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# THREE EASY PIECES:
## IMAGINARY SEMINAR TALKS IN HONOUR OF BOB ROSEBRUGH

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Abstract. We first take a whimsical look at a couple of questions relating to categories as generalized posets. Then we study the question of functorial choice of pullbacks. Finally, we consider a simple question in basic category theory, with an elementary solution which is surprisingly difficult to generalize to 2-categories.

## Introduction

Most readers will know Bob as the instigator and moderator of the "categories" bulletin board and the founder and managing editor of "Theory and Applications of Categories", the highly successful and pioneering electronic journal. These jobs he carried out efficiently and diplomatically for almost 25 years. It is hard to overestimate his contribution to the categorical community.

Readers will also know of his passion for theoretical computer science dating back almost 50 years to his student days, long before it became fashionable amongst category theorists. This evolved into his long lasting collaboration with Mike Johnson on database management. Parallel to this was his career-long friendship with Richard Wood which resulted in their collaboration on the categorical treatment of lattice theory.

Perhaps less well-known is his contribution to @CAT, the Atlantic Category Seminar. From the day he arrived to start his PhD research until he took his retirement, he was one of the pillars of the seminar (along with Richard Wood and Dietmar Schumacher). He religiously attended all talks and, when he wasn't presenting one himself, taking notes in one of his black notebooks, familiar to anyone who knows him. When he was at Mount Allison, he would drive down and back ($\approx 500$ km) almost every week during the term, regardless of the weather ("Neither snow nor rain nor heat nor gloom of night..."). I figure he must have driven about a quarter of a gigameter over the years. I wouldn't be surprised if he knew the exact number. He was also a licensed pilot and owned (half of) a plane and, on occasion, he would fly down to the seminar. It was mainly to break the monotony of driving and to get some airtime in, because it actually took longer, once you added the time going back and forth between airports.

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I present here three "seminar talks" which were never physically given. In these days of virtual seminars using Zoom, this goes one step further, *thought seminars* using ESP. The following "notes" may be useful for those who don't have "the gift".

The first is a nod to his work on lattices. We ask to what extent the following results for posets generalize to categories.

(a) Every distributive lattice is codistributive.

(b) A pair of functions between posets satisfying the adjointness bijections are automatically order-preserving, and give a Galois connection.

In the second, we consider the question of whether there is a functorial choice of pullbacks in **Set**. This has its roots in the theory of indexed categories, a recurrent theme in the early days of the seminar.

The last thought seminar relates to our joint paper (with RJW) on idempotents in bicategories [7]. We examine a simple question in basic category theory, whose generalization to 2-categories is surprisingly difficult (for me at least).

## 1. Ask a silly question

A distributive lattice, i.e. one satisfying $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ is also codistributive, i.e. satisfies the dual identity $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. The proof is easy, a one-liner, as any lattice theorist will tell you. But for a non-lattice theorist this may seem a bit surprising, and the one-line proof is not very illuminating, and certainly not in the spirit of category theory. So one might ask:

1.1. Silly question 1. Is a distributive category also codistributive?

Well, of course the answer is "no". It doesn't hold for the most basic distributive category, the category of finite sets. In fact it never holds unless the category is a lattice. But, undaunted by this wave of negativity, we nonetheless propose the following.

**Silly answer:** A distributive category is 80% codistributive.

Let **A** be a distributive category, i.e. a category with finite products and finite coproducts, such that for all $A$, $B$, and $C$, the canonical morphism

$$A \times B + A \times C \longrightarrow A \times (B + C)$$

is an isomorphism. (There's also a nullary part, but let's not worry about that now.)

For **A** to be codistributive we would need morphisms $\alpha$ and $\beta$

$$A + (B \times C) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} (A + B) \times (A + C)$$

such that $\alpha\beta = 1$ and $\beta\alpha = 1$.

In any category with finite sums and finite products, a morphism from a sum into a product, like $\alpha$, is a $2 \times 2$ matrix, and for the case at hand we have

$$\alpha = \begin{bmatrix} j_1 & j_1 \\ j_2\pi_1 & j_2\pi_2 \end{bmatrix}$$

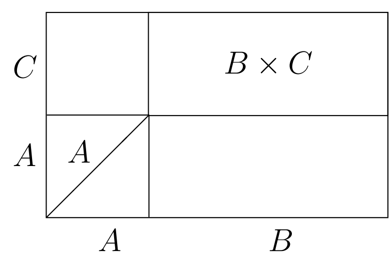
where the $j_i$ are coproduct injections and the $\pi_i$ product projections. It's this $\alpha$ that we would like to be invertible.

If **A** is distributive, then we have

$$(A+B) \times (A+C) \cong A \times A + A \times C + B \times A + B \times C$$

and the $\alpha$ corresponds to

$$[j_1\Delta, j_4]\colon A + (B \times C) \longrightarrow A \times A + A \times C + B \times A + B \times C.$$




$(A+B) \times (A+C)$

For $\beta$ we can take $[j_1\pi_1, j_1\pi_1, j_1\pi_2, j_2]$ and we have $\beta\alpha = 1_{A+(B\times C)}$. For codistributivity we need two morphisms $\alpha$ and $\beta$ and two equations $\alpha\beta = 1$ and $\beta\alpha = 1$. We have $\alpha$ and $\beta$ and one equation, so that's 75%.

We always have two morphisms, and one equation

$$A + 1 \underset{j_2}{\overset{!}{\rightleftarrows}} 1$$

in the nullary case, so I figure that's another 5%, giving 80%.

Note that the above provides a "conceptual", though admittedly complicated, proof of the lattice case, as all endos are identities.

If a distributive category is codistributive, then taking $B = C = 0$ we get that $\alpha$ is $\Delta\colon A \longrightarrow A \times A$ (now is the time to worry about nullary distributivity, $A \times 0 \cong 0 \cong 0 \times A$). $\Delta$ is an isomorphism only for posets, so we're reduced to the lattice case.

**A less silly answer albeit to a different question.**

Extensive categories are distributive but coproducts are nice. They are disjoint and universal (see [2]). There is a kind of fiberwise codistributivity. For any object $A$ we have that $A + (\ )$ preserves pullbacks or, put differently, for objects $B$ and $C$ over $D$ we have,

$$A + B \times_D C \xrightarrow{\cong} (A+B) \times_{(A+D)} (A+C)\ .$$

Indeed, not only is an extensive category $\mathbf{A}$ distributive but it is locally so, i.e. $\mathbf{A}/X$ is distributive for every $X$. So the codomain of the above morphism is isomorphic to

$$(A \times_{(A+D)} A) + (A \times_{(A+D)} C) + (B \times_{(A+D)} A) + (B \times_{(A+D)} C)$$

which in turn is isomorphic to

$$A + 0 + 0 + B \times_D C.$$

Lattices are "never" extensive, but we nevertheless ask:
**Question 1:** Is there a general theorem here?

1.2. Silly question 2. Let $\mathbf{A}$ and $\mathbf{B}$ be categories and $F\colon \mathrm{Ob}(\mathbf{A}) \longrightarrow \mathrm{Ob}(\mathbf{B})$ and $U\colon \mathrm{Ob}(\mathbf{B}) \longrightarrow \mathrm{Ob}(\mathbf{A})$ be object *functions*. Suppose that for every $A$ in $\mathrm{Ob}(\mathbf{A})$ and $B$ in $\mathrm{Ob}(\mathbf{B})$ we are given isomorphisms

$$\theta_{A,B}\colon \mathbf{B}(FA, B) \longrightarrow \mathbf{A}(A, UB).$$

Are $F$ and $G$ automatically adjoint *functors*?

Why would we even ask such an outrageous question? It's well-known that if $\mathbf{A}$ and $\mathbf{B}$ are posets, this is true. And why is it outrageous? Well, if $\mathbf{A}$ and $\mathbf{B}$ are groups (one object categories), then $F$ and $U$ are trivial and $\theta$ gives a bijection between the groups. A homomorphism with an adjoint is an isomorphism of groups. So we're asking whether two groups with the same cardinality are isomorphic. Which is ridiculous. Nevertheless, we propose the following.

**Silly answer:** It is 20% true.

Let's see how far we can get.

(1) For every $A$,

$$\theta_{A,FA}\colon \mathbf{B}(FA, FA) \longrightarrow \mathbf{A}(A, UFA)$$

and we let $\eta_A\colon A \longrightarrow UFA$ be $\theta_{A,FA}(1_{FA})$.

(2) For every $B$,

$$\theta_{UB,B}\colon \mathbf{B}(FUB, B) \longrightarrow \mathbf{A}(UB, UB)$$

and we let $\epsilon_B\colon FUB \longrightarrow B$ be $\theta^{-1}_{UB,B}(1_{UB})$.

(3) For $g\colon B \longrightarrow B'$ we apply

$$\theta_{UB,B'}\colon \mathbf{B}(FUB, B') \longrightarrow \mathbf{A}(UB, UB')$$

to

$$FUB \xrightarrow{\epsilon B} B \xrightarrow{g} B'$$

and get

$$Ug := \theta_{UB,B'}(g \cdot \epsilon B).$$

(4) For $f\colon A \longrightarrow A'$, we have

$$\theta_{A,FA'}\colon \mathbf{B}(FA, FA') \longrightarrow \mathbf{A}(A, UFA')$$

so we can apply $\theta^{-1}_{A,FA'}$ to

$$A \xrightarrow{f} A' \xrightarrow{\eta A'} UFA'$$

to get

$$Ff := \theta^{-1}_{A,FA'}(\eta A' \cdot f).$$

So we get all the data for functors $F$ and $U$ and for natural transformations $\eta\colon 1_{\mathbf{A}} \longrightarrow UF$ and $\epsilon\colon FU \longrightarrow 1_{\mathbf{b}}$, but the only equations we have are that $F$ and $U$ preserve identities, $F(1_A) = 1_{FA}$ and $U(1_B) = 1_{UB}$.

