

LATENT FIBRATIONS: FIBRATIONS FOR CATEGORIES OF PARTIAL MAPS

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ABSTRACT. Latent fibrations are an adaptation, appropriate for categories of partial maps (as presented by restriction categories), of the usual notion of fibration. This paper initiates the development of the basic theory of latent fibrations and explores some key examples. Latent fibrations cover a wide variety of examples, some of which are partial versions of standard fibrations, and some of which are particular to partial map categories (particularly those that arise in computational settings). Latent fibrations with various special properties are identified: hyperconnected latent fibrations, in particular, are shown to support the construction of a fibrational dual – important to reverse differential programming and, more generally, in the theory of lenses.

Contents

1	Introduction	423
2	Restriction Categories	427
3	Latent Fibrations	432
4	Types of Latent Fibrations	451
5	M-category Fibrations	465
6	Latent fibrations as fibrations in $SRest$	472
7	The Dual of a Latent Hyperfibration	478
8	Conclusion	488
A	The original definition of prone	490

1. Introduction

Fibrations play a fundamental role in categorical logic: in particular, they provide models of type theories [15] used in the semantics of logical and computational systems. The purpose of this paper is to develop the analog of fibrations for settings – such as those describing computation – which are based on partial maps. Because categories of partial

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maps admit an abstract and completely algebraic description as *restriction categories*, it is more general and, indeed, more convenient to develop a theory of fibrations for restriction categories. As the structure of a restriction category is more nuanced than that of a mere category – due to the necessity to support the partiality of maps – the restriction analogue of a fibration, which is called a *latent fibration*, is a necessarily more subtle notion. The purpose of this paper is to initiate a careful development of the theory of latent fibrations.

There are many reasons why an abstract theory of fibrations, into which partiality of maps has been baked, might be useful. Recalling that computation is fundamentally partial, there is an obvious motive to consider such settings in the semantics of computation and, in particular, to have partiality built into type theories describing computation. Notably the first use of latent fibrations was in Chad Nester’s MSc. thesis [21], where they were used in the study of realizability. More recently the desire to understand the semantics of differential programming, which has received increased attention due to its connection to machine learning ¹, has further stimulated the development of a general theory of partiality in fibrations.

In differential programming one needs to calculate the derivative of partially-defined smooth functions (that is, smooth functions defined on some open subset of their domain). The categorical structure of the so-called (total) “forward” derivative was developed in [3]: this characterized the operation which takes a smooth map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and produces the map

$$D[f]: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$$

whose value at a pair (x, v) is $J(f)(x) \cdot v$: the Jacobian of f , at x , in the direction v . The structure of differentials for partially defined smooth functions is described in [4] which introduced differential *restriction categories*. In this case, the structure is axiomatized by an operation which still takes a map $f: A \rightarrow B$ and produces a map $D[f]: A \times A \rightarrow B$, but now with additional axioms relating the partiality of $D[f]$ to that of f : in particular, one asks that $\overline{D[f]} = \overline{f} \times 1$, so that the partiality of $D[f]$ is entirely determined by the partiality of f . This structure, while being interesting in its own right, is also of key importance in understanding how one can build differential manifolds at this level of generality [6, Section 6].

Differentials are tightly linked to fibrations because the derivative, both in the total and the restriction case, can be regarded as a section of the “simple fibration” over \mathbb{X} . The simple fibration over \mathbb{X} is given by the category whose objects are pairs (Σ, A) , to be thought of as an object A in the context Σ , with a map from (Σ, A) to (Σ', B) consisting of a pair of maps

$$f: \Sigma \rightarrow \Sigma' \text{ and } g: \Sigma \times A \rightarrow B$$

where the second component g uses the context. Then, given a map $f: \Sigma \rightarrow \Sigma'$ in the base category, the pair $(f, D[f])$ gives a map in the simple slice category, and the functoriality of this operation is precisely the chain rule.

¹For example, Facebook’s Chief AI scientist has argued that deep learning be renamed differential programming (see [19]) and Tesla’s director of AI has mirrored this sentiment (see [17]).

In the restriction case, the construction of the ordinary simple fibration does not inherit a restriction structure; however, a restriction structure is inherited when we require pairs (f, g) such that $\bar{g} = \bar{f} \times 1$. This new construction does not provide an ordinary fibration over \mathbb{X} , and, indeed, there is no way to obtain an ordinary fibration. However, what it does provide is a *latent fibration*, which is, of course, the main subject of this paper. Intriguingly, sections of these latent simple fibrations recapture the additional axioms of differential restriction categories because the equation $\overline{D[f]} = \bar{f} \times 1$ is then forced.

There is an important addendum to this story: the backpropagation algorithm, which is widely used in machine learning, is based on computing the *reverse* differential [9] as computationally this can be much more efficient. The reverse derivative takes a smooth map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and produces the map

$$\mathbb{R}[f]: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$$

whose value at a pair (x, v) is $J^T(f)(x) \cdot v$, the *transpose* of the Jacobian of f , at x , in the direction v (note its difference in type from the forward derivative $D[f]: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$). Understanding the abstract properties of the reverse differential allows one to apply these machine learning techniques to different settings: for example, they have already been used to develop machine-learning algorithms for Boolean circuits [24]. Abadi and Plotkin introduced a simple differential programming language whose main feature is the introduction of a language primitive for computing reverse derivatives [1]. Reverse differential restriction categories were introduced in [12] and shown to provide a categorical model for the operational semantics of Abadi and Plotkin’s language.

When we look at the fibrational structure of the reverse derivative operation, it can also be seen as giving a section to a fibration, but instead of being a section of the simple slice over \mathbb{X} , now it is a section of the *fibrational dual* of the simple slice over \mathbb{X} . The fibrational dual can be defined for any ordinary fibration by taking the opposite category in each fibre. For the simple fibration, its dual is the category whose objects are pairs (A, A') as before, but now a map from (A, A') to (B, B') consists of a pair of maps

$$f: A \rightarrow B, f^*: A \times B' \rightarrow A'.$$

Fibrational duals have also received renewed interest due to their role in the theory of lenses. Lenses were originally developed for database theory [16] but are now used more widely in learning systems, and elsewhere [14, 22]. The research in this paper opens the door to the formal study of partial lenses.

Thus, not only is it useful for the theory of reverse differential *restriction* categories to be able to form the fibrational dual of a latent fibration, but they are also, more generally, important in the theory of partial lenses. However, in general, it is simply not the case that a latent fibration will have a fibrational dual! This is basically because the opposite of a restriction category is not generally a restriction category. Thus, taking the “opposite in each fibre” of a latent fibration will not necessarily produce a latent fibration. However, in the particular cases of interest, which use the simple latent fibration 3.3.2 and the codomain latent fibration 3.5.1, it is clear that it *is* possible to define a suitable fibrational

dual. Thus, one of the goals of this paper is to develop the theory of latent fibrations sufficiently so that the circumstances under which it is possible to define the fibrational dual of a latent fibration are fully understood.

Of course, aside from the aforementioned motivations, latent fibrations are of intrinsic mathematical interest in their own right. The definition of a latent fibration (Definition 3.2) involves a subtle change of the normal notion of Cartesian map to what, in order to clearly distinguish the notion, we call a *prone map*. Latent fibrations first appeared in [21] and were defined in a more complicated – albeit equivalent – manner: see Definition A.1 in Appendix A. It is reasonable to wonder whether these definitions are ad hoc. A theoretically convincing argument that the notion has a solid basis can be found in Section 6.6 where it is shown that the notion of latent fibration corresponds precisely to the 2-categorical notion introduced by Street [23] for the (carefully chosen) 2-category of restriction semi-functors and restriction transformations.

While many results about ordinary fibrations are true of latent fibrations, there are some subtle aspects to the theory. For example, while prone arrows always compose and isomorphisms are always prone, one might expect that partial isomorphisms should be prone as these generalize isomorphisms in a partial setting. However, they are not in general. This failure, in fact, leads to the investigation of important additional properties that a latent fibration can satisfy (see Section 4). The first additional property we study is that of being an *admissible* latent fibration (see Section 4.1): in admissible latent fibrations restriction idempotents have prone liftings which are restriction idempotents. An important consequence of being admissible is that one can split the idempotents to obtain an *r-split* latent fibration: these then can be linked to fibrations of partial map categories and \mathbf{M} -categories (see Section 5). The next additional property we consider is that of being *separated* (see Section 4.9), which is the requirement that the projection functor separates restriction idempotents. This turns out to be equivalent to asking that all restriction idempotents in the domain category be prone. When both these conditions hold the latent fibration is a *hyperfibration* (see Section 4.19), and this is in turn equivalent to asking that the projection functor of the latent fibration be a hyperconnection ([10, pg. 39]). We provide separating examples for each of these additional requirements, and develop their properties: in particular, we show that latent hyperfibrations have fibrational duals.

After developing the basic theory, we turn to obtaining an explicit description of latent fibrations as categories of partial maps (as \mathbf{M} -categories), Section 5. Indeed, we completely characterize the *r-split* latent fibrations of partial map categories, showing that they are equivalent to giving a fibration between the domain categories which is “*M*-plentiful” – this is a requirement, that *M*-maps lift in an appropriate manner. Finally, in the last section, Section 7, we describe how to define the fibrational dual of a latent hyperfibration.

The authors are grateful for Bob Rosebrugh’s many contributions to the category theory research community, through his interesting and enlightening talks and papers, his many years of service with TAC, and, on a personal level, his discussions with each of us. We hope that the present paper may serve as a continuation of his work on lenses and

fibrations (e.g., [16]) into settings which involve partial maps.

2. Restriction Categories

In this section we give a brief introduction to restriction categories concentrating on some of the less well-known aspects that are relevant to this paper. For further details see [5, 8].

2.1. RESTRICTION CATEGORIES. A **restriction category** (see [5] for details) is a category equipped with a **restriction** combinator which given a map $f: A \rightarrow B$, returns an endomorphism on the domain $\bar{f}: A \rightarrow A$ which satisfies just four identities²:

$$[\text{R.1}] \bar{f}f = f \quad [\text{R.2}] \bar{f} \bar{g} = \bar{g} \bar{f} \quad [\text{R.3}] \bar{f} \bar{g} = \overline{fg} \quad [\text{R.4}] f\bar{g} = \overline{fgf}$$

The prototypical restriction category is the category of sets and partial maps, **Par**. The restriction of a partial map in **Par** is the partial identity on the domain which is defined precisely when the partial map is defined.

In any restriction category \bar{f} is always an idempotent and any idempotent $e = ee$ with $e = \bar{e}$ is called a **restriction idempotent**. It is not the case that every idempotent need be a restriction idempotent. The restriction idempotents on an object A form a meet semi-lattice, with the meet given by composition, which we denote by $\mathcal{O}(A)$: the elements of $\mathcal{O}(A)$ may be regarded as distinguished predicates on the object A .

Restriction categories are always full subcategories of partial map categories (see [5, Proposition 3.3]) and this means that parallel maps in a restriction category can be partially ordered: the partial order is defined using the restriction by $f \leq g$ if and only if $\bar{f}g = f$ and is called the **restriction order**, and in fact gives an enrichment.

A map $f: A \rightarrow B$ in a restriction category \mathbb{X} is said to be **total** in case $\bar{f} = 1_A$. Total maps compose and include identities and, thus, form a (non-full) subcategory denoted **Total**(\mathbb{X}). Any category can be endowed with a trivial restriction which takes each map to the identity on its domain: thus, every category occurs as the total maps of some restriction category.

A map $s: A \rightarrow B$ in a restriction category is a **partial isomorphism** if there is a map $s^{(-1)}: B \rightarrow A$ – the **partial inverse** of s – with $ss^{(-1)} = \bar{s}$ and $s^{(-1)}s = \overline{s^{(-1)}}$ which we shall often write more succinctly as $\widehat{s} := \overline{s^{(-1)}}$. The partial inverse of a map is unique. Partial isomorphisms include all the restriction idempotents and are closed to composition. A restriction category in which all the maps are partial isomorphisms is called an **inverse category**. Inverse categories are to restriction categories what groupoids are to ordinary categories.

2.2. RESTRICTION (SEMI)FUNCTORS AND TRANSFORMATIONS. There are various sorts of morphisms between restriction categories which can be considered: the most basic is that of a **restriction functor**, $F: \mathbb{X} \rightarrow \mathbb{Y}$, which is a functor between the categories which in addition preserves the restriction structure, that is $F(\bar{f}) = \overline{F(f)}$. Restriction

²Composition is written in diagrammatic order in this paper.

functors so defined preserve total maps, restriction idempotents, and partial isomorphisms. As we shall see, the re-indexing or substitution functors for a latent fibration generally only satisfy the conditions for the slightly weaker notion of a restriction *semifunctor*; thus, it will be important to briefly review these and the related notion of a restriction transformation between restriction semifunctors.

2.3. DEFINITION. *Let \mathbb{X} and \mathbb{Y} be restriction categories.*

- A **restriction semifunctor** F from \mathbb{X} to \mathbb{Y} is a semifunctor from $\mathbb{X} \rightarrow \mathbb{Y}$ (that is, a map on objects and arrows which preserves domains, codomains, and composition, but not necessarily identities) which preserves restrictions. Note that for a restriction semifunctor, while $F(1_X)$ is not the identity, it is still a restriction idempotent since

$$\overline{F(1_X)} = F(\overline{1_X}) = F(1_X).$$

- If $F, G: \mathbb{X} \rightarrow \mathbb{Y}$ are restriction semifunctors, a **restriction transformation** $\alpha: F \Rightarrow G$ is a natural transformation from F to G such that for each $X \in \mathbb{X}$, $\overline{\alpha_X} = F(1_X)$.

Restriction functors and transformations organize themselves into a 2-category. Similarly, restriction semifunctors and their transformations organize themselves into a 2-category, which we will denote **SRest**.

2.4. **M-CATEGORIES AND r -SPLIT RESTRICTION CATEGORIES.** A restriction category is **r -split** if all its restriction idempotents split. Given an arbitrary restriction category, \mathbb{X} , one may always split its restriction idempotents to obtain $\mathbf{Split}_r(\mathbb{X})$, an r -split restriction category. The 2-category of r -split restriction categories, restriction functors, and total natural transformations is 2-equivalent to the 2-category of **M-categories** [5]. **M-categories** are categories with a system of monics which is closed to composition and pullbacks along any map. Functors between **M-categories** must not only preserve the **M**-maps but also the pullbacks along **M**-maps. Natural transformations between **M-functors** are natural transformations which, in addition, are Cartesian (or tight) for transformations between **M**-maps – that is the naturality squares for **M**-maps are pullbacks. The 2-equivalence is given on the one hand, by moving from an **M-category** $(\mathbb{X}, \mathbf{M}_{\mathbb{X}})$ to its partial map category $\mathbf{Par}(\mathbb{X}, \mathbf{M}_{\mathbb{X}})$ and on the other hand by moving to the **M-category** consisting of the total map category of the r -split restriction category, $\mathbf{Total}(\mathbb{E})$, with the restriction monics, $\mathbf{Monic}(\mathbb{E})$, $(\mathbf{Total}(\mathbb{E}), \mathbf{Monic}(\mathbb{E}))$. The restriction monics can be variously described as partial isomorphisms which are total, restriction sections, or, more interestingly, as left adjoints with respect to the partial order enrichment.

2.5. **PRECISE DIAGRAMS.** In a restriction category, we shall call a commuting diagram (viewed as a morphism from a directed graph) *precise* in case the restriction of the overall map that the diagram describes is equal to the restriction of every map leaving the start node.

2.6. DEFINITION. *In a restriction category, a commuting triangle*

$$\begin{array}{ccc} A & & \\ \downarrow k & \searrow g & \\ B & \xrightarrow{f} & C \end{array}$$

is **precise** in case $\bar{g} = \bar{k}$. We refer to k as the **left factor** (and f the **right factor**) of the triangle.

The following are some useful observations on precise triangles:

2.7. LEMMA. *In any restriction category:*

(i) *A commuting triangle*

$$\begin{array}{ccc} A & & \\ \downarrow k & \searrow g & \\ B & \xrightarrow{f} & C \end{array}$$

is precise if and only if $k\bar{f} = k$.

(ii) *A commuting triangle with right factor a restriction idempotent*

$$\begin{array}{ccc} A & & \\ \downarrow k & \searrow g & \\ B & \xrightarrow{e=\bar{e}} & C \end{array}$$

is precise if and only if $g = k$.

(iii) *A commuting triangle with right factor a partial isomorphism*

$$\begin{array}{ccc} A & & \\ \downarrow k & \searrow g & \\ B & \xrightarrow{\alpha} & C \end{array}$$

is precise if and only if $k = g\alpha^{(-1)}$.

(iv) *The commuting triangle*

$$\begin{array}{ccc} A & & \\ \downarrow \bar{f} & \searrow f & \\ A & \xrightarrow{f} & B \end{array}$$

is precise.

PROOF.

(i) We have the calculations:

$$(\Rightarrow) \text{ If } \bar{g} = \bar{k} \text{ then as } g = kf \text{ we have } k\bar{f} = \overline{kfk} = \bar{g}k = \bar{k}k = k.$$

$$(\Leftarrow) \text{ If } k\bar{f} = k \text{ then } \bar{g} = \overline{kf} = \overline{k\bar{f}} = \bar{k}.$$

(ii) We shall use (i) above: when the triangle is precise we have $g = ke = k\bar{e} = k$. Conversely, if $k = g$ then $k\bar{e} = ke = k$ so the triangle is precise.

(iii) Again we use (i) above: when the triangle is precise we have $g\alpha^{(-1)} = k\alpha\alpha^{(-1)} = k\bar{\alpha} = k$. Conversely, if $k = g\alpha^{(-1)}$ we have

$$k\bar{\alpha} = \overline{k\alpha k} = \overline{g\alpha^{(-1)}\alpha k} = \overline{g\alpha^{(-1)}}k = \overline{g\alpha^{(-1)}}k = \bar{k}k = k$$

showing the triangle is precise.

(iv) $\bar{f}f = f$ is precise with left factor \bar{f} because $\overline{\bar{f}} = \bar{f}$.

■

2.8. **CARTESIAN RESTRICTION CATEGORIES.** Cartesian restriction categories are described in [8]. They are restriction categories with a restriction terminal object and restriction products. A **restriction terminal object** in a restriction category, \mathbb{X} , is an object $1 \in \mathbb{X}$ such that for every object $X \in \mathbb{X}$ there is a unique total map $!_X: X \rightarrow 1$ such that every map $f: X \rightarrow 1$ has $f = \bar{f}!_X$. Given two objects $X, Y \in \mathbb{X}$ a **restriction product** of X and Y is an object $X \times Y$ together with two projections $\pi_0: X \times Y \rightarrow X$ and $\pi_1: X \times Y \rightarrow Y$ which are total and are such that given any other object Z with two maps $a: Z \rightarrow X$ and $b: Z \rightarrow Y$ there is a unique map $\langle a, b \rangle: Z \rightarrow X \times Y$ such that $\langle a, b \rangle\pi_0 = \bar{b}a$ and $\langle a, b \rangle\pi_1 = \bar{a}b$.

2.9. **LATENT PULLBACKS.** The notion of a latent pullback in a restriction category was introduced in [11, pg. 459]; here we give a slight modification of the original definition:

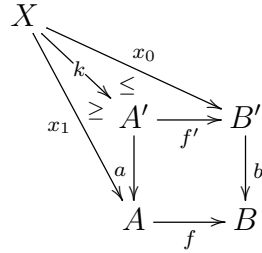
2.10. **DEFINITION.** *A commuting square in a restriction category*

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{f} & B \end{array}$$

is a **latent pullback** in case $f'\bar{b} = f'$ and $a\bar{f} = a$ (that is, it is precise) and given any e -commuting square (where $e = \bar{e}$ is a restriction idempotent)

$$\begin{array}{ccc} X & \xrightarrow{x_0} & B' \\ x_1 \downarrow & \stackrel{=}{=} e & \downarrow b \\ A & \xrightarrow{f} & B \end{array}$$

that is $ex_1f = ex_0b$, where $e \leq \overline{x_1f}$ and $e \leq \overline{x_0b}$ there is a unique map $k: X \rightarrow A'$



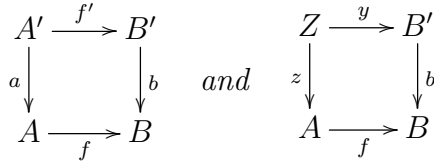
such that $\bar{k} = e$, $kf' \leq x_0$, $ka \leq x_1$, and $kf' = k\bar{a} = k$.

The original definition asked that for any (ordinary) commuting square $x_0b = x_1f$ there was a unique k with the same requirements above, except that $\bar{k} = \overline{x_1f} = \overline{x_0b}$ instead of $\bar{k} = \bar{e}$. It is readily seen that the two definitions are equivalent, as if e is a restriction idempotent on X such that $e \leq \overline{x_1f}$ and $e \leq \overline{x_0b}$, then $\bar{e} = \overline{ex_1f} = \overline{ex_0b}$.

We recall some basic properties of latent pullbacks:

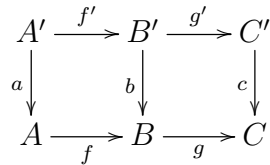
2.11. LEMMA.

(i) The two commuting squares



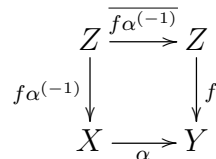
are latent pullbacks if and only if there is a unique mediating partial isomorphism $\alpha: A' \rightarrow Z$ with $\alpha z = a$, $\alpha y = f'$, $\bar{\alpha} = \bar{a} = \bar{f}'$, and $\overline{\alpha^{(-1)}} = \bar{z} = \bar{y}$ and one of the squares is a latent pullback.

(ii) If the two smaller squares below are latent pullbacks



then the outer square is a latent pullback. Furthermore, if the right square is a latent pullback and the perimeter is a latent pullback then the left square is a latent pullback.

(iii) If $\alpha: X \rightarrow Y$ is a partial isomorphism and $f: Z \rightarrow Y$ is any map, then



is a latent pullback.

(iv) If the latent pullback square

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{f} & B \end{array}$$

has f a partial isomorphism then f' is a partial isomorphism.

The last part of the lemma follows by combining parts (i) and (iii).

3. Latent Fibrations

We begin with the definition of a latent fibration before looking at examples and developing some of their basic properties.

3.1. THE DEFINITION.

3.2. DEFINITION. Let \mathbb{E} and \mathbb{B} be restriction categories and $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ a restriction semifunctor.

(i) An arrow $f': X' \rightarrow X$ in \mathbb{E} is **p-prone** in case whenever we have $g: Y \rightarrow X$ in \mathbb{E} and $h: \mathbf{p}(Y) \rightarrow \mathbf{p}(X')$ in \mathbb{X} such that $h\mathbf{p}(f') = \mathbf{p}(g)$ is a precise triangle (Definition 2.6) then there is a unique **lifting** of h to $\tilde{h}: Y \rightarrow X'$ so that $\mathbf{p}(\tilde{h}) = h$ and $\tilde{h}f' = g$ is a precise triangle:

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \tilde{h} \downarrow & & \\ X' & \xrightarrow{f'} & X \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathbf{p}(Y) & \xrightarrow{\mathbf{p}(g)} & \mathbf{p}(X) \\ h \downarrow & & \\ \mathbf{p}(X') & \xrightarrow{\mathbf{p}(f')} & \mathbf{p}(X) \end{array}$$

(ii) $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a **latent fibration** if for each $X \in \mathbb{E}$ and each $f: A \rightarrow \mathbf{p}(X)$ in \mathbb{B} such that $f = f\mathbf{p}(1_X)$, there is a **p-prone map** $f': X' \rightarrow X$ sitting over f (that is, $\mathbf{p}(f') = f$).

Note that if \mathbf{p} is a restriction functor (so that \mathbf{p} preserves identities) then the condition $f = f\mathbf{p}(1_X)$ is vacuous. However, if \mathbf{p} is a genuine semifunctor, then that condition is necessary, as if there is an $f': X' \rightarrow X$ sitting over f , then since $f' = f'1_X$, $\mathbf{p}(f') = \mathbf{p}(f')\mathbf{p}(1_X)$; i.e., f must satisfy $f = f\mathbf{p}(1_X)$.

Most of our examples of latent fibrations will be restriction functors; however, there is at least one important example of a latent fibration which is a genuine semifunctor (the forgetful functor from the restriction idempotent splitting of \mathbb{X} to itself: see Proposition 3.12).

More importantly, however, the main reason we have chosen to work with restriction semifunctors is that latent fibrations are precisely fibrations in the 2-category \mathbf{SRest} of restriction categories, semifunctors, and restriction transformations: see Section 6.

3.3. EXAMPLES OF LATENT FIBRATIONS. It is clear that, for any restriction category \mathbb{X} , the identity functor is a latent fibration. Furthermore, for any restriction categories \mathbb{X} and \mathbb{Y} , the projection $\pi_1: \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$ is a latent fibration. Moreover, there are a variety of other examples of latent fibrations: some are immediately seen to be restriction versions of a normal fibration counterpart; others are particular, however, to restriction categories. We present an overview of some of the basic examples before going into a more detailed description of them.

- (3.2.1) For any restriction category \mathbb{X} , the identity functor $1_{\mathbb{X}}: \mathbb{X} \rightarrow \mathbb{X}$ and projections $\pi_1: \mathbb{Y} \times \mathbb{X} \rightarrow \mathbb{X}$ are a latent fibrations.
- (3.2.2) If \mathbb{X} is a Cartesian restriction category, there are two latent versions of the simple slice fibration: the “lax” simple slice $\mathbb{X}(\mathbb{X})$ and the “strict” simple slice $\mathbb{X}[\mathbb{X}]$ mentioned in the Introduction. See Definition 3.4 and Proposition 3.5.
- (3.2.3) If \mathbb{X} is a restriction category with latent pullbacks, there are strict and lax versions of the codomain fibration: see Definition 3.6 and Proposition 3.7.
- (3.2.4) For any functor $F: \mathbb{X} \rightarrow \mathbf{Set}$ one may form the category of elements as a (normal) discrete fibration $\partial_F: \mathbf{Elt}(F) \rightarrow \mathbb{X}$. If \mathbb{X} is a restriction category then $\mathbf{Elt}(F)$ is also a restriction category and ∂_F is a latent fibration: see Section 3.7.1.
- (3.2.5) For any restriction category \mathbb{X} , there is a latent fibration of “propositions” (restriction idempotents) over \mathbb{X} : see Definition 3.8 and Proposition 3.9.
- (3.2.6) For any restriction functor $F: \mathbb{A} \rightarrow \mathbb{X}$, there is a latent fibration of “assemblies”, denoted $\mathbf{Asm}(F)$: see Definition 3.10 and 3.11.
- (3.2.7) For any restriction category \mathbb{X} , the forgetful functor from the restriction idempotent splitting of \mathbb{X} to itself, $\mathbf{Split}_r(\mathbb{X}) \rightarrow \mathbb{X}$ is a latent fibration which is a genuine semifunctor: see Proposition 3.12.

3.3.1. IDENTITY AND PROJECTION FUNCTORS. Clearly identity functors and projections are latent fibrations. The prone map over a map for the identity functor is just itself. For a projection $\pi_1: \mathbb{Y} \times \mathbb{X} \rightarrow \mathbb{X}$ the prone map over a map $f: X \rightarrow X'$ at (Y, X') is $(1, f): (Y, X) \rightarrow (Y, X')$.

