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L'-LOCALIZATION IN AN ∞-TOPOS

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ABSTRACT. We prove that, given any reflective subfibration $L_\bullet$ on an ∞-topos $\mathcal{E}$, there exists a reflective subfibration $L'_\bullet$ on $\mathcal{E}$ whose local maps are the L-separated maps, that is, the maps whose diagonals are L-local.

Contents

1. Introduction

This paper complements the work of [Ver19] by proving the following theorem, which is one of our main results in the theory of reflective subfibrations on an ∞-topos $\mathcal{E}$.

1.1. THEOREM. [Theorem 4.3 & Corollary 4.4] *Let $L_\bullet$ be a reflective subfibration on an ∞-topos $\mathcal{E}$. Then, there exists a reflective subfibration $L'_\bullet$ on $\mathcal{E}$ for which the L'-local maps are exactly the L-separated maps.*

In [Ver19], we took from [RSS17] the notion of *reflective subfibration* on an ∞-topos $\mathcal{E}$, and studied its properties. A reflective subfibration $L_\bullet$ on $\mathcal{E}$ is a pullback-compatible system of reflective subcategories $\mathcal{D}_X$ of $\mathcal{E}_{/X}$, for every $X \in \mathcal{E}$. The collection of all objects in $\mathcal{D}_X$, as X varies in $\mathcal{E}$, forms the class of *L-local maps*. Reflective subfibrations provide a suitable framework for the study of localizations in an ∞-topos. Indeed, all the most common examples of localizations from classical homotopy theory can be recovered in this setting: stable factorization systems ([Ver19, Thm. 4.8]), left exact reflective subcategories of an ∞-topos ([Ver19, Prop. 4.11]), and localizations at sets of maps ([Ver19, Prop. 5.11]). For the reader's convenience, in Section 2, we briefly gather from [Ver19] the general aspects of reflective subfibrations that we need.

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For a reflective subfibration $L_\bullet$ on $\mathcal{E}$, one can consider the *L-separated maps*, that is, those maps in $\mathcal{E}$ whose diagonal is L-local. For example, for the reflective subfibration $L^n_\bullet$ having the n-truncated maps as local maps, the $L^n_\bullet$-separated maps are the $(n+1)$-truncated maps, and they are themselves the local maps for a reflective subfibration, $L^{n+1}_\bullet$. It turns out that this behavior is completely general: for *any* $L_\bullet$, there exists an $L'_\bullet$ such that the L'-local maps are the L-separated maps (Theorem 4.3 and Corollary 4.4).

In this paper, we focus on the proof of this existence result, leaving the study of its consequences to [Ver19, §7]. To this end, one needs to carefully examine some connections between L-local and L-separated maps. We develop the study of these relationships in Section 3. Our main result there is the following characterization of *L'-localization maps*, that is, those maps out of a fixed object X (or, more generally, out of a map p) and into an L-separated object, which are universal among maps with this property.

1.2. Theorem. *The following are equivalent, for a map $\eta' \colon X \to X'$ in $\mathcal{E}$:*

1. *η' is an L'-localization of X;*
2. *η' is an effective epimorphism and*

$$\begin{tikzcd} X \arrow[r, "\Delta\eta'"] \arrow[dr, "\Delta X"'] & X \times_{X'} X \arrow[d] \\ & X \times X \end{tikzcd}$$

is an L-localization of ΔX.

The existence result for $L'_\bullet$, together with a few auxiliary lemmas needed in its proof, is the content of Section 4. The results in both Section 3 and Section 4 require some facts about locally cartesian closed ∞-categories that we collect in the Appendix (Section 5). Some of the results there are well known, but for others we could not find any reference in the literature. Examples of the results in the latter group are Proposition 5.2.1, where we prove the topos-theoretic version of the function extensionality axiom from HoTT, and Proposition 5.3.4, which provides a criterion for unique extensions of maps that is crucial for the proof of Theorem 3.10.

Our approach to localization is inspired by the work in homotopy type theory (HoTT) developed in [CORS18]. The notion of L-separated map, as well as Proposition 3.6 and Theorem 4.3, are expressed in HoTT in [CORS18, §2.2-2.3]. We take from there the main ideas for the proofs of Theorem 3.10 and Theorem 4.3. However, proof details and techniques have been modified to apply to the "term-free" exposition we work with. This is particularly evident in the proof of Theorem 3.10, and in the results of Section 4. All the proofs of the results in the Appendix are also specific to the higher-topos theoretic setting we work with. For a more detailed description of how our work relates to the study of localization in HoTT, we refer the reader to the Introduction of [Ver19].

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Notation and Conventions. By an ∞-*topos* we mean an ∞-category $\mathcal{E}$ with the following properties.

(a) $\mathcal{E}$ is a locally presentable ∞-category ([Lur09, Def. 5.5.0.1]).

(b) Colimits in $\mathcal{E}$ are universal ([Lur09, Def. 6.1.1.2]).

(c) The class of all maps in $\mathcal{E}$ is a local class of maps (see [Lur09, Thm. 6.1.3.8]).

This characterization of ∞-topoi follows from [Lur09, Thm. 6.1.6.8].

Given an ∞-category $\mathcal{C}$, we often depict a map $m\colon p \to q$ in a slice category $\mathcal{C}_{/Z}$ as a commuting triangle in $\mathcal{C}$ of the form

$$\begin{tikzcd} E \arrow[rr, "m"] \arrow[dr, "p"'] & & M \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

leaving the interior 2-simplex implicit. We will often carry over this implicitness to other maps in slice categories that are constructed from m, at least as long as the context is enough to disambiguate. For example, if the implicit 2-simplex of m above is σ, then (σ, σ) is the implicit 2-simplex of the map in $\mathcal{C}_{/Z^2}$ given by

$$\begin{tikzcd} E \arrow[rr, "m"] \arrow[dr, "{(p,p)}"'] & & M \arrow[dl, "{(q,q)}"] \\ & Z^2 & \end{tikzcd}$$

If p and q are objects in a slice category $\mathcal{C}_{/Z}$, we write $p \times^Z q$ to mean the product object of p and q in $\mathcal{C}_{/Z}$.

2. Reflective Subfibrations

We gather here some background material on reflective subfibrations in an ∞-topos $\mathcal{E}$ from the companion paper [Ver19].

2.1. Definition. [RSS17, §A.2] *Let $\mathcal{E}$ be an ∞-topos.*

1. *A* reflective subfibration *$L_\bullet$ on $\mathcal{E}$ is the assignment, for each $X \in \mathcal{E}$, of an ∞-category $\mathcal{D}_X$ such that:*

   (a) *Each $\mathcal{D}_X$ is a reflective ∞-subcategory of $\mathcal{E}_{/X}$, with associated localization functor $L_X\colon \mathcal{E}_{/X} \to \mathcal{E}_{/X}$. This is the composite of the reflector of $\mathcal{E}_{/X}$ into $\mathcal{D}_X$ and the inclusion of $\mathcal{D}_X$ into $\mathcal{E}_{/X}$. If $X = 1$, we write $\mathcal{D}$ for $\mathcal{D}_1$ and L for L_1.*

   (b) *For every map $f\colon X \to Y$ in $\mathcal{E}$, and any $p \in \mathcal{E}_{/Y}$, the map $L_X(f^*p) \to f^*(L_Y p)$ is an equivalence. In particular, the pullback functor $f^*\colon \mathcal{E}_{/Y} \to \mathcal{E}_{/X}$ restricts to a functor $\mathcal{D}_Y \to \mathcal{D}_X$ which we still denote by f^*.*

2. *A* modality *on $\mathcal{E}$ is a reflective subfibration $L_\bullet$ on $\mathcal{E}$ which is composing, in that, whenever $p\colon X \to Y$ is in $\mathcal{D}_Y$ and $q\colon Y \to Z$ is in $\mathcal{D}_Z$, the composite qp is in $\mathcal{D}_Z$.*

2.2. Example. In [Ver19, §4.2], we show that stable factorization systems in an ∞-topos $\mathcal{E}$ correspond to modalities on $\mathcal{E}$. In particular, for every $n \geq -2$, there is a modality $L^n_\bullet$ on $\mathcal{E}$, for which the L-local maps are the n-truncated maps. We call this modality the *n-truncated* modality on $\mathcal{E}$. Given an object $X \in \mathcal{E}$, we will also adopt the more conventional notation $\|X\|_n$ for $L^n(X)$.

2.3. Example. (Not all reflective subfibrations are modalities.) In [Ver19, §5], we show that every map $f \colon A \to B$ in $\mathcal{E}$ gives rise to a reflective subfibration on $\mathcal{E}$, called *f-localization*, whose local objects are all $X \in \mathcal{E}$ for which the map of internal homs $X^f \colon X^B \to X^A$ is an equivalence. Similarly, a map $p \colon E \to X$ in $\mathcal{E}_{/X}$ is an f-local map if the map in $\mathcal{E}_{/X}$ obtained by taking internal homs of p with $(X \to 1)^*(f)$ is an equivalence. When $\mathcal{E} = \infty\mathsf{Gpd}$, an f-local map is simply a map which has f-local fibers. As long as $B \neq 1$, f-localization is typically *not* a modality. A simple example is given in [CORS18, Ex. 4.8].

2.4. Remark. For every $X \in \mathcal{E}$ and every map $f \colon Y \to X$, we have that $(\mathcal{E}_{/X})_{/f} \simeq \mathcal{E}_{/Y}$ (this follows from [Lur09, Prop. 2.1.2.5], but see also [Kap14, Lemma 4.18] for a more explicit proof). Hence, for each $X \in \mathcal{E}$, a reflective subfibration $L_\bullet$ induces a reflective subfibration $L_\bullet^{/X}$ of $\mathcal{E}_{/X}$ by taking $\mathcal{D}_f^{/X}$ to be $\mathcal{D}_Y$. It follows that results about reflective subfibrations on an ∞-topos also hold "locally" in the ∞-topos $\mathcal{E}_{/X}$, for $X \in \mathcal{E}$.

From now on, we fix a reflective subfibration $L_\bullet$ on our favorite ∞-topos $\mathcal{E}$.

2.5. Notation. We adopt the following notation for the rest of this work.

- A morphism $p \colon E \to X$ is called *L-local* if, seen as an object of $\mathcal{E}_{/X}$, it is in $\mathcal{D}_X$. We call $E \in \mathcal{E}$ an *L-local object* if $E \to 1$ is an L-local map.
- For $X \in \mathcal{E}$, S_X denotes the class of all L_X*-equivalences*, i.e., maps α in $\mathcal{E}_{/X}$ such that $L_X(\alpha)$ is an equivalence. Equivalently, $S_X = {}^\perp\mathcal{D}_X$, where ${}^\perp\mathcal{D}_X$ denotes the class of maps in $\mathcal{E}_{/X}$ which are left orthogonal to maps in $\mathcal{D}_X$. When it is clear that α is a map in $\mathcal{E}_{/X}$, we often drop the explicit reference to the object X, and just talk about *L-equivalences*.
- Given $p \in \mathcal{E}_{/X}$, we write $\eta_X(p) \colon p \to L_X(p)$ for the reflection (or localization) map of p into $\mathcal{D}_X$. Note that $\eta_X(p) \in S_X$. For $X \in \mathcal{E}$, we set $\eta(X) := \eta_1(X)$.

