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A NOTE ON THE CATEGORICAL NOTIONS OF NORMAL SUBOBJECT AND OF EQUIVALENCE CLASS

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A tribute to Bob Rosebrugh

Nothing makes the earth seem so spacious as to have friends at a distance. They make the latitudes and longitudes.

H. D. Thoreau

Abstract. In a non-pointed category $\mathbb{E}$, a subobject which is normal to an equivalence relation is not necessarily an equivalence class. We elaborate this categorical distinction, with a special attention to the Mal'tsev context. Moreover, we introduce the notion of *fibrant equipment*, and we use it to establish some conditions ensuring the uniqueness of an equivalence relation to which a given subobject is normal, and to give a description of such a relation.

Introduction

It is unnecessary to insist on the importance in Algebra of the notion of normal subgroup. And it is well known that any normal subgroup $U \rightarrowtail X$ is actually the equivalence class of the identity element with respect to the equivalence relation R_U on the group X defined by: $xR_Uy \iff x.y^{-1} \in U$, that this equivalence relation is internal to the category Gp of groups (namely, a congruence), and that, moreover, it is the unique congruence R on X such that $\bar{1}^R = U$.

Categorically speaking, this can be expressed in the category Gp by the following two conditions:
1) the equivalence relation on U obtained by restriction of R_U coincides with the indiscrete

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equivalence relation ∇_U, which means that U is included in the equivalence class $\bar{1}$;
2) the induced morphism of equivalence relations:

$$\begin{tikzcd}
U \times U \arrow[r, tail, "\tilde{u}"] \arrow[d, shift right=2, "p_0^U"'] \arrow[d, shift left=2, "p_1^U"] & R_U \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
U \arrow[r, tail, "u"'] \arrow[u, "s_0^U"] & X \arrow[u, "s_0"]
\end{tikzcd}$$

is a fibrant morphism between equivalence relations (i.e. the downward square indexed by 0 is a pullback, which implies that any commutative square in the previous diagram is a pullback), which means that U coincides with $\bar{1}$.

This is the way how, in [2], the categorical notion of a subobject u normal to an equivalence relation R is defined in any category $\mathbb{E}$.

It was already clear that, in the category Set of sets, the empty set $\emptyset$ is then normal to any equivalence relation R, and that this categorical notion of normal subobject characterized an equivalence class of the equivalence relation R in Set if and only if we have moreover $U \neq \emptyset$.

So, let us introduce now the following precision:

0.1. Definition. *We say that a morphism $f : X \to Y$ in a category $\mathbb{E}$ is a plain morphism when there is some commutative diagram:*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\
& W &
\end{tikzcd}$$

where g is an extremal epimorphism.

So, any strong epimorphism and any split monomorphism is a plain morphism. When the ground category $\mathbb{E}$ is regular [1], saying that f is a plain morphism is equivalent to saying that the objects X and Y have same supports, namely that the canonical decompositions of their terminal maps factorize through the same subobject J of the terminal object 1:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\
& J \arrow[r, tail] & 1
\end{tikzcd}$$

which, in the case of the category Set, exactly means: $Y \neq \emptyset \Rightarrow X \neq \emptyset$. Therefore, in a regular category $\mathbb{E}$, plain morphisms are stable under pullbacks along regular epimorphirms.

In order to better understand the notion of plain map, let us briefly present another example. We just noticed that in Set, when $Y \neq \emptyset$, there is only one morphism with codomain Y which is not plain, namely the initial map $\alpha_Y : \emptyset \rightarrowtail Y$. In the topos Set^2 of ordered pairs of sets, when $Y \neq \emptyset$ and $Y' \neq \emptyset$, there are infinitely many, namely any (α_Y, f') and any $(f, \alpha_{Y'})$, provided the codomain of f is Y and the codomain of f' is Y'.

Clearly, when the ground category $\mathbb{E}$ is pointed, any morphism is a plain one.

0.2. Definition. *Given any subobject $u : U \rightarrowtail X$ and any equivalence relation R on X in a category $\mathbb{E}$, we say that u is an* equivalence class *of R when:*
1) u is normal to R;
2) u is plain monomorphism.

So, any point $\gamma : 1 \to X$ is an equivalence class of Δ_X, the discrete equivalence relation on X (i.e. the smallest one), and $1_X : X \to X$ the only equivalence class of ∇_X the undiscrete equivalence relation on X (i.e. the largest one). Recall that, when the ground category $\mathbb{E}$ is protomodular [3], a monomorphism u is normal to at most one equivalence relation R, see [2], and then, in this context, the fact to be a normal subobject or an equivalence class becomes a property.

Inspired by some recent results of [23] and [10], the aim of this work is to investigate the properties of the normal subobjects and of the equivalence classes in a non-protomodular context, namely when the uniqueness of the equivalence relation R is no longer ensured. A special attention will be given to the Mal'tsev context [16] [17] which is strictly laxer than the protomodular one, see [3]. We shall be specially interested in specifying further conditions implying the uniqueness of R and in understanding how they disappear in the protomodular case. This will lead, among other things, to an explicit description of the (unique) equivalence relation to which a split equivalence class is normal in the protomodular context (see Proposition 5.2). In the pointed case, some aspects of the relationship between split equivalence class and direct product decompositions will be involved, as already observed in [13], but with another kind of tools. On the way, we shall produce a remarkable situation: a monad (T, λ, μ) such that the canonical comparison functor $Kl^T \to AlgT$ between the Kleisli category of this monad [22] and its category of algebras is an equivalence of categories, see Section 1.18.

The article is organized along the following lines:
Section 1 is devoted to notations and to basic properties of normal subobjects and equivalence classes. We investigate three emblematic situations, the first and the second one being very elementary. The third one is based on the introduction of the notion of *fibrant equipment* of a map $f : X \to Y$ which makes explicit many hidden aspects (see Section 1.20) of the notion of connected pairs of equivalence relations in the Mal'tsev context, as defined in [11].
Section 2 is devoted to the unital context and briefly describes a first example of a further condition ensuring the uniqueness of the equivalence relation associated with an equivalence class.
Sections 3 and 4 give other examples of such conditions in the Mal'tsev and the pointed Mal'tsev contexts.
Section 5 is devoted to the same question in the protomodular context, and in a context

strictly located between the Mal'tsev and protomodular ones, namely that of categories which are protomodular on the only class of split monomorphisms [9].
Section 6 is devoted to the naturally Mal'tsev context and to the related non-pointed additive ones.
Section 7 deals with the case when monomorphisms in the base category admit cocartesian liftings along the underlying-object functor $(\ )_0 \colon \mathrm{Equ}\mathbb{E} \to \mathbb{E}$. This feature is used to characterize those monomorphisms (called *normalizing*) for which there exists an equivalence relation to which they are normal.

1. Notations and elementary properties

1.1. Notations. Any category $\mathbb{E}$ will be supposed finitely complete. Recall that an internal relation S on an object X is a subobject of the product $X \times X$:

$$S \xrightarrow{(d_0^S, d_1^S)} X \times X$$

We shall often use the simplicial notation for describing internal equivalence relations:

$$\begin{tikzcd} S \arrow[r, shift left=2, "d_1^S"] \arrow[r, shift right=2, "d_0^S"'] & X \arrow[l, "s_0^S" description] \end{tikzcd}$$

We denote by $\mathrm{Equ}\mathbb{E}$ the category of internal equivalence relations in $\mathbb{E}$ whose morphisms are given by the pairs $(f, \bar{f})$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "\bar{f}"] \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] & R \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] \\ X \arrow[u] \arrow[r, "f"'] & Y \arrow[u] \end{tikzcd}$$

We denote by $(\ )_0 : \mathrm{Equ}\mathbb{E} \to \mathbb{E}$ the forgetful functor associating with any equivalence relation its ground object; since $\mathbb{E}$ has finite limits, this functor $(\ )_0$ is a fibration whose cartesian maps are given by the inverse images along the ground morphisms in $\mathbb{E}$. Each fibre of this fibration is a preorder, which is equivalent to the fact that the functor $(\ )_0$ is faithful. It is clear that any internal equivalence relation S on an object X determines an equivalence relation on any hom-set $Hom_{\mathbb{E}}(Z, X)$ defined by kSh if and only if the pair (k, h) factorizes through the subobject S:

$$\begin{tikzcd} Z \arrow[rr, dotted] \arrow[dr, "{(k,h)}"'] & & S \arrow[dl, tail, "{(d_0^S, d_1^S)}"] \\ & X \times X & \end{tikzcd}$$

We denote by Δ_X the discrete equivalence on the object X (i.e. the smallest one) and by ∇_X the undiscrete equivalence relation (i.e. the largest one). The *kernel equivalence relation* $R[f]$ of a map $f : X \to Y$ is then the domain of the cartesian map above f with codomain Δ_Y, it is given by the following pullback in Equ$\mathbb{E}$:

$$\begin{tikzcd} R[f] \arrow[r, "\tilde{f}^Y"] \arrow[d, tail, "(d_0^f, d_1^f)"'] & \Delta_Y \arrow[d, tail, "(1_Y, 1_Y)"] \\ \nabla_X \arrow[r, "\nabla_f"'] & \nabla_Y \end{tikzcd}$$

An equivalence relation R on X is said to be *effective* when there is some map $f : X \to Y$ such that $R = R[f]$.

Let us recall that fibrant morphisms and cartesian maps are stable under pullback along any morphism in Equ$\mathbb{E}$.

For two equivalence relations R and S on a common object X, let us consider the pullback square:

$$\begin{tikzcd} R \times_X S \arrow[r, "p_1"] \arrow[d, "p_0"'] & S \arrow[d, "d_0^S"] \\ R \arrow[r, "d_1^R"'] & X \end{tikzcd}$$

A connector [11] between R and S is a map $p\colon R \times_X S \to X$ internally satisfying the following identities

$$1) : xSp(x, y, z) \qquad 2) : p(x, x, y) = y \qquad 3) : p(x, y, p(y, u, v)) = p(x, u, v)$$

$$1') : p(x, y, z)Rz \qquad 2') : p(x, y, y) = x \qquad 3') : p(p(x, y, u), u, v) = p(x, y, v)$$

In presence of a connector, the pullback above determines a *cartesian double relation* on the pair (R, S), i.e. a double relation such that all the commuting squares in the following diagram are pullbacks:

$$\begin{tikzcd}[row sep=large, column sep=large] R \times_X S \arrow[r, shift left=2, "p_1"] \arrow[r, leftarrow] \arrow[r, shift right=2, "(d_0^R.p_0, p)"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, leftarrow] \arrow[d, shift left=2, "(p, d_1^S.p_1)"] & S \arrow[d, shift right=2, "d_0^S"'] \arrow[d, leftarrow] \arrow[d, shift left=2, "d_1^S"] \\ R \arrow[r, shift left=2, "d_1^R"] \arrow[r, leftarrow] \arrow[r, shift right=2, "d_0^R"'] & X \end{tikzcd}$$

We say that the connector p presents R and S as a connected pair of equivalence relations and we write $[R, S]_p = 0$. Of course, one has $[R, S]_p = 0 \iff [S, R]_{p^{op}} = 0$, where $p^{op}\colon S \times_X R \to X$ is defined by $p^{op}(x, y, z) = p(z, y, x)$. Recall also the following straigtforward:

1.2. Lemma. *Let $(f, \bar{f}) : S \to R$ be any fibrant morphism between equivalence relations. Then:*
1) we have $R[f] \cap S = \Delta_X$,
2) the left hand part of following diagram produces a cartesian double equivalence relation:

$$\begin{tikzcd}[column sep=large, row sep=large]
R[\bar{f}\,] \arrow[r, shift left=2, "d_1^{\bar{f}}"] \arrow[r, shift right=2, "d_0^{\bar{f}}"'] \arrow[d, shift right=2, "R(d_0^S)"'] \arrow[d, shift left=2, "R(d_1^S)"] & S \arrow[l] \arrow[r, "\bar{f}"] \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] & R \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] \\
R[f] \arrow[u] \arrow[r, shift left=2, "d_1^f"] \arrow[r, shift right=2, "d_0^f"'] & X \arrow[u] \arrow[l] \arrow[r, "f"'] & Y \arrow[u]
\end{tikzcd}$$

where the three maps between the kernel pairs $R[f]$ and $R[\bar{f}\,]$ are those induced by universality. Accordingly, the map $p = d_1^S.d_0^{\bar{f}}$ produces a connector for the pair $(R[f], S)$ and we get $[R[f], S]_p = 0$.

Internal Mal'tsev operations, internal group structures

Given any object X, recall that an internal Mal'tsev operation on X is an internal ternary operation $p : X \times X \times X \to X$ satisfying 2) and 2'). It is right (resp. left) associative when it satisfies 3) (resp. 3'), and it is associative when both 3) and 3') hold. Accordingly we get $[\nabla_X, \nabla_X]_p = 0$ if and only if the object X is endowed with an associative Mal'tsev operation p. An internal group structure on X is just a pointed (by a point $0 : 1 \to X$) associative Mal'tsev operation p on X. Denote by R_p the relation on $X \times X$ in $\mathbb{E}$ defined by:

$$(x, y)R_p(t, z) \iff t = p(x, y, z)$$

It is called the Chasles relation associated with p. Then R_p is reflexive as soon as p is a Mal'tsev operation, and an equivalence relation as soon as p is moreover left associative. In this last case the diagonal $s_0 : X \rightarrowtail X \times X$ is an equivalence class of the Chasles relation R_p.