3.3.2. SIMPLE LATENT FIBRATIONS. We consider simple fibrations for Cartesian restriction categories:

3.4. DEFINITION. *If \mathbb{X} is a Cartesian restriction category, the domain category of the **lax simple fibration**, $\mathbb{X}(\mathbb{X})$, is described as follows:*

Objects: *Pairs of objects, (Σ, X) , of \mathbb{X} , where Σ is called the “context”;*

Maps: *$(f, f'): (\Sigma, X) \rightarrow (\Sigma', X')$ where $f: \Sigma \rightarrow \Sigma'$ and $f': \Sigma \times X \rightarrow X'$ are maps in \mathbb{X} such that $\pi_0 f \geq \overline{f'}$;*

Composition: *Identities are* $(1_\Sigma, \pi_1): (\Sigma, X) \rightarrow (\Sigma, X)$ *and given* $(f, f'): (\Sigma, X) \rightarrow (\Sigma', X')$ *and* $(g, g'): (\Sigma', X') \rightarrow (\Sigma'', X'')$ *the composite is the map* $(f, f')(g, g') := (fg, \langle \pi_0 f, f' \rangle g')$;

Restriction: $\overline{(f, f')} := (\overline{f}, \overline{f'}\pi_1)$.

The **strict simple fibration** is the subcategory $\mathbb{X}[\mathbb{X}]$ determined by the maps (f, f') for which $\overline{\pi_0 f} = \overline{f'}$.

The strict simple fibration was discussed in the introduction with respect to the differential of smooth partial maps as this differential may be seen as a section of the strict simple fibration.

3.5. PROPOSITION. *If \mathbb{X} is a Cartesian restriction category, then $\mathbb{X}(\mathbb{X})$ and $\mathbb{X}[\mathbb{X}]$ are Cartesian restriction categories, and the obvious projections to \mathbb{X} are latent fibrations.*

PROOF. We begin by showing that $\mathbb{X}(\mathbb{X})$ is a Cartesian restriction category. For identity maps to be well-defined we need $\overline{\pi_0 1} \geq \overline{\pi_1}$, which is obviously true as both sides are restrictions of total maps. To show composition is well defined consider $(f, f')(g, g') = (fg, \langle \pi_0 f, f' \rangle g')$: we must show $\overline{\pi_0 fg} \geq \overline{\langle \pi_0 f, f' \rangle g'}$ which is so by:

$$\overline{\langle \pi_0 f, f' \rangle g'} \leq \overline{\langle \pi_0 f, f' \rangle \pi_0 g} = \overline{f' \pi_0 f g} = \overline{f' \pi_0 f \pi_0 f g} = \overline{\pi_0 f \pi_0 f g} = \overline{\pi_0 f g}.$$

For the restriction product structure, set $(\Sigma, X) \times (\Sigma', X') := (\Sigma \times \Sigma', X \times X')$,

$$\pi_0 := (\pi_0, \pi_1 \pi_0): (\Sigma, X) \times (\Sigma', X') \rightarrow (\Sigma, X), \pi_1 := (\pi_1, \pi_1 \pi_1): (\Sigma, X) \times (\Sigma', X') \rightarrow (\Sigma', X')$$

and

$$\langle (f, f'), (g, g') \rangle := (\langle f, g \rangle, \langle \langle \pi_0 \pi_0 f, \pi_1 \pi_0 \rangle f', \langle \pi_0 \pi_1 g, \pi_1 \pi_1 \rangle g' \rangle).$$

We leave the remaining details of checking this is a Cartesian restriction category to the reader.

There is an obvious restriction functor

$$\pi: \mathbb{X}(\mathbb{X}) \rightarrow \mathbb{X}; \quad \begin{array}{ccc} (\Sigma, X) & & \Sigma \\ (f, f') \downarrow & \mapsto & \downarrow f \\ (\Sigma', X') & & \Sigma' \end{array}$$

We set the prone arrow of $f: \Sigma \rightarrow \Sigma'$ at (Σ', X) to be $(f, \overline{\pi_0 f} \pi_1): (\Sigma, X) \rightarrow (\Sigma', X)$; the lifting property is given by

$$\begin{array}{ccc} (\Gamma, Y) & \xrightarrow{(g, g')} & (\Sigma', X) \\ \tilde{h} := (h, g') \downarrow \text{dotted} & & \downarrow \\ (\Sigma, X) & \xrightarrow{(f, \overline{\pi_0 f} \pi_1)} & (\Sigma', X) \end{array} \quad \mapsto \quad \begin{array}{ccc} \Gamma & & \\ h \downarrow & \searrow g & \\ \Sigma & \xrightarrow{f} & \Sigma' \end{array}$$

The lifting \tilde{h} is well-defined as $\overline{g'} \leq \overline{\pi_0 g} = \overline{\pi_0 h}$ since the base triangle is precise. The top triangle commutes since

$$\begin{aligned} & (h, g')(f, \overline{\pi_0 f \pi_1}) \\ &= (hf, \langle \pi_0 h, g' \rangle \overline{\pi_0 f \pi_1}) \\ &= (g, \langle \pi_0 h, g' \rangle \pi_0 f \langle \pi_0 h, g' \rangle \pi_1) \\ &= (g, \overline{g' \pi_0 h f \pi_0 h g'}) \\ &= (g, \overline{g' \overline{\pi_0 g} \pi_0 h g'}) \\ &= (g, g') \quad (\text{since } \overline{g'} \leq \overline{\pi_0 g} = \overline{\pi_0 h}) \end{aligned}$$

The top triangle is precise since the bottom triangle is. The uniqueness of \tilde{h} follows from a similar calculation to showing the top triangle commutes. Thus, the lax slice $\mathbb{X}(\mathbb{X})$ is a latent fibration, and the result for the strict slice $\mathbb{X}[\mathbb{X}]$ follows similarly. ■

3.5.1. CODOMAIN LATENT FIBRATIONS. When \mathbb{X} has latent pullbacks, restriction variants of the codomain fibration form another source of examples. In fact, just as in the ordinary case (for example, see [15]), a restriction category has latent pullbacks if and only if the underlying functor from the (strict) arrow category is a latent fibration.

3.6. DEFINITION. For any restriction category \mathbb{X} , the restriction category $\mathbb{X}^{\rightsquigarrow}$ is defined as follows:

Objects: are maps $a: A' \rightarrow A$ of \mathbb{X} ;

Maps: are pairs of maps $(f, f'): a \rightarrow b$ such that $f'b \leq af$ and $f'\bar{b} = f'$:

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ a \downarrow & \geq & \downarrow b \\ A & \xrightarrow{f} & B \end{array}$$

we will refer to such lax squares as being **semi-precise**;

Composition: if $(f, f'): a \rightarrow b$ and $(g, g'): b \rightarrow c$ then the composite is $(fg, f'g'): a \rightarrow c$;

Restriction: the restriction of $(f, f'): a \rightarrow b$ is $(\bar{f}, \bar{f}'): a \rightarrow a$.

The restriction category \mathbb{X}^{\rightarrow} has the same definition except that the maps require the squares commute (and are still semi-precise). Thus there is an embedding $\mathbb{X}^{\rightarrow} \subseteq \mathbb{X}^{\rightsquigarrow}$ which is the identity on objects but, in general, a strict inclusion on maps.

We define the functor $\partial^{\rightsquigarrow} : \mathbb{X}^{\rightsquigarrow} \rightarrow \mathbb{X}$ by

$$\partial^{\rightsquigarrow} : \mathbb{X}^{\rightsquigarrow} \rightarrow \mathbb{X}; \quad \begin{array}{ccc} & a & A \\ & \downarrow (f, f') & \downarrow f \\ & b & B \end{array} \mapsto$$

it is clearly a restriction functor. The restriction functor $\partial^{\rightarrow} : \mathbb{X}^{\rightarrow} \rightarrow \mathbb{X}$ is defined similarly.

3.7. PROPOSITION. *For any restriction category, \mathbb{X} , the categories $\mathbb{X}^{\rightsquigarrow}$ and \mathbb{X}^{\rightarrow} are restriction categories, and the restriction functors, $\partial : \mathbb{X}^{\rightsquigarrow} \rightarrow \mathbb{X}$ and $\partial : \mathbb{X}^{\rightarrow} \rightarrow \mathbb{X}$ are latent fibrations if and only if \mathbb{X} has latent pullbacks.*

PROOF. In $\mathbb{X}^{\rightsquigarrow}$, composites are well-defined as

$$afg \geq f'bg \geq f'g'c \quad \text{and} \quad f'g'\bar{c} = f'(g'\bar{c}) = f'g'$$

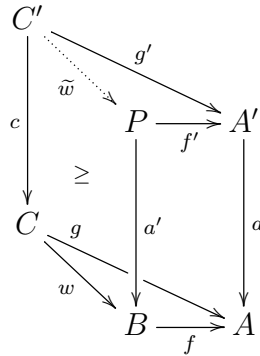
and restrictions are well-defined as

$$a\bar{f} = \overline{afa} \geq \overline{f'ba} = \overline{f'ba} = \overline{f'a} \quad \text{and} \quad \overline{f' \bar{a}} = \overline{f'b \bar{a}} = \overline{f'b \bar{a}} = \overline{f'b \bar{a} f \bar{a}} = \overline{f'b \bar{a} f} = \overline{f'b} = \overline{f'}$$

Similarly composites and restrictions are well-defined in \mathbb{X}^{\rightarrow} .

First we show that, when \mathbb{X} has latent pullbacks, we have ∂ -prone liftings, making it a latent fibration.

Latent pullback gives a prone arrow in $\mathbb{X}^{\rightsquigarrow}$ by considering



where the back face is a lax semi-precise square (so $cg \geq g'a$), the bottom triangle is precise (so $\bar{w} = \bar{g}$ and it commutes) and the square with apex P is a latent pullback. The back face $g'a = g'$ -commutes so there is a unique \tilde{w} with $\tilde{w} = \bar{g}$, $\tilde{w}f' \leq g'$, $\tilde{w}a' \leq cw$. The top triangle is now clearly precise as $\tilde{w}f' = g'$ since $\tilde{w} = \overline{\tilde{w}f'} = \bar{g}$. Finally, we must show the left square is semi-precise:

$$\overline{\tilde{w}a'} = \overline{\tilde{w}a'\tilde{w}} = \overline{\tilde{w}a'f\tilde{w}} = \overline{\tilde{w}a'f\tilde{w}} = \overline{\tilde{w}f'a\tilde{w}} = \overline{\tilde{w}f'a\tilde{w}} = \overline{\tilde{w}f'\tilde{w}} = \overline{\tilde{w}f'\tilde{w}} = \overline{\tilde{w}\tilde{w}} = \tilde{w}.$$

This shows that latent pullbacks are prone arrows; thus, if \mathbb{X} has latent pullbacks $\partial : \mathbb{X}^{\rightsquigarrow} \rightarrow \mathbb{X}$ is a latent fibration.

We need to show that \mathbb{X}^{\rightarrow} also has prone liftings given by the latent pullback. For this we need to show that if the back square is an equality the front must also be an equality. This is given by the following calculation:

$$\overline{\widehat{w}a'} = \overline{\widehat{w}} = \overline{g'} = \overline{g'a} = \overline{c\overline{g}} = \overline{c\overline{g}} = \overline{c\overline{w}} = \overline{c\overline{w}}$$

Conversely, suppose $\partial: \mathbb{X}^{\rightsquigarrow} \rightarrow \mathbb{X}$ is a latent fibration and we have a cospan

$$f : B \rightarrow A, a : A' \rightarrow A.$$

Then we get a prone lift of f ; that is, is a semi-precise square

$$\begin{array}{ccc} P & \xrightarrow{f'} & A' \\ a' \downarrow & \geq & \downarrow a \\ B & \xrightarrow{f} & A \end{array}$$

We can then modify a' by the idempotent $\overline{f'} = \overline{f'a}$ to get a precise square

$$\begin{array}{ccc} P & \xrightarrow{f'} & A' \\ \overline{f'a'} \downarrow & = & \downarrow a \\ B & \xrightarrow{f} & A \end{array}$$

and we claim this is a latent pullback of f and a . Indeed, if we have an e -commuting square

$$\begin{array}{ccc} Z & \xrightarrow{z_0} & A' \\ z_1 \downarrow & & \downarrow a \\ B & \xrightarrow{f} & A \end{array}$$

Then we can make a lax semi-precise square

$$\begin{array}{ccc} Z & \xrightarrow{ez_0} & A' \\ z_1 \downarrow & \geq & \downarrow a \\ B & \xrightarrow{f} & A \end{array}$$

and then the universal property of the prone lift of f gives a unique map from Z to P which one can check satisfies the required properties to make the precise square a latent pullback.

If $\partial: \mathbb{X}^{\rightarrow} \rightarrow \mathbb{X}$ is a latent fibration, then the existence of latent pullbacks follows similarly to above, except that last lax semi-precise square above is replaced by the semi-precise square

$$\begin{array}{ccc} Z & \xrightarrow{e_{z_0}} & A' \\ e_{z_1} \downarrow & = & \downarrow a \\ B & \xrightarrow{f} & A \end{array}$$

and one uses the universal property of the prone lift of f relative to this square instead. ■

If \mathbb{X} has latent pullbacks, it is not hard to see that the (strict/lax) simple latent fibration can be embedded into the (strict/lax) codomain latent fibration in a manner analogous to that for ordinary fibrations:

$$\begin{array}{ccc} \mathbb{X}(\mathbb{X}) & \xrightarrow{\quad} & \mathbb{X}^{\rightsquigarrow} \\ \pi \downarrow & & \downarrow \partial \\ \mathbb{X} & & \mathbb{X} \end{array} \quad ; \quad \begin{array}{ccc} (\Sigma, X) & & \\ \downarrow (f, f') & \mapsto & \\ (\Sigma', X') & & \end{array} \quad \mapsto \quad \begin{array}{ccc} \Sigma \times X & \xrightarrow{\pi_0} & \Sigma \\ \langle \pi_0 f, f' \rangle \downarrow & \geq & \downarrow f \\ \Sigma' \times X' & \xrightarrow{\pi_0} & \Sigma' \end{array}$$

3.7.1. DISCRETE LATENT FIBRATIONS. Let \mathbb{X} be a restriction category and $F: \mathbb{X}^{\text{op}} \rightarrow \mathbf{Set}$ any functor (which ignores the restriction structure), then we may form the category of elements of F , $\mathbf{Elt}(F)$:

Objects: (X, x) where X is an object of \mathbb{X} and x is an element of $F(X)$;

Maps: $f: (X, x) \rightarrow (Y, y)$ where $f: X \rightarrow Y$ in \mathbb{X} and $x = F(f)(y)$.

The functor $\partial_F: \mathbf{Elt}(F) \rightarrow \mathbb{X}$ has $\partial_F(X, x) = X$ and $\partial_F(f) = f$ is a discrete fibration in the usual sense. It is also a discrete latent fibration, in the sense that all the fibres are discrete.

The restrictions in $\mathbf{Elt}(\mathbb{X})$ are just the restrictions in \mathbb{X} . We must check that this is well-defined; that is, that if $f: (X, x) \rightarrow (Y, y)$ that $\bar{f}: (X, x) \rightarrow (X, x)$ is a map of $\mathbf{Elt}(F)$. For this we need $F(\bar{f})(x) = x$ which follows as $F(\bar{f})(x) = F(\bar{f})(F(f)(y)) = F(\bar{f}f)(y) = F(f)(y) = x$. The restriction identities are then immediate.

This is a latent fibration as one can easily check that the prone map above $f: X \rightarrow Y$ at (Y, y) is the usual Cartesian map, namely, $f: (X, F(f)(y)) \rightarrow (Y, y)$.

3.7.2. LATENT FIBRATIONS OF PROPOSITIONS. One way in which restriction idempotents can be used to construct a discrete latent fibration is as follows:

3.8. DEFINITION. Let \mathbb{X} be a restriction category, and define the restriction category $\mathcal{O}(\mathbb{X})$ by:

Objects: pairs (X, e) where X is an object of \mathbb{X} , and $e \in \mathcal{O}(X)$ is a restriction idempotent on X .

Maps: $f: \underline{(X, e)} \rightarrow (X', e')$ are maps $f: X \rightarrow X'$ of \mathbb{X} such that $e \leq \overline{f e'}$, or equivalently $e = e f e'$.

Composition: is composition in \mathbb{X} . This is well-defined since if

$$(X, e) \xrightarrow{f} (X', e') \xrightarrow{f'} (X'', e'')$$

are maps of $\mathcal{O}(\mathbb{X})$ given by $f: X \rightarrow X'$ and $f': X' \rightarrow X''$ in \mathbb{X} , then we have

$$\overline{e f f' e''} = \overline{e f e' f f' e''} = \overline{e f e' f f' e''} = \overline{e f e' f' e''} = \overline{e f e' f' e''} = \overline{e f e'} = e$$

so that $f f'$ gives a map $(X, e) \rightarrow (X'', e'')$ in $\mathcal{O}(\mathbb{X})$.

Identities: as in \mathbb{X} . That is, $1_{(X, e)} = 1_X$: this is well-defined as $e = e \overline{1_X} e$.

Restriction: also as in \mathbb{X} , with $\overline{f}: (X, e) \rightarrow (X, e)$ for $f: (X, e) \rightarrow (X', e')$ given by $\overline{f}: X \rightarrow X$. This is well-defined as

$$e = \overline{e f e'} = \overline{e f f e'} = \overline{e e f f e'} = \overline{e f e f e'} = \overline{e f e f e'} = \overline{e f e}$$

3.9. PROPOSITION. The canonical map $\mathbf{p}_{\mathcal{O}}: \mathcal{O}(\mathbb{X}) \rightarrow \mathbb{X}$ is a latent fibration.

We refer to $\mathbf{p}_{\mathcal{O}}$ as the **latent fibration of propositions**.

PROOF. Associativity of composition, the restriction combinator axioms, and the requirements on identity maps all hold in $\mathcal{O}(\mathbb{X})$ immediately since they hold in \mathbb{X} .

There is an obvious restriction functor

$$\mathbf{p}_{\mathcal{O}}: \mathcal{O}(\mathbb{X}) \rightarrow \mathbb{X}; \quad \begin{array}{ccc} (X, e) & & X \\ f \downarrow & \mapsto & f \downarrow \\ (Y, d) & & Y \end{array}$$

Suppose $f: X' \rightarrow X$ in \mathbb{X} and let (X, e) be an object of $\mathcal{O}(\mathbb{X})$. Then $(X', \overline{f e})$ is also an object of $\mathcal{O}(\mathbb{X})$, and $f: (X', \overline{f e}) \rightarrow (X, e)$ is a map in $\mathcal{O}(\mathbb{X})$ since $\overline{f e} = \overline{f e f e}$. We shall show that this map is prone over f . To that end, suppose that $g: (Y, e') \rightarrow (X, e)$ and $h: Y \rightarrow X'$ are maps in $\mathcal{O}(\mathbb{X})$ and \mathbb{X} respectively such that

$$\begin{array}{ccc} Y & & \\ h \downarrow & \searrow g & \\ X' & \xrightarrow{f} & X \end{array}$$

is precise in \mathbb{X} , then $h: (Y, e') \rightarrow (X', \overline{f e})$ is a map in $\mathcal{O}(\mathbb{X})$ since $\overline{h f e} = \overline{h f e} = \overline{g e} \geq e'$. Furthermore this gives a precise triangle in $\mathcal{O}(\mathbb{X})$ showing $f: (X', \overline{f e}) \rightarrow (X, e)$ is prone. ■

3.9.1. ASSEMBLY CATEGORIES. Another example comes from [21] and is constructed from a category of assemblies, $\mathbf{Asm}(F)$, associated to a restriction functor, $F: \mathbb{A} \rightarrow \mathbb{X}$, from any restriction category, \mathbb{A} , called the category of **realizers**, into a Cartesian restriction category, \mathbb{X} , the base.

3.10. DEFINITION. *The category of assemblies, $\mathbf{Asm}(F)$, is defined as follows:*

Objects: $\varphi \in \mathcal{O}(F(A) \times X)$ for all objects $A \in \mathbb{A}$ and $X \in \mathbb{X}$.

Maps: $f: \varphi \rightarrow \varphi'$, where $\varphi \in \mathcal{O}(F(A) \times X)$ and $\varphi' \in \mathcal{O}(F(A') \times X')$, are maps $f: X \rightarrow X'$ for which there is a “tracking” map $\gamma: A \rightarrow A'$ in \mathbb{A} satisfying:

$$[\mathbf{Tk.1}] \quad \varphi(F(\gamma) \times f) = \varphi(F(\gamma) \times f)\varphi'$$

$$[\mathbf{Tk.2}] \quad \overline{\varphi(1 \times f)} = \overline{\varphi(F(\gamma) \times f)}$$

Restriction: As in \mathbb{X} .

The proof that $\mathbf{Asm}(F)$ is a restriction category – and, indeed, when \mathbb{A} is a Cartesian restriction category and F preserves this Cartesian structure, then $\mathbf{Asm}(F)$ is a Cartesian restriction category – may be found in [21, Prop. 5.2 and 5.3].

If \mathbb{X} is a Cartesian restriction category and $F: \mathbb{A} \rightarrow \mathbb{X}$ is a restriction functor, the forgetful functor $\mathbf{p}: \mathbf{Asm}(F) \rightarrow \mathbb{X}$ is a latent fibration. Specifically, if $\varphi \in \mathcal{O}(F(A) \times X)$, $\psi \in \mathcal{O}(F(B) \times Y)$, and $f: \varphi \rightarrow \psi$ a map in $\mathbf{Asm}(F)$, then $\mathbf{p}(f)$ is $f: X \rightarrow Y$, the underlying map in \mathbb{X} . Clearly \mathbf{p} is a Cartesian restriction functor.

3.11. PROPOSITION. $\mathbf{p}: \mathbf{Asm}F \rightarrow \mathbb{X}$ as defined above is a latent fibration.

PROOF. Suppose $f: X \rightarrow Y$ is a map in \mathbb{X} , and let $\psi \in \mathcal{O}(F(B) \times Y)$ be an assembly. Then $(1 \times f)\psi \in \mathcal{O}(F(B) \times X)$ is also an assembly, and f can be viewed as a map $(1 \times f)\psi \rightarrow \psi$ in $\mathbf{Asm}(F)$ which is readily seen to be prone. ■

3.11.1. IDEMPOTENT SPLITTING. We end these examples with a latent fibration which is a genuine semifunctor.

3.12. PROPOSITION. *For any restriction category \mathbb{X} , the forgetful functor from the restriction idempotent splitting of \mathbb{X} back to \mathbb{X} , $U: \mathbf{Split}_r(\mathbb{X}) \rightarrow \mathbb{X}$, is a latent fibration.*

PROOF. Recall that in $\mathbf{Split}_r(\mathbb{X})$:

- an object is a pair (X, e) with e a restriction idempotent on \mathbb{X} ;
- a map $f: (X, e) \rightarrow (X', e')$ is a map $f: X \rightarrow X'$ such that $efe' = f$;
- composition and restriction are as in \mathbb{X} ;
- the identity of (X, e) is e itself.

Thus in general the forgetful functor $U: \mathbf{Split}_r(\mathbb{X}) \rightarrow \mathbb{X}$ is a genuine semifunctor (that is, it preserves composition and restriction but not necessarily identities).

To see that it is a latent fibration, suppose (X, e) is an object of $\mathbf{Split}_r(\mathbb{X})$, and $f: X' \rightarrow \mathbf{p}(X)$ in \mathbb{X} is such that $f = fU(1_{(X,e)}) = fe$. We want to show that $f: (X', \overline{fe}) \rightarrow (X, e)$ is a prone lift of f . Indeed, f is well-defined since

$$\overline{fe}f = f\overline{ee} = fe = f$$

If we have a precise factorization $hf = g$, then h is the unique lift:

$$\begin{array}{ccc} (X'', e') & & X'' \\ \downarrow h & \searrow g & \downarrow h \\ (X', \overline{fe}) & \xrightarrow{f} & (X, e) \end{array} \quad \mapsto \quad \begin{array}{ccc} X'' & & X \\ \downarrow h & \searrow g & \downarrow h \\ X' & \xrightarrow{f} & X \end{array}$$

It is well-defined since

$$e''h\overline{fe} = \overline{e''hfe} = \overline{e''geh} = \overline{gh} = \overline{hh} = h,$$

and is clearly unique. ■

3.13. BASIC THEORY OF LATENT FIBRATIONS AND PRONE ARROWS. A simple observation is that a latent fibration reduces to an ordinary fibration when it is between total map categories. To see this note that (i) a restriction semifunctor between such categories must be an ordinary functor, and (ii) the requirement of preciseness is automatically satisfied by total maps and so the condition of having enough prone maps reduces to the definition of a Cartesian arrow in an ordinary fibration.

3.14. LEMMA. *A latent fibration between restriction categories in which all maps are total is a fibration.*

Just as for ordinary fibrations we can define morphisms of latent fibrations:

3.15. DEFINITION. *A morphism of latent fibrations*

$$F = (F_1, F_0): (\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}) \rightarrow (\mathbf{p}': \mathbb{E}' \rightarrow \mathbb{B}')$$

is a pair of restriction semifunctors, between the domain categories $F_1: \mathbb{E} \rightarrow \mathbb{E}'$, and the codomain categories $F_0: \mathbb{B} \rightarrow \mathbb{B}'$, making

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{F_1} & \mathbb{E}' \\ \mathbf{p} \downarrow & & \downarrow \mathbf{q} \\ \mathbb{B} & \xrightarrow{F_0} & \mathbb{B}' \end{array}$$

commute, and such that F_1 preserves prone maps.

Transformations of fibrations are “pillows” of restriction transformations $(\alpha, \beta): (F_1, F_0) \rightarrow (G_1, G_0)$ with $\alpha\mathbf{q} = \mathbf{p}\beta$.

Many results for Cartesian arrows for an ordinary functor have a restriction analogue for prone arrows for a restriction functor. For example:

3.16. LEMMA. *For any restriction semifunctor, $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$, composites of \mathbf{p} -prone arrows in \mathbb{E} are \mathbf{p} -prone and identity maps are \mathbf{p} -prone.*

PROOF. Suppose f_1 and f_2 are prone in \mathbb{E} and $k\mathbf{p}(f_1f_2) = \mathbf{p}(g)$ is a precise triangle. Then $(k\mathbf{p}(f_1))\mathbf{p}(f_2)$ is also precise as:

$$k\mathbf{p}(f_1)\overline{\mathbf{p}(f_2)} = \overline{k\mathbf{p}(f_1f_2)}k\mathbf{p}(f_1) = \overline{\overline{k\mathbf{p}(f_1f_2)}}k\mathbf{p}(f_1) = \overline{\bar{k}}k\mathbf{p}(f_1) = k\mathbf{p}(f_1)$$

Now as f_2 is prone we have a unique $\widetilde{k\mathbf{p}(f_1)}$ sitting above making $\widetilde{k\mathbf{p}(f_1)}f_2$ precise. But then as $k\mathbf{p}(f_1) = \mathbf{p}(\widetilde{k\mathbf{p}(f_1)})$ is precise this gives a unique \widetilde{k} with $\widetilde{k}f_1 = \widetilde{k\mathbf{p}(f_1)}$ precise. This certainly means $\widetilde{k}(f_1f_2) = \widetilde{k\mathbf{p}(f_1)}f_2 = g$ but also this is precise as $\widetilde{k} = \overline{\widetilde{k\mathbf{p}(f_1)}}$ as $\widetilde{k}f_1 = \widetilde{k\mathbf{p}(f_1)}$ precisely commutes, and $k\mathbf{p}(f_1) = \overline{\widetilde{k}}$ as $\widetilde{k\mathbf{p}(f_1)}f_2 = g$ precisely commutes, so $\widetilde{k} = \overline{\widetilde{k}}$ showing $\widetilde{k}(f_1f_2) = g$ precisely commutes.

Finally \widetilde{k} is unique as supposing h was an alternate then $(hf_1)f_2 = g$ precisely commutes with $\mathbf{p}(hf_1) = k\mathbf{p}(f_1)$ making $hf_1 = \widetilde{k\mathbf{p}(f_1)}$ but then by similar reasoning $h = \widetilde{k}$.