2.6. Remark. [1] (Why reflective *subfibrations*?) Let $\mathcal{D}$ be the full subcategory of $\mathcal{E}^{\bullet\to\bullet}$ on the L-local maps. (Here, $\mathcal{E}^{\bullet\to\bullet}$ is the arrow category of $\mathcal{E}$.) Since $\mathcal{E}$ has pullbacks, the codomain functor $\mathrm{cod} \colon \mathcal{E}^{\bullet\to\bullet} \longrightarrow \mathcal{E}$ is a cartesian fibration (see [RV18, Prop. 5.1.26]). Since, given an L-local map $p \colon E \to Y$ and any map $f \colon X \to Y$ in $\mathcal{E}$, the pullback map $f^*(p)$ is an L-local map, the codomain functor restricts to a cartesian fibration

[1] We thank the anonymous referee for the suggestion of considering $\mathcal{D}$ as a cartesian fibration over $\mathcal{E}$. This turned out to be the decisive observation for the author to finally be able to figure out the proper relationship between reflective subfibrations and reflective subcategories of arrow categories.

cod: $\mathcal{D} \longrightarrow \mathcal{E}$. In other words, the inclusion functor $i\colon \mathcal{D} \hookrightarrow \mathcal{E}^{\bullet\to\bullet}$ gives a cartesian functor between cartesian fibrations (see [RV18, Def. 5.1.18]):

$$\begin{tikzcd} \mathcal{D} \arrow[rr, hook, "i"] \arrow[dr, "\mathrm{cod}"'] & & \mathcal{E}^{\bullet\to\bullet} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{E} & \end{tikzcd}.$$

Taking fibers over $X \in \mathcal{E}$, the diagram above restricts to $i_X\colon \mathcal{D}_X \hookrightarrow \mathcal{E}_{/X}$, the inclusion functor of $\mathcal{D}_X$ into $\mathcal{E}_{/X}$. Since each $\mathcal{D}_X$ is a reflective subcategory of $\mathcal{E}_{/X}$, [Lur09, Lemma 6.5.2.4] applies to give that $\mathcal{D}$ is a reflective subcategory of $\mathcal{E}$, i.e., $i\colon \mathcal{D} \hookrightarrow \mathcal{E}^{\bullet\to\bullet}$ has a left adjoint $L_\bullet\colon \mathcal{E}^{\bullet\to\bullet} \longrightarrow \mathcal{D}$. The pullback-compatibility condition of the localizations on the various slice categories, given in Definition 2.1.1(b), says that $L_\bullet$ is actually a cartesian functor of cartesian fibrations over $\mathcal{E}$:

$$\begin{tikzcd} \mathcal{E}^{\bullet\to\bullet} \arrow[rr, "L_\bullet"] \arrow[dr, "\mathrm{cod}"'] & & \mathcal{D} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{E} & \end{tikzcd}.$$

In other words, a reflective subfibration determines a fibered reflective subcategory of $\mathcal{E}^{\bullet\to\bullet}$ over $\mathcal{E}$ (via the codomain projection), in which both the inclusion and the reflector are cartesian functors of cartesian fibrations. If we reverse the above reasoning, we can see that the converse of the previous statement is also true: a fibered reflective subcategory of $\mathcal{E}^{\bullet\to\bullet}$ over $\mathcal{E}$, in which both the inclusion and the reflector are cartesian functors of cartesian fibrations, determines a reflective subfibration. (One key point for this converse statement is that pullbacks of fibered adjunctions are fibered adjunctions [RV18, Lemma 3.6.7], so that a fibered reflective subcategory of $\mathcal{E}^{\bullet\to\bullet}$ over $\mathcal{E}$ determines a reflective subcategory (fibered over the point) on each slice of $\mathcal{E}$.)

Given a map f in $\mathcal{E}$, we denote by Σ_f and by Π_f the left and right adjoint to the pullback functor f^*, respectively.

2.7. Lemma. [Ver19, Lemma 3.4] *Given $f\colon X \to Y$, we have:*

(i) $f^*(S_Y) \subseteq S_X$, that is, if $\alpha\colon p \to q$ is an L_Y-equivalence, then the induced map $f^*(p) \to f^*(q)$ on pullbacks is an L_X-equivalence;

(ii) $\Sigma_f(S_X) \subseteq S_Y$.

One of the characterizing features of ∞-topoi that plays a crucial role in the proof of our main result (Theorem 4.3) is the existence of classifying maps for local classes of maps.

2.8. DEFINITION. [Lur09, Def. 6.1.3.8, Prop. 6.2.3.14] *A class S of maps in $\mathcal{E}$ is called* local *if it closed under small coproducts in $\mathcal{E}^{\bullet\to\bullet}$, it is stable under pullbacks and satisfies the following condition. Given any pullback square in $\mathcal{E}$*

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where f is an effective epimorphism, p is in S if and only if q is in S.

2.9. DEFINITION. *Let S be a pullback-stable class of maps in an ∞-topos $\mathcal{E}$. Let* $\mathrm{Cart}(S)$ *be the sub-∞-category of $\mathcal{E}^{\bullet\to\bullet}$ having the maps in S as objects and pullback squares as morphisms. A* classifying map *for S is a terminal object of* $\mathrm{Cart}(S)$.

Thus, a classifying map $p\colon E \to X$ for S is a map in S such that every other map in S is a pullback of p in an essentially unique way.

2.10. PROPOSITION. [Lur09, Prop. 6.1.6.7] *Let S be a local class of maps in an ∞-topos $\mathcal{E}$. Then, there are arbitrarily large regular cardinals κ such that the class S_κ of maps in S that are relatively κ-compact ([Lur09, Def. 6.1.6.4]) is local and has a classifying map.*

Classifying maps enjoy an important property, called univalence, which characterizes equivalences between fibers of classifying maps. The notion of univalent map uses the construction of objects of equivalences in an ∞-topos, which we briefly recall from [GK17].

Let $J(\mathcal{E})$ be the *core* of $\mathcal{E}$, that is, the (strict) pullback of ∞-categories

$$\begin{tikzcd} J(\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d] \\ J(\mathrm{Ho}(\mathcal{E})) \arrow[r] & \mathrm{Ho}(\mathcal{E}) \end{tikzcd} .$$

Here, $\mathrm{Ho}(\mathcal{E})$ is the homotopy category of $\mathcal{E}$, and $J(\mathrm{Ho}(\mathcal{E}))$ is the usual core groupoid of the 1-category $\mathrm{Ho}(\mathcal{E})$.

2.11. PROPOSITION. [GK17, Thm. 2.10] *For every $X, Y \in \mathcal{E}$, there is a subobject* $\mathrm{Eq}_{\mathcal{E}}(X,Y)$ *of Y^X such that, for every $T \in \mathcal{E}$, there is an equivalence of ∞-groupoids*

$$\mathcal{E}(T, \mathrm{Eq}_{\mathcal{E}}(X,Y)) \simeq J(\mathcal{E}_{/T})(X \times T, Y \times T),$$

natural in $T \in \mathcal{E}$. Furthermore, this is also true "locally", that is, for every two objects p, q in a slice category $\mathcal{E}_{/X}$.

2.12. NOTATION. For $p\colon E \to X$ and $q\colon M \to X$, we write $\mathrm{Eq}_{/X}(E,M)$ for the domain of $\mathrm{Eq}_{\mathcal{E}_{/X}}(p,q)$. We will often just write $\mathrm{Eq}(p,q)$ for $\mathrm{Eq}_{\mathcal{E}_{/X}}(p,q)$.

2.13. Definition. [GK17, §3.1] *The* object of equivalences *for* $p\colon E \to X$ *is the object of* $\mathcal{E}_{/X\times X}$ *given by*

$$\mathrm{Eq}_{\mathcal{E}_{/X\times X}}(p \times \mathrm{id}_X, \mathrm{id}_X \times p)\colon\ \mathrm{Eq}_{/X\times X}(p \times \mathrm{id}_X, \mathrm{id}_X \times p) \to X \times X$$

where $p \times \mathrm{id}_X\colon E \times X \to X \times X$ *and similarly for* $\mathrm{id}_X \times p$. *We write the object of equivalences for* p *as* $\mathrm{Eq}_{/X}(p)\colon\ \mathrm{Eq}_{/X}(E) \to X \times X$

By the definition of Eq, it follows that the identity map $\mathrm{id}_p \in J(\mathcal{E}_{/X})(p,p)$ induces a map $\mathsf{idtoequiv}\colon X \to \mathrm{Eq}_{/X}(E)$ over $X \times X$.

2.14. Definition. [GK17, §3.2] *A* univalent map *is a map* $p\colon E \to X$ *in* $\mathcal{E}$ *for which the associated map* $\mathsf{idtoequiv}\colon X \to \mathrm{Eq}_{/X}(E)$ *is an equivalence in* $\mathcal{E}_{/X\times X}$.

2.15. Proposition. [GK17, Prop. 3.8] *Every classifying map* p *is univalent.*

The next result from our companion paper [Ver19] links the theory of univalent classifying map to reflective subfibrations.

2.16. Proposition. [Ver19, Prop. 3.12 & Thm. 3.15] *The class* $\mathcal{M}^L$ *of all* L*-local maps is a local class of maps of* $\mathcal{E}$. *In particular, there are arbitrarily large regular cardinals* κ *such the class of relatively* κ*-compact* L*-local maps admits a univalent classifying map* $u^L_\kappa\colon \widetilde{\mathcal{U}^L_\kappa} \to \mathcal{U}^L_\kappa$.

2.17. Definition. $f \in \mathcal{E}_{/X}$ *is said to be an* L-connected map (in $\mathcal{E}$) *if* $L_X(f) \simeq \mathrm{id}_X$. *Equivalently,* f *is* L*-connected if*

$$(f \xrightarrow{\eta_X(f)} L_X(f)) \simeq (f \xrightarrow{f} \mathrm{id}_X)$$

in $\mathcal{E}_{/X}$, *where the equivalence is given by* id_f *and* $L_X(f) \to \mathrm{id}_X$. *We refer to this fact by saying that an* L*-connected map* f *is* its own reflection map.

In particular, an L-connected map $f\colon E \to X$ is an L_X-equivalence when seen as a map $f\colon f \to \mathrm{id}_X$ in $\mathcal{E}_{/X}$.

2.18. Remark. By taking the reflection of $f \in \mathcal{E}_{/X}$ into $\mathcal{D}_X$ and using stability under pullbacks of reflection maps (see Definition 2.1(1b)), it follows that L-connected maps are stable under pullbacks along arbitrary maps.

We now give the core notion of this paper.

2.19. Definition. *A map* $p\colon E \to X$ *in* $\mathcal{E}$ *is called* L-separated *or* L'-local *if the object* $\Delta p \in \mathcal{E}_{/E\times_X E}$ *is in* $\mathcal{D}_{E\times_X E}$, *i.e., if* Δp *is an* L*-local map.*

2.20. Example. For the reflective subfibration $L^n_\bullet$ having the n-truncated maps as local maps (see Example 2.2), the $L^n_\bullet$-separated maps are the $(n+1)$-truncated maps.

2.21. Proposition. [Ver19, Prop. 6.5 & Prop. 6.7] *Let $L_\bullet$ be a reflective subfibration on an ∞-topos $\mathcal{E}$. Then the following hold.*

1. *Let $f\colon Y \to X$ be a map in $\mathcal{E}$, and let $p\colon E \to X$ and $q\colon M \to Y$ be L-separated maps. Then $f^*(p) \in \mathcal{E}_{/Y}$ and $\prod_f q \in \mathcal{E}_{/X}$ are L-separated. Furthermore, the internal hom p^f is L-separated.*

2. *The class $\mathcal{M}'$ of all L-separated maps is a local class of maps.*

3. Interactions between L-local and L-separated maps

We study here some relationships between L-local and L-separated maps and prove a characterization result for L'-localization maps which will be used in the next section as a fundamental step for the proof of Theorem 4.3.

3.1. Lemma. [CORS18, Lemma 2.21] *Suppose given a commutative triangle*

$$\begin{tikzcd} E \arrow[rr, "\alpha"] \arrow[dr, "p"'] & & M \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

in which $\Delta q \in \mathcal{D}_{M\times_X M}$ and $\alpha \in \mathcal{D}_M$, that is, q is L-separated and α is L-local. Then Δp is in $\mathcal{D}_{E\times_X E}$, i.e., p is L-separated.

Proof. The map $(\mathrm{id}_E \times_X \alpha\colon E\times_X E \to E\times_X M) = (E\times_X M \to M)^*(\alpha)$ is in $\mathcal{D}_{E\times_X M}$, since $\alpha\colon E \to M$ is in $\mathcal{D}_M$. Similarly, the map $((\mathrm{id}_E, \alpha)\colon E \to E\times_X M) = (\alpha\times_X \mathrm{id}_M)^*(\Delta q)$ is in $\mathcal{D}_{E\times_X M}$. But $(\mathrm{id}_E \times_X \alpha)\circ \Delta p = (\mathrm{id}_E, \alpha)$, so Δp is L-local, by [Ver19, Prop. 3.7]: if both f and $f\circ g$ are $L-$local maps, then so is g. ∎

3.2. Definition. *A map $\alpha\colon p \to p'$ in $\mathcal{E}_{/X}$ is called an L'*-localization map *of p if p' is L-separated and $\mathcal{E}_{/X}(\alpha, q)\colon \mathcal{E}_{/X}(p', q) \to \mathcal{E}_{/X}(p, q)$ is an equivalence of ∞-groupoids for every L-separated $q \in \mathcal{E}_{/X}$. In other words, for every map $\beta\colon p \to q$, there is a unique $\psi\colon p' \to q$ with $\psi\circ\alpha = \beta$.*

3.3. Remark. Given an L-separated $r \in \mathcal{E}_{/X}$ and any $t \in \mathcal{E}_{/X}$, $r^t \in \mathcal{E}_{/X}$ is again L-separated. It follows that, for a map $\alpha\colon p \to p'$ in $\mathcal{E}_{/X}$ with p' L-separated, the above definition can be rephrased internally, by asking that q^α is an equivalence in $\mathcal{E}_{/X}$ for every L-separated map $q\colon Y \to X$.