I figure that's about 20% of what's needed. (I chose 20% to be complementary to the previous silly answer. Something wrong with this?)

Let's dig a bit deeper. We know that if $U$ is a functor and

$$\theta_{A,B}\colon \mathbf{B}(FA, B) \longrightarrow \mathbf{A}(A, UB)$$

is natural in $B$, then $F$ is automatically a functor, and $\epsilon$ and $\eta$ are natural, and then $F$ is left adjoint to $U$. A dual result holds if instead $F$ is a functor. So perhaps we could get a more symmetric result generalizing both of these facts.

There's something intriguing about the display for $\theta_{A,B}$ above. The domain is functorial in $B$ though not in $A$ and the codomain is functorial in $A$ though not in $B$, yet the isomorphism $\theta_{A,B}$ is saying that they are "the same".

Below and in the next section we will make use of the following, well-known result, which we state explicitly for completeness.

1.3. Lemma. *(Transport of functoriality)*
*Let $\Phi\colon \mathbf{X} \longrightarrow \mathbf{Y}$ be a functor and assume that for each object $X$ in $\mathbf{X}$ we are given an object $\Theta X$ in $\mathbf{Y}$ and an isomorphism $\theta X\colon \Theta X \longrightarrow \Phi X$. Then $\Theta$ can be made into a functor in a unique way such that $\theta X$ is a natural isomorphism.*

Proof. For any $x\colon X \longrightarrow X'$ we want $\Theta x$ such that

$$\begin{array}{ccc} \Theta X & \xrightarrow{\theta X} & \Phi X \\ {\scriptstyle \Theta x}\downarrow & & \downarrow{\scriptstyle \Phi x} \\ \Theta X' & \xrightarrow[\theta X']{} & \Phi X' \end{array} .$$

So take $\Theta x = (\theta X')^{-1}\Phi x(\theta X)$. It works.

■

So let's transport the functorial structure of $\mathbf{A}(A, UB)$ on the right to the left along $\theta$. Let

$$P(A,B) := \mathbf{B}(FA, B).$$

$P$ is clearly functorial in $B$. For $f\colon A \longrightarrow A'$, define $P(f,B)$ to be the unique morphism such that

$$\begin{array}{ccc}
P(A',B) = \mathbf{B}(FA',B) & \xrightarrow{\theta_{A',B}} & \mathbf{A}(A',UB) \\
{\scriptstyle P(f,B)}\downarrow & & \downarrow{\scriptstyle \mathbf{A}(f,UB)} \\
P(A,B) = \mathbf{B}(FA,B) & \xrightarrow[\theta_{A,B}]{} & \mathbf{A}(A,UB)
\end{array}$$

commutes, i.e.

$$P(f,B) = \theta_{A,B}^{-1}\mathbf{A}(f,UB)\theta_{A',B}\ .$$

For fixed $B$, $P(A,B)$ is now functorial in $A$ and $\theta_{A,B}$ natural in $A$.

To get a functor $P\colon \mathbf{A}^{op} \times \mathbf{B} \longrightarrow \mathbf{Set}$ we need that the two functorialities commute: for $f\colon A \longrightarrow A'$ and $g\colon B \longrightarrow B'$

$$\begin{array}{ccc}
P(A',B) & \xrightarrow{P(f,B)} & P(A,B) \\
{\scriptstyle P(A',g)}\downarrow & & \downarrow{\scriptstyle P(A,g)} \\
P(A',B') & \xrightarrow[P(f,B')]{} & P(A,B')\ .
\end{array}$$

When we put in the definition of $P(f,-)$ we get the following *octagon condition*

$$\begin{array}{ccccccc}
 & & \mathbf{A}(A',UB) & \xrightarrow{\mathbf{A}(f,UB)} & \mathbf{A}(A,UB) & & \\
 & {\scriptstyle \theta_{A',B}}\nearrow & & & & \searrow{\scriptstyle \theta_{A,B}^{-1}} & \\
\mathbf{B}(FA',B) & & & & & & \mathbf{B}(FA,B) \\
{\scriptstyle \mathbf{B}(FA',g)}\downarrow & & & (*) & & & \downarrow{\scriptstyle \mathbf{B}(FA,g)} \\
\mathbf{B}(FA',B') & & & & & & \mathbf{B}(FA,B') \\
 & {\scriptstyle \theta_{A',B'}}\searrow & & & & \nearrow{\scriptstyle \theta_{A,B'}^{-1}} & \\
 & & \mathbf{A}(A',UB') & \xrightarrow[\mathbf{A}(f,UB')]{} & \mathbf{A}(A,UB') & & .
\end{array}$$

If the $\theta$'s satisfy $(*)$ then we get a functor $P\colon \mathbf{A}^{op} \times \mathbf{B} \longrightarrow \mathbf{Set}$. For fixed $A$, $P(A,-)$ is representable, $P(A,-) = \mathbf{B}(FA,-)$, so $F$ is a functor. And for fixed $B$, $P(-,B)$ is also representable: by construction

$$\theta_{-,B}\colon P(-,B) \longrightarrow \mathbf{A}(-,UB)$$

is a natural isomorphism, so $U$ is also a functor. Finally,

$$\theta_{A,B}\colon \mathbf{B}(FA,B) \longrightarrow \mathbf{A}(A,UB)$$

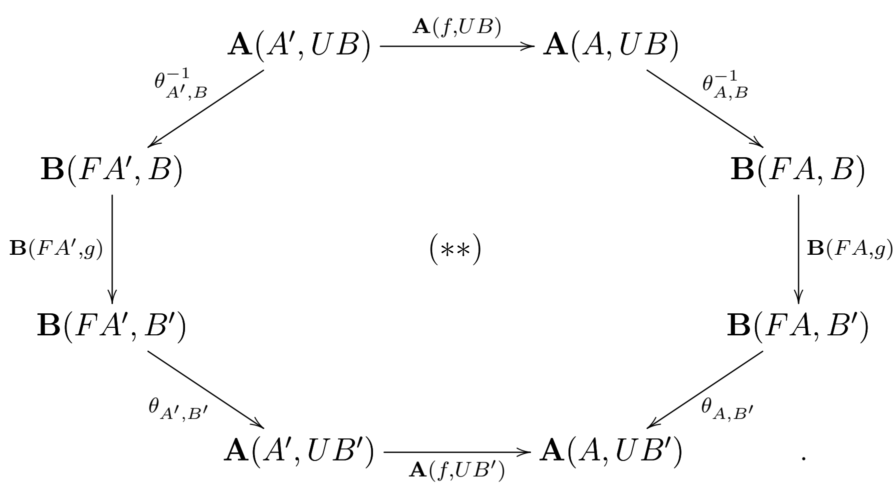
is a natural isomorphism, so $F$ is left adjoint to $U$. We have the following.

1.4. Theorem. *Let* $F\colon \mathrm{Ob}(\mathbf{A}) \longrightarrow \mathrm{Ob}(\mathbf{B})$ *and* $U\colon \mathrm{Ob}(\mathbf{B}) \longrightarrow \mathrm{Ob}(\mathbf{A})$ *be object functions and*

$$\theta_{A,B}\colon \mathbf{B}(FA,B) \longrightarrow \mathbf{A}(A,UB)$$

*be bijections satisfying the octagon condition* $(*)$. *Then* $F$ *and* $U$ *are functors, and* $F \dashv U$.

1.5. Remark. Instead of taking $P(A,B) = \mathbf{B}(FA,B)$ and using $\theta$ to make it functorial in $A$, we could have taken $Q(A,B) = \mathbf{A}(A,UB)$ and made it functorial in $B$. The octagon we get in this case looks different, not just $(*)$ with $\theta$ replaced with $\theta^{-1}$, which seems strange. Do we get different functors this way? No, what we get is




But it can be seen to be equivalent to $(*)$ by pre-composing $(**)$ by $\theta_{A',B}$ and postcomposing by $\theta^{-1}_{A,B'}$.

Note in passing that

$$\theta\colon P \longrightarrow Q$$

is a natural isomorphism now.

**I should have known!** I was quite surprised that something new could be said about such a central concept as adjunction which has been around for over sixty years. I guess I shouldn't have been so surprised. The referee has pointed out that it's not new at all. It had all been done by Street in his 2012 TAC paper "The core of adjoint functors" [11].

Street's motivation was the same as ours above and he gives the same definition of $\eta_A$, $\epsilon_B$, $Ug$ and $Ff$, but his context is that of enriched categories. Not distracted by the temptation to use elements he discovers simple conditions that insure that an adjunction core, the family of isomorphisms $\theta_{A,B}$, gives an actual adjunction. When specialized to **Set**-enriched categories (i.e. ordinary, garden variety categories) his theorem

3.2 is basically our theorem 1.4, though his conditions are different, being expressed as commutative squares, rather than our octagons. His conditions are simple and the proof of theorem 3.2 is “short and sweet”.

He goes on from there to study adjunction cores between monads in a bicategory, which specialize to adjunction cores between internal categories (for the bicategory $\mathcal{S}pan$). Not content with this level of generality, everything is extended to adjunction cores for lax $D$-algebras, for a doctrine $D$ (= pseudo-monad) on a bicategory.

It’s a nice paper and well worth reading. I guess it wasn’t such a silly question after all.