Identity maps are clearly always prone when \mathbf{p} is a restriction functor, but this is not completely immediate if \mathbf{p} is only a semifunctor. However, it still works in this case as if we have

$$\begin{array}{ccc} Y & & \mathbf{p}(Y) \\ \downarrow \text{dotted} & \searrow g & \downarrow h \\ X & \xrightarrow{1_X} & X \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathbf{p}(Y) & & \mathbf{p}(X) \\ \downarrow h & \searrow \mathbf{p}(g) & \downarrow \mathbf{p}(1_X) \\ \mathbf{p}(X) & \xrightarrow{\mathbf{p}(1_X)} & \mathbf{p}(X) \end{array}$$

with the base triangle precise, then since $\mathbf{p}(1_X)$ is a restriction idempotent, by Lemma 2.7.ii, $h = \mathbf{p}(g)$. Thus g is the unique fill-in for the triangle in \mathbb{E} . ■

As we shall see, however, restriction idempotents are *not* always prone: see Section 4 for details, and Proposition 4.13 for a characterization of which semifunctors have all restriction idempotents prone.

In the total case, it is well-known that any two Cartesian arrows over the same arrow (with a common codomain) have a unique mediating isomorphism. For prone arrows we now show that the analogous situation induces a mediating *partial* isomorphism.

3.17. DEFINITION. *In a restriction category, a **mediating** map between two maps f and f' with a common codomain, is a partial isomorphism, α , such that $\alpha f' = f$ and $\alpha^{(-1)}f = f'$ are both precise.*

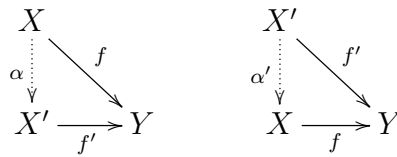
3.18. LEMMA. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a restriction semifunctor, then:*

- (i) *If $f: X \rightarrow Y$ and $f': X' \rightarrow Y$ are \mathbf{p} -prone maps of \mathbb{E} with $\mathbf{p}(f) = \mathbf{p}(f')$ then there is a unique mediating partial isomorphism $\alpha: X \rightarrow X'$ (so that $\mathbf{p}(\alpha) = \overline{\mathbf{p}(f)}$, $\alpha f' = f$, and $\overline{\alpha} = \overline{f}$, $\alpha^{(-1)}f = f'$, and $\overline{\alpha^{(-1)}} = \overline{f'}$).*

- (ii) If α is a mediating partial isomorphism in \mathbb{E} between f and f' (so that $\alpha f' = f$, and $\bar{\alpha} = \bar{f}$, $\alpha^{(-1)} f = f'$, and $\overline{\alpha^{(-1)}} = \bar{f}'$ as above) then, if either f or f' is \mathbf{p} -prone, then both f and f' are \mathbf{p} -prone.
- (iii) If $e, e': X \rightarrow X$ are both prone restriction idempotents in \mathbb{E} such that $\mathbf{p}(e) = \mathbf{p}(e')$, then $e = e'$.

PROOF.

- (i) Let α, α' be the liftings for the precise triangle $\overline{\mathbf{p}(f)}\mathbf{p}(f) = \mathbf{p}(f)$ (recalling $\mathbf{p}(f) = \mathbf{p}(f')$):



The fact that they are precise triangles implies immediately that $\bar{\alpha} = \bar{f}$ and $\bar{\alpha}' = \bar{f}'$. The composite of these precise triangles sits over the same precise triangle, $\overline{\mathbf{p}(f)}\mathbf{p}(f) = \mathbf{p}(f)$, and so, using Lemma 2.7.iv, $\alpha\alpha' = \bar{f}$ and $\alpha'\alpha = \bar{f}'$ making them partial inverses.

- (ii) Suppose f and f' are mediated by α , then the precise triangles on f are in bijective correspondence to those on f' , as a precise $kf = g$ is carried to a precise $(k\alpha)f' = g$ from which the result is immediate.
- (iii) By (i), there is a mediating partial isomorphism $\alpha: X \rightarrow X$ between e and e' . In particular, $\alpha e = e'$ is precise. But then by Lemma 2.7.ii, this means $\alpha = e'$, and similarly $\alpha^{(-1)} = e$. But then α is a restriction idempotent, so its partial inverse is itself, so we have $e = \alpha^{(-1)} = \alpha = e'$.

■

As noted above, restriction idempotents are not always prone. However, isomorphisms, and, more generally, restriction retractions are:

3.19. LEMMA. If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a restriction semifunctor, then:

- (i) Isomorphisms are always \mathbf{p} -prone and, if f is total and \mathbf{p} -prone with $\mathbf{p}(f)$ an isomorphism and \mathbf{p} a restriction functor, then f is itself an isomorphism.
- (ii) All restriction retractions in \mathbb{E} are \mathbf{p} -prone.
- (iii) If $f = rf'$ is \mathbf{p} -prone in \mathbb{E} and r is a restriction retraction, then f' is \mathbf{p} -prone.
- (iv) If \mathbf{p} is a restriction functor, and f is a \mathbf{p} -prone map such that \bar{f} splits, then $\mathbf{p}(f)$ is a restriction retraction if and only if f is a restriction retraction.

PROOF.

- (i) That isomorphisms are prone is immediate (and follows for example from part (ii), see below). Suppose $f: E' \rightarrow E$ is total, prone, and $\mathbf{p}(f)$ is an isomorphism with \mathbf{p} a restriction functor, then $\mathbf{p}(f)^{-1}$ lifts:

$$\begin{array}{ccc}
 E & & \mathbf{p}(E) \\
 \downarrow \widetilde{\mathbf{p}(f)^{-1}} & \searrow & \downarrow \mathbf{p}(f)^{-1} \\
 E' & \xrightarrow{f} & E \\
 & & \searrow \\
 & & \mathbf{p}(E') \xrightarrow{\mathbf{p}(f)} \mathbf{p}(E)
 \end{array}
 \quad \mapsto \quad
 \begin{array}{ccc}
 \mathbf{p}(E) & & \\
 \downarrow \mathbf{p}(f)^{-1} & \searrow & \\
 \mathbf{p}(E') & \xrightarrow{\mathbf{p}(f)} & \mathbf{p}(E)
 \end{array}$$

and then we have

$$\begin{array}{ccc}
 E' & & \mathbf{p}(E) \\
 \downarrow \widetilde{\mathbf{p}(f)^{-1}} & \searrow f & \downarrow \\
 E' & \xrightarrow{f} & E \\
 & & \searrow \\
 & & \mathbf{p}(E') \xrightarrow{\mathbf{p}(f)} \mathbf{p}(E)
 \end{array}
 \quad \mapsto \quad
 \begin{array}{ccc}
 \mathbf{p}(E) & & \\
 \parallel & \searrow f & \\
 \mathbf{p}(E') & \xrightarrow{\mathbf{p}(f)} & \mathbf{p}(E)
 \end{array}$$

showing $f \widetilde{\mathbf{p}(f)^{-1}}$ is a lifting of the identity map. However, an alternate lifting is the identity map so $f \widetilde{\mathbf{p}(f)^{-1}} = 1_{E'}$, showing f is an isomorphism. This is the same argument for why Cartesian arrows above isomorphisms are isomorphisms in an ordinary fibration.

- (ii) A restriction retraction, r , is a partial isomorphism whose inverse is a section m , thus $rm = \bar{r}$ and $mr = 1$. Suppose $\mathbf{p}(g) = h\mathbf{p}(r)$ then, by Lemma 2.7.iii, $h = \mathbf{p}(g)\mathbf{p}(m) = \mathbf{p}(gm)$ so that gm is a possible lifting and will necessarily be so provided $gmr = g$ but this is the case as $gmr = g1 = g$.
- (iii) Suppose the restriction retraction $r: Y \rightarrow X'$ has section $m: X' \rightarrow Y$, so $rm = \bar{r}$ and $mr = 1_{X'}$. Suppose that there is a $g: Z \rightarrow X$ so that $h\mathbf{p}(f') = \mathbf{p}(g)$ is precise. Then we claim that there is a lifting \tilde{h} over $h\mathbf{p}(m)$ via the prone-ness of f :

$$\begin{array}{ccc}
 Z & & \mathbf{p}(Z) \\
 \downarrow \tilde{h} & \searrow g & \downarrow h\mathbf{p}(m) \\
 Y & \xrightarrow{f} & X \\
 & & \searrow \\
 & & \mathbf{p}(Y) \xrightarrow{\mathbf{p}(f)} \mathbf{p}(X)
 \end{array}
 \quad \mapsto \quad
 \begin{array}{ccc}
 \mathbf{p}(Z) & & \\
 \downarrow h\mathbf{p}(m) & \searrow \mathbf{p}(g) & \\
 \mathbf{p}(Y) & \xrightarrow{\mathbf{p}(f)} & \mathbf{p}(X)
 \end{array}$$

For this, we need to check that the base triangle is precise. (It obviously commutes by definition of f). For preciseness, note that since $h\mathbf{p}(f') = \mathbf{p}(g)$ is precise,

$$h = \overline{h\mathbf{p}(f')} = \overline{h\mathbf{p}(1_{X'}f')} = \overline{h\mathbf{p}(1_{X'})\mathbf{p}(f')} \leq \overline{h\mathbf{p}(1_{X'})},$$

but we always have the opposite inequality, so $h = \overline{h\mathbf{p}(1_{X'})}$. Thus the base triangle above is precise since

$$\overline{h\mathbf{p}(m)} = \overline{h\mathbf{p}(\bar{m})} = \overline{h\mathbf{p}(1_{X'})} = \bar{h} = \overline{\mathbf{p}(g)}.$$

We now claim that $\tilde{h}r: Z \rightarrow X'$ is the unique lift for h , demonstrating the prone-ness of f' . The triangle commutes since $\tilde{h}rf' = \tilde{h}f = g$, is precise since

$$\tilde{h}r\overline{f'} = \overline{\tilde{h}rf'}\tilde{h}r = \overline{\tilde{h}f}\tilde{h}r = \overline{\tilde{h}}\tilde{h}r = \tilde{h}r,$$

and is over h since

$$\mathbf{p}(\tilde{h}r) = h\mathbf{p}(m)\mathbf{p}(r) = h\mathbf{p}(mr) = h\mathbf{p}(1_{X'}) = h$$

with the final equality proven above.

Finally, suppose we have some other $k: Z \rightarrow X'$ so that $kf' = g, \overline{k} = \overline{g}$, and $\mathbf{p}(k) = h$. Then km is a precise lifting of $h\mathbf{p}(m)$ as

$$\mathbf{p}(km) = \mathbf{p}(k)\mathbf{p}(m) = h\mathbf{p}(m), \quad kmf = kmrf' = kf' = g,$$

and

$$km\overline{f} = \overline{kmf}km = \overline{g}km = \overline{k}km = km.$$

Thus $km = \tilde{h}$, so $kmr = \tilde{h}r$, so $k = \tilde{h}r$. Thus f' is indeed \mathbf{p} -prone.

- (iv) If f is a restriction retraction then $\mathbf{p}(f)$ is certainly a restriction retraction since \mathbf{p} is a restriction functor. For the converse, suppose $f: Y \rightarrow X$ is \mathbf{p} -prone with $\mathbf{p}(f)$ a restriction retraction, so that there is an $m: \mathbf{p}(X) \rightarrow \mathbf{p}(Y)$ with $m\mathbf{p}(f) = 1_{\mathbf{p}(X)}$ and $\overline{\mathbf{p}(f)} = \mathbf{p}(f)m$. Then f is a retraction with \tilde{m} its section:

$$\begin{array}{ccc} X & & \mathbf{p}(X) \\ \tilde{m} \downarrow \text{dotted} & \searrow & m \downarrow \\ Y & \xrightarrow{f} & X & \mapsto & \mathbf{p}(Y) & \xrightarrow{\mathbf{p}(f)} & \mathbf{p}(X) \end{array}$$

so this means, by splitting \overline{f} , that f can be factorized into rf' where $r: Y \rightarrow Y'$ is the coequalizer of \overline{f} and 1_Y and lies over the equalizer of $\mathbf{p}(f)$ and $1_{\mathbf{p}(X)}$ which means f' is prone over an isomorphism. However, f' is also total as using the fact that r is epic we have $r\overline{f'} = \overline{rf'}r = \overline{f}r = r = r1_{Y'}$. It follows that $\mathbf{p}(f)^{-1}f' = 1_X$ and so f' is an isomorphism from part (i). This means, in turn, that $f = rf'$ is a restriction retraction. ■

In the case when \mathbf{p} is a restriction functor and all restriction idempotents in \mathbb{E} split, Cartesian arrows give prone arrows:

3.20. LEMMA. *Suppose that $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a restriction functor, and all restriction idempotents in \mathbb{E} split. If $f: X \rightarrow Y$ a total map in \mathbb{E} which is Cartesian for the functor $\mathbf{Total}(\mathbf{p}): \mathbf{Total}(\mathbb{E}) \rightarrow \mathbf{Total}(\mathbb{B})$, then f is \mathbf{p} -prone.*

PROOF. Suppose we have

$$\begin{array}{ccc} Z & & \mathfrak{p}(Z) \\ & \searrow g & \downarrow h \quad \searrow \mathfrak{p}(g) \\ X & \xrightarrow{f} & Y & \quad \mathfrak{p}(X) \xrightarrow{\mathfrak{p}(f)} \mathfrak{p}(Y) \end{array} \quad \mapsto$$

with the triangle in \mathbb{B} precise, so $\bar{h} = \overline{\mathfrak{p}(g)}$. Let $m: Z' \rightarrow Z$, $r: Z \rightarrow Z'$ be a splitting of \bar{g} , so $rm = \bar{g}$ and $mr = 1$. Then in particular $mg: Z' \rightarrow Y$ is total. Moreover, the pair $(\mathfrak{p}(m), \mathfrak{p}(r))$ gives a splitting of $\mathfrak{p}(g)$ and hence of \bar{h} , so $\mathfrak{p}(m)h$ is also total. Thus, we have diagrams in $\text{Total}(\mathbb{E})$ and $\text{Total}(\mathbb{B})$, and hence get a unique total $h': Z' \rightarrow X$:

$$\begin{array}{ccc} Z' & & \mathfrak{p}(Z) \\ \downarrow h' \quad \searrow mg & & \downarrow \mathfrak{p}(m)h \quad \searrow \mathfrak{p}(m)\mathfrak{p}(g) \\ X & \xrightarrow{f} & Y & \quad \mathfrak{p}(X) \xrightarrow{\mathfrak{p}(f)} \mathfrak{p}(Y) \end{array} \quad \mapsto$$

We now claim that the composite

$$Z \xrightarrow{r} Z' \xrightarrow{h'} X$$

gives the required unique arrow in the first triangle. Indeed, we have

$$rh'f = rmg = \bar{g}g = g,$$

and since h' is total,

$$\overline{rh'} = \bar{r} = \bar{g},$$

and

$$\mathfrak{p}(rh') = \mathfrak{p}(r)\mathfrak{p}(m)h = \mathfrak{p}(rm)h = \mathfrak{p}(\bar{g})h = \overline{\mathfrak{p}(g)}h = \bar{h}h = h,$$

so rh' has all the required properties. Moreover, if there is some other $k: Z \rightarrow X$ with these properties, then mk is total (as $\overline{mk} = \overline{m\bar{g}} = \overline{m\bar{g}} = 1$) and has the same other properties as h' , so $h' = mk$. Thus, $rh' = rmk = \bar{g}k = \bar{k}k = k$, so rh' is unique. ■

It follows from the general theory of fibrations of 2-categories that latent fibrations behave well with respect to composition and pullback. However, it is still helpful to see how this behaviour works out concretely.

3.21. LEMMA. *If $\mathfrak{q}: \mathbb{F} \rightarrow \mathbb{E}$ and $\mathfrak{p}: \mathbb{E} \rightarrow \mathbb{B}$ are restriction semifunctors, f' is \mathfrak{p} -prone over f , and f'' is \mathfrak{q} -prone over f' , then f'' is \mathfrak{qp} -prone over f .*

PROOF. This is given by a straightforward two step lifting:

$$\begin{array}{ccc} \begin{array}{ccc} F & & \\ \downarrow \tilde{h} \quad \searrow g & & \\ B & \xrightarrow{f''} & F \end{array} & \mapsto & \begin{array}{ccc} \mathfrak{q}(F) & & \\ \downarrow \tilde{h} \quad \searrow \mathfrak{q}(g) & & \\ \mathfrak{q}(B) & \xrightarrow{f'} & \mathfrak{q}(F') \end{array} & \mapsto & \begin{array}{ccc} \mathfrak{p}(\mathfrak{q}(F)) & & \\ \downarrow h \quad \searrow \mathfrak{p}(\mathfrak{q}(g)) & & \\ \mathfrak{p}(\mathfrak{q}(B)) & \xrightarrow{f} & \mathfrak{p}(\mathfrak{q}(F')) \end{array} \end{array}$$

■

3.22. COROLLARY. *Latent fibrations are closed under composition.*

Unfortunately, it is not the case that splitting the idempotents of a latent fibration $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$, to obtain $\mathbf{Split}_r(\mathbf{p}): \mathbf{Split}_r(\mathbb{E}) \rightarrow \mathbf{Split}_r(\mathbb{B})$, will yield in general a latent fibration. However, it is the case that splitting the idempotents of the domain category will yield a latent fibration over the same base, $U\mathbf{p}: \mathbf{Split}_r(\mathbb{E}) \rightarrow \mathbb{B}$, as we may precompose with the latent fibration $U: \mathbf{Split}_r(\mathbb{E}) \rightarrow \mathbb{E}$ of subsection 3.11.1. Shortly we shall see that $\mathbf{Split}_r(\mathbf{p})$ is a latent fibration when \mathbf{p} is an *admissible* latent fibration (see Proposition 5.23).

Pullbacks in the category of restriction categories and restriction functors are described in [5], and work similarly for restriction semifunctors:

3.23. DEFINITION. *If $p: \mathbb{E} \rightarrow \mathbb{B}$ and $F: \mathbb{X} \rightarrow \mathbb{B}$ are restriction semifunctors, then their pullback is*

$$\begin{array}{ccc} \mathbb{W} & \xrightarrow{p_1} & \mathbb{E} \\ p_0 \downarrow & & \downarrow \mathbf{p} \\ \mathbb{X} & \xrightarrow{F} & \mathbb{B} \end{array}$$

in which the category \mathbb{W} is defined by

Objects: are pairs (X, E) where X and E are objects of \mathbb{X} and \mathbb{E} respectively which satisfy $F(X) = \mathbf{p}(E)$.

Maps: of type $(X, E) \rightarrow (X', E')$ are pairs (f, g) where f and g are maps of \mathbb{X} and \mathbb{E} respectively which satisfy $F(f) = \mathbf{p}(g)$.

Composition and identities: are defined pointwise.

Restriction: is given by $\overline{(f, g)} = (\overline{f}, \overline{g})$, which is well defined since, if $F(f) = \mathbf{p}(g)$, then $F(\overline{f}) = \overline{F(f)} = \overline{\mathbf{p}(g)} = \mathbf{p}(\overline{g})$.

and the pullback maps p_0, p_1 are the first and second projections.

3.24. LEMMA. *If $p: \mathbb{E} \rightarrow \mathbb{B}$ and $F: \mathbb{X} \rightarrow \mathbb{B}$ are restriction semifunctors and \mathbb{W} is their pullback as defined above, then $(f, g): (X, E) \rightarrow (X', E')$ in \mathbb{W} is \mathbf{p}_0 -prone if and only if g is \mathbf{p} -prone.*

PROOF. Suppose we have

$$\begin{array}{ccc} (X'', E'') & \xrightarrow{(f', g')} & (X', E') \\ (X, E) & \xrightarrow{(f, g)} & (X', E') \end{array} \quad \mapsto \quad \begin{array}{ccc} X'' & & \\ h \downarrow & \searrow f' & \\ X & \xrightarrow{f} & X' \end{array}$$

then since g is p -prone and $F(f) = \mathbf{p}(g), F(f') = \mathbf{p}(g')$, etc., we also have

$$\begin{array}{ccc}
 E'' & & F(X'') \\
 \tilde{h} \downarrow & \searrow^{g'} & \downarrow^{F(h)} \quad \searrow^{F(f')} \\
 E & \xrightarrow{g} & X' \\
 & & \downarrow^{F(f)} \\
 & & F(X) \xrightarrow{F(f)} F(X')
 \end{array}
 \quad \mapsto$$

so (h, \tilde{h}) uniquely fills in the first triangle. Thus, (f, g) is \mathbf{p}_0 -prone; a similar proof shows the converse. ■

3.25. COROLLARY. *The pullback of a latent fibration along any restriction semifunctor is a latent fibration.*

Recall that in an ordinary fibration, any map in the domain category can be factored as a vertical map followed by a Cartesian map, and this factorization is unique up to a unique vertical isomorphism. A similar result holds for latent fibrations, with vertical replaced by *subvertical*:

3.26. DEFINITION. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a restriction semifunctor, then a map $v: X \rightarrow Y$ in \mathbb{E} is said to be **subvertical** if $\mathbf{p}(v)$ is a restriction idempotent.*

3.27. PROPOSITION. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a latent fibration, then for any map $f: X \rightarrow Y$ in \mathbb{E} , there is a subvertical map $v: X \rightarrow X'$ and a prone map $c: X' \rightarrow Y$ such that*

$$\begin{array}{ccc}
 X & & Y \\
 v \downarrow & \searrow^f & \\
 X' & \xrightarrow{c} & Y
 \end{array}$$

is a precise triangle, and, moreover, $\overline{\mathbf{p}(c)} = \mathbf{p}(v)$. Such a factorization is unique up to unique subvertical partial isomorphism.

PROOF. For existence, let $c: X' \rightarrow Y$ be a prone arrow over $\mathbf{p}(f): \mathbf{p}(X) \rightarrow \mathbf{p}(Y)$, and let v be the induced unique map

$$\begin{array}{ccc}
 X & & \mathbf{p}(X) \\
 \vdots \downarrow & \searrow^f & \downarrow^{\mathbf{p}(f)} \\
 X' & \xrightarrow{c} & Y \\
 & & \downarrow^{\mathbf{p}(f)} \\
 & & \mathbf{p}(X') \xrightarrow{\mathbf{p}(f)} \mathbf{p}(Y)
 \end{array}
 \quad \mapsto$$

This satisfies all the required conditions; uniqueness follows from using Lemma 3.18.i. ■

3.28. **SUBSTITUTION SEMIFUNCTORS.** Analogous to the fiber of an object in a fibration, is the notion of a *strand* of an object in a latent fibration.

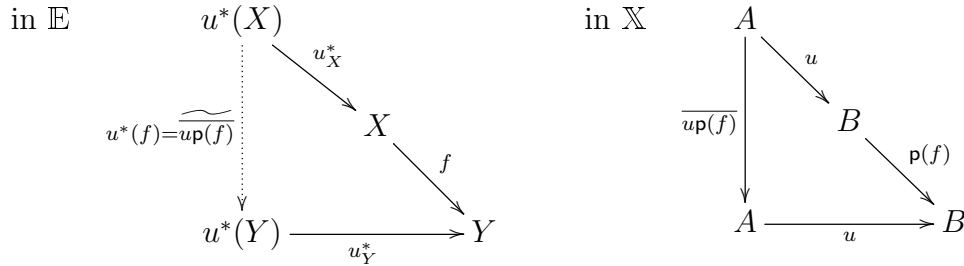
3.29. **DEFINITION.** Given a restriction functor $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$, the **strand** of $B \in \mathbb{B}$, written $\mathbf{p}^{(-1)}(B)$, is the category whose objects are objects E of \mathbb{E} such that $\mathbf{p}(E) = B$, and whose maps are the f of \mathbb{E} such that $\mathbf{p}(f) \leq 1_B$ (i.e. $\mathbf{p}(f) \in \mathcal{O}(B)$); that is, each f is subvertical.

A latent fibration may be *cloven* in the same sense as for ordinary fibrations:

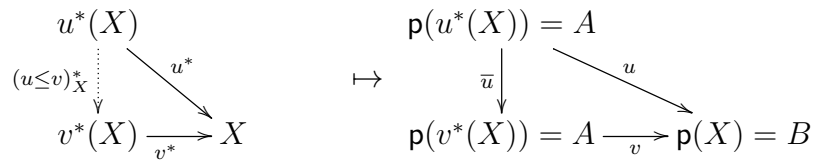
3.30. **DEFINITION.** A latent fibration **has a cleavage** (or is **cloven**) in case there is a chosen prone arrow $f_E^*: f^*(E) \rightarrow E$ over each $f: A \rightarrow \mathbf{p}(E)$ for each $E \in \mathbb{E}$.

We now construct, for a cloven latent fibration, the analogue of a reindexing or substitution functor. In general, the substitution functors we obtain between the strands of a latent fibration will only be restriction semifunctors.

Let $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ be a cloven latent fibration, and let $u: A \rightarrow B$ be a map in \mathbb{B} . We define the *substitution semifunctor* $u^*: \mathbf{p}^{(-1)}(B) \rightarrow \mathbf{p}^{(-1)}(A)$, from the strand above B to the strand above A as follows: on arrows $f: X \rightarrow Y$ in $\mathbf{p}^{(-1)}(B)$, $u^*(f)$ is the arrow $u^*(X) \rightarrow u^*(Y)$ in $\mathbf{p}^{(-1)}(A)$ given by the lifting



where $u^*(X)$ is the domain of the prone map above u with codomain X . Furthermore, if $u \leq v$ then there is for each $X \in \mathbf{p}^{(-1)}(B)$ a map $(u \leq v)_X^*: u^*(X) \rightarrow v^*(X)$ given by:



We then have:

3.31. **PROPOSITION.** If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a latent fibration with a cleavage and $u: A \rightarrow B$ is a map in \mathbb{X} , then $u^*: \mathbf{p}^{(-1)}(B) \rightarrow \mathbf{p}^{(-1)}(A)$ as defined above is a restriction semifunctor. Furthermore, the assignment $(-)^*: \mathbb{B}^{\text{op}} \rightarrow \mathbf{SRest}$ is a pseudofunctor where we regard \mathbb{B} as a 2-category whose 2-cells are given by the restriction ordering of maps.

Recall that a pseudofunctor $P: \mathbb{B}^{\text{op}} \rightarrow \mathbf{SRest}$ associates to each map (1-cell) $f: A \rightarrow B$ in \mathbb{B} a functor (1-cell) $P(f): P(B) \rightarrow P(A)$ in \mathbf{SRest} such that there are (2-cell)

natural isomorphisms $\alpha_{f,g}: P(g)P(f) \rightarrow P(fg)$ and $\alpha_X: 1_{P(X)} \rightarrow P(1_X)$ satisfying:

$$\begin{array}{ccccc}
 P(f) & \xrightarrow{P(f)\alpha_Y} & P(f)P(1_Y) & & P(f) & \xrightarrow{\alpha_X P(f)} & P(1_X)P(f) & & P(h)P(g)P(f) & \xrightarrow{\alpha_{g,h}P(f)} & P(gh)P(f) \\
 & \searrow & \downarrow \alpha_{f,1_x} & & & \searrow & \downarrow \alpha_{f,1_x} & & P(h)\alpha_{f,g} \downarrow & & \downarrow \alpha_{f,gh} \\
 & & P(f) & & & & P(f) & & P(h)P(fg) & \xrightarrow{\alpha_{h,fg}} & P(fgh)
 \end{array}$$

Similarly if $(f \leq g): f \rightarrow g$ is a 2-cell in \mathbb{B} then $P(f \leq g): P(f) \rightarrow P(g)$ must be a 2-cell (or transformation) in \mathbf{SRest} . This assignment must preserve the vertical (2-cell) composition, in the sense that $P(f \leq g)P(g \leq h) = P(f \leq h)$, and for the 1-cell composition we require

$$\begin{array}{ccc}
 P(g); P(f) & \xrightarrow{\alpha_{f,g}} & P(fg) \\
 \downarrow P(g \leq h); P(f \leq k) & & \downarrow P(fg \leq hk) \\
 P(h); P(k) & \xrightarrow{\alpha_{h,k}} & P(hk)
 \end{array}$$

where the semicolon emphasizes that we are using the horizontal composition in \mathbf{SRest} .