1.3. Elementary properties. The first very elementary observation about the normal subobjects and the equivalence classes is the following:

1.4. Lemma. *Suppose the subobject $j : U \rightarrowtail X$ normal to the equivalence relation R. Given any equivalence relation S on X such that $j^{-1}(S) = \nabla_U$ and $S \subset R$, the monomorphism j is normal to S as well. So, given any S such that $j^{-1}(S) = \nabla_U$, the monomorphism j is normal to $R \cap S$. In particular, if j is normal to R and S, it is normal to $R \cap S$. The same observations hold when j is an equivalence class.*

Proof. The conditions on S produce the following diagram:

$$\begin{tikzcd}
U \times U \arrow[r, tail, "\tilde{j}'"'] \arrow[rr, tail, bend left, "\tilde{j}"] \arrow[d, shift right=3, "p_0^U"'] \arrow[d, shift left=3, "p_1^U"] & S \arrow[r, tail, "\iota"'] \arrow[d, shift right=3, "d_0"'] \arrow[d, shift left=3, "d_1"] & R \arrow[d, shift right=3, "d_0"'] \arrow[d, shift left=3, "d_1"] \\
U \arrow[r, tail, "j"'] \arrow[u, "s_0^U" description] & X \arrow[r, equal] \arrow[u, "s_0" description] & X \arrow[u, "s_0" description]
\end{tikzcd}$$

Since j is normal to R, the commutative rectangles are pullbacks. Now, since ι is a monomorphism, so are the commutative left hand side squares. Accordingly j is normal to S. ∎

1.5. Proposition. *Given any map $f : X \to Y$ in $\mathbb{E}$, if the monomorphism $k : V \rightarrowtail Y$ is normal to the equivalence relation S, then $f^{-1}(k) : U \rightarrowtail X$ is normal to the equivalence relation $f^{-1}(S)$:*

$$\begin{tikzcd}
 & f^{-1}(S) \arrow[r] \arrow[d, shift right=3, "d_0"'] \arrow[d, shift left=3, "d_1"] & S \arrow[d, shift right=3, "d_0"'] \arrow[d, shift left=3, "d_1"] \\
 & X \arrow[r, "f"'] \arrow[u, "s_0" description] & Y \arrow[u, "s_0" description] \\
U \arrow[ur, tail, "f^{-1}(k)"] \arrow[r] & V \arrow[ur, tail, "k"'] &
\end{tikzcd}$$

In a regular category $\mathbb{E}$, the equivalence classes are stable under pullbacks along regular epimorphisms.

Proof. Consider the following right hand side pullback in $\mathrm{Equ}\mathbb{E}$ which produces the inverse image of S:

$$\begin{tikzcd}
\nabla_U \arrow[r, tail, "(f^{-1}(k), \tilde{f}^{-1}(\tilde{k}))"'] \arrow[rr, tail, bend left, "\nabla_{f^{-1}(k)}"] \arrow[d] & f^{-1}(S) \arrow[r, tail] \arrow[d, "(f, \tilde{f})"'] & \nabla_X \arrow[d, "\nabla_f"] \\
\nabla_V \arrow[r, tail, "(k, \tilde{k})"] \arrow[rr, tail, bend right, "\nabla_k"'] & S \arrow[r, tail] & \nabla_Y
\end{tikzcd}$$

The whole rectangle is a pullback as well, so that the lower factorization $(k, \tilde{k})$ produces the upper one, and makes the left hand side square a pullback. Therefore, since $(k, \tilde{k})$ is fibrant, so is $(f^{-1}(k), \tilde{f}^{-1}(\tilde{k}))$, which means that $f^{-1}(k) : U \rightarrowtail X$ is normal to $f^{-1}(S)$. Finally, we already noticed that in a regular category, the plain monomorphims are stable under pullbacks along regular epimorphisms. ∎

Whence the following classical observations:

1.6. Corollary. *Given any morphism $f : X \to Y$ in $\mathbb{E}$ and any point $\gamma : 1 \rightarrowtail Y$ of the object Y, the inclusion $w : f^{-1}(\gamma) \rightarrowtail X$ of the fibre of f above the point γ is normal to the kernel equivalence relation $R[f]$. Accordingly:*
1) in any regular category $\mathbb{E}$, when f is a regular epimorphism, the inclusion $w : f^{-1}(\gamma) \rightarrowtail X$ is an equivalence class of $R[f]$
2) in any pointed category $\mathbb{E}$, the kernel $k_f : \mathrm{Ker} f \rightarrowtail X$ of any map $f : X \to Y$ is the only equivalence class of its kernel equivalence relation $R[f]$.

Proof. Apply the previous proposition to following diagram:

$$\begin{tikzcd}
f^{-1}(\gamma) \arrow[r, tail, "w"] \arrow[d, "\tau"'] & X \arrow[d, "f"] \\
1 \arrow[r, tail, "\gamma"'] & Y
\end{tikzcd}$$

Since γ is a normal subobject of Δ_Y, the subobject w is a normal subobject of $f^{-1}(\Delta_Y) = R[f]$. Whence the first assertion and the point 1).

Suppose now $\mathbb{E}$ pointed, and denote by $0_Y : 1 \rightarrowtail Y$ the initial map associated with Y. Then $\mathrm{Ker} f = f^{-1}(0_Y)$ is the equivalence class of 0_X with respect to $R[f]$. If $j : U \rightarrowtail X$ is normal to $R[f]$, then consider the following diagram:

$$\begin{tikzcd}
U \arrow[r, tail, "0_U \times 1_U"'] \arrow[d] \arrow[rrr, tail, bend left, "j"] & U \times U \arrow[r, tail, "\tilde{j}"'] \arrow[d, shift right=2, "p_0^U"'] \arrow[d, dotted, shift left=2, "p_1^U"] & R[f] \arrow[r, "d_1^f"'] \arrow[d, shift right=2, "d_0^f"'] \arrow[d, dotted, shift left=2, "d_1^f"] & X \arrow[d, "f"] \\
1 \arrow[r, tail, "0_U"'] & U \arrow[r, tail, "j"'] \arrow[u, "s_0^U"'] & X \arrow[r, "f"'] \arrow[u, "s_0^f"'] & Y
\end{tikzcd}$$

Each square indexed by 0 being a pullback, the whole rectangle is a pullback and j is a kernel of f. ∎

1.7. Three emblematic situations. **1)** For an equivalence relation R on an object X and a point $\gamma : 1 \rightarrowtail X$ of X, we consider the subobject of X defined by the upper arrow obtained by the following left hand side pullback:

$$\begin{tikzcd}
{[\gamma]_R} \arrow[r, tail, "w"'] \arrow[d] \arrow[rr, tail, bend left, "\bar{\gamma}^R"] & R \arrow[r, "d_1^R"'] \arrow[d, "d_0^R"] & X \\
1 \arrow[r, tail, "\gamma"'] & X &
\end{tikzcd}$$

1.8. Lemma. *Given any equivalence relation R on X and any point $\gamma : 1 \rightarrowtail X$ of the object X in $\mathbb{E}$, the subobject $\bar{\gamma}^R : [\gamma]_R \rightarrowtail X$ is an equivalence class of R, called* the *equivalence class of γ modulo R.*

Proof. Since d_0^R is a split epimorphism, so is the terminal map of $[\gamma]_R$, which ensures that $\bar{\gamma}^R$ is a plain monomorphism. Let us show that it is normal to R. To this end, let us consider the following diagram:

$$\begin{tikzcd}
{[\gamma]_R \times [\gamma]_R} \arrow[r, tail, "R(w)"] \arrow[d, shift left=3, "p_1"] \arrow[d, shift right=3, "p_0"'] & R[d_0^R] \arrow[r, "d_2^{d_0}"] \arrow[d, shift right=3, "d_0^{d_0}"'] \arrow[d, shift left=3, "d_1^{d_0}"] & R \arrow[d, shift right=3, "d_0^R"'] \arrow[d, shift left=3, "d_1^R"] \\
{[\gamma]_R} \arrow[u, "s_0"] \arrow[r, tail, "w"'] \arrow[d, "\tau"'] & R \arrow[u, "s_0^{d_0}"] \arrow[r, "d_1^R"'] \arrow[d, "d_0^R"] & X \arrow[u, "s_0^R"] \\
1 \arrow[r, tail, "\gamma"'] & X &
\end{tikzcd}$$

where $d_2^{d_0}(xRy, xRz) = yRz$. The morphism of equivalence relations on the left is fibrant by Corollary 1.6, the one on the right is fibrant by construction, the pasting of the two makes $d_1^R.w = \bar{\gamma}^R$ normal to R. ■

When $\mathbb{C}$ is pointed, the construction above defines the *normalization* functor:

$$\begin{tikzcd}
\mathrm{Equ}\mathbb{E} \arrow[r, "\mathrm{nor}"] \arrow[d, "(\ )_0"'] & \mathrm{Mon}\mathbb{E} \arrow[d, "cod"] \\
\mathbb{E} \arrow[r, equal] & \mathbb{E}
\end{tikzcd}$$

Where Mon$\mathbb{E}$ is the category whose objects are the monomorphisms and whose maps are the commutative squares between them. It is defined by nor$R = \bar{\alpha}_X^R$, where $\alpha_X : 1 \rightarrowtail X$ is the initial map. The normalization functor is left exact and respects the cartesian maps of the two vertical fibrations. One easily checks that, if the monomorphism $n : U \rightarrowtail X$ is normal to an equivalence relation R on X, then n is isomorphic to nor$R$: Nor$R \rightarrowtail X$. More generally we get:

1.9. Lemma. *Let $\mathbb{E}$ be any pointed category and R any equivalence relation on X. Given any monomorphism $u : U \rightarrowtail X$ such that $u^{-1}(R) = \nabla_U$, we get $U \subset \mathrm{Nor}R$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
U \arrow[r, tail, "0_U \times 1_U"] \arrow[d, tail, dotted, "\theta_u"] \arrow[dd, bend right=60] & U \times U \arrow[r, "p_1^U"] \arrow[d, "\tilde{u}"'] & U \arrow[d, tail, "u"] \\
\mathrm{Nor}R \arrow[r, tail, "w"'] \arrow[d] & R \arrow[r, "d_1^R"'] \arrow[d, "d_0^R"] & X \\
1 \arrow[r, tail, "\alpha_X"'] & X &
\end{tikzcd}$$

The lower square and the left hand side vertical rectangle being pullbacks, there is a factorization θ_u making the upper left hand side square a pullback. Moreover the upper horizontal composition is 1_U, while the middle one is nor$R$. ■

2) Consider now a split epimorphism $(f,s): X \rightleftarrows Y$:

1.10. Lemma. [13] *Let (f,s) be a split epimorphism in a category $\mathbb{E}$ and S an equivalence relation on X such that $S \cap R[f] = \Delta_X$. Then the monomorphism s is an equivalence class of S if and only if we have $s^{-1}(S) = \nabla_Y$.*

Proof. The monomorphism s, being split, is an equivalence class of S if and only if it is normal to S, and in this case we trivially get $s^{-1}(S) = \nabla_Y$. Any assumption being of left exact nature, by the Yoneda embedding, it is enough to check the converse in *Set*. Let us show that the following square of split epimorphisms (where the factorization $\tilde{s}$ characterizes the equality $s^{-1}(S) = \nabla_Y$):

$$\begin{tikzcd}
Y \times Y \arrow[r, tail, "\tilde{s}"] \arrow[d, shift right, "p_0^Y"'] & S \arrow[d, shift right, "d_0^S"'] \\
Y \arrow[r, tail, "s"'] \arrow[u, shift right, "s_0^Y"'] & X \arrow[u, shift right, "s_0^S"']
\end{tikzcd}$$

is a pullback. Suppose $s(y)Sx$. By $s^{-1}(S) = \nabla_Y$, we have $s(y)Ssf(x)$ as well; whence $xSsf(x)$. But we have also $xR[f]sf(x)$, whence $x = sf(x)$ by $S \cap R[f] = \Delta_X$. ∎

3) Let us now introduce the third point:

1.11. Definition. *Let $f: X \to Y$ be any morphism in a category $\mathbb{E}$. We say that a map $\psi: X \times Y \to X$ is a fibrant equipment for f, when we have:*
(1) $\psi.(1_X, f) = 1_X$, (2) $f.\psi = p_1^Y : X \times Y \to Y$ and (3) $\psi.(\psi \times Y) = \psi.(X \times p_1^Y)$. If, in addition, f is split by s, we say that it is a fibrant equipment for the split epimorphism (f,s), when moreover we have (4) $\psi.(s \times Y) = s.p_1^Y$.

Set-theoretically: (1) $\psi(x, f(x)) = x$, (2) $f(\psi(x,y)) = y$, (3) $\psi(\psi(x,y),y') = \psi(x,y')$ and (4) is $\psi(s(y),y') = s(y')$. The previous terminology is justified by the following proposition which makes explicit the diagrammatic representations of the axioms:

1.12. Proposition. *Axiom (1) makes the vertical left hand side part of the following diagram a reflexive graph:*

$$\begin{tikzcd}
X \times Y \arrow[r, "f \times Y"] \arrow[d, shift right=2, "p_0^X"'] \arrow[d, shift left=2, dotted, "\psi"] & Y \times Y \arrow[d, shift right=2, "p_0^Y"'] \arrow[d, shift left=2, "p_1^Y"] \\
X \arrow[r, "f"'] \arrow[u, "{(1,f)}"'] & Y \arrow[u, "s_0^Y"']
\end{tikzcd}$$

Axiom (2) makes it a relation Σ_ψ on X which is reflexive and such that we have $\Sigma_\psi \cap R[f] = \Delta_X$ as soon as we have (1). In this case, the hereabove diagram becomes a morphism of reflexive relations. Then Σ_ψ is an equivalence relation on X if and only if we have (3), which makes our diagram a fibrant morphism between equivalence relations. Accordingly we have $[R[f], \Sigma_\psi]_{p_\psi} = 0$, with $p_\psi = \psi.(d_0^f \times Y)$.

