V[n](A_1,\dots,A_n)$, we have that the counit

$$\varepsilon\colon L_!(V[n](A_1,\dots,A_n)) \to [n](A_1,\dots,A_n)$$

is an isomorphism, as desired. ∎

1.2.7. Corollary. The adjunction

$$L_!\colon \widehat{\Theta[\mathcal{C}]} \rightleftarrows \widehat{\Delta}\int\widehat{\mathcal{C}}\colon \square$$

is idempotent.

Proof. Since $\mathcal{C}$ is regular Cartesian Reedy, finite products of representables are connected, so every object $[t]\in\Theta[\mathcal{C}]$ viewed as an object of $\widehat{\Delta}\int\widehat{\mathcal{C}}$ is normalized. Therefore, for any presheaf X on $\Theta[\mathcal{C}]$, we also have that $L_!X$ is normalized by Proposition 1.2.5. Then by the previous proposition, we see that $\varepsilon\colon L_!\square L_!X \to L_!X$ is an isomorphism, which proves that the adjunction is idempotent. ∎

1.2.8. Definition. We call a presheaf of sets on $\Theta[\mathcal{C}]$ a *$\mathcal{C}$-cellular set.*

1.2.9. Note. Although the case when $\mathcal{C}=\Theta_{n-1}$ (respectively $\mathcal{C}=\Theta=\Theta_\omega$) are not strictly the focus of this paper, note that $\Theta[\Theta_{n-1}]=\Theta_n$ (respectively $\Theta[\Theta]=\Theta$). In these cases, we call presheaves of sets on $\Theta[\mathcal{C}]$ *n-cellular sets* (respectively, *cellular sets*).

1.2.10. Definition. We say that a $\mathcal{C}$-cellular set X is *sober* if it belongs to the image of the intertwiner. Since the intertwiner adjunction is idempotent, we see that the category of sober $\mathcal{C}$-cellular sets is equivalent to the category of normalized $\widehat{\mathcal{C}}$-labeled simplicial sets.

1.2.11. Proposition. All representable $\mathcal{C}$-cellular sets are sober, and moreover, the full subcategory of sober $\mathcal{C}$-cellular sets is closed under small limits in $\widehat{\Theta[\mathcal{C}]}$.

Proof. By construction, we see that representables are sober. The closure of sober objects under limits follows from the fact that they span the image of an idempotent right adjoint. ∎

1.2.12. Proposition. Given a labeled simplicial set (S,Ω) and $f\colon S'\to S$ a map of simplicial sets, then given a Cartesian lift $\tilde{f}\colon (S',f^*\Omega)\to(S,\Omega)$, we have a canonical isomorphism

$$S'\square f^*\Omega \to S'\square * \times_{S\square *} S\square\Omega.$$

Proof. By computing the limit of the cospan $(S',*)\to(S,*)\leftarrow(S,\Omega)$ in $\widehat{\Delta}\int\widehat{\mathcal{C}}$, we see that the pullback is given by $(S',f^*\Omega\times_* *)\cong(S',f^*\Omega)$. Since the intertwiner commutes with limits, the claim follows. ∎

1.2.13. NOTE. If $\iota : S \hookrightarrow \Delta^n$ is the inclusion of a subcomplex (for instance, the inclusion of a horn or a boundary), and (Δ^n, Ω) is some labeling $(A_1, \dots, A_n)$, we let $V_S(A_1, \dots, A_n) \subseteq V[n](A_1, \dots, A_n)$ denote the intertwiner $S\Box\iota^*\Omega$ of the labeling Ω along ι. It follows from the previous proposition that this is canonically isomorphic to the pullback $V[n](A_1, \dots, A_n) \times_{\Delta^n\Box *} S\Box *$.

1.3. THE HORIZONTAL JOYAL MODEL STRUCTURE.

We define a Cisinski model structure on $\widehat{\Theta[\mathcal{C}]}$ and state several results that will be proven over the next few sections.

1.3.1. DEFINITION. There is a Cisinski model structure called the *horizontal Joyal model structure* on $\widehat{\Theta[\mathcal{C}]}$ where the separating interval is given by

$$E^1 = \mathscr{H}(\operatorname{cosk}_0 \Delta^1),$$

which is also isomorphic to $N(G_2)\Box *$, the nerve of the freestanding isomorphism G_2 equipped with the terminal labeling. The set of generating anodynes is given in terms of the corner-intertwiner (see Definition 1.6.10 for the precise construction) by the set

$$\mathscr{J} = \{\Box_n^{\lrcorner}(\lambda_k^n, \delta^{c_1}, \dots, \delta^{c_n}) \colon 0 < k < n \text{ and } c_1, \dots, c_n \in \operatorname{Ob}\mathcal{C}\},$$

where $\lambda_k^n \colon \Lambda_k^n \hookrightarrow \Delta^n$ is the simplicial horn inclusion, and where $\delta^c \colon \partial c \hookrightarrow c$ is the inclusion of the boundary of c (recall that $\mathcal{C}$ was taken to be a regular Cartesian Reedy category, so this makes sense).

We call $\operatorname{rlp}(\mathscr{J})$ the class of *horizontal inner fibrations*, and we call $\operatorname{llp}(\operatorname{rlp}(\mathscr{J}))$ the class of *horizontal inner anodynes*.

1.3.2. REMARK. *The precise definition and construction of the corner-intertwiner $\Box_n^{\lrcorner}$ is deferred to §1.4, but in this particular case, we can compute it by hand in terms of the intertwiner to be*

$$\Lambda_k^n\Box(\lambda_k^n)^*(c_1, \dots, c_n) \cup \left(\bigcup_{i=1}^n \Delta^n\Box(c_1, \dots, \partial c_i \dots, c_n)\right) \hookrightarrow \Delta^n\Box(c_1, \dots, c_n).$$

1.3.3. DEFINITION. We call an object with the right lifting property with respect to $\mathscr{J}$ a *formal $\mathcal{C}$-quasicategory.*

1.3.4. NOTE. In the case where $\mathcal{C}$ is the terminal category, these are precisely the quasicategories, since the horns in the definition above become exactly the simplicial inner horn inclusions. We call these objects formal $\mathcal{C}$-quasicategories because they are only an intermediate step. We aren't entirely sure what they model, but they behave like categories enriched in the minimal Cartesian closed model localization on $\widehat{\mathcal{C}}$.

The following results are stated here without proof. All proofs are heavily inspired by [Our10] and provided in full in §1.5, §1.6, and §1.8.

1.3.5. Proposition. The class of all monomorphisms of $\widehat{\Theta[\mathcal{C}]}$ is exactly $\mathrm{Cell}(\mathscr{M})$, where

$$\mathscr{M} = \{\square_n^{\lrcorner}(\delta^n, \delta^{c_1}, \dots, \delta^{c_n}) \colon n \geq 0 \text{ and } c_1, \dots, c_n \in \operatorname{Ob}\mathcal{C}\},$$

where $\delta^n \colon \partial\Delta^n \hookrightarrow \Delta^n$ is the inclusion of the boundary.

1.3.6. Proposition. For any inner anodyne inclusion $\iota \colon K \hookrightarrow \Delta^n$ and any family $f_1, \dots, f_n$ of monomorphisms of $\widehat{\mathcal{C}}$, the map

$$\square_n^{\lrcorner}(\iota, f_1, \dots, f_n)$$

is horizontal inner anodyne.

1.3.7. Theorem. The horizontal Joyal model structure is Cartesian closed, and in particular,

$$\mathrm{Cell}(\mathscr{M}) \times^{\lrcorner} \mathrm{Cell}(\mathscr{J}) \subseteq \mathrm{Cell}(\mathscr{J}).$$

1.3.8. Theorem. A horizontal inner fibration between formal $\mathcal{C}$-quasicategories is a fibration for the horizontal Joyal model structure if and only if it has the right lifting property with respect to the map $\Delta^0 \hookrightarrow E^1$. In particular, the formal $\mathcal{C}$-quasicategories are exactly the fibrant objects for the horizontal Joyal model structure.

1.4. The corner tensor construction. The overwhelming majority of this section is due to Oury, although we had to redo some of the proofs, since they contained mistakes. Following [Our10, 3.1] we define the corner tensor, a vast generalization of the corner product.

1.4.1. Definition. Suppose we have a category $\mathcal{T}$ and an n-ary functor

$$\wedge \colon \mathcal{T}^n \to \mathcal{T}.$$

Let $(\mathcal{A}_i)_{i=1}^n$ be a family of categories and let $\mathcal{D}$ be a category admitting enough colimits such that all tensors with $\mathrm{Hom}_{\mathcal{T}}$ exist and coends over $\mathcal{T}$ exist. Let

$$\square \colon \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{D}$$

be a functor. Then we define the following functor:

$$\square^{\lrcorner} \colon \mathcal{A}_1^{\mathcal{T}} \times \cdots \times \mathcal{A}_n^{\mathcal{T}} \to \mathcal{D}^{\mathcal{T}}$$

by the Day convolution, for example,

$$\square^{\lrcorner}(M_1, \dots, M_n)(t) = \int^{u_1, \dots, u_n \in \mathcal{T}} \mathcal{T}(\wedge(u_1, \dots, u_n), t) \cdot \square(M_1(u_1), \dots, M_n(u_n)).$$

1.4.2. Lemma. The functor $\square^{\lrcorner}$ preserves all colimits preserved by $\square$ in each variable.

Proof. By coend manipulation. ∎

We specialize now to the case where $\mathcal{T} = [1]$ is the categorical 1-simplex. The functor $\wedge\colon [1]^n \to [1]$ is given by taking the infimum.

1.4.3. NOTE. We will often consider arrows in a category $\mathcal{D}$ as functors $[1] \to \mathcal{D}$. By abuse of notation, we will denote the functor $[1] \to \mathcal{D}$ classifying an arrow $f\colon A \to B$ simply by f. We will denote the evaluation of this functor on the objects of $x \in [1]$ by $f(x)$, such that in the case of a map $f\colon A \to B$,

$$f(0) = A \quad \text{and} \quad f(1) = B \tag{1}$$

1.4.4. DEFINITION. Given $\Box\colon \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{D}$, with $\mathcal{D}$ cocomplete, we define the *corner tensor* $\Box^\lrcorner\colon \mathcal{A}_1^{[1]} \times \cdots \times \mathcal{A}_n^{[1]} \to \mathcal{D}^{[1]}$ by the formula

$$\Box^\lrcorner(f_1, \ldots, f_n)(t) = \int^{u_1,\ldots,u_n \in [1]} [1](u_1 \wedge \cdots \wedge u_n, t) \cdot \Box(f_1(u_1), \ldots, f_n(u_n)).$$

If $(g, h)\colon f_i \to f_i'$ is a commutative square, let

$$(g^\lrcorner, h^-)\colon \Box^\lrcorner(f_1, \ldots, f_i, \ldots, f_n) \to \Box^\lrcorner(f_1, \ldots, f_i', \ldots, f_n)$$

be the induced commutative square.

1.4.5. NOTE. Let $[1]^n$ be the n-fold power of the poset $[1]$, which is a cube, and let $C_n = [1]^n - \{(1, \ldots, 1)\}$ be the subposet of the cube removing the terminal vertex. To unwind the coend, notice that the set $[1](u_1 \wedge \cdots \wedge u_n, 0)$ vanishes when all of the $u_i = 1$. We can therefore evaluate the domain of the corner tensor as the colimit of the restriction

$$\Box^\lrcorner(f_1, \ldots, f_n)(0) = \operatorname{colim} \Box(f_1, \ldots, f_n)|_{C_n}.$$

The codomain of the corner tensor can be computed by noticing that the set

$$[1](u_1 \wedge \cdots \wedge u_n, 1)$$

is always a singleton, and therefore the colimit can be computed simply as the colimit of the functor $\Box(f_1, \ldots, f_n)\colon [1]^n \to \mathcal{D}$. However, this colimit is indexed by a category with a terminal object and therefore agrees with the the evaluation at that terminal object. That is, we have

$$\Box^\lrcorner(f_1, \ldots, f_n)(1) = \Box(f_1(1), \ldots, f_n(1)).$$

1.4.6. EXAMPLE. *If we take $\Box$ to be a bifunctor $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$ in an appropriately cocomplete category, then given $f_1\colon A \to B$ and $f_2\colon C \to D$, their corner tensor is the familiar corner product:*

$$f_1 \Box^\lrcorner f_2 = \left(A\Box D \coprod_{A\Box C} B\Box C \to B\Box D \right).$$

1.4.7. Remark. *The previous example is in some sense universal, and it will allow us to reduce certain questions about big corner tensors to binary ones. In particular, we will frequently use the observation that, in the situation of the example the map*

$$f_1(0)\square f_2(1) \to (f_1\square^\lrcorner f_2)(0)$$

is a pushout of the map

$$f_1(0)\square f_2(0) \to f_1(1)\square f_2(0).$$

1.4.8. Example. *In the category of n-fold multisimplicial sets $\widehat{(\Delta)^n}$, we have an n-fold exterior product functor sending an n-tuple of simplicial sets $(S_1, \dots, S_n)$ to the exterior product $\square(S_1, \dots, S_n)$. It can be seen that the exterior product preserves colimits argument-by-argument, so applying the corner tensor, we can compute exterior corner products of maps. It is a fact beyond the scope of this paper that the Cisinski model structure on multisimplicial sets that models the homotopy theory of spaces has cellular generating cofibrations given by*

$$\square^\lrcorner(\delta^{m_1}, \dots, \delta^{m_n})$$

where $\delta^m\colon \partial\Delta^m \hookrightarrow \Delta^m$ denotes the boundary inclusion. The generating anodynes are given by

$$\square^\lrcorner(\delta^{m_1}, \dots, \lambda_k^{m_i}, \dots, \delta^{m_n})$$

where $i \in \{1, \dots, n\}$, $m_i > 0$, and $k \in \{0, \dots, m_i\}$, and where $\lambda_k^{m_i}\colon \Lambda_k^{m_i} \hookrightarrow \Delta^{m_i}$ is any horn inclusion. This generalizes the description of the generating anodynes and cofibrations for the Cisinski model structure on bisimplicial sets that models the homotopy theory of spaces.

1.4.9. Lemma. If for any $i \in \{0, \dots, n\}$, the map $f_i\colon A_i \to B_i$ is an identity map, then the corner tensor $\square^\lrcorner(f_1, \dots, f_n)$ is an identity map.

Proof. Assume for simplicity that $i = 1$, which is without loss of generality by reindexing. Then we proceed by setting

$$U(s,t) = \int^{u_1,\dots,u_n} ([1](u_1, s) \times [1](u_2 \wedge \cdots \wedge u_n, t)) \cdot \square(f_1(u_1), f_2(u_2), \dots, f_n(u_n)).$$

First, we observe that the maps $U(0,0) \to U(1,0)$ and $U(0,1) \to U(1,1)$ are identities, as they are given by evaluation of the partial coend on an identity. Now, notice that by Yoneda reduction,

$$\int^{s,t} [1](s \wedge t, x) \times [1](u_1, s) \times [1](u_2 \wedge \cdots \wedge u_n, t) = [1](u_1 \wedge u_2 \wedge \cdots \wedge u_n, x),$$

so we have that

$$\square^\lrcorner(f_1, \dots, f_n)(x) = \int^{s,t} [1](s \wedge t, x) \cdot U(s,t),$$

which exhibits

$$\square^{\lrcorner}(f_1,\ldots,f_n)(0) = U(0,1) \coprod_{U(0,0)} U(1,0),$$

but this tells us that the map $U(0,1) \to \square^{\lrcorner}(f_1,\ldots,f_n)(0)$ is the pushout of an identity map. Then by 3-for-2, we see that $\square^{\lrcorner}(f_1,\ldots,f_n)$ must also be the identity. ■

1.4.10. Note. We will suppress the objects in the next few diagrams for readability.

1.4.11. Lemma. Suppose that $\mathcal{A}_1$ admits pushouts and $\square$ preserves pushouts in its first argument. Suppose we have a coCartesian square in $\mathcal{A}_1$

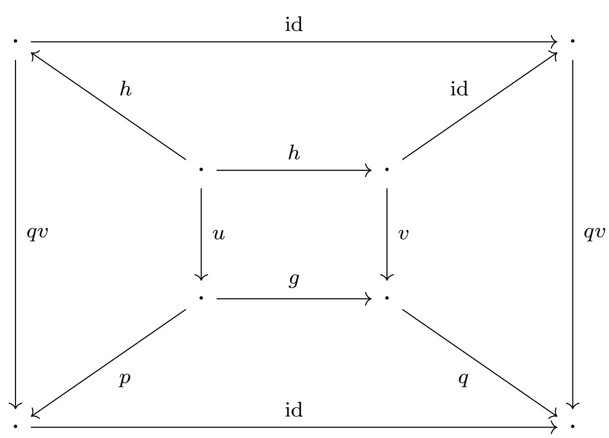
$$\begin{array}{ccc} \cdot & \xrightarrow{h} & \cdot \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ \cdot & \xrightarrow{g} & \cdot \end{array},$$

and suppose that we have a commutative square

$$\begin{array}{ccc} \cdot & \xrightarrow{g} & \cdot \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \cdot & \xrightarrow{\mathrm{id}} & \cdot \end{array}.$$

Given a family of maps $\mathbf{f} = (f_i \in \mathcal{A}_i^{[1]})_{i=2}^n$, let $Q_{\bullet,\mathbf{f}}$ denote the evaluation of $\square^{\lrcorner}(\bullet,\mathbf{f})$ at 0. Then $g^{\lrcorner}\colon Q_{p,\mathbf{f}} \to Q_{q,\mathbf{f}}$ is a pushout of $\square^{\lrcorner}(h,\mathbf{f})$.

Proof. The data allow us to construct a commutative cube

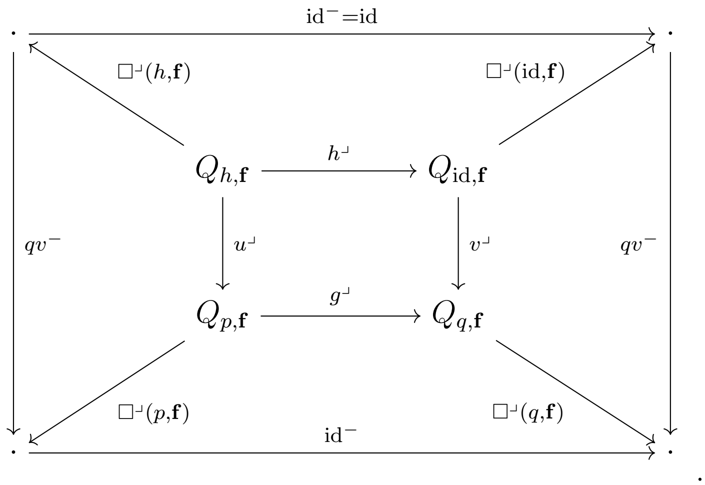


.

Since the front and back faces of this cube are coCartesian, it gives a coCartesian square in $\mathcal{A}_1^{[1]}$,

$$\begin{array}{ccc} h & \xrightarrow{(h,\mathrm{id})} & \mathrm{id} \\ {\scriptstyle (u,qv)}\downarrow & & \downarrow{\scriptstyle (v,qv)} \\ p & \xrightarrow{(g,\mathrm{id})} & q \end{array}.$$

Then applying $\square^{\lrcorner}(\bullet, \mathbf{f})$, we have a commutative cube



Its front and back faces remain pushouts, since $\square^{\lrcorner}$ preserves all colimits in each argument preserved by $\square$. Then the map $g^{\lrcorner}\colon Q_{p,\mathbf{f}} \to Q_{q,\mathbf{f}}$ is a pushout of the map $h^{\lrcorner}\colon Q_{h,\mathbf{f}} \to Q_{\mathrm{id},\mathbf{f}}$, but by Lemma 1.4.9, we see that $\square^{\lrcorner}(\mathrm{id}, \mathbf{f}) = \mathrm{id}$, so by commutativity, it follows that $h^{\lrcorner} = \square^{\lrcorner}(h, \mathbf{f})$, and therefore, $g^{\lrcorner}$ is a pushout of $\square^{\lrcorner}(h, \mathbf{f})$. ■

Assume in the sequel that each of the categories $(\mathcal{A}_i)_{i=1}^n$ admits connected colimits and that $\square$ preserves connected colimits in each argument

1.4.12. Lemma. [Our10, Lemma 3.10] Let $(\mathscr{J}_i)_{i=1}^n$ be a family of sets of morphisms of each $\mathcal{A}_i$. Then

$$\square^{\lrcorner}(\mathscr{J}_1, \ldots, \mathrm{Cell}(\mathscr{J}_k), \ldots, \mathscr{J}_n) \subseteq \mathrm{Cell}(\square^{\lrcorner}(\mathscr{J}_1, \ldots, \mathscr{J}_k, \ldots, \mathscr{J}_n)).$$

Proof. Without loss of generality, we may assume $k = 1$ by symmetry of the Cartesian product in **Cat**. Let f_1 be a map in $\mathcal{A}_1$ that belongs to $\mathrm{Cell}(\mathscr{J}_1)$. Then this gives the data of a cocontinous diagram $D\colon \alpha \to \mathcal{A}_1$ for an ordinal α with colimit C and structure maps $\phi_i\colon D(i) \to C$ such that $f_1 = \phi_0$. Additionally, for any $i < \alpha$, the maps $g_i\colon D(i) \to D(i+1)$ are pushouts of maps belonging to $\mathscr{J}_1$. Let $D^+\colon \alpha^{\triangleright} \to \mathcal{A}_1$ be the extension of D sending α to C and such that $D^+(i \to \alpha) = \phi_i$ for $i < \alpha$. Define $D^2\colon \alpha^{\triangleright} \to \mathcal{A}_1^{[1]}$ by the rule $i \mapsto \phi_i$ for $i < \alpha$ and $\alpha \mapsto \mathrm{id}_C$ and sending $i \to i+1$ to the commutative square $(g_i, \mathrm{id}_C)\colon \phi_i \to \phi_i{+}1$ and sending the map $i \to \alpha$ to the commutative square $(\phi_i, \mathrm{id}_C)\colon \phi_i \to \mathrm{id}_C$. Then the colimit of D^2 is id_C as pictured below.

$$\begin{array}{ccccccc} D(0) & \xrightarrow{g_0} & D(1) & \xrightarrow{g_1} & \cdots & \longrightarrow & C \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_i} & & \| \\ C & = & C & = & \cdots & = & C \end{array}.$$

Let $f_j \colon X_i \to Y_j \in \mathscr{J}_j$ for each $2 \leq j \leq n$. Denote $\Box^{\lrcorner}(\bullet, f_2, \ldots, f_n)$ by $\Box(\bullet, \mathbf{f})$ and the domain by $Q_{\bullet,\mathbf{f}}$. Then since $\Box^{\lrcorner}$ preserves connected colimits argument by argument, it follows that

$$\operatorname{colim} \Box^{\lrcorner}(D^2, \mathbf{f}) = \Box^{\lrcorner}(\mathrm{id}_C, \mathbf{f}),$$

and so we have the structure maps of the colimiting cocone

$$(\phi_i^{\lrcorner}, \mathrm{id}_C) \colon \Box^{\lrcorner}(\phi_i, \mathbf{f}) \to \Box^{\lrcorner}(\mathrm{id}_C, \mathbf{f}).$$

In particular, this demonstrates that the map

$$(\phi_0^{\lrcorner}, \mathrm{id}_C) \colon \Box^{\lrcorner}(\phi_0, \mathbf{f}) \to \Box^{\lrcorner}(\mathrm{id}_C, \mathbf{f})$$

is the transfinite composite of the maps

$$(g_i^{\lrcorner}, \mathrm{id}_C) \colon \Box^{\lrcorner}(\phi_i, \mathbf{f}) \to \Box^{\lrcorner}(\phi_{i+1}, \mathbf{f})$$

But by commutativity of the square

$$\begin{array}{ccc} Q_{\phi_0,\mathbf{f}} & \xrightarrow{\phi_0^{\lrcorner}} & Q_{\mathrm{id}_C,\mathbf{f}} \\ \downarrow{\scriptstyle \Box^{\lrcorner}(\phi_0,\mathbf{f})} & & \downarrow{\scriptstyle \Box^{\lrcorner}(\mathrm{id}_C,\mathbf{f})} \\ \Box(C, \mathbf{Y}) & \xrightarrow{\mathrm{id}} & \Box(C, \mathbf{Y}) \end{array},$$

we can see that

$$\Box^{\lrcorner}(\phi_0, \mathbf{f}) = \Box^{\lrcorner}(\mathrm{id}_C, \mathbf{f}) \circ \phi_0^{\lrcorner},$$

but from Lemma 1.4.9, we see

$$\Box^{\lrcorner}(\mathrm{id}_C, \mathbf{f}) = \mathrm{id},$$

so it immediately follows that $\Box^{\lrcorner}(\phi_0, \mathbf{f}) = \phi_0^{\lrcorner}$. This exhibits $\Box^{\lrcorner}(\phi_0, \mathbf{f})$ as a transfinite composite of the $g_i^{\lrcorner}$. Then it suffices to show that the $g_i^{\lrcorner}$ are pushouts of maps belonging to

$$\Box^{\lrcorner}(\mathscr{J}_1, , \ldots, \mathscr{J}_n).$$

Since each g_i was a pushout of a morphism $h_i \in \mathscr{J}_1$, and since we have a commutative square $(g_i, \mathrm{id}) \colon \phi_i \to \phi_{i+1}$, we are exactly in the situation of Lemma 1.4.11, which implies that each $g_i^{\lrcorner}$ is a pushout of $\Box^{\lrcorner}(h_i, \mathbf{f})$, which proves the proposition. ∎

1.4.13. Corollary. There is an inclusion

$$\square^{\lrcorner}(\mathrm{Cell}(\mathscr{J}_1),\dots,\mathrm{Cell}(\mathscr{J}_n)) \subseteq \mathrm{Cell}(\square^{\lrcorner}(\mathscr{J}_1,\dots,\mathscr{J}_n)).$$

Proof. Apply the previous lemma n times, using the fact that Cell is idempotent, since it is a closure operator. ∎

1.4.14. Definition. Let Rex_c denote the symmetric sub-multi-category of **Cat** whose objects are the categories admitting all finite colimits and whose k-morphisms are the k-ary functors that preserve finite connected colimits.

1.4.15. Observation. The corner tensor construction, sending a multimorphism

$$F\colon \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{D}$$

to its corner tensor

$$F^{\lrcorner}\colon \mathcal{A}_1^{[1]} \times \cdots \times \mathcal{A}_n^{[1]} \to \mathcal{D}^{[1]}$$

is a morphism of multicategories from Rex_c to itself. In particular, it is functorial with respect to the composition in Rex_c.

1.5. The regular Reedy structure of $\Theta[\mathcal{C}]$. In this section, we will prove some useful and interesting properties about regular Cartesian Reedy categories (see Definition 1.1.7). In particular, we demonstrate that the class of monomorphisms is exactly the class of relative cell complexes for a set of maps $\mathscr{M}$, which we will show coincides with the set of boundary maps for the regular skeletal Reedy category $\Theta[\mathcal{C}]$.

1.5.1. Proposition. The category $\Theta[\mathcal{C}]$ is a regular skeletal Reedy category whenever $\mathcal{C}$ is a regular Cartesian Reedy category with the following data:

- The dimension function of this regular Reedy category is given by

$$\dim[n](c_1,\dots,c_n) \overset{\text{def}}{=} n + \dim_{\mathcal{C}} c_1 + \cdots + \dim_{\mathcal{C}} c_n.$$

- A map $f\colon [n](c_1,\dots,c_n) \to [m](d_1,\dots,d_m)$ belongs to $\Theta[\mathcal{C}]^+$ if the underlying map of simplices $f_\Delta\colon [n] \to [m]$ belongs to Δ^+ and if for all $1 \le i \le n$, the section

$$h_{c_i} \to \prod_{j=f_\Delta(i-1)+1}^{f_\Delta(i)} h_{d_j}$$

  is nondegenerate (see Definition A.4.7).

- A map $f\colon [n](c_1,\dots,c_n) \to [m](d_1,\dots,d_m)$ belongs to $\Theta[\mathcal{C}]^-$ if the underlying map of simplices $f_\Delta\colon [n] \to [m]$ belongs to Δ^- and if for all i, j with $f_\Delta(i-1) < f_\Delta(i)$ and $f_\Delta(i-1)+1 \le j \le f(i)$, each of the maps $f_{ij}\colon c_i \to d_j$ belongs to $\mathcal{C}^-$.

Proof. As noted earlier, a regular Cartesian Reedy category admits a canonical Reedy multicategory structure. It is automatically also EZ-Reedy in the sense of [BR11, Definition 4.1] by the axioms for skeletal categories (see Definition A.4.1).

It follows therefore by [BR11, Proposition 4.4] that $\Theta[\mathcal{C}]$ is normal skeletal Reedy with the desired dimension function (see Proposition A.4.4). To prove that $\Theta[\mathcal{C}]$ is regular, it suffices to show that any nondegenerate section

$$f\colon [n](c_1,\dots,c_n)\to [m](d_1,\dots,d_m)$$

is monic. Let $g,g'\colon [\ell](a_1,\dots,a_\ell)\to [n](c_1,\dots,c_n)$ be two maps such that $f\circ g = f\circ g'$. Since the map f_Δ is injective, it follows that $g_\Delta = g'_\Delta$. Since they are equal, let $s \overset{\text{def}}{=} g_\Delta = g'_\Delta$. Then it suffices to show for each $1\leq k\leq \ell$, the maps

$$g_k, g'_k\colon h_{a_k}\to \prod_{i=s(k-1)+1}^{s(k)} h_{c_i}$$

are equal. However, since $f\circ g = f\circ g'$, we have the composite

$$(f)_{i=s(k-1)+1}^{s(k)}\circ g_k\colon h_{a_k}\to \prod_{i=s(k-1)+1}^{f_\Delta(i)}\ \prod_{j=f_\Delta(i-1)+1}^{f_\Delta(i)} h_{d_j}$$

and the composite

$$(f)_{i=s(k-1)+1}^{s(k)}\circ g'_k\colon h_{a_k}\to \prod_{i=s(k-1)+1}^{f_\Delta(i)}\ \prod_{j=f_\Delta(i-1)+1}^{f_\Delta(i)} h_{d_j}$$

are equal. But the map $(f)_{i=s(k-1)+1}^{s(k)}$ is monic because it is a product of the maps

$$f_i\colon h_{c_i}\to \prod_{j=f_\Delta(i-1)+1}^{f_\Delta(i)} h_{d_j},$$

which are nondegenerate because f belongs to $\Theta[\mathcal{C}]^+$ and monic because $\mathcal{C}$ is regular Cartesian Reedy. It follows therefore that $g_k = g'_k$ for all $1\leq k\leq \ell$. Therefore $g = g'$, which proves that the map f is monic. ∎

1.5.2. Proposition. The boundary $\partial[n](c_1,\dots,c_n)$ can be computed using the corner-intertwiner (see 1.3.1)

$$Q = \square_n^{\lrcorner}(\delta^n,\delta^{c_1},\dots,\delta^{c_n})(0).$$

Proof. To unwind the definition of Q, let

$$U_i = V_{\delta^n(\nu_{0i})}(\delta^{c_1}(\nu_{1i}), \dots, \delta^{c_n}(\nu_{ni}))$$

for $0 \leq i \leq n$, where

$$\nu_{ki} = \begin{cases} 0 & \text{if} \quad k = i \\ 1 & \text{otherwise.} \end{cases}$$

Since the intertwiner is a right adjoint, it preserves monomorphisms, so the canonical map $U_i \to [n](c_1, \dots, c_n)$ is injective. Let $U = \coprod_{i=0}^n U_i$. Then we have

$$Q \cong \operatorname{colim}(U \times_{[n](c_1,\dots,c_n)} U \rightrightarrows U).$$

In other words, we have

$$Q \cong V_{\partial\Delta^n}(c_1, \dots, c_n) \cup \left(\bigcup_{i=1}^n V[n](c_1, \dots, \partial c_i \dots, c_n) \right) \hookrightarrow [n](c_1, \dots, c_n),$$

which exhibits Q as a proper subpresheaf of the representable presheaf $[n](c_1, \dots, c_n)$. Since $\partial([n](c_1, \dots, c_n))$ is the maximal proper subpresheaf (by regularity), we see that $Q \subseteq \partial([n](c_1, \dots, c_n))$. Suppose conversely that $[m](d_1, \dots, d_m) \to [n](c_1, \dots, c_n)$ is a nondegenerate section with $\dim[m](d_1, \dots, d_m) < \dim[n](c_1, \dots, c_n)$. As it is nondegenerate, we see immediately that $m \leq n$.

Suppose $m < n$. Then the map $[m] \to [n]$ factors through $V_{\partial\Delta^n}(c_1, \dots, c_n) \subset Q$. Therefore, suppose $m = n$.

By the strictness of the inequality

$$\dim[m](d_1, \dots, d_m) < \dim[n](c_1, \dots, c_n),$$

we see that there exists some k with $1 \leq k \leq n$ such that $\dim d_k < \dim c_k$, as otherwise the dimensions would be equal, since $\dim d_i \leq dimc_i$ for all $1 \leq i \leq n$ by nondegeneracy. Then it follows immediately that

$$[m](d_1, \dots, d_m) \subset V[n](c_1, \dots, \partial c_k, \dots, c_n),$$

which proves the proposition. ∎

1.5.3. Corollary. We define the set

$$\mathscr{M} = \{\partial[t] \to [t] \mid [t] \in \operatorname{Ob}(\Theta[\mathcal{C}])\}.$$

Then the class $\operatorname{Cell}(\mathscr{M})$ is exactly the the class of monomorphisms of $\widehat{\Theta[\mathcal{C}]}$.

Proof. This is an immediate consequence of [Cis06, Proposition 8.1.37] or [BR11, 4.4]. ∎

1.5.4. Proposition. The category $\Theta[\mathcal{C}]$ is regular Cartesian Reedy when $\mathcal{C}$ is.

Proof. We treat the case of binary products of representables. The case of more general finite products of representables is similar, albeit more notation-heavy. We leave the details to the reader.

By our calculations in §1.2, since all involved objects are sober, we see that for a section

$$(\alpha,\beta)\colon [n](c_1^n,\dots,c_n^n) \to [m_1](c_1^{m_1},\dots,c_{m_1}^{m_1}) \times [m_2](c_1^{m_2},\dots,c_{m_2}^{m_2})$$

to be nondegenerate, the associated map of simplicial sets $[n] \to [m_1]\times[m_2]$ is monic. The map (α,β) therefore factors through the pullback of the labeling on $[m_1](c_1^{m_1},\dots,c_{m_1}^{m_1}) \times [m_2](c_1^{m_2},\dots,c_{m_2}^{m_2})$ to $[n]$, denoted by $([n],(\alpha,\beta)^*\Omega)$. The map

$$([n],(\alpha,\beta)^*\Omega) \to [m_1](c_1^{m_1},\dots,c_{m_1}^{m_1}) \times [m_1](c_1^{m_2},\dots,c_{m_2}^{m_2})$$

is monic, so it will be enough to show that the map

$$\iota\colon [n](c_1^n,\dots,c_n^n) \to ([n],(\alpha,\beta)^*\Omega)$$

must also be monic. The section ι is certainly nondegenerate, since if it were not, the map (α,β) could not be either.