Thus, for $P: \mathbb{B}^{\text{op}} \rightarrow \mathbf{SRest}$ the inequalities (which are covariant) become transformations between semifunctors whose composites must be preserved.

PROOF. We must first show that u^* is a semifunctor, that is $u^*(\bar{f}) = \overline{u^*(f)}$ and $u^*(fg) = \overline{u^*(f)u^*(g)}$.

First, suppose $\underline{f}: X \rightarrow Y$ is a map in $\mathfrak{p}^{(-1)}(B)$. Then $u^*(\bar{f})$ is defined as the lifting of $\underline{up(\bar{f})}$. However, $\underline{u^*_X(\bar{f})}$ also makes the triangle precise and $\mathfrak{p}(u^*_X(\bar{f})) = \mathfrak{p}(u^*_X(\underline{f})) = \underline{up(\bar{f})}$ so $u^*(\bar{f}) = \overline{u^*_X(\bar{f})}$ and, thus u^* preserves the restriction.

Next, suppose $\underline{f}: X \rightarrow Y$ and $\underline{g}: Y \rightarrow Z$ are maps of $\mathfrak{p}^{-1}(B)$. Then $u^*(fg)$ is defined as the lifting of $\underline{up(fg)} = \underline{up(f)up(g)}$ while $\overline{u^*(f)u^*(g)}$ is the lifting of $\mathfrak{p}(u^*(f)u^*(g)) = \mathfrak{p}(u^*(f))\mathfrak{p}(u^*(g)) = \underline{up(f)} \underline{up(g)} = \underline{up(f)up(g)} = \underline{up(f)up(g)} = \underline{up(f)up(g)}$ so that they are equal. Thus u^* preserves composition, and is therefore a restriction semifunctor.

To show that $(-)^*$ is a pseudo functor we use the fact that composites of prone maps are unique up to a unique mediating partial isomorphism (combine Lemma 3.16 with Lemma 3.18) and the unit data is provided by the fact that the identity map 1_X is prone so there is a mediating map $X \rightarrow 1_{\mathfrak{p}(X)}^*(X)$.

We must show $\overline{(u \leq v)_X^*} = u^*(1_X)$ and that the transformation is natural; that is, $u^*(f)(u \leq v)_Y^* = \overline{(u \leq v)_X^* v^*(f)}$ (so $(u \leq v)_f^*$ is the identity). The first requirement is immediate from the preciseness of the triangle defining $(u \leq v)_X^*$ as $u^*(1_X) = \overline{u^*_X} = \overline{(u \leq v)_X^*}$. Suppose for the second that $f: X \rightarrow Y$ in the strand over B then $\mathfrak{p}(u^*(f))(u \leq v)_Y^* = \underline{up(f)} \bar{u} = \underline{up(f)}$ and $\mathfrak{p}(\overline{(u \leq v)_X^* v^*(f)}) = \bar{u} \underline{vp(f)} = \bar{u} \underline{vp(f)} = \underline{up(f)}$ so both arrows sit above $\underline{up(f)} \in \mathbb{B}$. As all the components are precise liftings their composites are as well; thus, they are equal. This shows that the transformation is natural as required.

The assignment is immediately functorial with respect to the restriction preorder as the components of the transformations lift restriction idempotents which compose.

The coherence properties with the associator are similarly straightforward to prove. ■

We will see that in an r -split latent fibration $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$, whenever f is total, there is a total prone map above it (see Lemma 5.3). If the cleavage is such that, for every X , f_X^* is total whenever f is total, then substitution semifunctors along total maps become restriction functors.

3.32. PROPOSITION. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a latent fibration with a cleavage which preserves total maps then $f^*: \mathbf{p}^{(-1)}(B) \rightarrow \mathbf{p}^{(-1)}(A)$ is a restriction functor whenever f is total.*

PROOF. We have already shown that f^* is a restriction semifunctor. Since the cleavage of \mathbf{p} preserves total maps, we have that $f_X^*: u^*(X) \rightarrow X$ is total, when f is. This gives

$$\begin{aligned} f^*(1_X) &= f^*(\overline{1_X}) = \overline{f^*(1_X)} = \overline{f^*(1_X)f_X^*} \\ &= \overline{f^*(1_X)f_X^*} = \overline{f_X^*} = 1_{f^*(X)} \end{aligned}$$

and so f^* is a restriction functor. ■

4. Types of Latent Fibrations

In the previous section we described some properties of latent fibrations. However, it is important to note two things that do *not* hold in an arbitrary latent fibration $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$. Both involve the behaviour of restriction idempotents.

The first is that a restriction idempotent in the base need not lift: that is, given a restriction idempotent $e: A \rightarrow A$ in \mathbb{B} and an object X over A in \mathbb{E} , there need not be a prone restriction idempotent $e': X \rightarrow X$ over e . For an example of this, consider the latent fibration of propositions of a restriction category \mathbb{X} (Example 3.8). Suppose $e: A \rightarrow A$ is a restriction idempotent in \mathbb{X} , and (A, e') is an object in $\mathcal{O}(\mathbb{X})$ over A . Since the projection in this case simply sends a map to itself, a restriction idempotent over e must be e itself; thus, we would need e to be a map from (X, e') to (X, e') in $\mathcal{O}(\mathbb{X})$. But this would mean that $e = e'e$, which requires $e \leq e'$. Thus, for any restriction idempotent e' which is not $\geq e$, this is not possible. Of course, being a latent fibration, there is *a* lift of e : as per Proposition 3.9, it is the map

$$(X, ee') \xrightarrow{e} (X, e'),$$

it is just that this map is not a restriction idempotent in $\mathcal{O}(\mathbb{X})$. In fact, just as for ordinary fibrations, it is not even guaranteed to be an endomorphism.

Thus, in this case, there isn't even a restriction idempotent over e (let alone a prone restriction idempotent). As we shall see, however, the ability to lift restriction idempotents to prone restriction idempotents is useful, and is true for most latent fibrations. Thus, in the next section we consider such latent fibrations; we term these *admissible*.

The second issue is the behaviour of restriction idempotents in the domain category of a latent fibration. In particular, while identities are always prone, restriction idempotents (i.e., “partial identities”) need not be. To see the problem in general, suppose $e: X \rightarrow X$ is a restriction idempotent in \mathbb{E} , and suppose we have $g: Y \rightarrow X$ in \mathbb{E} and $h: B \rightarrow A$ in \mathbb{B} such that $hp(e) = p(g)$:

$$\begin{array}{ccc} Y & & A \\ \vdots \downarrow & \searrow g & \downarrow h \\ X & \xrightarrow{e} & X \end{array} \quad \mapsto \quad \begin{array}{ccc} A & & A \\ \downarrow h & \searrow p(g) & \\ A & \xrightarrow{p(e)} & A \end{array}$$

If e were the identity, we could obviously choose g as the unique lift. However, if e is partial, there is no obvious lift that will make the triangle in \mathbb{E} commute.

However, note that we do have some control given that the commutativity of the bottom triangle tells us something about how defined g is relative to e . Thus, if the restriction idempotents in \mathbb{E} were closely related to those in \mathbb{B} , then we could hope to have a lift. In fact, this is the case in the “strict” versions of the simple and codomain fibrations. In these examples, one can check that restriction idempotents are prone (while they are not generally in the “lax” versions).

Thus, restriction idempotents being prone is clearly an important condition that is true in some latent fibrations but not all. We shall see that this condition is equivalent to p being monic on restriction idempotents, or *separated*. Thus, in Section 4.9 we consider the theory and examples of separated latent fibrations.

Many examples (including the motivating example of the strict simple fibration) are both admissible and separated. The combination of these conditions has some very useful consequences (for example, see Proposition 4.23). Moreover, we will see that the combination of the admissible and separated conditions is equivalent to the restriction semifunctor p being *hyperconnected*; that is, p is a bijection on restriction idempotents.

Thus, in the next few sections, we consider the theory and examples of admissible, separated, and hyperconnected latent fibrations.

4.1. ADMISSIBLE LATENT FIBRATIONS.

4.2. DEFINITION. A restriction semifunctor $p: \mathbb{E} \rightarrow \mathbb{B}$ is **admissible** if for every $X \in \mathbb{E}$ and every restriction idempotent, e on $p(X)$ in the base such that $ep(1_X) = e$, there is a prone restriction idempotent e^* on X over e ; that is, with $p(e^*) = e$.

4.3. EXAMPLE. Most of the classes of latent fibrations we have met are admissible:

- (i) The identity $1_{\mathbb{X}}: \mathbb{X} \rightarrow \mathbb{X}$ is admissible.
- (ii) The lax and strict simple fibrations are admissible, as the prone lifting given in Proposition 3.5 is a restriction idempotent.
- (iii) The lax and strict codomain fibrations are admissible; for proof, see the result below.

- (iv) The product latent fibration is admissible; the prone lifting $(1, e)$ of a restriction idempotent e is again a restriction idempotent.
- (v) The assemblies fibration is admissible: given a restriction idempotent $e: X \rightarrow X$ and an assembly ϕ over X , it is easy to check that e (with identity tracking map) is prone in the assemblies category.
- (vi) The splitting fibration $\mathbf{Split}_r(\mathbb{X}) \rightarrow \mathbb{X}$ is admissible. If we are given an object (X, e') in $\mathbf{Split}_r(\mathbb{X})$ and a restriction idempotent $e: X \rightarrow X$ such that $e = ep(1_{(X, e')}) = ee'$, then

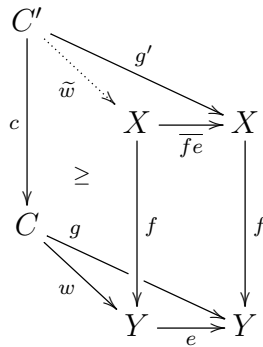
$$(X, e') \xrightarrow{e} (X, e')$$

is a well-defined map in $\mathbf{Split}_r(\mathbb{X})$, and it is straightforward to check that it is prone (since this fibration is separated, this also follows from Proposition 4.13).

- (vii) As noted in the introduction to this section, the latent fibration of propositions is **not** generally admissible. Nor is a discrete fibration in general admissible: the prone arrow above a restriction idempotent is not necessarily an endomorphism unless the point is fixed by the function induced by the idempotent.

4.4. PROPOSITION. *For any restriction category \mathbb{X} with latent pullbacks, the latent fibrations $\mathbb{X}^{\rightsquigarrow}$ and \mathbb{X}^{\rightarrow} are admissible.*

PROOF. Given a restriction idempotent $e: Y \rightarrow Y$ and a map $f: X \rightarrow Y$ in X , we need to define a prone lifting of e to $\mathbb{X}^{\rightsquigarrow}$ which is also a restriction idempotent. We claim that the pair (\overline{fe}, e) does this. Indeed, consider the diagram



Since $g = we$ is precise, by Lemma 2.7.iii, $g = w$. Similarly, for the top triangle to be precise, we must have $\tilde{w} = g'$. It remains to check that $\tilde{w} = g'$ satisfies the required conditions.

Indeed, it gives a lax square since the outer square is lax by assumption:

$$g'f \leq cg = cwe = cw,$$

and the top triangle commutes since

$$g'\overline{fe} = \overline{g'feg'} = \overline{cheeg'} = \overline{cheg'} = \overline{g'f} = g'\overline{f} = g'$$

with the last equality holding since the outer square is assumed precise.

Thus (fe, e) is prone, and so $\mathbb{X}^{\rightsquigarrow}$ is admissible; a similar proof shows that \mathbb{X}^{\rightarrow} is admissible. ■

The definition of admissible only asks for the existence of a prone restriction idempotent over a restriction idempotent in the base. However, by Lemma 3.18.iii, such a restriction idempotent is unique; thus, one in fact gets a section:

4.5. LEMMA. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is an admissible latent fibration, then the semilattice map*

$$\mathbf{p}|_{\mathcal{O}(X)}: \mathcal{O}(X) \rightarrow \{e \in \mathcal{O}(\mathbf{p}(X)): e \leq \mathbf{p}(1_X)\}$$

has a section $(-)^$, where e^* is the (unique) prone restriction idempotent above e at X . Furthermore, for any $d \in \mathcal{O}(X)$, $d \leq \mathbf{p}(d)^*$, so that the prone arrows are a reflective subposet of $\mathcal{O}(X)$.*

PROOF. Note that $(e_1e_2)^* = e_1^*e_2^*$, as both are prone over e_1e_2 , and $\mathbf{p}(1_X)^* = 1_X$, as 1_X is trivially prone. Thus, the section is also a semilattice morphism. Moreover, $\mathbf{p}(d)\mathbf{p}(d) = \mathbf{p}(d)$ is a trivial precise triangle with $d\mathbf{p}(d)^* = e$ precise above it: but this implies $d \leq \mathbf{p}(d)^*$ as required. ■

The following are useful properties of admissible latent fibrations:

4.6. LEMMA. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is an admissible latent fibration:*

- (i) *If $m: E' \rightarrow E$ is prone over a restriction monic, then m is a partial isomorphism;*
- (ii) *If $r: B \rightarrow \mathbf{p}(E)$ is a restriction retraction, then there is a (prone) restriction retraction $r': E' \rightarrow E$ with $\mathbf{p}(r') = r$.*

PROOF.

- (i) *If $m: E' \rightarrow E$ is over a restriction monic, then there is a map $r: \mathbf{p}(E) \rightarrow \mathbf{p}(E')$ such that $r\mathbf{p}(m) = \bar{r}$ and $\mathbf{p}(m)r = 1_{\mathbf{p}(E')}$. By admissibility, there is a prone restriction idempotent $\bar{r}^*: E \rightarrow E$ over \bar{r} . Let r' be defined as the unique lift*

$$\begin{array}{ccc} \begin{array}{ccc} E & & \\ \downarrow r' & \searrow \bar{r}^* & \\ E' & \xrightarrow{m} & E \end{array} & \mapsto & \begin{array}{ccc} \mathbf{p}(E) & & \\ \downarrow r & \searrow \bar{r} & \\ \mathbf{p}(E') & \xrightarrow{\mathbf{p}(m)} & \mathbf{p}(E) \end{array} \end{array}$$

Thus, $r'm = \bar{r}^* = \bar{r} = \bar{r}'$. Thus, r' satisfies one half of being a partial inverse to m ; we also need $mr' = \bar{m}$.

For this, first, since \bar{r}^* is itself prone, we have some unique $k: E' \rightarrow E$ such that

$$\begin{array}{ccc} \begin{array}{ccc} E' & & \\ \downarrow k & \searrow m & \\ E & \xrightarrow{\bar{r}^*} & E \end{array} & \mapsto & \begin{array}{ccc} \mathbf{p}(E') & & \\ \downarrow \mathbf{p}(m) & \searrow \mathbf{p}(m) & \\ \mathbf{p}(E) & \xrightarrow{\bar{r}} & \mathbf{p}(E) \end{array} \end{array}$$

However, by Lemma 2.7.ii, $k = m$. Thus, in particular, since the top triangle is precise, we have $m\bar{r}^* = m$, so $mr'm = m\bar{r}^* = m$. Thus we have the triangle

$$\begin{array}{ccc} E' & & \mathfrak{p}(E') \\ \downarrow mr' & \searrow m & \downarrow 1 \\ E' & \xrightarrow{m} & E \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathfrak{p}(E') & & \mathfrak{p}(E) \\ \downarrow 1 & \searrow \mathfrak{p}(m) & \downarrow \mathfrak{p}(m) \\ \mathfrak{p}(E') & \xrightarrow{\mathfrak{p}(m)} & \mathfrak{p}(E) \end{array}$$

But \bar{m} also fits precisely into the top triangle in place of mr' (and is over 1 since $\mathfrak{p}(m)$ is total), so since m is prone, $mr' = \bar{m}$. Thus m is a partial isomorphism with partial inverse r' .

(ii) Suppose $r : B \rightarrow \mathfrak{p}(E)$ is a restriction retraction so there is an s with $sr = 1_{\mathfrak{p}(E)}$ and $rs = \bar{r}$. There are prone arrows $r^* : E_1 \rightarrow E$ with $\mathfrak{p}(r^*) = r$ and $s^* : E_2 \rightarrow E_1$ with $\mathfrak{p}(s^*) = s$ – by (i) we know s^* is a partial isomorphism. The composite s^*r^* is then prone over the identity map so there is a partial isomorphism $\alpha : E \rightarrow E_2$ with $\bar{\alpha} = 1_E$, $\hat{\alpha} = \overline{s^*r^*}$ and $\alpha s^*r^* = 1_E$.

We have $\alpha^{(-1)} = s^*r^*$ so $(s^*)^{(-1)}\alpha^{(-1)} = (s^*)^{(-1)}s^*r^* = \widehat{s^*}r^*$ so $r' := \widehat{s^*}r^*$ is a partial isomorphism with $\alpha s^*r' = \alpha s^*\widehat{s^*}r^* = \alpha s^*r^* = 1_E$. So r' is a restriction retraction – which is automatically prone by virtue of being a restriction retraction – which, furthermore, lies over r as $\mathfrak{p}(r') = \mathfrak{p}((s^*)^{(-1)}s^*r^*) = s^{(-1)}sr = rsr = r$.

■

4.7. LEMMA. *If $\mathfrak{p} : \mathbb{E} \rightarrow \mathbb{B}$ is an admissible latent fibration then any mediating partial isomorphism α between two \mathfrak{p} -prone maps f' and f over the same map such that $\bar{f} = \mathfrak{p}(f)^*$ is itself prone.*

PROOF. Consider maps f and f' which are both \mathfrak{p} -prone above the same map, $\mathfrak{p}(f) = \mathfrak{p}(f')$. Let α be the mediating map between f and f' so in particular $\mathfrak{p}(\alpha) = \mathfrak{p}(f)$. Now consider $g : C \rightarrow B$ with $\mathfrak{p}(g) = h\mathfrak{p}(f)$. By Lemma 2.7.ii, it follows that $h = \mathfrak{p}(g)$. We wish to show that there is a lifting \tilde{h} so that $\tilde{h}\alpha = g$ is precise, however, the best we can do is to lift h , using the fact that f is prone, making $\tilde{h}f' = gf$:

$$\begin{array}{ccc} C & \xrightarrow{\tilde{h}} & B' \\ \downarrow g & & \downarrow \alpha \\ B & \xrightarrow{f} & A \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathfrak{p}(C) & \xrightarrow{h} & \mathfrak{p}(B) \\ \downarrow \mathfrak{p}(g) & & \downarrow \mathfrak{p}(f) \\ \mathfrak{p}(B) & \xrightarrow{\mathfrak{p}(f)} & \mathfrak{p}(A) \end{array}$$

Now we observe that $\tilde{h}\alpha$ sits over $\mathfrak{p}(g)$ as $\mathfrak{p}(\tilde{h}\alpha) = \mathfrak{p}(\tilde{h})\mathfrak{p}(\alpha) = h\overline{\mathfrak{p}(f)} = \mathfrak{p}(g)$; furthermore, $\tilde{h}\alpha f = \tilde{h}f' = gf$ and $\tilde{h}\alpha\bar{f} = \tilde{h}\alpha$ so the triangle $(\tilde{h}\alpha)f = (\tilde{h}f')$ is precise. However, as $\bar{g}\bar{f} = g\overline{\mathfrak{p}(f)^*}$ as $\mathfrak{p}(g) = \mathfrak{p}(g)\overline{\mathfrak{p}(f)}$, the triangle $gf = (\tilde{h}f')$ is precise. So we may conclude that $g = \tilde{h}\alpha$ and, as $\tilde{h}\alpha = \tilde{h}f'$, it is precise. The triangle is, furthermore, unique as α is a partial isomorphism, and thus α is prone.

■

4.8. PROPOSITION. *Admissible latent fibrations are closed under composition and pullback.*

PROOF. This follows from Lemma 3.21 and Lemma 3.24. ■

Perhaps the most useful property of admissible latent fibrations is that they can be r -split; we prove this in Proposition 5.23.

4.9. SEPARATED LATENT FIBRATIONS. We now consider a different property a restriction semifunctor can have.

4.10. DEFINITION. *A restriction semifunctor $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is said to be **separated** if for any object $X \in \mathbb{E}$ and restriction idempotents e, e' on X , if $\mathbf{p}(e) = \mathbf{p}(e')$ then $e = e'$.*

4.11. EXAMPLE.

- (i) The identity fibration $1_{\mathbb{X}}: \mathbb{X} \rightarrow \mathbb{X}$ is separated.
- (ii) The strict simple fibration $\mathbb{X}[\mathbb{X}]$ is separated: since the restriction of a map in $\mathbb{X}[\mathbb{X}]$ is entirely determined by its restriction in the base, the functor $\partial: \mathbb{X}[\mathbb{X}] \rightarrow \mathbb{X}$ is separated.
- (iii) Conversely, in general the lax simple fibration $\mathbb{X}(\mathbb{X})$ is not separated, as the inequality allowed in the definition of an arrow permits multiple possible restriction idempotents over a fixed one in the base. Thus, this is an example of a latent fibration which is admissible but not separated.
- (iv) For any restriction category \mathbb{X} with latent pullbacks, $\mathbb{X}^{\rightarrow} \rightarrow \mathbb{X}$ is separated: if

$$\begin{array}{ccc} X & \xrightarrow{e'} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{e} & Y \end{array}$$

is a restriction idempotent in \mathbb{X}^{\rightarrow} , then since the square must strictly commute and be precise, we have

$$e' = \overline{e'} = \overline{e'f} = \overline{e'f} = \overline{fe}.$$

Thus, the restriction idempotent (e', e) is entirely determined by e , and so the functor $\partial: \mathbb{X}^{\rightarrow} \rightarrow \mathbb{X}$ is a separated.

- (v) By lemma 2.11.ii, one may be tempted to think that $\mathbb{X}^{\rightsquigarrow}$ is separated; however, for this to be the case partially inverting the top and bottom arrows of a latent pullback would have to produce a semi-precise square: this is not the case in general.
- (vi) The propositions fibration is clearly separated, as in this case $\mathbf{p}(e) = e$. Thus, this is an example of a latent fibration which is separated but not admissible.

- (vii) For assemblies, notice that all restriction idempotents $e \in \mathbb{X}$ are tracked by identity realizers and so every restriction idempotent is tracked. This means that the restriction idempotents in $\mathbf{Asm}(F)$ are the same as those in \mathbb{X} making \mathbf{p} separated.
- (viii) The splitting fibration $\mathbf{Split}_r(\mathbb{X}) \rightarrow \mathbb{X}$ is clearly separated, as in this case $\mathbf{p}(e) = e$. Similarly, a discrete fibration is always separated.

An immediate property of a separated semifunctor is the following:

4.12. LEMMA. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is separated, then $\mathbf{p}(f)$ total implies f is total.*

PROOF. If we have such an f , then we also have

$$1_{\mathbf{p}(X)} = \overline{\mathbf{p}(f)} = \mathbf{p}(\overline{f}) = \mathbf{p}(1_X)\mathbf{p}(\overline{f}) \leq \mathbf{p}(1_X),$$

but we always have the opposite inequality, so $\mathbf{p}(1_X) = 1_{\mathbf{p}(X)}$. Then $\mathbf{p}(\overline{f}) = 1_{\mathbf{p}(X)} = \mathbf{p}(1_X)$, so since \mathbf{p} is separated, $\overline{f} = 1_X$. Thus f is total. ■

More importantly, however, being separated is precisely the right condition to assure that all restriction idempotents (and, more generally, partial isomorphisms) are prone.

4.13. PROPOSITION. *For any restriction semifunctor $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$, the following are equivalent:*

- (i) \mathbf{p} is separated;
- (ii) all partial isomorphisms in \mathbb{E} are \mathbf{p} -prone;
- (iii) all restriction idempotents in \mathbb{E} are \mathbf{p} -prone.

PROOF. For (i) \Rightarrow (ii), suppose that \mathbf{p} is separated. Suppose we have a partial isomorphism $f: X \rightarrow Y$ (with partial inverse $f^{(-1)}$) and a map $g: Z \rightarrow Y$ such that $h\mathbf{p}(f) = \mathbf{p}(g)$ is precise:

$$\begin{array}{ccc} \begin{array}{ccc} Z & & \\ \vdots \downarrow & \searrow g & \\ X & \xrightarrow{f} & Y \end{array} & \mapsto & \begin{array}{ccc} \mathbf{p}(Z) & & \\ h \downarrow & \searrow \mathbf{p}(g) & \\ \mathbf{p}(X) & \xrightarrow{\mathbf{p}(f)} & \mathbf{p}(Y) \end{array} \end{array}$$

We want to show that $gf^{(-1)}$ precisely fits in the left triangle (it is the only possible choice by Lemma 2.7.iii). First, note that

$$\mathbf{p}(\overline{gf^{(-1)}}) = \overline{h\mathbf{p}(f)\overline{\mathbf{p}(f^{(-1)})}} = \overline{h\mathbf{p}(f)} = \overline{\mathbf{p}(g)} = \mathbf{p}(\overline{g})$$

so since \mathbf{p} is separated, $\overline{gf^{(-1)}} = \overline{g}$. Then the triangle commutes since

$$gf^{(-1)}f = \overline{gf^{(-1)}}f = \overline{gf^{(-1)}}g = \overline{g}g,$$

and is precise. Moreover, $gf^{(-1)}$ is over h since

$$\mathbf{p}(gf^{(-1)}) = h\mathbf{p}(f)\mathbf{p}(f^{(-1)}) = h\overline{\mathbf{p}(f)} = h$$

since $h\mathbf{p}(f) = \mathbf{p}(g)$ is precise.

(ii) \Rightarrow (iii) is immediate, since restriction idempotents are partial isomorphisms.

For (iii) \Rightarrow (i), if all restriction idempotents are prone, then \mathbf{p} is separated by Lemma 3.18.iii. ■

When a restriction semifunctor is separated, the prone condition is a bit simpler to check:

4.14. LEMMA. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is separated, then an arrow $f: X \rightarrow X'$ in \mathbb{E} is prone if and only if for any $g: Y \rightarrow X'$ and any $h: \mathbf{p}(Y) \rightarrow \mathbf{p}(X)$ such that $h\mathbf{p}(f) = \mathbf{p}(g)$ is precise, there is a unique $h': Y \rightarrow X$ such that $h'f = g$.*

PROOF. The only change in the ordinary definition is the requirement that the triangle in \mathbb{E} be precise. However, if $h'f = g$, then $\overline{\mathbf{p}(h')} = \overline{h} = \overline{\mathbf{p}(g)}$, so since \mathbf{p} is separated, $\overline{h'} = \overline{g}$ is guaranteed. ■

These functors also have a useful factorization property of prone arrows:

4.15. LEMMA. *Suppose $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is separated. If*

$$\begin{array}{ccc} X & & \\ f_1 \downarrow & \searrow f & \\ X' & \xrightarrow{f_2} & X'' \end{array}$$

is a precise triangle in \mathbb{E} and f and f_2 are prone, then f_1 is prone.

PROOF. Throughout this proof we will use the fact that when looking at whether an arrow is prone for a separated semifunctor, we don't need to consider precise-ness of the top triangle (see Lemma 4.14).

Thus, we need to show that for $g: Y \rightarrow X'$ with $h: \mathbf{p}(Y) \rightarrow \mathbf{p}(X)$ such that $h\mathbf{p}(f_1) = \mathbf{p}(g)$ is precise, we have a unique fill-in $h': Y \rightarrow X$. Consider this diagram, extended to the right by post-composing with f_2 :

$$\begin{array}{ccc} Y & \xrightarrow{gf_2} & X'' \\ \downarrow h' & \searrow g & \\ X & \xrightarrow{f_1} X' \xrightarrow{f_2} X'' & \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathbf{p}(Y) & \xrightarrow{\mathbf{p}(gf_2)} & \mathbf{p}(X'') \\ \downarrow h & \searrow \mathbf{p}(g) & \\ \mathbf{p}(X) & \xrightarrow{\mathbf{p}(f_1)} \mathbf{p}(X') \xrightarrow{\mathbf{p}(f_2)} \mathbf{p}(X'') & \end{array}$$

Now, the outer triangle in the base is also precise since

$$\overline{\mathfrak{p}(gf_2)} = \overline{h\mathfrak{p}(f_1)\mathfrak{p}(f_2)} = \overline{h\mathfrak{p}(f_1)} = \overline{\mathfrak{p}(g)} = \overline{h}$$

where we have used the fact that $f = f_1f_2$ is precise to give $f_1\overline{f_2} = f_1$.