Proof. We have (2) if and only if the following diagram commute:

$$\begin{tikzcd}
X \times Y \arrow[rr, "(p_0^X, \psi)"] \arrow[dr, equal, "(p_0^X, p_1^Y)"'] & & X \times X \arrow[dl, "X \times f"] \\
& X \times Y &
\end{tikzcd}$$

So, (2) implies that (p_0^X, ψ) is jointly monomorphic and produces a relation. This relation Σ_ψ can be made explicit in set-theoretical terms by $x \Sigma_\psi x'$ if and only if $\psi(x, f(x')) = x'$ from which it is clear that $\Sigma_\psi \cap R[f] = \Delta_X$, by $x' = \psi(x, f(x')) = \psi(x, f(x)) = x$ when (1) holds. A reflexive relation S is an equivalence relation if and only if xSx' and xSx" imply $x'Sx$", which, when it is translated here, from $x' = \psi(x, y')$ and $x" = \psi(x, y")$ gives us $\psi(\psi(x, y'), y") = \psi(x, y")$, namely (3). Then the hereabove morphism of equivalence relations is fibrant and the last assertion is a straightforward consequence of Lemma 1.2. ■

The relation Σ_ψ will be called the *Chasles relation* induced by the fibrant equipment ψ, since, as we shall see in Corollary 1.22, any left associative Mal'sev operation on an object X is nothing but a fibrant equipment ψ for the projection $p_1^X : X \times X \to X$ and since, in this case, the equivalence relation Σ_ψ coincides with the Chasles equivalence relation R_p of the end of Section 1.1. Here is our third emblematic situation:

1.13. Proposition. *Let ψ be a fibrant equipment for f. When moreover f is split by a monomorphism $s : Y \rightarrowtail X$, this monomorphism s is an equivalence class of the Chasles equivalence relation Σ_ψ if and only if we have (4) $\psi.(s \times Y) = s.p_1^Y$, namely ψ is a fibrant equipment for (f, s).*

Proof. When f is split by s, consider the following diagram:

$$\begin{tikzcd}
Y \times Y \arrow[r, tail, "s \times Y"] \arrow[d, shift right=3, "p_0^Y"'] \arrow[d, shift left=3, "p_1^Y"] & X \times Y \arrow[r, "f \times Y"] \arrow[d, shift right=3, "p_0^X"'] \arrow[d, shift left=3, dotted, "\psi"] & Y \times Y \arrow[d, shift right=3, "p_0^Y"'] \arrow[d, shift left=3, "p_1^Y"] \\
Y \arrow[r, tail, "s"'] \arrow[u, "s_0^Y" description] & X \arrow[r, "f"'] \arrow[u, "(1,f)" description] & Y \arrow[u, "s_0^Y" description]
\end{tikzcd}$$

Axiom (4) holds if and only if the left hand side part of the diagram is a morphism of equivalence relations, which means that $s^{-1}(\Sigma_\psi) = \nabla_Y$. According to Lemma 1.10 and Proposition 1.12, this is equivalent to the fact that the split monomorphism s is normal to Σ_ψ. ■

1.14. Proposition. *Let ψ be a fibrant equipment for f and $j : U \rightarrowtail X$ any equivalence class of the Chasles relation Σ_ψ. Then any of the following conditions makes $f.j$ an isomorphism and produces, up to isomorphism, a splitting to f:*
1) $\mathbb{E}$ is a regular category and f is a regular epimorphism,
2) $\mathbb{E}$ is a pointed category.

Proof. 1) Suppose $\mathbb{E}$ regular. Consider the following diagram where J is the common support of U and X

$$\begin{tikzcd}
U\times U \arrow[r, tail, "\tilde{j}"] \arrow[d, shift right=3, "p_0"'] \arrow[d, shift left=3, "p_1"] & X\times Y \arrow[r, "f\times Y"] \arrow[d, shift right=3, "p_X"'] \arrow[d, shift left=3, dotted, "\psi"] & Y\times Y \arrow[d, shift right=3, "p_0"'] \arrow[d, shift left=3, "p_1"] \\
U \arrow[r, tail, "j"'] \arrow[u, "s_0" description] \arrow[d, two heads] & X \arrow[r, "f"'] \arrow[u, "(1,f)" description] & Y \arrow[u, "s_0" description] \arrow[d, two heads] \\
J \arrow[rr, equal] & & J
\end{tikzcd}$$

Since f is a regular epimorphism, the support of Y is J as well. The upper rows produce a composable pair of fibrant morphisms. The map $U \twoheadrightarrow J$ being a regular epimorphism, the Barr-Kock theorem (see [15]) makes the lower rectangle a pullback. So the map $f.j : U \to Y$ an isomorphism.
2) Suppose $\mathbb{E}$ pointed. The same proof holds with $J = 1$. ■

Examples
1) the unique fibrant equipment of the terminal map $\tau_X : X \to 1$ is 1_X and $\Sigma_{1_X} = \Delta_X$,
2) the unique fibrant equipment of 1_X is $p_1^X : X \times X \to X$ and $\Sigma_{p_1^X} = \nabla_X$.
3) Another example of fibrant equipment is given by the following:

1.15. Proposition. *Given any category $\mathbb{E}$ and any projection $p_1^Y : X \times Y \to Y$, there is a bijection between:*
1) the fibrant equipments ψ of this projection,
2) the maps $p : X \times Y \times Y \to X$ such that $p(x, u, u) = x$
and $p(p(x, u, v), v, w) = p(x, u, w)$.
Then we shall denote by Σ_p the Chasles relation associated with ψ. Accordingly we get $R[p_1^Y] \cap \Sigma_p = \Delta_{X\times Y}$ and an induced connected pair $[R[p_1^Y], \Sigma_p] = 0$. Given any map $\gamma : Y \to X$, the monomorphism $(\gamma, 1) : Y \rightarrowtail X \times Y$ is an equivalence class of the Chasles relation Σ_p if and only if $p(\gamma(u), u, v) = \gamma(v)$.

Proof. Let $\psi : X \times Y \times Y \to X \times Y$ be a fibrant equipment for p_1^Y, then axiom (2) means $\psi(x, u, v) = (p(x, u, v), v)$. Then (1) is equivalent to $p(x, u, u) = x$ and (3) to $p(p(x, u, v), v, w) = p(x, u, w)$. Finally (4) is equivalent to $p(\gamma(u), u, v) = \gamma(v)$. The last assertion follows from Proposition 1.13. ■

1.16. The monad of fibrant equipments. The notion of fibrant equipment comes from [8]. The functor $dom : \mathbb{E}/Y \to \mathbb{E}$ has a right adjoint $G : \mathbb{E} \to \mathbb{E}/Y$ associating with any object X the product projection $p_1^Y : X \times Y \to Y$; this adjunction produces the following monad (T, λ, μ) on the slice category $\mathbb{E}/Y$:

$$\begin{tikzcd}
X \arrow[r, tail, "(1_X,f)"] \arrow[d, "f"'] & X\times Y \arrow[d, "p_1^Y"] & X\times Y\times Y \arrow[l, "X\times p_1^Y"'] \arrow[d, "p_2^Y"] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

1.17. Proposition. *An algebra for the monad (T, λ, μ) on an object $f : X \to Y$ coincides with a fibrant equipment for f.*

Proof. Straightforward checking. ■

The bijection of Proposition 1.15 associates the map $p : X \times Y \times Y \to X$ defined by $p(x, u, v) = x$ (namely the projection on X) with the fibrant equipment given by the free algebra μ_f on the object $p_1^Y : X \times Y \to Y$ of $\mathbb{E}/Y$. The adjoint pair (G, dom) enters in a larger table. Consider the left hand side commutative downward square:

$$\begin{tikzcd} Y/\mathbb{E} \arrow[r, "cod"] \arrow[d, shift right, "\bar{G}"'] & \mathbb{E} \arrow[d, shift right, "G"'] \\ Pt_Y\mathbb{E} \arrow[r, "W"'] \arrow[u, shift right, "spl"'] & \mathbb{E}/Y \arrow[u, shift right, "dom"'] \end{tikzcd} \qquad \begin{tikzcd} Alg\bar{T} \arrow[r, "\tilde{W}"] \arrow[d, "U^{\bar{T}}"'] & AlgT \arrow[d, "U^T"] \\ Pt_Y\mathbb{E} \arrow[r, "W"'] & \mathbb{E}/Y \end{tikzcd}$$

where $Pt_Y\mathbb{E}$ is the category of split epimorphisms above Y, $Y/\mathbb{E}$ the coslice category of maps with domain Y, W the functor forgetting the monomorphic part of a split epimorphism (f, s) and *spl* the functor forgetting its epimorphic part. This last functor has a right adjoint $\bar{G}$ associating with a map $\gamma : Y \to X$ the split epimorphism $(p_1^Y, (\gamma, 1_Y)) : X \times Y \to Y$.
1) This downward square is a pullback.
2) The left hand side upward square commutes as well.
3) The monad $(\bar{T}, \overline{\lambda}, \overline{\mu})$ induced by the left hand side adjunction is given by the following diagram:

$$\begin{tikzcd} X \arrow[r, tail, "{(1,f)}"] \arrow[d, shift right, "f"'] & X \times Y \arrow[d, shift right, "p_1^Y"'] & X \times Y \times Y \arrow[l, "X \times p_1^Y"'] \arrow[d, shift right, "p_2^Y"'] \\ Y \arrow[u, shift right, "s"'] \arrow[r, equal] & Y \arrow[u, shift right, "{(s,1_Y)}"'] \arrow[r, equal] & Y \arrow[u, shift right, "{(s,s_0^Y)}"'] \end{tikzcd}$$

4) The hereabove right hand side commutative square of functors at the level of the categories of algebras, is a pullback as well. This means that any T-algebra on f in $\mathbb{E}/Y$ becomes a $\bar{T}$-algebra for any splitting s of f. In other words, a $\bar{T}$-algebra is characterized by the first three axioms of Definition 1.11; a $\bar{T}$-algebra satisfying axiom 4) is called a special $\bar{T}$-algebra in [8].

1.18. Fibrant equipment and direct product decomposition. When the category $\mathbb{E}$ is pointed or exact, the fibrant equipments are strongly related with the direct product decompositions. However we shall see that in the Mal'tsev context, this notion of fibrant equipment has a lower degree of classification power:

1.19. Proposition. *When the category $\mathbb{E}$ is pointed or exact, the only morphisms which have a fibrant equipment are the product projections, and this equipment is the free one.*

Proof. Let $f : X \to Y$ be a map endowed with a fibrant equipment ψ.
1) Suppose $\mathbb{E}$ pointed. Then $f\psi(x, 0) = 0$; so that we get a map $g = X \to \mathrm{Ker} f$ such that $(\ker f).g = \psi.(1_X, 0)$. Now $(g, f) : X \to \mathrm{Ker} f \times Y$ is a monomorphism since, for any pair (x, x') such that $(g, f)(x) = (g, f)(x')$ we get: $x' = \psi(x', f(x')) = \psi(x', f(x)) = \psi(\psi(x', 0), f(x)) = \psi(\psi(x, 0), f(x)) = \psi(x, f(x)) = x$. We have also the restriction $\phi = \psi.(k_f \times Y) : \mathrm{Ker} f \times Y \to X$ of ψ. Moreover $g\phi(k, y) = \psi(\psi(k, y), 0) = \psi(k, 0) = \psi(k, f(k)) = k$ and $f\phi(k, y) = f\psi(k, y) = y$; so that ϕ is a right inverse of the monomorphism $(g, f) : X \to \mathrm{Ker} f \times Y$ which is consequently an isomorphism which makes $f \simeq p_1^Y$.
2) Let $\mathbb{E}$ be exact. Denote by $q : X \twoheadrightarrow Q$ the quotient of the Chasles equivalence relation Σ_ψ and consider the following diagram:

$$\begin{tikzcd}
X \times Y \arrow[rr, "f \times Y"] \arrow[d, shift right=3, "p_X"'] \arrow[d, dotted, shift left=3, "\psi"] & & Y \times Y \arrow[d, shift right=3, "p_0^X"'] \arrow[d, shift left=3, "p_1"] \\
X \arrow[u, "(1,f)"] \arrow[rr, "f"'] \arrow[d, "q"'] & & Y \arrow[u, "s_0"] \arrow[d] \\
Q \arrow[rr] & & 1
\end{tikzcd}$$

Since the upper squares are pullbacks, so is the lower one by the Barr-Kock theorem; therefore $(q, f) : X \to Q \times Y$ is an isomorphism which makes $f \simeq p_1^Y$.

In both cases, the induced fibrant equipment is the free one. In first case, the transfert by isomorphism of the equipment ψ is given by:
$\chi((k, y), y') = (g, f).\psi(\psi(k, y), y') = (g, f).\psi(k, y') = (k, y')$
while in the second one, we get: $\chi((\bar{x}, y), y') = (q, f).\psi(\psi(x, y), y')$
$= (q, f).\psi(x, y')) = (\overline{\psi(x, y')}, y') = (\bar{x}, y')$. ∎

Accordingly, the previous proposition provides us with a remarkable situation: here, the monad (T, λ, μ) on $\mathbb{E}/Y$ of the last section is such that the canonical comparison functor $Kl^T \to AlgT$ from the Kleisli category of this monad [22] to its category of algebras is an equivalence of categories, this comparison functor being, in any case, fully faithful.