3.4. Lemma. [CORS18, Prop. 2.30] *Let $\eta'\colon p \to p'$ in $\mathcal{E}_{/Y}$ be an L'-localization map of $p \in \mathcal{E}_{/Y}$, with $\eta'\colon X \to X'$ as a map in $\mathcal{E}$. Then η' is an L-connected map (Definition 2.17).*

Proof. Let $\eta_{X'}(\eta')\colon \eta' \to L_{X'}(\eta')$ be the reflection map of $\eta' \in \mathcal{E}_{/X'}$ into $\mathcal{D}_{X'}$. Set $r := p' \circ L_{X'}(\eta')$, and consider $\eta_{X'}(\eta')\colon p \to r$ and $L_{X'}(\eta')\colon r \to p'$ as maps in $\mathcal{E}_{/Y}$. By Lemma 3.1 applied to $L_{X'}(\eta')$, r is L-separated. Hence, there is a unique $q\colon p' \to r$ with $q\eta' = \eta_{X'}(\eta')$ as maps $p \to r$ in $\mathcal{E}_{/Y}$. Since $L_{X'}(\eta')q\eta' = L_{X'}(\eta')\eta_{X'}(\eta') = \eta'$, the universal property of η' gives $L_{X'}(\eta')q = \mathrm{id}_{p'}$. Thus, we can consider $qL_{X'}(\eta')$ as a map $L_{X'}(\eta') \to L_{X'}(\eta')$ in $\mathcal{E}_{/X'}$ and $qL_{X'}(\eta')\eta_{X'}(\eta') = q\eta' = \eta_{X'}(\eta')$, so that $qL_{X'}(\eta') = \mathrm{id}_r$. Hence, η' is L-connected. ■

3.5. Lemma. *Let κ be a regular cardinal such that the class of relatively κ-compact L-local maps has a classifying map $u^L_\kappa\colon \widetilde{\mathcal{U}^L_\kappa} \to \mathcal{U}^L_\kappa$. Then $\mathcal{U}^L_\kappa$ is L-separated.*

Proof. We drop κ from our notation. Since u^L is univalent, we have an equivalence $\Delta(\mathcal{U}^L) \simeq \mathrm{Eq}_{/\mathcal{U}^L}(u^L)$ over $\mathcal{U}^L \times \mathcal{U}^L$ (see [Ver19, Def. 2.10]). By definition, $\mathrm{Eq}_{/\mathcal{U}^L}(u^L)$ is the object of equivalences in $\mathcal{E}_{/\mathcal{U}^L \times \mathcal{U}^L}$ between $\mathrm{id}_{\mathcal{U}^L} \times u^L$ and $u^L \times \mathrm{id}_{\mathcal{U}^L}$, both of which are L-local since u^L is. By [Ver19, Lemma 2.8], such an object of equivalences is then the pullback of a cospan of objects in $\mathcal{D}_{\mathcal{U}^L \times \mathcal{U}^L}$ and it is therefore in $\mathcal{D}_{\mathcal{U}^L \times \mathcal{U}^L}$. ■

3.6. Proposition. *Let $X \in \mathcal{E}$ and let $\eta'\colon X \to X'$ be an L'-localization of X. Then a map $p\colon E \to X$ is L-local if and only if there is a pullback square in $\mathcal{E}$*

$$\begin{tikzcd} E \arrow[r, "\eta_{X'}(\eta' p)"] \arrow[d, "p"'] & L_{X'}E \arrow[d, "L_{X'}(\eta' p)"] \\ X \arrow[r, "\eta'"'] & X' \end{tikzcd} .$$

Proof. For the non-trivial implication, assume p is L-local. Let κ be a regular cardinal such that p is relatively κ-compact and the class of relatively κ-compact L-local maps has a classifying map $u^L\colon \widetilde{\mathcal{U}^L_\kappa} \to \mathcal{U}^L_\kappa$. Let $P\colon X \to \mathcal{U}^L_\kappa$ be such that we have a pullback square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}^L_\kappa} \arrow[d, "u^L"] \\ X \arrow[r, "P"'] & \mathcal{U}^L_\kappa \end{tikzcd} . \tag{†}$$

Since $\mathcal{U}^L_\kappa$ is L-separated, there is a unique map $P'\colon X' \to \mathcal{U}^L_\kappa$ with $P = P'\eta'$. Let $p'\colon E' \to X'$ be the pullback map in

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}^L_\kappa} \arrow[d, "u^L"] \\ X' \arrow[r, "P'"'] & \mathcal{U}^L_\kappa \end{tikzcd} .$$

By definition of P', $\eta'\colon X \to X'$ induces a map $n\colon E \to E'$ such that the composite square

$$\begin{tikzcd} E \arrow[r, "n"] \arrow[d, "p"'] & E' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}^L_\kappa} \arrow[d, "u^L"] \\ X \arrow[r, "\eta'"'] & X' \arrow[r, "P'"'] & \mathcal{U}^L_\kappa \end{tikzcd} \tag{‡}$$

is the square (†). It follows that the left square in (‡) is also a pullback. Thanks to Lemma 3.4, η' is L-connected. Thus, so is n, by Remark 2.18. In particular, n is an L-equivalence (i.e., $n\colon n \to \mathrm{id}_{E'}$ is in $S_{E'}$). By composing domain and codomain of $n\colon n \to \mathrm{id}_E$ with p', Lemma 2.7 (ii) gives that $n\colon \eta' p \to p'$ is an L-equivalence. Since p' is L-local, it follows that n is the L-localization map of $\eta' p$, as required. ∎

3.7. Remark. As explained in Lemma 2.4, Proposition 3.6 is also true "locally", i.e., when we take our ground ∞-topos to be $\mathcal{E}_{/X}$ instead of $\mathcal{E}$. For the result above, this means specifically that, if

$$\begin{tikzcd} E \arrow[rr, "\eta'_X(p)"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

is an L'-localization of p in $\mathcal{E}_{/X}$, a map

$$\begin{tikzcd} Y \arrow[rr, "m"] \arrow[dr, "q"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

is $L^{/X}$-local (as an object in $(\mathcal{E}_{/X})_{/p}$, so m is in $\mathcal{D}_E$) if and only if

$$\begin{tikzcd} Y \arrow[r, "\eta_{E'}(\eta'_X(p)\circ m)"] \arrow[d, "m"'] & L_{E'}Y \arrow[d, "L_{E'}(\eta'_X(p)m)"] \\ E \arrow[r, "\eta'_X(p)"'] & E' \end{tikzcd}$$

is a pullback square in $\mathcal{E}_{/X}$. (Note that, in the above, $L_{E'}$ should be $L^{/X}_{p'}$, where $L^{/X}_{p'}$ is the reflector of $(\mathcal{E}_{/X})_{/p'}$ onto $\mathcal{D}^{/X}_{p'}$ and $L^{/X}_{\bullet}$ is the reflective subfibration on $\mathcal{E}_{/X}$ induced by $L_\bullet$, as in Remark 2.4. But, by its own definition, $L^{/X}_{p'} = L_{E'}$.)

The following corollary is probably well-known, though the only explicit reference we could find in the literature is [Rez10, Lemma 8.6], where the statement is proved in the context of model topoi. Note that our proof is completely internal and does not use the description of ∞-topoi as left exact localizations of presheaf categories. The reader might remember from Example 2.2 that, given $X \in \mathcal{E}$ and $n \geq -2$, we denote as $\|X\|_n$ the n-truncation object of X. This is the value at X of the n-truncation modality $L^n_\bullet$ (i.e., $L^n(X) = \|X\|_n$).

3.8. Corollary. *For $n \geq -2$, a map $p\colon E \to X$ is n-truncated if and only if $\|p\|_{n+1}$ is n-truncated and there is a pullback square*

$$\begin{tikzcd} E \arrow[r, "|\cdot|_{n+1}"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \|E\|_{n+1} \arrow[d, "\|p\|_{n+1}"] \\ X \arrow[r, "|\cdot|_{n+1}"'] & \|X\|_{n+1} \end{tikzcd} .$$

Proof. By [Ver19, Ex. 4.6 & Ex. 6.4], we can apply Proposition 3.6 where $L_\bullet$ is the n-truncation modality and get a pullback square

$$\begin{tikzcd} E \arrow[r, "m"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & L_{\|X\|_{n+1}}(E) \arrow[d, "L_{\|X\|_{n+1}}(|\cdot|_{n+1}p)"] \\ X \arrow[r, "|\cdot|_{n+1}"'] & \|X\|_{n+1} \end{tikzcd} .$$

Since $\|X\|_{n+1}$ is $(n+1)$-truncated and $L_{\|X\|_{n+1}}(|\cdot|_{n+1}p)$ is n-truncated, $L_{\|X\|_{n+1}}(E)$ is $(n+1)$-truncated. (This is an instance of Lemma 3.1.) Since m is a pullback of the $(n+1)$-connected map $|\cdot|_{n+1}\colon X \to \|X\|_{n+1}$, it is $(n+1)$-connected. Finally, any $(n+1)$-connected map $k\colon A \to B$ where B is $(n+1)$-truncated is an $(n+1)$-truncation of A. ∎

3.9. Proposition. [CORS18, Prop. 2.26] *Let*

$$\begin{tikzcd} E \arrow[rr, "\eta'_X(p)"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

be an L'-localization of $p \in \mathcal{E}_{/X}$. Let

$$\begin{tikzcd} E \arrow[rr, "\eta_{E\times_X E}(\Delta p)"] \arrow[dr, "\Delta p"'] & & R \arrow[dl, "r"] \\ & E \times_X E & \end{tikzcd}$$

be the L-localization of $\Delta p \in \mathcal{E}_{/E\times_X E}$ and consider r' defined by the pullback square

$$\begin{tikzcd} E \times_{E'} E \arrow[r] \arrow[d, "r'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[d, "\Delta p'"] \\ E \times_X E \arrow[r, "\eta'_X(p)\times_X \eta'_X(p)"'] & E' \times_X E' \end{tikzcd} . \tag{†}$$

Then there is a natural equivalence $\varphi\colon R \xrightarrow{\simeq} E \times_{E'} E$ over $E \times_X E$ as in

$$\begin{tikzcd} & E \arrow[dl, "\eta_{E\times_X E}(\Delta p)"'] \arrow[dr, "\Delta(\eta'_X(p))"] & \\ R \arrow[rr, "\varphi"] \arrow[dr, "r"'] & & E \times_{E'} E \arrow[dl, "r'"] \\ & E \times_X E & \end{tikzcd} .$$

PROOF. For sake of readability, we write η' and η for $\eta'_X(p)$ and $\eta_{E\times_X E}(\Delta p)$, respectively. The natural map φ is given by the universal property of η, since r' is L-local. (By definition, r' is the pullback of the L-local map $\Delta p'$.) Now, since $\eta' \times_X \eta'$ is the L'-localization map of the product object $p \times^X p$ of $\mathcal{E}_{/X}$, Proposition 3.6 applied in $\mathcal{E}_{/X}$ gives that there is a pullback square

$$\begin{tikzcd} R \arrow[r, "n"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "q"] \\ E \times_X E \arrow[r, "\eta' \times_X \eta'"] & E' \times_X E' \end{tikzcd}$$

where $n\colon (\eta' \times_X \eta')r \to q$ is the L-localization map of $(\eta' \times_X \eta')r$. Set $m := n\eta\colon E \to T$ and $l := \pi q$, where $\pi\colon E' \times_X E' \to X$ is given by the composite map $E' \times_X E' \to E' \xrightarrow{p'} X$. Note that π is L-separated, because it is the product in $\mathcal{E}_{/X}$ of the L-separated map p' with itself. Hence, since q is L-local, l is L-separated by Lemma 3.1. Since $m = n\eta$ is naturally a map $m\colon p \to l$ in $\mathcal{E}_{/X}$, there is a unique $s\colon E' \to T$ over X with commuting triangles

$$\begin{tikzcd} & E \arrow[dl, "\eta'"'] \arrow[dr, "m"] & \\ E' \arrow[rr, "s"] \arrow[dr, "p'"'] & & T \arrow[dl, "l"] \\ & X & \end{tikzcd} .$$

Now, $qs\eta' = qm = qn\eta = (\eta' \times_X \eta')\Delta p = \Delta p'\eta'$ so that $qs = \Delta p'$ and we can write $s\colon \Delta p' \to q$ as a map over $E' \times_X E'$. Hence, s induces the comparison map ψ of pullback squares in

$$\begin{tikzcd} E \times_{E'} E \arrow[rr] \arrow[dr, dashed, "\psi"] \arrow[dd, "r'"'] & & E' \arrow[dr, "s"] \arrow[dd, "\Delta p'" near end] & \\ & R \arrow[rr, "n" near start] \arrow[dd, "r" near start] & & T \arrow[dd, "q"] \\ E \times_X E \arrow[rr] \arrow[dr, "\mathrm{id}"] & & E' \times_X E' \arrow[dr, "\mathrm{id}"] & \\ & E \times_X E \arrow[rr, "\eta' \times_X \eta'"'] & & E' \times_X E' \end{tikzcd} .$$

Since the front face is a pullback, it follows that $\psi \circ \Delta\eta' = \eta$, from which we get $\psi\varphi\eta = \eta$, so that $\psi \circ \varphi = \mathrm{id}$. We now show that s is an equivalence, so that ψ (and therefore also φ) is an equivalence. Since $s\colon \Delta p' \to q$ is a map between L-local maps over $E' \times_X E'$, it is enough to show that $s \in S_{E' \times_X E'}$. Now, $\eta'\colon p \to p'$ is L-connected so it is an $L_{E'}$-equivalence (more precisely, $\eta'\colon \eta' \to \mathrm{id}_{E'}$ is in $S_{E'}$). By Lemma 2.7 (ii), composing $\eta'\colon \eta' \to \mathrm{id}_{E'}$ with $\Delta p'$ gives that $\eta'\colon (\Delta p')\eta' \to \Delta p'$ is in $S_{E' \times_X E'}$. Similarly, composing domain and codomain of η with $\eta' \times_X \eta'$ turns η into a map in $S_{E' \times_X E'}$ and then $m = n\eta$ is in $S_{E' \times_X E'}$, since n is an L-equivalence. Since $s\eta' = m$, $s \in S_{E' \times_X E'}$, as needed. ■

Our next result characterizes L'-localization maps in terms of their diagonal maps. We will use here some results from the Appendix (Section 5.3).