Thus, since $f = f_1f_2$ is prone, there is a unique $h' : Y \rightarrow X$ such that $\overline{h'} = \overline{gf_2}$ and $h'f_1f_2 = gf_2$. We need to show that h' makes the inner triangle commute; that is, we need $h'f = g$.

However, $\mathfrak{p}(h'f_1) = h\mathfrak{p}(f_1) = \mathfrak{p}(g)$, so since we have $h'f_1f_2 = gf_2$ and f_2 is prone, $h'f = g$. Uniqueness of h' then follows by uniqueness of factorizations for f_1f_2 . ■

4.16. COROLLARY. *If $\mathfrak{p} : \mathbb{E} \rightarrow \mathbb{B}$ is separated, then any restriction monic in \mathbb{E} is prone.*

PROOF. Suppose we have a restriction monic $m : X \rightarrow Y$ in \mathbb{E} . Then by assumption there is a restriction retraction $r : Y \rightarrow X$ so that $mr = 1$; thus,

$$\begin{array}{ccc} X & & \\ m \downarrow & \searrow 1 & \\ Y & \xrightarrow{r} & X \end{array}$$

is a precise triangle. By Lemma 3.19, identities and restriction retractions are always prone, so by the previous result, m is also prone. ■

Separated latent fibrations have the useful property that one can take the latent pullback of a prone arrow along a subvertical arrow:

4.17. LEMMA. *Suppose $\mathfrak{p} : \mathbb{E} \rightarrow \mathbb{B}$ is a separated latent fibration. Then every cospan*

$$B \xrightarrow{c} C \xleftarrow{v} A$$

with v subvertical and c prone has a corresponding latent pullback:

$$\begin{array}{ccc} U & \xrightarrow{c'} & A \\ w \downarrow & & \downarrow v \\ B & \xrightarrow{c} & C \end{array}$$

where w is subvertical and c' is prone.

PROOF. Since v is subvertical, $\mathfrak{p}(A) = \mathfrak{p}(C)$, and so the composite

$$\mathfrak{p}(B) \xrightarrow{\mathfrak{p}(c)} \mathfrak{p}(A) \xrightarrow{\mathfrak{p}(v)} \mathfrak{p}(C)$$

is well-defined; we take $c' : U \rightarrow A$ to be a prone lift of this map to A .

Then let $w: U \rightarrow B$ be the unique arrow from the universal property of c :

$$\begin{array}{ccc}
 \begin{array}{ccc}
 U & & \\
 \downarrow w & \searrow c'v & \\
 B & \xrightarrow{c} & C
 \end{array} & \mapsto &
 \begin{array}{ccc}
 \mathfrak{p}(U) & & \\
 \downarrow \mathfrak{p}(c)\mathfrak{p}(v) & \searrow \mathfrak{p}(c'v) & \\
 \mathfrak{p}(B) & \xrightarrow{\mathfrak{p}(c)} & \mathfrak{p}(C)
 \end{array}
 \end{array}$$

(Note that the triangle in \mathbb{B} is precise since $\overline{\mathfrak{p}(c'v)} = \overline{\mathfrak{p}(c)\mathfrak{p}(v)\mathfrak{p}(v)} = \overline{\mathfrak{p}(c)\mathfrak{p}(v)}$.)

Thus, w as above exists, and has the property that $\overline{w} = \overline{c'v}$. Moreover, since $\overline{\mathfrak{p}(w)} = \overline{\mathfrak{p}(c)\mathfrak{p}(v)} = \overline{\mathfrak{p}(c)}$, $\overline{w} = \overline{c'}$ as \mathfrak{p} is separated. Thus, the square is a potential candidate to be a latent pullback.

We will check that the original definition (see [11, pg. 460]) of latent pullback holds, as it involves one fewer piece of data. So, assume we have $a: X \rightarrow A$ and $b: X \rightarrow B$ so that $av = bc$. We need to find a unique b' as indicated:

$$\begin{array}{ccccc}
 X & & & & \\
 \downarrow b' & \searrow a & & & \\
 U & \xrightarrow{c'} & A & & \\
 \downarrow w & & \downarrow v & & \\
 B & \xrightarrow{c} & C & &
 \end{array}$$

with the additional properties that $\overline{b'} = \overline{bc} = \overline{av}$ and $b'\overline{w} = b'\overline{c'} = b'$.

Let b' be the unique arrow arising from the universal property of c' :

$$\begin{array}{ccc}
 \begin{array}{ccc}
 X & & \\
 \downarrow b' & \searrow a\overline{v} & \\
 U & \xrightarrow{c'} & A
 \end{array} & \mapsto &
 \begin{array}{ccc}
 \mathfrak{p}(X) & & \\
 \downarrow \mathfrak{p}(b)\mathfrak{p}(w) & \searrow \mathfrak{p}(a\overline{v}) & \\
 \mathfrak{p}(U) & \xrightarrow{\mathfrak{p}(c)\mathfrak{p}(v)} & \mathfrak{p}(C)
 \end{array}
 \end{array}$$

Note that $\mathfrak{p}(b)\mathfrak{p}(w)$ is well-defined since w is subvertical. The triangle in \mathbb{B} commutes since $\mathfrak{p}(b)\mathfrak{p}(w)\mathfrak{p}(c)\mathfrak{p}(v) = \mathfrak{p}(b)\overline{\mathfrak{p}(c)\mathfrak{p}(v)}\mathfrak{p}(c)\mathfrak{p}(v) = \mathfrak{p}(b)\mathfrak{p}(c)\mathfrak{p}(v) = \mathfrak{p}(a)\mathfrak{p}(v)\mathfrak{p}(v) = \mathfrak{p}(a)\mathfrak{p}(v) = \mathfrak{p}(a\overline{v})$ and is precise since

$$\overline{\mathfrak{p}(b)\mathfrak{p}(w)} = \overline{\mathfrak{p}(b)\overline{\mathfrak{p}(c)\mathfrak{p}(v)}} = \overline{\mathfrak{p}(b)\mathfrak{p}(c)\mathfrak{p}(v)} = \overline{\mathfrak{p}(a)\mathfrak{p}(v)\mathfrak{p}(v)} = \overline{\mathfrak{p}(a)\mathfrak{p}(v)}.$$

Thus, such a b' does exist, and has the property that $\overline{b'} = \overline{a\overline{v}} = \overline{av}$.

It immediately satisfies many of the required properties:

$$b'c' = a\overline{v} \leq a,$$

and

$$\overline{b'} = \overline{a\overline{v}}$$

and

$$b'\bar{c}' = \overline{b'c'b'} = \overline{a\bar{v}b'} = \overline{b'b'} = b'.$$

However, checking that $b'w \leq b$, that is, $\overline{b'wb} = b'w$, takes a bit more work. For this, we will use the cancellation property of the prone map c : if $\overline{b'wbc} = b'wc$, $\mathfrak{p}(\overline{b'wb}) = \mathfrak{p}(b'w)$, and the relevant triangle in the base is precise, then $\overline{b'wb} = b'w$ follows.

For the first requirement, $b'wc = b'c'v = a\bar{v}v = av$ while $\overline{b'wbc} = \overline{b'wav}$. However, we also have

$$\mathfrak{p}(\overline{b'w}) = \overline{\mathfrak{p}(b')\mathfrak{p}(w)} = \overline{\mathfrak{p}(b)\mathfrak{p}(w)\mathfrak{p}(w)} = \overline{\mathfrak{p}(b)\mathfrak{p}(w)} = \overline{\mathfrak{p}(a)\mathfrak{p}(v)}$$

by above; thus since \mathfrak{p} is separated we get $\overline{b'wav} = \overline{a\bar{v}av} = av$. For the second requirement,

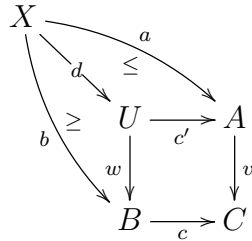
$$\mathfrak{p}(\overline{b'wb}) = \overline{\mathfrak{p}(b)\mathfrak{p}(w)\mathfrak{p}(b)} = \mathfrak{p}(b)\overline{\mathfrak{p}(w)} = \mathfrak{p}(b)\mathfrak{p}(w)\mathfrak{p}(w) = \mathfrak{p}(b'w).$$

For preciseness of the triangle in the base,

$$\overline{\mathfrak{p}(b'w)} = \overline{\mathfrak{p}(b)\mathfrak{p}(w)} = \overline{\mathfrak{p}(a)\mathfrak{p}(v)}$$

by above. Thus, by proneness of c , $\overline{b'wb} = b'w$, and so $b'w \leq b$.

Thus b' satisfies all the required properties; it remains to show that it is unique. Thus, suppose we have



with $\bar{d} = \bar{bc} = \bar{a\bar{v}}$ and $d = \bar{dc}' = \bar{d\bar{w}}$.

We want to show that $d = b'$; we will use the universal property of c' . Thus, we need to show that $dc' = b'c'$, $\mathfrak{p}(d) = \mathfrak{p}(b')$, and the relevant triangle in the base is precise. For the first requirement, since $dc' \leq a$,

$$dc' = \overline{dc'a} = \overline{da} = \overline{a\bar{v}a} = a\bar{v} = b'c'.$$

For the second requirement, since $d = \bar{d\bar{w}}$, and $d\bar{w} \leq b$,

$$\mathfrak{p}(d) = \mathfrak{p}(d\bar{w})\mathfrak{p}(w) = \mathfrak{p}(\overline{d\bar{w}b})\mathfrak{p}(w) = \mathfrak{p}(\overline{bcb})\mathfrak{p}(w) = \mathfrak{p}(b\bar{c})\mathfrak{p}(w) = \mathfrak{p}(b)\mathfrak{p}(w) = \mathfrak{p}(b'),$$

with the second-last equality since $w\bar{c} = w$. Finally, since $\bar{d} = \bar{a\bar{v}}$, the relevant triangle in \mathbb{E} and hence in \mathbb{B} is precise.

Thus, proneness of c' gives $d = b'$, and we have completed the proof that the square is a latent pullback. ■

4.18. PROPOSITION. *Separated latent fibrations are closed under composition and pullback.*

PROOF. This follows from Lemma 3.21 and Lemma 3.24. ■

4.19. HYPERCONNECTED LATENT FIBRATIONS.

4.20. DEFINITION. A restriction semifunctor $\mathfrak{p}: \mathbb{E} \rightarrow \mathbb{B}$ is said to be **hyperconnected** (and \mathfrak{p} is said to be a **hyperconnection**) if for any $X \in \mathbb{E}$, the map

$$\mathfrak{p}|_{\mathcal{O}(X)}: \mathcal{O}(X) \rightarrow \{d \in \mathcal{O}(\mathfrak{p}(X)) : d\mathfrak{p}(1_X) = d\}$$

sending $e \in \mathcal{O}(X)$ to $\mathfrak{p}(e)$ is an isomorphism. We say that a latent fibration \mathfrak{p} is a **hyperfibration** if \mathfrak{p} is a hyperconnection.

This is a generalization of the definition of hyperconnection (see [10, pg. 39]) to restriction semifunctors.

Before looking at examples, it will be useful to know the following:

4.21. PROPOSITION. A restriction semifunctor $\mathfrak{p}: \mathbb{E} \rightarrow \mathbb{B}$ is hyperconnected if and only if it is separated and admissible.

PROOF. If \mathfrak{p} is hyperconnected, then it is monic on restriction idempotents, and so is separated. Moreover, if $e: \mathfrak{p}(X) \rightarrow \mathfrak{p}(X)$ is a restriction idempotent in \mathbb{B} , then there is a restriction idempotent $e': X \rightarrow X$ over it (since \mathfrak{p} is hyperconnected), and this is prone by Proposition 4.13.

Conversely, if \mathfrak{p} is separated and admissible, then for any X , the map

$$\mathfrak{p}|_{\mathcal{O}(X)}: \mathcal{O}(X) \rightarrow \{d \in \mathcal{O}(\mathfrak{p}(X)) : d\mathfrak{p}(1_X) = e\}$$

is monic and has a section (by Lemma 4.5) thus is an isomorphism. ■

Thus, by the results of the previous two sections, we have the following examples (and non-examples) of hyperfibrations:

4.22. EXAMPLE.

- (i) The identity fibration $1_{\mathbb{X}}: \mathbb{X} \rightarrow \mathbb{X}$ is a hyperfibration.
- (ii) The strict simple fibration $\mathbb{X}[\mathbb{X}]$ is a hyperfibration, while in general the lax simple fibration $\mathbb{X}(\mathbb{X})$ is not.
- (iii) For any restriction category \mathbb{X} with latent pullbacks, $\mathbb{X}^{\rightarrow} \rightarrow \mathbb{X}$ is a hyperfibration, while in general $\mathbb{X}^{\rightsquigarrow} \rightarrow \mathbb{X}$ is not.
- (iv) The assemblies fibration is a hyperfibration.
- (v) The splitting fibration $\mathbf{Split}_r(\mathbb{X}) \rightarrow \mathbb{X}$ is a hyperfibration.
- (vi) Neither the propositions fibration nor the discrete fibrations are hyperfibrations (in general) as they are not admissible.

Restriction semifunctor	Restriction Functor	Admissible	Separated	Hyperfibration
$\text{Split}_r(\mathbb{X}) \rightarrow \mathbb{X}$	\times	\checkmark	\checkmark	\checkmark
$\mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$	\checkmark	\checkmark	\times	\times
(lax) $\mathbb{X}(\mathbb{X}) \rightarrow \mathbb{X}$	\checkmark	\checkmark	\times	\times
(strict) $\mathbb{X}[\mathbb{X}] \rightarrow \mathbb{X}$	\checkmark	\checkmark	\checkmark	\checkmark
(lax) $\mathbb{X}^{\rightsquigarrow} \rightarrow \mathbb{X}$	\checkmark	\checkmark	\times	\times
(strict) $\mathbb{X}^{\rightarrow} \rightarrow \mathbb{X}$	\checkmark	\checkmark	\checkmark	\checkmark
$\mathcal{O}(\mathbb{X}) \rightarrow \mathbb{X}$	\checkmark	\times	\checkmark	\times
$\text{Elt}(F) \rightarrow \mathbb{X}$	\checkmark	\times	\checkmark	\times
$\text{Asm}(F) \rightarrow \mathbb{X}$	\checkmark	\checkmark	\checkmark	\checkmark

Table 1: Properties of latent fibrations

Table 1 presents an overview of the examples of latent fibrations we have considered with their properties.

Hyperconnections are particularly well-behaved when looking at prone arrows. For a general restriction semifunctor, there can be prone maps over total maps or restriction monics which are themselves neither total nor restriction monic. However, for a hyperconnection this is no longer the case, as *all* prone maps above a total map are total, and, similarly, *all* maps prone above a restriction monic are restriction monics. In addition, any prone maps above a restriction retraction is a restriction retraction:

4.23. PROPOSITION. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a hyperconnection, then:*

- (i) \mathbf{p} reflects total maps;
- (ii) if α in \mathbb{E} is prone over a partial isomorphism, then α is itself a partial isomorphism;
- (iii) if s in \mathbb{E} is prone over a restriction monic then s is itself a restriction monic;
- (iv) if r in \mathbb{E} is prone over a restriction retraction, then r is itself a restriction retraction;
- (v) if \mathbf{p} is a hyperfibration, then the total maps and restriction monics of the domain and codomain restriction categories of \mathbf{p} each determine a subfibration of \mathbf{p} .

PROOF.

- (i) We saw in Lemma 4.12 that separated semifunctors reflect total maps.
- (ii) By Proposition 4.21, \mathbf{p} is admissible, so there is a prone restriction idempotent e over $\widehat{\mathbf{p}(\alpha)} = \overline{\mathbf{p}(\alpha)^{(-1)}}$. Let $\alpha': X \rightarrow Y$ be the lift of $\mathbf{p}(\alpha)^{(-1)}$:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 X & & \\
 \alpha' \downarrow \cdots & \searrow e & \\
 Y & \xrightarrow{\alpha} & X
 \end{array} & \mapsto & \begin{array}{ccc}
 \mathbf{p}(X) & & \\
 \mathbf{p}(\alpha)^{(-1)} \downarrow & \searrow \widehat{\mathbf{p}(\alpha)} & \\
 \mathbf{p}(Y) & \xrightarrow{\mathbf{p}(\alpha)} & \mathbf{p}(X)
 \end{array}
 \end{array}$$

Thus $\alpha'\alpha = e = \bar{e} = \overline{\alpha'}$ since the triangle is precise. We want to show that α' is the partial inverse of α , and so we also need $\alpha\alpha' = \bar{\alpha}$.

By definition, e and α are prone, so by Lemma 4.15, α' is also prone, so by Lemma 3.16, $\alpha\alpha'$ is prone. Thus $\alpha\alpha'$ and $\bar{\alpha}$ are prone over the same map $(\mathbf{p}(\bar{\alpha}))$, and so by Lemma 3.18.i, there is a mediating partial isomorphism $\beta: Y \rightarrow Y$

$$\begin{array}{ccc} Y & \xrightarrow{\beta} & Y \\ & \searrow \bar{\alpha} & \downarrow \alpha\alpha' \\ & & Y \end{array}$$

with $\bar{\beta} = \bar{\alpha}$ and $\overline{\beta^{(-1)}} = \overline{\alpha\alpha'}$. But then we have

$$\beta = \beta\beta^{(-1)}\beta = \beta\overline{\beta^{(-1)}} = \beta\alpha\alpha' = \bar{\alpha}.$$

So β is a restriction idempotent, and so its partial inverse $\beta^{(-1)}$ is itself, namely $\bar{\alpha}$. Thus $\alpha\alpha^{(-1)} = \beta^{(-1)}\bar{\alpha} = \bar{\alpha}$, as required.

- (iii) Suppose $s: Y \rightarrow X$ is prone over a restriction monic $\mathbf{p}(s)$. Then in particular s is prone over a partial isomorphism, so by (ii), it has a partial inverse r , so that $sr = \bar{s}$ and $rs = \bar{r}$. But since $\mathbf{p}(s)$ is a restriction monic, it is total, so by (i), s is total, and thus $sr = \bar{s} = 1$. Thus s is a restriction monic.
- (iv) Similar proof to (iii).
- (v) Extracting the total maps from the fibration turns the latent fibration into an ordinary fibration. Prone maps become Cartesian: so, in particular, restriction monics, being partial isomorphisms and therefore prone, are Cartesian.

■

Given any admissible latent fibration $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ we may always extract a latent hyperfibration $\widehat{\mathbf{p}}: \widehat{\mathbb{E}} \rightarrow \mathbb{B}$ by restricting the restriction idempotents to the prone restriction idempotents and the maps to those which pullback prone restrictions to prone restrictions: that is, $f \in \widehat{\mathbb{E}}$ in case for every prone restriction idempotent, e on the codomain of f , $\bar{f}e$ is a prone restriction idempotent (recall from Lemma 4.5 that the prone restrictions for an admissible semifunctor form a (reflective) subsemilattice). We then have:

4.24. PROPOSITION. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is an admissible latent fibration then*

$$\begin{array}{ccc} \widehat{\mathbb{E}} & \xrightarrow{\quad} & \mathbb{E} \\ & \searrow \widehat{\mathbf{p}} & \swarrow \mathbf{p} \\ & & \mathbb{B} \end{array}$$

$\widehat{\mathbf{p}}$ is a hyperfibration which is a latent subfibration of \mathbf{p} .

PROOF. $\widehat{\mathbf{p}}$ is clearly still admissible, and by definition, all restriction idempotents in it are prone. Thus, by Proposition 4.13 and Proposition 4.21, $\widehat{\mathbf{p}}$ is a hyperfibration. ■

4.25. PROPOSITION. *Hyperconnected latent fibrations are closed under composition and pullback.*

PROOF. Follows from Lemma 3.21 and Lemma 3.24. ■

Another key property of a hyperfibration (see Section 7) is that one can construct its fibrational dual.

5. M-category Fibrations

The purpose of this section is to provide a series of results describing the relation between fibrations of M-categories and latent fibrations. In particular, we completely characterize, as an M-category, a certain type of latent fibration which we call *r-split* (see Definition 5.2). This allows us to similarly characterize *r-split* admissible latent fibrations and *r-split* hyperfibrations.

Another key result in this story is that *admissible* latent fibrations are stable under the process of splitting restriction idempotents: this means that any admissible latent fibration can be embedded into an *r-split* latent fibration and hence into a latent fibration between the partial map categories of M-categories. In general, it is not the case that, when one splits the restriction idempotents of a latent fibration, one gets a latent fibration: necessary and sufficient conditions for this will be provided in Proposition 6.4.

In this section we shall use restriction functors rather than semifunctors. In particular, we shall need functors which preserve the total maps in order to access the equivalence between *r-split* restriction categories and M-categories.

5.1. *r-SPLIT LATENT FIBRATIONS.* In this section we will focus on restriction functors between *r-split* restriction categories, and demand that the fibres of these functors behave well with respect to splitting.

5.2. DEFINITION. *Let $\mathfrak{p}: \mathbb{E} \rightarrow \mathbb{B}$ be a restriction functor.*

- \mathfrak{p} is said to be **well-fibred** if for all B in \mathbb{B} , whenever a restriction idempotent in the fibre³ $\mathfrak{p}^{-1}(B)$ splits in \mathbb{E} , it also splits in $\mathfrak{p}^{-1}(B)$.
- \mathfrak{p} is said to be **r-split** in case the restriction categories \mathbb{E} and \mathbb{B} are *r-split*, and \mathfrak{p} is well-fibred.

One reason to ask that the projection functor be well-fibred is the following result:

5.3. LEMMA. *If $\mathfrak{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a restriction functor in which \mathbb{E} is *r-split*, then every total map, $f: B \rightarrow \mathfrak{p}(E)$, having a prone map $f': E' \rightarrow E$ above it whose idempotent \bar{f}' splits in $\mathfrak{p}^{-1}(B)$, has a prone map, which is total, above f at E .*

³Note that we reserve the notation $\mathfrak{p}^{-1}(B)$ for the classical fibre, containing only arrows that are mapped to 1_B by \mathfrak{p} , whereas $\mathfrak{p}^{(-1)}(B)$ is used for the strand containing all arrows that are mapped to a restriction idempotent on B .

PROOF. Suppose $f: B \rightarrow \mathfrak{p}(E)$ is total in \mathbb{B} with an $f': E' \rightarrow E$ which is prone above it. Note that $\overline{f'} \in \mathfrak{p}^{-1}(B)$ as $\mathfrak{p}(\overline{f'}) = \overline{\mathfrak{p}(f')} = \overline{f} = 1_B$. Thus, since \mathfrak{p} is well-fibred, there is a splitting $m: E'' \rightarrow E, r: E' \rightarrow E''$ of $\overline{f'}$ with both m and r in $\mathfrak{p}^{-1}(B)$ also. Let $f'' = fm'$; we claim that f'' is the required total prone map over f .

First, since $m \in \mathfrak{p}^{-1}(B)$,

$$\mathfrak{p}(f'') = \mathfrak{p}(m)\mathfrak{p}(f') = 1_B f = f,$$

so f'' is over f . Next,

$$\overline{f''} = \overline{mf'} = \overline{m}\overline{f'} = \overline{mr}\overline{m} = \overline{m} = 1_{E''},$$

so f'' is total. Finally, we have $f' = \overline{f'}f'' = rmf = rf''$, so since f' is prone and r is a restriction retraction, f'' is itself prone by Lemma 3.19.iii. ■

When $\mathfrak{p}: \mathbb{E} \rightarrow \mathbb{B}$ is an r -split latent fibration the total maps form a fibration because, as observed previously, prone maps in a category in which all maps are total, are just Cartesian maps. Thus, we have the important observation:

5.4. COROLLARY. *The total maps of an r -split latent fibration form a fibration.*

5.5. LATENT FIBRATIONS AND M-CATEGORIES. Next, it will be helpful to understand precise triangles and prone maps in a partial map category.

5.6. LEMMA. *Suppose that*

$$X \xleftarrow{m} X' \xrightarrow{f} Y, \quad X \xleftarrow{m'} X'' \xrightarrow{f'} Z, \quad Y \xleftarrow{n} Y' \xrightarrow{g} Z$$

are representatives of maps in a partial map category $\text{Par}(\mathbb{X}, \mathbb{M})$. Then the triangle

$$\begin{array}{ccc} X & & \\ (m,f) \downarrow & \searrow (m',f') & \\ Y & \xrightarrow{(n,g)} & Z \end{array}$$

precisely commutes in $\text{Par}(\mathbb{X}, \mathbb{M})$ if and only if there is a map $k: X'' \rightarrow Y'$ such that $kg = f'$ and $(m, f) \cong (m', kn)$.

PROOF. First, suppose that the triangle precisely commutes. Then $\overline{(m, f)} \cong \overline{(m', f')}$, so there is an isomorphism $\alpha: X'' \rightarrow X'$ such that $\alpha m = m'$. Let (am, bg) be a representative of $(m, f) \circ (n, g)$:

$$\begin{array}{ccccc} & & P & & \\ & & \swarrow a & \searrow b & \\ & X' & & & Y' \\ m \swarrow & & & & \searrow g \\ X & & & & Z \\ & \nwarrow f & \swarrow n & & \\ & Y & & & \end{array}$$

then since $(m, f) \circ (n, g) \cong (m', f')$, there is an isomorphism $\beta: X'' \rightarrow P$ such that $\beta am = m'$ and $\beta bg = f'$. Let $k = \beta b$, so we have $kg = f'$. Moreover, we claim that α witnesses $(m, f) \cong (m', kn)$. Indeed, we have $\alpha m = m'$. We also need $\alpha f = kn = \beta bn = \beta af$, so it suffices to show $\alpha = \beta a$. But $\alpha m = m' = \beta am$, so since m is monic, $\alpha = \beta a$, as required.