1.20. Fibrant equipment and connected pair of relations. The following section investigates the relationship between fibrant equipment and connected pair of equivalence relations:

1.21. Proposition. *Given any category $\mathbb{E}$ and any equivalence relation R on X, there is bijection between:*
1) the fibrant equipment ψ of the split epimorphism $(d_1^R, s_0^R) : R \rightleftarrows X$
2) the maps $\pi : R \times X \to X$ such that: i) $\pi(xRy, z)Rz$,
ii) $\pi(xRy, y) = x$, iii) $\pi(\pi(xRy, z)Rz, t) = \pi(xRy, t)$ and iv) $\pi(xRx, y) = y$.
When any of these two conditions holds, we shall call Chasles relation of π and denote

by Σ_π *the equivalence relation on* R *associated with this fibrant equipment* ψ*. Now, the subdiagonal* $s_0^R : X \rightarrowtail R$ *is an equivalence class of this Chasles relation* Σ_π*.*

Such a map π *produces a connected pair* $[R, \nabla_X]_\pi = 0$ *for* R *if and only if we have moreover: (v)* $\pi(xRy, \pi(yRu, t)) = \pi(xRu, t)$*.*

In this case, we shall say that the split epimorphism (d_1^R, s_0^R) *is endowed with an* associative equipment. *So, a connector for the pair* (R, ∇_X) *is exactly an associative equipment* π *for the split epimorphism* (d_1^R, s_0^R)*.*

Proof. Starting with a fibrant equipment ψ, we set $\pi(xRy, z) = d_0^R(\psi(xRy, z))$. Condition (2) on the fibrant equipment $\psi : R \times X \to R$ makes necessarily $\psi(xRy, z) = \pi(xRy, z)Rz$. Condition (1) is then equivalent to $\pi(xRy, y) = x$, Condition (3) to $\pi(\pi(xRy, z)Rz, t) = \pi(xRy, t)$, and Condition (4) to $\pi(xRx, y) = y$. Then the map $\pi : R \times X \to X$ satisfies all the axioms of a connector, except the axiom (3) which is (v). ■

In this way, we can enlighten a previous observation:

1.22. Corollary. *Given any category* $\mathbb{E}$ *and any object* X*, there is bijection between:*
1) the fibrant equipments of the split epimorphism $(p_1^X, s_0^X) : X \times X \rightleftarrows X$
2) the ternary operations $p : X \times X \times X \to X$ *on* X *such that*

$$p(x, z, z) = x, \quad p(p(x, z, v), v, w) = p(x, z, w) \quad \textit{and} \quad p(x, x, z) = z$$

namely the left associative Mal'tsev operations.

The Chasles equivalence relation Σ_p *on* $X \times X$ *is defined by* $(x, y)\Sigma_p(t, z)$ *if and only if* $t = p(x, y, z)$ *and we know that the diagonal* $s_0^X : X \rightarrowtail X \times X$ *is an equivalence class of* Σ_p*.*

Accordingly, a fibrant equipment p *for* (p_1^X, s_0^X) *is an associative one (or equivalently gives rise to a connected pair* $[\nabla_X, \nabla_X]_p = 0$*) if and only if the left associative Mal'tsev operation* p *on* X *is right associative as well, and therefore if and only is* p *is an associative Mal'tsev operation.*

1.23. Remark. *Suppose moreover* $\mathbb{E}$ *exact. Section 1.18 asserts that an equipment of the previous kind is isomorphic to the canonical equipment for the projection* $p_1^X : Q \times X \to X$*, where* Q *is the quotient of the Chasles equivalence relation. The isomorphism* $\chi : X \times X \to Q \times X$ *is defined by* $\chi(u, v) = (\overline{uv}, v)$ *with* $\chi^{-1}(\overline{uv}, t) = p(u, v, t)$*.*

From the previous corollary, we recover the following well known observation:

1.24. Corollary. *Given any internal group object* $(X, \circ, 0)$ *in* $\mathbb{E}$*, the map* p *defined by* $p(x, y, z) = x \circ y^{-1} \circ z$ *produces an associative equipment* $p : X \times X \times X \to X$ *for the split epimorphism* (p_1^X, s_0^X)*; so the diagonal* $s_0^X : X \rightarrowtail X \times X$ *is an equivalence class of the Chasles equivalence relation* Σ_p *on* $X \times X$ *which coincides with the kernel equivalence relation* $R[d]$*, where* $d : X \times X \to X$ *is the "division" map defined by* $d(x, y) = x \circ y^{-1}$*.*

Given any category $\mathbb{E}$, suppose now that the monomorphism $j : U \rightarrowtail X$ is normal to an equivalence relation R on the object X, and that moreover we have a connected pair $[\nabla_X, R]_p = 0$. Then consider the following diagram of equivalence relations where the right hand side fibrant morphism is produced by the connector p, and the total rectangle is a fibrant morphism as well since j is normal to R:

$$\begin{tikzcd}
U\times U \arrow[r, tail, "(j,1)\times U"'] \arrow[rrr, tail, bend left, "\tilde{j}"] \arrow[d, shift right=2ex, "p_0^U"'] \arrow[d, shift left=2ex, "p_1^U"] & X\times U\times U \arrow[r, tail, "X\times\tilde{j}"'] \arrow[d, shift right=2ex, "X\times p_0^U"'] \arrow[d, shift left=2ex, "(p.(X\times\tilde{j}),p_1^U)"] & X\times R \arrow[r, "(p_0^X,p)"'] \arrow[d, shift right=2ex, "X\times d_0^R"'] \arrow[d, shift left=2ex, "(p,d_1^R.p_1^R)"] & R \arrow[d, shift right=2ex, "d_0^R"'] \arrow[d, shift left=2ex, "d_1^R"] \\
U \arrow[u, "s_0^U"] \arrow[r, tail, "(j,1)"] \arrow[rrr, tail, bend right, "j"'] & X\times U \arrow[u, "X\times s_0^U"] \arrow[r, tail, "X\times j"] & X\times X \arrow[u, "X\times s_0^R"] \arrow[r, "p_0^X"] & X \arrow[u, "s_0^R"]
\end{tikzcd}$$

So, the two other morphisms of equivalence relations are fibrant as well. From that we get:

1.25. Theorem. *Given any monomorphism j normal to an equivalence relation R endowed with a connector p such that $[\nabla_X, R]_p = 0$:*
1) the monomorphism $(j,1)$ is an equivalence class of the vertical equivalence relation Σ_p^U on $X\times U$
2) this diagram produces a fibrant morphism of equivalence relations $(p_0^X, \bar{p}_0^X) : \Sigma_p^U \to R$ and therefore we get $\Sigma_p^U \cap R[p_0^X] = \Delta_{X\times U}$ and $[R[p_0^X], \Sigma_p^U] = 0$ in a canonical way
3) the equivalence relation Σ_p^U comes from the fibrant equipment on the split epimorphism $(p_1^U, (j,1))$ produced (according to Proposition 1.15) by the map $p.(X\times\tilde{j}) : X\times U\times U \to X$; accordingly we have $R[p_1^U]\cap\Sigma_p^U = \Delta_{X\times U}$ and $[R[p_1^U], \Sigma_p^U] = 0$
4) the restriction of this map to U, given by the map $p.(j\times\tilde{j}) : U\times U\times U \to X$, factorizes through U and gives it an internal associative Mal'tsev operation p_U.

Proof. The whole rectangle is a fibrant morphism since j is normal to R. The vertical diagram above $X\times U$ produces a subequivalence relation Σ_p^U of the equivalence relation $R\times X$ on $X\times X$, and the central part of the diagram is fibrant since the square indexed by 0 is clearly a pullback, again since j is normal to R. Accordingly, 1) the morphism $(p_0^X, \bar{p}_0^X) : \Sigma_p^U \to R$ is fibrant, and 2) so is the left and side morphism of equivalence relations, which means that the monomorphism $(j,1)$ is an equivalence class of Σ_p^U. In set-theoretical terms we have $(x,u)\Sigma_p^U(z,v)$ if and only if $z = p(x, j(u)Rj(v))$. The third point is a straightforward checking.

Now we have $p(j(u), j(v)Rj(w))Rj(u)$ by definition of p, and since j is normal to R, we have $p(j(u), j(v), j(w)) \in j(U)$, whence a unique $p_U(u,v,w)$ such that $j(p_U(u,v,w)) = p(j(u), j(v)Rj(w))$. This restriction map clearly gives U an associative Mal'tsev operation. ∎

When the equivalence relation R is a Chasles equivalence relation Σ_ψ, we have a kind of converse:

1.26. Corollary. *Let ψ be a fibrant equipment for the split epimorphism $(f,s) : X \rightleftarrows Y$ in $\mathbb{E}$. If the object Y is endowed with an associative Mal'tsev operation p_Y, then the Chasles equivalence relation Σ_ψ is such that $[\nabla_X, \Sigma_\psi]_p = 0$, by the connector $p : X\times X\times Y \to X$ with $p = \psi.(p_0^X, p_Y.(f\times f\times Y))$.*

Proof. Set $p = \psi.(p_0^X, p_Y.(f \times f \times Y)) : X \times X \times Y \to Y$, namely $p(x,z,y) = \psi(x, p_Y(f(x), f(z), y))$. Checking that it gives rise to a connector for the Chasles equivalence relation Σ_ψ is straightforward. Moreover, this connector p is such that we get $fp(x,z,y) = p_Y(f(x), f(z), y)$. ∎

1.27. Stability properties of fibrant equipments.

1.28. Proposition. *In any category $\mathbb{E}$, the fibrant equipments of morphisms (resp. split epimorphisms) are stable under products and pullbacks. Moreover, with the notations of the diagram below, we have $\Sigma_{\psi'} = g^{-1}(\Sigma_\psi)$.*

Proof. The stability under products is straightforward. Now consider the following pullback in $\mathbb{E}$:

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, shift right, "f'"'] & X \arrow[d, shift right, "f"'] \\ Y' \arrow[r, "h"'] \arrow[u, dotted, shift right, "s'"'] & Y \arrow[u, dotted, shift right, "s"'] \end{tikzcd}$$

such that $\psi : X \times Y \to X$ is a fibrant equipment for f. We get: $f.\psi.(g \times h) = p_1^Y.(g \times h) = h.p_1^{Y'}$, whence a unique factorization $\psi' : X' \times Y' \to X'$ such that $g.\psi' = \psi.(g \times h)$ and $f'.\psi' = p_1^{Y'}$ (=Axiom (2)). Axioms (1) and (3) easily follow. When f is split by s, there is an induced splitting s'. Once again Axiom (4) for (f', s') easily follows.

Now we get $(x,u)\Sigma_{\psi'}(z,v)$ if and only if $(z,v) = \psi'((x,u),v) = (\psi(x, h(v)), v)$, namely if and only if $z = \psi(x, h(v)) = \psi(x, f(z))$, which is $x\Sigma_\psi z$. Whence the last assertion. ∎

1.29. Proposition. *Given any category $\mathbb{E}$ and any split epimorphism (f, s) endowed with a fibrant equipment ψ, consider the following pullback:*

$$\begin{tikzcd} Y' \arrow[r, "h"] \arrow[d, tail, "s'"'] & Y \arrow[d, tail, shift right, "s"'] \\ X' \arrow[r, "g"'] & X \arrow[u, shift right, "f"'] \end{tikzcd}$$

Then the monomorphism $s' : Y' \rightarrowtail X'$ is normal to the equivalence relation $S = g^{-1}(\Sigma_\psi)$ on X' which is such that $S \cap R[f.g] = R[g]$.

Proof. As an inverse image of s along g, the monomorphism s' is normal to $S = g^{-1}(\Sigma_\psi)$. From $\Sigma_\psi \cap R[f] = \Delta_X$, we get $S \cap R[f.g] = R[g]$ (which implies that S does not come from a fibrant equipment on $f.g$). This equivalence relation $S \rightarrowtail X' \times X'$ is obtained as the equalizer of $g.p_1^{X'}$ and $\psi.(g \times f.g)$, namely, in set-theoretical term s: aSb if and only if $g(b) = \psi(g(a), fg(b))$. ∎

1.30. Corollary. *Given any commutative triangle in a category* $\mathbb{E}$*:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

if $(d_1^h, s_0^h) : R[h] \rightleftarrows Y$ *has a fibrant equipment, the subobject* $R[f] \rightarrowtail R[g]$ *is an equivalence class of an equivalence relation* S *on* $R[g]$ *such that* $S \cap R[f.d_1^g] = R[R(f)]$*, where* $R(f) : R[g] \to R[h]$ *is the factorization induced by* f *thanks to the commutation of the triangle.*

Accordingly, given any map $f : X \to Y$*, the subobject* $R[f] \rightarrowtail X \times X$ *is an equivalence class of an equivalence relation* Σ *on* $X \times X$ *such that* $\Sigma \cap R[f.p_1^X] = R[f \times f]$ *as soon as the object* Y *is endowed with an internal left associative Mal'tsev operation* p*.*

Proof. Apply the previous proposition and Proposition 1.21 to the following pullback:

$$\begin{tikzcd} R[f] \arrow[r, "\phi"] \arrow[d, tail] & Y \arrow[d, tail, shift right, "s_0^h"'] \\ R[g] \arrow[r, "R(f)"'] & R[h] \arrow[u, shift right, "d_1^h"'] \end{tikzcd}$$

Then observe that the subobject in question:

$$\begin{tikzcd} R[f] \arrow[rr, tail] \arrow[dr, shift right, "d_0^f"'] & & R[g] \arrow[dl, shift right, "d_0^g"'] \\ & Z \arrow[ul, shift right, "s_0^f"'] \arrow[ur, shift right, "s_0^g"'] & \end{tikzcd}$$

is a plain subobject. ∎

2. Unital context

Recall that a category $\mathbb{E}$ is unital [4], when it is pointed and such that for any pair (X, Y) of objects the following canonical injections are jointly strongly epic:

$$X \xrightarrow{j_0^X = (1_X, 0)} X \times Y \xleftarrow{j_1^Y = (0, 1_Y)} Y$$

The main examples of such categories are the categories $UMag$ of unitaty magmas, and the category Mon of monoids.

Given two maps $f : X \to Z$ and $g : Y \to Z$, we say that they *commute* when there exists a (necessarily unique) morphism $\varphi : X \times Y \to Z$ such that $\varphi.j_0^X = f$ and $\varphi.j_1^Y = g$. The uniqueness of this map makes the commutation a property inside the unital category $\mathbb{E}$. This map φ is then called the *cooperator* of f and g. A map $f : X \to Z$ is said to be central when it commute with 1_Z.

So, the pair $(1_X, 1_X)$ commutes if and only if there is a binary operation $m : X \times X \to X$ on the object X which gives it a (unique) internal commutative monoid structure; we then say that the object X is commutative in $\mathbb{E}$.

2.1. Proposition. *When $\mathbb{E}$ is a unital category and (R, S) a connected pair of equivalence relations on the object X, the pair of normalizations* $(\mathrm{nor}R, \mathrm{nor}S)$ *does commute in* $\mathbb{E}$.