About Street’s conditions (3.1) and (3.2) versus our (*) and (**). They are quite different, though of necessity equivalent, but it’s not that trivial to see this. At least it’s not transparent. In terms of elements (we’re only comparing them in the **Set** case, of course) (*) says that for all morphisms

$$f: A \longrightarrow A', \quad x: FA' \longrightarrow B, \quad g: B \longrightarrow B'$$

we have

$$\theta^{-1}(\theta(g \cdot x) \cdot f) = g \cdot \theta^{-1}(\theta(x) \cdot f)$$

whereas in Street’s (3.1) (which is the one corresponding to (*)) we have for all $x$ and $g$ as above

$$\theta(g \cdot x) = \theta(g \cdot \theta^{-1}(1_{UA})) \cdot \theta(x)$$

These conditions are quite different. Okay, the second one is simpler, but I still have a soft spot for the octagons. They are my babies after all!

## 2. Dead horse

In his mini-course “Homotopy-theoretic models of type theory”, given at the Fields Institute in May 2016, Peter Lumsdaine asked in passing, whether it might be possible to choose pullbacks in sets in such a way that the pullback functors

$$f^*: \mathbf{Set}/I \longrightarrow \mathbf{Set}/J,$$

which typically give a pseudo-functor

$$\mathbf{Set}/(\ ): \mathbf{Set}^{op} \longrightarrow \mathcal{CAT},$$

give an actual functor.

This was something I had thought about a long time ago in connection with indexed categories, so after his talk I said it could be done by transferring the functorial change of base for families of sets

$$f^*: \mathbf{Set}^I \longrightarrow \mathbf{Set}^J$$

along the equivalences $\mathbf{Set}/I \xrightarrow{\sim} \mathbf{Set}^I$. He was skeptical (and rightly so). After an exchange of some vague ideas, he concluded with “it’s a frustrating problem”.

At a later conference, I had refined my argument. I said the equivalences

$$\mathbf{Set}^I \longrightarrow \mathbf{Set}/I$$

could be deformed to isomorphisms, and then the functorial change of base on $\mathbf{Set}^{(\ )}$ could be transported to a functorial one on $\mathbf{Set}/(\ )$. He still seemed a bit skeptical. (I hadn't written anything down.) He wanted to know if the process respected the coherence isomorphisms, a valid point. He concluded with "It's a frustrating problem, and what makes it even more frustrating is that if you do settle it, no one will be interested".

**"Added in press:"** Peter has alerted me to the following exchange with Tim Campion on MathOverflow [5, 1] which mirrors pretty much my discussion below.

This is a set theoretical question, and the solution uses set theory in a way that the seasoned category theorist won't like at all. And the average set theorist won't even care.

With this introduction, what's left to be said? As they say in the movies "Damn the torpedoes, full speed ahead!"

2.1. Local equivalence. First, let's look at the equivalence

$$\Phi_I\colon \mathbf{Set}^I \longrightarrow \mathbf{Set}/I.$$

An object on the left is an $I$-indexed family of sets $\langle A_i\rangle_{i\in I}$, i.e. a function $A_{(\ )}$, from $I$ into the class of all sets. And a morphism $\langle f_i\rangle\colon \langle A_i\rangle \longrightarrow \langle B_i\rangle$ is an $I$-indexed family of functions $f_i\colon A_i \longrightarrow B_i$. On the other side, an object is a function $p\colon A \longrightarrow I$ and a morphism a commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & I\,. & \end{array}$$

I'm being precise here because we category theorists tend to identify the two.

$\Phi_I$ takes a family $\langle A_i\rangle$ to its coproduct

$$\begin{array}{c} \sum_{i\in I} A_i \\ \big\downarrow {\scriptstyle p} \\ I \end{array}$$

where $\sum_{i\in I} A_i = \{(i,a) | i \in I, a \in A_i\}$ and $p(i,a) = i$. For a morphism $\langle f_i\rangle\colon \langle A_i\rangle \longrightarrow \langle B_i\rangle$, $\Phi_I\langle f_i\rangle$ is the function

$$\sum_{i\in I} A_i \longrightarrow \sum_{i\in I} B_i$$

$$(i,a) \longmapsto (i, f_i(a)).$$

$\Phi_I$ is easily seen to be a functor. We've chosen a very specific representation of the coproduct, and it's important that $\Phi_I$ is one-to-one on objects.

$\Phi_I$ has a pseudo-inverse

$$\Psi_I \colon \mathbf{Set}/I \longrightarrow \mathbf{Set}^I$$

$$\begin{array}{c} A \\ \big\downarrow{\scriptstyle p} \\ I \end{array} \longmapsto \langle p^{-\text{`}}\{i\}\rangle_{i\in I}$$

and for a morphism

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & I & \end{array}$$

$\Psi_I(f) = \langle f|_{p^{-1}\{i\}}\rangle$.
$\Psi_I$ is also easily seen to be a functor and one-to-one on objects.

$\Psi_I\Phi_I\langle A_i\rangle = \langle\{i\} \times A_i\rangle$ so $\Psi_I\Phi_I$ is not the identity but we have a natural isomorphism

$$\alpha_I \colon \Psi_I\Phi_I \longrightarrow 1_{\mathbf{Set}^I}.$$

Similarly, $\Phi_I\Psi_I(A \xrightarrow{p} I) = \{(p(a), a)|a \in A\}$ with projection onto first factor. And we have another natural isomorphism

$$\beta_I \colon \Phi_I\Psi_I \longrightarrow 1_{\mathbf{Set}/I}.$$

Thus we have our equivalence of categories

$$\mathbf{Set}^I \xrightarrow{\sim} \mathbf{Set}/I.$$

This is the usual equivalence. The only thing to emphasize is that, as we constructed it, both $\Phi_I$ and $\Psi_I$ are one-to-one on objects.

2.2. Theorem. *(Schröder-Bernstein for categories)*

*Suppose we have an equivalence of categories* $\Phi \colon \mathbf{A} \longrightarrow \mathbf{B}$ *with pseudo-inverse* $\Psi \colon \mathbf{B} \longrightarrow \mathbf{A}$ *and isomorphims* $\alpha \colon \Psi\Phi \longrightarrow 1_{\mathbf{A}}$ *and* $\beta \colon \Phi\Psi \longrightarrow 1_{\mathbf{B}}$*. Further assume that* $\Phi$ *and* $\Psi$ *are both one-to-one on objects. Then there is an isomorphism of categories*

$$\Theta \colon \mathbf{A} \xrightarrow{\cong} \mathbf{B}$$

*and a natural isomorphism* $\gamma \colon \Phi \longrightarrow \Theta$*. Furthermore,* $\gamma A$ *is either an identity or* $\beta B$ *for some* $B$ *in* $\mathbf{B}$*.*

PROOF. We simply give an outline. It's basically the standard proof as can be found in any of the classic books on set theory, e.g. [3].

Define the sequence of subclasses of Ob$\mathbf{A}$ to be the set differences

$$C_n = (\Psi\Phi)^n(\mathrm{Ob}\mathbf{A}) - (\Psi\Phi)^n\Psi(\mathrm{Ob}\mathbf{B}), \quad n \geq 0.$$

So $C_0 = \mathrm{Ob}\mathbf{A} - \Psi(\mathrm{Ob}\mathbf{B})$ consists of all objects of $\mathbf{A}$ that are not $\Psi$ of any $B$ in $\mathbf{B}$. Define $\Theta$ on an object $A$ by

$$\Theta A = \begin{cases} \Phi A & \text{if } A \in C_n \text{ for some } n \\ \Psi^{-1} A & \text{otherwise} \end{cases}$$

Note that if $A$ is not in any $C_n$, it's not in $C_0$ so is $\Psi$ of some, necessarily unique $B$. And the definition makes sense.

Also define $\gamma A$ in

$$\gamma A = \begin{cases} 1_{\Phi A} & \text{if } A \in C_n \text{ for some } n \\ \beta\Psi^{-1} A & \text{otherwise.} \end{cases}$$

Note that $\beta\Psi^{-1}A\colon \Phi\Psi\Psi^{-1}A \longrightarrow \Psi^{-1}A$, i.e. $\Phi A \longrightarrow \Theta A$.

Now we extend $\Theta$ to a functor in the unique way that makes

$$\gamma\colon \Phi \longrightarrow \Theta$$

natural.

That $\Theta$ is a bijection on objects is the standard Schröder-Bernstein argument. Since $\Theta$ is isomorphic to $\Phi$ which is full and faithful, then $\Theta$ is also, and thus an isomorphism of categories.

■

Apply this to our equivalences $\Phi_I$, $\Psi_I$ to get an isomorphism of categories

$$\Theta_I\colon \mathbf{Set}^I \longrightarrow \mathbf{Set}/I$$

and a natural isomorphism $\gamma_I\colon \Phi_I \longrightarrow \Theta_I$.

Now we use the $\Theta_I$ to transport the functorial change of base for $\mathbf{Set}^I$ to a functorial one for $\mathbf{Set}/I$, i.e. for $f\colon J \longrightarrow I$ we define

$$f^\bullet := (\mathbf{Set}/I \xrightarrow{\Theta_f^{-1}} \mathbf{Set}^I \xrightarrow{f^*} \mathbf{Set}^J \xrightarrow{\Theta_J} \mathbf{Set}/J).$$

$(\ )^\bullet$ is definitely functorial, $(fg)^\bullet = g^\bullet f^\bullet$ and $1_J^\bullet = 1_{\mathbf{Set}/I}$, but is it change of base? This was Peter's question. And my answer was, yes because it's isomorphic to $f^*$, a good example of fuzzy thinking. Trying to make this clear is not that easy, so we take a more global approach, using fibrations (*pace* Bénabou).