3.10. THEOREM. [CORS18, Thm. 2.34] *The following are equivalent for a map in $\mathcal{E}_{/Z}$*

$$\begin{tikzcd} X \arrow[rr, "\eta'"] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & Z & \end{tikzcd}$$

1. *η' is an L'-localization map of p.*

2. *η' is an effective epimorphism and*

$$\begin{tikzcd} X \arrow[rr, "\Delta\eta'"] \arrow[dr, "\Delta p"'] & & X \times_{X'} X \arrow[dl] \\ & X \times_Z X & \end{tikzcd}$$

is an L-localization map of Δp.

PROOF. We prove the theorem when $Z = 1$; the general statement follows from this one by Remark 2.4. We show first that (1) implies (2). If $\eta' \colon X \to X'$ is an L'-localization of X, then, by Proposition 3.9, we only need to show that η' is an effective epimorphism. Let (π, i) be the (effective epi,mono)-factorization of η', with $i \colon W \to X'$. Since i is a mono, and X' is L-separated, so is W. (This is because i being a mono is equivalent to the fact that $\Delta W = i^*(\Delta X)$, where $i^*(\Delta X)$ is the pullback map of ΔX along i . Hence, if ΔX is an equivalence, so is ΔW.) Therefore there is a unique $s \colon X' \to W$ with $s\eta' = \pi$. From $is\eta' = i\pi = \eta'$, we get that $is = \mathrm{id}_{X'}$. Thus, i is both a mono and an effective epi, so it is an equivalence.

Conversely, assume η' is an effective epimorphism and $\Delta\eta'$ is the L-localization of ΔX. In the pullback square

$$\begin{tikzcd} X \times_{X'} X \arrow[r] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "\Delta X'"] \\ X \times X \arrow[r, "\eta' \times \eta'"'] & X' \times X' \end{tikzcd} \qquad (*)$$

$\eta' \times \eta'$ is also an effective epi and t is L-local by hypothesis. Thus, $\Delta X'$ is L-local since L-local maps are a local class of maps in $\mathcal{E}$ (Proposition 2.16). This shows that X' is L-separated. We now verify that η' has the universal property of an L'-localization map. Let $f \colon X \to Y$ be a map into an L-separated object Y. We show that f extends uniquely along η', by applying Proposition 5.3.4 to f and η'. We want to show that

$$E := \sum_{X' \times Y \to X'} \left(\prod_{X \times X' \times Y \to X' \times Y} (\mathrm{pr}_X, X' \times f)^{(\mathrm{pr}_X, \eta' \times Y)} \right)$$

is contractible in $\mathcal{E}_{/X'}$. Applying Lemma 5.3.1 and the Beck-Chevalley condition (Lemma 5.1.3) to the pullback squares

$$\begin{tikzcd} X \times X \times Y \arrow[r, "\mathrm{pr}_{X\times Y}"] \arrow[d, "X\times\eta'\times Y"'] \arrow[dr, phantom, "\lrcorner"] & X \times Y \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\eta'\times Y"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\eta'"'] \\ X \times X' \times Y \arrow[r] & X' \times Y \arrow[r] & X' \end{tikzcd}$$

we can instead show that

$$E' := \sum_{X\times Y\to X}\left(\prod_{X\times X\times Y\to X\times Y}(X\times\eta'\times Y)^*\left((\mathrm{pr}_X, X'\times f)^{(\mathrm{pr}_X,\eta'\times Y)}\right)\right)$$

is contractible in $\mathcal{E}_{/X}$. We will show that this object of $\mathcal{E}_{/X}$ is equivalent to the object id_X, which is contractible in $\mathcal{E}_{/X}$. Lemma 5.1.1 gives that

$$(X\times\eta'\times Y)^*\left((\mathrm{pr}_X, X'\times f)^{(\mathrm{pr}_X,\eta'\times Y)}\right)\simeq$$
$$\simeq((X\times\eta'\times Y)^*(\mathrm{pr}_X, X'\times f))^{(X\times\eta'\times Y)^*((\mathrm{pr}_X,\eta'\times Y))}$$

Notice that

$$(\mathrm{pr}_X, X'\times f) = (f\times\mathrm{pr}_Y)^*(\Delta Y), \quad (\mathrm{pr}_X,\eta'\times Y) = (\eta'\times\mathrm{pr}_{X'})^*(\Delta X')$$

and $(f\times\mathrm{pr}_Y)(X\times\eta'\times Y) = (f\times Y)(\mathrm{pr}_1,\mathrm{pr}_3)$, where $\mathrm{pr}_1\colon X\times X\times Y\to X$ and $\mathrm{pr}_3\colon X\times X\times Y\to Y$ are appropriate projections. One can then see that

$$(X\times\eta'\times Y)^*\left((\mathrm{pr}_X, X'\times f)\right) = (\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1)\colon X\times X\to X\times X\times Y,$$
$$(X\times\eta'\times Y)^*\left((\mathrm{pr}_X,\eta'\times Y)\right) = t\times Y\colon (X\times_{X'}X)\times Y\to X\times X\times Y$$

where t is defined in the pullback square $(*)$ above. Therefore,

$$(X\times\eta'\times Y)^*\left((\mathrm{pr}_X, X'\times f)^{(\mathrm{pr}_X,\eta'\times Y)}\right)\simeq(\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1)^{t\times Y}.$$

Now, since t is the localization of ΔX in $\mathcal{E}_{/X\times X}$, taking pullbacks along the projection $X\times X\times Y\to X\times X$ gives that $t\times Y$ is the localization of $\Delta X\times Y$ in $\mathcal{E}_{/X\times X\times Y}$. Since $(\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1)$ is L-local (as the pullback of the L-local map ΔY), we further have

$$(\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1)^{t\times Y}\simeq(\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1)^{\Delta X\times Y}\simeq$$
$$\simeq\prod_{\Delta X\times Y}(\Delta X\times Y)^*(\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1)\simeq\prod_{\Delta X\times Y}(\mathrm{id}_X, f),$$

where $(\mathrm{id}_X, f)\colon X\to X\times Y$. We can now finally conclude because

$$E'\simeq\sum_{X\times Y\to X}\left(\prod_{\mathrm{pr}_{X\times Y}\colon X\times X\times Y\to X\times Y}\left(\prod_{\Delta X\times Y}(\mathrm{id}_X, f)\right)\right)\simeq$$
$$\simeq\sum_{X\times Y\to X}\left(\prod_{\mathrm{pr}_{X\times Y}\circ(\Delta X\times Y)}(\mathrm{id}_X, f)\right)=\sum_{X\times Y\to X}(\mathrm{id}_X, f)=\mathrm{id}_X.$$

■

4. Existence of L'-localization

We prove here our main result that the class of L-separated maps is the class of local maps for a reflective subfibration on $\mathcal{E}$, and we start by proving a few preliminary results.

Recall that, if p, q are objects in a slice category $\mathcal{E}_{/Z}$, we write $p \times^Z q$ to mean the product object of p and q in $\mathcal{E}_{/Z}$.

The first result we need is an internal Yoneda lemma involving diagonal maps.

4.1. LEMMA. *Let $t\colon E \to X$ be a map in $\mathcal{E}$ and form the pullback square*

$$\begin{tikzcd} X \times E \arrow[r] \arrow[d, "X\times t"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "t"] \\ X \times X \arrow[r, "\mathrm{pr}_2"'] & X \end{tikzcd}.$$

Then there is a map in $\mathcal{E}_{/X^2}$

$$\begin{tikzcd} E \arrow[rr, "(t,\mathrm{id})"] \arrow[dr, "(\Delta X)t"'] & & X \times E \arrow[dl, "X\times t"] \\ & X \times X & \end{tikzcd},$$

inducing an equivalence

$$\beta\colon t \xrightarrow{\simeq} \prod_{\mathrm{pr}_1} (X \times t)^{\Delta X}$$

in $\mathcal{E}_{/X}$, where $\mathrm{pr}_1\colon X \times X \to X$ is the projection onto the first component.

PROOF. For any $k\colon M \to X$, the product object $(k \times X) \times^{X^2} (\Delta X)$ in $\mathcal{E}_{/X^2}$ is given by $(\Delta X)k$. In fact, $(\Delta X)k$ is also the product object $(X \times k) \times^{X^2} (\Delta X)$ in $\mathcal{E}_{/X^2}$. Taking $k = t$, we get that $(t, \mathrm{id})\colon (\Delta X)t \to X \times t$ gives a map

$$\beta\colon t \longrightarrow \prod_{\mathrm{pr}_1} (X \times t)^{\Delta X}$$

by adjointness. Using the fact that ΔX is a section of pr_2, and considering the adjoint pairs $\Sigma_{\mathrm{pr}_2} \dashv \mathrm{pr}_2^*$, $\mathrm{pr}_1^* \dashv \prod_{\mathrm{pr}_1}$, we get a chain of natural equivalences

$$\mathcal{E}_{/X}(k, t) \simeq \mathcal{E}_{/X}(\mathrm{pr}_2(\Delta X)k, t) \simeq \mathcal{E}_{/X^2}((\Delta X)k, X \times t) \simeq$$

$$\simeq \mathcal{E}_{/X^2}\left(k \times X, (X \times t)^{\Delta X}\right) \simeq \mathcal{E}_{/X}\left(k, \prod_{\mathrm{pr}_1} (X \times t)^{\Delta X}\right)$$

where the composite map is given by composition with β. ∎

4.2. Lemma. *Let $X \in \mathcal{E}$ and let $r\colon R \to X^2$ be an object in $\mathcal{E}_{/X^2}$. Let also $\widetilde{X \times r}$ be the composite map $(\tau \times X) \circ (X \times r)$, where $\tau\colon X^2 \simeq X^2$ is the canonical involution. Then the following hold.*

(i) There is a natural equivalence in $\mathcal{E}_{/X^2}$

$$\beta\colon r \xrightarrow{\simeq} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)} .$$

(ii) There is a map $\rho\colon \Delta X \to \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)}$ such that, given any map $\eta\colon \Delta X \to r$ in $\mathcal{E}_{/X^2}$, there is a commutative square

$$\begin{tikzcd} \Delta X \arrow[r, dashed, "\rho"] \arrow[d, "\eta"'] & \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)} \arrow[d, "\prod_{\mathrm{pr}_{23}} (\widetilde{X\times r})^{(\eta \times X)}"] \\ r \arrow[r, "\beta"'] & \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)} \end{tikzcd} . \tag{1}$$

Proof. The first claim is a special case of Lemma 4.1 applied to $r = (r_1, r_2)\colon R \to X^2$, seen as a map $r\colon r_2 \to \mathrm{pr}_2$ in $\mathcal{E}_{/X^2}$. Indeed, the following pullback square in $\mathcal{E}$

$$\begin{tikzcd} X^3 \arrow[r, "\mathrm{pr}_{13}"] \arrow[d, "\mathrm{pr}_{23}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X^2 \arrow[d, "\mathrm{pr}_2"] \\ X^2 \arrow[r, "\mathrm{pr}_2"'] & X \end{tikzcd}$$

witnesses that $\mathrm{pr}_3\colon X^3 \to X$ is the product object of $\mathrm{pr}_2\colon X^2 \to X$ with itself in $\mathcal{E}_{/X}$ and the displayed maps pr_{13} and pr_{23} give the projection maps out of this product. The map $\Delta X \times X\colon X^2 \to X^3$, seen as a map $\mathrm{pr}_3 \to \mathrm{pr}_3$, is the diagonal of the object $\mathrm{pr}_3 \in \mathcal{E}_{/X}$. Since $\widetilde{X \times r} = \mathrm{pr}_{13}^*(r)$, Lemma 4.1 gives the desired natural equivalence $\beta\colon r \simeq \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)}$.