Proof. Let p be the connector of the pair (R, S). Then define the cooperator $\varphi : \mathrm{nor}R \times \mathrm{nor}S \to X$ by $\varphi(u, v) = p(u, 0, v)$. ■

It is clear that j_0^X is the kernel of $p_1^Y : X \times Y \to Y$, and that accordingly j_0^X is normal to $R[p_1^Y]$.

2.2. Proposition. *In a unital category $\mathbb{E}$, the equivalence relation $R[p_1^Y]$ is the smallest one to which the monomorphism j_0^X is normal.*

Proof. Let Σ be another equivalence relation to which j_0^X is normal. Consider $S = R[p_1^Y] \cap \Sigma$ to which j_0^X is normal as well. We have to show that $S = R[p_1^Y]$. For that consider the following diagram:

$$\begin{tikzcd}
X \times X \arrow[r, tail, "\tilde{j}"'] \arrow[rr, tail, bend left, "j_0^{X\times X}"] \arrow[d, shift right=3, "p_0^X"'] \arrow[d, shift left=3, "p_1^X"] & S \arrow[r, tail, "\iota"'] \arrow[d, shift right=3, "d_0^S"'] \arrow[d, shift left=3, "d_1^S"] & R[p_1^Y] = X \times X \times Y \arrow[d, shift right=3, "p_0^X \times Y"'] \arrow[d, shift left=3, "p_1^X \times Y"] \\
X \arrow[r, tail, "j_0^X"'] \arrow[u, "s_0^X"] & X \times Y \arrow[r, equal] \arrow[u, "s_0^S"] & X \times Y \arrow[u, "s_0^X \times Y"]
\end{tikzcd}$$

We have $(s_0^X \times Y).j_1^Y = j_1^Y : Y \rightarrowtail X \times X \times Y$. Accordingly the factorizations through ι of the two injections $j_0^{X\times X}$ and j_1^Y make ι an isomorphism in the unital category $\mathbb{E}$. ■

2.3. Corollary. *In a unital category $\mathbb{E}$, the equivalence relation $R[p_1^Y]$ is the unique equivalence relation S on $X \times Y$ with j_0^X normal to S and $S \cap R[p_0^X] = \Delta_{X\times Y}$.*

Proof. By the previous proposition, j_0^X normal to S implies $R[p_1^Y] \subset S$. From that we get $(x, y)S(x', y)$. Suppose $(x, y)S(x', y')$. So, we get $(x', y)S(x', y')$. Then, from $S \cap R[p_0^X] = \Delta_{X\times Y}$, we get $y = y'$ and $S \subset R[p_1^Y]$. ■

Let us now make explicit an example of an equivalence relation Σ on $X \times Y$ of which, on the one hand, j_0^X is the normalized equivalence class and which, on the other hand, is strictly larger than $R[p_1^Y]$. Actually we shall give it in the strictly stronger context of a pointed Mal'tsev category, namely a category in which any reflexive relation is an equivalence relation, see [16] and [17].

Define $Malo_*$ as the variety of pointed Mal'tsev operations, namely ternary operations $p : X \times X \times X \to X$ satisfying the Mal'tsev identities $p(x, y, y) = x = p(y, y, x)$.

Let us consider the set $\mathbb{W} = \{0, a, b\}$ endowed with the Mal'tsev operation defined by the following 12 equalities, besides the 15 other ones demanded by a Mal'tsev operation:

p(0,a,0)=a	p(0,b,0)=b	p(a,0,a)=a	p(b,0,b)=b
p(a,0,b)=a	p(b,0,a)=b	p(a,b,0)=0	p(0,a,b)=0
p(b,a,0)=0	p(0,b,a)=0	p(a,b,a)=b	p(b,a,b)=a

Denote by $\mathbb{W}_a$ the subalgebra defined on $\{0, a\}$, and by $t_a : \mathbb{W}_a \rightarrowtail \mathbb{W}$ the inclusion homomorphism. We get a retraction homomorphism $g_a : \mathbb{W} \twoheadrightarrow \mathbb{W}_a$ as well, defined by $g(a) = a = g(b)$. The point, here, is that we get $\mathrm{Ker} g_a = \{0\}$.

Accordingly, given any other algebra M in $Malo_*$, the canonical injection $j_0^M : M \rightarrowtail M \times \mathbb{W}$ is the kernel of $g_a.p_1^{\mathbb{W}}$ as well, and is consequently the normalized equivalence class of $\Sigma = R[g_a.p_1^{\mathbb{W}}]$ which is strictly larger than $R[p_1^{\mathbb{W}}]$.

3. Mal'tsev context

We shall investigate now the case where the ground category $\mathbb{E}$ is a Mal'tsev category, namely, as we just recalled, a category in which any reflexive relation is an equivalence relation [16], [17].

Denote by $Pt\mathbb{E}$ the category whose objects are the split epimorphisms in $\mathbb{E}$ and whose morphisms are the commutative squares between such data. Let $\P_{\mathbb{E}} : Pt\mathbb{E} \to \mathbb{E}$ be the functor associating with any split epimorphism its codomain. It is a fibration called the *fibration of (generalized) pointed objects* whose cartesian maps are the pullbacks of split epimorphisms. It is shown in [4] that the category $\mathbb{E}$ is a Mal'tsev one if and only if the (pointed) fibers of this fibration are unital.

This last property implies that, given any pair (R, S) of equivalence relations on an object X, we get [11] that:
1) there is at most one map $p : R \times_X S \to X$ satisfying axioms 2) and 2') of a connector;
2) when there is such a p, it necessarily satisfies the four other axioms;
3) so, being connected becomes a property denoted by $[R, S] = 0$; we shall then speak of *centralizing double relation* on the pair (R, S) instead of *cartesian double relation*; we shall also say that the equivalence relations R and S centralize each other;
4) $R \cap S = \Delta_X$ implies $[R, S] = 0$;
5) an object X is called *affine* when we get: $[\nabla_X, \nabla_X] = 0$; in this case the associated associative Mal'tsev operation p is necessarily commutative, namely such that $p(x, y, z) = p(z, y, x)$.

In fact, connected equivalence relations in a Mal'tsev category are characterized, in the unital fibers of $\P_{\mathbb{E}}$, in terms of commuting subobjects. More precisely, (R, S) is a connected pair of equivalence relations in $\mathbb{E}$ if and only if the split epimorphisms $(d_1^R, s_0^R) : R \rightleftarrows X$ and $(d_1^S, s_0^S) : S \rightleftarrows X$ commute as subobjects of $(p_1^X, s_0^X) : X \times X \rightleftarrows X$ in $Pt_X\mathbb{E}$.

Finally, recall that we gave in the previous section an example which can be seen as a split monomorphism $(0, \tau_{\mathbb{W}}) : \mathbf{1} \rightleftarrows \mathbb{W}$ in the Mal'tsev variety $Malo$ of Mal'tsev operations which is both normal to $R[g_a]$ and to $\Delta_{\mathbb{W}}$. So that in a Mal'tsev category, there exist subobjects which are equivalence classes of two distinct equivalence relations.

We shall now give specifications in the Mal'tsev context of some results of the first section. Let us begin by Lemma 1.2:

3.1. Lemma. *Suppose given any split epimorphism of equivalence relations in a category $\mathbb{E}$ as on the left hand side:*

$$
\begin{tikzcd}
R_X \arrow[d, shift right=2, "d_1^X"'] \arrow[d, shift left=2, "d_0^X"] \arrow[r, shift right, "R_f"'] & R_Y \arrow[l, shift right, "R_s"'] \arrow[d, shift right=2, "d_1^Y"'] \arrow[d, shift left=2, "d_0^Y"] \\
X \arrow[u] \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, "s"'] \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R_X \arrow[r, shift right, "R_f"'] & R_Y \arrow[l, shift right, "R_s"'] \\
X \arrow[u, "s_0^X"] \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, "s"'] \arrow[u, "s_0^Y"']
\end{tikzcd}
$$

then the following two conditions are equivalent:
i) the upward right hand side diagram is a pullback of split epimorphisms
ii) we have $R_X \cap R[f] = \Delta_X$
and they hold whenever we have:
iii) (f, R_f) is a fibrant morphism of equivalence relations.
When $\mathbb{E}$ is a Mal'tsev category, the three conditions are equivalent.

Proof. In any category $\mathbb{E}$, the domain of the pullback of s_0^Y along R_f is nothing but $R_X \cap R[f]$. So that $i) \iff ii)$ holds in any category. Obviously if (f, R_f) is fibrant, the upward square with s_0 is a pullback and we get i). Conversely, consider the following diagram where the lower square is a pullback of split epimorphisms:

$$
\begin{tikzcd}
R_X \arrow[d, dotted, "\phi"'] \arrow[dr, shift left, "R_f"] \arrow[dd, bend right=60, "d_0^X"'] & \\
P \arrow[r, shift left, "\pi_f"] \arrow[d, shift right, "\pi_X"'] & R_Y \arrow[l, shift left, "\sigma"] \arrow[ul, shift left] \arrow[d, shift right, "d_0^Y"'] \\
X \arrow[u, shift right] \arrow[uu, bend left=40, "s_0^X"] \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "s"] \arrow[u, shift right, "s_0^Y"']
\end{tikzcd}
$$

Let ϕ be the natural factorization. Thanks to the Yoneda embedding, it is easy to check that, in any category $\mathbb{E}$, $R_X \cap R[f] = \Delta_X$ (namely ii)) implies that ϕ is a monomorphism. When, in addition, the category $\mathbb{E}$ is a Mal'tsev category, the factorization ϕ, being involved in a pullback of split epimorphisms, is necessarily a strong epimorphism, since the fibre $Pt_Y\mathbb{E}$ is unital. Accordingly ϕ is an isomorphism and the morphism (f, R_f) is fibrant. ∎

Now, let us specify the second emblematic situation:

3.2. Proposition. *Suppose $\mathbb{E}$ is a Mal'tsev category. Given any split monomorphism (s, f), there is at most one equivalence relation S on its codomain X such that $s^{-1}(S) = \nabla_Y$ and $S \cap R[f] = \Delta_X$. When such an equivalence relation S does exists:*
1) the split monomorphism s is an equivalence class of S
2) S is the smallest equivalence relation R on X such that $s^{-1}(R) = \nabla_Y$.
Accordingly any split monomorphism (s, f) is the equivalence class of most one equivalence relation S such that $S \cap R[f] = \Delta_X$.

Proof. Suppose $s^{-1}(S) = \nabla_Y$. Then consider the following diagram:

$$\begin{tikzcd}
Y \times Y \arrow[r, tail, "\tilde{s}"] \arrow[d, shift right=2ex, "p_0^Y"'] \arrow[d, shift left=2ex, "p_1^Y"] & S \arrow[r, "(f.d_0^S, f.d_1^S)"] \arrow[d, shift right=2ex, "d_0^S"'] \arrow[d, shift left=2ex, "d_1^S"] & Y \times Y \arrow[d, shift right=2ex, "p_0^Y"'] \arrow[d, shift left=2ex, "p_1^Y"] \\
Y \arrow[r, tail, "s"'] \arrow[u] & X \arrow[r, "f"'] \arrow[u] & Y \arrow[u]
\end{tikzcd}$$

So, we get a split epimorphism of equivalence relations $S \rightleftarrows \nabla_Y$. If, in addition, we have $S \cap R[f] = \Delta_X$ and $\mathbb{E}$ is a Mal'tsev category, $S \to \nabla_Y$ is a fibrant morphism according to the previous lemma, and this is also the case of its splitting $\nabla_Y \rightarrowtail S$. This last point means that s is normal to S.

On the other hand, the pair $(s_0^S, \tilde{s})$ is jointly strongly epic since any of the downward right hand side squares is a pullback. If R is another equivalence on X such that $s^{-1}(R) = \nabla_Y$, then $s^{-1}(R \cap S) = \nabla_Y$, consider the following factorizations:

$$\begin{tikzcd}
& R \cap S \arrow[d, tail, "j"'] & \\
X \arrow[r, tail, "s_0^S"'] \arrow[ur, tail, dotted, "s_0^{R\cap S}"] & S & Y \times Y \arrow[l, tail, "\tilde{s}"] \arrow[ul, tail, dotted, "\tilde{s}_{R\cap S}"']
\end{tikzcd}$$

they show that j is an isomorphism and $S \subset R$. Accordingly S is the smallest among the equivalence relations R on X such that $s^{-1}(R) = \nabla_Y$. Whence its uniqueness as an equivalence relation to which s is normal. ∎

The following proposition gives us the description of this unique equivalence relation S and makes an explicit relationship with the third emblematic situation:

3.3. Proposition. *Suppose $\mathbb{E}$ is a Mal'tsev category. Given any split monomorphism (s, f), the following conditions are equivalent:*
1) the split monomorphism s is normal to a (unique) equivalence relation S such that $S \cap R[f] = \Delta_X$
2) there is a (unique) map $\psi : X \times Y \to X$ satisfying: (1) $\psi.(1, f) = 1_X$ and (4) $\psi.(s \times Y) = s.p_1^Y$ which actually produces a fibrant equipment for (f, s).
Accordingly this unique equivalence relation S is nothing but the Chasles equivalence relation Σ_ψ.