2.3. Global equivalence. All of the categories $\mathbf{Set}^I$ with their change of base functors fit into one category of families, given by the Grothendieck semi-direct product construction

$$\mathbf{Fam}.$$

An object is a pair $(I, \langle A_i \rangle_{i \in I})$ consisting of a set $I$ and an $I$-indexed family of sets $\langle A_i \rangle$. A morphism $(I, \langle A_i \rangle) \longrightarrow (J, \langle B_j \rangle)$ is a pair $(\alpha, \langle f_i \rangle)$ where $\alpha \colon I \longrightarrow J$ is a function, and $\langle f_i \rangle$ is a family of functions $\langle f_i \colon A_i \longrightarrow B_{\alpha i} \rangle_{i \in I}$. Projection onto the first factor

$$\begin{array}{c} \mathbf{Fam} \\ \downarrow{\scriptstyle P} \\ \mathbf{Set} \end{array}$$

is a fibration, with the cartesian morphisms being those $(\alpha, \langle f_i \rangle)$ with all $f_i$ isomorphisms.

The Grothendieck construction applied to the categories $\mathbf{Set}/I$ with their change of base gives the arrow category

$$\begin{array}{c} \mathbf{Set}^{\mathbf{2}} \\ \downarrow{\scriptstyle D_1} \\ \mathbf{Set}\ . \end{array}$$

where $D_1$ is the codomain functor. It's also a fibration (of course) and the cartesian morphisms

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle q} \\ I & \xrightarrow[\alpha]{} & J \end{array}$$

are pullback squares (aha!).

The functors $\Phi_I$ and $\Psi_I$ fit together to give an equivalence pair over $\mathbf{Set}$

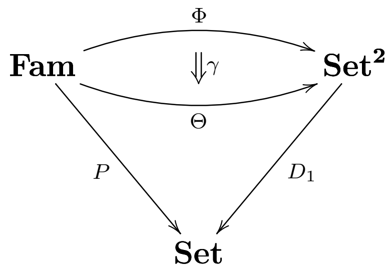
$$\begin{array}{ccc} \mathbf{Fam} & \overset{\Phi}{\underset{\Psi}{\rightleftarrows}} & \mathbf{Set}^{\mathbf{2}} \\ & {\scriptstyle P}\searrow \quad \swarrow{\scriptstyle D_1} & \\ & \mathbf{Set} & \end{array}$$

with the isomorphisms $\alpha \colon \Psi\Phi \longrightarrow 1_{\mathbf{Fam}}$ and $\beta \colon \Phi\Psi \longrightarrow 1_{\mathbf{Set}^{\mathbf{2}}}$ given fiberwise by the $\alpha_I$ and $\beta_I$.

Now, $\Phi$ and $\Psi$ are both one-to-one on objects so by Theorem 2.2 there is an isomorphism of categories

$$\Theta \colon \mathbf{Fam} \longrightarrow \mathbf{Set}^{\mathbf{2}}$$

and a natural isomorphism $\gamma\colon \Phi \longrightarrow \Theta$ with $\gamma$ either an identity or an instance of $\beta$. It follows that $D_1\gamma = \mathrm{id}_P$




that is, $\Theta$ is an isomorphism *over* **Set**. As such it preserves cartesian morphisms (as they are defined by a universal property).

**Fam** has a functorial choice of cartesian morphisms, namely those for which the $f_i$ are identities: for $\alpha\colon I \longrightarrow J$ and $(J, \langle B_j \rangle)$,

$$(I, \langle B_{\alpha(i)} \rangle_i) \xrightarrow{(\alpha, \langle 1_{B_{\alpha(i)}} \rangle)} (J, \langle B_j \rangle_j).$$

Then $\Theta$ takes them to a functorial choice of cartesian morphisms in $\mathbf{Set}^{\mathbf{2}}$, i.e. pullbacks. We summarize.

2.4. Theorem. **Set** *has a functorial choice of pullbacks.*

One can use this to get a functorial choice of pullbacks for any category where pullbacks are constructed in **Set**, for example any Grothendieck topos, algebraic category, or more generally any locally presentable category. However, a good case can be made that this is not the right way to go, and dealing with pseudo-functors or fibrations is. There is something inherently pseudo with large categories like **Set** or Grothendieck toposes. However, on occasion it may be convenient, or even necessary, to be dealing with actual functors rather than mere pseudo-functors. Instead of going through set theoretical contortions to get functorial pullbacks, it would be more natural to replace the pseudo-functor by a canonically constructed pseudo-equivalent functor, work with that, and afterwards "descend" the results to the original pseudo-functor. Fortunately, there has been a ready-made theorem for this, around for forty years, namely Street's strictification theorem ([10], Theorem 7.5). But that's a whole different story.

2.5. Counterexample. On the other hand small categories, like Lawvere theories for example, are inherently strict and some of them have pullbacks. So there it might make more sense to look for a functorial choice of pullbacks, or for a counterexample there.

The initial Lawvere theory, the category of finite cardinals with all functions has pullbacks, because it's equivalent to the category of finite sets. Does it have a functorial choice of pullbacks?

For a natural number $n \in \mathbb{N}$, let

$$[n] = \{1, 2, ..., n\}$$

(with $[0] = \emptyset$), and **FinCard** the full subcategory of **Set** determined by the $[n]$, $n \in \mathbb{N}$. Our Schröder-Bernstein argument doesn't work here, at least not in any obvious way. For example, the number of morphisms $[3] \longrightarrow [2]$ is 8 whereas the number of $[2]$-indexed families of cardinals summing to 3 is 4. In fact we have:

2.6. THEOREM. **FinCard** *does not have a functorial choice of pullbacks.*

PROOF. Suppose it did. Then for any function $f\colon [m] \longrightarrow [n]$ and any bijection $\sigma\colon [n] \longrightarrow [n]$ we would have a chosen pullback

$$\begin{array}{ccc} [m] & \xrightarrow{f_\sigma} & [n] \\ {\scriptstyle f^*(\sigma)}\downarrow & & \downarrow{\scriptstyle \sigma} \\ [m] & \xrightarrow[f]{} & [n] \end{array}$$

which gives a function $f^*\colon S_n \longrightarrow S_m$ on the symmetric groups. Just a function, not a homomorphism. But this gives a functor

$$\mathbf{FinCard}^{op} \longrightarrow \mathbf{Set}_0$$

whose value on objects is $S_n$ of cardinality $n!$. By the second last paragraph of [9], no such functor exists.

■

The argument of [9] is quite involved and the last step involving $n!$ uses a Maple calculation. I checked again, using Excel this time, and it's still true, 21 years later. It's remarkable that you have to go to the $12^{th}$ "difference" before a negative number appears, showing that there is indeed no functor. The argument is less than transparent, so we pose:

**Problem 1:** Find a simple proof that **FinCard** doesn't have a functorial choice of pullbacks.

Now that it's known that there isn't a functorial choice, the work is cut down by (at least) half. It may be instructive to look a bit deeper.

Pullbacks in **FinCard** are constructed as follows. For $f$ and $g$ as below, take the pullback $P$ in $\mathbf{Set}_0$

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & [m] \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ [n] & \xrightarrow[g]{} & [r]\ . \end{array}$$

To be precise, take $P = \{(x, y) | 1 \leq x \leq n, 1 \leq y \leq m, \text{ and } g(x) = f(y)\}$, and $\pi_1, \pi_2$ the projections. $P$ will have

$$p = \sum_{i=1}^{r} |f^{-1}(i)||g^{-1}(i)|$$

elements. (We don't need the actual number but it's a nice formula.) We choose a bijection $\phi\colon [p] \longrightarrow P$, a counting function. Then

$$\begin{array}{ccc} [p] & \xrightarrow{\pi_2\phi} & [m] \\ {\scriptstyle \pi_1\phi}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ [n] & \xrightarrow[g]{} & [r] \end{array}$$

is a pullback in **FinCard**. Where we have some elbow room is in the choice of $\phi$, which comes down to totally ordering $P$. The first thing that comes to mind is the lexicographic ordering

$$(x, y) < (x', y') \iff (x < x') \vee (x = x' \wedge y < y')$$

and there, something interesting happens.

2.7. Proposition. *With the lexicographic ordering, pullbacks in* **FinCard** *are semi-functorial, i.e. lexicographic pullbacks paste horizontally, but pulling back along the identity is not the identity.*

Proof. The lexicographic pullback is characterized by the following

(a)

$$\begin{array}{ccc} [p] & \xrightarrow{\psi} & [n] \\ {\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ [m] & \xrightarrow[\phi]{} & [r] \end{array} \qquad \text{is a pullback, and}$$

(b) for every $1 \le i < j \le p$,

$$g(i) < g(j)$$

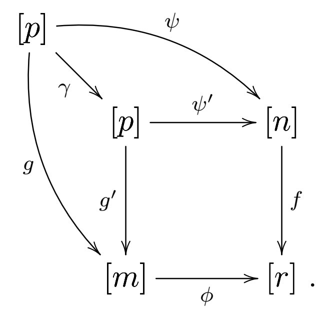
or

$$g(i) = g(j) \quad \text{and} \quad \psi(i) < \psi(j).$$

It's clear that the lexicographic pullback satisfies these conditions. Suppose we have another diagram

$$\begin{array}{ccc} [p] & \xrightarrow{\psi'} & [n] \\ {\scriptstyle g'}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ [m] & \xrightarrow[\phi]{} & [r] \end{array}$$

satisfying (a) and (b). Then there exists an isomorphism $\gamma\colon [p] \longrightarrow [p]$ such that




We claim that $\gamma$ is order-preserving. Suppose not. Then there are $1 \leq i < j \leq p$ such that $\gamma(i) \not< \gamma(j)$, so $\gamma(j) < \gamma(i)$ (they can't be equal because $\gamma$ is a bijection). Then we have

$$g(i) < g(j) \quad \text{or} \quad (g(i) = g(j) \quad \text{and} \quad \psi(i) < \psi(j))$$

but we also have

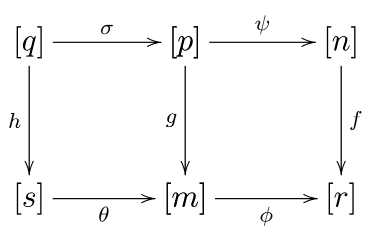
$$g'(\gamma(j)) < g'(\gamma(i)) \quad \text{or} \quad (g'(\gamma(j)) = g'(\gamma(i)) \quad \text{and} \quad \psi'(\gamma(j)) < \psi'(\gamma(i))$$

i.e.

$$g(j) < g(i) \quad \text{or} \quad (g(j) = g(i) \quad \text{and} \quad \psi(j) < \psi(i)).$$

There are four possibilities for the conjunction of both disjunctions and inspection readily shows none of them can hold. We conclude that $\gamma$ is order-preserving and so is the identity.