Conversely, suppose that we have such a k , and an $\alpha: X'' \rightarrow X$ that witnesses $(m, f) \cong (m', kn)$. Then, by definition, α witnesses $\overline{(m, f)} \cong \overline{(m', f')}$. Moreover,

$$\begin{array}{ccc} X'' & \xrightarrow{k} & Y' \\ \alpha \downarrow & & \downarrow n \\ X' & \xrightarrow{f} & Y \end{array}$$

commutes by assumption, and it is straightforward to check that it is a pullback since α is an isomorphism and n is monic. Thus

$$\begin{array}{ccccc} & & X'' & & \\ & & \swarrow \alpha & \searrow k & \\ & X' & & & Y' \\ m \swarrow & & & & \searrow n \\ X & & Y & & Z \\ & \nearrow f & \nwarrow g & & \end{array}$$

is a representative of $(m, f) \circ (n, g)$, and equals (m', f') by assumption on α and k . ■

The following result allows us to build prone maps in a partial map category from Cartesian maps in the original category:

5.7. LEMMA. *Suppose $\mathbf{p}: (\mathbb{E}, \mathbf{M}_{\mathbb{E}}) \rightarrow (\mathbb{B}, \mathbf{M}_{\mathbb{B}})$ is an \mathbf{M} -category functor. If $f: X' \rightarrow Y$ is \mathbf{p} -Cartesian in \mathbb{E} , then for any $m: X \rightarrow X'$ in $\mathbf{M}_{\mathbb{E}}$, (the equivalence class of) (m, f) is $\text{Par}(\mathbf{p})$ -prone in the restriction category $\text{Par}(\mathbb{E}, \mathbf{M}_{\mathbb{E}})$.*

PROOF. Suppose $Z \xleftarrow{n} Z' \xrightarrow{g} Y$ is a map from Z to Y :

$$\begin{array}{ccc} Z & & \\ & \searrow (n,g) & \\ X & \xrightarrow{(m,f)} & Y \end{array}$$

Then by the previous lemma, to have a precise triangle with $\mathbf{p}(m, f): X \rightarrow Y$ in the base means that there is an $h: Z' \rightarrow \mathbf{p}(X')$ with $h\mathbf{p}(g) = \mathbf{p}(f)$, with the triangle of the form

$$\begin{array}{ccc} \mathbf{p}(Z) & & \\ (\mathbf{p}(n), h\mathbf{p}(m)) \downarrow & \searrow (\mathbf{p}(n), \mathbf{p}(g)) & \\ \mathbf{p}(X) & \xrightarrow{(\mathbf{p}(m), \mathbf{p}(f))} & \mathbf{p}(Y) \end{array}$$

Then since f is \mathbf{p} -Cartesian, there is a unique $\tilde{h}: Z' \rightarrow X'$ in \mathbb{E} such that $\tilde{h}f = g$ and $\mathbf{p}(\tilde{h}) = h$. Then by the previous lemma, $(n, \tilde{h}m)$ is a precise fill-in for the first triangle:

$$\begin{array}{ccc} Z & & \\ \downarrow (n, \tilde{h}m) & \searrow (n, g) & \\ X & \xrightarrow{(m, f)} & Y \end{array}$$

and is over $(\mathbf{p}(n), \tilde{h}\mathbf{p}(m))$ by definition.

It remains to show it is unique. If we have another precise fill-in, by the previous lemma we can take it to be of the form (n, km) for some $k: Z' \rightarrow X'$ such that $kf = g$. Since it must be over $(\mathbf{p}(n), h\mathbf{p}(m))$, we must have some isomorphism $\alpha: \mathbf{p}(Z') \rightarrow \mathbf{p}(Z')$ such that $\mathbf{p}(n) = \alpha\mathbf{p}(n)$ and $\mathbf{p}(k)\mathbf{p}(m) = \alpha h\mathbf{p}(m)$. But since $\mathbf{p}(n)$ is monic, the first equality means $\alpha = 1$; then since $\mathbf{p}(m)$ is monic the second equality gives $\mathbf{p}(k) = h$. Thus we have $kf = g$ and $\mathbf{p}(k) = h$, so since \tilde{h} was unique with these properties, $k = \tilde{h}$. So $(n, \tilde{h}m)$ is unique. ■

Thus, having an \mathbf{M} -category functor $\mathbf{p}: (\mathbb{E}, \mathbf{M}_{\mathbb{E}}) \rightarrow (\mathbb{B}, \mathbf{M}_{\mathbb{B}})$ being an ordinary fibration is close to $\text{Par}(\mathbf{p}): \text{Par}(\mathbb{E}, \mathbf{M}_{\mathbb{E}}) \rightarrow \text{Par}(\mathbb{B}, \mathbf{M}_{\mathbb{B}})$ being a latent fibration. However, something is missing: if we are given a span

$$\begin{array}{ccc} & Y' & \\ m \swarrow & & \searrow f \\ Y & & \mathbf{p}(X) \end{array}$$

in $\text{Par}(\mathbb{B}, \mathbf{M}_{\mathbb{B}})$, then by the above result, we can lift f to a Cartesian arrow; however, there is no reason why there should be an $\mathbf{M}_{\mathbb{E}}$ monic over m in \mathbb{E} . Thus, we make the following definitions.

5.8. DEFINITION. Let $\mathbf{p}: (\mathbb{E}, \mathbf{M}_{\mathbb{E}}) \rightarrow (\mathbb{B}, \mathbf{M}_{\mathbb{B}})$ be an \mathbf{M} -category functor.

- \mathbf{p} is said to be **M-plentiful** in case for every object $X \in \mathbb{E}$ and $\mathbf{M}_{\mathbb{B}}$ -map $m: \mathbf{p}(X) \rightarrow Y$ there is an $\mathbf{M}_{\mathbb{E}}$ -map $n: X \rightarrow X'$ in \mathbb{E} with $\mathbf{p}(n) = m$.
- \mathbf{p} is said to be an **M-fibration** if it is a fibration (in the ordinary sense) and it is **M-plentiful**.

This is enough to get an (r -split) latent fibration:

5.9. LEMMA. If $\mathbf{p}: (\mathbb{E}, \mathbf{M}_{\mathbb{E}}) \rightarrow (\mathbb{B}, \mathbf{M}_{\mathbb{B}})$ is an \mathbf{M} -fibration then $\text{Par}(\mathbf{p}): \text{Par}(\mathbb{E}, \mathbf{M}_{\mathbb{E}}) \rightarrow \text{Par}(\mathbb{B}, \mathbf{M}_{\mathbb{B}})$ is an r -split latent fibration.

PROOF. Let $(m, f): Y \rightarrow \mathbf{p}(X)$ be a map in $\text{Par}(\mathbb{B}, \mathbf{M}_{\mathbb{B}})$; i.e., a span:

$$Y \xleftarrow{m} Y' \xrightarrow{f} \mathbf{p}(X)$$

We lift this to

$$Y_0 \xleftarrow{n} Y'_1 \xrightarrow{f^*} X$$

where f^* is a Cartesian arrow above f and n is any $\mathbf{M}_{\mathbb{E}}$ -map above m , the existence of which is guaranteed by the plentiful requirement. Then by Lemma 5.7, (n, f^*) is prone in $\mathbf{Par}(\mathbb{T}, \mathbf{M}_{\mathbb{T}})$.

We also need to show that $\mathbf{Par}(\mathbf{p})$ is well-fibred. An idempotent is a span (m, m) where $m: X' \rightarrow X$ is in $\mathbf{M}_{\mathbb{E}}$: its splitting (as an equalizer with the identity) is $(m, 1_{X'})$. Now (m, m) could have $(\mathbf{p}(m), \mathbf{p}(m)) = (1_X, 1_X)$ and yet $\mathbf{p}(X') \neq \mathbf{p}(X)$ although, as both are limits of the trivial equalizer, they must be isomorphic, say by $\gamma: \mathbf{p}(X') \rightarrow \mathbf{p}(X)$. But that means there is a Cartesian arrow above γ^{-1} , $(\gamma^{-1})^*: X_0 \rightarrow X'$ which is an isomorphism and has $\mathbf{p}(X_0) = \mathbf{p}(X)$. This means that $(\gamma^{-1})^*m: X_0 \rightarrow X$ is also a splitting and it is now in the fibre over $\mathbf{p}(X)$. ■

Conversely we have:

5.10. LEMMA. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is an r -split latent fibration, then the induced functor between the total map categories, with their restriction monics, $\mathbf{Total}(\mathbf{p}): (\mathbf{Total}(\mathbb{E}), \mathbf{Monic}(\mathbb{E})) \rightarrow (\mathbf{Total}(\mathbb{B}), \mathbf{Monic}(\mathbb{B}))$, is an \mathbf{M} -fibration.*

PROOF. The fact that \mathbf{p} is a latent fibration ensures that $\mathbf{Total}(\mathbf{p})$ is a fibration as discussed in Corollary 5.4. It remains, therefore, to show that $\mathbf{Total}(\mathbf{p})$ is \mathbf{M} -plentiful. Toward this end let $m: \mathbf{p}(E) \rightarrow X$ be a restriction monic in \mathbb{B} , so that by definition there is a restriction retraction $r: X \rightarrow \mathbf{p}(E)$ such that $mr = 1_{\mathbf{p}(E)}$ and $rm = \bar{r}$. Let $r': X' \rightarrow E$ be a prone arrow over r . Then since \mathbb{E} is r -split, we can use Lemma 3.19.iv to get that r' is itself a restriction retraction. Thus, there is a restriction monic $m': E \rightarrow X'$ which is its partial inverse. Since restriction functors preserve partial inverses, we have

$$\mathbf{p}(m') = \mathbf{p}(r'^{(-1)}) = \mathbf{p}(r')^{(-1)} = r^{(-1)} = m,$$

showing that \mathbf{p} is \mathbf{M} -plentiful. ■

As a consequence we now have:

5.11. THEOREM. *r -split latent fibrations, in the equivalence between r -split restriction categories and \mathbf{M} -categories, correspond precisely to \mathbf{M} -fibrations.*

This gives us a further source of examples of latent fibrations. Any category with finite limits \mathbb{X} is an \mathbf{M} -category with respect to all its monics. The arrow category gives the standard fibration over \mathbb{X} , $\partial: \mathbb{X}^{\rightarrow} \rightarrow \mathbb{X}$. Then moving to $\mathbf{Par}(\partial)$ always gives a latent fibration as above any monic $m: X \rightarrow Y$ in \mathbb{X} and any object $x: X' \rightarrow X$ in \mathbb{X}^{\rightarrow} lies the map (which is a square):

$$\begin{array}{ccc} X' & \xlongequal{\quad} & X' \\ x \downarrow & & \downarrow xm \\ X & \xrightarrow{\quad m \quad} & Y \end{array}$$

which is clearly monic. Note when m is an isomorphism this is an isomorphism in \mathbb{X}^{\rightarrow} and thus this ∂ is \mathbf{M} -conservative. This means the functor $\mathbf{Par}(\partial)$ is \mathbf{M} -plentiful and the $\mathbf{Par}(\partial)$ is a latent fibration – and in a few moments we will see, because pullbacks of monics along any map are monic, that this is an *admissible* latent fibration. Furthermore, it is not hard to see that this is actually the full subfibration of $\mathbf{Par}(\mathbb{X}, \mathbf{Monic})^{\rightsquigarrow} \rightarrow \mathbf{Par}(\mathbb{X}, \mathbf{Monic})$ determined by the objects $a: A' \rightarrow A \in \mathbb{E}$ such that a is total. This means, in particular, that it is not hyperconnected.

It is worth noting that this is not the only choice of \mathbf{M} -maps possible for the domain category \mathbb{X}^{\rightarrow} , as we could have chosen squares which are pullbacks. These are clearly monics in \mathbb{X}^{\rightarrow} and still give an \mathbf{M} -plentiful fibration using the codomain map as the above square for demonstrating \mathbf{M} -plentitude is a pullback. It is not hard then to see that passing to the partial map categories for this fibration gives a subfibration of the strict codomain fibration, $\mathbf{Par}(\mathbb{X}, \mathbf{Monic})^{\rightarrow} \rightarrow \mathbf{Par}(\mathbb{X}, \mathbf{Monic})$ determined by objects a which are total maps.

5.12. **ADMISSIBLE, SEPARATED, AND HYPERCONNECTED \mathbf{M} -FIBRATIONS.** In this section we characterize the admissible, separated, and hyperconnected latent fibrations between partial map categories.

5.13. **DEFINITION.** *An \mathbf{M} -fibration $\mathbf{q}: (\mathbb{T}, \mathbf{M}_{\mathbb{T}}) \rightarrow (\mathbb{S}, \mathbf{M}_{\mathbb{S}})$ is an **admissible \mathbf{M} -fibration** if Cartesian maps over \mathbf{M} -maps are themselves \mathbf{M} -maps.*

We can immediately observe:

5.14. **LEMMA.** *If $\mathbf{p}: (\mathbb{E}, \mathbf{M}_{\mathbb{E}}) \rightarrow (\mathbb{B}, \mathbf{M}_{\mathbb{B}})$ is an admissible \mathbf{M} -fibration then $\mathbf{Par}(\mathbf{p}): \mathbf{Par}(\mathbb{E}, \mathbf{M}_{\mathbb{E}}) \rightarrow \mathbf{Par}(\mathbb{B}, \mathbf{M}_{\mathbb{B}})$ is an admissible r -split latent fibration.*

PROOF. We must show that there is a prone restriction idempotent above every restriction idempotent. A restriction idempotent as a partial map is a span (m, m) . Thus, if we let m^* be a Cartesian map above m , by assumption m^* is in $\mathbf{M}_{\mathbb{E}}$, and by Lemma 5.7, (m^*, m^*) is prone over (m, m) . ■

Conversely, we also have:

5.15. **LEMMA.** *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is an admissible r -split latent fibration, then*

$$\mathbf{Total}(\mathbf{p}): (\mathbf{Total}(\mathbb{E}), \mathbf{Monic}(\mathbb{E})) \rightarrow (\mathbf{Total}(\mathbb{B}), \mathbf{Monic}(\mathbb{B}))$$

is an admissible \mathbf{M} -fibration.

PROOF. We already know by Lemma 5.10 that $\mathbf{Total}(\mathbf{p})$ is an \mathbf{M} -fibration; thus, it remains to show the \mathbf{M} -admissible property. Suppose that $m: X' \rightarrow X$ is a Cartesian map in $\mathbf{Total}(\mathbb{E})$ such that $\mathbf{p}(m)$ is a restriction monic in \mathbb{B} . By Lemma 3.20, m is prone in \mathbb{E} . Then by Lemma 4.6, m is a partial isomorphism, so there is an $r: X \rightarrow X'$ so that $rm = \bar{r}$ and $mr = \bar{m} = 1$. That is, m is a restriction monic. ■

5.16. THEOREM. *Admissible r -split latent fibrations, in the equivalence between r -split restriction categories and \mathbf{M} -categories, correspond precisely to admissible \mathbf{M} -fibrations.*

We next turn to characterizing separated latent fibrations between partial map categories.

5.17. DEFINITION. *An \mathbf{M} -fibration $\mathbf{q}: (\mathbb{T}, \mathbf{M}_{\mathbb{T}}) \rightarrow (\mathbb{S}, \mathbf{M}_{\mathbb{S}})$ is a **separated \mathbf{M} -fibration** if all $\mathbf{M}_{\mathbb{T}}$ -maps are Cartesian.*

5.18. LEMMA. *If $\mathbf{q}: (\mathbb{T}, \mathbf{M}_{\mathbb{T}}) \rightarrow (\mathbb{S}, \mathbf{M}_{\mathbb{S}})$ is a separated \mathbf{M} -fibration then*

$$\text{Par}(\mathbf{q}): \text{Par}(\mathbb{T}, \mathbf{M}_{\mathbb{T}}) \rightarrow \text{Par}(\mathbb{S}, \mathbf{M}_{\mathbb{S}})$$

is a separated r -split latent fibration.

PROOF. By Proposition 4.13, it suffices to show that restriction idempotents are prone, so suppose that (m, m) is a restriction idempotent in $\text{Par}(\mathbb{T}, \mathbf{M}_{\mathbb{T}})$. Then by assumption m is Cartesian, so by Lemma 5.7, (m, m) is prone. ■

5.19. LEMMA. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a separated r -split latent fibration, then*

$$\text{Total}(\mathbf{p}): (\text{Total}(\mathbb{E}), \text{Monic}(\mathbb{E})) \rightarrow (\text{Total}(\mathbb{B}), \text{Monic}(\mathbb{B}))$$

is a separated \mathbf{M} -fibration.

PROOF. Suppose $m: X \rightarrow Y$ is a restriction monic in \mathbb{E} . Then by Corollary 4.16, m is prone, and by definition it is total, hence it is Cartesian in $\text{Total}(\mathbb{E})$. ■

5.20. THEOREM. *Separated r -split latent fibrations, in the equivalence between r -split restriction categories and \mathbf{M} -categories, correspond precisely to separated \mathbf{M} -fibrations.*

By Proposition 4.21, we now also have a characterization of r -split latent hyperfibrations:

5.21. COROLLARY. *r -split latent hyperfibrations, in the equivalence between r -split restriction categories and \mathbf{M} -categories, correspond precisely to \mathbf{M} -fibrations which are admissible and separated.*

5.22. r -SPLITTING A LATENT FIBRATION. Next, we turn to the question of what happens when we split the restriction idempotents of a latent fibration. Unfortunately, in general, this will not again be a latent fibration. To see the issue, suppose we have a latent fibration $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ and a map $f: (Y, e_0) \rightarrow (\mathbf{p}(X), \mathbf{p}(e))$ in $\text{Split}_r(\mathbb{B})$. As a mere map in \mathbb{B} , we can lift it to a \mathbf{p} -prone map $f^*: X' \rightarrow X$ in \mathbb{E} . However, we need it to be a map in $\text{Split}_r(\mathbb{E})$, and in particular we need a restriction idempotent e' over e to serve as the domain for f^* as a map in $\text{Split}_r(\mathbb{E})$. As we have seen earlier (specifically, in the introduction to section 4), this need not always exist. Moreover, we will need this idempotent e' to itself be prone, so that when we compose it with f^* the result will again be prone.

Of course, asking for a prone restriction idempotent in \mathbb{E} over a restriction idempotent in \mathbb{B} is precisely what we demand of an admissible latent fibration. This property turns out to be sufficient to ensure that splitting the restriction idempotents of a latent fibration is still a latent fibration.

5.23. PROPOSITION. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is an admissible latent fibration then $\mathbf{Split}_r(\mathbf{p}): \mathbf{Split}_r(\mathbb{E}) \rightarrow \mathbf{Split}_r(\mathbb{B})$ is an admissible latent fibration.*

PROOF. It is easy to see that splitting idempotents always results in a well-fibred restriction functor: suppose $e = \bar{e}$ is an object of $\mathbf{Split}_r(\mathbb{B})$ and $e_0: e_1 \rightarrow e_1$ is in $\mathbf{p}^{-1}(e)$, so $\mathbf{p}(e_1) = e$ and $\mathbf{p}(e_0) = e$, then a splitting of the map e_0 is given by $e_1 \xrightarrow{e_0} e_0 \xrightarrow{e_0} e_1$ and this is all in $\mathbf{p}^{-1}(e)$.

Now suppose $f: e_0 \rightarrow \mathbf{p}(e)$ is a map in $\mathbf{Split}_r(\mathbb{B})$, with e a restriction idempotent in \mathbb{E} ; we must show that there is a prone lifting to some map in $\mathbf{Split}_r(\mathbb{E})$. As a mere map, $f: B \rightarrow \mathbf{p}(E)$ in \mathbb{B} has a prone arrow over it $f': E' \rightarrow E$ but also e_0 has a, prone, restriction idempotent e_0^* on E' over it (as \mathbf{p} is an admissible latent fibration): we claim that $e_0^*f'e: e_0^* \rightarrow e$ is prone over $f: e_0 \rightarrow \mathbf{p}(e)$: it is certainly over f . Suppose now that $g: e_1 \rightarrow e$ in \mathbb{E} and $\mathbf{p}(g) = hf$ is precise with $h: \mathbf{p}(e_1) \rightarrow e_0$ then there is a lifting of h, \tilde{h} , in \mathbb{E} to give a precise triangle $\tilde{h}(e_0^*f') = g$. The preciseness means $\tilde{h} = \bar{g} \leq e_1$ so it starts at the idempotent e_1 as desired. It also ends at e_0^* as $he_0 = h$ so $\tilde{h}e_0^* = \tilde{h}$. Finally we have $\tilde{h}e_0^*f'e = \tilde{h}f'e = ge = g$. Thus $e_0^*f'e: e_0^* \rightarrow e$ is prone over $f: e_0 \rightarrow \mathbf{p}(e)$ and $\mathbf{Split}_r(\mathbf{p}): \mathbf{Split}_r(\mathbb{E}) \rightarrow \mathbf{Split}_r(\mathbb{B})$ is a latent fibration.

Finally we must show that above each restriction idempotent e on $\mathbf{p}(e')$, in $\mathbf{Split}_r(\mathbb{B})$, which means $e \leq \mathbf{p}(e')$ in \mathbb{B} , there is a prone restriction idempotent on e' : it is of course e^*e' , which is prone by the argument above, and less than or equal to e' . ■

The property of being separated (thus, hyperconnected) is also preserved under this process:

5.24. COROLLARY. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a latent hyperfibration then $\mathbf{Split}_r(\mathbf{p}): \mathbf{Split}_r(\mathbb{E}) \rightarrow \mathbf{Split}_r(\mathbb{B})$ is a latent hyperfibration.*

PROOF. Partial isomorphisms in $\mathbf{Split}_r(\mathbb{E})$ are partial isomorphisms in \mathbb{E} so are prone there and this easily means they are prone in $\mathbf{Split}_r(\mathbb{E})$. The result thus follows from Proposition 4.13 and Proposition 4.21. ■

6. Latent fibrations as fibrations in SRest

The main result in this section shows that latent fibrations are fibrations in the 2-category **SRest** of restriction categories, restriction semifunctors, and transformations. This gives, from a 2-categorical perspective, a further justification of how latent fibrations are defined. Before proving this we describe another characterization of latent fibrations, which is also useful in obtaining the main result of the section: namely that $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a latent fibration if and only if the induced functor $\mathbf{Total}(\mathbf{Split}(\mathbf{p})) : \mathbf{Total}(\mathbf{Spit}(\mathbb{E})) \rightarrow \mathbf{Total}(\mathbf{Split}(\mathbb{B}))$ is an ordinary fibration. Regarding maps in $\mathbf{Total}(\mathbf{Split}_r(\mathbb{X}))$ as latent total maps in \mathbb{X} , one then has, in support of the nomenclature, that a latent fibration is a fibration between latent total maps. This, characterization, in particular, allows a precise characterization of when r -splitting a latent fibration is a latent fibration.

6.1. THE ORDINARY FIBRATION OF LATENT TOTAL MAPS. We start with a key observation, which justifies the term “latent” fibration, as we show that a latent fibration is precisely an ordinary fibration between the categories of latent total maps:

6.2. PROPOSITION. $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a latent fibration if and only if

$$\text{Total}(\text{Split}_r(\mathbf{p})): \text{Total}(\text{Split}_r(\mathbb{E})) \rightarrow \text{Total}(\text{Split}_r(\mathbb{B}))$$

is an ordinary fibration.

PROOF.

(\Rightarrow) Suppose \mathbf{p} is a latent fibration, we must show that, for any total map $\bar{f} \xrightarrow{f} \mathbf{p}(e)$, there is a Cartesian arrow $\bar{f}^* \xrightarrow{f^*} e$ lying over it. Since f is a map in $\text{Split}_r(\mathbb{B})$, we have $f\mathbf{p}(e) = f$, so f has a prone map f^* over it. It is straightforward to check that viewing this as a total map satisfies all the Cartesian requirements. In particular,

$$\begin{array}{ccc} \bar{g} & \searrow^g & \\ \tilde{h} \downarrow \text{dotted} & & \\ \bar{f}^* & \xrightarrow{f^*} & e \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathbf{p}(\bar{g}) & \searrow^{\mathbf{p}(g)} & \\ h \downarrow & & \\ \bar{f} & \xrightarrow{f} & \mathbf{p}(e) \end{array}$$

when the right triangle is precise then there is a unique precise lifting to \tilde{h} with $\tilde{h} = \bar{g}$ and so the desired diagram of total maps is obtained.

(\Leftarrow) Suppose now that we know that $\text{Total}(\text{Split}_r(\mathbf{p}))$ is an ordinary fibration, then we must show there is a prone map above $f: B \rightarrow \mathbf{p}(E)$. Of course, we may turn f into a total map $f: \bar{f} \rightarrow 1_{\mathbf{p}(E)}$ in $\text{Split}_r(\mathbb{X})$ and, by assumption this has a Cartesian map $f^*: e' \rightarrow 1_E$ over it. However, this means $\mathbf{p}(f^*) = f$ and so we have that f^* is certainly a map over f . To show it is prone we must show we have a lift \tilde{h} over any h making the right-hand triangle precise.

$$\begin{array}{ccc} Z & \searrow^g & \\ \tilde{h} \downarrow \text{dotted} & & \\ A & \xrightarrow{f^*} & E \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathbf{p}(Z) & \searrow^{\mathbf{p}(g)} & \\ h \downarrow & & \\ B & \xrightarrow{f} & \mathbf{p}(E) \end{array}$$

The preciseness of the right-hand diagram, however, gives the triangle of total maps in the first part of the proof which gives the commuting triangle over the Cartesian arrow, which unwinds to give the left-hand precise diagram above.

■

As a corollary of this we have:

6.3. COROLLARY. *For any restriction category \mathbb{X} , the semifunctor $\mathbf{u}: \mathbf{Total}(\mathbf{Split}_r(\mathbb{X})) \rightarrow \mathbb{X}$ (where $\mathbf{u}(f) = f$ and for an object $e: X \rightarrow X$, $\mathbf{u}(e) = X$) is a latent fibration.*

PROOF. This is so if and only if $\mathbf{u}' = \mathbf{Total}(\mathbf{Split}_r(\mathbf{u}))$ is a fibration: however, it is not hard to see that \mathbf{u}' is an equivalence of categories with a section \mathbf{v} with action on objects $\mathbf{v}(e) = e \xrightarrow{e} e$ with unit $\eta_e: e \rightarrow \mathbf{u}(\mathbf{v}(e)) = \text{id}$. In general an equivalence is *not* a fibration; however, when its unit at the base is the identity – as it is in this case – then it is a fibration.

Alternatively, one may observe directly that given $f: A \rightarrow B$ and $e = \bar{e}: B \rightarrow B$ such that $fe = f$, the prone lift of f is just $f: \bar{f} \rightarrow e \in \mathbf{Total}(\mathbf{Split}_r(\mathbb{X}))$. ■

This observation is more delicate than it might seem: it is certainly not the case that $\mathbf{p}: \mathbf{Total}(\mathbb{X}) \rightarrow \mathbb{X}$ is a latent fibration in general. A counter-example is given by the free restriction category over a directed graph (see [5]) the total map category of such a category is discrete so finding prone arrows over non-trivial maps is impossible just from type considerations. One may think that the problem is that the category \mathbb{X} does not have splittings of idempotents but even if restriction idempotents split in \mathbb{X} , provided \mathbb{X} has a non-total partial map, $\mathbf{p}: \mathbf{Total}(\mathbb{X}) \rightarrow \mathbb{X}$ will not be a latent fibration as there is no map sitting above any non-total partial map. Thus, the form of the semifunctor $\mathbf{p}: \mathbf{Total}(\mathbf{Split}_r(\mathbb{X})) \rightarrow \mathbb{X}$ was important: semifunctors do not preserve total maps and this \mathbf{p} covers all maps of \mathbb{X} , albeit with total maps.

We have already noted that even if $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a latent fibration $\mathbf{Split}_r(\mathbf{p})$ need not be (see Section 5.22). Of course, the functor $\mathbf{Total}(\mathbf{Split}_r(\mathbf{p}))$ must be an ordinary fibration, and, in fact, it also is an \mathbf{M} -functor with respect to sections of restriction idempotents. In the previous section we discovered that an \mathbf{M} -functor which is a fibration, under the creation of the partial map category, becomes a latent fibration if and only if it is \mathbf{M} -plentiful. The partial map construction on $\mathbf{Total}(\mathbf{Split}_r(\mathbb{X}))$, for any restriction category \mathbb{X} , just returns $\mathbf{Split}_r(\mathbb{X})$. This allows us to conclude that $\mathbf{Split}_r(\mathbf{p})$ is a latent fibration when \mathbf{p} is a latent fibration if and only if $\mathbf{Total}(\mathbf{Split}_r(\mathbf{p}))$ is plentiful for sections of restriction idempotents.