Proof. Suppose we have such an equivalence relation S, and consider the following commutative diagram:

$$\begin{tikzcd}
Y \times Y \arrow[r, tail, "\tilde{s}"'] \arrow[rr, tail, bend left, "s\times Y"] \arrow[d, shift right=2ex, "p_0^Y"'] \arrow[d, dotted, shift left=2ex, "p_1^Y"] & S \arrow[r, "(d_0^S, f.d_1^S)"'] \arrow[d, shift right=2ex, "d_0^S"'] \arrow[d, dotted, shift left=2ex, "d_1^S"] & X \times Y \arrow[d, shift right=1ex, "p_0^X"'] \\
Y \arrow[r, tail, "s"'] \arrow[u, "s_0^Y"] & X \arrow[r, equal] \arrow[u, "s_0^S"] & X \arrow[u, shift right=1ex, "(1,f)"']
\end{tikzcd}$$

We have $S \cap R[f] = \Delta_X$ if and only if the map $(d_0^S, f.d_1^S) : S \to X \times Y$ is a monomorphism. Since the following leftward square is a pullback of split epimorphisms and $\mathbb{E}$ is a Mal'tsev category:

$$\begin{tikzcd}
Y \times Y \arrow[d, shift right, "p_0^Y"'] & X \times Y \arrow[l, "f \times Y"] \arrow[d, shift right, "p_0^X"'] \\
Y \arrow[u, shift right, "s_0^Y"'] \arrow[r, shift left, "s"] & X \arrow[l, shift left, "f"] \arrow[u, shift right, "(1,f)"']
\end{tikzcd}
\quad Y \times Y \xrightarrow{s \times Y} X \times Y$$

the pair $(s \times Y, (1, f))$ is jointly extremally epic, and the factorizations $\tilde{s}$ and s_0^S make the monomorphism $(d_0^S, f.d_1^S)$ an isomorphism. Let us denote γ its inverse. Then the map $\psi = d_1^S.\gamma : X \times Y \to X$ satisfies axioms (1) and (4).

Conversely these two conditions are sufficient to get a fibrant equipment for (f, s): indeed, from them, we get: $f.\psi.(1, f) = f = p_1^Y.(1, f)$ and $f.\psi.(s \times Y) = f.s.p_1^Y = p_1^Y = p_1^Y.(s \times Y)$; now since, once again, the pair $(s \times Y, (1, f))$ is jointly extremally epic, we conclude $f.\psi = p_1^Y$, namely (2), which implies that Σ_ψ is a reflexive relation, and, $\mathbb{E}$ being a Mal'tsev category, an equivalence relation, whence (3). By Proposition 1.12 we have $\Sigma_\psi \cap R[f] = \Delta_X$ and by Proposition 1.13, s is normal to Σ_ψ. ■

3.4. Corollary. *Given any Mal'tsev category $\mathbb{E}$, and any equivalence relation R on X, the following conditions are equivalent:*
1) the split monomorphism $(s_0^R, d_1^R) : X \rightleftarrows R$ is an equivalence class of a (unique) equivalence relation S such that $S \cap R[d_1^R] = \Delta_R$
2) the split epimorphism (d_1^R, s_0^R) has a fibrant equipment
3) the split epimorphism (d_1^R, s_0^R) has an associative equipment
4) the equivalence relation R is central, namely we get $[R, \nabla_X] = 0$.

In particular, the following conditions are equivalent:
1) the split monomorphism $(s_0^X, p_1^X) : X \rightleftarrows X \times X$ is the equivalence class of a (unique) equivalence relation S such that $S \cap R[p_1^X] = \Delta_{X \times X}$
2) the object X is endowed with a (unique) Mal'tsev operation p
3) we get $[\nabla_X, \nabla_X] = 0$ or equivalently the object X is affine.

Proof. The four equivalences are consequences of Proposition 1.21 and the previous proposition. ■

3.5. Corollary. *Given any commutative triangle in a Mal'tsev category $\mathbb{E}$:*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\
& Z &
\end{tikzcd}$$

if the equivalence relation $R[h]$ is central, the subobject $R[f] \rightarrowtail R[g]$ is the equivalence class of an equivalence relation S on $R[g]$ such that $S \cap R[f.d_1^g] = R[R(f)]$, where $R(f) : R[g] \to R[h]$ is the factorization induced by f.

Accordingly, given any map $f : X \to Y$, the subobject $R[f] \rightarrowtail X \times X$ is the equivalence class of an equivalence relation Σ on $X \times X$ such that $\Sigma \cap R[f.p_1^X] = R[f \times f]$ as soon as the object Y is affine.

PROOF. Apply Corollary 1.30 in the Mal'tsev context. ∎

3.6. REGULAR MAL'TSEV CONTEXT. When $\mathbb{E}$ is a regular category and $f : X \twoheadrightarrow Y$ a regular epimorphism, the direct image of an equivalence relation S on X, i.e. the canonical decomposition of $(f \times f).(d_0^S, d_1^S)$:

$$\begin{tikzcd} S \arrow[r, two heads, "\check{f}"] \arrow[d, tail, "{(d_0^S,d_1^S)}"'] & f(S) \arrow[d, tail, "{(\delta_0,\delta_1)}"] \\ X \times X \arrow[r, two heads, "f\times f"'] & Y \times Y \end{tikzcd}$$

only produces a reflexive relation on Y. In a regular Mal'tsev one, it is necessarily an equivalence relation. So, we get direct images of equivalence relations. Any Mal'tsev variety $\mathbb{V}$ being an exact category is a fortiori a regular Mal'tsev category. The category $TopMalo$ of topological Mal'tsev operations is an example of a regular Mal'tsev category which is not exact, see [21].

So, let us recall from [7], the following important:

3.7. PROPOSITION. *Let $\mathbb{E}$ be a regular Mal'tsev category. Given any regular epimorphism $f : X \twoheadrightarrow Y$, if $u : U \rightarrowtail X$ is normal to an equivalence relation S, then the direct image $f(u) : f(U) \rightarrowtail Y$ is normal to the direct image $f(S)$.*

and from [11] the other important:

3.8. PROPOSITION. *Let $\mathbb{E}$ be a regular Mal'tsev category. Given any regular epimorphism $f : X \twoheadrightarrow Y$ and any pair (R, S) of equivalence relations on X, if we have $[R, S] = 0$, then we get $[f(R), f(S)] = 0$.*

The regular Mal'tsev context ensures the uniqueness of the result described in Theorem 1.25:

3.9. THEOREM. *Let $\mathbb{E}$ be a regular Mal'tsev category and $j : U \rightarrowtail X$ be a plain monomorphism. The following conditions are equivalent:*
1) the plain monomorphism j is the equivalence class of a (necessarily unique) central equivalence relation R on X
2) the split monomorphism $(j, 1) : U \rightarrowtail X \times U$ is the equivalence class of a (necessarily unique) equivalence relation S on $X \times U$ such that $S \cap R[p_1^U] = \Delta_{X \times U}$

PROOF. Suppose 1). Thanks to Theorem 1.25 the split epimorphism $(p_1^U, (j, 1)) : X \times U \rightleftarrows U$ has a fibrant equipment which produces an equivalence relation S on $X \times U$ such that

$S \cap R[p_1^U] = \Delta_{X\times U}$ and makes fibrant morphisms the two following ones, the left hand side part meaning that $(j, 1)$ is normal to S:

$$\begin{tikzcd}
U \times U \arrow[r, tail, "\widetilde{(j,1)}"] \arrow[d, shift right=6, "p_0^U"'] \arrow[d, shift left=6, "p_1^U"] & S \arrow[r, "\tilde{p}_0^X"] \arrow[d, shift right=6, "d_0^S"'] \arrow[d, shift left=6, "d_1^S"] & R \arrow[d, shift right=6, "d_0^R"'] \arrow[d, shift left=6, "d_1^R"] \\
U \arrow[r, tail, "{(j,1)}"'] \arrow[u, "s_0^U"] & X \times U \arrow[r, "p_0^X"'] \arrow[u, "s_0^S"] & X \arrow[u, "s_0^R"]
\end{tikzcd}$$

Since $\mathbb{E}$ is a Mal'tsev category, this equivalence relation S is unique.

Conversely suppose 2). We know that this S is unique and determined by a fibrant equipment for the split epimorphism $(p_1^U, (j, 1))$. On the other hand, since j is a plain monomorphism, the projection $p_0^X : X \times U \to X$ is a regular epimorphism as the pullback of the regular epimorphism $U \twoheadrightarrow J$ defining the common support J of U and X. Now consider the following diagram, where the vertical central part is S and the right hand side part is the direct image of S along the regular epimorphism p_0^X:

$$\begin{tikzcd}
U \times U \arrow[r, tail, "{(j,1)\times U}"] \arrow[d, shift right=6, "p_0^U"'] \arrow[d, shift left=6, "p_1^U"] & X \times U \times U \arrow[r, "\tilde{p}_0^X"] \arrow[d, shift right=6, "p_0^{X\times U}"'] \arrow[d, shift left=6, "\psi"] & p_0^X(S) \arrow[d, shift right=6, "\delta_0"'] \arrow[d, shift left=6, "\delta_1"] \\
U \arrow[r, tail, "{(j,1)}"'] \arrow[u, "s_0^U"] & X \times U \arrow[r, "p_0^X"'] \arrow[u, "{(1,p_1^U)}"] & X \arrow[u, "\sigma_0"]
\end{tikzcd}$$

Since $(j, 1)$ is normal to S, its direct image $p_0^X(j, 1) = j$ is normal to $p_0^X(S)$. Since we have $S \cap R[p_1^U] = \Delta_{X\times U}$, and consequently $[S, R[p_1^U]] = 0$ because we are in a Mal'tsev category, we get $0 = [p_0^X(S), p_0^X(R[p_1^U])] = [p_0^X(S), \nabla_X]$. Accordingly $p_0^X(S)$ is central. This is the unique central equivalence relation to which j is normal: if there is some other similar R, the diagram of our first part implies that $\tilde{p}_0^X$ is a regular epimorphism since the morphism $S \to R$ is a fibrant one. Accordingly we get $R = p_0^X(S)$. ∎

Similarly, we can ensure the uniqueness in the following results:

3.10. Proposition. *Let $\mathbb{E}$ be a regular Mal'tsev category, (f, s) a split epimorphism and $g : X' \twoheadrightarrow X$ a regular epimorphism. Consider the following pullback:*

$$\begin{tikzcd}
Y' \arrow[r, two heads, "h"] \arrow[d, tail, "\sigma"'] & Y \arrow[d, tail, shift right=2, "s"'] \\
X' \arrow[r, two heads, "g"'] & X \arrow[u, shift right=2, "f"']
\end{tikzcd}$$

the following conditions are equivalent:
1) the monomorphism σ is the equivalence class of an equivalence relation S on X' such that $S \cap R[f.g] = R[g]$, which is then the unique equivalence relation satisfying this condition
2) the monomorphism s is the equivalence class of a (unique) equivalence relation Σ on X such that $\Sigma \cap R[f] = \Delta_X$.

Proof. We know that 2) implies 1) by Proposition 1.29. Conversely suppose 1) and consider the previous pullback. Since g is a regular epimorphism, so is h, which makes s the direct image of σ along g. Since σ is normal to S, the monomorphism s is normal to the direct image $g(S)$. Since both S and $R[f.g]$ contain $R[g]$, their direct images along g preserve their intersection, so that $g(S) \cap R[f] = \Delta_X$. Then, according to Proposition 3.3, the monomorphism s satisfies Condition (2), so that $g(S) = \Sigma_\psi$. On the other hand, $R[g] \subset S$ implies $S = g^{-1}(g(S)) = g^{-1}(\Sigma_\psi)$, and consequently S is unique. ■

3.11. Corollary. *Let be given any commutative triangle in a regular Mal'tsev category* $\mathbb{E}$*:*

$$\begin{tikzcd} X \arrow[rr, two heads, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

where f is regular epimorphism. The following conditions are equivalent:
1) the equivalence relation $R[h]$ is central
2) the subobject $R[f] \rightarrowtail R[g]$ is the equivalence class of a (unique) equivalence relation S on $R[g]$ such that $S \cap R[f.d_1^g] = R[R(f)]$, where $R(f) : R[g] \to R[h]$ is the factorization induced by f.

Accordingly, given any regular epimorphism $f : X \twoheadrightarrow Y$, the subobject $R[f] \rightarrowtail X \times X$ is the equivalence class of a (unique) equivalence relation Σ on $X \times X$ such that $\Sigma \cap R[f.p_1^X] = R[f \times f]$ if and only if the object Y is affine.

Proof. We know that 1) implies 2) by Corollary 3.5. Conversely suppose 2). Apply Proposition 3.10 to the following pullback:

$$\begin{tikzcd} R[f] \arrow[r, two heads] \arrow[d, tail] & Y \arrow[d, tail, shift right, "s_0^h"'] \\ R[g] \arrow[r, two heads, "R(f)"'] & R[h] \arrow[u, shift right, "d_1^h"'] \end{tikzcd}$$

where the factorization $R(f)$ is a regular epimorphism, since so is f. ■

4. Pointed Mal'tsev context

In the pointed Mal'tsev context some of our observations become more radical:

4.1. Proposition. *Given any pointed Mal'tsev category $\mathbb{E}$ and any split epimorphism $(f, s) : X \rightleftarrows Y$, if s is the equivalence class of an equivalence relation S on X such that $S \cap R[f] = \Delta_X$, then there is a canonical isomorphism $\gamma : X \simeq \mathrm{Ker} f \times Y$ making $(f, s) \simeq (p_1^Y, j_1^Y)$.*

Proof. The Mal'tsev category being pointed is unital. In this context, the map ψ determined by the equivalence relation S of Proposition 3.3 produces a pair ($\phi = \psi.j_0^X, \sigma = \psi.j_1^Y$) of commutative maps: $\phi : X \to X$, $\sigma : Y \to X$.

The axiom (4) $\psi.(s \times Y) = s.p_1^Y$ gives us $\sigma = s$ and $\phi.s = 0$, while the axiom (2) $f.\psi = p_1^Y$ gives us $f.\phi = 0$. From (3), we get $\psi.(\phi \times Y) = \psi$, whence $\phi^2 = \phi$, and $\psi.(\phi, f) = 1_X$, so that the pair (ϕ, f) is jointly monic.