It's now easy enough to show that lexicographic pullbacks paste *horizontally*. If we have two, as in




then for $1 \leq i < j \leq q$ we have

$$(h(i) < h(j)) \vee (h(i) = h(j) \wedge (\sigma(i) < \sigma(j)))$$

and $\sigma(i) < \sigma(j)$ implies

$$(g\sigma(i) < g\sigma(j)) \vee ((g\sigma(i) = g\sigma(j)) \wedge (\psi\sigma(i) < \psi\sigma(j))$$

which gives us three possibilities

(i) $h(i) < h(j)$

(ii) $h(i) = h(j)$ and $g\sigma(i) < g\sigma(j)$

(iii) $h(i) = h(j)$ and $g\sigma(i) = g\sigma(j)$ and $\psi\sigma(i) < \psi\sigma(j)$.

Now, (ii) is impossible because $g\sigma(i) = \theta h(i) = \theta h(j) = g\sigma(j)$. In (iii), $g\sigma(i) = g\sigma(j)$ is redundant also because $g\sigma(i) = \theta h(i) = \theta h(j) = g\sigma(j)$. So we're left with

$$h(i) < h(j) \quad \text{or} \quad (h(i) = h(j) \quad \text{and} \quad \psi\sigma(i) < \psi\sigma(j))$$

which is exactly what we want, i.e. the pasted sequence is a lexicographic pullback.

Note that, in a lexicographic pullback as above, the $g$ is order-preserving, so if $f$ isn't then pulling back along the identity can't give $f$ back again.

■

2.8. Remark. Lexicographic pullbacks don't paste vertically. Another way of saying this is that taking lexicographic ordering from the right (rexicographic?) doesn't work.

2.9. Remark. There's a strange interplay between order-preserving and arbitrary functions. If we take a lexicographic pullback of an arbitrary $f$ along the identity

$$\begin{array}{ccc} [n] & \xrightarrow{\psi} & [n] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ [r] & \xrightarrow[1_{[r]}]{} & [r] \end{array}$$

then $g$ is order-preserving and $\psi$ is invertible, so we get a canonical factorization of $f$ as a bijection followed by an order-preserving map

$$f = g\psi^{-1}.$$

The two theorems of this section may not be of any practical use but may provide closure for those who worry about such things. We can now bury the horse!

## 3. The square root of adjoints

**Question:** Let $F\colon \mathbf{A} \longrightarrow \mathbf{B}$ be a functor such that $F \times F\colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{B} \times \mathbf{B}$ has a left adjoint. Does $F$ itself have a left adjoint?

The answer is "yes", but it's not as easy to prove as one might imagine. Well, of course it is, once you see the solution.

3.1. Start of a 2-categorical proof. Let $G$ be left adjoint to $F \times F$. If $F$ had a left adjoint $H$, then $G$ would be isomorphic to $H \times H$, so we could construct an adjoint hopeful as

$$H = (\mathbf{B} \xrightarrow{\Delta} \mathbf{B} \times \mathbf{B} \xrightarrow{G} \mathbf{A} \times \mathbf{A} \xrightarrow{P_1} \mathbf{A})$$

where $P_1$ is projecting onto the first factor. Let $\epsilon\colon G{\cdot}F{\times}F \longrightarrow 1_{\mathbf{A}\times\mathbf{A}}$ and $\eta\colon 1_{\mathbf{B}\times\mathbf{B}} \longrightarrow F{\times}F{\cdot}G$ be the adjunction transformations for $G \dashv F \times F$. Now we can define transformations $e\colon HF \longrightarrow 1_\mathbf{A}$ and $h\colon 1_\mathbf{B} \longrightarrow FH$ as

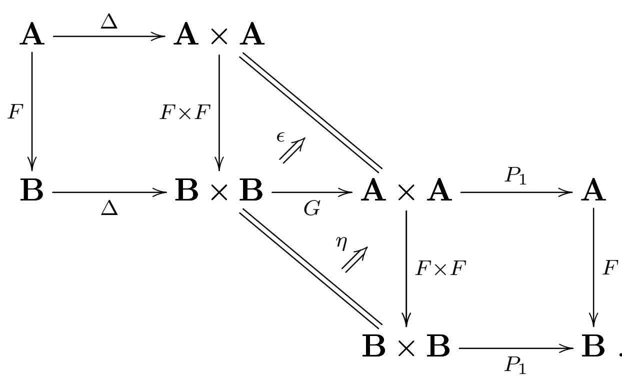
$$e = (HF = P_1 G\Delta F = P_1 G(F \times F)\Delta \xrightarrow{P_1\epsilon\Delta} P_1\Delta = 1_\mathbf{A})$$

and

$$h = (1_\mathbf{B} = P_1\Delta \xrightarrow{P_1\eta\Delta} P_1(F \times F)G\Delta = FP_1G\Delta = FH)$$

both pictured below.

Now for the triangle equalities. Consider:



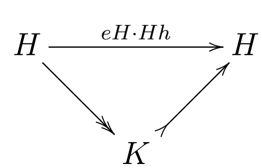
The parallelogram in the middle is the identity by one of the triangle equations for $G \dashv F \times F$, and since $P_1\Delta = 1$, we immediately get the corresponding triangle equation for $H \dashv F$.

This is going swimmingly well, just a "follow your nose" argument. One might expect the other triangle identity to be harder as it involves more arrows but straightforward nonetheless. But in fact it's much harder. It took me several weeks of trying to prove/or disprove it (it may be just me, of course). At the risk of annoying the impatient reader (should he or she have gotten this far), I will delay the proof for a while, to give the competitive reader a chance to have a go at it.

Here is an argument which may or may not lead to a proof. It uses a result from my thesis [8], and given as an exercise in Mac Lane's book [6], page 84. If

$$H \xrightarrow{Hh} HFH \xrightarrow{eH} H$$

doesn't turn out to be an identity as we had hoped, it is at least an idempotent, and if $\mathbf{A}$ has split idempotents, we can split it



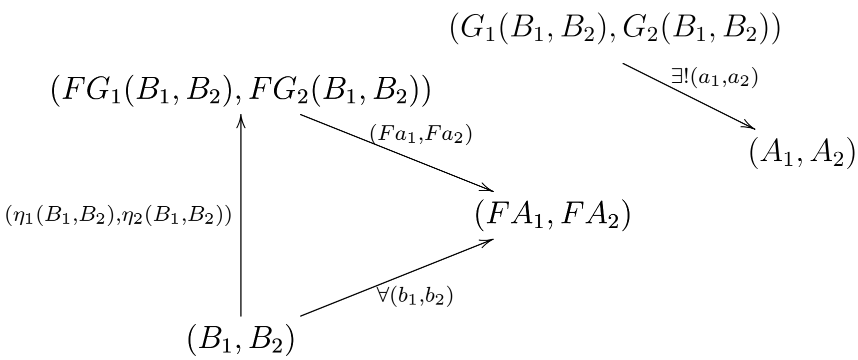
and then $K$ is left adjoint to $F$. But then $K \times K$ is left adjoint to $F \times F$, so $K \times K \cong G$, and then

$$\begin{aligned} K &= P_1(K \times K)\Delta \\ &\cong P_1 G \Delta \\ &= H. \end{aligned}$$

So $H$ turns out to be left adjoint to $F$ after all. And $H$ doesn't use split idempotents at all...

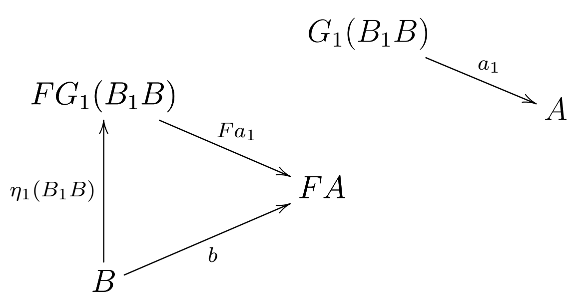
3.2. A proof in $\mathcal{C}at$. We give an elementary proof in terms of universal arrows.