We will directly calculate $(\alpha,\beta)^*\Omega$ in Observation 1.6.13, and it will be shown that the labeling, determined by its restriction to each edge $(e_i)_{i=1}^n$ of the spine of $[n]$ is given by the formula

$$e_i^*(\alpha,\beta)^*\Omega = \prod_{k_\alpha=\alpha(i-1)+1}^{\alpha(i)} c_{k_\alpha}^{m_1} \times \prod_{k_\beta=\beta(i-1)+1}^{\beta(i)} c_{k_\beta}^{m_2}.$$

Since the section ι is nondegenerate and the associated map on simplicial sets is an isomorphism, we see that each of the sections

$$\iota_i\colon c_i^n \to \prod_{k_\alpha=\alpha(i-1)+1}^{\alpha(i)} c_{k_\alpha}^{m_1} \times \prod_{k_\beta=\beta(i-1)+1}^{\beta(i)} c_{k_\beta}^{m_2}$$

must be nondegenerate as well, and therefore since $\mathcal{C}$ is regular Cartesian Reedy, it follows that each of the maps ι_i is monic, which proves the axiom (CR1) and also that

$$\dim c_i^n \leq \sum_{k_\alpha=\alpha(i-1)+1}^{\alpha(i)} \dim c_{k_\alpha}^{m_1} + \sum_{k_\beta=\beta(i-1)+1}^{\beta(i)} \dim c_{k_\beta}^{m_2}.$$

But it follows from this that

$$\begin{aligned}\dim[n](c_1^n,\dots c_n^n) &= n + \sum_{i=1}^{n} \dim c_i^n \\ &\le n + \sum_{i=1}^{n}\left(\sum_{k_\alpha=\alpha(i-1)+1}^{\alpha(i)} \dim c_{k_\alpha}^{m_1} + \sum_{k_\beta=\beta(i-1)+1}^{\beta(i)} \dim c_{k_\beta}^{m_2}\right) \\ &\le m_1 + \sum_{i=1}^{m_1} \dim c_i^{m_1} + m_2 + \sum_{j=1}^{m_2} \dim c_j^{m_2} \\ &= \dim[m_1](c_1^m,\dots,c_{m_1}^{m_1}) + \dim[m_2](c_1^m,\dots,c_{m_2}^{m_2}),\end{aligned}$$

which proves the axiom (CR2).

The axiom (CR3) is immediate, since $\Theta[\mathcal{C}]$ has a terminal object $*$ and $\dim * = 0$. ■

1.5.5. Remark. *The proposition above gives us a way to construct new regular Cartesian Reedy categories from known examples by applying the functor $\Theta[-]$, which we view as a kind of free-generation-under-suspension functor. Moreover, regular Cartesian Reedy categories are stable under certain kinds of filtered unions (at least filtered unions of fully faithful maps preserving the dimension grading). The list of such categories generated from the terminal category under the operation $\Theta[-]$ is exactly the family of Θ_n for $0 \le n < \omega$, and stabilizng under good-enough filtered unions, we also obtain the category $\Theta = \Theta_\omega$. These are ultimately the only examples we care about in this paper.*

Though we do not need it here, it can be also be shown that finite products of regular Cartesian Reedy categories are also regular Cartesian Reedy. Starting with the terminal category and taking finite products, $\Theta[-]$, and good-enough filtered unions, we obtain a large supply of regular Cartesian Reedy categories. We leave it as an open question as to whether or not these operations generate all examples.

1.6. The anodyne theorem for horizontal inner anodynes. In this section, following [Our10, 3.4.4], we will demonstrate that the horizontal inner anodynes are closed under corner products with monomorphisms. As a corollary of the analysis in this section, we will complete the proof that $\Theta[\mathcal{C}]$ is regular Cartesian Reedy. We make no claim to originality.

1.6.1. Definition. Given a simplicial set S define the functor

$$H_S\colon \left(\widehat{\Delta}\downarrow S\right) \times \left(\widehat{\Delta}\int\widehat{\mathcal{C}}\right)_S \to \widehat{\Delta}\int\widehat{\mathcal{C}}$$

by the rule

$$\left(S' \xrightarrow{f} S, (S,\Omega)\right) \mapsto (S', f^*\Omega),$$

and we define the *relative intertwiner over S*

$$\square_S \overset{\text{def}}{=} \square \circ H_S.$$

Notice that when $S = \Delta^n$, the fibre decomposes as

$$\left(\widehat{\Delta} \int \widehat{\mathcal{C}}\right)_{\Delta^n} \simeq \widehat{\mathcal{C}}^n \simeq \underbrace{\widehat{\mathcal{C}} \times \cdots \times \widehat{\mathcal{C}}}_{n \text{ times}}$$

So we can write

$$\square_n\colon \left(\widehat{\Delta} \downarrow \Delta^n\right) \times \widehat{\mathcal{C}}^n \to \widehat{\Theta[\mathcal{C}]}.$$

1.6.2. Observation. Given a labeled simplicial set (S, Ω), a map of simplicial sets

$$f\colon Y \to S,$$

and an object $[t] = [n](c_1, \ldots, c_n)$ of $\Theta[\mathcal{C}]$, we can compute the set $\square_S(f, \Omega)_t$ as follows: For any n-simplex $s \in S_n$, let $(W_{s,i})_{i=1}^n$ be the family of $\mathcal{C}$-sets obtained by evaluation of Ω on s. A map $[t] \to Y \square f^*\Omega$ is by definition a map $[t] \to (Y, f^*\Omega)$. Such a map is determined by giving an n-simplex $y \in Y_n$ together with a family of maps

$$(c_i \xrightarrow{\zeta_i} W_{fy,i})_{i=1}^n.$$

Then we can compute

$$\square_S(f, \Omega)_t \cong \coprod_{y \in Y_n} \prod_{i=1}^n W_{fy,i,c_i}.$$

1.6.3. Definition. Given a finite family of simplicial sets $\mathbf{S} = (S_i)_{i=1}^n$, we define a functor:

$$H_{\mathbf{S}}\colon \left(\widehat{\Delta} \downarrow \prod_{i=1}^n S_i\right) \times \prod_{i=1}^n \left(\widehat{\Delta} \int \widehat{\mathcal{C}}\right)_{S_i} \to \widehat{\Delta} \int \widehat{\mathcal{C}}$$

by the rule:

$$\left(S \xrightarrow{\prod_{i=1}^n f_i} \prod_{i=1}^n S_i, (\Omega_i)_{i=1}^n\right) \mapsto \left(S, \prod_{i=1}^n (f_i^*\Omega_i)\right),$$

where the expression

$$\left(S, \prod_{i=1}^n (f_i^*\Omega_i)\right)$$

means the product of the labels $f_i^*\Omega_i$ in the fibre over S.

As in the previous definition, we define the *relative multi-intertwiner* by the formula

$$\square_{\mathbf{S}} \overset{\text{def}}{=} \square \circ H_{\mathbf{S}}.$$

1.6.4. Remark. *Notice that if we are given a finite ordered family of labeled simplicial sets*

$$(\mathbf{S}, \mathbf{\Omega}) = (S_i, \Omega_i)_{i=1}^n$$

and a family of maps

$$\mathbf{f} = (f_i \colon S \to S_i)_{i=1}^n ,$$

we have a canonical isomorphism

$$H_{\mathbf{S}}(\mathbf{f}, \mathbf{\Omega}) \cong H_{S_1}(f_1, \Omega_1) \times^S \cdots \times^S H_{S_n}(f_n, \Omega_n),$$

where $\times^S$ *denotes the product in* $\left(\widehat{\Delta} \int \widehat{\mathcal{C}}\right)_S$.

1.6.5. Observation. Let

$$(\mathbf{S}, \mathbf{\Omega}) = (\Delta^{m_i}, \Omega_i)_{i=1}^n$$

be a family of labeled simplices, and let

$$\mathbf{f} = (f_i \colon \Delta^r \to \Delta^{m_i})_{i=1}^n ,$$

be a family of maps defining an r-simplex of the product. We may identify the Ω_i with families of $\mathcal{C}$-presheaves $(X_{i,\ell})_{\ell=1}^{m_i}$, so we compute $H_{\Delta^{m_i}}(f_i, \Omega_i)$ as the labeled simplex

$$[r]\left(\left(\prod_{k=f_i(j-1)+1}^{f_i(j)} X_{i,k}\right)_{j=1}^{r}\right),$$

and therefore, we can compute $H_{\mathbf{S}}(\mathbf{f}, \mathbf{\Omega})$ as the labeled simplex

$$[r]\left(\left(\prod_{i=1}^{n}\left(\prod_{k=f_i(j-1)+1}^{f_i(j)} X_{i,k}\right)\right)_{j=1}^{r}\right).$$

1.6.6. Lemma. The relative intertwiner $\square_S$ preserves colimits in the first variable.

Proof. Since colimits are computed objectwise in presheaves, it suffices to show that the functor $\square_S(\bullet, \Omega)_t$ preserves colimits for all $[t] = [n](c_1, \ldots, c_n) \in \Theta[\mathcal{C}]$ and all labels Ω of S. Therefore, it suffices by 1.6.2 to show this in the case where $\mathcal{C}$ is the terminal category, since we may fix the family of objects $(c_1, \ldots, c_n)$. For each $s \in S_n$ let $(W_{s,i})_{i=1}^n$ be the evaluation of Ω on s. Then given $f \colon Y \to S$, we have a Cartesian square

$$\begin{array}{ccc} \coprod_{y \in Y_n} \prod_{i=1}^n W_{fy,i} & \longrightarrow & Y_n \\ \downarrow{\scriptstyle \tau^* f} & & \downarrow{\scriptstyle f} \\ \coprod_{s \in S_n} \prod_{i=1}^n W_{s,i} & \xrightarrow{\tau} & S_n \end{array},$$

exhibiting $\coprod_{y \in Y_n} \prod_{i=1}^n W_{fy,i}$ as the pullback of f along τ, but by the universality of colimits in the category of sets, we are done. ∎

1.6.7. Note. This is Oury's proof, but this statement can also be seen to follow immediately from Proposition 1.2.12. To see this, notice that for any simplicial set Y, we have that $Y\Box * \cong \mathscr{H}(Y)$. Moreover, since $\mathscr{H}$ is a left adjoint, it commutes with colimits.

Then, notice that

$$\Box_S(f\colon S' \to S, (S,\Omega)) \cong S'\Box f^*\Omega.$$

The lemma tells us then that

$$\Box_S(f\colon S' \to S, (S,\Omega)) \cong S'\Box * \times_{S\Box *} S\Box\Omega,$$

which is equivalently

$$\mathscr{H}(S') \times_{\mathscr{H}(S)} S\Box\Omega.$$

But colimits in $\widehat{\Theta[\mathcal{C}]}$ are universal, so given a diagram $F\colon I \to \left(\widehat{\Delta} \downarrow S\right)$, we have an isomorphism

$$\begin{aligned}\operatorname*{colim}_{i\in I}(\Box_S(F(i) \to S, (S,\Omega))) &\cong \operatorname*{colim}_{i\in I}(\mathscr{H}(F(i)) \times_{\mathscr{H}(S)} S\Box\Omega)\\ &\cong (\operatorname*{colim}_{i\in I}\mathscr{H}(F(i))) \times_{\mathscr{H}(S)} S\Box\Omega\\ &\cong \Box_S(\operatorname*{colim}_{i\in I}F(i) \to S, (S,\Omega)),\end{aligned}$$

as desired.

1.6.8. Lemma. The relative intertwiner $\Box_n$ preserves connected colimits in each variable.

Proof. We saw from the previous lemma that this functor preserves colimits in the first variable, so representing $f\colon Y \to \Delta^n$ as the canonical colimit over its category of elements $(\Delta \downarrow Y)$, we immediately reduce to the case where $Y = [p]$ is a simplex. But we know in this case that any map $f\colon [p] \to [n]$ factors as a degeneracy followed by a face map. In the case that f is a face map, we can compute the pullback of $V[n](X_1,\dots,X_n) \to V[n](*,\dots,*)$ along $f\Box *\colon [p](*,\dots,*) \to [n](*,\dots,*)$ to be

$$V[p]\left(\prod_{i=f(0)+1}^{f(1)} X_i, \dots, \prod_{i=f(p-1)}^{f(p)} X_i\right).$$

By universality of colimits in $\widehat{\mathcal{C}}$, we see that it suffices to show that

$$V[p]\,(\bullet,\dots,\bullet) = \Box_p(\mathrm{id}_{\Delta^p},\bullet,\dots,\bullet)$$

preserves connected colimits in each variable. In the case where f is a degeneracy map, we can compute the pullback over $[p]$ to be

$$V[p](*,\dots,X_1,\dots,*,\dots,X_n,\cdots *),$$

where we fill in the terminal object of $\widehat{\mathcal{C}}$ in each argument i where $f(i-1) = f(i)$. In this case again, it again suffices to show that $V[p]$ preserves connected colimits in each variable, but this is precisely the content of [Rez10, Proposition 4.5], where the first step of the proof is to show that when we set $X_k = \varnothing$, we have a natural isomorpism of functors

$$V[p+1+q](A_1, \dots, A_p, \varnothing, B_1, \dots, B_q) \cong V[p](A_1, \dots, A_p) \coprod V[q](B_1, \dots, B_q),$$

and the second step exhibits the obvious parametric right adjoint

$$\left(\left(V[p](A_1, \dots, A_p) \coprod V[q](B_1, \dots, B_q)\right) \downarrow \widehat{\Theta[\mathcal{C}]}\right) \to \widehat{\mathcal{C}}.$$

■

1.6.9. Note. This proof is substantially easier than Oury's proof, which relies on a long direct computation.

1.6.10. Definition. Since the categories $\widehat{\mathcal{C}}$, $\widehat{\Theta[\mathcal{C}]}$, $\widehat{\mathcal{C}}^n$, and $\left(\widehat{\Delta} \downarrow \Delta^n\right)$ are all cocomplete (since they are all presheaf categories), and since the intertwiner preserves connected colimits argument-by-argument, we can use 1.4 to define the functor

$$\square^{\lrcorner}_n \colon \left(\widehat{\Delta} \downarrow \Delta^n\right)^{[1]} \times \underbrace{\widehat{\mathcal{C}}^{[1]} \times \cdots \times \widehat{\mathcal{C}}^{[1]}}_{n \text{ times}} \to \widehat{\Theta[\mathcal{C}]}^{[1]},$$

called the *corner intertwiner*.

More generally, for any finite family of simplices $(\Delta^{m_i})_{i=1}^n$, we can do the same trick and define the *corner-multi-intertwiner*

$$\square^{\lrcorner}_{m_1,\dots,m_n} \colon \left(\widehat{\Delta} \downarrow \Delta^{m_1} \times \cdots \times \Delta^{m_n}\right)^{[1]} \times \left(\widehat{\mathcal{C}}^{[1]}\right)^{m_1} \times \cdots \times \left(\widehat{\mathcal{C}}^{[1]}\right)^{m_n} \to \widehat{\Theta[\mathcal{C}]}^{[1]}.$$

Following [Our10, 3.85], we begin with the following observations:

1.6.11. Observation. We saw from the definition of $\square$ and the definition of products in $\widehat{\Delta} \int \widehat{\mathcal{C}}$ that the diagram

$$\begin{array}{ccc} \widehat{\Delta} \int \widehat{\mathcal{C}} \times \widehat{\Delta} \int \widehat{\mathcal{C}} & \xrightarrow{\square \times \square} & \widehat{\Theta[\mathcal{C}]} \times \widehat{\Theta[\mathcal{C}]} \\ \downarrow{\scriptstyle \times} & & \downarrow{\scriptstyle \times} \\ \widehat{\Delta} \int \widehat{\mathcal{C}} & \xrightarrow{\quad \square \quad} & \widehat{\Theta[\mathcal{C}]} \end{array}$$

commutes. We also compute that the diagram

$$\begin{array}{ccc}
\left(\widehat{\Delta}\downarrow\Delta^n\right)\times\widehat{\mathcal{C}}^n\times\left(\widehat{\Delta}\downarrow\Delta^m\right)\times\widehat{\mathcal{C}}^m & \xrightarrow{H_n\times H_m} & \widehat{\Delta}\int\widehat{\mathcal{C}}\times\widehat{\Delta}\int\widehat{\mathcal{C}} \\
\Big\downarrow{\scriptstyle \varsigma} & & \\
\left(\widehat{\Delta}\downarrow\Delta^n\right)\times\left(\widehat{\Delta}\downarrow\Delta^m\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m & & \Big\downarrow{\scriptstyle \times} \\
\Big\downarrow{\scriptstyle P\times\mathrm{id}\times\mathrm{id}} & & \\
\left(\widehat{\Delta}\downarrow\Delta^n\times\Delta^m\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m & \xrightarrow{H_{n,m}} & \widehat{\Delta}\int\widehat{\mathcal{C}}
\end{array}$$

commutes as well where ς permutes the factors and P is the functor sending a pair of simplicial sets $f\colon S\to\Delta^n$ and $g\colon S'\to\Delta^m$ over Δ^n and Δ^m respectively to the simplicial set

$$f\times g\colon S\times S'\to\Delta^n\times\Delta^m$$

over the product $\Delta^n\times\Delta^m$. Taking these two diagrams together, we see that the diagram

$$\begin{array}{ccc}
\left(\widehat{\Delta}\downarrow\Delta^n\right)\times\widehat{\mathcal{C}}^n\times\left(\widehat{\Delta}\downarrow\Delta^m\right)\times\widehat{\mathcal{C}}^m & \xrightarrow{\Box_n\times\Box_m} & \widehat{\Theta[\mathcal{C}]}\times\widehat{\Theta[\mathcal{C}]} \\
\Big\downarrow{\scriptstyle \varsigma} & & \\
\left(\widehat{\Delta}\downarrow\Delta^n\right)\times\left(\widehat{\Delta}\downarrow\Delta^m\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m & & \Big\downarrow{\scriptstyle \times} \\
\Big\downarrow{\scriptstyle P\times\mathrm{id}\times\mathrm{id}} & & \\
\left(\widehat{\Delta}\downarrow\Delta^n\times\Delta^m\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m & \xrightarrow{\Box_{n,m}} & \widehat{\Theta[\mathcal{C}]}
\end{array}$$

also commutes.

Then by 1.6.8, every functor appearing in this diagram preserves connected colimits in each argument, the intertwiners by the lemma, and the functors P and $\times$, since they are products in presheaf categories and therefore preserve colimits in both arguments. Then by the functoriality of the corner tensor functor 1.4.15, we obtain a commutative diagram

$$\begin{array}{ccc}
\left(\widehat{\Delta}\downarrow\Delta^n\right)^{[1]}\times\left(\widehat{\mathcal{C}}^{[1]}\right)^n\times\left(\widehat{\Delta}\downarrow\Delta^m\right)^{[1]}\times\left(\widehat{\mathcal{C}}^{[1]}\right)^m & \xrightarrow{\square_n^{\lrcorner}\times\square_m^{\lrcorner}} & \widehat{\Theta[\mathcal{C}]}^{[1]}\times\widehat{\Theta[\mathcal{C}]}^{[1]} \\
\downarrow{\scriptstyle \varsigma} & & \\
\left(\widehat{\Delta}\downarrow\Delta^n\right)^{[1]}\times\left(\widehat{\Delta}\downarrow\Delta^m\right)^{[1]}\times\left(\widehat{\mathcal{C}}^{[1]}\right)^n\times\left(\widehat{\mathcal{C}}^{[1]}\right)^m & & \downarrow{\scriptstyle \times^{\lrcorner}} \\
\downarrow{\scriptstyle P^{\lrcorner}\times\mathrm{id}\times\mathrm{id}} & & \\
\left(\widehat{\Delta}\downarrow\Delta^n\times\Delta^m\right)^{[1]}\times\left(\widehat{\mathcal{C}}^{[1]}\right)^n\times\left(\widehat{\mathcal{C}}^{[1]}\right)^m & \xrightarrow{\square_{n,m}^{\lrcorner}} & \widehat{\Theta[\mathcal{C}]}^{[1]}
\end{array}$$

also commutes, where $P^{\lrcorner} = \times^{\lrcorner}$ is the corner product of simplicial sets.

1.6.12. Observation. Consider the corner product of a simplicial inner horn inclusion with a simplicial boundary inclusion

$$\lambda_j^n \times^{\lrcorner} \delta^m \colon \Lambda_j^n \times \Delta^m \cup \Delta^n \times \partial\Delta^m \hookrightarrow \Delta^n \times \Delta^m.$$

Then it is a standard fact of quasicategory theory that we can factor this map as a sequence

$$\Lambda_j^n \times \Delta^m \cup \Delta^n \times \partial\Delta^m = X_0 \subseteq X_1 \subseteq \cdots \to X_{k-1} \subseteq X_k = \Delta^n \times \Delta^m$$

where each inclusion $X_{i-1} \hookrightarrow X_i$ is the pushout of an inner horn inclusion $\Lambda_{\ell_i}^{r_i} \to \Delta^{r_i}$ along an inclusion $\Lambda_{\ell_i}^{r_i} \hookrightarrow X_{i-1}$. By the construction of the sequence, each $[r_i] \to X_i \to \Delta^n \times \Delta^m$ is nondegenerate and does not factor through X_{i-1}, so in particular, it does not factor through X_0, and therefore the components $\alpha_i \colon \Delta^{r_i} \to \Delta^n$ and $\beta_i \colon \Delta^{r_i} \to \Delta^m$ do not factor through Λ_j^n or $\partial\Delta^m$. In particular, the image of α_i is either $\partial_j\Delta^n$ or all of Δ^n, and the image of β_i must be all of Δ^m, so all three maps α_i, β_i, and (α_i, β_i) send the initial and terminal vertices of Δ^{r_i} to the initial and terminal vertices of Δ^n, Δ^m, and $\Delta^n \times \Delta^m$ respectively.

1.6.13. Observation. Let $(\alpha, \beta) \colon \Delta^r \to \Delta^n \times \Delta^m$ be an injective map preserving initial and terminal elements. Let $\mathbf{A} = (A_i)_{i=1}^n$ and $\mathbf{B} = (B_i)_{i=1}^m$ be objects of $\widehat{\mathcal{C}}^n$ and $\widehat{\mathcal{C}}^m$ respectively. Let

$$K_{\alpha,\beta} \colon \widehat{\mathcal{C}}^n \times \widehat{\mathcal{C}}^m \to \widehat{\mathcal{C}}^r$$

be the functor defined by the rule

$$(\mathbf{U}, \mathbf{V}) \mapsto \alpha^*\mathbf{U} \times \beta^*\mathbf{V},$$

taking the product of the pullbacks to the fibre over Δ^r. Then we have a diagram:

$$\begin{array}{ccc}
\left(\widehat{\Delta}\downarrow\Delta^r\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m & \xrightarrow{\mathrm{id}\times K_{\alpha,\beta}} & \left(\widehat{\Delta}\downarrow\Delta^r\right)\times\widehat{\mathcal{C}}^r \\
\Big\downarrow{\scriptstyle ((\alpha,\beta)\circ(-))\times\mathrm{id}\times\mathrm{id}} & & \Big\downarrow{\scriptstyle H_r} \\
\left(\widehat{\Delta}\downarrow\Delta^n\times\Delta^m\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m & \xrightarrow{H_{n,m}} & \widehat{\Delta}\int\widehat{\mathcal{C}}
\end{array}$$

To show that the diagram commutes, let $p\colon X\to\Delta^r$ be a map. Then evaluating $H_r(p,K_{\alpha,\beta}(\mathbf{U},\mathbf{V})=H_r(p,\alpha^*\mathbf{U}\times\beta^*\mathbf{V})$ on a simplex $x\colon\Delta^s\to X$ is

$$\begin{aligned}
(px)^*(\alpha^*\mathbf{U}\times\beta^*\mathbf{V}) &= (px)^*\alpha^*\mathbf{U}\times(px)^*\beta^*\mathbf{U}\\
&= (\alpha px)^*\mathbf{U}\times(\beta px)^*\mathbf{V}\\
&= H_{n,m}((\alpha px,\beta px),\mathbf{U},\mathbf{V})\\
&= H_{n,m}\circ((\alpha,\beta)\circ\times\mathrm{id}\times\mathrm{id})(px,\mathbf{U},\mathbf{V}),
\end{aligned}$$

which demonstrates that the diagram commutes. Let $(t_i)_{i=1}^r$ such that $t_i=\alpha(i)-\alpha(i-1)+\beta(i)-\beta(i-1)$. Note that the sum of the t_i is exactly $n+m$, since α and β preserve initial and terminal objects. We define a functor

$$\tau_i\colon\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m\to\widehat{\mathcal{C}}^{t_i}$$

by the rule

$$(\mathbf{A},\mathbf{B})\mapsto(A_{\alpha(i-1)+1},\dots,A_{\alpha(i)},B_{\beta(i-1)+1},\dots,B_{\beta(i)}).$$

Then define

$$\tau\colon\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m\to\prod_{i=1}^r\widehat{\mathcal{C}}^{t_i}.$$

It is clear that τ is a permutation of variables and therefore an isomorphism. Then let

$$P_i\colon\widehat{\mathcal{C}}^{t_i}\to\widehat{\mathcal{C}}$$

be the functor defined by the rule

$$(X_1,\dots,X_{t_i})\mapsto X_1\times\cdots\times X_{t_i}.$$

Then the P_i assemble to a map $(P_1,\dots,P_r)$ such that

$$(P_1,\dots,P_r)\circ\tau=K_{\alpha,\beta}.$$

Then the diagram

$$\begin{array}{ccc}
\left(\widehat{\Delta}\downarrow\Delta^r\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m & \xrightarrow{\ \mathrm{id}\times\tau\ } & \left(\widehat{\Delta}\downarrow\Delta^r\right)\times\prod_{i=1}^r\widehat{\mathcal{C}}^{t_i} \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}\times(P_i)_{i=1}^r} \\
\Big\downarrow{\scriptstyle ((\alpha,\beta)\circ(-))\times\mathrm{id}\times\mathrm{id}} & & \left(\widehat{\Delta}\downarrow\Delta^r\right)\times\widehat{\mathcal{C}}^r \\
 & & \Big\downarrow{\scriptstyle H_r} \\
\left(\widehat{\Delta}\downarrow\Delta^n\times\Delta^m\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m & \xrightarrow{\ H_{n,m}\ } & \widehat{\Delta}\int\widehat{\mathcal{C}}
\end{array}$$

commutes, and therefore, composing the bottom horizontal and right vertical maps with $\square$, we have another commutative diagram

$$\begin{array}{ccc}
\left(\widehat{\Delta}\downarrow\Delta^r\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m & \xrightarrow{\ \mathrm{id}\times\tau\ } & \left(\widehat{\Delta}\downarrow\Delta^r\right)\times\prod_{i=1}^r\widehat{\mathcal{C}}^{t_i} \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}\times(P_i)_{i=1}^r} \\
\Big\downarrow{\scriptstyle ((\alpha,\beta)\circ(-))\times\mathrm{id}\times\mathrm{id}} & & \left(\widehat{\Delta}\downarrow\Delta^r\right)\times\widehat{\mathcal{C}}^r \\
 & & \Big\downarrow{\scriptstyle \square_r} \\
\left(\widehat{\Delta}\downarrow\Delta^n\times\Delta^m\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m & \xrightarrow{\ \square_{n,m}\ } & \widehat{\Theta[\mathcal{C}]}
\end{array}.$$

The bottom horizontal and lower right vertical maps preserve connected colimits, as we have seen. Each of the components of the left vertical map preserves connected colimits because colimits are computed in the domain for comma categories (and the identity preserves all colimits). The map $\prod_{i=1}^r P_i$ preserves colimits in each argument because colimits are universal in toposes. Then applying the corner tensor functor, we have the commutative diagram

$$\begin{array}{ccc}
\left(\widehat{\Delta}\downarrow\Delta^r\right)^{[1]}\times\left(\widehat{\mathcal{C}}^{[1]}\right)^n\times\left(\widehat{\mathcal{C}}^{[1]}\right)^m & \xrightarrow{\mathrm{id}\times\tau} & \left(\widehat{\Delta}\downarrow\Delta^r\right)^{[1]}\times\prod_{i=1}^r\left(\widehat{\mathcal{C}}^{[1]}\right)^{t_i} \\
 & & \big\downarrow{\scriptstyle \mathrm{id}\times(P_i^{\lrcorner})_{i=1}^r} \\
\big\downarrow{\scriptstyle ((\alpha,\beta)\circ(-))^{\lrcorner}\times\mathrm{id}\times\mathrm{id}} & & \left(\widehat{\Delta}\downarrow\Delta^r\right)\times\left(\widehat{\mathcal{C}}^{[1]}\right)^r \\
 & & \big\downarrow{\scriptstyle \square_r^{\lrcorner}} \\
\left(\widehat{\Delta}\downarrow\Delta^n\times\Delta^m\right)^{[1]}\times\left(\widehat{\mathcal{C}}^{[1]}\right)^n\times\left(\widehat{\mathcal{C}}^{[1]}\right)^m & \xrightarrow{\square_{n,m}^{\lrcorner}} & \widehat{\Theta[\mathcal{C}]}^{[1]}
\end{array}$$

.

1.6.14. Lemma. Let $(\alpha,\beta)\colon \Delta^r\to\Delta^n\times\Delta^m$ be a nondegenerate section with $r\geq 2$ and such that α and β preserve initial and terminal vertices. Let $\mathbf{f}=\{f_i\colon \partial c_i\hookrightarrow c_i\}_{i=1}^n$ and $\mathbf{g}=\{g_i\colon \partial d_i\hookrightarrow d_i\}_{i=1}^m$ be families of boundary inclusions for $\mathcal{C}$. Then for any inner horn inclusion $\lambda_k^r\colon \Lambda_k^r\hookrightarrow\Delta^r$ (viewed as a map over $\Delta^n\times\Delta^m$ by composing with (α,β)), the map

$$\square_{n,m}^{\lrcorner}(\lambda_k^r,\mathbf{f},\mathbf{g})$$

is horizontal inner anodyne.

Proof. By Observation 1.6.13, we see that

$$\square_{n,m}^{\lrcorner}(\lambda_k^r,\mathbf{f},\mathbf{g})\cong\square_r^{\lrcorner}(\lambda_k^r,(P_1^{\lrcorner},\dots,P_r^{\lrcorner})\circ\tau(\mathbf{f},\mathbf{g})),$$

but the value of the argument in position $1\leq j\leq r$ is

$$P_j^{\lrcorner}\circ\tau_j(\mathbf{f},\mathbf{g})=f_{\alpha(j-1)+1}\times^{\lrcorner}\cdots\times^{\lrcorner}f_{\alpha(j)}\times^{\lrcorner}g_{\beta(j-1)+1}\times\dots^{\lrcorner}\times^{\lrcorner}g_{\beta(j)},$$

which belongs to the class $\mathrm{Cell}(\mathscr{B})$ (where $\mathscr{B}$ denotes the set of boundary inclusions for $\mathcal{C}$). That is, the map

$$\square_r^{\lrcorner}(\lambda_k^r,(P_1^{\lrcorner},\dots,P_r^{\lrcorner})\circ\tau(\mathbf{f},\mathbf{g}))$$

belongs to

$$\square_r^{\lrcorner}(\lambda_k^r,\mathrm{Cell}(\mathscr{B}),\dots,\mathrm{Cell}(\mathscr{B})).$$

By Lemma 1.4.12, it follows therefore that this map belongs to

$$\mathrm{Cell}(\square_r^{\lrcorner}(\lambda_k^r,\mathscr{B},\dots,\mathscr{B})).$$

But the set of maps

$$\square_r^{\lrcorner}(\lambda_k^r,\mathscr{B},\dots,\mathscr{B})$$

is a subset of the generating horizontal inner anodynes, and therefore the map

$$\square_{n,m}^{\lrcorner}(\lambda_k^r,\mathbf{f},\mathbf{g})\cong\square_r^{\lrcorner}(\lambda_k^r,(P_1^{\lrcorner},\dots,P_r^{\lrcorner})\circ\tau(\mathbf{f},\mathbf{g}))$$

is horizontal inner anodyne. ■

Finally, we reach our destination.

1.6.15. THEOREM. [Anodyne Theorem [Our10, 3.88]] The class of horizontal anodynes is closed under corner products with monomorphisms. In particular, if we let

$$\mathscr{J} = \{\square_n^\lrcorner(\lambda_k^n, \delta^{c_1}, \dots, \delta^{c_n}) | \text{ for } n \geq 2, 0 < k < n\}.$$

Then we have

$$\mathscr{M} \times^\lrcorner \mathscr{J} \subseteq \mathrm{Cell}(\mathscr{J}).$$

PROOF. Let $f_0 \colon \partial\Delta^n \hookrightarrow \Delta^n$, and let $\mathbf{f} = (f_i)_{i=1}^n$ be a family of boundary inclusions in $\widehat{\mathcal{C}}$. Let $g_0 \colon \Lambda_k^m \hookrightarrow \Delta^n$ be an inner horn inclusion, and let $\mathbf{g} = (g_i)_{i=1}^n$ be a family of boundary inclusions in $\widehat{\mathcal{C}}$. By 1.6.11, we have

$$\square_n^\lrcorner(f_0, \dots, f_n) \times^\lrcorner \square_m^\lrcorner(g_0, \dots, g_m) \cong \square_{n,m}^\lrcorner(f_0 \times^\lrcorner g_0, f_1, \dots, f_n, g_1, \dots, g_n).$$

By 1.6.12, we know that $f_0 \times^\lrcorner g_0$ can be factored as a finite sequence of pushouts of inner horn inclusions. By 1.4.12, it follows that

$$\square_{n,m}^\lrcorner(f_0 \times^\lrcorner g_0, f_1, \dots, f_n, g_1, \dots, g_n)$$

is a finite composite of pushouts of maps

$$\square_{n,m}^\lrcorner(h_i, f_1, \dots, f_n, g_1, \dots, g_n)$$

where $(h_i \colon \Lambda_{\ell_i}^{r_i} \to \Delta^{r_i})_{i=1}^k$ are inner horn inclusions and the implicit maps

$$(\alpha_i, \beta_i) \colon \Delta^{r_i} \to \Delta^n \times \Delta^m$$

are initial and terminal vertex preserving.

But by the previous lemma, we see that each of these maps is horizontal inner anodyne, so we are done. ∎

1.7. COMPARISON WITH REZK'S COMPLETE $\Theta[\mathcal{C}]$-SPACES. Since Rezk's complete Segal model structure on $\mathrm{Psh}_\Delta(\Theta[\mathcal{C}])$ is Cartesian, since $* \hookrightarrow E$ is one of the generators of the localization (see [Rez10]), and since $\Theta[\mathcal{C}]$ is regular skeletal Reedy, it follows by several results of Cisinski [Cis06, Proposition 8.2.9, Theorem 3.4.36, Proposition 2.3.30] that Rezk's localizer for complete $\Theta[\mathcal{C}]$-Segal spaces is the simplicial completion of a localizer on $\Theta[\mathcal{C}]$. We denote the functor $\widehat{\Theta[\mathcal{C}]} \to \mathrm{Psh}_\Delta(\Theta[\mathcal{C}])$ sending a cellular set to the associated discrete cellular space by $(-) \boxtimes \Delta^0$.