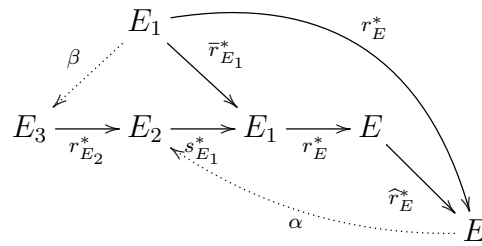
Explicitly, for $\mathbf{Total}(\mathbf{Split}_r(\mathbf{p}))$ to be **plentiful for restriction sections** means that for every $e = \bar{e} \in \mathbb{E}$ and partial isomorphism $s: \mathbf{p}(e) \rightarrow d$ with $\bar{s} = \mathbf{p}(e)$ and $sd = s$ there is a total partial isomorphism $s': e \rightarrow d'$ in \mathbb{E} with $\bar{s}' = e$ so that $\mathbf{p}(s') = s$ and $\mathbf{p}(d') = d$. It is more natural to expect liftings for maps into $\mathbf{p}(e)$ and thus it is useful to re-express these requirements using the partial inverse of s . The requirement then becomes, equivalently, that for every $e = \bar{e} \in \mathbb{E}$ and partial isomorphism $r: d \rightarrow \mathbf{p}(e)$ with $\widehat{r} = \mathbf{p}(e)$ there is a partial isomorphism $r': d' \rightarrow e$ with $\widehat{r}' = e$, $\mathbf{p}(d') = d$ and $\mathbf{p}(r') = r$. We shall simply say that \mathbf{p} is **plentiful** in this case. Also, when this condition is satisfied for a given idempotent $e \in \mathbb{E}$ and partial isomorphism $r: d \rightarrow \mathbf{p}(e)$ with $\widehat{r} = \mathbf{p}(e)$, we say that \mathbf{p} is **plentiful at r** . This allows us to state:

6.4. PROPOSITION. *A latent fibration \mathbf{p} has $\mathbf{Split}_r(\mathbf{p})$ a latent fibration if and only if \mathbf{p} is plentiful.*

Proposition 5.23 established directly that when \mathbf{p} is an admissible latent fibration then $\mathbf{Split}_r(\mathbf{p})$ is an admissible latent fibration. However, that proof did not use the plentifulness of \mathbf{p} and, indeed, it is not entirely obvious why an admissible latent fibration \mathbf{p} should be plentiful.

6.5. COROLLARY. *Admissible latent fibrations are plentiful.*

To see why an admissible fibrations is plentiful directly, consider $r : d \rightarrow \mathbf{p}(e)$, where $e \in \mathcal{O}(E)$ and $\widehat{r} = \mathbf{p}(e)$, a partial isomorphism in \mathbb{B} . We wish to show that at the partial inverse $s := r^{(-1)}$ the fibration \mathbf{p} is plentiful. For this consider the prone liftings of s, r , and their restriction idempotents:



This gives α and β mediating partial isomorphisms from the prone lifting of the restriction idempotents to the two composites of the liftings of s and r so that $\overline{\alpha} = \widehat{r}_E^*$, $\widehat{\alpha} = \overline{s_{E_1}^* r_E^*}$, $\overline{\beta} = \overline{r_{E_1}^*}$ and $\widehat{\beta} = \overline{r_{E_2}^* s_{E_1}^*}$. Also note that $r_E^* = \overline{r_{E_1}^*} r_E^* \widehat{r}_E^*$ as $\overline{r_E^*} \leq \overline{r_{E_1}^*}$ and $\widehat{r_E^*} \leq \widehat{r_E^*}$ by virtue of Lemma 4.5. For the same reason $e \leq \widehat{r_E^*}$. Set $s' := \alpha s_{E_1}^* \beta$ and $r' := \beta^{(-1)} r_E^* e$, then one can check that these are partial inverses with $s' : e \rightarrow d_{E_2}^*$ and $\mathbf{p}(s') = s$, so that s' is a witness to \mathbf{p} being plentiful at s .

6.6. FIBRATIONS IN \mathbf{SRest} . The purpose of this section is to prove:

6.7. THEOREM. *Latent fibrations are precisely fibrations in the 2-category \mathbf{SRest} .*

Recall that Street [23] provided a general definition of a fibration in a 2-category. The following is a version of Street’s definition from [20]:

6.8. DEFINITION. *A 1-cell $\mathbf{p} : E \rightarrow B$ in a 2-category \mathcal{K} is a **fibration** if and only if $\mathcal{K}(X, \mathbf{p})$ is a fibration for all $X \in \mathcal{K}$ and for all $x : Y \rightarrow X \in \mathcal{K}$*

$$\begin{array}{ccc} \mathcal{K}(X, E) & \xrightarrow{\mathcal{K}(x, E)} & \mathcal{K}(Y, E) \\ \mathcal{K}(X, \mathbf{p}) \downarrow & & \downarrow \mathcal{K}(Y, \mathbf{p}) \\ \mathcal{K}(X, B) & \xrightarrow{\mathcal{K}(x, B)} & \mathcal{K}(Y, B) \end{array}$$

is a fibred functor (in the sense of preserving Cartesian maps).

Equivalently, and more concretely (as pointed out in [20]), a 1-cell $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ in a 2-category is a fibration if every 2-cell $\alpha: G \Rightarrow F; \mathbf{p}$ has a **lift** $\alpha^*: \alpha^*F \Rightarrow F$

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{F} & \mathbb{E} \\ & \searrow \Rightarrow \alpha & \downarrow \mathbf{p} \\ & G & \mathbb{B} \end{array} = \begin{array}{ccc} \mathbb{X} & \xrightarrow{F} & \mathbb{E} \\ \uparrow \alpha^* & & \downarrow \mathbf{p} \\ \mathbb{X} & \xrightarrow{\alpha^*F} & \mathbb{E} \\ & \searrow G & \downarrow \mathbf{p} \\ & & \mathbb{B} \end{array}$$

which is \mathbf{p} -Cartesian in the following sense.

A 2-cell $\beta: H \Rightarrow F$ in a 2-category is **\mathbf{p} -Cartesian** if for all 2-cells ξ and γ such that

$$\begin{array}{ccc} \mathbb{Y} & \xrightarrow{D} & \mathbb{E} \\ & \searrow X & \downarrow \xi \\ & & \mathbb{X} \end{array} \begin{array}{ccc} & \nearrow F & \downarrow \mathbf{p} \\ & & \mathbb{B} \end{array} = \begin{array}{ccc} \mathbb{Y} & \xrightarrow{D} & \mathbb{E} \\ & \searrow X & \downarrow \gamma \\ & & \mathbb{X} \end{array} \begin{array}{ccc} & \nearrow H & \downarrow \mathbf{p} \\ & & \mathbb{E} \end{array} \begin{array}{ccc} & \nearrow F & \downarrow \mathbf{p} \\ & & \mathbb{B} \end{array}$$

there is a unique 2-cell, $\hat{\gamma}: D \Rightarrow X; H$, the **Cartesian lift**, such that:

$$\begin{array}{ccc} \mathbb{Y} & \xrightarrow{D} & \mathbb{E} \\ & \searrow X & \downarrow \hat{\gamma} \\ & & \mathbb{X} \end{array} \begin{array}{ccc} & \nearrow H & \downarrow \mathbf{p} \\ & & \mathbb{B} \end{array} = \begin{array}{ccc} \mathbb{Y} & \xrightarrow{D} & \mathbb{E} \\ & \searrow X & \downarrow \gamma \\ & & \mathbb{X} \end{array} \begin{array}{ccc} & \nearrow H & \downarrow \mathbf{p} \\ & & \mathbb{E} \end{array} \begin{array}{ccc} & \nearrow F & \downarrow \mathbf{p} \\ & & \mathbb{B} \end{array} \text{ and } \begin{array}{ccc} \mathbb{Y} & \xrightarrow{D} & \mathbb{E} \\ & \searrow X & \downarrow \hat{\gamma} \\ & & \mathbb{X} \end{array} \begin{array}{ccc} & \nearrow H & \downarrow \beta \\ & & \mathbb{E} \end{array} \begin{array}{ccc} & \nearrow F & \downarrow \mathbf{p} \\ & & \mathbb{B} \end{array} = \begin{array}{ccc} \mathbb{Y} & \xrightarrow{D} & \mathbb{E} \\ & \searrow H & \downarrow \xi \\ & & \mathbb{X} \end{array} \begin{array}{ccc} & \nearrow F & \downarrow \mathbf{p} \\ & & \mathbb{B} \end{array}$$

There is an alternative perspective we can take on this structure in the 2-category **SRest**. Observe that **SRest** has copowers: this means for any mere category \mathbb{C} there is an equivalence $\mathbf{SRest}(\mathbb{C} \bullet \mathbb{X}, \mathbb{Y}) \cong \mathbf{Cat}(\mathbb{C}, \mathbf{SRest}(\mathbb{X}, \mathbb{Y}))$ where $\mathbb{C} \bullet \mathbb{X}$ is given by the product $\mathbb{C} \times \mathbb{X}$ of \mathbb{C} viewed as a trivial restriction category (i.e. all restrictions are the identity) with \mathbb{X} . When a 2-category has copowers we may re-express the properties of having a lift and of being a Cartesian arrow more succinctly using copowers as two lifting properties⁴ – where the second lifting is required to be unique:

$$\begin{array}{ccc} 1 \bullet \mathbb{X} & \longrightarrow & \mathbb{E} \\ \partial_1 \bullet \mathbb{X} \downarrow & \nearrow \alpha^* & \downarrow \mathbf{p} \\ 2 \bullet \mathbb{X} & \xrightarrow{\alpha} & \mathbb{B} \end{array} \quad \begin{array}{ccc} 2 \bullet \mathbb{Y} & \xrightarrow{2 \times X} & 2 \bullet \mathbb{X} \\ \downarrow & & \searrow \alpha^* \\ 3' \bullet \mathbb{Y} & \longrightarrow & \mathbb{E} \\ \downarrow \iota & \nearrow \hat{\gamma} & \downarrow \mathbf{p} \\ 3 \bullet \mathbb{Y} & \xrightarrow{\gamma} & \mathbb{B} \end{array}$$

“lift” “Cartesian lift”

Here, $\mathbb{2}$ is the free living arrow, $\mathbb{3}$ is the free living composable pair, and $\mathbb{3}'$ is the category with two arrows with a common codomain. The functor $\iota: \mathbb{3}' \rightarrow \mathbb{3}$ is the inclusion and $\mathbb{2}$

⁴We are grateful to our referee for drawing our attention to this way of expressing a fibration.

$\rightarrow \mathfrak{B}$ picks out the second arrow of the composite. One can check that the lifting property across the lower square is exactly the (parameterized) property of being prone. While we will not make explicit use of this perspective in the results that follow, it is a useful alternate viewpoint to keep in mind.

We shall prove the theorem using a series of lemmas. First the easiest step:

6.9. LEMMA. *If \mathfrak{p} is a fibration in the 2-category \mathbf{SRest} , then $\mathfrak{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a latent fibration.*

PROOF. Notice that $\mathbf{SRest}(\mathbb{1}, \mathbb{X})$, where $\mathbb{1}$ is the final (restriction) category consisting of one object and one map, is precisely the same as $\mathbf{Total}(\mathbf{Split}(\mathbb{X}))$. Clearly a functor $\mathbb{1} \rightarrow \mathbb{X}$ corresponds to a restriction idempotent and transformations between such functors are total maps between the idempotents. Thus, the fact that $\mathbf{SRest}(\mathbb{1}, \mathfrak{p}) : \mathbf{SRest}(\mathbb{1}, \mathbb{E}) \rightarrow \mathbf{SRest}(\mathbb{1}, \mathbb{B})$ is an ordinary fibration implies by Proposition 6.2 that \mathfrak{p} is a latent fibration. ■

To establish that a latent fibration is a fibration in the 2-category \mathbf{SRest} we must show that $\mathbf{SRest}(\mathbb{X}, \mathfrak{p})$ is an ordinary fibration and that every $H : \mathbb{Y} \rightarrow \mathbb{X}$ induces a fibred functor $\mathbf{SRest}(H, _)$. As above this can be broken into two steps: showing that for each α there is a lift, α^* , and showing that the lift α^* is Cartesian in the 2-categorical sense. This latter is shown by arguing that α^* is Cartesian if and only if each component is prone, and precomposing with a functor does not change this property.

6.10. LEMMA. *If $\mathfrak{p} : \mathbb{E} \rightarrow \mathbb{B}$ is a latent fibration then $\mathbf{SRest}(\mathbb{X}, \mathfrak{p}) : \mathbf{SRest}(\mathbb{X}, \mathbb{E}) \rightarrow \mathbf{SRest}(\mathbb{X}, \mathbb{B})$ for each $\alpha : G \rightarrow F; \mathfrak{p}$ has a lift $\alpha^* : \alpha^*F \rightarrow F$.*

PROOF. The proof relies on having a cleavage so at this stage we are assuming the ability to have or, at least, choose a cleavage for a latent fibration.

We must show that if $\alpha : G \rightarrow F; \mathfrak{p}$ is a natural transformation between restriction semifunctors then there is a lift of this transformation to $\alpha^* : \alpha^*F \rightarrow F$ which is prone. The semifunctor α^*F and transformation α^* are defined as follows:

$$\begin{array}{ccc}
 \alpha^*F(X) \xrightarrow{\alpha_X^*} F(X) & & G(X) \xrightarrow{\alpha_X} \mathfrak{p}(F(X)) \\
 \alpha^*F(g) := \widehat{G(g)} \downarrow & \Downarrow F(g) & G(g) \downarrow \\
 \alpha^*F(Y) \xrightarrow{\alpha_Y^*} F(Y) & \mapsto & G(Y) \xrightarrow{\alpha_Y} \mathfrak{p}(F(Y)) \\
 & & \downarrow \mathfrak{p}(F(g))
 \end{array}$$

The lifting $\widehat{G(g)}$ exists as the right-hand square (reduced to a triangle with hypotenuse $\alpha_X \mathfrak{p}(F(g))$) is precise as

$$\overline{\alpha_X \mathfrak{p}(F(g))} = \overline{G(g) \alpha_Y} = \overline{G(g) \alpha_Y} = \overline{G(g) G(1_Y)} = \overline{G(g)}$$

The uniqueness of $\widehat{G(g)}$ now easily implies that $\widehat{G(g_1 g_2)} = \widehat{G(g_1)} \widehat{G(g_2)}$ showing α^*F is a semifunctor. Furthermore, as $\alpha^*F(\bar{e}) = \alpha^*F(e) = \alpha^*F(e) \alpha^*$, it follows that α^*F preserves restrictions and is therefore a restriction semifunctor. Clearly α^* then becomes a transformation of restriction semifunctors as $\overline{\alpha_X^*} = \overline{G(1_X)} = \overline{\alpha^*F(1_X)}$. ■

6.11. LEMMA. *In \mathbf{SRest} a 2-cell $\beta : H \rightarrow F : \mathbb{Y} \rightarrow \mathbb{X}$ is \mathbf{p} -Cartesian if and only if every component $\beta_X : H(X) \rightarrow F(X)$ is \mathbf{p} -prone.*

PROOF. If $\beta : H \rightarrow F : \mathbb{Y} \rightarrow \mathbb{X}$ is \mathbf{p} -Cartesian in the 2-categorical sense then we may probe at an object with $X : \mathbb{1} \rightarrow \mathbb{Y}$ to obtain the component β_X which must still be Cartesian. However, this is Cartesian in \mathbf{SRest} if and only if β_X is prone.

Conversely, if every component β_X is prone then we must show that for any $Q : \mathbb{Z} \rightarrow \mathbb{Y}$ that $Q; \beta$ is \mathbf{p} -Cartesian. Towards this end we suppose we have $\gamma : G; \mathbf{p} \rightarrow Q; H; \mathbf{p}$; we must show there is a unique $\hat{\gamma} : G \rightarrow Q; H$. We immediately have a $\hat{\gamma}_Z$ at each component which by preciseness has restriction $\overline{G(1_Z)}$ so all we need is that the naturality squares commute: but this is a straightforward consequence of the uniqueness of the lift. ■

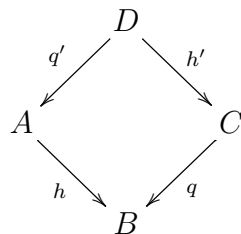
6.12. COROLLARY. α^* as defined in Lemma 6.10 is Cartesian in the 2-categorical sense.

This also completes the proof of the theorem.

7. The Dual of a Latent Hyperfibration

Latent hyperfibrations have the interesting property that one can form their fibred dual. This process is, for example, important in framing the semantics of reverse differential programming in which a fibrewise ability to take the dual is required. For an ordinary fibration, one can construct its fibred dual by using a span construction on the domain category of the fibration (for example, see [2, pg. 898], [15, Section 1.10.11], and [18]). In this section, we see how to generalize this construction to any latent hyperfibration. We begin by investigating certain conditions under which we can form a restriction category of spans.

7.1. HYPER-OPENS AND SPANS. Let \mathbb{X} be a restriction category and \mathcal{H} and \mathcal{Q} be two **downclosed systems** of maps, that is classes of maps which are downclosed, closed to composition, and contain the partial isomorphisms, which commute in the sense that for each $h : A \rightarrow B \in \mathcal{H}$ and $q : C \rightarrow B \in \mathcal{Q}$ there is a latent pullback



such that $h' \in \mathcal{H}$ and $q' \in \mathcal{Q}$. Note that if there is one latent pullback with this property then *all* latent pullbacks for the cospan (h, q) will have this property as the classes are certainly closed to mediation.

We shall be particularly interested in the situation in which the maps in \mathcal{H} are **hyper-open** in the sense that, for each $(h : X \rightarrow Y) \in \mathcal{H}$ the map

$$h^* : \exists_h(1_X)/\mathcal{O}(Y) \rightarrow \overline{h}/\mathcal{O}(X) : e \mapsto \overline{he}$$

is an isomorphism with inverse

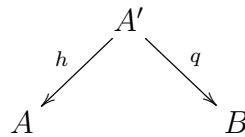
$$\exists_h: \bar{h}/\mathcal{O}(X) \rightarrow \exists_h(1_X)/\mathcal{O}(Y); e \mapsto \exists_h(e).$$

Note that any hyper-open map is automatically open (for a map to be open one only requires the existence of the adjoint, where hyper-open requires the adjoint to be an isomorphism; see [7]) and so in particular any hyper-open map $h: X \rightarrow Y$ has an associated map $\hat{h}: Y \rightarrow Y$ which satisfies the properties of a range combinator.

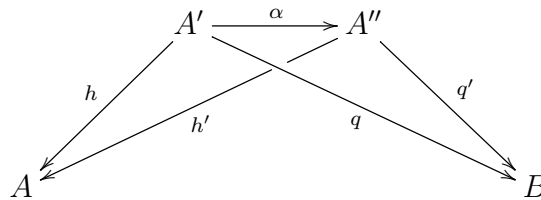
Given such a commuting pair of classes of maps we can form the span category $\text{Span}_{\mathbb{X}}(\mathcal{H}, \mathcal{Q})$ with the following data:

Objects: Those of \mathbb{X} ;

Maps: An arrow $A \rightarrow B$ consists of an equivalence class of spans



such that $h \in \mathcal{H}, q \in \mathcal{Q}$ with $\bar{h} = \bar{q}$ under the equivalence relation that $(f, g) \sim (f', g')$ when there is a mediating partial isomorphism $\alpha \in \mathcal{H}$ such that:



where $\alpha h' = h$ and $\alpha q' = q$, $\alpha^{-1}h = h'$ and $\alpha^{-1}q = q'$, and $\overline{\alpha^{-1}} = \bar{q}'$ and $\bar{\alpha} = \bar{q}$.

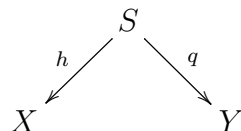
Identities: The identity span $A = A = A$;

Composition: By latent pullback.

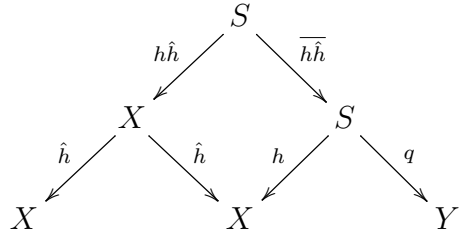
$\text{Span}_{\mathbb{X}}(\mathcal{H}, \mathcal{Q})$ is clearly a category but, in general, is not a restriction category. However, if \mathcal{H} is a hyper-open class of maps then it is a restriction category with $\overline{(h, q)} = (\hat{h}, \hat{h})$.

7.2. PROPOSITION. *If \mathcal{H} and \mathcal{Q} are two commuting classes of maps with \mathcal{H} a class of hyper-opens then $\text{Span}_{\mathbb{X}}(\mathcal{H}, \mathcal{Q})$ is a restriction category.*

PROOF. We will use \circ to denote composition (still in diagrammatic order) in $\text{Span}_{\mathbb{X}}(\mathcal{H}, \mathcal{Q})$. For [R.1], let



be a representative of an arrow in $\text{Span}_{\mathbb{X}}(\mathcal{H}, \mathcal{Q})$. Then since \hat{h} is a restriction idempotent (and hence a partial isomorphism), by Lemma 2.11.iii, the composite $\overline{(h, q)} \circ (h, q)$ is given by

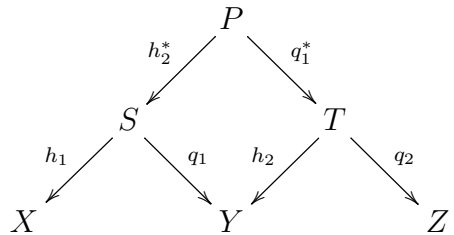


But using the range properties of open maps, we have

$$(h\hat{h}\hat{h}, \overline{h\hat{h}q}) = (h, \overline{h\hat{h}q}) = (h, \overline{q\hat{h}q}) = (h, q)$$

as required. Conditions [R.2] and [R.3] follow similarly.

For [R.4], let $(h_2^*h_1, q_1^*q_2)$ denote the composite $(h_1, q_1) \circ (h_2, q_2)$:

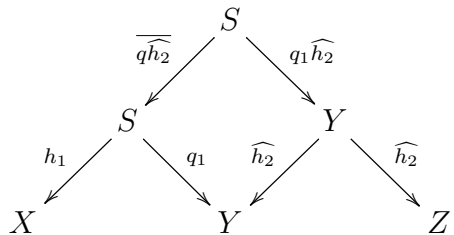


(where the diamond is a latent pullback).

Then $\overline{(h_1, q_1) \circ (h_2, q_2)} = \overline{(h_2^*h_1, h_2^*h_1)}$, so as for [R.1],

$$\overline{(h_1, q_1) \circ (h_2, q_2)} \circ (h_1, q_1) = \overline{(h_1\hat{h}_2^*h_1, h_1\hat{h}_2^*h_1q_1)} = \overline{(h_1\hat{h}_2^*h_1h_1, h_1\hat{h}_2^*h_1q_1)}.$$

On the other hand, $(h_1, q_1) \circ \overline{(h_2, q_2)}$ is given by



that is,

$$(h_1, q_1) \circ \overline{h_2, q_2} = \overline{(q_1\hat{h}_2h_1, q_1\hat{h}_2)} = \overline{(q_1\hat{h}_2h_1, q_1\hat{h}_2q_1)}$$

Thus, comparing the two expressions, it suffices to prove that

$$\overline{h_1\hat{h}_2^*h_1} = \overline{q_1\hat{h}_2}.$$

Note that these are both restriction idempotents on S . To prove they are equal, since h_2^* is hyper-open, pulling back along h_2^* is an isomorphism; thus, it suffices to instead prove that

$$\overline{\overline{h_2^* h_1 h_2^* h_1}} = \overline{\overline{h_2^* q_1 h_2}}.$$

Indeed,

$$\overline{\overline{h_2^* h_1 h_2^* h_1}} = \overline{\overline{h_2^* h_1 h_2^* h_1}} = \overline{\overline{h_2^* h_1}}$$

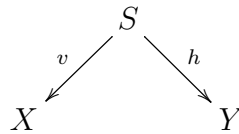
while

$$\overline{\overline{h_2^* q_1 h_2}} = \overline{\overline{h_2^* q_1 h_2}} = \overline{\overline{q_1^* h_2 h_2}} = \overline{\overline{q_1^* h_2}} = \overline{\overline{h_2^* q_1}} = \overline{\overline{h_2^* h_1}}$$

since $\overline{q_1} = \overline{h_1}$. ■

7.3. THE FIBRATIONAL DUAL. Our main construction of this section is the following:

7.4. PROPOSITION. *Suppose $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a latent hyperfibration. Then there is a restriction category \mathbb{E}^* whose objects are those of \mathbb{E} and whose maps are equivalence classes of spans*



where v is subvertical, h is prone, and $\overline{v} = \overline{h}$; the equivalence is up to subvertical partial isomorphism and as in Proposition 7.2, the restriction of $[(v, c)]$ is $[(\hat{v}, \hat{v})]$. Moreover, there is a restriction functor $\mathbf{p}^*: \mathbb{E}^* \rightarrow \mathbb{B}$ which is defined on objects as $\mathbf{p}^*(A) = \mathbf{p}(A)$ and on arrows as $\mathbf{p}^*(v, h) = \mathbf{p}(h)$.

PROOF. Using Lemma 4.17, the prones and subverticals in a hyperfibration form a commuting pair of classes of maps; moreover, the subverticals are hyper-open and the classes are downclosed systems of maps. Thus Proposition 7.2 tells us we can form a restriction category from the spans. We will use \circ to denote composition in \mathbb{E}^* .

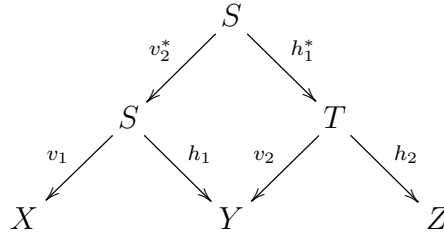
We need to show that \mathbf{p}^* is a well-defined restriction functor. To start, first note that for a subvertical partial isomorphism α , $\alpha\alpha^{(-1)} = \overline{\alpha}$ gives $\mathbf{p}(\alpha)\mathbf{p}(\alpha^{(-1)}) = \overline{\mathbf{p}(\alpha)} = \mathbf{p}(\alpha)$, so that $\mathbf{p}(\alpha) \leq \mathbf{p}(\alpha^{(-1)})$. The reverse inequality follows similarly, and so $\mathbf{p}(\alpha) = \mathbf{p}(\alpha^{(-1)})$. We will use this to check \mathbf{p}^* is well-defined.

Indeed, suppose that (v_1, h_1) and (v_2, h_2) represent the same map in \mathbb{E}^* , so that, in particular, there is a subvertical partial isomorphism α so that $h_1 = \alpha h_2$ and $\overline{\alpha^{(-1)}} h_2 = h_1$. Then

$$\mathbf{p}(h_1) = \mathbf{p}(\alpha)\mathbf{p}(h_2) = \mathbf{p}(\alpha^{(-1)})\mathbf{p}(h_2) = \overline{\mathbf{p}(\alpha^{(-1)})}\mathbf{p}(h_2) = \overline{\mathbf{p}(\alpha^{(-1)})}h_2 = \mathbf{p}(h_2),$$

so \mathbf{p}^* is well-defined.

For preservation of composition, suppose that we form the composite of (v_1, h_1) and (v_2, h_2) :



As in the previous lemma, we can choose h_1^* to be the prone arrow over $\mathbf{p}(h_1)\mathbf{p}(v_2)$ so $\mathbf{p}(h_1^*) = \mathbf{p}(h_1)\mathbf{p}(v_2)$. Then we have

$$\mathbf{p}(h_1^*h_2) = \mathbf{p}(h_1^*)\mathbf{p}(h_2) = \mathbf{p}(h_1)\mathbf{p}(v_2)\mathbf{p}(h_2) = \mathbf{p}(h_1)\mathbf{p}(\overline{v_2})\mathbf{p}(h_2) = \mathbf{p}(h_1)\mathbf{p}(h_2)$$

since $\overline{v_2} = \overline{h_2}$. Thus

$$\mathbf{p}^*[(v_1, h_1) \circ (v_2, h_2)] = \mathbf{p}^*[(v_2^*v_1, h_1^*h_2) = \mathbf{p}(h_1^*h_2) = \mathbf{p}(h_1)\mathbf{p}(h_2) = \mathbf{p}^*(v_1, h_1)\mathbf{p}^*(v_2, h_2),$$

so composition is preserved.