$G: \mathbf{B} \times \mathbf{B} \longrightarrow \mathbf{A} \times \mathbf{A}$ is given by two functors $\mathbf{B} \times \mathbf{B} \longrightarrow \mathbf{A}$, $G_1 = P_1 G$ and $G_2 = P_2 G$. The universal property of $\eta$ is then that, for every $(b_1, b_2): (B_1, B_2) \longrightarrow (FA_1, FA_2)$ there exists a unique $(a_1, a_2): (G_1(B_1, B_2), G_2(B_1, B_2)) \longrightarrow (A_1, A_2)$ such that




(∗)

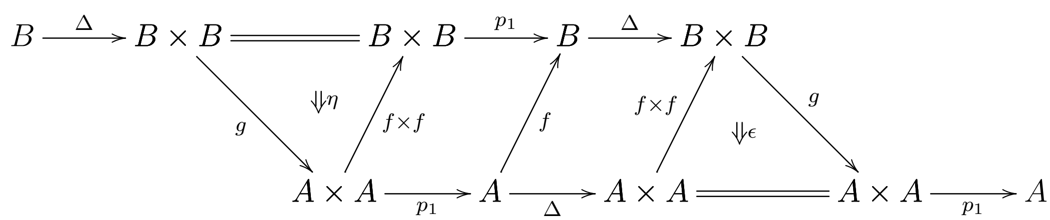
commutes. We'll show that $\eta_1(B, B): B \longrightarrow FG_1(B, B)$ is a universal arrow for $F$. Let $b: B \longrightarrow FA$. Specialize (∗) to the case $B_1 = B_2 = B$, $A_1 = A_2 = A$, $b_1 = b_2 = b$. Then we get $a_1$ and $a_2$ satisfying (∗) in that special case, and so we have




which gives existence. If we had another $a_1'$ satisfying this, then $(a_1, a_2)$ and $(a_1', a_2)$ both satisfy (∗) so $a_1 = a_1'$.

This gives a proof in the case of $\mathcal{C}at$, but a diagramatic proof as we started giving would hold in a 2-category with 2-products, a better result. Regardless of whether we are interested in this more general result, we are faced with an identity which holds in that situation, and as 2-category theorists, we should be able to prove it (a challenge for the 2-categoricians). Perhaps the above proof can be leveraged into this more general one.

To be clear, what we're trying to prove is that the composite 2-cell



is the identity on $p_1 g \Delta$.

3.3. A result for 2-monads. What we'll do is generalize and thus get a better result. Of course it could be harder to prove (it isn't) but it's also harder to make mistakes. Note that the functor $TA = A \times A$ is part of a 2-monad $\mathcal{A}$, whose unit is $\Delta \colon A \longrightarrow A \times A$ and multiplication $p_1 \times p_2 \colon (A \times A) \times (A \times A) \longrightarrow A \times A$, which looks promising .

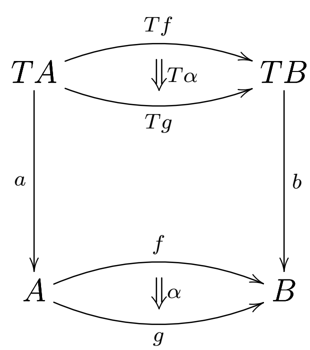
3.4. Theorem. *Let $\mathbb{T} = (T, \eta, \mu)$ be a 2-monad on a 2-category $\mathcal{A}$. If $f \colon A \longrightarrow B$ is a morphism of $\mathcal{A}$, then the following are equivalent:*

*(1) $Tf$ has a left adjoint in $\mathcal{A}$,*

*(2) $Tf$ has a left adjoint in $\mathcal{EM}(\mathbb{T})$, the Eilenberg-Moore 2-category of $\mathbb{T}$,*

*(3) $f_* = [\eta B \cdot f] \colon A \xrightarrow{\bullet} B$ has a left adjoint in $\mathcal{K}l(\mathbb{T})$, the Kleisli 2-category of $\mathbb{T}$.*

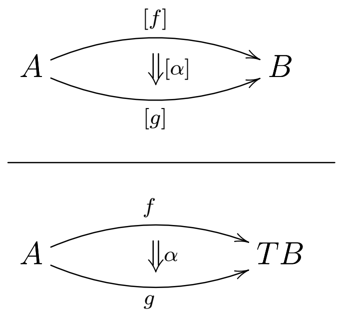
Before we give the proof, let's recall what the 2-categories $\mathcal{EM}(\mathbb{T})$ and $\mathcal{K}l(\mathbb{T})$ are (see for example [4]). We are referring to the strict versions given by the theory of monads enriched in the cartesian category **Cat**. The objects and morphisms of $\mathcal{EM}(\mathbb{T})$ are those of the underlying monad $\mathbb{T}_0$ on the category $\mathcal{A}_0$ (forget the 2-cells). So associativity and unit laws hold on the nose, and morphisms preserve the structures strictly. A 2-cell $\alpha \colon f \longrightarrow g$ between homomorphisms $f,\, g \colon (A, a) \longrightarrow (B, b)$ is a 2-cell $\alpha \colon f \longrightarrow g$ in $\mathcal{A}$ such that



commutes, i.e. $b \cdot T\alpha = \alpha a$. There's the obvious forgetful 2-functor $\mathcal{EM}(\mathbb{T}) \longrightarrow \mathcal{A}$, which has a left 2-adjoint given by

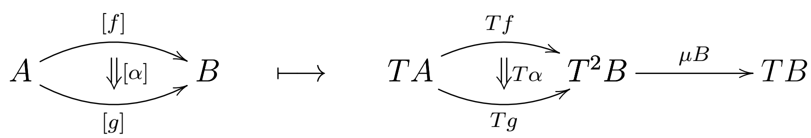
$$A \longmapsto (TA, \mu A).$$

The objects of $\mathcal{K}l(\mathbb{T})$ are those of $\mathcal{A}$ and a morphism $[f]: A \longrightarrow\!\!\!\!\!\bullet\ B$ in $\mathcal{K}l(\mathbb{T})$ is given by a morphism $f: A \longrightarrow TB$. A 2-cell $[\alpha]: [f] \longrightarrow [g]$ is given by a 2-cell $\alpha: f \longrightarrow g$ in $\mathcal{A}$



The usual embedding

$$\mathcal{K}l(\mathbb{T}) \longrightarrow \mathcal{EM}(\mathbb{T})$$



is of course full and faithful on arrows, but also on 2-cells, i.e. it is a 2-fully faithful 2-functor. That's all we need.

Proof. (Of theorem) Precisely because we have a 2-fully faithful embedding of $\mathcal{K}l(\mathbb{T}) \hookrightarrow \mathcal{EM}(\mathbb{T})$, (2) and (3) are equivalent.

We have a forgetful 2-functor, so an adjunction in $\mathcal{EM}(\mathbb{T})$ has an underlying one in $\mathcal{A}$, so (2) $\Rightarrow$ (1).

The hard part is (1) $\Rightarrow$ (3). Assume $Tf: TA \longrightarrow TB$ has a left adjoint $g: TB \longrightarrow TA$ in $\mathcal{A}$, with adjunction transformations $e: gTf \longrightarrow 1_{TA}$ and $h: 1_{TB} \longrightarrow Tf \cdot g$. We'll show that $\bar{g} = [g \cdot \eta B]$ is left adjoint to $f_* = [\eta B \cdot f] = [Tf \cdot \eta A]$.

An easy calculation will show that

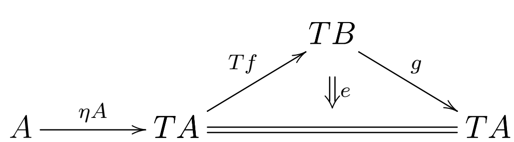
$$\bar{g} \cdot f_* = [g \cdot \eta B \cdot f] = [g \cdot Tf \cdot \eta A]$$

and

$$f_* \cdot \bar{g} = [Tf \cdot g \cdot \eta B].$$

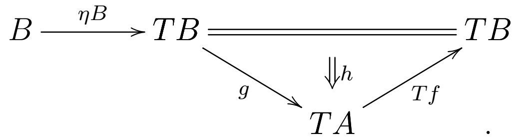
Now let $\bar{e}: \bar{g} \cdot f_* \longrightarrow \mathrm{id}_A$ be

$$[e \cdot \eta A]: [g \cdot Tf \cdot \eta A] \longrightarrow [\eta A]$$

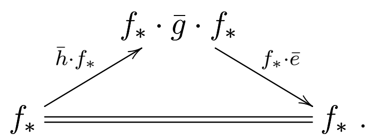


and let $\bar{h}: \mathrm{id}_B \longrightarrow f_* \cdot \bar{g}$ be

$$[h \cdot \eta B]: [\eta B] \longrightarrow [Tf \cdot g \cdot \eta B]$$




We first prove the triangle identity

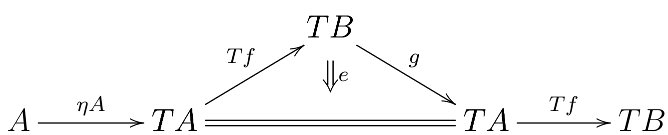



We can compute, first of all that

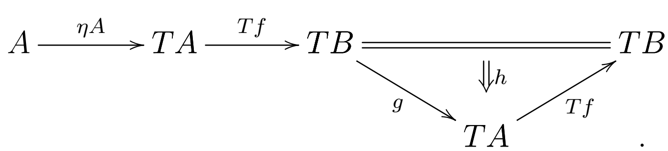
$$f_* \cdot \bar{g} \cdot f_* = [Tf \cdot g \cdot Tf \cdot \eta A],$$

and then that

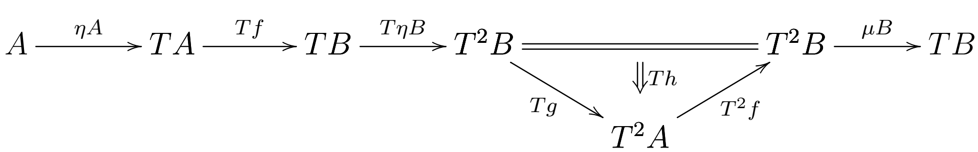
$$f_* \cdot \bar{e} = [Tf \cdot e \cdot \eta A],$$