1.7.1. OBSERVATION. To show that Rezk's localizer is the simplicial completion of the localizer generated by the horizontal inner anodynes, it suffices to show the following two properties hold:

- (i) The maps $\square_n^\lrcorner(\lambda_i^n, \delta^{c_1}, \dots, \delta^{c_n}) \boxtimes \Delta^0$ belong to the localizer for complete $\Theta[\mathcal{C}]$-Segal spaces for $n \geq 2$ and $0 < i < n$.
- (ii) The Segal maps $\mathrm{Se}[n](c_1, \dots, c_n) \colon \mathrm{Sp}[n](c_1, \dots, c_n) \hookrightarrow [n](c_1, \dots, c_n)$ are horizontal inner anodyne.

We will make use of the following lemma:

1.7.2. Lemma. For any inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$, and any presheaves $X_1, \ldots, X_n$ on $\mathcal{C}$, the map $V_{\Lambda^n_k}(X_1, \ldots, X_n) \boxtimes \Delta^0 \hookrightarrow V[n](X_1, \ldots, X_n) \boxtimes \Delta^0$ belongs to the localizer for complete Segal-$\Theta[\mathcal{C}]$-spaces.

Proof. We will suppress the $\boxtimes\Delta^0$ factor denoting discrete simplicial presheaves for the duration of this proof. By [Rez10, 5.2], we know that the maps $\mathrm{Se}[n](X_1, \ldots, X_n)$ are already weak equivalences. Then we proceed following the argument of [JT07, Lemma 3.5]. Notice that trivial cofibrations have the right-cancellation property with respect to monomorphisms. Then we show that since the class of trivial cofibrations contains the class of maps $\mathrm{Se}[n](X_1, \ldots, X_n)$, it also contains the class of maps

$$V_{\partial_0\Delta^n \cup \partial_n\Delta^n}(X_1, \ldots, X_n) \hookrightarrow V[n](X_1, \ldots, X_n)$$

by induction on n. Notice first that the map

$$V_{\Lambda^2_1}(X_1, X_2) \hookrightarrow V[n](X_1, X_2)$$

is automatically a trivial cofibration, since $\Lambda^2_1 = \mathrm{Sp}[2]$. For the case of $n > 2$, notice that by cancellation, it suffices to show that the maps

$$V_{\mathrm{Sp}[n]}(X_1, \ldots, X_n) \xrightarrow{i_n} V_{\partial_0\Delta^n \cup \mathrm{Sp}[n]}(X_1, \ldots, X_n) \xrightarrow{j_n} V_{\partial_0\Delta^n \cup \partial_n\Delta^n}(X_1, \ldots, X_n)$$

are trivial cofibrations. Then notice that

$$V_{\mathrm{Sp}[n]}(X_1, \ldots, X_n) \xrightarrow{i_n} V_{\partial_0\Delta^n \cup \mathrm{Sp}[n]}(X_1, \ldots, X_n)$$

is a pushout of the map

$$V_{\mathrm{Sp}[n-1]}(X_1, \ldots, X_n) \hookrightarrow V_{\partial_0\Delta^n}(X_1, \ldots, X_n),$$

and is therefore a trivial cofibration. Notice that for $d_0 \colon [n-1] \to [n]$, $d_0^{-1}(\mathrm{Sp}[n]) = \mathrm{Sp}[n-1]$ and $d_0^{-1}(\partial_n\Delta^n) = \partial_{n-1}\Delta^{n-1}$. Then the square

$$\begin{array}{ccc} V_{\mathrm{Sp}[n-1]\cup\partial_{n-1}\Delta^{n-1}}(X_1, \ldots, X_n) & \longrightarrow & V_{\mathrm{Sp}[n-1]\cup\partial_n\Delta^n}(X_1, \ldots, X_n) \\ \downarrow{\scriptstyle k_{n-1}} & & \downarrow{\scriptstyle j_n} \\ V_{\partial_0\Delta^n}(X_1, \ldots, X_n) & \xrightarrow{d_0'} & V_{\partial_0\Delta^n\cup\partial_n\Delta^n}(X_1, \ldots, X_n) \end{array}$$

is coCartesian, and k_{n-1} is a trivial cofibration by using the cancellation property with the map j_{n-1}. Therefore, it follows that j_n is also a trivial cofibration.

We now prove the lemma: By the cancellation property, it suffices to show that

$$V_{\mathrm{Sp}[n]}(X_1, \ldots, X_n) \hookrightarrow V_{\Lambda^n_k}(X_1, \ldots, X_n)$$

is a trivial cofibration for $n \geq 2$ and $0 < i < n$. The case $n = 2$ is obvious, so it suffices to show for the case $n > 2$. Given a set $S \subseteq [n]$, let

$$\Lambda^n_S = \bigcup_{i \notin S} \partial_i \Delta^n.$$

We will show that for $n > 2$ and S a nonempty subset of $\{1, \ldots, n-1\}$, the map

$$V_{\mathrm{Sp}[n]}(X_1, \ldots, X_n) \hookrightarrow V_{\Lambda^n_S}(X_1, \ldots, X_n)$$

is a trivial cofibration. We argue by induction on n and $s = n - \mathrm{Card}(S)$. If $s = 1$, $\Lambda^n_S = \partial_0 \Delta^n \cup \partial_n \Delta^n$, in which case we are done by the previous argument. If $s > 1$, let $T = S \cup \{b\}$ for some $b \in \{1, \ldots, n-1\} \setminus S$. Then by the inductive hypothesis,

$$V_{\mathrm{Sp}[n]}(X_1, \ldots, X_n) \hookrightarrow V_{\Lambda^n_T}(X_1, \ldots, X_n)$$

is a trivial cofibration. Then it suffices to show that

$$V_{\Lambda^n_T}(X_1, \ldots, X_n) \hookrightarrow V_{\Lambda^n_S}(X_1, \ldots, X_n)$$

is a trivial cofibration. We see that the diagram

$$\begin{array}{ccc} V_{\Lambda^n_T \cap \partial_b \Delta^n}(X_1, \ldots, X_n) & \longrightarrow & V_{\Lambda^n_T}(X_1, \ldots, X_n) \\ \downarrow & & \downarrow \\ V_{\partial_b \Delta^n}(X_1, \ldots, X_n) & \longrightarrow & V_{\Lambda^n_S}(X_1, \ldots, X_n) \end{array}$$

is a pushout, and therefore, it suffices to show that

$$V_{\Lambda^n_T \cap \partial_b \Delta^n}(X_1, \ldots, X_n) \hookrightarrow V_{\partial_b \Delta^n}(X_1, \ldots, X_n)$$

is a trivial cofibration. But this is true by the inductive hypothesis on n. Therefore, we are done. ■

1.7.3. Proposition. The map

$$\square^{\lrcorner}_n(\lambda^n_k, \delta^{c_1}, \ldots, \delta^{c_n}) \boxtimes \Delta^0$$

is a trivial cofibration in the model structure for complete $\Theta[\mathcal{C}]$-Segal spaces.

Proof. We again suppress the $\boxtimes\Delta^0$ factor. Let $Q = \square_n^{\lrcorner}(\lambda_k^n, \delta^{c_1}, \dots, \delta^{c_n})$. Evaluation of Q on 0 is the source and evaluation on 1 is the target. We must show that the monomorphism $Q\colon Q(0) \hookrightarrow Q(1)$ is a trivial cofibration. Notice first that

$$\square_n(\Lambda_k^n, c_1, \dots, c_n) \hookrightarrow Q(0) \hookrightarrow Q(1) = [n](c_1, \dots, c_n)$$

is a weak equivalence by the lemma. Then by right-cancellation, it suffices to show that

$$\square_n(\Lambda_k^n, c_1, \dots, c_n) \hookrightarrow Q(0)$$

is a trivial cofibration. Let

$$U(s,t) = \int^{u_0,\dots,u_n} ([1](u_0, s) \times [1](u_1 \wedge \dots \wedge u_n, t)) \cdot \square(\lambda_k^n(u_0), \delta^{c_1}(u_1), \dots, \delta^{c_n}(u_n)),$$

where evaluation on $u_i \in [1]$ denotes taking the source or target. Then we see by coend reduction that

$$\int^{s,t} [1](s \wedge t, x) \times [1](u_0, s) \times [1](u_1 \wedge \dots \wedge u_n, t) = [1](u_0 \wedge u_1 \wedge \dots \wedge u_n, x),$$

so by commutation of coends, we see that

$$Q(x) = \int^{s,t} [1](s \wedge t, x) U(s,t),$$

which proves that

$$Q(0) = U(1,0) \coprod_{U(0,0)} U(0,1),$$

but $U(0,1) = \square_n(\Lambda_k^n, c_1, \dots, c_n)$, so the map

$$\square_n(\Lambda_k^n, c_1, \dots, c_n) \hookrightarrow Q(0)$$

is a pushout of $U(0,0) \to U(1,0)$, which we will show is a trivial cofibration. Notice that in $U(0,0)$, everything vanishes when $u_0 = 1$, so we have that

$$U(0,0) \cong \int^{u_1 \dots, u_n} [1](u_1 \wedge \dots \wedge u_n, 0) \cdot \square(\lambda_k^n(0), \delta^{c_1}(u_1), \dots, \delta^{c_n}(u_n)).$$

Notice also that by cofinality, we have that

$$U(1,0) = \int^{u_0,\dots,u_n} ([1](u_0, 1) \times [1](u_1 \wedge \dots \wedge u_n, 0)) \cdot \square(\lambda_k^n(u_0), \delta^{c_1}(u_1), \dots, \delta^{c_n}(u_n)),$$

is isomorphic to

$$\int^{u_1,\dots,u_n} [1](u_1 \wedge \dots \wedge u_n, t) \cdot \square(\lambda_k^n(1), \delta^{c_1}(u_1), \dots, \delta^{c_n}(u_n)).$$

Then the map $U(0,0) \hookrightarrow U(1,0)$ is induced by the natural maps

$$\square(\lambda_k^n(0), \delta^{c_1}(u_1), \dots, \delta^{c_n}(u_n)) \hookrightarrow \square(\lambda_k^n(1), \delta^{c_1}(u_1), \dots, \delta^{c_n}(u_n)).$$

But $\lambda_k^n(0) \hookrightarrow \lambda_k^n(1)$ is the inner horn inclusion $\Lambda_k^n \hookrightarrow \Delta^n$, and therefore, by Lemma 1.7.2, these are all trivial cofibrations. But $U(0,0)$ and $U(0,1)$ are homotopy coends.

To see this, notice that each of these objects can be computed as colimits over cubical diagrams with the terminal vertex removed. Equipping these finite directed categories with the degree-raising Reedy structure, we see that a diagram is projectively cofibrant if and only if it is Reedy-cofibrant. To see that the diagrams in question are Reedy-cofibrant, it suffices to notice that the latching object at any vertex is a union of subobjects, which implies that the latching map is monic at each vertex, and consequently that the diagram is Reedy-cofibrant.

Therefore, the map $U(0,0) \hookrightarrow U(0,1)$ is a monic weak equivalence and therefore a trivial cofibration, which proves the proposition. ∎

This proves one direction of the theorem; now we prove the converse.

1.7.4. Proposition. The maps

$$\mathrm{Se}[n](c_1, \dots, c_n)\colon \mathrm{Sp}[n](c_1, \dots, c_n) \hookrightarrow [n](c_1, \dots, c_n)$$

are horizontal inner anodyne.

Proof. The statement is obvious in the case $n = 0$ and $n = 1$, so assume $n \geq 2$. We define $\Lambda^n = \partial_0\Delta^n \cup \partial_n\Delta^n \subset \Delta^n$ and let $\lambda^n\colon \Lambda^n \hookrightarrow \Delta^n$ denote the inclusion.

Since the map λ^n is inner anodyne, and since the empty maps $e^{c_i}\colon \varnothing \hookrightarrow c_i$ are monic, it follows by Lemma 1.4.12 that the corner-intertwiner

$$\square_n^{\lrcorner}(\lambda^n, e^{c_1}, \dots, e^{c_n})$$

is horizontal inner anodyne. However, it is easy to see[1] that this map is exactly

$$\square_n(\Lambda^n, c_1, \dots, c_n) \hookrightarrow [n](c_1, \dots, c_n).$$

Therefore, it suffices to show that the map

$$\square_n(\mathrm{Sp}[n], c_1, \dots, c_n) \hookrightarrow \square_n(\Lambda^n, c_1, \dots, c_n)$$

[1] This is a very special case. We emphasize, for example, that

$$\square_n^{\lrcorner}(\mathrm{Se}[n], e^{c_1}, \dots, e^{c_n})$$

is *not*

$$\square_n(\mathrm{Se}[n], c_1, \dots, c_n) \hookrightarrow [n](c_1, \dots, c_n)$$

for $n > 2$, since we can see that

$$V[n](c_1, c_2, \dots, \varnothing) \subset \square_n^{\lrcorner}(\mathrm{Se}[n], e^{c_1}, \dots, e^{c_n})(0).$$

is horizontal inner anodyne. We will first show that the map

$$\Box_n(\mathrm{Sp}[n] \cup \partial_0\Delta^n, c_1, \dots, c_n) \hookrightarrow \Box_n(\Lambda^n, c_1, \dots, c_n)$$

is horizontal inner anodyne. To see this, we proceed by induction on n. This is immediate for $n \leq 2$. Suppose $n > 2$. Then the map

$$\Box_n(\mathrm{Sp}[n] \cup \partial_0\Delta^n, c_1, \dots, c_n) \hookrightarrow \Box_n(\Lambda^n, c_1, \dots, c_n)$$

is horizontal inner anodyne, as it is a pushout of the map

$$\Box_{n-1}(\mathrm{Sp}[n-1] \cup \partial_0\Delta^{n-1}, c_1, \dots, c_{n-1}) \hookrightarrow \Box_{n-1}(\partial_n\Delta^n, c_1, \dots, c_{n-1}),$$

which is horizontal inner anodyne by the induction hypothesis. Then it suffices to show that

$$\Box_n(\mathrm{Sp}[n], c_1, \dots, c_n) \hookrightarrow \Box_n(\mathrm{Sp}[n] \cup \partial_0\Delta^n, c_1, \dots, c_n)$$

is horizontal inner anodyne. Again, we proceed by induction on n and notice that this is clear for $n \leq 2$, but we see immediately that

$$\Box_n(\mathrm{Sp}[n], c_1, \dots, c_n) \hookrightarrow \Box_n(\mathrm{Sp}[n] \cup \partial_0\Delta^n, c_1, \dots, c_n)$$

is a pushout of

$$\Box_{n-1}(\mathrm{Sp}[n-1], c_2, \dots, c_n) \hookrightarrow \Box_{n-1}n(\partial_0\Delta^n, c_2, \dots, c_n),$$

which is horizontal inner anodyne by the induction hypothesis, which concludes the proof. ■

1.7.5. Corollary. Consider the functor

$$Y \otimes E^\bullet\colon \Theta[\mathcal{C}] \times \Delta \to \widehat{\Theta[\mathcal{C}]}$$

(where Y is the Yoneda embedding) defined by the rule

$$([n](c_1, \dots, c_n), [m]) \mapsto [n](c_1, \dots, c_n) \times E^m.$$

Then by cocontinuous extension and precomposition with $Y \otimes E^\bullet$, then the adjunction induced by cocontinuous extension and precomposition gives a Quillen equivalence

$$\mathrm{Psh}_\Delta(\Theta[\mathcal{C}])_{\mathrm{CSS}} \underset{\mathrm{Sing}_E}{\overset{\mathrm{Real}_E}{\rightleftarrows}} \widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}}$$

between the model structure for complete $\Theta[\mathcal{C}]$-Segal spaces and the horizontal Joyal model structure, and the left Kan extension of the functor

$$d\colon \Theta[\mathcal{C}] \to \mathrm{Psh}_\Delta(\Theta[\mathcal{C}])$$

defined by the rule

$$[n](c_1, \dots, c_n) \mapsto [n](c_1, \dots, c_n) \times \Delta^0$$

induces a Quillen equivalence

$$\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}} \underset{d^*}{\overset{d_!}{\rightleftarrows}} \mathrm{Psh}_\Delta(\Theta[\mathcal{C}])_{\mathrm{CSS}}.$$

That is to say, the two model categories are both left and right Quillen-equivalent.

PROOF. This is an immediate consequence of the previous proposition together with [Cis06, Proposition 2.3.27]. ■

1.8. RECOGNITION OF HORIZONTAL JOYAL FIBRATIONS. In this section, we will prove the analogue of Joyal's pseudofibration theorem for the horizontal model structure on $\Theta[\mathcal{C}]$. We will need to set up some definitions.

1.8.1. DEFINITION. Recall from Definition 1.6.3, we defined the functor $H_{\mathbf{S}}$ for a finite family of simplicial sets $\mathbf{S}$. Consider the case of the functor $H_{S,n}$, where the family is made up of two simplicial sets Δ^n and some simplicial set S. Then we define the functor

$$H_{S,\Delta^n}\colon \left(\widehat{\Delta}\downarrow(S\times\Delta^n)\right)\times\left(\widehat{\Delta}\int\widehat{\mathcal{C}}\right)_n\to\widehat{\Delta}\int\widehat{\mathcal{C}},$$

to be the restriction of $H_{S,n}$ to the terminal labeling of S, the unique labeling of S that sends all edges of S to the terminal object of $\widehat{\mathcal{C}}$. Composing $\Box$ with $H_{S,n}$ is denoted by $\Box_{S,n}$. For the remainder of this section, we also, by abuse of notation, define

$$\Box_S\colon \left(\widehat{\Delta}\downarrow S\right)\to\widehat{\Theta[\mathcal{C}]}$$

to be the composite of $\Box$ with the restriction of H_S to the terminal labeling of S.

1.8.2. OBSERVATION. Observe that from 1.6.8 the functor

$$\Box_{S,n}\colon \left(\widehat{\Delta}\downarrow(S\times\Delta^n)\right)\times\widehat{\mathcal{C}}^n\to\widehat{\Theta[\mathcal{C}]}$$

preserves connected colimits in each argument and therefore can be corner tensored. By the same argument as 1.6.11, given $h\colon Y\to S$ a map of simplicial sets, we can compute

$$\begin{aligned} h\times^{\lrcorner}\Box_n^{\lrcorner}(f_0,\dots,f_n) &= \Box_S^{\lrcorner}(h)\times^{\lrcorner}\Box_n^{\lrcorner}(f_0,\dots,f_n)\\ &= \Box_{S,n}^{\lrcorner}(h\times^{\lrcorner} f_0,f_1,\dots,f_n).\end{aligned}$$

In what follows, we will use a very nice observation of Danny Stevenson [Ste18]. Consider the case of the simplicial set E^1 and the map $e\colon\Delta^0\hookrightarrow E^1$.

1.8.3. LEMMA. [Ste18, Lemma 2.19] The map of simplicial sets

$$e\times^{\lrcorner}\delta^n\colon E^1\times\partial\Delta^n\cup\Delta^0\times\Delta^n\hookrightarrow E^1\times\Delta^n,$$

is inner anodyne for all $n>0$.

1.8.4. OBSERVATION. It follows from the small object argument that we can factor $e\times^{\lrcorner}\delta^n$ as a composite

$$E^1\times\partial\Delta^n\coprod_{\Delta^0\times\partial\Delta^n}\Delta^0\times\Delta^n\xrightarrow{\iota}E\xrightarrow{\varepsilon}E^1\times\Delta^n,$$

where ι is a relative cell complex for the inner horn inclusions and where ε is an inner fibration between quasicategories. Since $e\times^{\lrcorner}\delta^n$ is inner anodyne, it follows that we have a lift $E^1\times\Delta^n\xrightarrow{\eta}E$ by the lifting property that exhibits $e\times^{\lrcorner}\delta^n$ as a retract of ι.

1.8.5. Observation. A simplex $(\alpha,\beta)\colon \Delta^r \to E^1 \times \Delta^n$ is determined by the destination of its vertices. We label the vertices by (a,i) and (b,i) for $0 \le i \le n$, where e is the inclusion of $a \in E^1$. Given a simplex $(\alpha,\beta)\colon \Delta^r \to E^1 \times \Delta^n$ and a family of presheaves $\mathbf{U} = (U_i)_{i=1}^n$ on $\mathcal{C}$, we define

$$K_\beta \colon \widehat{\mathcal{C}}^n \to \widehat{\mathcal{C}}^r$$

to be the functor sending

$$\mathbf{U} \mapsto \beta^*\mathbf{U}.$$

Similar to 1.6.13, we have a diagram

$$\begin{array}{ccc}
\left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^n & \xrightarrow{\mathrm{id} \times K_\beta} & \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^r \\
\Big\downarrow {\scriptstyle ((\alpha,\beta)\circ(-))\times \mathrm{id} \times \mathrm{id}} & & \Big\downarrow {\scriptstyle H_r} \\
\left(\widehat{\Delta} \downarrow E^1 \times \Delta^n\right) \times \widehat{\mathcal{C}}^n & \xrightarrow{H_{E^1,n}} & \widehat{\Delta} \textstyle\int \widehat{\mathcal{C}}
\end{array},$$

and the diagram commutes by a direct computation. By abuse of notation, set $\beta(-1) = 0$ and $\beta(r+1) = n$. Then we define the family

$$(t_i = \beta(i) - \beta(i-1))_{i=0}^{r+1}$$

and let

$$\tau_i \colon \widehat{\mathcal{C}}^n \to \widehat{\mathcal{C}}^{t_i}$$

be the map defined by sending

$$\mathbf{U} \mapsto (U_{\beta(i-1)+1}, \dots, U_{\beta(i)}).$$

Then this family of maps defines a map

$$\widehat{\mathcal{C}}^n \to \prod_{i=0}^{r+1} \widehat{\mathcal{C}}^{t_i}$$

which is a permutation and therefore an isomorphism. Then for each $0 < i < r+1$ let

$$P_i \colon \widehat{\mathcal{C}}^{t_i} \to \widehat{\mathcal{C}}$$

be the functor defined by the rule

$$(X_1, \dots, X_{t_i}) \mapsto X_1 \times \cdots \times X_{t_i},$$

and for $i = 0$ or $i = r+1$, define

$$P_i \colon \widehat{C}^{t_i} \to *$$

to be the terminal functor. Then these P_i assemble to a map

$$(P_0, \ldots, P_{r+1})\colon \prod_{i=0}^{r+1} \widehat{\mathcal{C}}^{t_i} \to \widehat{\mathcal{C}}^r,$$

such that $(P_0, \ldots, P_{r+1}) \circ \tau = K_\beta$. Then we have a commutative diagram

$$\begin{array}{ccc}
\left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^n & \xrightarrow{\mathrm{id} \times \tau} & \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \prod_{i=0}^{r+1} \widehat{\mathcal{C}}^{t_i} \\
 & & \downarrow{\scriptstyle \mathrm{id} \times (P_i)_{i=0}^{r+1}} \\
\downarrow{\scriptstyle ((\alpha,\beta)\circ(-))\times \mathrm{id}} & & \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^r \\
 & & \downarrow{\scriptstyle H_r} \\
\left(\widehat{\Delta} \downarrow E^1 \times \Delta^n\right) \times \widehat{\mathcal{C}}^n & \xrightarrow{H_{E^1,n}} & \widehat{\Delta} \int \widehat{\mathcal{C}}
\end{array},$$

and therefore, composing the bottom horizontal and right vertical maps with $\square$, we have another commutative diagram

$$\begin{array}{ccc}
\left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^n & \xrightarrow{\mathrm{id} \times \tau} & \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \prod_{i=0}^{r+1} \widehat{\mathcal{C}}^{t_i} \\
 & & \downarrow{\scriptstyle \mathrm{id} \times (P_i)_{i=0}^{r+1}} \\
\downarrow{\scriptstyle ((\alpha,\beta)\circ(-))\times \mathrm{id}} & & \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^r \\
 & & \downarrow{\scriptstyle \square_r} \\
\left(\widehat{\Delta} \downarrow E^1 \times \Delta^n\right) \times \widehat{\mathcal{C}}^n & \xrightarrow{\square_{E^1,n}} & \widehat{\Theta[\mathcal{C}]}
\end{array}.$$

We see that each of the components of the left vertical arrow preserves colimits and similarly for the upper right vertical arrow as well as for the upper horizontal arrow. The bottom right vertical and bottom horizontal maps both preserve connected colimits argument-by-argument, so applying the corner tensor functor, we obtain a commutative diagram

$$\begin{array}{ccc}
\left(\widehat{\Delta}\downarrow\Delta^r\right)^{[1]}\times(\widehat{\mathcal{C}}^{[1]})^n & \xrightarrow{\mathrm{id}\times\tau} & \left(\widehat{\Delta}\downarrow\Delta^r\right)^{[1]}\times\prod_{i=0}^{r+1}(\widehat{\mathcal{C}}^{[1]})^{t_i} \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}\times(P_i^{\lrcorner})_{i=0}^{r+1}} \\
\Big\downarrow{\scriptstyle ((\alpha,\beta)\circ(-))^{\lrcorner}\times\mathrm{id}} & & \left(\widehat{\Delta}\downarrow\Delta^r\right)^{[1]}\times(\widehat{\mathcal{C}}^{[1]})^r \\
 & & \Big\downarrow{\scriptstyle \square_r^{\lrcorner}} \\
\left(\widehat{\Delta}\downarrow E^1\times\Delta^n\right)^{[1]}\times(\widehat{\mathcal{C}}^{[1]})^n & \xrightarrow{\square^{\lrcorner}_{E^1,n}} & \widehat{\Theta[\mathcal{C}]}^{[1]}
\end{array}$$

.

1.8.6. Lemma. Let $(\alpha,\beta)\colon\Delta^r\to E^1\times\Delta^n$ be a nondegenerate section with $r\geq 2$, let

$$\mathbf{f}=\{f_i\colon\partial c_i\hookrightarrow c_i\}_{i=1}^n$$

be a family of boundary inclusions. Then for any inner horn inclusion $\lambda^r_k\colon\Lambda^r_k\hookrightarrow\Delta^r$, the map

$$\square^{\lrcorner}_{E^1,n}(\lambda^r_k,\mathbf{f})$$

is horizontal inner anodyne.

Proof. The proof is practically identical to that of Lemma 1.6.14 using Observation 1.8.5 in place of Observation 1.6.13. ∎

1.8.7. Theorem. Set $e\colon\Delta^0\to E^1$. Then for any boundary inclusion

$$\square^{\lrcorner}_n(\delta^n,\delta^{c_1},\dots,\delta^{c_n}),$$

with $n>0$, the map

$$e\times^{\lrcorner}\square^{\lrcorner}_n(\delta^n,\delta^{c_1},\dots,\delta^{c_n})$$

is a horizontal inner anodyne.

Proof. By 1.8.4 we see that $e\times^{\lrcorner}\delta^n$ can be factored as $\varepsilon\circ\iota$ such that it is a retract of ι, which is a relative cell complex of inner horn inclusions. By Lemma 1.4.12, it follows that

$$\square^{\lrcorner}_{E^1,n}(\iota,\delta^{c_1},\dots,\delta^{c_n})$$

is transfinite composite of pushouts of inner horn inclusions

$$\square^{\lrcorner}_{E^1,n}(h_i,\delta^{c_1},\dots,\delta^{c_n}),$$

where the $h_i\colon\Lambda^{r_i}_{\ell_i}\to\Delta^{r_i}$ are inner horn inclusions in $\left(\widehat{\Delta}\downarrow E^1\times\Delta^n\right)$. By the previous lemma, each of these maps is horizontal inner anodyne, and it follows therefore that the map

$$\square^{\lrcorner}_{E^1,n}(\iota,\delta^{c_1},\dots,\delta^{c_n})$$

is as well. But the map

$$\square^{\lrcorner}_{E^1,n}(e \times^{\lrcorner} \delta^n, \delta^{c_1}, \dots, \delta^{c_n})$$

is a retract of

$$\square^{\lrcorner}_{E^1,n}(\iota, \delta^{c_1}, \dots, \delta^{c_n}),$$

which we have just shown to be horizontal inner anodyne. Then from 1.8.2, we see that

$$\square^{\lrcorner}_{E^1,n}(e \times^{\lrcorner} \delta^n, \delta^{c_1}, \dots, \delta^{c_n}) \cong e \times^{\lrcorner} \square^{\lrcorner}_{n}(\delta^n, \delta^{c_1}, \dots, \delta^{c_n}),$$

which proves the theorem. ■

1.8.8. Corollary. The formal $\mathcal{C}$-quasicategories are the fibrant objects of the horizontal Joyal model structure.

Proof. By Cisinski's theorem (see Theorem A.1.6), an object is fibrant in a Cisinski model structure if it has the right lifting property with respect to all of the anodyne maps generated by pushout products of the generating cofibrations with the inclusion of either endpoint of interval object as well as pushout-products of the generating anodynes with pushout-product powers of the boundary inclusion into the interval object. By Theorem 1.6.15, we see that an object has the right lifting property with respect to the second set of maps if and only if it has the right lifting property with respect to the horizontal inner anodynes, since such maps all belong to the saturated class generated only by the horizontal inner horn inclusions, which are the generating anodynes in this situation.

Since the formal $\mathcal{C}$-quasicategories by definition have the right lifting property with respect to all inner anodynes, which are closed under pushout-products with arbitrary monomorphisms, and since the maps $e \times^{\lrcorner} \square^{\lrcorner}_{n}(\delta^n, \delta^{c_1}, \dots, \delta^{c_n})$ are inner anodyne for all $n > 0$, it suffices to show that any formal $\mathcal{C}$-quasicategory has the right lifting property with respect to the single map $e \times^{\lrcorner} \delta^0$, but this map is isomorphic to the map $e\colon \Delta^0 \hookrightarrow E^1$, and such a lift always exists by choosing the lift through the retraction $E^1 \to \Delta^0$. ■

1.8.9. Corollary. The fibrations between fibrant objects in the horizontal Joyal model structure are horizontal inner fibrations having the right lifting property with respect to the map $e\colon \Delta^0 \hookrightarrow E^1$.

Proof. A fibration between fibrant objects must have the right lifting property with respect to all horizontal inner anodynes and all maps of the form

$$e \times^{\lrcorner} \square^{\lrcorner}_{n}(\delta^n, \delta^{c_1}, \dots, \delta^{c_n}).$$

Since every inner fibration has the right lifting property with respect to all of those maps for $n > 0$, it follows that an inner fibration between fibrant objects need only have the right lifting property with respect to the case where $n = 0$, which is exactly the map e. ■

2. The coherent nerve, horizontal case

In [Lur09], Lurie reintroduces an important adjunction

$$\widehat{\Delta} \underset{\mathfrak{N}_\Delta}{\overset{\mathfrak{C}_\Delta}{\rightleftarrows}} \mathbf{Cat}_{\widehat{\Delta}},$$

coming originally from work of Cordier and Porter, where the left adjoint is called the *coherent realization* and the right adjoint is called the *coherent nerve*. One of the significant early theorems in [Lur09] demonstrates that this adjunction is in fact a Quillen equivalence between the Joyal model structure on the one hand and the Bergner model structure on the other.

We find that it is useful to instead consider this adjunction as one of the form

$$\widehat{\Theta[*]} \underset{\mathfrak{N}_\Delta}{\overset{\mathfrak{C}_\Delta}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(*)},$$

where we have obvious isomorphisms $\Theta[*] \cong \Delta$ and $\mathrm{Psh}_\Delta(*) \cong \widehat{\Delta}$. This is suggestive of a generalization to a new case where we replace $*$ with a small regular Cartesian Reedy category $\mathcal{C}$. We will develop this adjunction throughout the current chapter, and we will demonstrate that an analogous Quillen equivalence indeed holds.

2.1. The coherent realization for $\Theta[\mathcal{C}]$.

The goal of this section is to show that for any small regular Cartesian Reedy category $\mathcal{C}$, we can construct a new adjunction

$$\widehat{\Theta[\mathcal{C}]} \underset{\mathfrak{N}}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$$

generalizing the coherent nerve and realization. We will also give a useful computation of $\mathfrak{C}$ in some special cases.

2.1.1. Definition. A *$\mathcal{C}$-precategory* is a simplicial presheaf F on $\mathcal{C}$ such that F_0 is a constant presheaf on $\mathcal{C}$. Then we define the category of $\mathcal{C}$-precategories to be the full subcategory

$$\mathbf{PCat}(\mathcal{C}) \subseteq \widehat{\Delta \times \mathcal{C}}$$

spanned by the precategory objects on $\mathcal{C}$.

2.1.2. Definition. The functor $k\colon \Delta \times \mathcal{C} \to \Theta[\mathcal{C}]$ defined by the rule

$$([n], c) \mapsto [n](c, \ldots, c)$$

induces a colimit-preserving functor $k^*\colon \widehat{\Theta[\mathcal{C}]} \to \widehat{\Delta \times \mathcal{C}}$ that lands in $\mathbf{PCat}(\mathcal{C})$ called the *associated precategory*. For each $c \in \mathcal{C}$, the functor $k^*(-)_c$ also admits a left adjoint by cocontinuous extension of the functor $\Delta \to \Theta[\mathcal{C}]$ sending a simplex $[n]$ to $[n](c, \ldots, c)$. Given a simplicial set S, we denote this cocontinuous extension by $S \oslash c$.

2.1.3. Definition. The *pointwise realization* functor $\mathfrak{C}_{\Delta,\bullet}\colon \widehat{\Delta\times\mathcal{C}} \to \mathbf{Cat}_{\widehat{\Delta}}^{\mathcal{C}^{\mathrm{op}}}$ defined by the rule

$$\mathfrak{C}_{\Delta,\bullet}(X)_c \overset{\mathrm{def}}{=} \mathfrak{C}_{\Delta}(X_c)$$

restricts to colimit preserving functor $\mathfrak{C}_{\Delta,\bullet}\colon \mathbf{PCat}(\mathcal{C}) \to \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$ in the obvious way, which by abuse of notation, we also refer to as the *pointwise realization*.

2.1.4. Definition. The *coherent realization* $\mathfrak{C}_{\Theta[\mathcal{C}]}\colon \widehat{\Theta[\mathcal{C}]} \to \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$ (also denoted by $\mathfrak{C}$ by abuse of notation when $\mathcal{C}$ is fixed) is defined as the composite:

$$\widehat{\Theta[\mathcal{C}]} \xrightarrow{k^*} \mathbf{PCat}(\mathcal{C}) \xrightarrow{\mathfrak{C}_{\Delta,\bullet}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}.$$

It is immediate from the cocontinuity of each functor in this composite that the functor $\mathfrak{C}_{\Theta[\mathcal{C}]}$ is cocontinuous and therefore determined on representables. Therefore, it admits a right adjoint given by the Kan construction

$$\mathfrak{N}\colon \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})} \to \widehat{\Theta[\mathcal{C}]}$$

where for any category C enriched in simplicial presheaves on $\mathcal{C}$, we have $\mathfrak{N}(C)_t \cong \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}(\mathfrak{C}([t]), C)$. We call this right adjoint $\mathfrak{N}$ the *coherent nerve*.

2.1.5. Note. In what follows, we will give an explicit computation of $\mathfrak{C}$ on representables and in fact more generally on cellular sets of the form $V[n](A_1,\dots,A_n)$.

2.1.6. Remark. *It is easy to see that if we substitute the terminal category for $\mathcal{C}$, this specializes precisely to the usual coherent realization $\mathfrak{C}_\Delta = \mathfrak{C}_{\Theta[*]}$.*

We will extensively abuse notation in what follows by identifying a simplicial set with its associated constant simplicial presheaf on $\mathcal{C}$ and identifying a presheaf on $\mathcal{C}$ with its associated discrete simplicial presheaf.