For the second part, we describe the map ρ and how it makes the square (1) commute by looking at its adjunct. Under the adjunction $\mathrm{pr}_{23}^* \dashv \prod_{\mathrm{pr}_{23}}$, giving a square as (1) is the same as giving a square

$$\begin{tikzcd} X \times \Delta X \arrow[r, dashed, "\rho'"] \arrow[d, "X \times \eta"'] & (\widetilde{X \times r})^{(r \times X)} \arrow[d, "(\widetilde{X\times r})^{(\eta \times X)}"] \\ X \times r \arrow[r, "\beta'"'] & (\widetilde{X \times r})^{(\Delta X \times X)} \end{tikzcd}$$

since $X \times \Delta X = \mathrm{pr}_{23}^*(\Delta X)$ and similarly for $X \times r$. Taking further adjoints along $(-) \times^{X^2} (\Delta X \times X) \dashv (-)^{\Delta X \times X}$, we need to exhibit a square

$$\begin{tikzcd}
(X \times \Delta X) \times^{X^3} (\Delta X \times X) \arrow[r, "(X\times\eta)\times^{X^3}(\Delta X\times X)"] \arrow[d, "(X\times\Delta X)\times^{X^3}(\eta\times X)"'] & (X \times r) \times^{X^3} (\Delta X \times X) \arrow[d, "\beta^\sharp"] \\
(X \times \Delta X) \times^{X^3} (r \times X) \arrow[r, dashed, "\rho^\sharp"'] & \widetilde{X \times r}
\end{tikzcd}.$$

The products $(X\times\Delta X)\times^{X^3}(\Delta X\times X)$, $(X\times r)\times^{X^3}(\Delta X\times X)$ and $(X\times\Delta X)\times^{X^3}(r\times X)$ in $\mathcal{E}_{/X^3}$, together with their projections onto the factors, are represented, in order, by the following pullback squares in $\mathcal{E}$

$$\begin{tikzcd}
X \arrow[r, "\Delta X"] \arrow[d, "\Delta X"'] \arrow[dr, "(\mathrm{id},\mathrm{id},\mathrm{id})" description] \arrow[dr, phantom, very near start, "\lrcorner"] & X^2 \arrow[d, "\Delta X\times X"] \\
X^2 \arrow[r, "X\times\Delta X"'] & X^3
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "r"] \arrow[d, "(r_1,\mathrm{id})"'] \arrow[dr, "(r_1,r_1,r_2)" description] \arrow[dr, phantom, very near start, "\lrcorner"] & X^2 \arrow[d, "\Delta X\times X"] \\
X \times R \arrow[r, "X\times r"'] & X^3
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "(\mathrm{id},r_2)"] \arrow[d, "r"'] \arrow[dr, "(r_1,r_2,r_2)" description] \arrow[dr, phantom, very near start, "\lrcorner"] & R \times X \arrow[d, "r\times X"] \\
X^2 \arrow[r, "X\times\Delta X"'] & X^3
\end{tikzcd}.$$

Using Lemma 4.1 as in the first part, we know the map $\beta^\sharp$ is given by

$$\begin{tikzcd}
R \arrow[r, "\beta^\sharp=(r_1,\mathrm{id})"] \arrow[d, "r"'] \arrow[dr, "(r_1,r_1,r_2)" description] & X \times R \arrow[d, "\widetilde{X\times r}"] \\
X^2 \arrow[r, "\Delta X\times X"'] & X^3
\end{tikzcd}.$$

We take $\rho^\sharp$ to be given by

$$\begin{tikzcd}
R \arrow[r, "\rho^\sharp=(r_2,\mathrm{id})"] \arrow[d, "r"'] \arrow[dr, "(r_1,r_2,r_2)" description] & X \times R \arrow[d, "\widetilde{X\times r}"] \\
X^2 \arrow[r, "X\times\Delta X"'] & X^3
\end{tikzcd}.$$

Then the composite maps $\beta^\sharp\Big((X\times\eta)\times^{X^3}(\Delta X\times X)\Big)$ and $\rho^\sharp\Big((X\times\Delta X)\times^{X^3}(\eta\times X)\Big)$ are given by the following composite maps in $\mathcal{E}_{/X^3}$, respectively:

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "(\mathrm{id},\mathrm{id},\mathrm{id})"'] & R \arrow[r, "(r_1,\mathrm{id})"] \arrow[d, "(r_1,r_1,r_2)"] & X \times R \arrow[dl, "\widetilde{X\times r}"] \\
& X^3 &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "(\mathrm{id},\mathrm{id},\mathrm{id})"'] & R \arrow[r, "(r_2,\mathrm{id})"] \arrow[d, "(r_1,r_2,r_2)"] & X \times R \arrow[dl, "\widetilde{X\times r}"] \\
& X^3 &
\end{tikzcd}.$$

By using properties of the product $X \times R$ and since η is a section of both r_1 and r_2, one can see that these composite maps are equal since they are both equal to the morphism $(\mathrm{id},\eta)\colon (\mathrm{id},\mathrm{id},\mathrm{id}) \to \widetilde{X \times r}$ in $\mathcal{E}_{/X^3}$. (The needed homotopies are obtained by using either degenerate 2-simplices or the 2-simplices defining $\eta\colon \Delta X \to r$.) ∎

4.3. Theorem. [CORS18, Thm. 2.25] *For any $Y \in \mathcal{E}$, each $f \in \mathcal{E}_{/Y}$ has an L'-localization $\eta'_Y(f)\colon f \to f'$.*

Proof. We prove the result for $Y = 1$. Fix $X \in \mathcal{E}$ and let $\eta\colon \Delta X \to r$ be the L-reflection map of $\Delta X \in \mathcal{E}_{/X^2}$. Let κ be a regular cardinal such that r is relatively κ-compact and the relatively κ-compact L-local maps have a classifying map $u_L^\kappa\colon \widetilde{\mathcal{U}_L^\kappa} \to \mathcal{U}_L^\kappa$. Omitting κ from our notation, we then have pullback squares

$$\begin{tikzcd} R \arrow[r, "\tilde{r}"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}_L} \arrow[d, "u_L"] \\ X \times X \arrow[r, "\overline{r}"'] & \mathcal{U}_L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{\mathcal{U}_L} \arrow[r] \arrow[d, "u_L"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \arrow[d, "u"] \\ \mathcal{U}_L \arrow[r, tail, "\iota"'] & \mathcal{U} \end{tikzcd} .$$

We denote the composite pullback square as

$$\begin{tikzcd} R \arrow[r, "\tilde{r}"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \arrow[d, "u"] \\ X \times X \arrow[r, "\ulcorner r \urcorner"'] & \mathcal{U} \end{tikzcd} . \tag{2}$$

Let (η', i) be the (effective epi, mono)-factorization of $\ulcorner r \urcorner^\sharp\colon X \to \mathcal{U}^X$, the adjunct map to $\ulcorner r \urcorner$. Set $X' := \mathrm{cod}(\eta')$, so that we can picture the situation as:

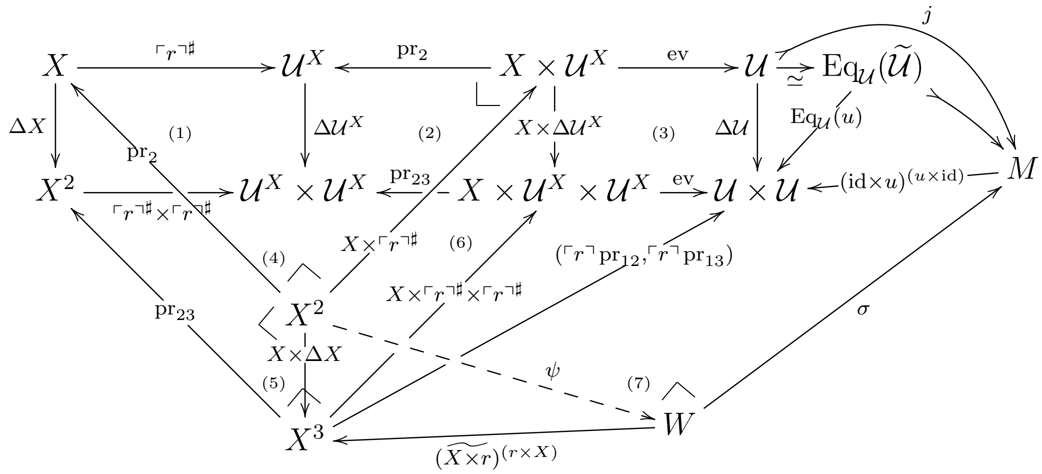
$$\begin{tikzcd} X \arrow[rr, two heads, "\eta'"] \arrow[dr, "\ulcorner r \urcorner"'] & & X' \arrow[dl, tail, "i"] \\ & \mathcal{U}^X & \end{tikzcd}$$

together with the implicit 2-simplex witnessing the commutativity of the diagram. Note that, if (η'_L, i_L) is the (effective epi,mono)-factorization of $\overline{r}^\sharp$, then $\eta' = \eta'_L$ and $i = \iota^X \circ i_L$ since ι^X is a mono.

Our goal is to apply Theorem 3.10 to η'. The map η' is an effective epi by definition. To show that X' is L-separated, note first that $\mathcal{U}_L$ is L-separated by Lemma 3.5, hence so is $\mathcal{U}_L^X$, by Proposition 2.21(1). Since i is a mono, this implies that X' is L-separated. It remains to show that $\Delta\eta'$ is the L-localization map of ΔX. We can see $\Delta\eta'$ as a map $\Delta\eta'\colon \Delta X \to t$ in $\mathcal{E}_{/X^2}$, where t is the pullback map $(\eta' \times \eta')^*(\Delta X')$ and it is therefore L-local. Hence, there is a unique map $\varphi\colon r \to t$ with $\varphi\eta = \Delta\eta'$ as maps in $\mathcal{E}_{/X^2}$. We will show that φ is an equivalence.

The strategy we adopt is to, first, construct a monomorphism $\varphi'\colon t \rightarrowtail r$ and, then, show that $\varphi'\varphi\colon r \to r$ is an equivalence by showing that we have $\varphi'\varphi\eta = \eta$. This will imply that φ itself is an equivalence. Note that, by definition of φ, showing that $\varphi'\varphi\eta = \eta$ is the same as showing that $\varphi'\Delta\eta' = \eta$.

Step 1. Construction of φ' and description of $\varphi'\Delta\eta'$ We construct φ' as a composite of some equivalences and a monomorphism. Consider the diagram:



. (D)

The maps labelled as ev are appropriate counits of product − internal-hom adjunctions. We proceed to explain this diagram, show how it defines φ', and give a description of $\varphi'\Delta\eta'$.

(i) Recall that $\Delta\eta'$ is a map $\Delta X \to t$ in $\mathcal{E}_{/X^2}$, and one can show that t is the pullback map of the cospan in (1) of (D). Because of this, the square (1) determines $\Delta\eta'$.

(ii) Thanks to Function Extensionality (Proposition 5.2.1), $\Delta\mathcal{U}^X \simeq \Pi_{\mathrm{pr}_{23}}\mathrm{ev}^*(\Delta\mathcal{U})$. Hence, $t \simeq (\ulcorner r \urcorner^\sharp \times \ulcorner r \urcorner^\sharp)^* \left(\Pi_{\mathrm{pr}_{23}}\mathrm{ev}^*(\Delta\mathcal{U})\right)$.