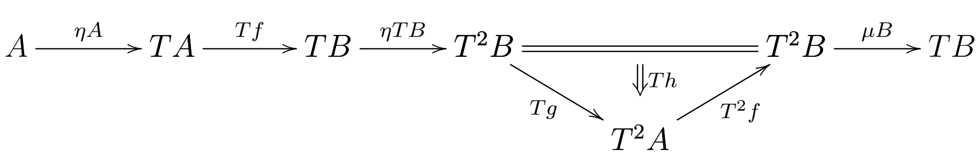

$$\bar{h} \cdot f_* = [h \cdot Tf \cdot \eta A],$$



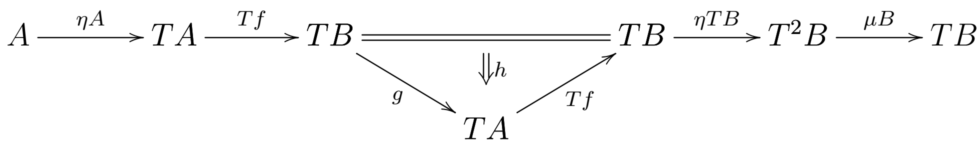

The first is straightforward but the second uses 2-naturality of $\eta$. Indeed, $\bar{h} \cdot f_*$ is, by definition, the composite



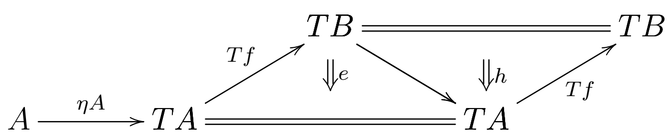

which, just using naturality of $\eta$, is equal to



and now we can use the 2-naturality of $\eta$ to rewrite it as

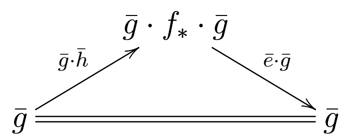


and $\mu B \cdot \eta TB = 1_{TB}$. Now the composite $(f_* \cdot \bar{e})(\bar{h} \cdot f_*)$ is given by



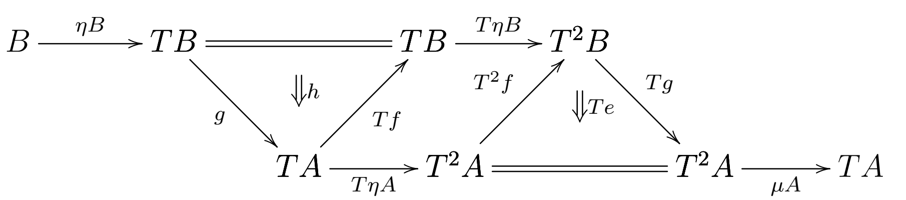
which is the identity on $Tf \cdot \eta A$ or $\mathrm{id}_{f_*}$.

The other triangle identity

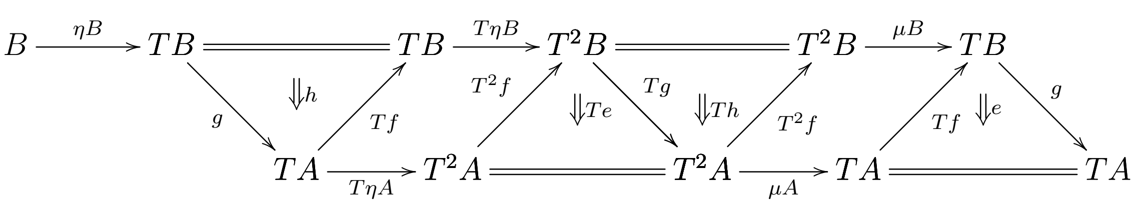


is harder as it involves mostly $\bar{g}$'s which are not as nice as the $f_*$. It's what makes the hard part of the theorem "hard", and corresponds to the identity we couldn't prove at the start of the section.

A calculation like the one we just did shows that our triangle identity comes down to showing that the following composite 2-cell is $\mathrm{id}_{g \cdot \eta B}$:

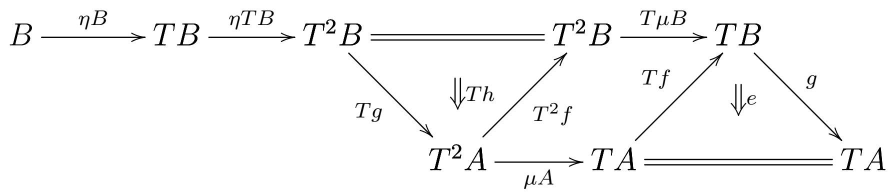


Tack a piece onto it (the two triangles and parallelogram on the right below) to get

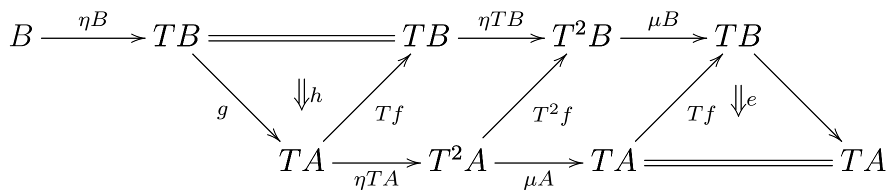


which by simple inspection we see is the identity $\mathrm{id}_{g \cdot \eta B}$. Indeed, the $Th$ and $Te$ in the middle cancel, then the $\mu$ and $T\eta$ cancel, and finally the $h$ and $e$ cancel. But the thing we tacked on (including the tail $T\eta B \cdot \eta B$) is also $\mathrm{id}_{g \cdot \eta B}$. That tail $T\eta B \cdot \eta B$ is equal to

$\eta TB \cdot \eta B$ so our add-on is




and now we can use the 2-naturality of $\eta$ to rewrite it as




and then, the parallelograms cancel leaving $e$ next to $h$ to also cancel. That completes the proof.

■

3.5. Back to the original question. So how does this relate to our original question? As mentioned above, if $\mathcal{A}$ is a 2-category with binary 2-products, the functor $T: \mathcal{A} \longrightarrow \mathcal{A}$ given by $TA = A \times A$ supports a 2-monad structure, with $\eta A = \Delta: A \longrightarrow A \times A$ and $\mu A = p_1 \times p_2: (A \times A) \times (A \times A) \longrightarrow A \times A$. In fact $\mathbb{T}$ is induced by the 2-adjunction

$$\mathcal{A} \underset{\times}{\overset{\Delta}{\rightleftarrows}} \mathcal{A} \times \mathcal{A} \, .$$

The Kleisli 2-category $\mathcal{K}l(\mathbb{T})$ is the 2-category with the same objects as $\mathcal{A}$ (of course) but with morphisms $A \longrightarrow\!\!\!\!\!\!\bullet\;\; B$, pairs $f_1, f_2: A \longrightarrow B$ of morphisms with coordinate-wise composition. The 2-cells are pairs of natural transformations, with everything coordinate-wise as well. It is the 2-full sub-2-category of $\mathcal{A} \times \mathcal{A}$ determined by the diagonal objects $(A, A)$.

An adjoint pair in $\mathcal{K}l(\mathbb{T})$, $(g_1, g_2) \dashv (f_1, f_2)$, is simply a pair of adjoints $g_1 \dashv f_1$ and $g_2 \dashv f_2$. Since $f_*$ is $(f, f)$, then the equivalence (1) $\Leftrightarrow$ (3) from Theorem 3.4 says that $f \times f: A \times A \longrightarrow B \times B$ has a left adjoint if and only if $f$ has one, which is where the whole story started.

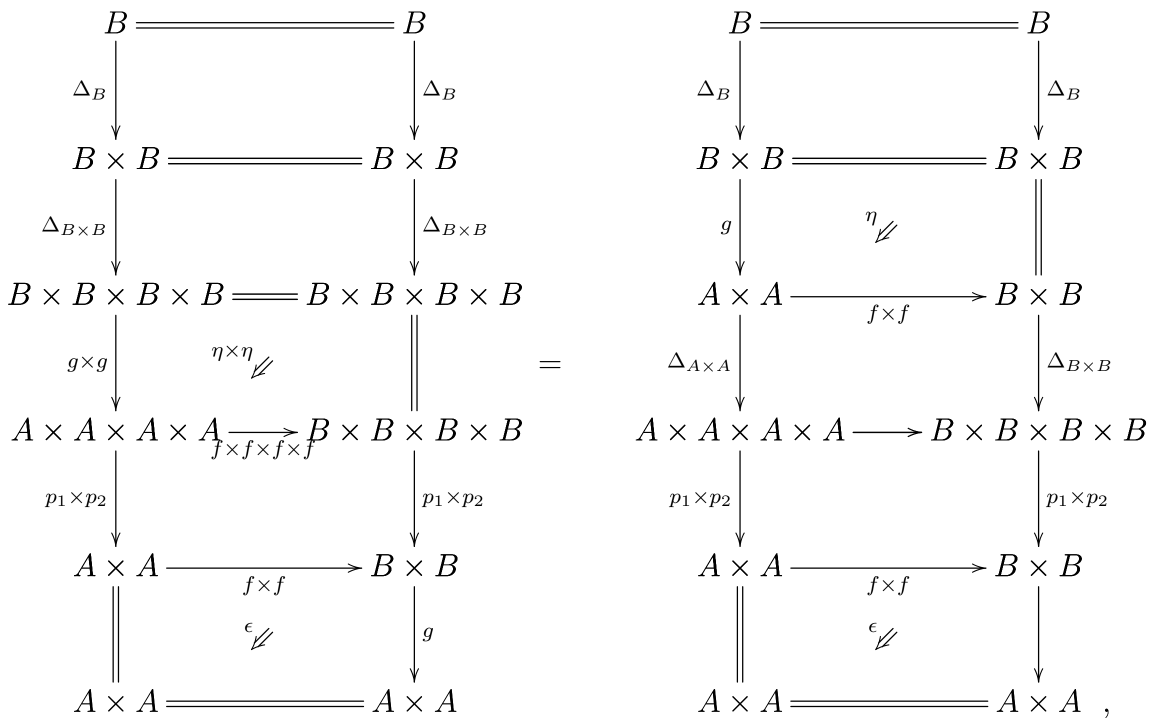
Getting back to the original question, of whether $f \times f$ having a left adjoint implies that $f$ does, we can now run the proof of Theorem 3.4, in that special case, and hope for some simplifications. I wasn't able to simplify it, but give it anyway lest the reader not believe the monad proof. We write it out in terms of quintets which makes things clearer, for me at least. Also we write them vertically, because of line-width considerations. Viewed this way, what we are trying to prove is that the composite 2-cell