Finally, \mathbf{p}^* is a restriction functor since $\mathbf{p}(\hat{h}) = \mathbf{p}(h)$ by definition of \hat{h} . ■

The following lemma is useful when working with maps in \mathbb{E}^* .

7.5. LEMMA. *Suppose that $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a hyperfibration.*

(i) *For any $[(v, c)]: X \rightarrow Y$ in \mathbb{E}^* and any $c': Y \rightarrow Z$ prone in \mathbb{E} ,*

$$[(v, c)] \circ [(\overline{c'}, c)] = [(\overline{cc'}v, cc')].$$

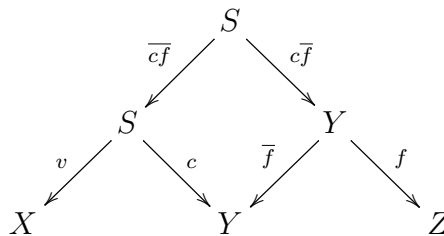
(ii) *For any $v: Y \rightarrow X$ subvertical in \mathbb{E} and any $[(v', c)]: Y \rightarrow Z$ in \mathbb{E}^* ,*

$$[(v, \overline{v})] \circ [(v', c)] = [(v'v, \overline{v'vc})].$$

(iii) *Any $[(v, c)]: X \rightarrow Y$ in \mathbb{E}^* can be factored as:*

$$(v, c) = (v, \overline{v}) \circ (\overline{c}, c).$$

PROOF. For (i), by Lemma 2.11.iii, $(v, c) \circ (\overline{f}, f)$ is given by



Thus, the composite is $(v, c) \circ (\overline{f}, f) = (\overline{cfv}, cf)$; part (ii) follows similarly.

For (iii), by (i), $(v, \overline{v}) \circ (\overline{c}, c) = (\overline{cv}, \overline{vc})$, which equals (v, c) since $\overline{v} = \overline{c}$. ■

7.6. LEMMA. *If $p: \mathbb{E} \rightarrow \mathbb{B}$ is a hyperfibration, and $f: X \rightarrow Y$ is prone in \mathbb{E} , then (\bar{f}, f) is prone in \mathbb{E}^* .*

PROOF. Suppose we are given

$$\begin{array}{ccc} Z & & \\ & \searrow^{(v,c)} & \\ X & \xrightarrow{(\bar{f},f)} & Y \end{array} \quad \mapsto \quad \begin{array}{ccc} p(Z) & & \\ & \searrow^{p^*(v,c)=p(c)} & \\ h \downarrow & & \\ p(X) & \xrightarrow{p(f)} & p(Y) \end{array}$$

such that $\bar{h} = \overline{p(c)}$. Then we claim that (v, h') is the required unique fill-in, where h' is the unique fill-in from the universal property of f as a prone arrow in E :

$$\begin{array}{ccc} S & & \\ \downarrow h' & \searrow^c & \\ X & \xrightarrow{f} & Y \end{array} \quad \mapsto \quad \begin{array}{ccc} p(Z) & & \\ & \searrow^{p(c)} & \\ h \downarrow & & \\ p(X) & \xrightarrow{p(f)} & p(Y) \end{array}$$

First, (v, h') is a well-defined arrow in \mathbb{E}^* : by Lemma 4.15, h' is prone in \mathbb{E} , and by definition of h' and (v, c) , $\bar{h}' = \bar{c} = \bar{v}$.

Second, (v, h') does make the triangle commute: using Lemma 7.5.i,

$$(v, h') \circ (\bar{f}, f) = (\bar{h}'\bar{f}v, h'f) = (\bar{c}v, c) = (v, c)$$

since $\bar{c} = \bar{v}$.

It remains to show that (v, h') is unique with these properties. Thus, suppose we have an arrow $(w: T \rightarrow Z, k: T \rightarrow X)$ in \mathbb{E}^* such that

$$\begin{array}{ccc} Z & & \\ \downarrow (w,k) & \searrow^{(v,c)} & \\ X & \xrightarrow{(\bar{f},f)} & Y \end{array} \quad \mapsto \quad \begin{array}{ccc} p(Z) & & \\ & \searrow^{p(c)} & \\ h \downarrow & & \\ p(X) & \xrightarrow{p(f)} & p(Y) \end{array}$$

Since the triangle commutes, using Lemma 7.5.i, $(\bar{k}fw, kf) = (v, c)$. Thus, by definition of arrows in \mathbb{E}^* , there is some subvertical partial isomorphism $\alpha: S \rightarrow T$ such that

$$\alpha k f = c, \quad \alpha \bar{k} f w = v, \quad \bar{\alpha} = \bar{c}, \quad \text{and} \quad \overline{\alpha^{(-1)}} = \bar{k}f.$$

We claim that this same α witnesses that $(w, k) = (v, h')$ in \mathbb{E}^* ; that is, we claim that

$$\alpha k = h', \quad \alpha w = v, \quad \bar{\alpha} = \bar{h}', \quad \text{and} \quad \overline{\alpha^{(-1)}} = \bar{w}.$$

Indeed, for the first requirement, we have $\alpha k f = c = h' f$. However, we also have

$$p(\alpha k) = \overline{p(\alpha)}p(k) = \overline{p(c)}p(k) = \bar{h}p(k) = \bar{h}h = h = p(h'),$$

so by proneness of f , $\alpha k = h'$. For the second requirement,

$$v = \alpha \overline{k} f w = \overline{\alpha k} f \alpha w = \overline{c} \alpha w = \overline{\alpha} \alpha w = w.$$

For the third requirement,

$$\overline{\alpha} = \overline{c} = \overline{h'}.$$

For the fourth requirement, we use hyperconnectedness:

$$\overline{\mathbf{p}(\alpha^{(-1)})} = \overline{\mathbf{p}(k)\mathbf{p}(f)} = \overline{h\mathbf{p}(f)} = \overline{h} = \overline{\mathbf{p}(k)} = \overline{\mathbf{p}(w)},$$

with the third equality since $h\mathbf{p}(f) = c$ is precise. Thus, since \mathbf{p} is a hyperconnection, $\overline{\alpha^{(-1)}} = \overline{w}$.

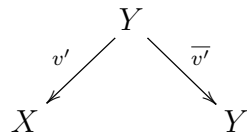
Thus α witnesses $(w, k) = (v, h')$ in \mathbb{E}^* , and so (\overline{f}, f) is prone, as required. ■

7.7. THEOREM. *If $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a hyperfibration then so is $\mathbf{p}^*: \mathbb{E}^* \rightarrow \mathbb{B}$.*

PROOF. Given an $f: A \rightarrow B$ in \mathbb{B} and a Y over B let $f': X \rightarrow Y$ be its \mathbf{p} -prone lift. Then the previous lemma tells us that $(\overline{f'}, f')$ is \mathbf{p}^* -prone, and so \mathbf{p}^* is a latent fibration. Moreover, it is immediately admissible as this lift is a restriction idempotent if f is. Finally, it is easy to check that \mathbf{p}^* is a hyperconnection since \mathbf{p} is, so \mathbf{p}^* is a hyperfibration. ■

Our goal in the rest of this section is to further understand this latent fibration. We begin by characterizing the subvertical maps:

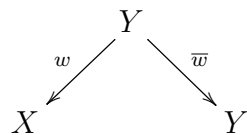
7.8. LEMMA. *Let $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ be a hyperfibration. A map $[(v, c)]: X \rightarrow Y$ is subvertical in \mathbb{E}^* if and only if c is a partial isomorphism, if and only if $[(v, c)]$ has a unique representative of the form*



PROOF. If (v, c) is subvertical, then c is prone and subvertical, so by Proposition 4.23.ii, c is a partial isomorphism. Then define $v' := c^{(-1)}v$. Then the pair $(v', \overline{v'})$ represents the same map as (v, c) via the subvertical partial isomorphism c ; note that since $\overline{v} = \overline{c}$,

$$\overline{v'} = \overline{c^{(-1)}v} = \overline{c^{(-1)}}\overline{v} = \overline{c^{(-1)}}\overline{c} = \overline{c^{(-1)}}.$$

Moreover, such a representative is unique. If we have any other equivalent span



then there is some subvertical partial isomorphism $\alpha: Y \rightarrow Y$ such that (in particular) $\alpha \overline{v'} = \overline{w}$, $\overline{\alpha} = \overline{v'}$, and $\alpha v = w$. But then $\alpha \leq \alpha \overline{v'} = \overline{w}$, so α is a restriction idempotent, and so we have

$$w = \alpha v' = \overline{\alpha} v' = \overline{v'} v' = v'$$

If we have a map $[(v, c)]$ with such a representative, then it is clearly subvertical. ■

Before we look at prone maps, it will be helpful to understand certain partial isomorphisms in \mathbb{E}^* .

7.9. LEMMA. *Let $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ be a hyperfibration. If $(v, \bar{v}): X \rightarrow S$ represents a partial isomorphism in \mathbb{E}^* , then v is a partial isomorphism in \mathbb{E} .*

PROOF. Since (v, \bar{v}) is a subvertical partial isomorphism, its partial inverse is also subvertical (and over the same map, $\mathbf{p}(\bar{v})$). Then by the previous lemma, its partial inverse can be taken to be of the form

$$\begin{array}{ccc} & X & \\ v' \swarrow & & \searrow \bar{v}' \\ S & & X \end{array}$$

We want to show that v' is the partial inverse of v . Since (v', \bar{v}') represents a partial inverse of (v, \bar{v}) , we have that their composite is equivalent to $(\bar{v}, \bar{v}) = (\hat{v}, \hat{v})$; by Lemma 7.5.ii, we have that the composite is represented by

$$(v'v, \bar{v}'v\bar{v}') = (v'v, \bar{v}'v).$$

Thus, there is a subvertical partial isomorphism $\alpha: X \rightarrow X$ so that in particular

$$\alpha\bar{v}'v = \hat{v}, \alpha v'v = \hat{v}, \bar{\alpha} = \bar{\hat{v}} = \hat{v}.$$

Then $\alpha \leq \alpha\bar{v}'v = \hat{v}$, so α is a restriction idempotent, and so $\alpha = \bar{\alpha} = \hat{v}$. Moreover, as noted above $\mathbf{p}(\bar{v}') = \mathbf{p}(v) = \mathbf{p}(\hat{v})$, so since \mathbf{p} is a hyperconnection, $\hat{v} = \bar{v}'$. So then

$$\bar{v}' = \hat{v} = \alpha v'v = \bar{v}'v'v = v'v.$$

The other equality ($vv' = \bar{v}$) follows similarly, and so v is indeed a partial isomorphism. ■

We now characterize prone maps in \mathbb{E}^* :

7.10. LEMMA. *Let $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ be a hyperfibration. A map $[(v, c)]: X \rightarrow Y$ is prone in \mathbb{E}^* if and only if v is a partial isomorphism, if and only if $[(v, c)]$ has a unique representative of the form*

$$\begin{array}{ccc} & X & \\ \bar{c}' \swarrow & & \searrow c' \\ X & & Y \end{array}$$

PROOF. By Lemma 7.5.iii,

$$[(v, c)] = [(v, \bar{v})] \circ [(\bar{c}, c)].$$

However, by Lemma 7.6, $[(\bar{c}, c)]$ is prone, by assumption $[(v, c)]$ is prone, and this factorization is precise since $[(v, c)]$ and $[(v, \bar{v})]$ both have restriction $[(\hat{v}, \hat{v})]$. Thus by Lemma 4.15, $[(v, \bar{v})]$ is also prone. Thus by Proposition 4.23.i, $[(v, \bar{v})]$ is a partial isomorphism in \mathbb{E}^* , and so by Lemma 7.9, v is a partial isomorphism.

Given that v is a partial isomorphism, define $c' : X \rightarrow Y$ by $c' := v^{(-1)}c$. Then it is straightforward to check that $v : S \rightarrow X$ witnesses that (\bar{c}', c') represents the same map as $[(v, c)]$. That this representative is unique is similar to the proof of a similar statement in Lemma 7.8.

The result that such maps are prone in \mathbb{E}^* was Lemma 7.6. ■

By Proposition 3.27, any map $f : X \rightarrow Y$ in the domain category of a latent fibration $\mathbf{p} : \mathbb{E} \rightarrow \mathbb{B}$ has a subvertical/prone factorization, and this factorization is unique up to vertical partial isomorphism. We will need a slight strengthening of this result in the case when \mathbf{p} is a latent hyperfibration:

7.11. LEMMA. *Suppose $\mathbf{p} : \mathbb{E} \rightarrow \mathbb{B}$ is a hyperfibration. Then for any $f : X \rightarrow Y$ in \mathbb{E} , there is a factorization*

$$X \xrightarrow{v} S \xrightarrow{c} Y$$

where $\bar{c} = \hat{v}$. Such a factorization is unique up to subvertical partial isomorphism.

PROOF. Given such an f , let $c : S \rightarrow Y$ be a prone lift of $\mathbf{p}(f)$, and let $v : X \rightarrow S$ be the induced map over $\overline{\mathbf{p}(f)}$:

$$\begin{array}{ccc} X & & \mathbf{p}(X) \\ \downarrow v & \searrow f & \downarrow \overline{\mathbf{p}(f)} \\ S & \xrightarrow{c} & Y \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathbf{p}(X) & & \mathbf{p}(X) \\ \downarrow \overline{\mathbf{p}(f)} & \searrow \mathbf{p}(f) & \downarrow \mathbf{p}(f) \\ \mathbf{p}(X) & \xrightarrow{\mathbf{p}(f)} & \mathbf{p}(Y) \end{array}$$

Then we have

$$\mathbf{p}(\bar{c}) = \overline{\mathbf{p}(c)} = \overline{\mathbf{p}(f)} = \mathbf{p}(\bar{v}) = \mathbf{p}(\hat{v})$$

(with the last by definition of \hat{v}) and so since \mathbf{p} is a hyperconnection, $\bar{c} = \hat{v}$. Uniqueness of such a factorization follows from the uniqueness property of prone arrows. ■

With the above results in hand, we can now prove a result that can be used to prove that $(\mathbb{E}^*)^*$ is isomorphic to \mathbb{E} , but is useful in its own right.

7.12. PROPOSITION. *Suppose that $\mathbf{p} : \mathbb{X} \rightarrow \mathbb{B}$ and $\mathbf{q} : \mathbb{Y} \rightarrow \mathbb{B}$ are hyperfibrations. Then there is a bijection between morphisms of latent fibrations (i.e., restriction functors which preserve the projection and prone arrows)*

$$\{F : \mathbb{X} \rightarrow \mathbb{Y}^*\} \cong \{G : \mathbb{X}^* \rightarrow \mathbb{Y}\}.$$

PROOF. We will sketch the construction and leave most of the details to the reader. Given a morphism of latent fibrations $F : \mathbb{X} \rightarrow \mathbb{Y}^*$, define $\hat{F} : \mathbb{X}^* \rightarrow \mathbb{Y}$ as follows. On objects, \hat{F} is defined as F . On a representative arrow

$$\begin{array}{ccc} & S & \\ v \swarrow & & \searrow c \\ X & & Y \end{array}$$

since F is a restriction functor, $F(v)$ is a subvertical in \mathbb{Y}^* , and so by Lemma 7.8 corresponds to a unique subvertical arrow which we denote as $F_v(v): FX \rightarrow FS$. Similarly since F preserves prone arrows, $F(c)$ is prone in \mathbb{Y}^* , and so by Lemma 7.10, corresponds to a unique prone arrow $F_c(c): F(S) \rightarrow F(Y)$. We then define $\hat{F}([(v, c)])$ to be the composite

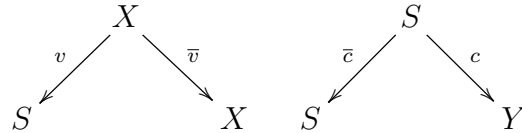
$$F(X) \xrightarrow{F_v(v)} F(S) \xrightarrow{F_c(c)} F(Y).$$

This is well-defined since if (v, c) is equivalent to some other (v', c') , then there is a partial isomorphism taking v to v' and c to c' ; thus $F(v)$ and $F(v')$ go to the same map under F , and so $F_v(v) = F_v(v')$ and similarly $F_c(c) = F_c(c')$.

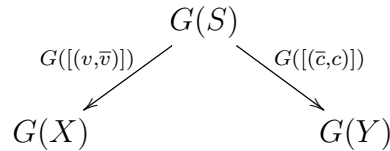
Conversely, suppose we have a morphism of latent fibrations $G: \mathbb{X}^* \rightarrow \mathbb{Y}$. We define $\hat{G}: \mathbb{X} \rightarrow \mathbb{Y}^*$ on objects as G . For an arrow $f: X \rightarrow Y$ in \mathbb{X} , by Lemma 7.11 there is a factorization

$$X \xrightarrow{v} S \xrightarrow{c} Y$$

such that $\hat{v} = \bar{c}$ (and this factorization is unique up to subvertical partial isomorphism). Then



are respectively subvertical and prone in \mathbb{X}^* . Thus we define $\hat{G}(f)$ to be the equivalence class of the span



Note that this has the required restriction property since

$$\overline{[(v, \bar{v})]} = \hat{v} = \bar{c} = \hat{c} = \overline{[(\bar{c}, c)]}.$$

Also note that \hat{G} is independent of the choice of factorization since such factorizations are unique up to subvertical partial isomorphism. ■

7.13. COROLLARY. *If $p: \mathbb{E} \rightarrow \mathbb{B}$ is a latent hyperfibration, then \mathbb{E}^{**} is isomorphic to \mathbb{E} .*

PROOF. Using Proposition 7.12, from the identity functor $\mathbb{E}^* \xrightarrow{1} \mathbb{E}^*$ we get a functor $\mathbb{E} \xrightarrow{\eta} \mathbb{E}^{**}$ and a functor $\mathbb{E}^{**} \xrightarrow{\varepsilon} \mathbb{E}$; using the previous results it is straightforward to check that these are inverse to one another. ■

8. Conclusion

Aside from developing the applications of latent fibrations to, for example, restriction differential categories and partial lenses, there is much theoretical work to be done to bring the theory of latent fibrations to a state of maturity. We certainly hope to return to this task in future papers and we also hope that others will join us in this development. While there are some aspects of this theory for which we already have a partial understanding there are other aspects which are completely open. For example, traditionally fibrations are used to model logical features such as existential and universal quantification and as a semantics of type theories, however, at this stage, we have little idea of what a type theory corresponding to a latent fibration might look like.

Turning to aspects for which we do have some understanding we have shown above that every latent fibration has associated re-indexing (or substitution) restriction *semifunctors*. Nonetheless, we have not here provided a full equivalence of such a collection of such re-indexing semifunctors to a latent fibration. In other words, we have not provided the analogue of the normal Grothendieck construction for latent fibrations. In fact, we do have an understanding of this construction and we hope to publish its description separately. Notably the construction requires more structure than for the usual Grothendieck construction but, as in the usual case, has the potential of providing a significantly different and important perspective. The additional structure is actually a “categorification” of the structure used to define restriction presheaves introduced in [13].

A further, rather separate issue, concerns the behaviour of factorization systems over latent fibrations. It is well-known that given an ordinary fibration, $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$, any orthogonal factorization system in \mathbb{B} can be lifted to one in \mathbb{E} . A similar result can be proven for latent fibrations. However, there is some delicacy at the outset of this project, for defining what is meant by a factorization system for a restriction category has some unexpected features. This also we hope to return to in future work.

At this stage in the development of the theory of latent fibrations, what we know still seems dwarfed by what we still do not know. Perhaps the major open issues concern the understanding of how logical features and, indeed, restriction category features (joins, meets, discreteness, latent limits, classifiers, etc.) interact with the structure of latent fibrations and how these, in turn, translate into type theoretic features.

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A. The original definition of prone

The notion of a latent fibration was first introduced in [21] in order to describe the structures encountered while exploring realizability at the level of restriction categories. In this paper we have used a more convenient notion of “prone” which is a considerable simplification of the original definition. The purpose of this appendix is to show that the original definition of prone from [21] and the one used in this paper are indeed equivalent.

Here is the original definition of prone:

A.1. DEFINITION. *Let $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ be a restriction functor, with \mathbb{E} and \mathbb{B} restriction categories. An arrow $f: X' \rightarrow X$ in \mathbb{E} is **p-prone** in case whenever we have $g: Y \rightarrow X$ in \mathbb{E} and $h: \mathbf{p}(Y) \rightarrow \mathbf{p}(X')$ in \mathbb{X} such that $h\mathbf{p}(f) \geq \mathbf{p}(g)$,*

$$\begin{array}{ccc}
 \text{in } \mathbb{E} & \begin{array}{ccc} Y & & \\ \exists \tilde{h} \downarrow & \searrow g & \\ X' & \xrightarrow{f} & X \end{array} & \text{in } \mathbb{B} & \begin{array}{ccc} \mathbf{p}(Y) & & \\ h \downarrow & \searrow \mathbf{p}(g) & \\ \mathbf{p}(X') & \xrightarrow{\mathbf{p}(f)} & \mathbf{p}(X) \end{array}
 \end{array}$$

there is a **lifting** $\tilde{h}: Y \rightarrow X'$ in \mathbb{X} such that

- (a) \tilde{h} is a **candidate lifting**: that is, $\tilde{h}f \geq g$ and $\mathbf{p}(\tilde{h}) \leq h$.
- (b) \tilde{h} is the **smallest candidate lifting**: that is, for any other candidate lifting $k: Y \rightarrow X'$ in \mathbb{X} , $\tilde{h} \leq k$.

Our objective is to show that this notion of a prone map is equivalent to Definition 3.2.i which has been used in this paper. A first step in this direction is to show that the lifting of h is unique having $\tilde{h}f \geq g$ minimal is precisely to ask for the apparently stronger requirement that $\tilde{h}f = g$ and $\tilde{h} = \bar{g}$, in other words that the lifted triangle is precise:

A.2. LEMMA. (see [21] Lemma 6.2) *For a restriction functor $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$, if f is a **p-prone** arrow (in the sense of Definition A.1) and $h\mathbf{p}(f) \geq \mathbf{p}(g)$ then \tilde{h} is the smallest lifting of h*

$$\begin{array}{ccc}
 \text{in } \mathbb{E} & \begin{array}{ccc} A & & \\ \tilde{h} \downarrow & \searrow g & \\ B & \xrightarrow{f} & C \end{array} & \text{in } \mathbb{B} & \begin{array}{ccc} A & & \\ h \downarrow & \searrow \mathbf{p}(g) & \\ B & \xrightarrow{\mathbf{p}(f)} & C \end{array}
 \end{array}$$

if and only if \tilde{h} is the unique map such that $\tilde{h} = \bar{g}$ and $\tilde{h}f = g$ and $\mathbf{p}(\tilde{h}) \leq h$.

PROOF.

(\Rightarrow) Note that for the lifting \tilde{h} we have $\tilde{h} \leq \overline{\tilde{h}f} \leq \bar{g}$ and yet $\bar{g}\tilde{h} \leq \tilde{h}$ is certainly a candidate lifting so, as \tilde{h} is the smallest such, $\bar{g}\tilde{h} = \tilde{h}$, so $\bar{g} \leq \tilde{h}$ and, thus, $\tilde{h} = \bar{g}$. It then follows that $\tilde{h}f \leq g$ as $\overline{\tilde{h}f} \leq \bar{g}g = \bar{g} = \tilde{h}$ and so $\tilde{h}f = g$.

(\Leftarrow) Suppose that \tilde{h} is the unique map such that $\tilde{h} = \bar{g}$, $\tilde{h}f = g$, and $\mathbf{p}(\tilde{h}) \leq h$, then certainly it is a candidate lifting. Suppose also that v is some other candidate lifting, so that $vf \geq g$ and $\mathbf{p}(v) \leq w$, then $\bar{g}v$ has $\bar{g}v \geq \overline{gvf} \geq \overline{gg} = \bar{g} \geq \overline{gv}$ so $\bar{g}v = \bar{g}$, as $vf \geq g$ so $\bar{g}vf = g$, and $\mathbf{p}(\bar{g}v) = \overline{\mathbf{p}(g)}\mathbf{p}(v) \leq h$. So $\bar{g}v$ satisfies the specification of \tilde{h} and so $\bar{g}v = \tilde{h}$ and, consequently, $v \geq \tilde{h}$ showing the minimality of \tilde{h} . ■

Of course once one realizes the lifted triangle is always going to be precise then one can reduce the lax triangle in the base to a precise triangle with $h' = \overline{\mathbf{p}(g)}h$ (by Lemma 2.7.v). This means:

A.3. PROPOSITION. *Definition A.1 of being prone is equivalent to Definition 3.2.*

More precisely, if $\mathbf{p}: \mathbb{E} \rightarrow \mathbb{B}$ is a restriction functor, $f: B \rightarrow C$ is \mathbf{p} -prone – in the sense of Definition A.1 – in \mathbb{E} if and only if every precise triangle $h\mathbf{p}(f) = \mathbf{p}(g)$ with left factor h in \mathbb{B} has a unique \tilde{h} with $\mathbf{p}(\tilde{h}) = h$ making $\tilde{h}f = g$ a precise triangle in \mathbb{E} :

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & & \\
 \tilde{h} \downarrow & \searrow g & \\
 B & \xrightarrow{f} & C
 \end{array} & \xrightarrow{\mathbf{p}} & \begin{array}{ccc}
 \mathbf{p}(A) & & \\
 \mathbf{p}(\tilde{h})=h \downarrow & \searrow \mathbf{p}(g) & \\
 \mathbf{p}(B) & \xrightarrow{\mathbf{p}(f)} & \mathbf{p}(C)
 \end{array}
 \end{array}$$

PROOF.

(\Rightarrow) If f is prone in the sense of Definition A.1, and $h\mathbf{p}(f) = \mathbf{p}(g)$ is a precise triangle with left factor h then, by Lemma A.2, $\tilde{h}f = g$ is a precise triangle with $\mathbf{p}(\tilde{h}) \leq h$. However, $\mathbf{p}(\tilde{h}) = \overline{\mathbf{p}(h)} = \overline{\mathbf{p}(g)} = \overline{\mathbf{p}(g)}h = h$ so $\mathbf{p}(\tilde{h}) = h$.

(\Leftarrow) Let $h\mathbf{p}(f) \geq \mathbf{p}(g)$ be a lax triangle with lax left factor h , then, by Lemma 2.7.v, $\overline{\mathbf{p}(g)}h\mathbf{p}(f) = \mathbf{p}(g)$ is a precise triangle and so there is, by assumption, a unique left factor making a precise triangle $\overline{\mathbf{p}(g)}hf = g$ with $\mathbf{p}(\overline{\mathbf{p}(g)}h) = \overline{\mathbf{p}(g)}h$. This $\overline{\mathbf{p}(g)}h$ we claim satisfies all the requirements of being a lifting of h . It is a candidate lifting as $\overline{\mathbf{p}(g)}hf = g \geq g$ and $\mathbf{p}(\overline{\mathbf{p}(g)}h) = \overline{\mathbf{p}(g)}h \leq h$.

To show that $\overline{\mathbf{p}(g)}h$ is the least candidate lifting, consider another candidate lifting, k , with $kf \geq g$ and $\mathbf{p}(k) \leq h$ then $kf \geq g$ then $(\bar{g}k)f = g$ is precise as is $(\overline{\mathbf{p}(g)}\mathbf{p}(k))\mathbf{p}(f) = \mathbf{p}(g)$. Now $\mathbf{p}(\bar{g}k) = \overline{\mathbf{p}(g)}\mathbf{p}(k) \leq \overline{\mathbf{p}(g)}h$ as $\mathbf{p}(k) \leq h$, however, this inequality is an equality as $\overline{\mathbf{p}(g)}\mathbf{p}(k) = \overline{\mathbf{p}(g)}h$ so that by the uniqueness of lifting (under Definition 3.2) we have $\bar{g}k = \overline{\mathbf{p}(g)}h$ so that $\overline{\mathbf{p}(g)}h = \bar{g}k \leq k$ as required. ■

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