(iii) Since the bottom square (5) in (D) is a pullback, we can use the Beck-Chevalley condition (Lemma 5.1.3) to get an equivalence $t \simeq \Pi_{\mathrm{pr}_{23}}(\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})^*(\Delta\mathcal{U})$. Since the pullback of $X \times \Delta\mathcal{U}^X$ along $X \times \ulcorner r \urcorner^\sharp \times \ulcorner r \urcorner^\sharp$ is $X \times t$, the square (6) in (D) determines the map $X \times \Delta\eta' \colon X \times \Delta X \to X \times t$ in $\mathcal{E}_{/X^3}$. It follows that the map $X \times \Delta X \to (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})^*(\Delta\mathcal{U})$ determined by the square given as the composite of (3) and (6) is the adjunct of the composite map

$$\Delta X \xrightarrow{\Delta\eta'} t \simeq \prod_{\mathrm{pr}_{23}} (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})^*(\Delta\mathcal{U}) \ .$$

(iv) We now consider the map j in $\mathcal{E}_{/\mathcal{U}^2}$ displayed in the top-right corner of (D). Here, M is simply a name for the domain of the map $(\mathrm{id} \times u)^{(u \times \mathrm{id})}$. The map j is defined as the composite of the equivalence $\Delta\mathcal{U} \simeq \mathrm{Eq}_{\mathcal{U}}(u)$, given by univalence ([Ver19, Def. 2.10]), and the monomorphism $\mathrm{Eq}_{\mathcal{U}}(u) \rightarrowtail (\mathrm{id} \times u)^{(u \times \mathrm{id})}$. Thus, j is a mono as well. Using the fact that (7) in (D) is a pullback square, we obtain a monomorphism

$$\prod_{\mathrm{pr}_{23}} (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})^*(\Delta\mathcal{U}) \xrightarrow{\prod_{\mathrm{pr}_{23}} (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})^*(j)} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)} \ .$$

Here, $\widetilde{X \times r}$ is the pullback map $\mathrm{pr}_{13}^*(r) = (\tau \times X)(X \times r)$, where $\tau \colon X^2 \simeq X^2$ is the swapping equivalence, and W is simply a name for the domain of the map $(\widetilde{X \times r})^{(r\times X)}$. Note that the map displayed above is indeed a monomorphism because, being right adjoints, pullback and dependent-product functors preserve monomorphisms. Therefore, we get a composite monomorphism

$$t \rightarrowtail \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r\times X)}.$$

The map ψ in $\mathcal{E}_{/X^3}$ given in (D) is determined, as a map $X \times \Delta X \to (\widetilde{X \times r})^{(r\times X)}$, by the composite of the squares (3) and (6) with the 2-simplex representing the map $j \colon \Delta(\mathcal{U}) \rightarrowtail (\mathrm{id} \times u)^{(u\times \mathrm{id})}$. It follows that ψ is the adjunct to the composite

$$\Delta X \xrightarrow{\Delta\eta'} t \rightarrowtail \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r\times X)} .$$

This means that this latter map is the composite

$$\Delta X \xrightarrow{\gamma} \prod_{\mathrm{pr}_{23}} X \times \Delta X \xrightarrow{\prod_{\mathrm{pr}_{23}} \psi} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r\times X)},$$

where γ is the unit of the adjunction $\mathrm{pr}_{23}^* \dashv \prod_{\mathrm{pr}_{23}}$ at ΔX.

(v) Since $\widetilde{X \times r} = \mathrm{pr}_{13}^*(r)$, $\widetilde{X \times r}$ is L-local. Hence, because $\eta \times X \colon \Delta X \times X \to r \times X$ is an L-localization map (it is the pullback along pr_{12} of η), we have an equivalence

$$(\widetilde{X \times r})^{(\eta\times X)} \colon (\widetilde{X \times r})^{(r\times X)} \xrightarrow{\simeq} (\widetilde{X \times r})^{(\Delta X\times X)} .$$

Whence, we have a composite monomorphism

$$t \rightarrowtail \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r\times X)} \simeq \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X\times X)} .$$

(vi) Finally, we have an equivalence $\beta \colon r \xrightarrow{\simeq} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X\times X)}$ as in Lemma 4.2. Composing the monomorphism obtained in (v) with the inverse of β we obtain the needed monomorphism $\varphi' \colon t \rightarrowtail r$. Using what we found in (iv) above, the composite $\varphi'\Delta\eta'$ is then given as the composite

$$\Delta X \xrightarrow{\gamma} \prod_{\mathrm{pr}_{23}} X \times \Delta X \xrightarrow{\prod_{\mathrm{pr}_{23}} \psi} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r\times X)} \xrightarrow{\simeq} r,$$

where the displayed equivalence is $\beta^{-1} \prod\limits_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\eta\times X)}$.

Step 2. Proof that $\varphi'\Delta\eta' = \eta$. By the work above, it suffices to show that the maps

$$\Delta X \xrightarrow{\eta} r \xrightarrow[\simeq]{\beta} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)}$$

and

$$\Delta X \xrightarrow{\gamma} \prod_{\mathrm{pr}_{23}} X \times \Delta X \xrightarrow{\prod_{\mathrm{pr}_{23}} \psi} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)} \xrightarrow[\simeq]{\prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\eta \times X)}} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)}$$

are equal in $\mathcal{E}_{/X^2}$. By Lemma 4.2 (ii), there is a map $\rho\colon \Delta X \to \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)}$ making the following diagram commute in $\mathcal{E}_{/X^2}$:

$$\begin{tikzcd}
\Delta X \arrow[r, dashed, "\rho"] \arrow[d, "\eta"'] & \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)} \arrow[d, "\prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\eta \times X)}"] \\
r \arrow[r, "\beta"'] & \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)}
\end{tikzcd} .$$

Thus, we only need to show that $\rho = \left(\prod_{\mathrm{pr}_{23}} \psi\right) \gamma$. Equivalently, we can show that the adjunct maps $\rho', \psi\colon (X \times \Delta X) \to (\widetilde{X \times r})^{(r \times X)}$ are equal in $\mathcal{E}_{/X^3}$. Since the square (7) in the diagram (D) is a pullback, we only need to show that ρ' and ψ are equal after composing with $g := (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})$ and

$$\sigma\colon g(\widetilde{X \times r})^{(r \times X)} \to (\mathrm{id} \times u)^{(u \times \mathrm{id})},$$

that is, as maps $g(X \times \Delta X) \to (\mathrm{id} \times u)^{(u \times \mathrm{id})}$. Finally, we can further show that $\sigma\rho'$, $\sigma\psi$ are equal in $\mathcal{E}_{/\mathcal{U}^2}$ by showing their adjuncts along the adjunction $(-) \times^{\mathcal{U}^2} (u \times \mathrm{id}) \dashv (-)^{(u \times \mathrm{id})}$ are equal.

In order to describe the adjunct of $\sigma\rho'$, we use Lemma 5.1.2 with $f = X \times \Delta X$, $g := (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})$, $p = \mathrm{id} \times u$ and $q = u \times \mathrm{id}$. Consequently, $g^* q = r \times X$, $g^* p = \widetilde{X \times r}$ and the adjunct of $\sigma\rho'$ is given as the composite map

$$g((X \times \Delta X) \times^{X^3} (r \times X)) \xrightarrow{\rho^\sharp} g(\widetilde{X \times r}) = gg^*(\mathrm{id} \times u) \xrightarrow{\epsilon_{(\mathrm{id} \times u)}} \mathrm{id} \times u.$$

Recall the pullback square (2) defining $\ulcorner r \urcorner$. Since $(X \times \Delta X) \times^{X^3} (r \times X) = (X \times \Delta X) r$, using the proof of Lemma 4.2 and the fact that $g^*(\mathrm{id} \times u) = \widetilde{X \times r}$, we have that the maps $\rho^\sharp\colon (X \times \Delta X) r \to \widetilde{X \times r}$ and $\epsilon_{(\mathrm{id} \times u)}\colon g(\widetilde{X \times r}) \to \mathrm{id} \times u$ are described by the two squares below

$$\begin{tikzcd} R \arrow[r, "\rho^\sharp=(r_2,\mathrm{id})"] \arrow[d, "r"'] \arrow[dr, "(r_1,r_2,r_2)" description] & X\times R \arrow[d, "\widetilde{X\times r}"] \\ X^2 \arrow[r, "X\times\Delta X"'] & X^3 \end{tikzcd} \qquad \begin{tikzcd} X\times R \arrow[rr, "\epsilon_{(\mathrm{id}\times u)}=(\ulcorner r\urcorner(r_1\,\mathrm{pr}_2,\mathrm{pr}_1),\widetilde{r}\,\mathrm{pr}_2)"] \arrow[d, "\widetilde{X\times r}"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathcal{U}\times\widetilde{\mathcal{U}} \arrow[d, "\mathrm{id}\times u"] \\ X^3 \arrow[rr, "g=(\ulcorner r\urcorner\mathrm{pr}_{12},\ulcorner r\urcorner\mathrm{pr}_{13})"'] & & \mathcal{U}^2 \end{tikzcd}.$$

Hence, the composite $\epsilon_{(\mathrm{id}\times u)}\rho^\sharp$ (the adjunct of $\sigma\rho'$ in $\mathcal{E}_{/\mathcal{U}^2}$) is given by the map

$$\begin{tikzcd} R \arrow[rr, "(\ulcorner r\urcorner r,\widetilde{r})"] \arrow[dr, "(\ulcorner r\urcorner,\ulcorner r\urcorner)r"'] & & \mathcal{U}\times\widetilde{\mathcal{U}} \arrow[dl, "\mathrm{id}\times u"] \\ & \mathcal{U}^2 & \end{tikzcd}. \tag{3}$$

To describe the adjunct of $\sigma\psi$, note that, from the squares (6), (7) and (3) and the definition of j in the diagram (D), $\sigma\psi$ is given as the map in $\mathcal{E}_{/\mathcal{U}^2}$ described by the diagram

$$\begin{tikzcd} X^2 \arrow[rr, "\ulcorner r\urcorner"] \arrow[d, "X\times\Delta X"'] \arrow[drr, "(\ulcorner r\urcorner,\ulcorner r\urcorner)" description] & & \mathcal{U} \arrow[r, "j"] \arrow[d, "\Delta\mathcal{U}"] & M \arrow[dl, "(\mathrm{id}\times u)^{(u\times\mathrm{id})}"] \\ X^3 \arrow[rr, "g=(\ulcorner r\urcorner\mathrm{pr}_{12},\ulcorner r\urcorner\mathrm{pr}_{13})"'] & & \mathcal{U}^2 & \end{tikzcd}.$$

Then, the adjunct of $j\ulcorner r\urcorner$ in $\mathcal{E}_{/\mathcal{U}^2}$ is the composite $j^\sharp(\ulcorner r\urcorner\times^{\mathcal{U}^2}(u\times\mathrm{id}))$, where the map $j^\sharp\colon \Delta\mathcal{U}\times^{\mathcal{U}^2}(u\times\mathrm{id})\to\mathrm{id}\times u$ is the adjunct of j. Using that there are pullback squares

$$\begin{tikzcd} R \arrow[r, "(\widetilde{r},\ulcorner r\urcorner r)"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}}\times\mathcal{U} \arrow[d, "u\times\mathrm{id}"] \\ X^2 \arrow[r, "(\ulcorner r\urcorner,\ulcorner r\urcorner)"'] & \mathcal{U}^2 \end{tikzcd} \qquad \begin{tikzcd} \widetilde{\mathcal{U}} \arrow[r, "(\mathrm{id},u)"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}}\times\mathcal{U} \arrow[d, "u\times\mathrm{id}"] \\ \mathcal{U} \arrow[r, "\Delta\mathcal{U}"'] & \mathcal{U}^2 \end{tikzcd}$$

we get that $j^\sharp(\ulcorner r\urcorner\times^{\mathcal{U}^2}(u\times\mathrm{id}))$ is the composite map

$$\begin{tikzcd} R \arrow[r, "\widetilde{r}"] \arrow[dr, "(\ulcorner r\urcorner,\ulcorner r\urcorner)r"'] & \widetilde{\mathcal{U}} \arrow[r, "j^\sharp=(u,\mathrm{id})"] \arrow[d, "(u,u)"] & \mathcal{U}\times\widetilde{\mathcal{U}} \arrow[dl, "\mathrm{id}\times u"] \\ & \mathcal{U}^2 & \end{tikzcd} = \begin{tikzcd} R \arrow[rr, "(u\widetilde{r},\widetilde{r})"] \arrow[dr, "(\ulcorner r\urcorner,\ulcorner r\urcorner)r"'] & & \mathcal{U}\times\widetilde{\mathcal{U}} \arrow[dl, "\mathrm{id}\times u"] \\ & \mathcal{U}^2 & \end{tikzcd}. \tag{4}$$

One can now see that the maps (3) and (4) are equal by using the square (2) defining $\ulcorner r\urcorner$ (including the implicit given homotopies). Our proof is then complete. ∎

Once we know that every map in $\mathcal{E}$ has an L'-localization, we can also show that L'-localization form a reflective subfibration on $\mathcal{E}$. The crucial point here is to show pullback-compatibility of L'-reflections. This is necessary when working in higher topos theory, but it is superfluous in homotopy type theory as reflections are automatically stable under pullbacks in that setting.