$$\begin{array}{ccc}
B & = & B \\
{\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\
B \times B & = & B \times B \\
{\scriptstyle g}\downarrow & \overset{\eta}{\swarrow} & \| \\
A \times A & \xrightarrow[f\times f]{} & B \times B \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\
A & \xrightarrow{f} & B \\
{\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\
A \times A & \xrightarrow{f\times f} & B \times B \\
\| & \overset{\epsilon}{\swarrow} & \downarrow{\scriptstyle g} \\
A \times A & = & A \times A \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\
A & = & A
\end{array} \qquad (*)$$

is $\mathrm{id}_{p_1 g \Delta}$. It's clear from the proof of Theorem 3.4, and also the universal arrow proof in $\mathcal{C}at$, that we have to involve the other candidate for left adjoint, viz. $(*)$ with $p_1$ replaced by $p_2$. We use 2-products in $\mathcal{A}$ to combine them into the column on the left below. After we show that that's the identity we'll project out. Consider

$$
\begin{array}{ccccc}
B & = & B & = & B \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\
B \times B & = & B \times B & = & B \times B \\
\downarrow{\scriptstyle g} & \overset{\eta}{\Swarrow} & \| & & \| \\
A \times A & \xrightarrow[f \times f]{} & B \times B & = & B \times B \\
\downarrow{\scriptstyle \Delta \times \Delta} & & \downarrow{\scriptstyle \Delta \times \Delta} & & \downarrow{\scriptstyle \Delta \times \Delta} \\
A \times A \times A \times A & \xrightarrow{f \times f \times f \times f} & B \times B \times B \times B & = & B \times B \times B \times B \\
\| & \overset{\epsilon \times \epsilon}{\Swarrow} & \downarrow{\scriptstyle g \times g} & \overset{\eta \times \eta}{\Swarrow} & \| \\
A \times A \times A \times A & = & A \times A \times A \times A & \xrightarrow[f \times f \times f \times f]{} & B \times B \times B \times B \\
\downarrow{\scriptstyle p_1 \times p_2} & & \downarrow{\scriptstyle p_1 \times p_2} & & \downarrow{\scriptstyle p_1 \times p_2} \\
A \times A & = & A \times A & \xrightarrow{f \times f} & B \times B \\
\| & & \| & \overset{\epsilon}{\Swarrow} & \downarrow{\scriptstyle g} \\
A \times A & = & A \times A & = & A \times A \quad .
\end{array}
\qquad (**)
$$

If we first compose horizontally, then the $\epsilon \times \epsilon$ and $\eta \times \eta$ cancel, then $p_1 \times p_2$ and $\Delta \times \Delta$ cancel, leaving $\eta$ and $\epsilon$ to cancel too. So the whole composite is the identity. But the right column is also an identity. Indeed, $\Delta_B \times \Delta_B \cdot \Delta_B$ is equal to $\Delta_{B\times B} \cdot \Delta_B$ and



and then the $p_1 \times p_2$ cancel with $\Delta$ and $\epsilon$ with $\eta$.

From this we get that the left column of $(**)$ is an identity, and if we post compose it by $p$, we get $(*)$.

3.6. A PROBLEM. This is related to our joint paper (Paré, Rosebrugh, Wood) [7] on idempotents in bicategories, where we answer in the affirmative the following question of Lawvere's (posed orally): Is the category of fixed points of a left exact idempotent functor on a topos again a topos?

We now might ask the following, more elementary, question.

**Question:** Does taking fixed points preserve adjointness?

To be precise, let $\mathbf{A}$ and $\mathbf{B}$ be categories, $E$ and $F$ idempotent functors on $\mathbf{A}$ and $\mathbf{B}$ respectively and let $\Phi\colon \mathbf{A} \longrightarrow \mathbf{B}$ be such that $\Phi E = F\Phi$. Then $\Phi$ restricts to a functor

$$\mathbf{Fix}(\Phi)\colon \mathbf{Fix}(E) \longrightarrow \mathbf{Fix}(F).$$

If $\Phi$ has a left adjoint, does $\mathbf{Fix}(\Phi)$ also have a left adjoint?

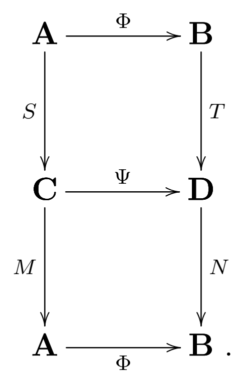
At first glance, it doesn't seem likely, but the splitting of idempotents is a lot better than one might imagine.

To simplify things, let's introduce some more notation. Let $\mathbf{Fix}(E)$, $\mathbf{Fix}(F)$, $\mathbf{Fix}(\Phi)$ be $\mathbf{C}$, $\mathbf{D}$ and $\Psi$ respectively. Split $E$ and $F$ as

$$E = MS \quad , \; SM = 1_{\mathbf{C}}$$

$$F = NT \quad , \; TN = 1_{\mathbf{D}}$$

so we have a commutative diagram



Now assume $\Phi$ has a left adjoint $\Lambda$ with unit $\eta\colon 1_{\mathbf{B}} \longrightarrow \Phi\Lambda$ and counit $\epsilon\colon \Lambda\Phi \longrightarrow 1_{\mathbf{A}}$.

Based on what we've done above (note that $\Delta P_1$ was an idempotent), let's define

$$\Theta\colon \mathbf{D} \longrightarrow \mathbf{C}$$

to be the composite

$$\mathbf{D} \xrightarrow{N} \mathbf{B} \xrightarrow{\Lambda} \mathbf{A} \xrightarrow{S} \mathbf{C}$$

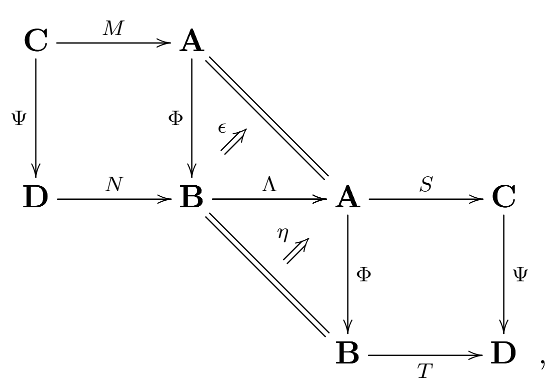
and natural transformations $h\colon 1_{\mathbf{D}} \longrightarrow \Phi\Theta$

$$h = (1_{\mathbf{D}} = TN \xrightarrow{T\eta N} T\Phi\Lambda N = \Psi S\Lambda N = \Psi\Theta)$$

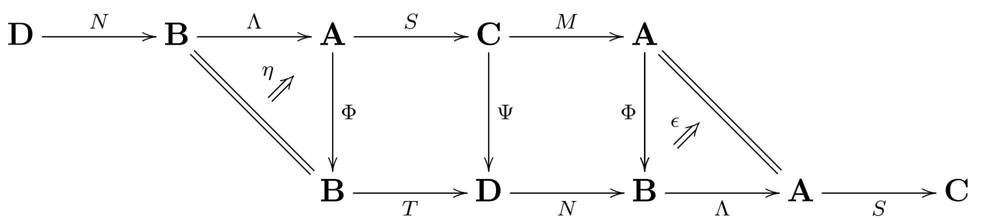
and $e\colon \Theta\Phi \longrightarrow 1_C$

$$e = (\Theta\Psi = S\Lambda N\Psi = S\Lambda\Phi M \xrightarrow{S\epsilon M} SM = 1_{\mathbf{C}}).$$

Then we have $\Psi e \cdot h\Psi = \mathrm{id}_{\Psi}$:



where the parallelogram is the identity because $\Lambda \dashv \Phi$. The other triangle identity, $e\Theta \cdot \Theta h = \mathrm{id}_\Theta$,




I have no idea how to deal with this.

However, using the same result as before ([8] or [6], p. 84), $e\Theta \cdot \Theta h$ is an idempotent $\Theta \longrightarrow \Theta$ and if it splits, the splitting will give a left adjoint to $\Psi$, so $\Psi$ almost has a left adjoint. If, e.g. $\mathbf{C}$ has split idempotents, a rather weak condition, then the answer to our question is "yes".

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