2.1.7. Definition. We define a construction on objects

$$Q\colon \Delta \int \widehat{\mathcal{C}} \to \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}.$$

Suppose $[n](X_1,\dots,X_n)$ is any object of $\Delta\int\widehat{\mathcal{C}}$. Then we define $Q([n](X_1,\dots,X_n))$ as follows:

- The objects are the vertices $\{0,\dots,n\}$
- The Hom-object

$$\mathrm{Hom}(i,j) = \begin{cases} \varnothing & \text{for } i>j \\ \Delta^0 & \text{for } i=j \\ X_{i+1}\times\Delta^1\times X_{i+2}\times\cdots\times\Delta^1\times X_j & \text{for } i<j \end{cases}$$

- The associative composition law, $\mathrm{Hom}(i,j) \times \mathrm{Hom}(j,k) \to \mathrm{Hom}(i,k)$ which is the inclusion on the bottom face with respect to j:

$$\begin{gathered} X_{i+1} \times \Delta^1 \times \cdots \times \Delta^1 \times X_j \times \{1\} \times X_{j+1} \times \Delta^1 \times \cdots \times \Delta^1 \times X_k \\ \downarrow \\ X_{i+1} \times \Delta^1 \times \cdots \times \Delta^1 \times X_j \times \Delta^1 \times X_{j+1} \times \Delta^1 \times \cdots \times \Delta^1 \times X_k \end{gathered}$$

2.1.8. Proposition. The construction Q can be extended to morphisms in $\Delta \int \widehat{\mathcal{C}}$ and be made functorial.

Proof. Recall that a map

$$[n](X_1, \ldots, X_n) \to [m](Y_1, \ldots Y_m)$$

in $\Delta \int \widehat{\mathcal{C}}$ is given by a pair $(\gamma, \mathbf{f})$, where $\gamma\colon [n] \to [m]$ is a map of simplices together with a family of maps

$$\mathbf{f} = \left(f_i\colon X_i \to \prod_{j=\gamma(i-1)+1}^{\gamma(i)} Y_j \right)_{i=1}^{n}.$$

The category $\Delta \int \widehat{\mathcal{C}}$ is both fibred and opfibred over Δ. Every map $\gamma\colon [n] \to [m]$ in Δ admits a unique factorization

$$[n] \xrightarrow{\sigma} [r] \xrightarrow{\delta} [m]$$

where σ is surjective and δ is injective. Using both the fibration and opfibration structure (since it is a presentable fibration), we obtain (up to choice of cleavage and opcleavage) a unique factorization of any map in $\Delta \int \widehat{\mathcal{C}}$ as a composite of three maps

$$([n], \Omega) \xrightarrow{\widetilde{\sigma}} ([r], \sigma_! \Omega) \xrightarrow{\mathrm{id}, \mathbf{f}} ([r], \delta^* \Omega') \xrightarrow{\widetilde{\delta}} ([m], \Omega'),$$

where $\widetilde{\sigma}$ is a coCartesian lift of the degeneration $\sigma\colon [n] \to [r]$ and $\widetilde{\delta}$ is a Cartesian lift of the injective map $\delta\colon [r] \to [m]$. In order to extend the construction Q to morphisms, we will therefore extend the definition of Q to each of these three classes individually.

Given a coCartesian lift $\widetilde{\sigma}$ of a surjective map $\sigma\colon [n] \to [r]$, we can factor σ as a composite of degeneracy maps of relative dimension 1. By choosing coCartesian lifts for each of the degeneracy maps of relative dimension 1 factoring σ, we can factor $\widetilde{\sigma}$ as a composite of coCartesian lifts of relative dimension 1. Therefore, it will suffice to define Q on such maps, which allows us to extend to all coCartesian lifts of surjective maps.

Suppose we have a coCartesian lift of a degeneracy $\sigma\colon [n+1] \to [n]$ such that $\sigma(i) = \sigma(i+1)$.

$$\widetilde{\sigma}\colon [n+1](X_1, \ldots, X_n) \to [n](X_1, \ldots, \widehat{X_{i+1}}, \ldots, X_{n+1}).$$

Then $Q(\widetilde{\sigma})_{ab}\colon \operatorname{Hom}(a,b)\to\operatorname{Hom}(\sigma(a),\sigma(b))$ is defined on the homs as follows:

$$Q(\widetilde{\sigma})_{ab}=\begin{cases}\tau_{X_{i+1}} & \text{if } a=i=b-1\\ \operatorname{id}\times(\tau_{\Delta^1\times X_{i+1}}) & \text{if } a<i=b-1\\ (\tau_{X_{i+1}\times\Delta^1})\times\operatorname{id} & \text{if } a=i<b-1\\ \operatorname{id}\times\max\circ(\operatorname{id}\times\tau_{X_{i+1}}\times\operatorname{id})\times\operatorname{id} & \text{if } a<i<b-1\\ \operatorname{id} & \text{otherwise}\end{cases}$$

where $\max\colon\Delta^1\times\Delta^1\to\Delta^1$ is induced by the map of posets sending $(x,y)\mapsto\max(x,y)$ and $\tau_A\colon A\to *$ is the terminal map for $A\in\operatorname{Psh}_\Delta(\mathcal{C})$. Specifically, in the case where $a<i<b-1$, the map is given by the composite:

$$\begin{array}{c} X_{a+1}\times\Delta^1\times\cdots\times(\Delta^1\times X_{i+1}\times\Delta^1)\times\cdots\times\Delta^1\times X_b\\ \big\downarrow{\scriptstyle \operatorname{id}\times(\operatorname{id}\times\tau_{X_{i+1}}\times\operatorname{id})\times\operatorname{id}}\\ X_{a+1}\times\Delta^1\times\cdots\times(\Delta^1\times *\times\Delta^1)\times\cdots\times\Delta^1\times X_b\\ \big\|{\scriptstyle\cong}\\ X_{a+1}\times\Delta^1\times\cdots\times(\Delta^1\times\Delta^1)\times\cdots\times\Delta^1\times X_b\\ \big\downarrow{\scriptstyle \operatorname{id}\times(\max)\times\operatorname{id}}\\ X_{a+1}\times\Delta^1\times\cdots\times(\Delta^1)\times\cdots\times\Delta^1\times X_b \end{array}$$

Similarly, given a Cartesian lift $\widetilde{\delta}$ of an injective map $\delta\colon[r]\to[m]$, we can factor this map as a composite of injective maps, each of which is the inclusion of a codimension 1 face. By Cartesian lifting now, we can factor $\widetilde{\delta}$ as a composite of Cartesian lifts of inclusions of codimension 1 faces and can similarly reduce the question of defining Q on the Cartesian lifts of injective maps to this special case. We subdivide this further to the case of a Cartesian lift of an outer face and a Cartesian lift of an inner face.

Suppose we have a Cartesian lift of the inclusion of a codimension 1 outer face (say the inclusion of the face opposite the $n+1$st vertex.

$$\widetilde{\delta}\colon[n](X_1,\dots,X_n)\to[n+1](X_1,\dots,X_{n+1}).$$

In this case, the map $Q(\widetilde{\delta})_{ab}$ can just be taken to be the obvious isomorphism.

So now suppose we have a Cartesian lift

$$\widetilde{\delta}\colon[n](X_1,\dots,X_i\times X_{i+1},\dots,X_{n+1})\to[n+1](X_1,\dots,X_{n+1}).$$

of the inclusion of a codimension 1 inner face $\delta\colon[n]\to[n+1]$ such that $\delta^{-1}(i)=\varnothing$ for some $0<i<n+1$. Then $Q(\widetilde{\delta})_{ab}\colon\operatorname{Hom}(a,b)\to\operatorname{Hom}(\delta(a),\delta(b))$ is defined on the homs as follows:

$$Q(\widetilde{\delta})_{ab}=\begin{cases}\operatorname{id}\times(\operatorname{id}_{X_i}\times i_0\times\operatorname{id}_{X_{i+1}})\times\operatorname{id} & \text{if } a<i\le b\\ \operatorname{id} & \text{otherwise}\end{cases},$$

where $i_0\colon * \hookrightarrow \Delta^1$ is the inclusion of the initial vertex. We illustrate the case where $a < i \leq b$:

$$\begin{array}{c} X_{a+1} \times \Delta^1 \times \cdots \times (X_i \times X_{i+1}) \times \cdots \times \Delta^1 \times X_b \\ \Big\| \cong \\ X_{a+1} \times \Delta^1 \times \cdots \times (X_i \times * \times X_{i+1}) \times \cdots \times \Delta^1 \times X_b \\ \Big\downarrow {\scriptstyle \mathrm{id} \times (\mathrm{id}_{X_i} \times i_0 \times \mathrm{id}_{X_{i+1}}) \times \mathrm{id}} \\ X_{a+1} \times \Delta^1 \times \cdots \times (X_i \times \Delta^1 \times X_{i+1}) \times \cdots \times \Delta^1 \times X_b \end{array}$$

Finally, consider the case of a morphism

$$(\gamma, \mathbf{f})\colon [n](X_1, \ldots, X_n) \to [n](Y_1, \ldots, Y_n)$$

where $\gamma = \mathrm{id}$. The map on objects is just the identity, and the map on homs is just the obvious induced map

$$\begin{array}{c} X_{a+1} \times \Delta^1 \times \cdots \times \Delta^1 \times X_i \times \Delta^1 \times \cdots \times \Delta^1 \times X_b \\ \Big\downarrow {\scriptstyle f_{a+1} \times \mathrm{id} \times \cdots \times \mathrm{id} \times f_i \times \mathrm{id} \cdots \times \mathrm{id} \times f_b} \\ Y_{a+1} \times \Delta^1 \times \cdots \times \Delta^1 \times Y_i \times \Delta^1 \times \cdots \times \Delta^1 \times Y_b \end{array}$$

by functoriality of the product. ■

2.1.9. NOTE. We specify the following abuse of notation: Given an ordered family of sets $S_1, \ldots, S_n$, we denote by

$$[n](S_1, \ldots, S_n)$$

the nerve of the free category on the directed graph specified as follows:

- The set of vertices is $\{0, \ldots, n\}$, and
- The set of arrows from the vertex $i-1$ to i is S_i.

2.1.10. PROPOSITION. The functor $Q\colon \Delta \int \widehat{\mathcal{C}} \to \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$ factors as the composite

$$\Delta \int \widehat{\mathcal{C}} \xrightarrow{V} \widehat{\Theta[\mathcal{C}]} \xrightarrow{\mathfrak{C}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$$

PROOF. Since for any $[m](X_1, \ldots, X_m) \in \Delta \int \widehat{\mathcal{C}}$, the enriched categories

$$Q([m](X_1, \ldots, X_m)) \quad \text{and} \quad \mathfrak{C}(V[m](X_1, \ldots, X_m))$$

have object sets in natural bijection with the set of vertices of $[m]$ it suffices to produce, for each pair of vertices $i, j \in [m]$ a natural isomorphism

$$\mathfrak{C}(V[m](X_1, \ldots, X_m))(i, j) \to Q([m](X_1, \ldots, X_m))(i, j),$$

which amounts to the data of a natural (in c) isomorphism of simplicial sets

$$\mathfrak{C}(V[m](X_1,\dots,X_m))(x,y)_c \to Q([m](X_1,\dots,X_m))(x,y)_c.$$

First, observe that specifying a simplex

$$\Delta^n \to Q([m](X_1,\dots,X_m))(i,j)_c$$

is equivalent to specifying a morphism of simplicial presheaves on $\mathcal{C}$

$$\begin{aligned} c\times\Delta^n &\to X_{i+1}\times\Delta^1\times\cdots\times\Delta^1\times X_j \\ &= (\Delta^1)^{j-i-1}\times\prod_{k=i+1}^{j} X_k, \end{aligned}$$

which is itself equivalent to specifying a simplex

$$\Delta^n \to (\Delta^1)^{j-i-1}\times\prod_{k=i+1}^{j} X_k(c),$$

so in particular, we may make the identification

$$Q([m](X_1,\dots,X_m))(i,j)_c \cong \mathfrak{C}_\Delta([m](X_1(c),\dots,X_m(c)))(i,j),$$

and therefore it follows that we have an isomorphism, natural in c:

$$Q([m](X_1,\dots,X_m))_c \cong \mathfrak{C}_\Delta([m](X_1(c),\dots,X_m(c))).$$

It therefore suffices to demonstrate a natural isomorphism of simplicial sets

$$k^*(V[m](X_1,\dots,X_m))_c \cong [m](X_1(c),\dots,X_m(c)),$$

but it can be observed that specifying a simplex

$$\Delta^n \to k^*(V[m](X_1,\dots,X_m))_c$$

corresponds to specifying a map

$$[n](c,\dots,c) \to V[m](X_1,\dots,X_m),$$

which in turn is given by the data of a map

$$\gamma\colon [n]\to[m]$$

together with maps

$$c \to \prod_{k=\gamma(0)+1}^{\gamma(n)} X_k,$$

or equivalently, an element

$$x \in \prod_{k=\gamma(0)+1}^{\gamma(n)} X_k(c),$$

which specifies a unique simplex

$$x\colon \Delta^n \to [m](X_1(c), \ldots, X_m(c)),$$

which proves the claim. ∎

2.2. The Bergner-Lurie model structure on $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}}$. In this section, we cite important results from [Lur09, Appendix A.3] concerning the generalized Bergner model structure for categories enriched in an excellent monoidal model category.

2.2.1. Definition. Let $\mathbf{S}$ be a monoidal model category, and let X be an $\mathbf{S}$-enriched category. Then we define the *homotopy category* $\mathbf{h}X$ to be the ordinary category underlying the the associated $\mathbf{hS}$-enriched category also denoted by $\mathbf{h}X$.

2.2.2. Definition. Let $\mathbf{S}$ be a monoidal model category, and let $f\colon X \to Y$ be an $\mathbf{S}$-enriched functor of $\mathbf{S}$-enriched categories.

- We say that f is *weakly fully faithful* if for every pair of objects x, x' of X, the component map
$$f_{x,x'}\colon X(x,x') \to Y(f(x), f(x'))$$
is a weak equivalence of $\mathbf{S}$.
- We say that f is *weakly essentially surjective* if the induced functor on homotopy categories $\mathbf{h}X \to \mathbf{h}Y$ is essentially surjective, that is, if for every object y of $\mathbf{h}Y$ there exists an object x of $\mathbf{h}X$ and an isomorphism $y \cong f(x)$ in $\mathbf{h}Y$.
- We say that f is an $\mathbf{S}$-*enriched weak equivalence* if it is weakly fully faithful and weakly essentially surjective.

2.2.3. Definition. Let $\mathbf{S}$ be a monoidal model category.

We say that an $\mathbf{S}$-enriched category X is *locally fibrant* if for every pair of objects x, x' of X, the object of morphisms $X(x, x')$ is a fibrant object of $\mathbf{S}$.

We say that an $\mathbf{S}$-enriched functor of $\mathbf{S}$-enriched categories $f\colon X \to Y$ be an is a *local fibration* if the following two conditions hold:

1. For every pair of objects x, x' of X, the component map
$$f_{x,x'}\colon X(x,x') \to Y(f(x), f(x'))$$
is a fibration.
2. The induced functor on homotopy categories $\mathbf{h}X \to \mathbf{h}Y$ is an isofibration of ordinary categories.

2.2.4. Definition. Let $\mathbf{S}$ be a monoidal model category. We will define the *categorical suspension* functor $\mathbf{2}\colon \mathbf{S} \to \mathbf{Cat}_{\mathbf{S}}$. Given an object S of $\mathbf{S}$, we define $\mathbf{2}(S)$ as follows:

- The set of objects of $\mathbf{2}(S)$ is precisely the set $\{0,1\}$.
- The object of morphisms is defined by

$$\mathbf{2}(S)(i,j) = \begin{cases} \mathbf{1}_{\mathbf{S}} & \text{if} \quad i = j \\ S & \text{if} \quad i < j \\ \varnothing & \text{if} \quad i > j \end{cases},$$

where $\mathbf{1}_{\mathbf{S}}$ is the unit object of $\mathbf{S}$. The extension of the definition to morphisms is the obvious one. We also define the $\mathbf{S}$-enriched category $[0]_{\mathbf{S}}$ to be the enriched category with one object whose object of endomorphisms is exactly $\mathbf{1}_{\mathbf{S}}$.

2.2.5. Proposition. [Lur09, Proposition A.3.2.4] Suppose $\mathbf{S}$ is a symmetric monoidal combinatorial model category in which all objects are cofibrant. Then there exists a left-proper combinatorial model structure on $\mathbf{Cat}_{\mathbf{S}}$ with weak equivalences the $\mathbf{S}$-enriched weak equivalences as defined above and with cofibrations the weakly saturated class generated by the set

$$\{\varnothing \hookrightarrow [0]_{\mathbf{S}}\} \cup \{\mathbf{2}(f) \mid f \text{ is a generating cofibration of } \mathbf{S}\}$$

This model structure is called the Lurie-Bergner model structure on $\mathbf{S}$-enriched categories.

2.2.6. Definition. [Lur09, Definition A.3.2.16] A model category $\mathbf{S}$ equipped with a symmetric monoidal product $\otimes$ is called *excellent* if the following conditions hold:

(A1) The model category $\mathbf{S}$ is combinatorial.

(A2) Every monomorphism of $\mathbf{S}$ is a cofibration, and cofibrations are stable under products.

(A3) The collection of weak equivalences of $\mathbf{S}$ is stable under filtered colimits

(A4) The monoidal structure of $\mathbf{S}$ is compatible with the model structure. That is, the tensor product is a left-Quillen bifunctor.

(A5) The model category $\mathbf{S}$ satisfies the invertibility hypothesis.[2]

We quickly make use of the following lemma:

2.2.7. Lemma. [Lur09, Lemma A.3.2.20] Let $\mathbf{S}$ be an excellent symmetric monoidal model category, and let $\mathbf{S}'$ be a symmetric monoidal model category satisfying axioms (A1-A4). Then if there exists a monoidal left-Quillen functor $\mathbf{S} \to \mathbf{S}'$, it follows that $\mathbf{S}'$ is also excellent.

[2]Lawson showed in [Law16] that the invertibility hypothesis is always satisfied under the other four hypotheses, so we will refrain from going into too much detail.

2.2.8. Corollary. For any small category $\mathcal{C}$, the model category $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$ of simplicial presheaves on $\mathcal{C}$ with the injective model structure and the Cartesian product is excellent.

Proof. It is clear that axioms (A1), (A2), and (A4) are satisfied. Axiom (A3) is also satisfied by recalling that the injective model structure is regular by Proposition A.3.5 and therefore closed under filtered colimits by Proposition A.3.7. It therefore suffices to demonstrate axiom (A5).

If we take $p^*\colon \widehat{\Delta} \to \mathrm{Psh}_\Delta(\mathcal{C})$ to be the functor induced by the projection $p\colon \mathcal{C}\times\Delta \to \Delta$, it is clearly monoidal, as it preserves all limits. Moreover, it is also clear that it sends monomorphisms to monomorphisms and weak equivalences to weak equivalences. Since $\widehat{\Delta}$ is excellent, it follows therefore from Lemma 2.2.7 that $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$ is also excellent. ∎

2.2.9. Corollary. If $\mathscr{S}$ is a set of morphisms of $\mathrm{Psh}_\Delta(\mathcal{C})$ such that the left-Bousfield localization of $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$ at $\mathscr{S}$ is again a Cartesian monoidal model category, then the model category $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathscr{S}}$ obtained from this localization is also excellent.

Proof. As before, axioms (A1) and (A2) are obviously satisfied. Axiom (A4) is satisfied by hypothesis, and axiom (A3) follows from the fact that any localizer containing a regular localizer is again regular by Proposition A.3.2. It again suffices to demonstrate axiom (A5).

Notice now that the identity functor is a monoidal left-Quillen functor

$$\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}} \to \mathrm{Psh}_\Delta(\mathcal{C})_{\mathscr{S}}.$$

Then axiom (A5) again follows from Lemma 2.2.7. ∎

2.2.10. Theorem. [Lur09, Theorem A.3.2.24] Let $\mathbf{S}$ be an excellent monoidal model category. Then the following two results hold:

1. An $\mathbf{S}$-enriched category X is Bergner-Lurie fibrant if and only if it is locally fibrant.
2. An $\mathbf{S}$-enriched functor $f\colon X \to Y$ is a fibration for the Bergner-Lurie model structure if and only if it is a local fibration.

Tying this all together, we obtain the following characterization of the Bergner-Lurie model structure:

2.2.11. Corollary. For any small category $\mathcal{C}$ and any set $\mathscr{S}$ of morphisms of $\mathrm{Psh}_\Delta(\mathcal{C})$, there exists a left-proper combinatorial model structure on $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$ characterized by the following classes of maps

(C) The cofibrations are exactly the weakly saturated class generated by the set of maps

$$\{\varnothing \hookrightarrow [0]_{\mathrm{Psh}_\Delta(\mathcal{C})}\} \cup \{\mathbf{2}(f) \mid f \text{ is a generating cofibration of } \mathrm{Psh}_{\boldsymbol{\Delta}}(\mathcal{C})\}$$

(W) The weak equivalences are exactly the $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathscr{S}}$-enriched weak equivalences.

(F) The fibrant objects are the $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched categories whose Hom-objects are $\mathscr{S}$-local injectively-fibrant simplicial presheaves on $\mathcal{C}$, and the fibrations with fibrant target are exactly the local fibrations.

2.2.12. NOTE. We denote the Bergner-Lurie model structure with respect to a set of maps $\mathscr{S}$ of $\mathrm{Psh}_\Delta(\mathcal{C})$ by $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_\mathscr{S}}$. In the special case when $\mathscr{S}$ is empty, this reduces to the case where $\mathrm{Psh}_\Delta(\mathcal{C})$ is equipped with the injective model structure, and we denote its associated Bergner-Lurie model category by $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}}$.

2.2.13. REMARK. *The absence of hypotheses on $\mathcal{C}$ in this section is what leads us to believe that there may be a way to drop the hypothesis that $\mathcal{C}$ is regular Cartesian Reedy, but this goes beyond the scope of this paper.*

2.3. NECKLACES AND THE COHERENT REALIZATION. Necklaces were introduced by Dugger and Spivak in [DS11a] in order to understand the mapping objects $\mathfrak{C}_\Delta(X)(x, y)$. They prove a useful theorem that allows one to compute the coherent realization up to homotopy as a simplicially-enriched category whose hom-objects are the nerves of ordinary categories. We will demonstrate here how their theory generalizes to our setting. We begin by recalling the definition of a necklace:

2.3.1. DEFINITION. A *necklace* is a bi-pointed simplicial set $(T, (\alpha, \omega))$ of the form

$$T = \Delta^{m_1} \vee \cdots \vee \Delta^{m_n} \stackrel{\text{def}}{=} \operatorname{colim}\left(\Delta^{m_1} \stackrel{\top}{\leftarrow} \Delta^0 \stackrel{\bot}{\rightarrow} \ldots \stackrel{\top}{\leftarrow} \Delta^0 \stackrel{\bot}{\rightarrow} \Delta^{m_n}\right),$$

(where $\bot$ (resp. $\top$) denotes the inclusion of the initial (resp. terminal) vertex of a simplex) with specified vertices

$$(\alpha, \omega)\colon \Delta^0 \coprod \Delta^0 \xrightarrow{\bot \amalg \top} \Delta^{m_1} \coprod \Delta^{m_n} \xrightarrow{\iota_1 \amalg \iota_n} T.$$

By abuse of notation, we will simply refer to necklaces by the name of the simplicial set, suppressing the distinguished vertices (α, ω). We define the category $\mathcal{N}ec$ to be the full subcategory of bi-pointed simplicial sets spanned by the necklaces.

In [DS11a], Dugger and Spivak construct a functor

$$\mathfrak{C}_\Delta^{\mathcal{N}ec}\colon \widehat{\Delta} \to \mathbf{Cat}_{\widehat{\Delta}}$$

whose evaluation on a simplicial set X is given as follows:

- The set of objects of $\mathfrak{C}_\Delta^{\mathcal{N}ec}(X)$ is X_0.
- Given any two vertices $x, x' \in X_0$, the simplicial set of morphisms from x to x' is given by the formula

$$\mathfrak{C}_\Delta^{\mathcal{N}ec}(X)(x, x') \stackrel{\text{def}}{=} N(\mathcal{N}ec \downarrow X_{x,x'}),$$

  where $(\mathcal{N}ec \downarrow X_{x,x'})$ denotes the slice over the bi-pointed simplicial set $X_{x,x'}$ in the category of bi-pointed simplicial sets.

- The composition map is induced by concatenation of necklaces. That is, given a pair of necklaces $T \to X_{x,x'}$ and $T' \to X_{x',x''}$, their composite is given by the necklace

$$T \vee_{\omega_T, \alpha_{T'}} T' \to X_{x,x''}.$$

In order to compare $\mathfrak{C}_\Delta^{\mathcal{N}ec}$ and $\mathfrak{C}_\Delta$, Dugger and Spivak introduce an auxiliary functor $\mathfrak{C}_\Delta^{\mathcal{H}oc}$ that admits specified natural transformations to both. This leads to the main theorem:

2.3.2. THEOREM. [DS11a, Theorem 5.2] There is a specified natural zig-zag of weak equivalences of functors valued in simplicially-enriched categories:

$$\mathfrak{C}_\Delta^{\mathcal{N}ec} \leftarrow \mathfrak{C}_\Delta^{\mathcal{H}oc} \to \mathfrak{C}_\Delta.$$

The functors $\mathfrak{C}_\Delta^{\mathcal{N}ec}$ and $\mathfrak{C}_\Delta^{\mathcal{H}oc}$, much like the functor $\mathfrak{C}_\Delta$, send simplicial sets to simplicially enriched categories with set of objects equal to the set of 0-simplices. Ergo, they induce useful functors when applied pointwise to precategories.

2.3.3. DEFINITION. As in Definition 2.1.3, the *pointwise necklace realization*

$$\mathfrak{C}_{\Delta,\bullet}^{\mathcal{N}ec} \colon \mathbf{PCat}(\mathcal{C}) \to \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$$

is defined by the rule

$$\mathfrak{C}_{\Delta,\bullet}^{\mathcal{N}ec}(X)_c \overset{\mathrm{def}}{=} \mathfrak{C}_\Delta^{\mathcal{N}ec}(X_c).$$

The *pointwise homotopy coherent realization* $\mathfrak{C}_{\Delta,\bullet}^{\mathcal{H}oc}$ is defined similarly.

2.3.4. DEFINITION. The *necklace realization* functor $\mathfrak{C}^{\mathcal{N}ec} \colon \widehat{\Theta[\mathcal{C}]} \to \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$ is the composite

$$\widehat{\Theta[\mathcal{C}]} \xrightarrow{k^*} \mathbf{PCat}(\mathcal{C}) \xrightarrow{\mathfrak{C}_{\Delta,\bullet}^{\mathcal{N}ec}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}.$$

The *homotopy coherent realization* functor $\mathfrak{C}^{\mathcal{H}oc}$ is defined similarly.

From these definitions and Theorem 2.3.2 above, we deduce the following useful corollary.

2.3.5. COROLLARY. There is a specified natural zig-zag of weak equivalences of functors valued in $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$

$$\mathfrak{C}^{\mathcal{N}ec} \leftarrow \mathfrak{C}^{\mathcal{H}oc} \to \mathfrak{C}$$

2.4. GADGETS. To prove the equivalence between quasicategories and simplicially enriched categories, Dugger and Spivak make use of another intermediate construction called a category of gadgets. These are subcategories of bi-pointed simplicial sets that generalize necklaces while still retaining many of their useful properties. We begin by recalling the definition of a category of gadgets.

2.4.1. DEFINITION. [DS11a] A *category of gadgets* is a subcategory $\mathcal{G}$ of the category $\widehat{\Delta}_{*,*}$ satisfying the following properties:

- The category $\mathcal{G}$ contains $\mathcal{N}ec$.
- For all $G \in \mathcal{G}$ and all necklaces T, there is an equality

$$\mathcal{G}(T, G) = \widehat{\Delta}_{*,*}(T, G).$$

- For any $(G, \alpha, \omega) \in \mathcal{G}$, the simplicial set $\mathfrak{C}(G)(\alpha, \omega)$ is contractible.

The category $\mathcal{G}$ is moreover said to be *closed under wedges* if

- For any G, G' in $\mathcal{G}$, the wedge $G \vee G'$ is as well.

For any bi-pointed simplicial set $X_{x,x'}$ and any category of gadgets $\mathcal{G}$, we functorially define a simplicial set

$$\mathfrak{C}^{\mathcal{G}}_{\Delta}(X)(x, x') \overset{\text{def}}{=} N(\mathcal{G} \downarrow X_{x,x'}).$$

Moreover, if $\mathcal{G}$ is closed under wedges, the collection of simplicial sets

$$(\mathfrak{C}^{\mathcal{G}}_{\Delta}(X)(x, x'))_{x,x' \in X}$$

assembles to a simplicially enriched category $\mathfrak{C}^{\mathcal{G}}_{\Delta}(X)$ with composition induced by concatenation of gadgets.

Dugger and Spivak then prove the following useful proposition:

2.4.2. Proposition. [DS11a, Proposition 5.5] For any category of gadgets $\mathcal{G}$, the natural map

$$\mathfrak{C}^{\mathcal{N}ec}_{\Delta}(X)(x, x') \to \mathfrak{C}^{\mathcal{G}}_{\Delta}(X)(x, x')$$

induced by the inclusion $\mathcal{N}ec \hookrightarrow \mathcal{G}$ is a weak homotopy equivalence. Moreover, if $\mathcal{G}$ is closed under wedges, the natural transformation of functors valued in simplicially-enriched categories

$$\mathfrak{C}^{\mathcal{N}ec}_{\Delta} \to \mathfrak{C}^{\mathcal{G}}_{\Delta}$$

is a weak equivalence.

As in the previous section, given a category $\mathcal{G}$ of gadgets, we can extend the realization functor to a functor

$$\mathfrak{C}^{\mathcal{G}} \colon \widehat{\Theta[\mathcal{C}]}_{*,*} \to \mathrm{Psh}_{\Delta}(\mathcal{C}),$$

and when $\mathcal{G}$ is closed under wedges these specify a functor

$$\mathfrak{C}^{\mathcal{G}} \colon \widehat{\Theta[\mathcal{C}]} \to \mathbf{Cat}_{\mathrm{Psh}_{\Delta}(\mathcal{C})},$$

from which we obtain the following corollary:

2.4.3. Corollary. The inclusion $\mathcal{N}ec \hookrightarrow \mathcal{G}$ induces a natural equivalence of functors valued in $\mathrm{Psh}_{\Delta}(\mathcal{C})$

$$\mathfrak{C}^{\mathcal{N}ec}(X)(x, x') \to \mathfrak{C}^{\mathcal{G}}(X)(x, x'),$$

and when $\mathcal{G}$ is closed under wedges, these assemble to a natural equivalence of functors valued in $\mathbf{Cat}_{\mathrm{Psh}_{\Delta}(\mathcal{C})}$:

$$\mathfrak{C}^{\mathcal{N}ec} \to \mathfrak{C}^{\mathcal{G}}.$$

2.5. Quillen functoriality. In this section, we show that the adjunction

$$\widehat{\Theta[\mathcal{C}]}_{\text{hJoyal}} \underset{\mathfrak{N}}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}}$$

is a Quillen pair. We will extensively use the characterization of $\mathfrak{C}$ given in Proposition 2.1.10. We begin with the following observation:

2.5.1. Proposition. For any $n > 0$, let $K \subseteq \{1, \dots, n-1\}$ and define

$$\Lambda^n_K = \bigcup_{i \notin K} \partial_i \Delta^n,$$

and let

$$\lambda^n_K \colon \Lambda^n_K \hookrightarrow \Delta^n$$

denote the inclusion map. Then

$$\mathfrak{C}(\square^{\lrcorner}_n(\lambda^n_K, \delta^{c_1}, \dots, \delta^{c_n}))(i,j)$$

is an isomorphism whenever $i \neq 0$ or $j \neq n$. Moreover, the map

$$\mathfrak{C}(\square^{\lrcorner}_n(\lambda^n_K, \delta^{c_1}, \dots, \delta^{c_n}))(0,n)$$

is exactly

$$\delta^{c_1} \times^{\lrcorner} h^1_K \times^{\lrcorner} \cdots \times^{\lrcorner} h^{n-1}_K \times^{\lrcorner} \delta^{c_n},$$

where

$$h^k_K = \begin{cases} \lambda^1_1 \text{ if } k \in K \\ \delta^1 \text{ otherwise} \end{cases}.$$

Proof. Let $f \colon X \to Y$ be a monomorphism in $\widehat{\Theta[\mathcal{C}]}$, and let x, x' be a pair of vertices of X. We will show that $\gamma \colon \mathfrak{C}(X)(x,x') \to \mathfrak{C}(Y)(fx, fx')$ is also monic. Since these are simplicial presheaves on $\mathcal{C}$, it suffices to show that for all $c \in \mathcal{C}$, the component map of simplicial sets $\gamma_c \colon \mathfrak{C}(X)(x,x')_c \to \mathfrak{C}(Y)(fx,fx')_c$ is monic. However, by the definition of the coherent realization, we have that $\mathfrak{C}(X)(x,x')_c \cong \mathfrak{C}_\Delta(k^*(X)_c)(x,x')$ and similarly for Y. However, the functor k^* happens to be a right adjoint defined by precomposition, and therefore it preserves monomorphisms. Moreover, we know that $k^*(X) \to k^*(Y)$ being monic also implies that the map of simplicial sets $k^*(f)_c \colon k^*(X)_c \to k^*(Y)_c$ is also monic. Therefore, it suffices to show that given a monomorphism of simplicial sets $f \colon X \to Y$ and a pair of vertices x, x' in X, the induced map $\mathfrak{C}_\Delta(X)(x,x') \to \mathfrak{C}_\Delta(Y)(fx,fx')$ is monic, but this is clear using the skeletal filtration and working simplex-by-simplex. With that out of the way, we prove the first claim.