4.4. Corollary. *Given any reflective subfibration $L_\bullet$ of an ∞-topos $\mathcal{E}$, there exists a reflective subfibration $L'_\bullet$ of $\mathcal{E}$ such that the L'-local maps are exactly the L-separated maps. Furthermore, if $L_\bullet$ is a modality, then so is $L'_\bullet$.*

Proof. Let $\mathcal{D}'$ be the full subcategory of $\mathcal{E}$ spanned by the L-separated objects and let $\iota\colon \mathcal{D}' \to \mathcal{E}$ be the inclusion functor. Theorem 4.3 constructs, for every $X \in \mathcal{E}$, an L'-localization map $\eta(X)\colon X \to L'(X)$. By definition of L'-localization map, this means that, for every $X \in \mathcal{E}$, the ∞-category defined as the pullback

$$\begin{tikzcd} \mathcal{D}'_{X/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d, "\iota"] \\ \mathcal{E}_{X/} \arrow[r] & \mathcal{E} \end{tikzcd}$$

has an initial object. (Here, $\mathcal{D}'_{X/}$ is the slice category of $\mathcal{D}'$ under X, and similarly for $\mathcal{E}_{X/}$.) By [Joy08, §17.4], ι has a left adjoint $L'\colon \mathcal{E} \to \mathcal{D}'$, i.e., $\mathcal{D}'$ is a reflective subcategory of $\mathcal{E}$. The same construction performed on each slice category now gives that, for every $X \in \mathcal{E}$, the full subcategory $\mathcal{D}'_X$ of $\mathcal{E}_{/X}$ on the L-separated $p \in \mathcal{E}_{/X}$ is reflective. Since L-separated maps are closed under pullbacks (see Proposition 2.21), to conclude that we get a reflective subfibration $l'_\bullet$ on $\mathcal{E}$, we need to verify that the L'-reflection maps are compatible with pullbacks.

Let then $p\colon E \to X$ be an object in $\mathcal{E}_{/X}$ and $f\colon Y \to X$ a map in $\mathcal{E}$. Let

$$\begin{tikzcd} E \arrow[rr, "\eta' := \eta'_X(p)"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

be the L'-localization of p. We need to show that $m := f^*(\eta')\colon f^*(p) \to f^*(p')$ is the L'-localization of $f^*(p)$ in $\mathcal{E}_{/Y}$. To do so we use Theorem 3.10. Set $f^*(E) := Y \times_X E$, $q := f^*(p)$ and $f^*(E') := Y \times_X E'$. Since η' is an effective epimorphism and effective epimorphisms are closed under pullbacks, an application of the pasting lemma for pullbacks show that m is also an effective epimorphism. By Proposition 2.21(1), $f^*(p')$ is L-separated. Therefore, we only need to show that $\Delta(m)$, as a map in $\mathcal{E}_{/f^*(E)\times_Y f^*(E)}$, is the L'-localization map of Δq. In $\mathcal{E}_{/Y}$ we have the pullback square (products are products in $\mathcal{E}_{/Y}$)

$$\begin{tikzcd} q \times_{f^*(p')} q \arrow[r] \arrow[d, "(m\times m)^*(\Delta(f^*(p'))"'] \arrow[dr, phantom, "\lrcorner", very near start] & f^*(p') \arrow[d, "\Delta(f^*(p'))"] \\ q \times q \arrow[r, "m\times m"'] & f^*(p') \times f^*(p') \end{tikzcd}$$

and Δm is a map $\Delta q \to (m \times m)^*(\Delta(f^*(p'))$ in $\left(\mathcal{E}_{/Y}\right)_{/(q\times q)}$. Since $\left(\mathcal{E}_{/Y}\right)_{/(q\times q)} \simeq \mathcal{E}_{/f^*(E)\times_Y f^*(E)}$ and $m = f^*(\eta')$, one can see that Δm is the map

$$\begin{tikzcd} f^*(E) \arrow[rr, "\Delta m"] \arrow[dr, "\Delta q"'] & & f^*(E) \times_{f^*(E')} f^*(E) \arrow[dl, "t"] \\ & f^*(E) \times_Y f^*(E) & \end{tikzcd}$$

in $\mathcal{E}_{/f^*(E)\times_Y f^*(E)}$, where t corresponds to the map $(m\times m)^*(\Delta(f^*(p'))$ above. Similarly, $\Delta\eta'$ is a map $\Delta p \to s$ in $\mathcal{E}_{/E\times_X E}$ (where s is given by a suitable pullback) and it is the L-localization of Δp by Theorem 3.10. We want to show that Δm is a pullback of this L-localization and conclude because $L_\bullet$ is a reflective subfibration. Let $g\colon f^*(E)\to E$ and $g'\colon f^*(E')\to E'$ be the projection maps. As in the proof of Proposition 2.21, we see that the following are all pullback squares in $\mathcal{E}$

$$\begin{tikzcd}
f^*(E)\times_Y f^*(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E\times_X E \arrow[d] \\
f^*(E) \arrow[r, "g"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*(E')\times_Y f^*(E') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E'\times_X E' \arrow[d] \\
f^*(E') \arrow[r, "g'"'] & E'
\end{tikzcd}$$

$$\begin{tikzcd}
f^*(E) \arrow[r, "g"] \arrow[d, "\Delta(f^*(p))"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\Delta p"] \\
f^*(E)\times_Y f^*(E) \arrow[r] & E\times_X E
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*(E') \arrow[r, "g'"] \arrow[d, "\Delta(f^*(p'))"'] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[d, "\Delta p'"] \\
f^*(E')\times_Y f^*(E') \arrow[r] & E'\times_X E'
\end{tikzcd}.$$

Then in the diagram

$$\begin{tikzcd}
f^*(E)\times_{f^*(E')} f^*(E) \arrow[rr] \arrow[dr] \arrow[dd, "t"] & & E\times_{E'} E \arrow[dr] \arrow[dd, "s"] & \\
& f^*(E') \arrow[rr, "g'"] \arrow[dd, "\Delta(f^*(p'))"] & & E' \arrow[dd, "\Delta p'"] \\
f^*(E)\times_Y f^*(E) \arrow[rr] \arrow[dr, "m\times_Y m"'] & & E\times_X E \arrow[dr, "\eta'\times_X\eta'"] & \\
& f^*(E')\times_Y f^*(E') \arrow[rr] & & E'\times_X E'
\end{tikzcd}$$

the left and right sides are pullbacks (by definition of t and s) and the front square is a pullback by the above. Therefore, the back square is also a pullback. A final application of the pasting lemma now shows that there are pullback squares in $\mathcal{E}$

$$\begin{tikzcd}
f^*(E) \arrow[r, "\Delta m"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & f^*(E)\times_{f^*(E')} f^*(E) \arrow[r, "t"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & f^*(E)\times_Y f^*(E) \arrow[d] \\
E \arrow[r, "\Delta\eta'"'] & E\times_{E'} E \arrow[r, "s"'] & E\times_X E
\end{tikzcd}$$

completing the proof that $L'_\bullet$ is a reflective subfibration.

The final claim about L' being a modality when L is follows from the observation that, given composable maps $f\colon X\to Y$ and $g\colon Y\to Z$ in $\mathcal{E}$, we have $\Delta(gf) = p\Delta f$, where p is the leftmost vertical map in

$$\begin{tikzcd}
X\times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta g"] \\
X\times_Z X \arrow[r, "\mathrm{id}_X\times_Z f"'] & X\times_Z Y \arrow[r, "f\times_Z \mathrm{id}_Y"'] & Y\times_Z Y
\end{tikzcd}.$$

Therefore, if g is L-separated (so that Δg is L-local), p is L-local. If also f is L-separated and L is a modality, we can then conclude from $\Delta(gf) = p\Delta f$ that gf is L-separated. ■

5. Appendix: On locally cartesian closed ∞-categories

We prove here some miscellaneous facts about locally cartesian closed (lcc) ∞-categories that we need but we could not fit elsewhere. Some of these results are well-known, but others do not seem to appear or be proven in the literature.

In Section 5.1, we discuss some results about cartesian-closedness of pullback functors, and some interactions between their adjoints. In Section 5.2, we give a "term-free" version of the type-theoretic axiom known as *function extensionality*, and we prove that it holds in any lcc ∞-category. Finally, in Section 5.3, we prove a "fiberwise" criterion for extending a map along another one with the same domain.

We fix throughout an lcc ∞-category $\mathcal{C}$.

5.1. Pullback functor and its adjoints. The first set of results we need explore the behaviours of the pullback functors and of their adjoints in $\mathcal{C}$.

5.1.1. Lemma. *Let $\mathcal{C}$ be a locally cartesian closed ∞-category. Given any morphism $g\colon Y \to X$ in $\mathcal{C}$ the pullback functor $g^*\colon \mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ is cartesian closed, i.e., for every $p, q \in \mathcal{C}_{/X}$, $g^*(p^q)$ is the exponential object $g^*(p)^{g^*(q)}$ in $\mathcal{C}_{/Y}$.*

A proof of the above result for 1-categories can be found in [Joh02, Lemma A.1.5.2] and the same proof carries over to ∞-categories.

5.1.2. Lemma. *Let $\epsilon\colon gg^* \to \mathrm{id}_{\mathcal{C}_{/X}}$ be the counit of the adjunction $g \circ (-) \dashv g^*$. Given $X \in \mathcal{C}$, take $p, q \in \mathcal{C}_{/X}$. Suppose given a diagram in $\mathcal{C}$*

$$\begin{tikzcd}
A \arrow[r, "\rho"] \arrow[dr, bend right, "f"'] & W \arrow[r, "\sigma := \epsilon_{p^q}"] \arrow[d, "(g^*p)^{(g^*q)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "p^q"] \\
& Y \arrow[r, "g"'] & X
\end{tikzcd}.$$

Let $\rho^\sharp\colon f \times^Y g^*q \to g^*p$ be the adjunct to ρ in $\mathcal{C}_{/Y}$ and consider the map $\sigma\rho\colon gf \to p^q$ in $\mathcal{C}_{/X}$. Then, $g(f \times^Y g^*q) = gf \times^X q$ and the adjunct of $\sigma\rho$ is given by the composite map: $g(f \times^Y g^*q) \xrightarrow{\rho^\sharp} gg^*p \xrightarrow{\epsilon_p} p$.

Proof. The fact that $g(f \times^Y g^*q) = gf \times^X q$ is given by the pasting-lemma for pullbacks. By definition, the adjunct of $\sigma\rho$ is the composite

$$gf \times^X q \xrightarrow{\sigma\rho \times^X q} p^q \times^X q \xrightarrow{\mathrm{ev}_{p,q}} p$$

and the adjunct $\rho^\sharp$ is the composite

$$f \times^Y g^*q \xrightarrow{\rho \times^Y g^*q} (g^*p)^{(g^*q)} \times^Y g^*q \xrightarrow{\mathrm{ev}_{g^*p, g^*q}} g^*p.$$

Using that $(g^*p)^{(g^*q)} \times^Y g^*q = g^*(p^q \times^X q)$, the map ev_{g^*p,g^*q} is the map $g^*(\mathrm{ev}_{p,q})$. One then needs to show that the maps $\mathrm{ev}_{p,q}(\sigma\rho \times^X q)$ and $\epsilon_p g^*(\mathrm{ev}_{p,q})(\rho \times^Y g^*q)$ are equal. Consider the diagram below, where all squares are pullbacks

$$\begin{tikzcd}
& g^*(T\times_X Q) \arrow[rr, "\sigma'"] \arrow[dd] \arrow[dr] & & T\times_X Q \arrow[dd] \arrow[dr] & \\
(gf)^*Q \arrow[ur, "m"] \arrow[rr] \arrow[dd, "(gf)^*q"'] & & g^*Q \arrow[rr] \arrow[dd, "g^*q", near start] & & Q \arrow[dd, "q"] \\
& W \arrow[rr, "\sigma", near end] \arrow[dr] & & T \arrow[dr, "p^q"] & \\
A \arrow[ur, "\rho"] \arrow[rr, "f"'] & & Y \arrow[rr, "g"'] & & X
\end{tikzcd} .$$

Then m (as a map over Y) is $\rho \times^Y g^*q$ and $\sigma' m$ (as a map over X) is $\sigma\rho \times^X q$. The claim now follows thanks to the following commutative diagram, where the back, front and bottom faces of the cube (and, hence, also the top face) are pullbacks

$$\begin{tikzcd}
& g^*(T\times_X Q) \arrow[rr, "\sigma'"] \arrow[dd] \arrow[dr, dashed, "g^*(\mathrm{ev}_{p,q})"] & & T\times_X Q \arrow[dd] \arrow[dr, "\mathrm{ev}_{p,q}"] & \\
(gf)^*Q \arrow[ur, "\rho\times^Y g^*q"] \arrow[dd, "(gf)^*q"'] & & g^*P \arrow[rr, "\epsilon_p"] \arrow[dd, "g^*p", near start] & & P \arrow[dd, "p"] \\
& W \arrow[rr, "\sigma", near end] \arrow[dr] & & T \arrow[dr, "p^q"] & \\
A \arrow[ur, "\rho"] \arrow[rr, "f"'] & & Y \arrow[rr, "g"'] & & X
\end{tikzcd} .$$

∎

5.1.3. Lemma. [Beck–Chevalley condition] *Let $\mathcal{C}$ be a locally cartesian closed ∞-category and let*

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

be a pullback square in $\mathcal{C}$. Then there are canonical natural equivalences

$$\sum_k h^* \xrightarrow{\simeq} g^* \sum_f \quad \textit{and} \quad f^* \prod_g \xrightarrow{\simeq} \prod_h k^* \ .$$

Proof. The first map being an equivalence at every $p \in \mathcal{C}_{/C}$ is a restatement of the pasting lemma for pullbacks. The result for dependent products follows from the one for dependent sums by taking right adjoints, since adjoints compose. ∎

5.2. Function extensionality. In homotopy type theory, given types X and A, and morphisms $f, g\colon X \to A$, there is a map

$$(f =_{A^X} g) \longrightarrow \prod_{x:X} (f(x) =_A g(x))$$

evaluating a path between f and g at each $x : X$. The statement that this map is an equivalence (for all types A, X and all $f, g\colon A \to X$) is known as *function extensionality.* In our setting, function extensionality can be stated as follows.