Denote by $t : T \rightarrowtail X$ the equalizer of the pair $(1_X, \phi)$. Since (ϕ, f) is jointly monic, it coincides with Ker$f$. Denote by $g : X \to T = \text{Ker}f$ its canonical retraction such that $t.g = \phi$. From $\phi.s = 0$, we get $g.s = 0$; while from $f.\phi = 0$, we get $f.t = 0$. On the other hand, since $(\phi, f) = (t.g, f)$ is jointly monic, so is $(g, f) : X \to T \times Y$. Now the factorization t and s through X of the extremal epimorphic (j_0^T, j_1^Y) makes the monomorphism (g, f) an isomorphism. ■

Here, the assertion of the uniqueness of the equivalence relation S given by Proposition 3.3 brings some precision to Theorem 4.3 of [13] which asserts the same result with a quite different proof.

4.2. Corollary. *Let $\mathbb{E}$ be a regular pointed Mal'tsev category and R an equivalence relation on X. The following conditions are equivalent:*
1) the equivalence relation R is central
2) its normalization $j : \text{nor}R = \text{Kerd}_0^R \rightarrowtail X$ is central.

Proof. Since $\mathbb{E}$ is pointed any monomorphism is a plain one. Suppose (1). Then Theorem 3.9 implies that the split monomorphism $(j, 1) : \text{Kerd}_0^R \rightarrowtail X \times \text{Kerd}_0^R$ is normal. So the cooperating map $X \times \text{Kerd}_0^R \xrightarrow{X \times j_1^K} X \times \text{Kerd}_0^R \times \text{Kerd}_0^R \xrightarrow{p} X$ makes the monomorphism j a central one, where p is the fibrant equipment for the split epimorphism $(j, 1)$ given by Propositions 3.3 and 1.15.

Conversely suppose j is a central monomorphism and $\phi : X \times \text{Kerd}_0^R \to X$ is the associated cooperator. By composition with $\text{Kerd}_0^R \times j$, this cooperator gives Kerd_0^R a structure of abelian object in $\mathbb{E}$. Then consider the map $p : X \times \text{Kerd}_0^R \times \text{Kerd}_0^R \to X$ defined by $p(x, k, k') = \phi(x, k' - k)$. It gives a fibrant equipment for the split epimorphism $(j, 1)$, making $(j, 1)$ a normal monomorphism. Then by Theorem 3.9, the equivalence relation R is central. ■

This result was already established in the semi-abelian context in [18]. Here we get rid of the exactness and of the protomodularity. We can even say that we get rid of the zero object if we refer to the non-pointed version given by Theorem 3.9 where the result relies upon the fact that in a regular Mal'tsev category the direct image of equivalence relations along regular epimorphisms preserves the centralization of the pairs.

5. Protomodular and related contexts

A category $\mathbb{E}$ is protomodular [3] when the fibration of points $¶_{\mathbb{E}}$ is such that any base-change functor is conservative. The major examples are the categories Gp of groups and

K-Lie of K Lie algebras. In a protomodular category a monomorphism is normal to at most one equivalence relation [2].

Any protomodular category is a Mal'tsev one. So any of the previous results valid in a Mal'tsev category are still valid in a protomodular one. Actually some of them will bring many precisions about already known results in the protomodular setting, see Proposition 5.2. Rather than giving them immediately we prefer first to set them in an intermediate context located between Mal'tsev and protomodular and defined in [9]:

5.1. Definition. *Given any category $\mathbb{E}$ and any class J of morphisms in $\mathbb{E}$, this category will be said protomodular on the class J when the base-change functor with respect to the fibration of points $\P_{\mathbb{E}}$ along any map in J is conservative.*

It is clear that:
1) a pointed category is protomodular if and only if it is protomodular on the class $\mathbb{S}p\mathbb{M}$ of split monomorphisms
2) a category is protomodular on the class $\mathbb{S}p\mathbb{M}$ of split monomorphisms if and only if any fibre $Pt_Y\mathbb{E}$ is protomodular
3) any category $\mathbb{E}$ is protomodular on the class $\mathbb{S}p\mathbb{E}$ of split epimorphisms
4) any regular category $\mathbb{E}$ is protomodular on the class $\mathbb{R}eg$ of regular epimorphisms
6) accordingly a regular category $\mathbb{E}$ is protomodular if and only if it is protomodular on the class $\mathbb{M}$ of monomorphisms.

Examples. 1) Let us denote by LAM the subvariety of the variety $Malo$ whose Mal'tsev operations are the left associative ones. Then LAM is protomodular on $\mathbb{S}p\mathbb{M}$ without being protomodular.

Proof. Clearly the empty set prevents LAM from being protomodular. Now consider the following pullback in LAM with $Y' \neq \emptyset$ (when $Y' = \emptyset$, the map t being necessarily an isomorphism):

$$\begin{tikzcd} X' \arrow[r, tail, "k"] \arrow[d, shift right, "f'"'] & X \arrow[d, shift right, "f"'] \\ Y' \arrow[u, shift right, "s'"'] \arrow[r, tail, shift left, "t"] & Y \arrow[u, shift right, "s"'] \arrow[l, shift left, "g"] \end{tikzcd}$$

Then X' is the subobject of the elements x of X satisfying $f(x) = tgf(x)$. Given any x in X, we have $f(p(x, sf(x), stgf(x))) = p(f(x), f(x), tgf(x)) = tgf(x)$. Consequently the element $p(x, sf(x), stgf(x))$ belongs to X'. Now let $\bar{X}$ be any subobject of X containing X' and $s(Y)$. Then we have: $x = p(p(x, sf(x), stgf(x)), stgf(x), sf(x))$; therefore x is in $\bar{X}$ since any of the three terms belongs to $\bar{X}$. Accordingly $\bar{X} = X$ and the base-change along t is conservative. When $Y' = \emptyset$, the other sets in the diagram are empty, and t becomes an isomorphism; so the base-change t^* is trivially conservative. ∎

2) given any left exact conservative functor $U : \mathbb{E} \to \mathbb{E}'$, if the category $\mathbb{E}'$ is protomodular on $\mathbb{S}p\mathbb{M}$, so is the category $\mathbb{E}$
3) accordingly any left associative Mal'tsev variety $\mathbb{V}$, namely whose theory contains a left associative Mal'tsev operation, is protomodular on $\mathbb{S}p\mathbb{M}$.

PROOF. Take the forgetful functor $U : \mathbb{V} \to LAM$. ∎

4) the notion of category protomodular on $\mathbb{S}p\mathbb{M}$ is clearly (by the above observation 2)) stable under slicing and coslicing.

5.2. PROPOSITION. *Suppose $\mathbb{E}$ is protomodular on the class $\mathbb{S}p\mathbb{M}$ of split monomorphisms. Then: 1) $\mathbb{E}$ is a Mal'tsev category,*
2) a split monomorphism $(s, f) : Y \rightleftarrows X$ is normal to at most one equivalence relation S which is necessarily such that $S \cap R[f] = \Delta_X$; so, being an equivalence class for a split monomorphism becomes a property,
3) the following conditions are equivalent:
i) the split monomorphism s is normal to S
ii) there is a fibrant equipment for (f, s) and S is the associated Chasles equivalence relation.

When $\mathbb{E}$ is protomodular on the class $\mathbb{M}$ of monomorphisms, then it is a Mal'tsev category, any monomorphism is normal to at most one equivalence relation. So, being normal or an equivalence class for a monomorphism becomes a property.

PROOF. The proofs of the two first assertions are the same as the ones given for any protomodular categories in [2], since the only base-change functors which is used in the proof of Mal'tsevness is the base-change along the split monomorphism $s_0^S : X \to S$, when S is a reflexive relation and the only base-change functors which is used in the proof of the uniqueness of the equivalence relation S to which a monomorphism u is normal is the base-change along the monomorphism u. It remains to check that $S \cap R[f] = \Delta_X$. For that consider the following diagram:

$$\begin{tikzcd}
Y \times Y \arrow[r, tail, "\tilde{s}"'] \arrow[rr, tail, bend left, "s\times Y"] \arrow[d, shift right, "p_0^Y"'] \arrow[d, shift left=2, dotted, "p_1^Y"] & S \arrow[r, "(d_0^S, f.d_1^S)"'] \arrow[d, shift right, "d_0^S"'] \arrow[d, shift left=2, dotted, "d_1^S"] & X \times Y \arrow[d, shift right, "p_0^X"'] \\
Y \arrow[r, tail, "s"'] \arrow[u, shift right, "s_0^Y"'] & X \arrow[r, equal] \arrow[u, shift right, "s_0^S"'] & X \arrow[u, shift right, "(1,f)"']
\end{tikzcd}$$

Since the left hand side square and the whole rectangle are pullback of split epimorphisms, we get $s^*((d_0^S, f.d_1^S)) = 1_{Y\times Y}$, namely an isomorphism; accordingly, since s^* is conservative, the map $(d_0^S, f.d_1^S)$ is an isomorphism as well, which implies that $S \cap R[f] = \Delta_X$. Assertion 3) then follows from that fact that any category which is protomodular on $\mathbb{S}p\mathbb{M}$ is a Mal'tsev one. The last part concerning the categories which are protomodular on the class $\mathbb{M}$ of monomorphisms is checked in a similar way by using base-change along monomorphisms instead of base-change along split monomorphisms. ∎

When applied to a protomodular category $\mathbb{E}$, the previous result (and this is new) produces an explicit description of the unique equivalence relation to which a split monomorphism (s, f) could be normal.

5.3. COROLLARY. *Suppose $\mathbb{E}$ is protomodular on the class $\mathbb{S}p\mathbb{M}$ of split monomorphisms. The object X is affine if and only if the split monomorphism $(s_0^X, p_1^X) : X \rightleftarrows X \times X$ is normal.*

5.4. PROPOSITION. *Let $\mathbb{E}$ be a protomodular category on $\mathbb{S}p\mathbb{M}$, and $j : U \rightarrowtail X$ a monomorphism. If the monomorphism j is normal to a central equivalence relation S on X, then the split (by p_1^U) monomorphism $(j, 1) : U \rightarrowtail X \times U$ is normal, which makes U an affine object.*

When, in addition, $\mathbb{E}$ is regular and j is a plain monomorphism, then the following conditions are equivalent:
1) j is an equivalence class of a central equivalence relation S
2) the split monomorphism $(j, 1) : U \rightarrowtail X \times U$ is an equivalence class.
When such a central equivalence relation S exists, it is unique.

PROOF. Apply the Proposition 5.2, Theorem 1.25 and Theorem 3.9. ∎

5.5. COROLLARY. *Let be given commutative triangle in a regular category $\mathbb{E}$ which is protomodular on $\mathbb{S}p\mathbb{M}$:*

$$\begin{tikzcd} X \arrow[rr, two heads, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

where f is regular epimorphism. The following conditions are equivalent:
1) the equivalence relation $R[h]$ is central
2) the subobject $R[f] \rightarrowtail R[g]$ is an equivalence class.

Accordingly, given any regular epimorphism $f : X \twoheadrightarrow Y$, the subobject $R[f] \rightarrowtail X \times X$ is an equivalence class if and only if the object Y is affine.

PROOF. By Proposition 5.2, the category $\mathbb{E}$ is a Mal'tsev one, and we can apply Corollary 3.11. ∎

6. Naturally Mal'tsev and related contexts

A naturally Mal'tsev category [20] is a category $\mathbb{E}$ where any object X is endowed with a natural Mal'tsev operation π_X. It was shown that is equivalent to demanding that any fibre $Pt_X\mathbb{E}$ is additive [4], which implies that $\mathbb{E}$ is protomodular on $\mathbb{S}p\mathbb{M}$ and therefore a Mal'tsev category. Let us recall the following:

6.1. PROPOSITION. *Given a pointed category $\mathbb{E}$, the following conditions are equivalent:*
1) $\mathbb{E}$ is additive
2) any downward pullback of split epimorphism:

$$\begin{tikzcd} \mathrm{Ker} f \arrow[r, tail, "\ker f"] \arrow[d, shift right] & X \arrow[d, shift right, "f"'] \\ 1 \arrow[r, tail, "\alpha_Y"'] \arrow[u, shift right] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

is an upward pushout
3) any base-change functor $f^* : Pt_Y\mathbb{E} \to Pt_X\mathbb{E}$ is an equivalence of categories whose inverse equivalence is given by the pushout of split monomorphisms along the map f
4) $\mathbb{E}$ is protomodular and any monomorphism is normal.

Let us also recall:

6.2. Proposition. *Given any category $\mathbb{E}$, the following conditions are equivalent:*
1) it is a naturally Mal'tsev category;
2) it is a Mal'tsev category in which any pair (R, S) of equivalence relations centralize each other;
3) it is a Mal'tsev category in which any equivalence relation S is central;
4) any fibre $Pt_Y\mathbb{E}$ is additive;
5) the base-change functor $s^* : Pt_Y \to Pt_X\mathbb{E}$ along any split monomorphism $(s, f) : Y \rightleftarrows X$ is an equivalence of categories (and in particular a conservative functor).

From that we get:

6.3. Proposition. *Given any category $\mathbb{E}$ the following conditions are equivalent:*
1) it is a naturally Mal'tsev category;
2) it is Mal'tsev category such that any split monomorphism $(s, f) : Y \rightleftarrows X$ is the equivalence class of a (unique) equivalence relation S such that $S \cap R[f] = \Delta_X$.
In a regular naturally Mal'tsev category $\mathbb{E}$, any plain monomorphism $u : U \rightarrowtail X$ is an equivalence class of a unique (central) equivalence relation S on X.

Proof. Suppose $\mathbb{E}$ is a naturally Mal'tsev category and, given any split monomorphism (f, s), consider the following diagram:

$$\begin{tikzcd}
Y \times Y \arrow[r, tail, "s\times Y"] \arrow[d, shift right=2ex, "p_0^Y"'] \arrow[d, shift left=2ex, "p_1^Y"] & X \times Y \arrow[d, shift right=2ex, "p_0^X"'] \arrow[d, shift left=2ex, "\psi"] \\
Y \arrow[u, "s_0^Y"] \arrow[r, tail, shift left, "s"] & X \arrow[u, "(1,f)"] \arrow[l, dotted, shift left, "f"]
\end{tikzcd}$$

with $\psi = \pi_X.((1, sf) \times s)$, in set-theoretical terms: $\psi(x, y) = \pi_X(x, sf(x), s(y))$. It clearly satisfies the conditions (1) and (4) of Proposition 3.2; accordingly s is normal to the right hand side equivalence relation S. The uniqueness of this S is ensured by the fact that $\mathbb{E}$ is Mal'tsev category. Actually, here, we have much more: in the additive fibre $Pt_Y\mathbb{E}$, the downward square indexed by 0 is a pullback along s (namely a kernel) so that the upward square appears as the pushout of the split monomorphism (s_0^Y, p_0^Y) along s.