Let X denote the domain of $\square^{\lrcorner}_n(\lambda^n_K, \delta^{c_1}, \dots, \delta^{c_n})$. It is straightforward to see that the map $X \hookrightarrow [n](c_1, \dots, c_n)$ is bijective on vertices. By symmetry, we reduce to the case where we demand that $i \neq 0$. Therefore, we need to prove that for any $j \in \{0, \dots, n\}$ with $i \neq 0$, the map

$$\mathfrak{C}(X)(i,j) \to \mathfrak{C}([n](c_1, \dots, c_n))(i,j)$$

is an isomorphism. However, these are simplicial presheaves on $\mathcal{C}$, so it suffices to show that for all $c \in \mathcal{C}$, the map

$$\mathfrak{C}(X)(i,j)_c \to \mathfrak{C}([n](c_1, \dots, c_n))(i,j)_c$$

is an isomorphism of simplicial sets. However, as before, this becomes equivalent to proving that the map

$$\mathfrak{C}_\Delta(k^*(X)_c)(i,j) \to \mathfrak{C}_\Delta(k^*([n](c_1, \dots, c_n))_c)(i,j)$$

is an isomorphism. By [DS11a, Proposition 4.3], which says that for any simplicial set S with a pair of vertices s, s', we have

$$\mathfrak{C}_\Delta(S)(s,s') \cong \operatorname*{colim}_{T \in \left(\mathcal{N}ec \downarrow S_{s,s'}\right)} \mathfrak{C}_\Delta(T)(\alpha, \omega).$$

Unwinding the definitions, it suffices to show that the map

$$\operatorname*{colim}_{T \in (\mathcal{N}ec \downarrow ((k^* X)_c)_{i,j})} \mathfrak{C}_\Delta(T)(\alpha, \omega) \to \operatorname*{colim}_{T \in (\mathcal{N}ec \downarrow ((k^*[n](c_1, \dots, c_n))_c)_{i,j})} \mathfrak{C}_\Delta(T)(\alpha, \omega)$$

is an isomorphism. Unwinding things a bit more and passing through some adjunctions, it suffices to show that the functor

$$(\mathcal{N}ec \downarrow X_{i,j}) \to (\mathcal{N}ec \downarrow [n](c_1, \dots, c_n)_{i,j})$$

is an equivalence where the functor $\mathcal{N}ec \to \widehat{\Theta[\mathcal{C}]}_{*,*}$ is the functor sending a necklace T to the cellular set $T \oslash c$.

If $f \colon T \oslash c \to [n](c_1, \dots, c_n)_{i,j}$ is a bi-pointed map with T a necklace and $i \neq 0$, then f factors uniquely through the inclusion of the subobject $[n-1](c_2, \dots, c_n) \subseteq V_{\Lambda^n_K}(c_1, \dots, c_n) \subset X$. It follows that the categories of necklaces above are equivalent and that the colimits are equivalent, so we have $\mathfrak{C}(X)(i,j)_c = \mathfrak{C}([n](c_1, \dots, c_n)(i,j)_c$. Since c was arbitrary, this proves that the map $\mathfrak{C}(X)(i,j) = \mathfrak{C}([n](c_1, \dots, c_n)(i,j)$ is an isomorphism whenever $i \neq 0$. The other case with $j \neq n$ follows by symmetry.

The second part comes from the observation that when $K = \{1, \dots, n-1\}$,

$$\mathfrak{C}(V_{\Lambda^n_K}(c_1, \dots, c_n))(0,n) = \bigcup_{i=1}^{n-1} c_1 \times \Gamma^1_i \times \dots \times \Gamma^{n-1}_i \times c_n,$$

where

$$\Gamma^\ell_i = \begin{cases} \Lambda^1_1 \text{ for } \ell = i \\ \Delta^1 \text{ otherwise} \end{cases}.$$

To see this, notice that Λ^n_K is the union of the two outer faces, and attaching them along their common face gives a colimit in $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$ where $\mathfrak{C}(V_{\Lambda^n_K}(c_1,\ldots,c_n)(0,n)$ is freely generated by compositions

$$\mathfrak{C}([n-1](c_1,\ldots,c_{n-1}))(0,\ell)\times\{1\}\times\mathfrak{C}([n-1](c_2,\ldots,c_n))(\ell,n).$$

For when K is otherwise, each additional inner face gives the factor

$$\mathfrak{C}([n-1](c_1,\ldots,c_{n-1}))(0,\ell)\times\{0\}\times\mathfrak{C}([n-1](c_2,\ldots,c_n))(\ell,n),$$

so in general,

$$\mathfrak{C}(V_{\Lambda^n_K}(c_1,\ldots,c_n))(0,n)=\bigcup_{i=1}^{n-1} c_1\times\Gamma^1_{i,K}\times\cdots\times\Gamma^{n-1}_{i,K}\times c_n,$$

where

$$\Gamma^\ell_{i,K}=\begin{cases}\partial\Delta^1 \text{ for } \ell=i \text{ and } i\notin K\\ \Lambda^1_1 \text{ for } \ell=i \text{ and } i\in K\\ \Delta^1 \text{ otherwise}\end{cases}.$$

Each factor

$$V[n](c_1,\ldots,\partial c_j,\ldots,c_n)$$

contributes

$$\mathfrak{C}(V[n](c_1,\ldots,\partial c_j,\ldots,c_n))(0,n)=c_1\times\Delta^1\times\cdots\times\Delta^1\times\partial c_j\times\Delta^1\times\cdots\times\Delta^1\times c_n,$$

and taking the union of all of the factors gives exactly the domain of the inclusion

$$\delta^{c_1}\times^\lrcorner h^1_K\times^\lrcorner\cdots\times^\lrcorner h^{n-1}_K\times^\lrcorner\delta^{c_n}.$$

■

2.5.2. Proposition. The functor $\mathfrak{C}$ sends monomorphisms to cofibrations and horizontal inner anodynes to trivial cofibrations.

Proof. When $K=\varnothing$, $\lambda^n_K=\delta^n$, so the lemma tells us that

$$\mathfrak{C}(\square^\lrcorner_n(\delta^n,\delta^{c_1},\ldots,\delta^{c_n}))$$

is a pushout of the map

$$\mathbf{2}(\delta^{c_1}\times^\lrcorner\delta^1\times^\lrcorner\cdots\times^\lrcorner\delta^1\times^\lrcorner\delta^{c_n}),$$

which is a cofibration, which proves the claim.

Similarly, when K is a singleton, $\lambda^n_K=\lambda^n_k$ is the inclusion of an inner horn, so

$$\mathfrak{C}(\square^\lrcorner_n(\lambda^n_k,\delta^{c_1},\ldots,\delta^{c_n}))$$

is the pushout of the map

$$\mathbf{2}(\delta^{c_1}\times^\lrcorner h^1_k\times^\lrcorner\cdots\times^\lrcorner h^{n-1}_k\times^\lrcorner\delta^{c_n}),$$

where $h^k_k=\lambda^1_1$. This is a corner map where one factor is a trivial cofibration (because it is Kan anodyne), and therefore its image under $\mathbf{2}$ is a trivial cofibration. Since the pushout of a trivial cofibration is a trivial cofibration, we are done. ■

2.5.3. Corollary. The coherent nerve of a fibrant $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$-enriched category is a formal $\mathcal{C}$-quasicategory.

2.5.4. Lemma. The object $\mathfrak{C}(E^n)$ is weakly contractible for all n.

Proof. We notice immediately that $\mathfrak{C}(E^n)(i,j)_\bullet$ is a constant simplicial presheaf for all i, j, so it suffices to show that $\mathfrak{C}(E^n)(i,j)_*$ is contractible for all i, j, but then it follows immediately from the classical case. ∎

2.5.5. Proposition. The coherent nerve functor $\mathfrak{N}\colon \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})} \to \widehat{\Theta[\mathcal{C}]}$ sends fibrations between fibrant $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched categories to fibrations for the horizontal Joyal model structure.

Proof. Given a fibration between two fibrant $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched categories, $p\colon \mathcal{D} \to \mathcal{D}'$, we see immediately that the coherent nerve takes this fibration to a horizontal inner fibration between formal $\mathcal{C}$-quasicategories by Proposition 2.5.2. To show that it is a fibration for the horizontal Joyal model structure, it suffices by Theorem 1.3.8 to show that it has the right lifting property with respect to the inclusion $e\colon \Delta^0 \hookrightarrow E^1$. By Proposition 2.5.2, we see that $\mathfrak{C}$ takes the monomorphism e to a cofibration, and by the previous lemma, we see that $\mathfrak{C}(e)$ is a weak equivalence. It follows that $\mathfrak{N}(p)$ is a fibration for the horizontal Joyal model structure. ∎

2.5.6. Corollary. The adjunction

$$\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}} \underset{\mathfrak{N}}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}}$$

is a Quillen pair.

Proof. If $\mathfrak{C}$ takes cofibrations to cofibrations, and $\mathfrak{N}$ takes fibrations between fibrant objects to fibrations between fibrant objects, then the adjunction is a Quillen pair, but this is exactly what we proved in this section. ∎

The following lemma will be useful for later.

2.5.7. Lemma. If $A \hookrightarrow B$ is an inner anodyne map and of simplicial sets and $c \in \mathcal{C}$, the induced map

$$A \oslash c \hookrightarrow B \oslash c$$

is horizontal inner anodyne. Consequently, if X is a formal $\mathcal{C}$-quasicategory, the simplicial set $k^*(X)_c$ is a quasicategory for all $c \in \mathcal{C}$.

Proof. This follows directly by Lemma 1.7.2 applied to the maps

$$V_{\Lambda^n_i}(c, \dots, c) \hookrightarrow V_{\Delta^n}(c, \dots, c),$$

as these maps are exactly the maps

$$\Lambda^n_i \oslash c \hookrightarrow \Delta^n \oslash c.$$

∎

2.6. The Hom by cosimplicial resolutions for $\mathcal{C}$-precategories. In this section, we will see that if X is a $\mathcal{C}$-precategory such that X_c is a quasicategory for all $c \in \mathcal{C}$, the simplicial presheaves $\mathfrak{C}_{\Delta,\bullet}(X)(x,x')$ can be computed by resolutions. To show this, we will make use of some helpful results in [DS11b].

We define the following four cosimplicial bi-pointed simplicial sets:

$$C^\bullet_{\mathrm{cyl}} \overset{\mathrm{def}}{=} \operatorname{colim}\left(\Delta^0 \amalg \Delta^0 \leftarrow \Delta^\bullet \amalg \Delta^\bullet \hookrightarrow \Delta^\bullet \times \Delta^1\right)$$

$$C^\bullet_E \overset{\mathrm{def}}{=} \operatorname{colim}\left(\Delta^0 \amalg \Delta^0 \leftarrow E^\bullet \amalg E^\bullet \hookrightarrow E^\bullet \times \Delta^1\right)$$

$$C^\bullet_R \overset{\mathrm{def}}{=} \operatorname{colim}\left(\Delta^0 \amalg \Delta^0 \leftarrow \Delta^\bullet \amalg \Delta^0 \overset{\iota}{\to} \Delta^\bullet \star \Delta^0\right)$$

$$C^\bullet_L \overset{\mathrm{def}}{=} \operatorname{colim}\left(\Delta^0 \amalg \Delta^0 \leftarrow \Delta^0 \amalg \Delta^\bullet \overset{\iota}{\to} \Delta^0 \star \Delta^\bullet\right).$$

in which the $\star$ operation denotes the combinatorial join of simplicial sets. These cosimplicial bi-pointed simplicial sets fit in a natural diagram

$$\begin{array}{ccccc} C^\bullet_L & \longleftarrow & C^\bullet_{\mathrm{cyl}} & \longrightarrow & C^\bullet_R \\ & & \downarrow & & \\ & & C^\bullet_E & & \end{array},$$

wherein the maps are defined as follows

- The map $C^\bullet_{\mathrm{cyl}} \to C^\bullet_E$ is induced by the counit map $\Delta^\bullet \to E^\bullet = \operatorname{cosk}_0 \Delta^\bullet$.
- The map $C^\bullet_{\mathrm{cyl}} \to C^\bullet_R$ is induced by the natural map of posets $\Delta^\bullet \times \Delta^1 \to \Delta^\bullet \star \Delta^0 \cong \Delta^{\bullet+1}$ defined by the rule

$$(k,j) \mapsto \begin{cases} k & \text{if } j = 0 \\ \bullet + 1 & \text{otherwise}. \end{cases}$$

- Similarly, the map $C^\bullet_{\mathrm{cyl}} \to C^\bullet_R$ is induced by the natural map of posets $\Delta^\bullet \times \Delta^1 \to \Delta^0 \star \Delta^\bullet \cong \Delta^{\bullet+1}$ defined by the rule

$$(k,j) \mapsto \begin{cases} 0 & \text{if } j = 0 \\ k + 1 & \text{otherwise}. \end{cases}$$

As these are cosimplicial objects in a cocomplete category, they induce adjunctions

$$\operatorname{Map}^{(-)} \colon \widehat{\Delta}_{*,*} \to \widehat{\Delta},$$

where

$$\operatorname{Map}^{(-)}_X(x,x')_n \overset{\mathrm{def}}{=} \operatorname{Hom}(C^n_{(-)}, X_{x,x'})$$

2.6.1. Lemma. Each of the cosimplicial bi-pointed simplicial sets above is a Reedy-cofibrant cosimplicial resolution of Δ^1 with respect to the Joyal model structure on bi-pointed simplicial sets $\widehat{\Delta}_{*,*}$.

Proof. Reedy cofibrancy is trivial for all four cosimplicial objects. It is also trivial to see that $C_E^\bullet$ is a resolution of Δ^1.

To see that the map $C_{\text{cyl}}^\bullet \to C_R^\bullet$ is an equivalence of cosimplicial objects, consider the following natural morphism of spans:

$$\begin{array}{ccccc} \Delta^\bullet \diamond \Delta^0 & \hookleftarrow & \Delta^\bullet & \longrightarrow & \Delta^0 \\ \downarrow{\scriptstyle \sim} & & \| & & \| \\ \Delta^\bullet \star \Delta^0 & \hookleftarrow & \Delta^\bullet & \longrightarrow & \Delta^0 \end{array},$$

where $\diamond$ denotes the alternative join. Then by [Lur09, 4.2.1.2], the left vertical map is a natural weak equivalence in the Joyal model structure. The other vertical maps are identity morphisms, so this is a weak equivalence of spans and therefore the induced map on homotopy colimits is also a weak equivalence. However, since the Joyal model structure is left-proper and the left horizontal maps are cofibrations, the strict colimit of each row is actually a model for the homotopy colimit. But the pushout of the top row is none other than $C_{\text{cyl}}^\bullet$, and the pushout of the bottom row is none other than $C_R^\bullet$. By symmetry, the same statement holds for the map $C_{\text{cyl}}^\bullet \to C_L^\bullet$. It remains to show then that $C_R^\bullet$ is a resolution of Δ^1, but this is [DS11b, Lemma 9.3]. ∎

The following corollary is immediate from the fact that quasicategories are the fibrant objects of the Joyal model structure together with the general theory of cosimplicial resolutions:

2.6.2. Corollary. If $X_{x,x'}$ is a bi-pointed quasicategory, then there is a natural isomorphism in the homotopy category

$$\operatorname{Map}_X^{(-)}(x, x') \cong \operatorname{hMap}_X(x, x'),$$

where the $(-)$ on the lefthand side can take any of the values cyl, E, R or L, and where the righthand side denotes the homotopy function complex of maps of bi-pointed simplicial sets

$$\Delta^1 \to X_{x,x'}.$$

Since the choice of resolution doesn't matter, we abuse notation and let Map and $C^\bullet$ denote whichever choice of resolution and adjoint that is convenient. The following theorem is a key result in [DS11b].

2.6.3. Theorem. [DS11b, Corollary 5.3] There is a zig-zag, natural in $X_{x,x'}$

$$\mathfrak{C}_\Delta(X)(x,x') \leftsquigarrow\rightsquigarrow \mathrm{Map}_X(x,x'),$$

which becomes a zig-zag of natural weak homotopy equivalences upon restriction to bi-pointed quasi-categories.

From the naturality of this result, we obtain the following corollary:

2.6.4. Corollary. If $X_{x,x'}$ is a bi-pointed $\mathcal{C}$-precategory such that each X_c is a quasi-category, the component of the natural zig-zag of maps of simplicial presheaves on $\mathcal{C}$ at $X_{x,x'}$ is a zig-zag of weak equivalences in $\mathrm{Psh}_\Delta(\mathcal{C})$:

$$\mathfrak{C}_{\Delta,\bullet}(X)(x,x')_c \leftsquigarrow\rightsquigarrow \mathrm{Map}_{X_c}(x,x')$$

Combining this corollary with Lemma 2.5.7, we obtain the following:

2.6.5. Corollary. Upon restriction to bi-pointed formal $\mathcal{C}$-quasicategories, we have a natural zig-zag of weak equivalences in $\mathrm{Psh}_\Delta(\mathcal{C})$

$$\mathfrak{C}(X)(x,x')_c \leftsquigarrow\rightsquigarrow \mathrm{Map}_{k^*(X)_c}(x,x').$$

Unwinding the definitions, we note the following corollary:

2.6.6. Corollary. If $X_{x,x'}$ is a bi-pointed formal $\mathcal{C}$-quasicategory, we have zig-zags of weak equivalences in $\mathrm{Psh}_\Delta(\mathcal{C})$

$$\mathfrak{C}(X)(x,x')_c \leftsquigarrow\rightsquigarrow \mathrm{Hom}((C_R^\bullet \oslash c), X_{x,x'}),$$

and

$$\mathfrak{C}(X)(x,x')_c \leftsquigarrow\rightsquigarrow \mathrm{Hom}((C_L^\bullet \oslash c), X_{x,x'}).$$

We also make note of one more useful fact:

2.6.7. Lemma. The cosimplicial bi-pointed $\mathcal{C}$-cellular sets $C_R^\bullet \oslash c$ and $C_L^\bullet \oslash c$ are Reedy-cofibrant cosimplicial resolutions of $[1](c) = \Delta^1 \oslash c$ in $\widehat{\Theta[\mathcal{C}]}_{*,*}$.

Proof. First, Reedy cofibrancy is clear, so it remains to show that these are indeed resolutions of $[1](c)$. From [DS11b, Proposition 9.3], we see that the maps

$$C_R^n \to \Delta^1$$

and

$$C_L^n \to \Delta^1$$

admit inner anodyne sections

$$\Delta^1 \hookrightarrow C_R^n$$

and

$$\Delta^1 \hookrightarrow C_L^n$$

respectively. Then the maps

$$C_R^n \oslash c \to \Delta^1 \oslash c = [1](c)$$

and

$$C_L^n \oslash c \to \Delta^1 \oslash c = [1](c)$$

are equivalences, since by functoriality, they admit sections, and by Lemma 2.5.7, those sections are horizontal inner anodyne, so we conclude by 3-for2. ∎

2.7. The Hom by cosimplicial resolution. For every object $[1](c)$ in $\Theta[\mathcal{C}]$, we introduce two additional cosimplicial resolutions, which we can use to define simplicial presheaves that represent the mapping space between two vertices of a $\Theta[\mathcal{C}]$-set.

First, we define cosimplicial objects

$$\Delta^\bullet_{\triangleright(c)}, \text{ resp. } \Delta^\bullet_{\triangleleft(c)} \colon \Delta \to \Theta[\mathcal{C}],$$

which sends

$$[n] \mapsto [n+1](*, \dots, *, c), \text{ resp. } [n] \mapsto [n+1](c, *, \dots, *),$$

which is defined on maps by sending a morphism $f \colon [m] \to [n]$ to the map

$$[m+1](*, \dots, *, c) \to [m+1](*, \dots, *, c)$$

whose restriction to $[m] \subset [m+1]$ factors through $[n] \subset [n+1]$ as f and whose restriction to $\{m+1\}$ factors through $\{n+1\}$ as the identity (and similarly for the mirror image). This construction gives a canonical embedding of cosimplicial objects $\mathscr{H}\Delta^\bullet \hookrightarrow \Delta^\bullet_{\triangleright(c)}$ (and similarly for the mirror). Therefore, we can construct three bi-pointed cosimplicial objects

$$C_R^\bullet(c) \overset{\text{def}}{=} \operatorname{colim}\left(* \leftarrow \mathscr{H}\Delta^\bullet \hookrightarrow \Delta^\bullet_{\triangleright(c)}\right)$$

$$C_L^\bullet(c) \overset{\text{def}}{=} \operatorname{colim}\left(* \leftarrow \mathscr{H}\Delta^\bullet \hookrightarrow \Delta^\bullet_{\triangleleft(c)}\right).$$

$$C_E^\bullet(c) \overset{\text{def}}{=} \operatorname{colim}\left(* \amalg * \leftarrow E^\bullet \amalg E^\bullet \hookrightarrow E^\bullet \times [1](c)\right).$$

We also have canonical maps of cosimplicial objects

$$C_R^\bullet \oslash c \to C_R^\bullet(c)$$

and

$$C_L^\bullet \oslash c \to C_L^\bullet(c).$$

2.7.1. Proposition. All of the cosimplicial objects described above are Reedy-cofibrant objects of bi-pointed $\mathcal{C}$-cellular sets. Each is objectwise horizontal-Joyal equivalent to the constant bi-pointed cosimplicial object $[1](c)$ for any $c \in \mathcal{C}$. That is to say, each of these cosimplicial objects is a cosimplicial resolution of $[1](c)$ in $\widehat{\Theta[\mathcal{C}]}_{*,*}$.

Proof. That these cosimplicial objects are Reedy cofibrant is obvious. The proof that $C^\bullet_{E(c)}$ is a bi-pointed cosimplicial resolution of $[1](c)$ is identical to the proof that $C^\bullet_E$ is a bi-pointed cosimplicial resolution of Δ^1. It's simply because the pushout is a homotopy pushout and for any cellular set X, the map $E^n \times X \to X$ is a horizontal Joyal equivalence.

We will demonstrate that for each natural number n, the map $C^n_R(c) \to [1](c)$ can be exhibited as retract of the map $C^n_R \oslash c \to [1](c)$. First, notice that before we pass to quotients, we have a natural map of cosimplicial objects $(\Delta^\bullet \star \Delta^0) \oslash c \to \Delta^\bullet_{\triangleright(c)}$, since

$$(\Delta^n \star \Delta^0) \oslash c \cong [n+1](c, \ldots, c)$$

and keeping track of cosimplicial structure, the map is the obvious map $[n+1](c, \ldots, c) \to [n+1](*, \ldots, *, c)$, and it is clear that it is natural. It is straightforward that this transformation descends to the level of quotients, so we obtain a natural transformation $C^\bullet_R \oslash c \to C^\bullet_R(c)$. Then for every natural number n, we have a commutative diagram

$$\begin{array}{ccc} [1](c) & = & [1](c) \\ \downarrow & & \downarrow \\ C^n_R \oslash c & \longrightarrow & C^n_R(c) \end{array},$$

where the maps $[1](c) \to C^n_R \oslash c$ and $[1](c) \to C^n_R(c)$ are induced by the map $[0] \xrightarrow{\{n\}} [n]$. While the morphism of cosimplicial objects likely will not have a section, it just so happens that for each n, a section of the map $C^n_R \oslash c \to C^n_R(c)$ exists. In particular, since $\mathcal{C}$ is regular Cartesian Reedy and in particular satisfies (CR3), we can choose some point $q\colon * \to c$, which, for each natural number n, allows us to define a section

$$[n+1](*, \ldots, *, c) \xrightarrow{[n+1](q, \ldots, q, \mathrm{id}_c)} [n+1](c, \ldots, c, c).$$

This map is evidently a section as can be seen by composing it with the natural map. It is also straightforward to see that this map is compatible with passing to quotients. Moreover, composite of the induced map $C^n_R(c) \to C^\bullet_R \oslash c$ with the section $[1](c) \to C^n_R(c)$ induced by inclusion of the nth vertex can be calculated immediately to be the map $[1](c) \to C^n_R \oslash c$. That is to say, for every natural number n, we have a commmutative diagram

$$\begin{array}{ccccc} [1](c) & = & [1](c) & = & [1](c) \\ \downarrow & & \downarrow & & \downarrow \\ C^n_R(c) & \longrightarrow & C^n_R \oslash c & \longrightarrow & C^n_R(c) \end{array}.$$

whose bottom and top horizontal maps compose to the identity. It follows that the map $[1](c) \to C^n_R(c)$ is a horizontal inner anodyne, as it is a retract of the map $[1](c) \to C^n_R \oslash c$, which is horizontal inner anodyne. By 3-for-2, we have that for every natural number n, the maps $C^n_R(c) \to [1](c)$ are weak equivalences. It follows therefore that the cosimplicial object $C^\bullet_R(c)$ is a Reedy-cofibrant cosimplicial resolution of $[1](c)$, as desired. The case of $C^\bullet_L(c)$ is the same by symmetry. ∎

2.7.2. Proposition. For all natural numbers $n \geq 0$ and $c \in \mathcal{C}$, we have an isomorphism of simplicial presheaves

$$\mathfrak{C}(C^n_R(c))(\alpha, \omega) \cong Q^n \times c,$$

where $Q^n \overset{\text{def}}{=} \mathfrak{C}_\Delta(C^n_R)(\alpha, \omega)$. Moreover, the construction of this isomorphism is natural in c and n.

Proof. First, notice that since the functor $\mathfrak{C}$ commutes with colimits, we can compute $\mathfrak{C}(C^n_R(c))$ as the colimit of the span

$$\mathfrak{C}(\mathscr{H}(\Delta^0)) \leftarrow \mathfrak{C}(\mathscr{H}(\Delta^n)) \hookrightarrow \mathfrak{C}(\Delta^n_{\triangleright(c)}).$$

We can compute

$$\mathfrak{C}(\Delta^n_{\triangleright(c)})(i,j) \cong \begin{cases} [1]^{j-i-1} \times c & \text{if } j = n+1 \\ [1]^{j-i-1} & \text{otherwise} \end{cases}.$$

If we write the quotient map $F\colon \mathfrak{C}(\Delta^n_{\triangleright(c)}) \to \mathfrak{C}(C^n_R(c))$, then the quotient has two objects $\alpha < \omega$, with $\alpha = F(i)$ for all $i < n+1$ and $\omega = F(n+1)$. We immediately compute that

$$\mathfrak{C}(C^n_R(c))(\lambda, \rho) = \begin{cases} * & \text{if} \quad \lambda = \rho \\ \varnothing & \text{if} \quad \rho < \lambda \end{cases},$$

so it remains to consider the case $\lambda = \alpha < \omega = \rho$. Then we compute $\mathfrak{C}(C^n_R(c))(\alpha, \omega)$ by noticing that it is the quotient of the complex

$$\mathfrak{C}(\Delta^n_{\triangleright(c)})(0, n+1) = [1]^n \times c$$

obtained by collapsing, for each $0 \leq i \leq n$, the subcomplexes

$$\mathfrak{C}(\Delta^n_{\triangleright(c)})(0,i) \times \{1\} \times \mathfrak{C}(\Delta^n_{\triangleright(c)})(i, n+1) \cong [1]^{i-1} \times \{1\} \times [1]^{n-i} \times c \subset [1]^n \times c,$$

arising from the composition operation, to the factor

$$\mathfrak{C}(\Delta^n_{\triangleright(c)})(i, n+1) = [1]^{n-i} \times c.$$

Since colimits are universal in $\mathrm{Psh}_\Delta(\mathcal{C})$, we can pull out the $\times c$ factor. In other words, if we set $Q^n \overset{\text{def}}{=} \mathfrak{C}_\Delta(C^n)(\alpha, \omega)$, we have that $\mathfrak{C}(C^n_R(c))(\alpha, \omega) \cong Q^n \times c$. ∎

2.7.3. Definition. Given a bi-pointed $\mathcal{C}$-cellular set $X_{x,y}$ we define the *invariant mapping object* from x to y to the simplicial presheaf obtained by taking homotopy function complexes

$$\mathrm{hMap}_X(x,y)_c \overset{\mathrm{def}}{=} \mathrm{hFun}_{\widehat{\Theta[\mathcal{C}]}_{*,*}}([1](c), X_{x,y}).$$

2.7.4. Remark. *It is a general fact of abstract homotopy theory that if X is a fibrant object, we can compute the homotopy function complex with any Reedy-cofibrant cosimplicial resolution of* $[1](c)$*. Any two Reedy-cofibrant cosimplicial resolutions are related by a natural zig-zag of weak equivalences, so any one will do.*

2.7.5. Proposition. For any formal $\mathcal{C}$-quasicategory X and any pair of vertices x, y, there is a natural zig-zag of weak equivalences between $\mathrm{Map}_X^{(-)}(x,y)$ and $\mathfrak{C}(X)(x,y)$.

Proof. By Lemma 2.6.7, we see that any choice of cosimplicial resolution of $[1](c)$ will work, since they all compute $\mathrm{hMap}_X(x,y)_c$. But using the resolution $(C_R^\bullet \oslash c)$ of $[1](c)$, the associated mapping object is exactly the mapping object $\mathrm{Map}^{(R)}_{k^*(X)_c}(x,y)$, but $k^*(X)_c$ is a quasicategory, so the result now follows from Corollary 2.6.6. ∎

2.7.6. Lemma. For any bi-pointed formal $\mathcal{C}$-quasicategory $X_{x,y}$, the functor

$$\left(\Delta \downarrow \widehat{\Theta[\mathcal{C}]}_{*,*}(C_R^\bullet(c), X_{x,y})\right) \to \left(\Delta \downarrow \widehat{\Theta[\mathcal{C}]}_{*,*}((C_R^\bullet \oslash c), X_{x,y})\right)$$

induced by the map of cosimplicial resolutions of $[1](c)$

$$(C_R^\bullet \oslash c) \to C_R^\bullet(c)$$

induces a weak homotopy equivalence on nerves. Moreover, as we have a natural isomorphism

$$\left(\Delta \downarrow \widehat{\Theta[\mathcal{C}]}_{*,*}((C_R^\bullet \oslash c), X_{x,y})\right) \cong \left(\Delta \downarrow \widehat{\Delta}_{*,*}(C_R^\bullet, k^*(X_{x,y})_c)\right),$$

we have a commutative diagram in which the specified maps are weak homotopy equivalences

$$\begin{array}{ccc}
\operatorname*{colim}_{[n], C_R^n \to k^*(X_{x,y})_c} \mathfrak{C}_\Delta(C_R^n)(\alpha,\omega) & \longleftarrow & \operatorname*{colim}_{[n], C_R^n(c) \to X_{x,y}} \mathfrak{C}_\Delta(C_R^n)(\alpha,\omega) \\
\uparrow & & \uparrow \\
\operatorname*{hocolim}_{[n], C_R^n \to k^*(X_{x,y})_c} \mathfrak{C}_\Delta(C_R^n)(\alpha,\omega) & \underset{\sim}{\longleftarrow} & \operatorname*{hocolim}_{[n], C_R^n(c) \to X_{x,y}} \mathfrak{C}_\Delta(C_R^n)(\alpha,\omega) \\
\downarrow \wr & & \downarrow \wr \\
\operatorname*{hocolim}_{[n], C_R^n \to k^*(X_{x,y})_c} * & \underset{\sim}{\longleftarrow} & \operatorname*{hocolim}_{[n], C_R^n(c) \to X_{x,y}} *
\end{array}$$

.

Proof. Any weak homotopy equivalence of simplicial sets induces a weak homotopy equivalence on nerves of their categories of elements. As the map $(C_R^\bullet \oslash c) \to C_R^\bullet(c)$ is a map between Reedy-cofibrant resolutions of $[1](c)$, it follows that the induced map

$$\widehat{\Theta[\mathcal{C}]}_{*,*}(C_R^\bullet(c), X_{x,y}) \to \widehat{\Theta[\mathcal{C}]}_{*,*}((C_R^\bullet \oslash c), X_{x,y})$$

is a weak homotopy equivalence whenever $X_{x,y}$ is fibrant. Taken together, these observations prove the first claim. To obtain the diagram in the statement, notice that the vertical maps are equivalences as each of the simplicial sets $\mathfrak{C}_\Delta(C_R^n)(\alpha, \omega)$ is weakly contractible and that the bottom horizontal map is an equivalence, since each of these homotopy colimits is naturally weakly equivalent to the nerve of the diagram category. The fact that the middle horizontal map is a weak homotopy equivalence follows by 3-for-2. ∎

2.7.7. Note. In what follows, we will make use of a special category of gadgets denoted by $\mathcal{Y}$. It is the full subcategory of $\widehat{\Delta}_{*,*}$ spanned by those bi-pointed simplicial sets $Y_{\alpha,\omega}$ such that $\mathfrak{C}_\Delta(Y)(\alpha, \omega)$ is contractible. The properties of this category of gadgets are spelled out in [DS11b, Section 5].

2.7.8. Lemma. Given a $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched category $\mathcal{D}$ and any pair of objects $x, y \in \mathcal{D}$, there is a commutative diagram natural in c.

$$\begin{array}{ccc}
\mathfrak{C}(\mathfrak{N}\mathcal{D})(x,y)_c & \longrightarrow & \mathcal{D}(x,y)_c \\
\cong\uparrow & \nwarrow & \uparrow \\
\operatorname*{colim}_{T \in (\mathcal{N}ec \downarrow k^*(\mathfrak{N}\mathcal{D}_{x,y})_c)} \mathfrak{C}_\Delta(T)(\alpha,\omega) & \longrightarrow & \operatorname*{colim}_{Y \in (\mathcal{Y} \downarrow k^*(\mathfrak{N}\mathcal{D}_{x,y})_c)} \mathfrak{C}_\Delta(Y)(\alpha,\omega)
\end{array},$$

in which the specified map is an isomorphism.

Proof. Let $Y_{\alpha,\omega} \in \mathcal{Y}$ and suppose we are given a bi-pointed (suppressing them notationally for now) map

$$f \colon Y \to k^*(\mathfrak{N}\mathcal{D})_c,$$

we can apply simplicial coherent realization to obtain a map

$$\mathfrak{C}_\Delta f \colon \mathfrak{C}_\Delta Y \to \mathfrak{C}_\Delta(k^*(\mathfrak{N}\mathcal{D})_c),$$

but we have

$$\mathfrak{C}_\Delta(k^*(\mathfrak{N}\mathcal{D})_c) = (\mathfrak{C}_{\Delta,\bullet} k^*(\mathfrak{N}\mathcal{D}))_c$$

by definition of the pointwise realization. But we have that $\mathfrak{C}_{\Delta,\bullet} k^* = \mathfrak{C}$ again by definition, so we have obtained a bi-pointed map of simplicially enriched categories

$$\mathfrak{C}_\Delta Y \to (\mathfrak{C}\mathfrak{N}\mathcal{D})_c.$$

We have a counit natural transformation $\mathfrak{C}\mathfrak{N} \xrightarrow{\varepsilon} \mathrm{id}$, so we may compose it with our map

$$\mathfrak{C}_\Delta Y \to (\mathfrak{C}\mathfrak{N}\mathcal{D})_c \xrightarrow{(\varepsilon_D)_c} \mathcal{D}_c$$

to arrive at the desired destination. After passing to homs and taking colimits, we obtain commutativity of the upper triangle. Commutativity of the lower triangle is just restriction of the long slanted map. Then it suffices to show that the left vertical map is an isomorphism. However, the lefthand vertical map is an isomorphism by [DS11a, Proposition 4.3] together with our observation that

$$(\mathfrak{C}\mathfrak{N}\mathcal{D})_c \cong \mathfrak{C}_\Delta((\mathfrak{N}\mathcal{D})_c).$$

∎

2.7.9. Theorem. For any fibrant $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$-enriched category $\mathcal{D}$, the counit map

$$\epsilon_{\mathcal{D}} \colon \mathfrak{C}\mathfrak{N}\mathcal{D} \to \mathcal{D}$$

is a weak equivalence of $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$-enriched categories.

Proof. The counit map is bijective on objects, so it suffices to show that for all

$$x, y \in \mathfrak{C}\mathfrak{N}\mathcal{D},$$

the induced map $\mathfrak{C}\mathfrak{N}\mathcal{D}(x,y) \to \mathcal{D}(x,y)$ is a weak homotopy equivalence in $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$. However, weak equivalences in that category are precisely the levelwise weak equivalences of simplicial sets, so it suffices to show that for every $c \in \mathcal{C}$, the map

$$\mathfrak{C}\mathfrak{N}\mathcal{D}(x,y)_c \to \mathcal{D}(x,y)_c$$

is a weak equivalence of simplicial sets. Let $X = k^*(\mathfrak{N}\mathcal{D})_c$ with x, y a pair of points of the simplicial set X as well as the corresponding pair of objects in $\mathcal{D}$. Observe first of all that X is a quasicategory by Lemma 2.5.7, since $\mathfrak{N}\mathcal{D}$ is a formal $\mathcal{C}$-quasicategory, being the image of a fibrant object under a right Quillen functor. Then consider the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname*{colim}_{T_{\alpha,\omega}\to X_{x,y}} \mathfrak{C}_\Delta(T)(\alpha,\omega) & \longrightarrow & \operatorname*{colim}_{Y\to X_{x,y}} \mathfrak{C}_\Delta(Y)(\alpha,\omega) & \longleftarrow & \operatorname*{colim}_{[n],C^n_R\to X_{x,y}} \mathfrak{C}_\Delta(C^n_R)(\alpha,\omega) \\
\uparrow & & \uparrow & & \uparrow \\
\operatorname*{hocolim}_{T_{\alpha,\omega}\to X_{x,y}} \mathfrak{C}_\Delta(T)(\alpha,\omega) & \xrightarrow{\sim} & \operatorname*{hocolim}_{Y_{\alpha,\omega}\to X_{x,y}} \mathfrak{C}_\Delta(Y)(\alpha,\omega) & \xleftarrow{\sim} & \operatorname*{hocolim}_{[n],C^n_R\to X_{x,y}} \mathfrak{C}_\Delta(C^n_R)(\alpha,\omega) \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
\operatorname*{hocolim}_{T_{\alpha,\omega}\to X_{x,y}} *, & \xrightarrow{\sim} & \operatorname*{hocolim}_{Y_{\alpha,\omega}\to X_{x,y}} * & \xleftarrow{\sim} & \operatorname*{hocolim}_{[n],C^n_R\to X_{x,y}} * \\
\| & & \| & & \| \\
\mathfrak{C}^{\mathcal{N}ec}_\Delta(X)(x,y) & \xrightarrow{\sim} & \mathfrak{C}^{\mathcal{Y}}_\Delta(X)(x,y) & \xleftarrow{\sim} & N\Big(\Delta\downarrow\widehat{\Delta}(C^\bullet_R, X_{x,y})\Big)
\end{array},$$

in which the left horizontal arrows are induced by the inclusion

$$(\mathcal{N}ec\downarrow X_{x,y}) \hookrightarrow (\mathcal{Y}\downarrow X_{x,y}),$$

which are the indexing categories of the colimits appearing in the left and central columns respectively, and in which the right horizontal arrows are coming from the inclusion

$$\Big(\Delta\downarrow\widehat{\Delta}_{*,*}(C^\bullet_R, X_{x,y})\Big) \hookrightarrow (\mathcal{Y}\downarrow X_{x,y}),$$

which are the indexing categories of the colimits appearing in the right and central columns respectively, where

$$\Big(\Delta\downarrow\widehat{\Delta}_{*,*}(C^\bullet_R, X_{x,y})\Big)$$

denotes the category of elements.

The double lines are equalities, using the nerve as a model of the homotopy colimit of a constant diagram of spaces, so they are literal equalities. The two horizontal maps in the bottom row are weak equivalences by [DS11b, Proposition 5.2] and the fact that X is a quasicategory. The downward-oriented vertical maps are equivalences because the spaces appearing in the colimits are all contractible. By two applications of 3-for-2, it follows that the horizontal maps in third and second rows are weak equivalences.

Cutting down the diagram to size and combining with Lemma 2.7.8, we have a commutative diagram

$$
\begin{array}{ccccc}
\mathfrak{C}\mathfrak{N}\mathcal{D}(x,y)_c & \longrightarrow & \mathcal{D}(x,y) & & \\
\cong\uparrow & \nwarrow & \uparrow & & \\
\operatorname*{colim}_{T_{\alpha,\omega}\to X_{x,y}} \mathfrak{C}_\Delta(T)(\alpha,\omega) & \longrightarrow & \operatorname*{colim}_{Y\to X_{x,y}} \mathfrak{C}_\Delta(Y)(\alpha,\omega) & \longleftarrow & \operatorname*{colim}_{[n],C^n_R\to X_{x,y}} \mathfrak{C}_\Delta(C^n_R)(\alpha,\omega) \\
\sim\uparrow a & & \uparrow & & \uparrow \\
\operatorname*{hocolim}_{T_{\alpha,\omega}\to X_{x,y}} \mathfrak{C}_\Delta(T)(\alpha,\omega) & \xrightarrow{\sim} & \operatorname*{hocolim}_{Y_{\alpha,\omega}\to X_{x,y}} \mathfrak{C}_\Delta(Y)(\alpha,\omega) & \xleftarrow{\sim} & \operatorname*{hocolim}_{[n],C^n_R\to X_{x,y}} \mathfrak{C}_\Delta(C^n_R)(\alpha,\omega)
\end{array},
$$

in which the left vertical map designated by a is an equivalence by [DS11a, Theorem 5.3], since by definition

$$\operatorname*{hocolim}_{T_{\alpha,\omega}\to X_{x,y}} \mathfrak{C}_\Delta(T)(\alpha,\omega) = \mathfrak{C}^{\mathcal{H}oc}_\Delta(X)(x,y).$$

We rewrite the relevant part of the diagram as:

$$
\begin{array}{ccccc}
\mathfrak{C}(\mathfrak{N}\mathcal{D})(x,y)_c & \longrightarrow & \mathcal{D}(x,y)_c & & \\
\sim\uparrow & & \uparrow & \overset{\gamma_c}{\nwarrow} & \\
\operatorname*{hocolim}_{T_{\alpha,\omega}\to X_{x,y}} \mathfrak{C}_\Delta(T)(\alpha,\omega) & \xrightarrow{\sim} & \operatorname*{hocolim}_{Y\to X_{x,y}} \mathfrak{C}_\Delta(Y)(\alpha,\omega) & \xleftarrow{\sim} & \operatorname*{hocolim}_{[n],C^n_R\to X_{x,y}} \mathfrak{C}_\Delta(C^n_R)(\alpha,\omega)
\end{array},
$$

where we let γ_c be the composite

$$\operatorname*{hocolim}_{[n],C^n_R\to X_{x,y}} \mathfrak{C}_\Delta(C^n_R)(\alpha,\omega) \to \operatorname*{colim}_{[n],C^n_R\to X_{x,y}} \mathfrak{C}_\Delta(C^n_R)(\alpha,\omega) \to \mathcal{D}(x,y)_c.$$

By 3-for-2, it suffices to show that the map γ_c is a weak equivalence in $\widehat{\Delta}$.

However, unwinding the definitions and applying adjunctions, we have an equivalence

$$\left(\Delta\downarrow \widehat{\Delta}_{*,*}(C^\bullet, X_{x,y})\right) \simeq \left(\Delta\downarrow (\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})})_{*,*}(\mathfrak{C}(C^\bullet\oslash c), D_{x,y})\right),$$

of categories of elements, so by precomposition along the map $C^\bullet_R\oslash c\to C^\bullet_R(c)$, we obtain a commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}(x,y)_c & & \\
\gamma_c\uparrow & \overset{\gamma'_c}{\nwarrow} & \\
\operatorname*{hocolim}_{[n],\mathfrak{C}(C^n_R\oslash c)\to D_{x,y}} \mathfrak{C}_\Delta(C^n_R)(\alpha,\omega) & \xleftarrow[\sim]{} & \operatorname*{hocolim}_{[n],\mathfrak{C}(C^n_R(c))\to D_{x,y}} \mathfrak{C}_\Delta(C^n_R)(\alpha,\omega) \\
\downarrow\sim & & \downarrow\sim \\
\operatorname*{hocolim}_{[n],\mathfrak{C}(C^n_R\oslash c)\to D_{x,y}} * & \xleftarrow[\sim]{} & \operatorname*{hocolim}_{[n],\mathfrak{C}(C^n_R(c))\to D_{x,y}} *,
\end{array}
$$

in which the designated maps are equivalences by Lemma 2.7.6. It therefore suffices by 3-for-2 to show that the map γ'_c is an equivalence.

Let $W^{n,c} \overset{\text{def}}{=} \mathfrak{C}(C^n_R(c))(\alpha,\omega)$. Then we immediately see that we have an isomorphism $\mathfrak{C}(C^n_R(c)) \cong \mathbf{2}(W^{n,c})$. so we have an equivalence by adjunction

$$\begin{aligned}\big(\Delta \downarrow (\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})})_{*,*}(\mathfrak{C}(C^\bullet_R(c))_{\alpha,\omega}, D_{x,y})\big) &\cong \big(\Delta \downarrow (\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})})_{*,*}(\mathbf{2}(W^{\bullet,c})_{\alpha,\omega}, D_{x,y})\big)\\ &\simeq (\Delta \downarrow \mathrm{Psh}_\Delta(\mathcal{C})(W^{\bullet,c}, \mathcal{D}(x,y)).\end{aligned}$$

This allows us to rewrite the map γ'_c as

$$\gamma'_c\colon \operatorname*{hocolim}_{[n],W^{n,c}\to\mathcal{D}(x,y)} \mathfrak{C}_\Delta(C^n_R)(\alpha,\omega) \to \operatorname*{colim}_{[n],W^{n,c}\to\mathcal{D}(x,y)} \mathfrak{C}_\Delta(C^n_R)(\alpha,\omega) \to \mathcal{D}(x,y)_c.$$

Then by Proposition 2.7.2, we see that $W^{n,c} \cong Q^n \times c$, so we can rewrite γ'_c as the composite

$$\gamma'_c\colon \operatorname*{hocolim}_{[n],Q^n\times c\to\mathcal{D}(x,y)} Q^n \to \operatorname*{colim}_{[n],Q^n\times c\to\mathcal{D}(x,y)} Q^n \to \mathcal{D}(x,y)_c,$$

which is equivalent by adjunction to the map

$$\gamma'_c\colon \operatorname*{hocolim}_{[n],Q^n\to\mathcal{D}(x,y)_c} Q^n \to \operatorname*{colim}_{[n],Q^n\to\mathcal{D}(x,y)_c} Q^n \to \mathcal{D}(x,y)_c.$$

However, since $\mathcal{D}(x,y)_c$ is a Kan complex by our assumption that $\mathcal{D}$ was fibrant, and since $Q^\bullet$ is a Reedy-cofibrant cosimplicial resolution of a point, it follows from [DS11b, Lemma 5.10] that the map γ'_c is a weak homotopy equivalence, as desired. ∎

2.7.10. Remark. *This result is even stronger than it first appears, because it implies that the counit map is a weak equivalence for fibrant categories enriched in any Cartesian closed left-Bousfield localization of* $\mathrm{Psh}_\Delta(\mathcal{C})_{inj}$. *It reduces proving comparison theorems for such localizations to showing that* $\mathfrak{C}$ *is a left-Quillen functor (something we already know for the horizontal Joyal model structure by Corollary 2.5.6) and reflects weak equivalences.*

2.8. The horizontal comparison theorem. Dugger and Spivak introduce a definition of a Dwyer-Kan equivalence as a stepping stone to proving the comparison theorem. They use the definition of DK-equivalence as an intermediate step to proving that $\mathfrak{C}_\Delta$ is homotopy-conservative. We give an analogous definition as follows:

2.8.1. Definition. A map $f\colon X \to Y$ of presheaves on $\Theta[\mathcal{C}]$ is called a *horizontal Dwyer-Kan equivalence* if the following two properties hold:

- The induced map

$$f_*\colon \mathrm{Ho}(\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}})(*,X) \to \mathrm{Ho}(\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}})(*,Y)$$

  is bijective, and
- For any two vertices $x, x' \in X_0$, the induced map

$$\mathrm{hMap}_X(x,x') \to \mathrm{hMap}_Y(f(x),f(x'))$$

  is a weak equivalence of simplicial presheaves on $\mathcal{C}$.

2.8.2. Proposition. A map $f\colon X \to Y$ of presheaves on $\Theta[\mathcal{C}]$ is a horizontal weak equivalence if and only if it is a horizontal Dwyer-Kan equivalence.

Proof. It is clear that any horizontal Joyal equivalence is automatically horizontally Dwyer-Kan since our constructions are all homotopy-invariant, so we prove that all horizontal Dwyer-Kan equivalences are horizontal Joyal equivalences. We first assume that $f\colon X \to Y$ is a horizontal Dwyer-Kan equivalence between formal $\mathcal{C}$-quasicategories. Then recall that the construction

$$\mathcal{Q}\colon \widehat{\Theta[\mathcal{C}]} \to \widehat{\Theta[\mathcal{C}] \times \Delta}$$

defined by the rule

$$\mathcal{Q}(X)_{[n](c_1,\ldots,c_n),m} \overset{\text{def}}{=} \operatorname{Hom}([n](c_1,\ldots,c_n) \times E^m, X).$$

is the right adjoint of a Quillen equivalence by §1.7, and therefore the map f between fibrant objects X and Y is a weak equivalence if and only if its image $\mathcal{Q}$ is.

In order to prove that the map $\mathcal{Q}(f)$ is an equivalence of complete $\Theta[\mathcal{C}]$-Segal spaces, the Segal condition reduces us to showing that the map $\mathcal{Q}(f)_0\colon \mathcal{Q}(X)_0 \to \mathcal{Q}(Y)_0$ is a weak homotopy equivalence and that for every $c \in \mathcal{C}$, the map $\mathcal{Q}(f)_{[1](c)}\colon \mathcal{Q}(X)_{[1](c)} \to \mathcal{Q}(Y)_{[1](c)}$ is a weak homotopy equivalence. Since our original map f was a Dwyer-Kan equivalence, we see by unwinding adjunctions that the map

$$\pi_0\mathcal{Q}(f)_0\colon \pi_0\mathcal{Q}(X)_0 \to \pi_0\mathcal{Q}(Y)_0$$

is bijective and that for every $c \in C$, the square

$$\begin{array}{ccc} \mathcal{Q}(X)_{[1](c)} & \longrightarrow & \mathcal{Q}(Y)_{[1](c)} \\ \downarrow & & \downarrow \\ \mathcal{Q}(X)_0 \times \mathcal{Q}(X)_0 & \longrightarrow & \mathcal{Q}(Y)_0 \times \mathcal{Q}(Y)_0 \end{array}$$

is homotopy Cartesian. It suffices therefore to show that the map $\mathcal{Q}(f)_0\colon \mathcal{Q}(X)_0 \to \mathcal{Q}(Y)_0$ is a weak homotopy equivalence. Since $\mathcal{Q}(X)$ and $\mathcal{Q}(Y)$ are complete $\Theta[\mathcal{C}]$-Segal spaces, we may restrict them along the inclusion functor $\eta \times \mathrm{id}\colon \Delta \times \Delta \hookrightarrow \Theta[\mathcal{C}] \times \Delta$ to obtain a map

$$\mathcal{Q}_*(f)\colon \mathcal{Q}_*(X) \to \mathcal{Q}_*(Y)$$

of complete Segal spaces in the classical sense. It is immediate that the map $\mathcal{Q}_*(f)$ is a Dwyer-Kan equivalence of ordinary complete Segal spaces, and therefore by [Rez01, Proposition 7.6], we see that the map

$$\mathcal{Q}_*(f)_0\colon \mathcal{Q}_*(X)_0 \to \mathcal{Q}_*(Y)_0$$

is a weak homotopy equivalence. But the map $\mathcal{Q}_*(f)_0$ is equal on the nose to the map $\mathcal{Q}(f)_0$, which is therefore a weak homotopy equivalence, as desired.

In general, given a horizontal Dwyer-Kan equivalence $f\colon X \to Y$ where X and Y are no longer assumed to be fibrant, we can take a fibrant replacement $\tilde{Y}$ of Y such that $Y \to Y'$ is a trivial cofibration for the horizontal Joyal model structure. Then we can also factor $X \to Y \to \tilde{Y}$ into a trivial Joyal cofibration $X \to \tilde{X}$ followed by a fibration $\tilde{X} \to \tilde{Y}$. But notice now that the condition of being horizontally DK-equivalent is homotopy invariant, so the map $\tilde{X} \to \tilde{Y}$ is also a horizontal DK-equivalence. Since $\tilde{Y}$ is fibrant and $\tilde{X} \to \tilde{Y}$ is a horizontal Joyal fibration, this is a horizontal Joyal equivalence. Then by 3-for-2 we see that f is also a horizontal Joyal equivalence, which concludes the proof. ∎

2.8.3. Proposition. A map $f\colon X \to Y$ of presheaves on $\Theta[\mathcal{C}]$ is a horizontal Joyal equivalence if and only if $\mathfrak{C}(f)$ is a weak equivalence of $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$-enriched categories.

Proof. We only need to check one direction, since the other direction is immediate by the fact that $\mathfrak{C}$ is left-Quillen. Assume $f\colon X \to Y$ has the property that $\mathfrak{C}(f)$ is an equivalence. Then as in the previous proposition, we can reduce to the case where X and Y are fibrant. However, in this case, we know from Proposition 2.7.5 that $\mathfrak{C}(X)(x, x')$ is connected by a natural zig-zag of weak equivalences to $\mathrm{Map}_X(x, x')$, so if the map $\mathfrak{C}(X)(x, x') \to \mathfrak{C}(Y)(f(x), f(x'))$ is a weak equivalence, it follows that the map $\mathrm{Map}_X(x, x') \to \mathrm{Map}_Y(f(x), f(x'))$ is also a weak equivalence.

Then it suffices to show that when $\mathfrak{C}(f)$ is an equivalence, the induced map on sets of homotopy classes

$$[*, X]_{E^1} \to [*, Y]_{E^1}$$

is a bijection. Notice that

$$[*, X]_{E^1} \cong \pi_0 \widehat{\Theta[\mathcal{C}]}(E^\bullet, X)$$

since $E^n = \mathscr{H} \operatorname{cosk}_0 \Delta^n$. By abuse of notation, we also denote the simplicial set $\operatorname{cosk}_0 \Delta^n$ by E^n. We noticed earlier that $\mathscr{H}$ has a right adjoint, which we now denote by $\mathscr{N}$. Using this, we can rewrite the question as asking for the induced map to give a bijection

$$\pi_0 \widehat{\Delta}(E^n, \mathscr{N} X) \to \pi_0 \widehat{\Delta}(E^n, \mathscr{N} Y),$$

which is the same as giving a bijection

$$[\Delta^0, \mathscr{N} X]_{E^1} \to [\Delta^0, \mathscr{N} Y]_{E^1}.$$

Notice also that the data classifying an equivalence in $\mathfrak{C}(X)$ all factor through the simplicial category $\mathfrak{C}(X)_{*_\mathcal{C}}$ obtained by evaluating each of the Hom objects at the terminal object $*_\mathcal{C}$ of $\mathcal{C}$. We have that

$$\mathfrak{C}(X)_{*_\mathcal{C}} \cong \mathfrak{C}_\Delta(\mathscr{N} X),$$

since

$$\mathfrak{C}(X)_{*_\mathcal{C}} \cong (\mathfrak{C}_{\Delta,\bullet}(k^*(X)))_{*_\mathcal{C}} \cong \mathfrak{C}_\Delta(k^*(X)_{*_\mathcal{C}}),$$

and $k^*(X)_{*_\mathcal{C}}$ is precisely $\mathscr{N}X$. Since $\mathscr{N}X$ is quite clearly a quasicategory, the claim follows immediately from the ordinary case. This implies that the map f is a horizontal Dwyer-Kan equivalence, and therefore by the previous proposition, a horizontal Joyal equivalence, which concludes the proof. ■

2.8.4. Theorem. The Quillen pair

$$\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}} \underset{\mathfrak{N}}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}}$$

is a Quillen equivalence.

Proof. As we have proven that the derived counit is always an equivalence in Theorem 2.7.9, all we have left to show is that the derived unit transformation

$$X \to \mathfrak{N}\mathfrak{C}(X) \to \mathfrak{N}\mathcal{D}$$

is a weak equivalence for all presheaves X on $\Theta[\mathcal{C}]$, where $\mathfrak{C}(X) \to \mathcal{D}$ is a weak equivalence and $\mathcal{D}$ is fibrant. However, by the previous proposition, we see that it suffices to show that the map

$$\mathfrak{C}(X) \to \mathfrak{C}\mathfrak{N}\mathfrak{C}(X) \to \mathfrak{C}\mathfrak{N}\mathcal{D}$$

is a weak equivalence. We obtain a naturality square from the counit

$$\begin{array}{ccc} \mathfrak{C}\mathfrak{N}\mathfrak{C}(X) & \longrightarrow & \mathfrak{C}\mathfrak{N}\mathcal{D} \\ \downarrow & & \downarrow \wr \\ \mathfrak{C}(X) & \xrightarrow{\sim} & \mathcal{D} \end{array},$$

in which the indicated arrows are equivalences (for the bottom horizontal, this was by choice, and for the righthand vertical, it comes from Theorem 2.7.9). But if we precompose with the unit map $\mathfrak{C}\eta_X\colon \mathfrak{C}(X) \to \mathfrak{C}\mathfrak{N}\mathfrak{C}(X)$, the lefthand arrow becomes the identity by the triangle identities, which proves the claim by applying 3-for-2 to the commutative diagram

$$\begin{array}{ccccc} \mathfrak{C}(X) & \longrightarrow & \mathfrak{C}\mathfrak{N}\mathfrak{C}(X) & \longrightarrow & \mathfrak{C}\mathfrak{N}\mathcal{D} \\ & \searrow\!\!= & \downarrow & & \downarrow \wr \\ & & \mathfrak{C}(X) & \xrightarrow{\sim} & \mathcal{D} \end{array}.$$

■

3. The Coherent Nerve, Local case

3.1. The $(\mathcal{C},\mathscr{S})$-enriched model structure. While our presentation of the horizontal Joyal model structure comes mainly from David Oury's thesis [Our10], what follows is independent, making use of the resolution technology we developed in the previous section to give a simple and satisfying story. Suppose $\mathcal{M}=(\mathcal{C},\mathscr{S})$ is a Cartesian presentation in the sense of Rezk (with the additional stipulation that $\mathcal{C}$ is regular Cartesian Reedy), where $\mathscr{S}$ is a set of monomorphisms of $\mathrm{Psh}_\Delta(\mathcal{C})$ such that the left-Bousfield localization of $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$ at $\mathscr{S}$ is a Cartesian model category.

Recall that we had a number of functorial cosimplicial objects

$$C^\bullet_{(-)}(\bullet)\colon \Delta\times\mathcal{C}\to\widehat{\Theta[\mathcal{C}]}_{*,*},$$

such that $C^\bullet_{(-)}(\bullet)$ was a cosimplicial resolution of $[1](\bullet)$, which is a Reedy cofibrant diagram $\mathcal{C}\to\widehat{\Theta[\mathcal{C}]}_{*,*}$. Since $\widehat{\Theta[\mathcal{C}]}_{*,*}$ is cocomplete, $C^\bullet_{(-)}(\bullet)$ extends to a cocontinuous functor

$$\Sigma_{(-)}\colon\ \mathrm{Psh}_\Delta(\mathcal{C})\to\widehat{\Theta[\mathcal{C}]}_{*,*}.$$

3.1.1. Proposition. The functor $\Sigma_{(-)}$ is left-Quillen when $\widehat{\Theta[\mathcal{C}]}_{*,*}$ is equipped with the horizontal Joyal model structure.

Proof. By Proposition A.4.11 and Corollary A.4.12, it suffices to show that the functor $\Sigma_{(-)}$ sends generating monomorphisms to cofibrations and that for all $n\geq 0$ and $c\in\mathcal{C}$, the map $\Sigma_{(-)}(\Delta^n\times c)\to\Sigma_{(-)}(\Delta^0\times c)$ is a horizontal Joyal equivalence. The first statement is clear by our construction, and the second statement follows from the fact that

$$\Sigma_{(-)}(\Delta^n\times c)\cong C^n_{(-)}(c),$$

in which case it follows from our proof that the cosimplicial objects $C^\bullet_{(-)}(c)$ are bi-pointed cosimplicial resolutions of $[1](c)$. ∎

3.1.2. Corollary. The functor $\Sigma_{(-)}$ is independent up-to-homotopy of choice of resolution $C^\bullet_{(-)}(\bullet)$.

Proof. Since a simplicial presheaf is always the homotopy colimit of its canonical diagram over its category of elements, and since left-Quillen functors preserve homotopy colimits, it suffices to show that the construction

$$\Sigma_{(-)}(\Delta^n\times c)$$

is independent up-to-homotopy. But this is clear since all $C^\bullet_{(-)}(\bullet)$ are connected by natural zig-zags of natural weak equivalences, since they are all cosimplicial resolutions of the same functor $[1](\bullet)$. ∎

We can therefore, without any worry, denote $\Sigma_{(-)}$ simply by Σ. Then we define the following model structure:

3.1.3. Definition. If $\mathcal{M} = (\mathcal{C}, \mathscr{S})$ is a Cartesian presentation, we define the model category $\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}}$ to be the left-Bousfield localization of $\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}}$ at the set $\Sigma(\mathscr{S})$, where we call the fibrant objects $\mathcal{M}$-*enriched quasicategories* or simply $\mathcal{M}$-*quasicategories*.

3.1.4. Proposition. A formal $\mathcal{C}$-quasicategory X is an $\mathcal{M}$-quasicategory if and only if $\mathrm{Map}_X(x, y)$ is $\mathscr{S}$-local for all pairs of vertices x, y in X.

Proof. By definition of left Bousfield localization, we see that a formal $\mathcal{C}$-quasicategory X is $\Sigma(\mathscr{S})$-local if and only if it for every $s\colon A \to B$ in $\mathscr{S}$, the induced map on homotopy function complexes

$$\mathrm{hFun}^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(s), X)\colon\ \mathrm{hFun}^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(B), X) \to \mathrm{hFun}^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(A), X)$$

is a weak homotopy equivalence. However, in the category $\widehat{\Theta[\mathcal{C}]}$, we have a universal and normalized Reedy-cofibrant cosimplicial resolution of terminal object, namely $E^{\bullet}$. It follows that since all of the maps $s \in \mathscr{S}$ are monomorphisms and since Σ sends monomorphisms to monomorphisms, we deduce that in fact a formal $\mathcal{C}$-quasicategory X is $\Sigma(\mathscr{S})$-local if and only if it for every $s\colon A \to B$ in $\mathscr{S}$, the induced map on homotopy function complexes

$$\mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(s), X)\colon\ \mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(B), X) \to \mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(A), X)$$

is a trivial Kan fibration, since if X is a formal $\mathcal{C}$-quasicategory, this functor sends monomorphisms in the first argument to Kan fibrations. By considering the monomorphism $\varnothing \hookrightarrow A$, we obtain a sequence of Kan fibrations

$$\mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(B), X) \to \mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(A), X) \to \mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(\varnothing), X).$$

Notice that a point of the simplicial set $\mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(\varnothing), X)$ gives a pair of vertices (x, y) in X, and it is obvious then that the map $\mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(s), X)$ is a trivial fibration if and only if for every point $\varepsilon_{(x,y)} \in \mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(\varnothing), X)_0$ classifying a pair of vertices (x, y) of X, the induced map on fibres over $\varepsilon_{(x,y)}$ is a trivial fibration.

Since both maps are Kan fibrations, the strict fibres are also the homotopy fibres, and we can compute them directly. We immediately see that the fibre of the map

$$\mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(A), X) \to \mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(\varnothing), X)$$

over a point $\varepsilon_{(x,y)}$ is exactly

$$\mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}_{*,*}}(\Sigma(A), X_{x,y})$$

the homotopy function complex of bi-pointed maps and similarly, the fibre of the composite

$$\mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(B), X) \to \mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}}(\Sigma(\varnothing), X)$$

is also

$$\mathrm{hFun}_E^{\widehat{\Theta[\mathcal{C}]}_{*,*}}(\Sigma(B), X_{x,y})$$

the homotopy function complex of bi-pointed maps. But passing to adjoints, we see that the fibres can equivalently be identified with

$$\mathrm{hFun}^{\mathrm{Psh}_\Delta(\mathcal{C})}(A, \mathrm{Map}_X(x,y))$$

and

$$\mathrm{hFun}^{\mathrm{Psh}_\Delta(\mathcal{C})}(B, \mathrm{Map}_X(x,y))$$

respectively, and that the induced map

$$\mathrm{hFun}^{\mathrm{Psh}_\Delta(\mathcal{C})}(B, \mathrm{Map}_X(x,y)) \to \mathrm{hFun}^{\mathrm{Psh}_\Delta(\mathcal{C})}(A, \mathrm{Map}_X(x,y))$$

is none other than the map

$$\mathrm{hFun}^{\mathrm{Psh}_\Delta(\mathcal{C})}(s, \mathrm{Map}_X(x,y)).$$

This proves the equivalence of both statements, since this map being an equivalence for all $s \in \mathscr{S}$ is exactly what it means for $\mathrm{Map}_X(x,y)$ to be $\mathscr{S}$-local. ∎

3.1.5. Corollary. Let $\mathscr{B}$ denote the set of simplicial boundary inclusions. Then a formal $\mathcal{C}$-quasicategory is an $\mathcal{M}$-quasicategory with respect to a Cartesian presentation $\mathcal{M} = (\mathcal{C}, \mathscr{S})$ if and only if it has the right lifting property with respect to

$$\Sigma(\mathscr{B} \times^{\lrcorner} \mathscr{S}).$$

Proof. Since $\mathrm{Psh}_\Delta(\mathcal{C})$ admits a Reedy-cofibrant universal normalized resolution of its terminal object by looking at the cosimplicial object $\Delta^\bullet$, we can compute the homotopy function complex

$$\mathrm{hFun}^{\mathrm{Psh}_\Delta(\mathcal{C})}(A, B)$$

for any injectively fibrant object B as

$$\mathrm{Hom}_{\mathrm{Psh}_\Delta(\mathcal{C})}(\Delta^\bullet \times A, B).$$

If s is an injective map in $\mathrm{Psh}_\Delta(\mathcal{C})$ and B is injectively fibrant, the map $\mathrm{hFun}_\Delta^{\mathrm{Psh}_\Delta(\mathcal{C})}(s, B)$ is a Kan fibration. We can see therefore that this map is a weak homotopy equivalence if and only if it is a trivial fibration, in which case, this is equivalent to it having the right lifting property with respect to all boundary inclusions of simplices.

If we unwind the construction $\mathrm{hFun}_\Delta^{\mathrm{Psh}_\Delta(\mathcal{C})}(s, B)$, we see that this implies that for every boundary inclusion $\delta^n \colon \partial\Delta^n \hookrightarrow \Delta^n$, the object B has the right lifting property with respect to the pushout product $s \times^{\lrcorner} \delta^n$. It follows that an injectively fibrant simplicial presheaf B is $\mathscr{S}$-local if and only if it has the right lifting property with respect to the set of maps $\mathcal{B} \times^{\lrcorner} \mathscr{S}$.

Let X be a formal $\mathcal{C}$-quasicategory. Then the proposition says that X is $\Sigma(\mathscr{S})$-local if and only if its mapping objects $\mathrm{Map}_X(x, y)$ are $\mathscr{S}$-local for all pairs of vertices (x, y) of X. The desired result now comes from the observation that given a problem where we'd like to extend a map

$$i \colon \Sigma(s \times^{\lrcorner} \delta^n(0)) \to X$$

to $\Sigma(s \times^{\lrcorner} \delta^n(1))$ the choice of the map i already determines the vertices in question, so we can pass back and forth over the adjunction with nary a worry. The result now follows. ∎

3.1.6. Theorem. For any Cartesian presentation $\mathcal{M} = (\mathcal{C}, \mathscr{S})$, the model category $\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}}$ is Cartesian closed, and the Quillen equivalences

$$\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}} \leftrightsquigarrow \widehat{\Theta[\mathcal{C}] \times \Delta}_{\mathrm{Seg,Cpt}}$$

remain Quillen equivalences between $\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}}$ and the left Bousfield localization $\widehat{\Theta[\mathcal{C}] \times \Delta}_{\mathscr{S}_\Theta}$ of the model structure for complete $\Theta[\mathcal{C}]$-Segal spaces $\widehat{\Theta[\mathcal{C}] \times \Delta}_{\mathrm{Seg,Cpt}}$ at the set of maps $V^{\Delta}[1](\mathscr{S})$, where V^{Δ} denotes the version of Rezk's intertwiner for simplicial presheaves on $\mathcal{C}$ and $\Theta[\mathcal{C}]$.

Proof. Recall that we have an adjunction

$$\mathcal{P} \colon \widehat{\Theta[\mathcal{C}] \times \Delta} \rightleftarrows \widehat{\Theta[\mathcal{C}]} \colon \mathcal{Q},$$

where the right adjoint $\mathcal{Q}$ is the functor defined by the rule

$$\mathcal{Q}(X)_{[n](c_1,\dots,c_n),m} \overset{\mathrm{def}}{=} \mathrm{Hom}([n](c_1, \dots, c_n) \times E^m, X).$$

Using §1.7, we see that this adjunction gives a Quillen equivalence between the model structure for complete $\Theta[\mathcal{C}]$-Segal spaces and the horizontal Joyal model structure.

Let us denote Rezk's simplicial version of the intertwiner by V^{Δ} to avoid confusion. Then we will show that the the composite functor $\mathcal{P}V^{\Delta}[1](-)$ is isomorphic to Σ_E.

First, notice that we can write down a functor $\mathcal{C} \times \Delta \to \widehat{\Theta[\mathcal{C}] \times \Delta}$ whose image under $\mathcal{P}$ is $\mathcal{C}^{\bullet}_E(\bullet)$. We define it as a colimit of functors:

$$C^{\bullet}_D(\bullet) \overset{\mathrm{def}}{=} \mathrm{colim}\left(\Delta^0 \amalg \Delta^0 \leftarrow \Delta^{\bullet} \amalg \Delta^{\bullet} \to [1](\bullet) \times \Delta^{\bullet}\right),$$

valued in $\widehat{\Theta[\mathcal{C}] \times \Delta}$, where here, the symbol Δ^n refers to the constant simplicial presheaf on $\Theta[\mathcal{C}]$ with value Δ^n. We will not, however, distinguish between the constant simplicial presheaf Δ^n on $\Theta[\mathcal{C}]$ and the constant simplicial presheaf Δ^n on $\mathcal{C}$ itself, since the context will always be clear. At any rate, we will exhibit an isomorphism, natural in c and n

$$C^n_D(c) \to V^{\Delta}[1](c \times \Delta^n).$$

Using the formula for V^Δ in [Rez10, 4.4], we have

$$V[1](c \times \Delta^n)([n](c_1, \dots, c_n)) = \coprod_{\rho \in \Delta([n],[1])} \prod_{i=1}^{n} \prod_{j=\rho(i-1)+1}^{\rho(i)} (\mathcal{C}(c_i, c) \times \Delta^n).$$

Observe that the set $\Delta([n],[1])$ is an interval, and that we can index the elements ρ by setting $\rho_\perp$ to be the map sending everything to 0, letting $\rho_\top$ be the map sending everything to 1, and for each $1 \leq i \leq n$, let ρ_i be the unique $\rho \in \Delta([n],[1])$ such that $\rho(i-1) < \rho(i)$. For each element $\rho \in \Delta([n],[1])$, let K_ρ be its associated component. Then we have

$$K_{\rho_i} = \begin{cases} \Delta^0 & \text{if} \quad i \in \{\perp, \top\} \\ \mathcal{C}(c_i, c) \times \Delta^n & \text{otherwise.} \end{cases}$$

On the other hand, we can easily compute the summands J_ρ of the simplicial set

$$([1](c) \times \Delta^n)\,([n](c_1, \dots, c_n))$$

by the formula

$$J_{\rho_i} = \begin{cases} \mathcal{C}(c_i, c) \times \Delta^n & \text{if} \quad i \notin \{\perp, \top\} \\ \Delta^n & \text{otherwise.} \end{cases}$$

Therefore, we have a natural map $[1](c) \times \Delta^n \to V[1](c \times \Delta^n)$ exhibiting $V[1](c \times \Delta^n)$ as the quotient of $[1](c) \times \Delta^n$ by the summands corresponding to the bottom and top elements of the interval $\Delta([n],[1])$. However, the coproduct of these summands corresponds precisely to the inclusion $\Delta^n \coprod \Delta^n \hookrightarrow [1](c) \times \Delta^n$, which is precisely what we killed in the definition of $C_D^\bullet(\bullet)$. Therefore, the canonical map $C_D^\bullet(\bullet) \to V^\Delta[1]|_{\mathcal{C} \times \Delta}$ is an isomorphism. As we saw above, $\mathcal{P}\mathcal{C}_D^\bullet(\bullet) \cong \mathcal{C}_E^\bullet(\bullet)$.