Conversely suppose $\mathbb{E}$ satisfying condition 2). Then the split epimorphism (s_0^X, p_1^X) is an equivalence class, and the object X is affine by Corollary 3.4. The fact that the associated internal Mal'tsev operation is natural is straightforward.

The last assertion is consequence of Theorem 3.9. ∎

We have a counterexample when the monomorphism u is not plain: consider the variety *AutMalo* of autonomous Mal'tsev operations, namely associative and commutative Mal'tsev operations. It is an exact naturally Mal'tsev category. Then the initial monomorphism $\emptyset \rightarrowtail A$ to any algebra A is normal to any equivalence relation S on A.

Recall that, in a non-pointed context, there are actually many steps inside the "additive" setting. Recall the following table given by decreasing order of generality, see [6]:

6.4. Definition. *A category $\mathbb{E}$ is:*
1) a naturally Mal'tsev one when any fibre of the fibration $\P_{\mathbb{C}}$ is additive
2) antepenessentially affine when any base-change functor is fully faithful
3) penessentially affine when, in addition, any base-change functor is fibrant on monomorphisms
4) essentially affine when any base-change functor is an equivalence of categories.

Clearly we are in a protomodular context as soon as level 2. By Theorem 2.7 in [6] we can show that, in a penessentially affine category, any monomorphism is normal to a (unique) central equivalence relation. Let $Grd_Y\mathbb{E}$ be the category of internal groupoids in $\mathbb{E}$ with Y as object of objects.
1) $Grd_Y\mathbb{E}$ is essentially affine as soon as $\mathbb{E}$ is additive
2) $Grd_Y\mathbb{E}$ is penessentially affine as soon as $\mathbb{E}$ is a Mal'tsev category
3) $Grd_Y\mathbb{E}$ is antepenessentially affine as soon as $\mathbb{E}$ is a Gumm category (in the varietal context, this means that the variety $\mathbb{V}$ is congruence modular, see [19] and [14] for more details).

7. When $(\ )_0 : \mathrm{Equ}\mathbb{E} \to \mathbb{E}$ is a bifibration

In *Set*, any reflexive relation $(d_0^W, d_1^W) : W \rightarrowtail X \times X$ on X generates an equivalence relation. It is the infimum of the family of the equivalence relations R containing W. This is a consequence of the fact that, in *Set*, there no finite restriction to the stability of the equivalence relations on X under intersection. The same property holds for any variety.

By existence of "infima of equivalence relations" in $\mathbb{E}$, we mean existence of infima in any preordered fibre $\mathrm{Equ}_X\mathbb{E}$ stable under base-change with respect to the fibration $(\ )_0 : \mathrm{Equ}\mathbb{E} \to \mathbb{E}$.

7.1. Proposition. *[10, Proposition 2.6] Given any finitely complete category $\mathbb{E}$ with infima of equivalence relations, the fibration $(\ )_0 : Equ\mathbb{E} \to \mathbb{E}$ is a cofibration as well. Therefore, it is a* bifibration*.*

There are other types of sufficient conditions, but of right exact nature, for the fibration $(\ )_0$ to be a bifibration or to get some cocartesian maps:

7.2. Proposition. *[23] Let $\mathbb{E}$ be a regular Mal'tsev category having pushout of split monomorphism. Then the fibration $(\ )_0$ is a bifibration.*

Proof. Start with any map $h : X \to Y$ and any equivalence relation R on X. Then the pushout of the split monomorphism (s_0^R, d_0^R) along h produces a reflexive graph G on Y. The category $\mathbb{E}$ being regular, take its regular image R_h in $Y \times Y$; it is a reflexive relation on Y, and so, an equivalence relation when $\mathbb{E}$ is moreover a Mal'tsev category. From that, it is straightforward to observe that the induced morphism $(h, \check{h})\colon R \to R_h$ is cocartesian above h. ■

7.3. Proposition. *[5] Let $\mathbb{E}$ be a topos. Then the fibration $(\ )_0$ is cofibrational on monomorphisms, namely it is such that there are cocartesian maps above any monomorphism.*

Proof. Let $u : U \rightarrowtail X$ be any monomorphism and R any equivalence relation on U. Then the following upward pushout:

$$\begin{tikzcd} R \arrow[r, tail, "\tilde{u}"] \arrow[d, dotted, shift right=3, "d_0^R"'] \arrow[d, dotted, shift left=3, "d_1^R"] & S \arrow[d, dotted, shift right=3, "d_0^S"'] \arrow[d, dotted, shift left=3, "d_1^S"] \\ U \arrow[r, tail, "u"'] \arrow[u, "s_0^R"] & X \arrow[u, "s_0^S"] \end{tikzcd}$$

produces, from R, an equivalence relation S on X by Corollary 3.3 in [5] which, in addition, is such that the induced morphism $(u, \tilde{u}) : R \rightarrowtail S$ is a fibrant morphism. By construction, this morphism is cocartesian above u. When R is ∇_U, the codomain of this cocartesian map above u is called the *disconnectedly equivalence relation generated* by u. ■

When $(\ )_0$ is a bifibration, any map $f : X \to Y$ in $\mathbb{E}$ and any equivalence relation R on X produces a cocartesian image $f_!(R)$, namely the codomain of the cocartesian map above f with domain R. When $(\ )_0$ is cofibrational on monomorphisms, given any subobject $u : U \rightarrowtail X$, we shall be specially interested in the cocartesian lifting above u:

$$\zeta_u = (u, \hat{u}) : \nabla_U \rightarrowtail u_!(\nabla_U)\,.$$

which is necessarily a cartesian morphism as well, since its domain is an undiscrete relation. So, let us set $\mathrm{Geq}(u) = u_!(\nabla_U)$. This construction produces a functor making the following diagram commute:

$$\begin{tikzcd} \mathrm{Mon}\mathbb{E} \arrow[r, "\mathrm{Geq}"] \arrow[d, "cod"'] & \mathrm{Equ}\mathbb{E} \arrow[d, "(\ )_0"] \\ \mathbb{E} \arrow[r, equal] & \mathbb{E} \end{tikzcd}$$

since any commutative square:

$$\begin{tikzcd} U \arrow[r, "\bar{h}"] \arrow[d, tail, "u"'] & V \arrow[d, tail, "v"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

produces a map $\nabla_U \overset{\nabla_{\bar{h}}}{\to} \nabla_V \overset{\zeta_v}{\to} \mathrm{Geq}(v)$ and a canonical factorization $\mathrm{Geq}(h, \bar{h}) : \mathrm{Geq}(u) \to \mathrm{Geq}(v)$ above h.

7.4. Proposition. *Let $\mathbb{E}$ be a pointed category such that the fibration $(\ )_0$ is cofibrational on monomorphisms, the following holds.*
1) Geq *is left adjoint to the functor* nor*:*

$$\begin{tikzcd} \mathrm{Mon}\mathbb{E} \arrow[r, shift right=2, "\mathrm{Geq}"'] \arrow[r, phantom, "\top"] & \mathrm{Equ}\mathbb{E} \arrow[l, shift right=2, "\mathrm{nor}"'] \end{tikzcd}$$

2) When the category $\mathbb{E}$ is pointed protomodular, the co-unit $\epsilon : \mathrm{Geq}\,\mathrm{nor} \rightarrowtail 1_{\mathrm{Equ}\mathbb{E}}$ of the adjoint pair is an isomorphism.
3) When $\mathbb{E}$ is topos, the pointed category $\mathbb{E}_* = Pt_1\mathbb{E}$ of pointed objects in $\mathbb{E}$ is such that the unit $\eta : 1_{\mathrm{Mon}\mathbb{E}} \rightarrowtail \mathrm{nor}\mathrm{Geq}$ of the adjoint pair is an isomorphism.

Proof. Starting with any monomorphism $u : U \rightarrowtail X$ we get a natural comparison $\eta_u : U \rightarrowtail \mathrm{Nor}\mathrm{Geq}(u)$ by Lemma 1.9. Since $(\ )_0$ is a fibration, we can reduce the proof about the adjunction in the following way: let be given another equivalence relation S on X and a map:

$$\begin{tikzcd} U \arrow[rr, tail, "\phi"] \arrow[dr, "u"'] & & \mathrm{Nor}S \arrow[dl, "\mathrm{nor}S"] \\ & X & \end{tikzcd}$$

It produces a morphism in $\mathrm{Equ}\mathbb{E}$:

$$\nabla_U \xrightarrow{\nabla_\phi} \nabla_{\mathrm{Nor}S} \xrightarrow{(\mathrm{nor}S,\tilde{\mathrm{nor}}S)} S$$

which induces the desired factorization $\mathrm{Geq}(u) \rightarrowtail S$. The point 2) is a consequence of the Proposition 7.5 below. The point 3) is a consequence of the fact that, in any topos, the monomorphism u is normal to $\mathrm{Geq}(u)$, see Proposition 7.3. ■

In the protomodular context, let us recall the following useful result from [12]:

7.5. Proposition. *When the category $\mathbb{E}$ is protomodular, any fibrant morphism of equivalence relations is cocartesian in* $\mathrm{Equ}\mathbb{E}$*.*

Proof. Let $(h, \tilde{h}) : R \to S$ be a fibrant morphism in $\mathrm{Equ}\mathbb{E}$ above h. Suppose we have another morphism $(h, \bar{h}) : R \to \Sigma$. Then we get the following diagram where the whole rectangle of split epimorphisms is a pullback by the fibrant morphism, and the left hand side as well since ι is a monomorphism:

$$\begin{tikzcd} R \arrow[r, "h'"'] \arrow[rr, bend left, "\tilde{h}"] \arrow[d, shift right=3, "d_0^R"'] \arrow[d, dotted, shift left=3, "d_1^R"] & S \cap \Sigma \arrow[r, tail, "\iota"'] \arrow[d, shift right=3, "d_0"'] \arrow[d, dotted, shift left=3, "d_1"] & S \arrow[d, shift right=3, "d_0^S"'] \arrow[d, dotted, shift left=3, "d_1^S"] \\ X \arrow[r, "h"'] \arrow[u, "s_0^U" description] & Y \arrow[r, equal] \arrow[u, "s_0" description] & Y \arrow[u, "s_0^S" description] \end{tikzcd}$$

This means that the image of the map ι in the fibre $Pt_Y\mathbb{E}$ by the base-change h^* is an isomorphism. Since $\mathbb{E}$ is protomodular, the map ι is itself an isomorphism. We get $S \cap \Sigma = S$ and $S \subset \Sigma$; whence the desired factorization. ■

We shall call *normalizing* any monomorphism u such that there exists an equivalence relation R with u normal to R. Let us denote by $\mathrm{No}\mathbb{E}$ the essential image of nor in $\mathrm{Mon}\mathbb{E}$; its objects are normalizing monomorphisms.

7.6. PROPOSITION. *[23] Whenever the functor $(\ )_0$ is a cofibrational on monomorphisms, a monomorphism $u : U \rightarrowtail X$ is normalizing if and only if its cocartesian lifting $\nabla_U \rightarrowtail \mathrm{Geq}(u)$ is a fibrant morphism in* $\mathrm{Equ}\mathbb{E}$*.*

PROOF. The condition is clearly sufficient. Now suppose u normal to R, then we get $\mathrm{Geq}(u) \subset R$, and u normal to $\mathrm{Geq}(u)$ by Lemma 1.4. ∎

In universal algebra, it can be convenient to reduce the study of the lattice of congruences in a pointed algebra X to the lattice of their zero classes. Actually, for pointed protomodular varieties, these lattices are isomorphic. The following proposition establishes that, even with fewer assumptions on the base category, normalizing subobjects and equivalence relations are still connected by an adjunction, that becomes an equivalence in the protomodular case.

7.7. PROPOSITION. *Let $\mathbb{E}$ be a pointed category such that the functor $(\ )_0$ is cofibrational on monomorphisms. Consider the restriction $\overline{\mathrm{Geq}}$ of the functor* Geq *to* $\mathrm{No}\mathbb{E}$*. Then the following holds.*
1) $\overline{\mathrm{Geq}}$ is left adjoint to nor*:*

$$\begin{tikzcd} \mathrm{No}\mathbb{E} \arrow[r, bend right, "\overline{\mathrm{Geq}}"'] \arrow[r, phantom, "\top"] & \mathrm{Equ}\mathbb{E} \arrow[l, bend right, "\mathrm{nor}"'] \end{tikzcd}$$

2) The unit of the adjunction is an isomorphism, the counit is monomorphic.
3) $\overline{\mathrm{Geq}}$ is fully faithful, so that the essential image of $\overline{\mathrm{Geq}}$ is a mono-coreflective subcategory of $\mathrm{Equ}\mathbb{E}$*.*
4) If $\mathbb{E}$ is furthermore protomodular, then the pair $(\overline{\mathrm{Geq}}, \mathrm{nor})$ forms an adjoint equivalence.
5) As a consequence, for any object X of $\mathbb{E}$, it determines an isomorphism of posets between subobjects in the fibers:

$$\begin{tikzcd} {[\mathrm{No}_X\mathbb{E}]} \arrow[r, bend right, "\overline{\mathrm{Geq}}"'] \arrow[r, phantom, "\cong"] & {[\mathrm{Equ}_X\mathbb{E}]} \arrow[l, bend right, "\mathrm{nor}"'] \end{tikzcd}$$

PROOF. Straightforward from Proposition 7.4 ∎

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