Observe that Σ_E preserves connected colimits, and since $V^\Delta[1](-)$ preserves connected colimits and the functor $\mathcal{P}$ preserves all small colimits, we have that the functors Σ_E and $\mathcal{P} \circ V^\Delta[1](-)$ both factor as cocontinuous functors followed by projection from an undercategory. That is to say, the functors

$$\Sigma_E \colon \operatorname{Psh}_\Delta(\mathcal{C}) \to \widehat{\Theta[\mathcal{C}]}_{\Sigma_E(\varnothing)/}$$

and

$$\mathcal{P} \circ V^\Delta[1](-) \colon \operatorname{Psh}_\Delta(\mathcal{C}) \to \widehat{\Theta[\mathcal{C}]}_{\mathcal{P} \circ V^\Delta[1](\varnothing)/}$$

preserve small colimits. However, it is easy to see that

$$\Sigma_E(\varnothing) \cong \mathcal{P} \circ V^\Delta[1](\varnothing) \cong * \coprod *,$$

so they both define colimit preserving functors $\operatorname{Psh}_\Delta(\mathcal{C}) \to \widehat{\Theta[\mathcal{C}]}_{*,*}$. But we know that these functors are cocontinuous and also isomorphic when restricted to representables, so they must themselves be isomorphic, as they are the universal cocontinuous extensions of isomorphic functors.

It follows that given a Cartesian presentation $\mathcal{M} = (\mathcal{C}, \mathscr{S})$ the left Bousfield localization of the model structure for complete Segal $\Theta[\mathcal{C}]$-spaces on $\widehat{\Theta[\mathcal{C}] \times \Delta}$ at the set of maps $V^{\Delta}[1](\mathscr{S})$ is precisely the simplicial completion of $\widehat{\Theta[\mathcal{C}]}$ at the set of maps $\Sigma_E(\mathscr{S})$, since $\Sigma_E(\mathscr{S}) \cong \mathcal{P}V^{\Delta}[1](\mathscr{S})$. By [Rez10, Proposition 8.5], the localized model structure on $\widehat{\Theta[\mathcal{C}] \times \Delta}$ is Cartesian. Then it follows from [Ara14, Corollary 2.21] that the left Bousfield localization of the horizontal Joyal model structure on $\widehat{\Theta[\mathcal{C}]}$ at the set $\Sigma_E(\mathscr{S})$ is Cartesian as well, as desired. ■

In what follows, let Σ be Σ_R.

3.1.7. Proposition. The adjoint pair $\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}} \underset{\mathfrak{N}}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_{\Delta}(\mathcal{C})_{\mathscr{S}}}$ is a Quillen pair.

Proof. It suffices to show that $\mathfrak{N}$ preserves fibrant objects by the properties of the left-Bousfield localization. Since the coherent nerve of any fibrant $\mathrm{Psh}_{\Delta}(\mathcal{C})$-enriched category $\mathcal{D}$ is already a formal $\mathcal{C}$-quasicategory, it suffices to show that $\mathfrak{N}\mathcal{D}$ has the right-lifting property with respect to $\Sigma(\mathscr{B} \times^{\lrcorner} \mathscr{S})$. This will be true so long as the maps belonging to the set $\mathfrak{C}(\Sigma(\mathscr{B} \times^{\lrcorner} \mathscr{S}))$ are all weak equivalences. Recall that we defined the object $\mathbf{2}(A)$ for any simplicial presheaf A on $\mathcal{C}$ and that it denotes the $\mathrm{Psh}_{\Delta}(\mathcal{C})$-enriched category whose objects are $\{0, 1\}$ and where

$$\mathbf{2}(A)(x,y) = \begin{cases} * \text{ if } x = y \\ A \text{ if } x < y \\ \varnothing \text{ otherwise} \end{cases} .$$

Recall from Proposition 2.7.2 that for all $n \geq 0$ and $c \in \mathcal{C}$, we have an isomorphism

$$\mathfrak{C}(\Sigma(\Delta^n \times c))(\alpha, \omega) \cong Q^n \times c,$$

where $Q^n = \mathfrak{C}_{\Delta}(C^n_R)(\alpha, \omega)$. Following [Lur09, Proposition 2.2.2.7], We define a realization

$$|\bullet|_Q : \mathrm{Psh}_{\Delta}(\mathcal{C}) \to \mathrm{Psh}_{\Delta}(\mathcal{C})$$

by left Kan extension of the functor $\Delta^n \times c \mapsto Q^n \times c$ along the Yoneda embedding. Let $\mathcal{A}$ denote the class of simplicial presheaves A on $\mathcal{C}$ such that the map

$$|A|_Q \to A$$

is an injective equivalence. This class is closed under filtered colimits, since injective weak equivalences are closed under filtered colimits, so it suffices to consider the case where A has finitely many nondegenerate representable cells $[n] \times c$. Since Δ and $\mathcal{C}$ are regular skeletal, so is their product by [Cis06, 8.2.7], and the boundary of a representable cell is given by the formula

$$\partial(\Delta^n \times c) = \partial\Delta^n \times c \cup \Delta^n \times \partial c.$$

We work by induction on Reedy dimension and number of cells. If $A = \varnothing$, we are done, since the map in question is the identity. Otherwise, suppose

$$A = A' \coprod_{\partial(\Delta^n \times c)} \Delta^n \times c.$$

This is a homotopy pushout since $\partial(\Delta^n \times c) \to \Delta^n \times c$ is an injective cofibration. Similarly,

$$|A|_Q = |A'| \coprod_{|\partial(\Delta^n \times c)|_Q} |\Delta^n \times c|$$

is also a homotopy-pushout since $|\bullet|_Q$ preserves monomorphisms. Then we see that the map

$$|\Delta^n \times c|_Q = Q^n \times c \to \Delta^n \times c$$

is already a weak equivalence since $Q^n \to \Delta^n$ is a weak equivalence and the injective model structure is Cartesian. The map

$$|\partial(\Delta^n \times c)|_Q \to \partial(\Delta^n \times c)$$

is a weak equivalence by the induction hypothesis, since the Reedy dimension of $\partial(\Delta^n \times c)$ is less than the dimension of $\Delta^n \times c$. Finally, we see that

$$|A'|_Q \to A'$$

is a weak equivalence since A' has one fewer nondegenerate cell than A and is therefore also covered in the induction hypothesis. Therefore, the natural map

$$\mathfrak{C}(\Sigma(A)) \cong \mathbf{2}(|A|_Q) \xrightarrow{\sim} \mathbf{2}(A)$$

is a weak equivalence in $\mathbf{Cat}_{\mathrm{Psh}_\Delta(C)_{\mathrm{inj}}}$ for all simplicial presheaves A on $\mathcal{C}$. From this, it follows that since $\mathbf{2}(b \times^\lrcorner f)$ is an $\mathcal{M}$-equivalence for any $f \in \mathscr{S}$, and since we have a natural equivalence of arrows

$$\mathfrak{C}(\Sigma(b \times^\lrcorner f)) \xrightarrow{\sim} \mathbf{2}(b \times^\lrcorner f),$$

then by 3-for-2, $\mathfrak{C}(\Sigma(b \times^\lrcorner f))$ is a weak equivalence, which proves that the functor is left Quillen. ∎

3.1.8. Theorem. The Quillen pair $\widehat{\Theta[\mathcal{C}]}_\mathcal{M} \underset{\mathfrak{N}}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_\mathscr{S}}$ is a Quillen equivalence.

Proof. It suffices to show that $\mathfrak{C}$ is homotopy-conservative, so let $f\colon X \to Y$ be a map in $\widehat{\Theta[\mathcal{C}]}$ such that $\mathfrak{C}(f)$ is an equivalence in $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_\mathcal{M}}$. Using the same argument as in §2.8, we reduce to the case where $f\colon X \to Y$ is a map between $\mathcal{M}$-quasicategories.

Since $\mathcal{M}$-quasicategories are formal $\mathcal{C}$-quasicategories, we can use Proposition 2.7.5 to obtain a natural zig-zag of weak equivalences

$$\mathrm{Map}_X(x, y) \leftrightsquigarrow \mathfrak{C}(X)(x, y)$$

for any pair of vertices x, y of X. By 3-for-2 and since

$$\mathfrak{C}(X)(x, y) \to \mathfrak{C}(Y)(fx, fy)$$

was assumed to be an $\mathcal{M}$-equivalence, we see that the map

$$\mathrm{Map}_X(x, y) \to \mathrm{Map}_Y(fx, fy)$$

must also be an $\mathcal{M}$-equivalence.

In fact, since both $\mathrm{Map}_X(x, y)$ and $\mathrm{Map}_Y(fx, fy)$ are local, this map is actually an equivalence for $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}}$. The argument showing that f is bijective on iso-components is the same as in the proof of Proposition 2.8.3 by passing to the underlying quasicategory. Therefore, it follows that f is a horizontal Dwyer-Kan equivalence, which concludes the proof. ∎

3.1.9. Definition. Let $f\colon X \to Y$ be a morphism of $\mathcal{C}$-cellular sets. Then we say that the map f is $\mathcal{M}$-locally *fully faithful* if for all pairs of vertices x, x' of X, the map of simplicial presheaves $\mathrm{hMap}_X(x, x') \to \mathrm{hMap}_Y(fx, fx')$ is a weak equivalence in $\mathcal{M}$.

3.1.10. Corollary. A morphism $f\colon X \to Y$ of $\mathcal{C}$-cellular sets is $\mathcal{M}$-locally fully faithful if and only if the associated map

$$\mathfrak{C}(f)\colon \mathfrak{C}(X) \to \mathfrak{C}(Y)$$

is a weakly fully faithful map of $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathscr{S}}}$-enriched categories.

Proof. Since the condition uses the homotopy-invariant mapping objects, we may assume that X and Y are formal $\mathcal{C}$-quasicategories. Then the claim follows immediately by the existence of the natural zig-zags of weak equivalences as in the theorem. ∎

3.2. The Yoneda embedding and Yoneda's lemma.

3.2.1. Definition. Let $\mathrm{op}\colon \Delta \cong \Delta$ be the automorphism of Δ induced by restriction of the automorphism $\mathrm{op}\colon \mathbf{Cat} \cong \mathbf{Cat}$ sending a category to its opposite and a functor to the opposite functor. Observe that automorphism is the identity on objects. Then pulling back the fibration

$$\Delta \int \widehat{\mathcal{C}} \to \Delta$$

along this automorphism induces an automorphism

$$\mathrm{op}\colon \Delta \int \widehat{\mathcal{C}} \cong \Delta \int \widehat{\mathcal{C}}.$$

Given a labeled n-simplex $[n](A_1, \ldots, A_n)$, we observe that

$$\mathrm{op}([n](A_1, \ldots, A_n)) = [n](B_1, \ldots, B_n),$$

since $\mathrm{op}([n]) = [n]$. However, we observe that $\mathrm{op}(\delta^{i-1,i})\colon \mathrm{op}([1]) \to \mathrm{op}([n])$ can be identified with $\delta^{n-i,n-i+1}\colon [1] \to [n]$, which shows by composition that we have $B_i = A_{n-i+1}$, or that is to say $\mathrm{op}([n](A_1,\ldots,A_n)) = [n](A_n,\ldots,A_1)$. It follows by definition of $\Theta[\mathcal{C}]$ that this automorphism restricts to an automorphism

$$\mathrm{op}\colon \Theta[\mathcal{C}] \cong \Theta[\mathcal{C}],$$

which extends to an automorphism of presheaf categories $\mathrm{op}\colon \widehat{\Theta[\mathcal{C}]} \to \widehat{\Theta[\mathcal{C}]}$. Given a $\mathcal{C}$-cellular set $X \in \widehat{\Theta[\mathcal{C}]}$, we denote $\mathrm{op}(X)$ by X^{op}.

3.2.2. Lemma. The $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched categories $\mathfrak{C}(X^{\mathrm{op}})$ and $\mathfrak{C}(X)^{\mathrm{op}}$ are naturally isomorphic.

Proof. The automorphism $\mathrm{op}\colon \Delta \to \Delta$ extends to an automorphism $\widehat{\mathrm{op}\times\mathrm{id}}\colon \widehat{\Delta\times\mathcal{C}} \cong \widehat{\Delta\times\mathcal{C}}$, which then restricts to an automorphism $\mathrm{op}\colon \mathbf{PCat}(\mathcal{C}) \cong \mathbf{PCat}(\mathcal{C})$. It is immediate by unwinding the definitions that we have an isomorphism $k^*(X^{\mathrm{op}}) \cong k^*(X)^{\mathrm{op}}$ natural in the $\mathcal{C}$-cellular set X. Moreover we have an easy chain of isomorphisms natural with with respect to $Y \in \mathbf{PCat}(\mathcal{C})$, and $c \in \mathcal{C}$

$$\begin{aligned}\mathfrak{C}_{\Delta,\bullet}(Y^{\mathrm{op}})_c &\cong \mathfrak{C}_\Delta((Y^{\mathrm{op}})_c)\\ &\cong \mathfrak{C}_\Delta((Y_c)^{\mathrm{op}})\\ &\cong \mathfrak{C}_\Delta(Y_c)^{\mathrm{op}}\\ &\cong \mathfrak{C}_{\Delta,\bullet}(Y)_c^{\mathrm{op}},\end{aligned}$$

which implies that we have an isomorphism natural in the precategory Y

$$\mathfrak{C}_{\Delta,\bullet}(Y^{\mathrm{op}}) \cong \mathfrak{C}_{\Delta,\bullet}(Y)^{\mathrm{op}}.$$

So taking these two isomorphisms together, we have a natural isomorphism

$$\begin{aligned}\mathfrak{C}(X^{\mathrm{op}}) &= \mathfrak{C}_{\Delta,\bullet}(k^*(X^{\mathrm{op}}))\\ &\cong \mathfrak{C}_{\Delta,\bullet}(k^*(X)^{\mathrm{op}})\\ &\cong \mathfrak{C}_{\Delta,\bullet}(k^*(X))^{\mathrm{op}}\\ &= \mathfrak{C}(X)^{\mathrm{op}},\end{aligned}$$

as desired. ■

3.2.3. Definition. [Lur09, A.3.4.1] Let $\mathbf{S}$ be an excellent symmetric monoidal model category, and let $\mathbf{A}$ be a combinatorial $\mathbf{S}$-enriched model category (that is to say, it is tensored and cotensored over $\mathbf{S}$ and the tensor functor $\mathbf{S}\otimes\mathbf{A} \to \mathbf{A}$ is a left-Quillen bifunctor). Then we say that a full subcategory $\mathcal{U} \subseteq \mathbf{A}$ is a *chunk* if the following statements hold:

- For any finite family of maps $(\phi_i\colon A \to B_i)_{i\in I}$ in $\mathcal{U}$, there exists a factorization

  $$A \xrightarrow{p} \overline{A} \xrightarrow{q} \prod_i B_i$$

  of the product map $\prod_i \phi_i$ where the map p is a trivial cofibration, the map q is a fibration, and $\overline{A}$ belongs to $\mathcal{U}$. Moreover, this factorization can be chosen to depend functorially on the family $(\phi_i\colon A \to B_i)_{i\in I}$ via an $\mathbf{S}$-enriched functor.

- For any finite family of maps $(\phi_i\colon B_i \to A)_{i\in I}$ in $\mathcal{U}$, there exists a factorization

  $$\coprod_i B_i \xrightarrow{p} \overline{A} \xrightarrow{q} A$$

  of the coproduct map $\coprod_i \phi_i$ where the map p is a cofibration, the map q is a trivial fibration, and $\overline{A}$ belongs to $\mathcal{U}$. Moreover, this factorization can be chosen to depend functorially on the family $(\phi_i\colon A \to B_i)_{i\in I}$ via an $\mathbf{S}$-enriched functor.

We let $\mathbf{A}^\circ$ denote the full subcategory of $\mathbf{A}$ spanned by the fibrant and cofibrant objects, and if $\mathcal{U} \subseteq \mathbf{A}$ is a chunk, we let $\mathcal{U}^\circ = \mathcal{U} \cap \mathbf{A}^\circ$.

If $\mathcal{D}$ is a small $\mathbf{S}$-enriched category we say that a full subcategory $\mathcal{U} \subseteq \mathbf{A}$ is a $\mathcal{D}$-chunk if $\mathcal{U}$ is a chunk of $\mathbf{A}$ and the subcategory $\mathcal{U}^{\mathcal{D}} \subseteq \mathbf{A}^{\mathcal{D}}$ is a chunk of $\mathbf{A}^{\mathcal{D}}$ when $\mathbf{A}^{\mathcal{D}}$ is regarded as a model category with respect to the projective model structure.

We fix a Cartesian presentation $\mathcal{M} = (\mathcal{C}, \mathscr{S})$ for the remainder of this section. By abuse of notation, we will call a $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched category an $\mathcal{M}$-enriched category. Then before we give a construction of the Yoneda embedding and a proof of Yoneda's lemma for $\mathcal{M}$-quasicategories, we will need two lemmas from [Lur09]. The first statement uses the language of chunks in order to avoid size issues.

3.2.4. Proposition. [Lur09, 4.2.4.4] Let $X \in \widehat{\Theta[\mathcal{C}]}$ be a small $\mathcal{C}$-cellular set, $\mathcal{D}$ a small $\mathcal{M}$-enriched category, and let $\phi\colon \mathfrak{C}(X) \to \mathcal{D}$ be a weak equivalence of $\mathcal{M}$-enriched categories. Suppose $\mathbf{A}$ is a combinatorial $\mathcal{M}$-enriched model category, and let $\mathcal{U}$ be $\mathcal{D}$-chunk. Then the induced map

$$\mathfrak{N}((\mathcal{U}^{\mathcal{D}})^\circ) \to \mathfrak{N}(\mathcal{U}^\circ)^X$$

is an equivalence of $\mathcal{M}$-quasicategories. In particular, since $\mathbf{A}$ is a $\mathcal{D}$-chunk of itself for any small $\mathcal{D}$, the statement holds for $\mathbf{A}$ itself.

Proof. Let κ be a cardinal such that X and $\mathcal{D}$ are κ-small. Then using [Lur09, Lemma A.3.4.15], the chunk $\mathcal{U}$ can be expressed as a κ-filtered colimit of small $\mathcal{D}$-chunks. Since the localizer of the model structure for $\mathcal{M}$-quasicategories is regular, it is closed under filtered colimits, it suffices to prove the claim in the case where $\mathcal{U}$ is a small $\mathcal{D}$-chunk.

Then before we proceed, we unwind how the induced map γ behaves precisely. Observe that for any $Z \in \widehat{\Theta[\mathcal{C}]}$, the induced map

$$\mathfrak{N}((\mathcal{U}^{\mathcal{D}})^\circ) \to \mathfrak{N}(\mathcal{U}^\circ)^X$$

corresponds to the composite

$$\begin{array}{ccc}
\operatorname{Ho}\mathbf{Cat}_{\mathcal{M}}(Z,\mathfrak{N}((\mathcal{U}^{\mathcal{D}})^\circ)) & & \\
\downarrow\cong & & \\
\operatorname{Ho}\mathbf{Cat}_{\mathcal{M}}(\mathfrak{C}Z,(\mathcal{U}^{\mathcal{D}})^\circ) & & \\
\downarrow\alpha & & \\
\operatorname{Ho}\mathbf{Cat}_{\mathcal{M}}(\mathfrak{C}Z\times\mathcal{D},\mathcal{U}^\circ) & & \\
\downarrow\gamma & & \\
\operatorname{Ho}\mathbf{Cat}_{\mathcal{M}}(\mathfrak{C}Z\times\mathfrak{C}X,\mathcal{U}^\circ) & \xrightarrow{\rho} & \operatorname{Ho}\mathbf{Cat}_{\mathcal{M}}(\mathfrak{C}(Z\times X),\mathcal{U}^\circ) \\
 & & \downarrow\psi \\
 & & \operatorname{Ho}\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}}(Z\times X,\mathfrak{N}(\mathcal{U}^\circ)) \\
 & & \downarrow\omega \\
 & & \operatorname{Ho}\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}}(Z,\mathfrak{N}(\mathcal{U}^\circ)^X).
\end{array}$$

It suffices to show that for all $Z \in \widehat{\Theta[\mathcal{C}]}$, the composite of these maps is bijective.

- Observe first that the map α is a bijection for all $Z \in \widehat{\Theta[\mathcal{C}]}$ by [Lur09, Corollary A.3.4.14], since we see that $(\mathcal{U}^{\mathcal{D}})^\circ$ represents the exponential $(\mathcal{U}^\circ)^{\mathcal{D}}$ in homotopy category of $\mathbf{Cat}_{\mathcal{M}}$.
- Notice now that the map γ is a bijection for all $Z \in \widehat{\Theta[\mathcal{C}]}$ because weak equivalences in $\mathbf{Cat}_{\mathcal{M}}$ are stable under Cartesian product.
- The map ψ is bijective for any $Z \in \widehat{\Theta[\mathcal{C}]}$ because it is the map induced on homotopy categories by a Quillen equivalence, because the object $Z \times X$ is cofibrant, and because the object $\mathcal{U}^\circ$ is fibrant.
- The map ω is bijective for any $Z \in \widehat{\Theta[\mathcal{C}]}$ because $\widehat{\Theta[\mathcal{C}]}$ is a Cartesian monoidal model category and $\mathfrak{N}(\mathcal{U}^\circ)$ is fibrant.

It remains to show that ρ is bijective. In fact, we will show that for any $\mathcal{C}$-cellular sets Z, Z' the map $\mathfrak{C}(Z \times Z') \to \mathfrak{C}(Z) \times \mathfrak{C}(Z')$ is a Bergner-Lurie equivalence. First, assume that there exist fibrant $\mathcal{M}$-enriched categories $\mathcal{Z}$ and $\mathcal{Z}'$ such that $Z = \mathfrak{N}\mathcal{Z}$ and $Z' = \mathfrak{N}\mathcal{Z}'$. Then counit map factors as

$$\mathfrak{C}(Z \times Z') \xrightarrow{f} \mathfrak{C}(Z) \times \mathfrak{C}(Z') \xrightarrow{g} \mathcal{Z} \times \mathcal{Z}'.$$

However, by Theorem 2.7.9, the maps g and gf are equivalences, since $\mathcal{Z}$ and $\mathcal{Z}'$ are fibrant, so by 3-for-2, the map f must be an equivalence as well.

To prove the general case, we use Theorem 3.1.8 to find weak equivalences $Z \xrightarrow{\sim} T$ and $Z' \xrightarrow{\sim} T'$ where T and T' are cellular nerves of fibrant $\mathcal{M}$-enriched categories. Then the map $Z \times Z' \to T \times T'$ is also an equivalence, and since the functor $\mathfrak{C}$ sends weak equivalences in $\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}}$ to Bergner-Lurie equivalences, and we are reduced to the previous case. ■

3.2.5. Lemma. Let $\mathcal{D}$ be a small fibrant $\mathcal{M}$-enriched category. Then the enriched Yoneda embedding

$$\mathcal{D} \hookrightarrow \mathcal{M}^{\mathcal{D}^{\mathrm{op}}}$$

factors through the full subcategory

$$(\mathcal{M}^{\mathcal{D}^{\mathrm{op}}}_{\mathrm{proj}})^{\circ}$$

spanned by the projectively fibrant and cofibrant $\mathcal{M}$-enriched functors $\mathcal{D}^{\mathrm{op}} \to \mathcal{M}$.

Proof. We must show that for every $d \in \mathcal{D}$, the representable functor h_d is projectively fibrant and cofibrant. To see that any such representable functor h_d is projectively fibrant, it suffices to demonstrate that for all $d' \in \mathcal{D}$, the object $h_d(d') = \mathcal{D}(d', d) \in \mathcal{M}$ is fibrant. However, this holds precisely because $\mathcal{D}$ is fibrant, since all of its hom objects are fibrant objects of $\mathcal{M}$.

To see that such a representable functor h_d is projectively cofibrant, we must show that the map $\varnothing \to h_d$ is a retract of a relative cell complex built from the generating projective cofibrations in $\mathcal{M}^{\mathcal{D}^{\mathrm{op}}}_{\mathrm{proj}}$. However, by [Lur09, Remark A.3.3.5], the generating cofibrations are precisely those of the form

$$h_d \otimes f \colon h_d \otimes A \to h_d \otimes A'$$

where $f \colon A \to A'$ is a cofibration in $\mathcal{M}$ and $d \in \mathcal{D}$. In particular, the map $\varnothing \to h_d$ is none other than the tensor of h_d with the map $\varnothing \hookrightarrow *$ in $\mathcal{M}$. ∎

3.2.6. Proposition. [Lur09, 4.2.4.7] Let $\mathcal{I}$ be a fibrant $\mathcal{M}$-enriched category, X an object of $\widehat{\Theta[\mathcal{C}]}$, and $p \colon \mathfrak{N}\mathcal{I} \to X$ be any map. Then we can find the following:

- A fibrant $\mathcal{M}$-enriched category $\mathcal{D}$.
- An enriched functor $P \colon \mathcal{I} \to \mathcal{D}$.
- A map $j \colon X \to \mathfrak{N}(\mathcal{D})$ that is a weak equivalence in $\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}}$.
- An equivalence between $j \circ p$ and $\mathfrak{N}(P)$ as objects of the $\mathcal{M}$-quasicategory $\mathfrak{N}(\mathcal{D})^{\mathfrak{N}(\mathcal{I})}$.

Proof. First, choose a fibrant replacement $i_0 \colon \mathfrak{C}(X) \xrightarrow{\sim} \mathcal{D}_0$ in the Bergner-Lurie model structure for $\mathcal{M}$-enriched categories. We see that for any pair of vertices x, x' of X, the map $\mathfrak{C}(X)(x, x') \to \mathcal{D}_0(i_0 x, i_0 x')$ is a weak equivalence in $\mathcal{M}$.

Let $\mathbf{A} = \mathcal{M}^{\mathcal{D}_0^{\mathrm{op}}}_{\mathrm{proj}}$ be the $\mathcal{M}$-enriched category of $\mathcal{M}$-enriched presheaves on $\mathcal{D}_0$ equipped with the projective model structure. Then the enriched Yoneda embedding $\mathcal{D}_0 \to \mathbf{A} = \mathcal{M}^{\mathcal{D}_0^{\mathrm{op}}}$ factors through $\mathbf{A}^{\circ}$ by Lemma 3.2.5.

Then let $Y_0 \colon \mathfrak{C}(X) \to \mathbf{A}^{\circ}$ be the composite of i_0 with the Yoneda embedding. Then we see as well that for any pair of vertices x, x' of X, the map $\mathfrak{C}(X)(x, x') \to \mathbf{A}^{\circ}(Y_0 x, Y_0 x')$ is a weak equivalence in $\mathcal{M}$. That is to say, the $\mathcal{M}$-enriched functor $Y_0 \colon \mathfrak{C}(X) \to \mathbf{A}^{\circ}$ is weakly fully faithful. Let $j_0 \colon X \to \mathfrak{N}(\mathbf{A}^{\circ})$ be the map corresponding to Y_0 under adjunction.

Since $\mathcal{I}$ is a fibrant $\mathcal{M}$-enriched category, the counit $\varepsilon\colon \mathfrak{C}\mathfrak{N}\mathcal{I} \to \mathcal{I}$ is a weak equivalence, which us allows us to apply Proposition 3.2.4 and show that the induced map

$$\mathfrak{N}((\mathbf{A}^{\mathcal{I}})^\circ) \to \mathfrak{N}(\mathbf{A}^\circ)^{\mathfrak{N}\mathcal{I}}$$

is a Bergner-Lurie equivalence. It follows that the composite

$$j_0 \circ p\colon \mathfrak{N}\mathcal{I} \to \mathfrak{N}(\mathbf{A}^\circ)$$

is equivalent to $\mathfrak{N}(P_0)$ for some $\mathcal{M}$-enriched functor $P_0\colon \mathcal{I} \to \mathbf{A}^\circ$. Then we can take $\mathcal{D}$ to be the essential image of $\mathfrak{C}(X)$ in $\mathbf{A}^\circ$ and it follows that the maps j_0 and P_0 factor uniquely through maps $j\colon X \to \mathfrak{N}(\mathcal{D})$ and $P\colon \mathcal{I} \to \mathcal{D}$ having the desired properties. ■

Now we proceed to construct the Yoneda embedding (and friends):

3.2.7. Definition. Let X be a small $\mathcal{C}$-cellular set, and let $\Phi\colon \mathfrak{C}(X) \xrightarrow{\sim} \mathcal{D}$ be an $\mathcal{M}$-enriched fibrant replacement of its coherent realization. Since $\mathcal{D}$ is fibrant, the functor

$$\mathrm{Hom}_{\mathcal{D}}\colon \mathcal{D} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{M}$$

factors through the full subcategory $\mathcal{M}^\circ$.

Then we have a map

$$\mathfrak{C}(X \times X^{\mathrm{op}}) \xrightarrow{\mathfrak{C}(p_1)\times\mathfrak{C}(p_2)} \mathfrak{C}(X) \times \mathfrak{C}(X)^{\mathrm{op}} \xrightarrow{\Phi\times\Phi^{\mathrm{op}}} \mathcal{D} \times \mathcal{D}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathcal{D}}} \mathcal{M}^\circ$$

which yields by adjunction the *Yoneda embedding*

$$j\colon X \to \mathfrak{N}(\mathcal{M}^\circ)^{X^{\mathrm{op}}}.$$

We will also need the following construction: By Lemma 3.2.5, the $\mathcal{M}$-enriched Yoneda embedding

$$\mathcal{D} \hookrightarrow \mathcal{M}^{\mathcal{D}^{\mathrm{op}}}$$

factors through the inclusion of the full subcategory $(\mathcal{M}^{\mathcal{D}^{\mathrm{op}}}_{\mathrm{proj}})^\circ$ of projectively fibrant and cofibrant $\mathcal{M}$-enriched presheaves on $\mathcal{D}$. Then we define the composite $\mathcal{M}$-enriched functor

$$J\colon \mathfrak{C}(X) \xrightarrow{\phi} \mathcal{D} \hookrightarrow (\mathcal{M}^{\mathcal{D}^{\mathrm{op}}}_{\mathrm{proj}})^\circ.$$

3.2.8. Note. We introduce some notation: First, we will denote the nerve $\mathfrak{N}(\mathcal{M}^\circ)$ of the large fibrant $\mathcal{M}$-enriched category $\mathcal{M}^\circ$ by $\mathcal{S}_{\mathcal{M}}$, and we call it the $\mathcal{M}$-quasicategory of $\mathcal{M}$-spaces. Moreover, for any small $\mathcal{C}$-cellular set X, we let $\mathcal{P}(X) \overset{\mathrm{def}}{=} \mathcal{S}_{\mathcal{M}}^{X^{\mathrm{op}}}$ and call it the $\mathcal{M}$-quasicategory of presheaves (of $\mathcal{M}$-spaces) on X.

In order to deal with some size issues in the next theorem, we also introduce the huge Cisinski model category $\mathcal{M}^+ \overset{\mathrm{def}}{=} \mathrm{Psh}_\Delta(\mathcal{C})^+_{\mathscr{S}}$, which is the extension of $\mathcal{M}$ in an obvious way to a bigger universe. That is to say, its localizer is generated by the same set of maps, but on the category of not-necessarily small simplicial presheaves on $\mathcal{C}$, denoted by $\mathrm{Psh}_\Delta(\mathcal{C})^+$. We denote the huge nerve $\mathfrak{N}((\mathcal{M}^+)^\circ)$ by $\mathcal{S}^+_{\mathcal{M}}$.

3.2.9. PROPOSITION. [Yoneda embedding][Lur09, 5.1.3.1] Let X be a small $\mathcal{C}$-cellular set, and let $\Phi\colon \mathfrak{C}(X) \to \mathcal{D}$ be an $\mathcal{M}$-enriched fibrant replacement of its coherent realization. Then the associated Yoneda embedding $j\colon X \to \mathcal{P}(X)$ is $\mathcal{M}$-locally fully faithful.

PROOF. First, observe that we have the following commutative diagram

$$\begin{array}{ccccccc}
X & & \longrightarrow & & & & (X \times X^{\mathrm{op}})^{X^{\mathrm{op}}} \\
\downarrow & & & & & & \downarrow \\
\mathfrak{N}(\mathcal{D}) & \longrightarrow & \mathfrak{N}((\mathcal{D} \times \mathcal{D}^{\mathrm{op}})^{\mathcal{D}^{\mathrm{op}}}) & \longrightarrow & \mathfrak{N}(\mathcal{D} \times \mathcal{D}^{\mathrm{op}})^{\mathfrak{N}(\mathcal{D}^{\mathrm{op}})} & \longrightarrow & \mathfrak{N}(\mathcal{D} \times \mathcal{D}^{\mathrm{op}})^{X^{\mathrm{op}}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{N}((\mathcal{M}^{\mathcal{D}^{\mathrm{op}}})^{\circ}) & \longrightarrow & \mathfrak{N}((\mathcal{M}^{\circ})^{\mathcal{D}^{\mathrm{op}}}) & \longrightarrow & \mathfrak{N}(\mathcal{M}^{\circ})^{\mathfrak{N}(\mathcal{D}^{\mathrm{op}})} & \longrightarrow & \mathfrak{N}(\mathcal{M}^{\circ})^{X^{\mathrm{op}}}
\end{array},$$

where the lower left square commutes by universality, as it is the unit of an adjunction, the middle square commutes by naturality of the morphism distributing $\mathfrak{N}$ over exponentials, and the lower right square commutes by naturality. The top rectangle also commutes by universality. The composite along the top and righthand side is precisely the Yoneda map j, while the composite of the lefthand vertical maps is the adjunct j' of the map J constructed above. The composite of the bottom horizontal maps is exactly the map appearing in Proposition 3.2.4, and since the map $\Phi^{\mathrm{op}}\colon \mathfrak{C}(X)^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is an equivalence, the hypotheses of Proposition 3.2.4 are satisfied and therefore the composite of the bottom horizontal arrows is also a weak equivalence. By Corollary 3.1.10, it suffices to show that the map $\mathfrak{C}(j')$ is a weakly fully faithful $\mathcal{M}$-enriched functor. However, observe that the composite

$$\mathfrak{C}(X) \xrightarrow{\mathfrak{C}(j')} \mathfrak{C}\mathfrak{N}(\mathcal{M}_{\mathrm{proj}}^{\mathcal{D}^{\mathrm{op}}})^{\circ} \xrightarrow{\varepsilon} (\mathcal{M}_{\mathrm{proj}}^{\mathcal{D}^{\mathrm{op}}})^{\circ}$$

is exactly the map J by adjointness, and since $(\mathcal{M}_{\mathrm{proj}}^{\mathcal{D}^{\mathrm{op}}})^{\circ}$ is a fibrant $\mathcal{M}$-enriched category, the counit ε is an equivalence. Therefore, it suffices to show that J is weakly fully faithful, but this is clear, because J factors as a composite of the Bergner-Lurie equivalence $\Phi\colon \mathfrak{C}(X) \to \mathcal{D}$ followed by the strictly fully faithful Yoneda embedding $\mathcal{D} \hookrightarrow (\mathcal{M}_{\mathrm{proj}}^{\mathcal{D}^{\mathrm{op}}})^{\circ}$, which proves the claim. ∎

3.2.10. DEFINITION. We say that a map $F\colon X^{\mathrm{op}} \to \mathcal{S}_{\mathcal{M}}$ is a *representable functor* if the corresponding presheaf $F \in \mathcal{S}_{\mathcal{M}}^{X^{\mathrm{op}}} = \mathcal{P}(X)$ belongs to the essential image of the Yoneda embedding $j\colon X \to \mathcal{P}(X)$. If $x\colon * \to X$ is a vertex of X, then the composite $j \circ x$ classifies a representable functor called h_x that corresponds intuitively to the functor that we might incoherently try to define as $\mathrm{hMap}_X(-, x)$.

3.2.11. PROPOSITION. [Yoneda's Lemma][Lur09, 5.5.2.1] Let X be a small $\mathcal{C}$-cellular set, and let $f\colon X^{\mathrm{op}} \to \mathcal{S}_{\mathcal{M}}$ be an object of $\mathcal{P}(X)$. Then let $F\colon \mathcal{P}(X)^{\mathrm{op}} \to \mathcal{S}_{\mathcal{M}}^{+}$ be the functor represented by f. Then the composite

$$X^{\mathrm{op}} \xrightarrow{j^{\mathrm{op}}} \mathcal{P}(X)^{\mathrm{op}} \xrightarrow{F} \mathcal{S}_{\mathcal{M}}^{+}$$

factors through the inclusion $\mathcal{S}_{\mathcal{M}} \subset \mathcal{S}^{+}_{\mathcal{M}}$ and is equivalent to f.

Proof. By Proposition 3.2.6, we can choose a small fibrant $\mathcal{M}$-enriched category $\mathcal{D}$ and an equivalence $\Phi\colon X^{\mathrm{op}} \to \mathfrak{N}(\mathcal{D})$ such that $f \sim \mathfrak{N}(f') \circ \Phi$ for some $\mathcal{M}$-enriched functor $f'\colon \mathcal{D} \to \mathcal{M}^{\circ}$. Without loss of generality, we can assume that f' is a projectively cofibrant diagram. Observe that since $\mathcal{D}$ is fibrant, the adjunct map $\mathfrak{C}(X) \to \mathcal{D}$ is also an equivalence, so by the construction in Proposition 3.2.9, we have a commutative diagram

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle j'} & \searrow^{j} & \\ \mathfrak{N}((\mathcal{M}^{\mathcal{D}}_{\mathrm{proj}})^{\circ}) & \xrightarrow[\Psi]{\sim} & \mathcal{P}(X) \end{array},$$

in which the map j is a Yoneda embedding, the map

$$\Psi\colon \mathfrak{N}((\mathcal{M}^{\mathcal{D}}_{\mathrm{proj}})^{\circ}) \xrightarrow{\sim} \mathcal{P}(X).$$

is an equivalence by Proposition 3.2.4, and the map j' is adjunct to the map J. Since this diagram commutes on the nose, we will compute $F \circ j^{\mathrm{op}} = F \circ \Psi^{\mathrm{op}} \circ j'^{\mathrm{op}}$.

We observe that $F \circ \Psi^{\mathrm{op}}$ can be identified with the coherent nerve of the the map

$$G\colon \left((\mathcal{M}^{\mathcal{D}}_{\mathrm{proj}})^{\circ}\right)^{\mathrm{op}} \to \left(\mathcal{M}^{+}\right)^{\circ}$$

represented by f'. We see that the map j'^{op} is exactly the composite

$$X^{\mathrm{op}} \xrightarrow{\Phi} \mathfrak{N}(\mathcal{D}) \to \mathfrak{N}((\mathcal{M}^{\mathcal{D}}_{\mathrm{proj}})^{\circ})^{\mathrm{op}},$$

where the second map is the nerve of (opposite) of the enriched Yoneda embedding of $\mathcal{D}^{\mathrm{op}}$. Then the composite $G \circ j'^{\mathrm{op}}$ is exactly $\mathfrak{N}(f') \circ \Phi$ by the ordinary enriched Yoneda lemma, and this is equivalent to f, as desired. ∎

3.2.12. Remark. *Working pointwise, this statement says that we have a natural equivalence of $\mathcal{M}$-spaces $f(x) \simeq F(h_x)$, but we have an equivalence $F = h_f \simeq \mathcal{P}(X)(-, f)$, so with a bit more unwinding, we see that the statement tells us that $f(x) \simeq \mathcal{P}(X)(h_x, f)$, which is the traditional statement of Yoneda's lemma, as desired.*

3.3. Examples. The only examples we really care about are the cases where $\mathcal{C} = \Theta_n$ for $0 \leq n \leq \omega$ and where $\mathscr{S}$ is the set of generating anodynes for the model structure on weak n-categories. We invite the reader to consider other applications. We expect that a simple application would be to consider the left-Bousfield localization of spaces at homology equivalences, but we aren't certain if this is a Cartesian model structure.

A. Appendix: Recollections on Cisinski Theory

We recall in the appendix a number of important technical facts from Cisinski theory, which comprises a large body of results on the construction of extremely tame model structures on presheaf categories. This is the big machine backing up Definition 1.3.1 as well as our reduction of §1.7 to checking properties of generating anodynes.

A.1. Cisinski model structures and localizers. In what follows, we will work with a fixed small category $\mathcal{A}$. These are stated in more generality in [Cis06, Chapter 1], but we will specialize to the case of a Cartesian cylinder functor, that is, a cylinder functor determined by taking the Cartesian product with an interval object.

A.1.1. Definition. A separating interval object of $\widehat{\mathcal{A}}$ is an object I together with two monic arrows $\partial_0, \partial_1 \colon * \to I$ such that the pullback $* \times_I * = \varnothing$. We call the induced map

$$\delta^I \colon \; * \coprod * \xrightarrow{(\partial_0, \partial_1)} I$$

the boundary map. We say that an interval is injective if the object I is an injective object of the category $\widehat{\mathcal{A}}$. That is to say, it has the right lifting property with respect to all monomorphisms in $\widehat{\mathcal{A}}$.

A.1.2. Definition. A cellular model $\mathscr{M}$ for $\widehat{\mathcal{A}}$ is a small set of monomorphisms such that llp(rlp($\mathscr{M}$)) is exactly the class of monomorphisms of $\widehat{\mathcal{A}}$.

A.1.3. Proposition. [Cis06, Proposition 1.2.27] Every category of presheaves on a small category $\mathcal{A}$ admits a cellular model in which the target of each map is a quotient of a representable.

A.1.4. Definition. A class of anodynes **An** for a separating interval I is a class of monomorphisms that satisfies the following properties:

- **An** is generated by a set S, that is, there exists a set of monomorphisms S such that $\mathbf{An} = \mathrm{llp}(\mathrm{rlp}(S))$.
- For any monomorphism g, the corner maps $\partial_i \times^\lrcorner g \in \mathbf{An}$ for $i \in \{0, 1\}$.
- For any map $f \in \mathbf{An}$, the map $\delta^I \times^\lrcorner f \in \mathbf{An}$.

A.1.5. Proposition. [Cis06, Proposition 1.3.13] Given any set S of monomorphisms and any separating interval object I, there exists a smallest class of anodynes $\mathbf{An}_I(S)$ for I. In particular, this class is generated by the set of maps $\Lambda_I(S, \mathscr{M})$ where $\mathscr{M}$ is a cellular model for $\mathcal{A}$ defined as follows:

- We define the set $\Lambda^0_I(S, \mathscr{M}) = S \cup \partial_0 \times^\lrcorner \mathscr{M} \cup \partial_1 \times^\lrcorner \mathscr{M}$
- Then we define for any set of maps T the set $\Lambda_I(T) = \delta^I \times^\lrcorner T$.
- Then we define $\Lambda_I(S, \mathscr{M}) = \bigcup_i^\infty \Lambda^i_I(\Lambda^0_I(S, \mathscr{M}))$.

A.1.6. Theorem. [Cis06, 1.3.22] Given a (small) set of monomorphisms $S \subset \widehat{A}$ and some separating interval I, there exists a model structure on $\widehat{A}$ in which:

- The cofibrations are exactly the monomorphisms.
- The fibrant objects are the objects $a \in \widehat{A}$ such that the terminal map $a \to *$ belongs to the class of maps
$$\mathrm{rlp}(\mathbf{An}_I(S)).$$
- A map $f\colon a \to a'$ with a' fibrant is a fibration if and only if f belongs to
$$\mathrm{rlp}(\mathbf{An}_I(S)).$$

A.1.7. Definition. A *Cisinski model structure* is any model structure constructed using Theorem A.1.6.

A.1.8. Corollary. Taking $S = \varnothing$ and I to be the subobject classifier $\mathfrak{L}$ of $\widehat{A}$ with the two canonical sections $\varnothing, \mathrm{id}\colon * \to \mathfrak{L}$, we obtain the minimal Cisinski model structure. More generally, we can replace $\mathfrak{L}$ with any injective separating interval.

A.1.9. Definition. An $\mathcal{A}$-localizer W is a class of maps of $\widehat{A}$ satisfying the following axioms

- The class W satisfies 3-for-2.
- Every trivial fibration belongs to W.
- The class of monomorphisms in W is closed under pushout and transfinite composition.

If S is a set of morphisms of $\widehat{\mathcal{A}}$, there exists a minimal localizer containing S, which we call the localizer generated by S and denote by $\mathsf{W}(S)$. We say that a localizer W is *accessible* if it generated by a set of morphisms.

A.1.10. Theorem. [Cis06, Theorem 1.4.3] Given any set of morphisms S of $\widehat{\mathcal{A}}$, the localizer $\mathsf{W}(S)$ is the class of weak equivalences for a Cisinski model structure. Moreover, this model structure is the left-Bousfield localization of the minimal Cisinski model structure at the set S.

A.2. Simplicial Completion. The localizers on $\mathcal{A}$ have a non-free component, namely that the class of trivial fibrations must always belong to W. The theory of simplicial completions allows us to embed the class of $\mathcal{A}$-localizers into a larger class of localizers that doesn't suffer from this defect. These are models for free homotopy theories modeled on $\mathcal{A}$. The idea here is to replace the interval object with an external interval object.

A.2.1. Definition. We define the free homotopy theory on $\mathcal{A}$ generated by a set S of maps in $\widehat{\mathcal{A}}$ to be the minimal $\mathcal{A} \times \Delta$-localizer containing the set of maps $S \boxtimes \Delta^0$ and the set of maps $\Lambda_{\Delta^1}(\varnothing, \mathscr{M})$, where $\mathscr{M}$ is any cellular model for $\mathcal{A} \times \Delta$ with respect to the interval object $\Delta^1 \overset{\text{def}}{=} * \boxtimes \Delta^1$, where $\boxtimes$ denotes the external product.

A.2.2. Remark. [Cis06, 3.4.50] *By well-known combinatorial arguments, it can be seen that taking Δ^1 to be the separating interval object forces all objects $\Delta^n = * \boxtimes \Delta^n$ to be weakly contractible. The free homotopy theory construction therefore adds new representables but homotopically nullifies all of them. We can therefore view it as a way to present a homotopy theory for presheaves on $\mathcal{A}$ without automatically forcing all of the trivial fibrations to be weak equivalences.*

The free homotopy theory on $\mathcal{A}$ is in general radically different from the homotopy theory given by the injective model structure on simplicial presheaves, which we will see later is its regular completion. Cisinski gives the example where $\mathcal{A} = B\mathbb{G}$ for a group $\mathbb{G}$. The difference between the free homotopy theory and its regular completion in this case is the difference between equivariant homotopy theory and higher Galois theory. That is, the free homotopy theory presents ordinary equivariant homotopy theory, while the regular completion of the free homotopy theory on $B\mathbb{G}$ models non-abelian $\mathbb{G}$-representations.

A.2.3. Definition. Given an $\mathcal{A}$-localizer W, we define the simplicial completion of W to be the $\mathcal{A} \times \Delta$-localizer W_Δ generated by the class of maps of simplicial objects $X \to X'$ such that $X_n \to X'_n$ belongs to W for each $i \geq 0$ together with the projection maps $X \times \Delta^1 \to X$ for all simplicial presheaves X on A. We say that an $\mathcal{A} \times \Delta$-localizer is *discrete* if it is the simplicial completion of a localizer on $\mathcal{A}$.

A.2.4. Proposition. If the localizer W is accessible, so is W_Δ.

A.2.5. Proposition. If the localizer W is the minimal $\mathcal{A}$-localizer, then W_Δ is the smallest localizer containing the localizer of the free homotopy theory on $\mathcal{A}$ together with set of maps $\Lambda_I(\varnothing, \mathscr{M})$ for any choice of injective separating interval object I and any cellular model $\mathscr{M}$.

A.2.6. Proposition. [Cis06, Proposition 2.3.27] If W is any accessible localizer, the functor $p^* \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A} \times \Delta}$ induced by the projection $\mathcal{A} \times \Delta \to \mathcal{A}$ is a left Quillen equivalence. Also, by choosing a Reedy-cofibrant cosimplicial resolution $D^\bullet$ of the terminal object $*$ of $\widehat{\mathcal{A}}$ with respect to the minimal localizer $\mathsf{W}_{\min}$, the functor

$$\mathrm{Real}_D \colon \widehat{\mathcal{A} \times \Delta} \to \widehat{\mathcal{A}}$$

induced by left Kan extension of the functor defined by the rule

$$(A, [n]) \mapsto A \times D^n$$

is also a left Quillen equivalence.

A.2.7. Corollary. The simplicial completion defines a bijective Galois connection between $\mathcal{A}$-localizers and discrete $\mathcal{A} \times \Delta$-localizers.

A.3. Regularity. An important property of simplicial sets is no longer present in the case of a general $\mathcal{A}$-localizer, namely the property that every object is the canonical homotopy colimit of its diagram of representables. This leads to the following definition:

A.3.1. Definition. A presheaf X on $\mathcal{A}$ is called W-*regular* with respect to a localizer W if the canonical map

$$\operatorname*{hocolim}_{A\to X\in(\mathcal{A}\downarrow X)} A \to \operatorname*{colim}_{A\to X\in(\mathcal{A}\downarrow X)} A \cong X$$

is a W-equivalence. A localizer W on $\mathcal{A}$ is called a regular localizer if every presheaf X on $\mathcal{A}$ is W-regular.

A.3.2. Proposition. [Cis06, Remark 3.4.14] If $\mathsf{W} \subseteq \mathsf{W}'$ is an inclusion of localizers and W is regular, then so too is W'.

A.3.3. Definition. The regular completion of a localizer W is the smallest regular localizer containing W. In particular, it follows from the preceding proposition that the regular completion is the smallest localizer generated by W and the regular completion of the minimal localizer.

A.3.4. Proposition. [Cis06, Corollary 3.4.24] The regular completion of an accessible localizer is accessible.

A.3.5. Proposition. [Cis06, Proposition 3.4.34] The localizer of the injective model structure on simplicial presheaves on $\mathcal{A}$ consisting of the maps of simplicial presheaves $X \to X'$ whose components are weak homotopy equivalences $X_A \to X'_A$ is the regular completion of the localizer of the free homotopy theory on $\mathcal{A}$.

A.3.6. Corollary. The Cisinski model structure obtained from the simplicial completion W_Δ of an accessible localizer W on $\mathcal{A}$ is a left-Bousfield localization of the injective model structure on simplicial presheaves if and only if it is regular. In particular, the Galois correspondence between localizers on $\mathcal{A}$ and localizers containing the simplicial completion of the minimal localizer restricts to a bijective Galois correspondence between regular localizers on $\mathcal{A}$ and discrete localizers on $\mathcal{A} \times \Delta$ containing the objectwise weak homotopy equivalences.

We also make note of the following technical fact:

A.3.7. Proposition. [Cis06, Corollary 3.4.41] Let $\mathcal{A}$ be a small category, and let W be a regular $\mathcal{A}$-localizer. Then W is closed under filtered colimits.

A.4. Skeletal categories. In this section, we recall Cisinski's theory of skeletal categories (catégories squelettiques). These are generalized Reedy categories $\mathcal{A}$ with a dimension grading and satisfying certain axioms. Under the strong condition of normality, the category $\widehat{\mathcal{A}}$ admits a canonical cellular model given by the boundary inclusions. Under a further strong assumption of regularity, every $\mathcal{A}$-localizer will be shown to be regular.

A.4.1. Definition. A skeletal category is given by the data of a small category $\mathcal{A}$, subcategories $\mathcal{A}^-$ and $\mathcal{A}^+$ together with a grading function $\dim\colon \operatorname{Ob}\mathcal{A} \to \mathbf{N}$ satisfying the following axioms:

- Every isomorphism belongs to both $\mathcal{A}^-$ and $\mathcal{A}^+$.
- If $f\colon A \to A'$ belongs to $\mathcal{A}^+$ (resp. $\mathcal{A}^-$), then $\dim(A) \leq \dim(A')$ (resp. $\dim(A') \leq \dim(A)$) with the inequality being strict if the map f is not an isomorphism.
- Every map f of $\mathcal{A}$ admits a factorization, unique up to unique isomorphism of factorizations, into a composite $\delta \circ \pi$ with $\delta \in \mathcal{A}^+$ and $\pi \in \mathcal{A}^-$.
- Two arrows $f, g\colon A \to A'$ of $\mathcal{A}^-$ are equal if and only if they have the same sections.

A.4.2. Definition. Given a natural number n and a presheaf X on $\mathcal{A}$, we define the n-skeleton to be the sieve

$$\operatorname{Sk}^n(X)_A \stackrel{\text{def}}{=} \{u\colon A \to X \mid \exists \alpha\colon A \to A', \quad \dim(A') \leq n, \quad \exists u'\colon A' \to X, \quad u = u' \circ \alpha\}.$$

If A is a representable object of $\mathcal{A}$, we define the boundary ∂A of A to be $\operatorname{Sk}^{\dim(A)-1}(A)$, and we denote its inclusion by $\delta^A\colon \partial A \hookrightarrow A$.

We take the following as a definition, but it is in fact a characterization from [Cis06, 8.1.37]

A.4.3. Definition. A skeletal category is called normal if its objects have no nontrivial automorphisms.

A.4.4. Proposition. [Cis06, Proposition 8.1.37] If $\mathcal{A}$ is a normal skeletal category, then the set of maps $\mathscr{M} \stackrel{\text{def}}{=} \{\delta^A\}_{A \in \mathcal{A}}$ gives a cellular model for $\widehat{\mathcal{A}}$. Moreover, the class of monomorphisms of $\mathcal{A}$ is exactly $\operatorname{Cell}(\mathscr{M})$.

A.4.5. Remark. *Cisinski shows that whenever X is a normal presheaf (we omit this definition, but in the case where $\mathcal{A}$ is normal skeletal, every presheaf satisfies this property), its n-skeleton can be computed as the image of X under the composite adjunction induced by the inclusion of the full subcategory $\mathcal{A}_{\leq n} \hookrightarrow \mathcal{A}$, similar to the case of Δ. In particular, normal skeletal categories have a well-behaved skeleton-coskeleton adjunction.*

A.4.6. Definition. Let $\mathcal{A}$ be a small category. Then we say a presheaf X on $\mathcal{A}$ is connected if its category of elements $(\mathcal{A} \downarrow X)$ is a connected category. Equivalently, viewing X as a functor $X\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$, we can take the colimit of X. This colimit is called the set of connected components of X and is denoted by $\tau_! X$ (that is to say, left Kan extension along the terminal functor $\mathcal{A} \to *$), and we can say that X is connected if $\tau_! X$ is the terminal set.

Since the colimit of a presheaf X is computed as the set of connected components of its category of elements $(\mathcal{A} \downarrow X)$, we see immediately that any representable presheaf is connected. Moreover, since the functor $\tau_!$ is a left adjoint, we see that any connected colimit of connected presheaves is connected.

A.4.7. Definition. Let $\mathcal{A}$ be a skeletal category, and let $A \in \mathcal{A}$ be an object. Then a degeneracy of A is a map $A \to A'$ such that $\dim(A') < \dim(A)$. We say that A is degenerate if it admits a degeneracy and nondegenerate if it is not degenerate.

Let X be a presheaf on $\mathcal{A}$. Then we say that an element $u \in X(A)$ is degenerate (resp. nondegenerate) if the corresponding object $(A, u) \in (\mathcal{A} \downarrow X)$ of the category of elements is degenerate (resp. nondegenerate) with respect to the induced skeletal structure on $(\mathcal{A} \downarrow X)$. We will call the corresponding map $h_A \to X$ a nondegenerate section of X.

A.4.8. Definition. We say that presheaf X on a skeletal category is regular if every nondegenerate element of X is monic. Additionally, we say that a skeletal category is regular if it is normal and every representable presheaf is regular. Equivalently, a normal skeletal category $\mathcal{A}$ is regular if every map belonging to $\mathcal{A}^+$ is monic.

A.4.9. Theorem. [Cis06, Proposition 8.2.9] Every localizer W on a regular skeletal category is regular.

A.4.10. Corollary. Let $\mathcal{A}$ be a regular skeletal category. Then the free homotopy theory generated by $\mathcal{A}$ is the injective model structure on $\widehat{\mathcal{A} \times \Delta}$ with respect to the Kan-Quillen model structure on $\widehat{\Delta}$.

Proof. Since $\mathcal{A}$ and Δ are both regular skeletal, their product with the product skeletal structure is also regular skeletal. It follows that the free homotopy theory generated by $\mathcal{A}$ on $\mathcal{A} \times \Delta$ is regular, and therefore it is its own regular completion. But the regular completion of the free homotopy theory is precisely the injective model structure. ∎

A.4.11. Proposition. Let $\mathcal{A}$ be a regular skeletal category. Suppose we are given a functor $F\colon \mathcal{A} \times \Delta \to \mathcal{M}$ where $\mathcal{M}$ is a model category with the property that weak equivalences in $\mathcal{M}$ are closed under colimits indexed by ordinals along transfinite sequences of cofibrations. Then the cocontinuous extension $F_!\colon \widehat{\mathcal{A} \times \Delta} \to \mathcal{M}$ is a left Quillen functor with respect to the free homotopy theory if and only if the following conditions hold:

- For all pairs (A, n), with $A \in \mathcal{A}$ and $n \geq 0$, the map

$$F_!(\partial A \boxtimes \Delta^n) \coprod_{F_!(\partial A \boxtimes \partial \Delta^n)} F_!(A \boxtimes \partial \Delta^n) \to F_!(A \boxtimes \Delta^n)$$

  is a cofibration.

- For all triples (A, n, k) where $n \geq 1$ and $0 \leq k \leq n$, the map

$$F_!(A \boxtimes \Lambda^n_k) \to F_!(A \boxtimes \Delta^n)$$

  is a weak equivalence.

Proof. The first condition implies that all monomorphisms of $\widehat{\mathcal{A} \times \Delta}$ are sent to cofibrations, since these maps are precisely the image under $F_!$ of the boundary inclusions $\partial(A \boxtimes \Delta^n) \hookrightarrow A \boxtimes \Delta^n$, which generate all monomorphisms under formation of relative cell complexes.

In order to show that the functor is left Quillen then, it suffices to show that for all triples (A, n, k) where $n \geq 1$ and $0 \leq k \leq n$, the map

$$F_!(\partial A \boxtimes \Delta^n) \coprod_{F_!(\partial A \boxtimes \Lambda^n_k)} F_!(A \boxtimes \Lambda^n_k) \to F_!(A \boxtimes \Delta^n)$$

is a weak equivalence.

However, this condition would imply that for any monomorphism $X \to Y$ in $\widehat{\mathcal{A}}$, the map

$$F_!(\pi^*(X) \boxtimes \Delta^n) \coprod_{F_!(\pi^*(X) \boxtimes \partial\Delta^n)} F_!(\pi^*(Y) \boxtimes \Lambda^n_k) \to F_!(\pi^*(Y) \boxtimes \Delta^n)$$

is also an equivalence, where $\pi^* \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A} \times \Delta}$ is the functor precomposing a presheaf with the projection, since every monomorphism in $\widehat{\mathcal{A}}$ is generated under formation of relative cell complexes by the boundary inclusions. However, this now implies that if the condition holds for the maps $\varnothing \hookrightarrow \partial A$ and $\varnothing \hookrightarrow A$, we have a pushout square

$$\begin{array}{ccc}
F_!(\partial A \boxtimes \Lambda^n_k) & \overset{\sim}{\hookrightarrow} & F_!(\partial A \boxtimes \Delta^n) \\
\downarrow & & \downarrow \\
F_!(A \boxtimes \Lambda^n_k) & \overset{\sim}{\hookrightarrow} & P \\
& \searrow^{\sim} & \vdots \\
& & F_!(A \boxtimes \Delta^n)
\end{array},$$

where the top horizontal arrow is a weak equivalence by applying the statement to the monomorphism $\varnothing \hookrightarrow \partial A$, the bottom horizontal arrow is a pushout of that map, and the bottom slanted map is an equivalence by applying the statement to the monomorphism $\varnothing \hookrightarrow A$. It follows that the map $P \to F_!(A \boxtimes \Delta^n)$ is an equivalence.

The next argument will proceed for each pair $n \geq 1$ and $0 \leq k \leq n$, so fix some such pair. We set $\Phi(-) \overset{\text{def}}{=} F_!(- \boxtimes \Lambda^n_k)$ and $\Psi(-) \overset{\text{def}}{=} F_!(- \boxtimes \Delta^n)$. Then we have a natural transformation

$$\alpha \colon \Phi \to \Psi.$$

We will assume that for all $A \in \mathcal{A}$, the map $\alpha_A \colon \Phi(A) \to \Psi(A)$ is a weak equivalence and demonstrate that for all $A \in \mathcal{A}$, the map $\alpha_{\partial A} \colon \Phi(\partial A) \to \Psi(\partial A)$ is also a weak equivalence.

We proceed by induction on the dimension of A. Suppose A is any object of minimal dimension. Then its boundary is empty, and this implies that the map $\alpha_{\partial A}$ is a weak equivalence (in fact an isomorphism), which proves the base case of the induction. Then suppose the claim holds for all B of dimension strictly smaller than A. Then we build ∂A up cell by cell, which clearly works for finitely many steps, since the pushout diagrams are pushouts of cofibrant objects along cofibrations, which are homotopy pushouts. The condition we assumed on $\mathcal{M}$ takes care of the transfinite steps. All cells are attached along boundaries of strictly smaller dimension, so we are done. ■

A.4.12. Corollary. In the situation above, we can replace the second condition with the statement that for all $A \in \mathcal{A}$, and all natural numbers $n \geq 0$, the map

$$\gamma_{A,n} \colon F(A, [n]) \to F(A, [0])$$

is a weak equivalence.

B. Some parametrized category theory

In this appendix, we record some useful results about indexed category theory.

B.1. Limits and colimits in indexed categories.

B.1.1. Proposition. Let $p \colon \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration and let D be a small category. Suppose that for any $b \in \mathcal{B}$, the fibre $\mathcal{E}_b$ admits limits of shape D and for any map $f \colon b' \to b$ in $\mathcal{B}$, the associated transition functor $f^* \colon \mathcal{E}_b \to \mathcal{E}_{b'}$ preserves those limits. Then if $F \colon D \to \mathcal{E}$ is a diagram such that the composite diagram $pF \colon D \to \mathcal{C}$ admits a limit, so too does the diagram F.

Proof. Let $\ell \in \mathcal{B}$ be a limit for pF. Then since $\pi \colon \ell \to pF$ is a cone over pF, we can use cartesianness to produce a diagram in $G \colon D \to \mathcal{E}_\ell$ such that $G(d) \simeq \pi_d^* F(d)$ (that is to say, the map $\widetilde{\pi} \colon G \to F$ in $\mathcal{E}^D$ is a p^D-Cartesian lift of the map $\pi \colon \ell \to pF$ in $\mathcal{B}^D$) (using exponentiability of Grothendieck fibrations). Since $\mathcal{E}_\ell$ admits limits of shape D, the diagram G admits a limit $\widetilde{\ell}$.

We show that $\widetilde{\ell}$ is a limit for F. First, notice that it is a cone over F, as we have a natural transformation $\widetilde{\ell} \to G \to F$. Therefore, we only need to show universality. Let $e \to F$ be some other cone. Then $p(e) \to p(F)$ is a cone over $p(F)$, and we obtain a map $f \colon p(e) \to \ell$ by the universal property of the limit. By Cartesianness of the lift $\widetilde{\pi} \colon G \to F$, we see that the cone $e \to F$ factors uniquely as $e \to G \to F$. Choosing a p^D-Cartesian lift $\widetilde{f} \colon H \to G$, we have a universal factorization $e \to H \to G$, or in other words, the transformation $e \to H$ exhibits e as a cone over f^*G. But by assumption, $f^*\widetilde{\ell} \to f^*G$ exhibits $f^*\widetilde{\ell}$ as the limit of f^*G, so there exists a universal map of cones $e \to f^*\widetilde{\ell}$ over f^*G, as desired. ■

B.1.2. Corollary. Let $p \colon \mathcal{E} \to \mathcal{B}$ be a Grothendieck opfibration and let D be a small category. Suppose that for any $b \in \mathcal{B}$, the fibre $\mathcal{E}_b$ admit colimits of shape D and for any map $f \colon b \to b'$ in $\mathcal{B}$, the associated transition functor $f_! \colon \mathcal{E}_b \to \mathcal{E}_{b'}$ preserves those colimits. Then if $F \colon D \to \mathcal{E}$ is a diagram such that the composite diagram $pF \colon D \to \mathcal{C}$ admits a colimit, so too does the diagram F.

Proof. Immediate by duality. ∎

B.2. Presentable fibrations.

B.2.1. Definition. We say that a category $\mathcal{C}$ is *(locally) presentable* if there exists a small category $\mathcal{D}$ together with an a fully faithful accessible functor $\mathcal{C} \hookrightarrow \widehat{\mathcal{D}}$ exhibiting $\mathcal{C}$ as a reflective subcategory of $\widehat{\mathcal{D}}$. A functor $\mathcal{C} \to \mathcal{C}'$ between presentable categories is called a *presentable morphism* if it admits a left adjoint.

B.2.2. Definition. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration. We say that the fibration p is a *presentable* fibration if for each $b \in \mathcal{B}$, the fibre $\mathcal{E}_b$ over b is presentable and if for every map $f\colon b' \to b$, the induced functor $f^*\colon \mathcal{E}_b \to \mathcal{E}_{b'}$ is a presentable morphism.

B.2.3. Note. Our definition of presentable fibration is dual to the one appearing in [Lur09]. We do this merely for simplicity of notation.

B.2.4. Definition. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration with $\mathcal{B}$ small, and let $a\colon \mathcal{B} \to \mathcal{B}'$ be a functor with $\mathcal{B}'$ locally small. Then the *right Kan extension* $a_*p\colon \mathcal{E}' \to \mathcal{B}'$ of p to $\mathcal{B}'$ along a is the Grothendieck fibration constructed by applying the Grothendieck construction to the functor

$$\mathcal{B}'^{\mathrm{op}} \to \mathbf{Cat}$$

defined by the rule

$$b' \mapsto \mathrm{Fib}_{/\mathcal{B}}\left((a \downarrow b'), \mathcal{E}\right)$$

B.2.5. Proposition. Let $\mathcal{B}$ be a small category and let $p\colon \mathcal{E} \to \mathcal{B}$ be a presentable fibration. Suppose $a\colon \mathcal{B} \to \mathcal{B}'$ is a functor with $\mathcal{B}'$ locally small. Then the right Kan extension $a_*p\colon \mathcal{E}' \to \mathcal{B}'$ is a presentable fibration. Moreover, if $\mathcal{B}'$ admits all limits (resp. colimits) of shape $\mathcal{D}$, so too does $\mathcal{E}'$.

Proof. First, notice that the expression $\mathrm{Fib}_{/\mathcal{B}}\left((a \downarrow b'), \mathcal{E}\right)$ is a formula for the pseudolimit

$$\lim_{a(b) \to b'} \mathcal{E}_b,$$

so by universal properties, in order to prove the first statement, it suffices to show that a small pseudolimit of presentable categories along presentable morphisms taken in the (huge) category $\mathbf{Cat}^+$ of locally small categories is also the pseudolimit in the (huge) category of presentable categories. However, this is precisely [Lur09, Theorem 5.5.3.18]. Then suppose we have a morphism $g\colon b' \to b''$ in $\mathcal{B}'$. Then the map g induces a Cartesian morphism of fibrations $(a \downarrow b') \to (a \downarrow b'')$ over $\mathcal{B}$, so we obtain a functor

$$(a_*\mathcal{E})_{b''} = \mathrm{Fib}_{/\mathcal{B}}\left((a \downarrow b''), \mathcal{E}\right) \to \mathrm{Fib}_{/\mathcal{B}}\left((a \downarrow b'), \mathcal{E}\right) = (a_*\mathcal{E})_{b'}$$

by precomposition. However, we can compute the target of this functor as

$$\mathrm{Fib}_{/\mathcal{B}}\left((a \downarrow b'), \mathcal{E}\right) \simeq \lim_{a(b) \to b'} \mathcal{E}_b,$$

so by the universal property, we obtain a cone

$$\ell_{(b,a(b)\to b')}\colon (a_*\mathcal{E})_{b'} \rightsquigarrow \mathcal{E}_b$$

over the diagram $(a \downarrow b')^{\mathrm{op}} \to \mathbf{Cat}$ defined by the rule $(b, a(b) \to b') \mapsto \mathcal{E}_b$. Then by universal properties, it suffices to show that each component of this cone is a morphism of presentable categories. But this functor is none other than the projection

$$\pi_{(b,a(b)\to b'\to b'')}\colon (a_*\mathcal{E})_{b''} \to \mathcal{E}_b,$$

which is presentable by merit of the fact that $(a_*\mathcal{E})_{b''}$ is a limit formed in presentable categories. It follows therefore that $a_*p\colon a_*\mathcal{E} \to \mathcal{B}'$ is a presentable fibration. The additional claim follows now from Proposition B.1.1 and its dual. ∎

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