5.2.1. Proposition. [Function Extensionality] *Let $\mathcal{C}$ be a locally cartesian closed ∞-category. Given $A, X \in \mathcal{C}$, let $\mathrm{ev}\colon A^X \times X \to A$ be the counit at A of the adjunction $(-) \times X \dashv (-)^X$ and form the pullback*

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\Delta A"] \\
A^X \times A^X \times X \arrow[r, "(\mathrm{ev}_1, \mathrm{ev}_2)"'] & A \times A
\end{tikzcd} .$$

Here ev_1 (resp. ev_2) is the composite of the projection $A^X \times A^X \times X \to A^X \times X$ onto the first (resp. second) and third components with the evaluation map. Consider the projection map $\mathrm{pr}\colon A^X \times A^X \times X \to A^X \times A^X$. Then, there is a canonical equivalence in $\mathcal{C}_{/A^X \times A^X}$

$$\Delta(A^X) \xrightarrow{\simeq} \prod_{\mathrm{pr}} q \, .$$

Proof. Let $k\colon E \to A^X \times A^X$ be an object in $\mathcal{C}_{/A^X \times A^X}$. By adjointness, there is a natural equivalence

$$\mathcal{C}_{/A^X \times A^X}\left(k, \prod_{\mathrm{pr}} q\right) \simeq \mathcal{C}_{/A^X \times A^X \times X}(k \times X, q).$$

By the description of hom-spaces in ∞-slice categories (see [Lur09, Lemma 5.5.5.12]) and since Q is a pullback, we get a homotopy pullback square of ∞-groupoids

$$\begin{tikzcd}
\mathcal{C}_{/A^X \times A^X \times X}(k \times X, q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(E \times X, A) \arrow[d, "\mathcal{C}(E \times X, \Delta A)"] \\
* \arrow[r, "(\mathrm{ev}_1, \mathrm{ev}_2) \circ (k \times X)"'] & \mathcal{C}(E \times X, A \times A)
\end{tikzcd} .$$

But $\mathcal{C}(E, \Delta(A^X)) \simeq \mathcal{C}(E \times X, \Delta A) \simeq \Delta_{\mathcal{C}(E \times X, A)}$, which means that

$$\mathcal{C}_{/A^X \times A^X \times X}(k \times X, q) \simeq \mathrm{hofib}_k(\mathcal{C}(E, \Delta(A^X))) \simeq \mathcal{C}_{/A^X \times A^X}(k, \Delta(A^X)),$$

where the last equivalence is again [Lur09, Lemma 5.5.5.12]. We then get the needed composite natural equivalence

$$\mathcal{C}_{/A^X \times A^X}\left(k, \prod_{\mathrm{pr}} q\right) \simeq \mathcal{C}_{/A^X \times A^X}(k, \Delta(A^X)).$$

∎

Proposition 5.2.1 can be promoted to a result about diagonals of dependent products. We now set up what we need to state this generalization of Proposition 5.2.1.

Let $p\colon E \to X$ be a map in $\mathcal{C}$ and let

$$\begin{tikzcd} (\prod_X p) \times X \arrow[rr, "\epsilon"] \arrow[dr, "\pi"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

be the component of the counit of the adjunction $(-) \times X \dashv \prod_X$ at $p \in \mathcal{C}_{/X}$. Here π is the projection map onto X. The projection map

$$\left(\prod_X p\right) \times \left(\prod_X p\right) \times X \to X$$

is the product object $\pi \times^X \pi$ in $\mathcal{C}_{/X}$. Hence, the product map $\epsilon \times^X \epsilon\colon \pi \times^X \pi \to p \times^X p$ in $\mathcal{C}_{/X}$ is the map over X given by

$$(\epsilon_1, \epsilon_2)\colon \left(\prod_X p\right) \times \left(\prod_X p\right) \times X \to E \times_X E,$$

where ϵ_1 (resp. ϵ_2) is the composite of the projection

$$\left(\prod_X p\right) \times \left(\prod_X p\right) \times X \to \left(\prod_X p\right) \times X$$

onto the first (resp. the second) and third components with the counit map. The pullback of Δp along $\epsilon \times^X \epsilon$ in $\mathcal{C}_{/X}$ can be described as the pullback square

$$\begin{tikzcd} Q' \arrow[r] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\Delta p"] \\ (\prod_X p) \times (\prod_X p) \times X \arrow[r, "(\epsilon_1,\epsilon_2)"'] & E \times_X E \end{tikzcd} \tag{5}$$

in $\mathcal{C}$ and Q' can be naturally regarded as an object over X.

5.2.2. Proposition. [Dependent Function Extensionality] *Let $\mathcal{C}$ be a locally cartesian closed ∞-category and let $p\colon E \to X$ be a map in $\mathcal{C}$. Construct q' as in (5) and let*

$$\mathrm{pr}\colon \left(\prod_X p\right) \times \left(\prod_X p\right) \times X \to \left(\prod_X p\right) \times \left(\prod_X p\right)$$

be the projection map. Then there is a canonical equivalence in $\mathcal{C}_{/(\prod_X p)\times(\prod_X p)}$

$$\Delta\left(\prod_X p\right) \xrightarrow{\simeq} \prod_{\mathrm{pr}} q' .$$

Mutatis mutandis, the proof is the same as for Proposition 5.2.1, so we omit it.

5.2.3. Remark. If $\mathcal{C}$ is a locally cartesian closed ∞-category, then so is $\mathcal{C}_{/X}$ for any $X \in \mathcal{C}$. Thus, Proposition 5.2.1 and Proposition 5.2.2 hold true also in $\mathcal{C}_{/X}$ and give, for maps $p\colon E \to X$, $f\colon Y \to X$ and $q\colon M \to Y$ in $\mathcal{C}$, an alternative description of the diagonal of $p^f \in \mathcal{C}_{/X}$ and of $\Delta\left(\prod_f q\right)$ as a map in $\mathcal{C}_{/Y}$.

5.3. Contractibility. We provide here a criterion for the existence and the uniqueness of extensions of one map along another one with the same domain. This result is linked to the notion of contractibility in $\mathcal{C}$.

Recall that an object $A \in \mathcal{C}$ is *contractible* if the map $A \to 1$ is an equivalence. When we apply this definition to an object $p \in \mathcal{C}_{/X}$, this means that p is contractible in $\mathcal{C}_{/X}$ exactly when, seen as a map in $\mathcal{C}$, it is an equivalence. Since equivalences in an ∞-topos form a local class of maps, we immediately get the following result.

5.3.1. Lemma. *Let $\mathcal{E}$ be an ∞-topos and let $f\colon Y \to X$ be an effective epimorphism in $\mathcal{E}$. For any $p \in \mathcal{E}_{/X}$, $f^*(p) \in \mathcal{E}_{/Y}$ is contractible if and only if p is.*

The following lemma is a standard exercise in 2-category theory since the notions of slice ∞-categories and of adjunctions between ∞-categories can be completely characterized in the 2-category of ∞-categories — see [RV18, §3 and 4].

5.3.2. Lemma. *Let $\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$ be an adjunction and let $D \in \mathcal{D}$. Then there is an induced adjunction on slice categories*

$$\mathcal{C}_{/GD} \underset{\bar{G}}{\overset{\bar{F}}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}_{/D}$$

where, for $p \in \mathcal{C}_{/GD}$ and $q \in \mathcal{D}_{/D}$, $\bar{F}(p) = \epsilon_D Fp$ and $\bar{G}(q) = Gq$.

5.3.3. Lemma. *Let $p\colon D \to B \times C$ be a map in a locally cartesian closed ∞-category $\mathcal{C}$. Consider the map $q\colon E \to B \times C^B$ given by the pullback square*

$$\begin{tikzcd} E \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "p"] \\ B \times C^B \arrow[r, "(\mathrm{pr}_1, \mathrm{ev})"'] & B \times C \end{tikzcd}.$$

Then there is an equivalence

$$\left(\prod_B \sum_{B\times C \to B} p\right) \simeq \left(\sum_{C^B} \prod_{B \times C^B \to C^B} q\right).$$

Proof. Let $\mathrm{pr}_B\colon B \times C \to B$ and $\mathrm{pr}_{C^B}\colon B \times C^B \to C^B$ be the projection maps. Note that $\Pi_{\mathrm{pr}_{C^B}} q$ is, by definition, a map $\Pi_{\mathrm{pr}_{C^B}} q\colon \Sigma_{C^B} \Pi_{\mathrm{pr}_{C^B}} q \longrightarrow C^B$. On the other hand, we can see p as a map $p\colon \Sigma_{\mathrm{pr}_B} p \to \mathrm{pr}_B$ in $\mathcal{C}_{/B}$. Setting $\alpha := \Pi_B p$, we then get a map

$$\alpha\colon \prod_B \sum_{\mathrm{pr}_B} p \longrightarrow \prod_B \mathrm{pr}_B = C^B.$$

It is therefore sufficient to show that $\alpha \simeq \Pi_{\mathrm{pr}_{C^B}} q$ in $\mathcal{C}_{/C^B}$. Let $k\colon Z \to C^B$ be an object in $\mathcal{C}_{/C^B}$. Using Lemma 5.3.2 applied to the adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "B\times(-)"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{/B} \arrow[l, shift left, "\Pi_B"] \end{tikzcd}$$

we get $\mathcal{C}_{/C^B}(k,\alpha) \simeq \left(\mathcal{C}_{/B}\right)_{/\,\mathrm{pr}_B}\left(\kappa^\sharp, p\right)$. Here $\kappa^\sharp$ is the composite $(\mathrm{pr}_1, \mathrm{ev})(B\times k)$, seen as a map from $(B \times Z \xrightarrow{\mathrm{pr}_1} B)$ to pr_B, and thus as an object in $\left(\mathcal{C}_{/B}\right)_{/\,\mathrm{pr}_B}$. Since $\left(\mathcal{C}_{/B}\right)_{/\,\mathrm{pr}_B} \simeq \mathcal{C}_{/B\times C}$ and using the definition of $q = (\mathrm{pr}_1, \mathrm{ev})^* p$, we obtain

$$\mathcal{C}_{/C^B}(k,\alpha) \simeq \mathcal{C}_{/B\times C}\left(\kappa^\sharp, p\right) = \mathcal{C}_{/B\times C}\left((\mathrm{pr}_1, \mathrm{ev})(B\times k), p\right) \simeq$$

$$\simeq \mathcal{C}_{/B\times C^B}(B\times k, q) = \mathcal{C}_{/B\times C^B}\left((\mathrm{pr}_{C^B})^* k, q\right) \simeq \mathcal{C}_{/C^B}\left(k, \prod_{\mathrm{pr}_{C^B}} q\right),$$

whence $\alpha \simeq \Pi_{\mathrm{pr}_{C^B}} q$, as needed. ∎

Intuitively, the following result is about the existence of a unique extension of a map f along another map g in terms of unique extensions along the fibers of g. Taking fibers out of the picture, we get the following odd-looking statement.

5.3.4. Proposition. [cf. [CORS18, Lemma 2.23]] *Let $f\colon A \to C$ and $g\colon A \to B$ be two maps in a locally cartesian closed ∞-category $\mathcal{C}$. Form the following pullback squares in $\mathcal{C}$:*

$$\begin{tikzcd} A\times C \arrow[r] \arrow[d, "(\mathrm{pr}_A, g\times C)"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\Delta B"] \\ A\times B\times C \arrow[r, "g\times \mathrm{pr}_B"'] & B\times B \end{tikzcd} \qquad \begin{tikzcd} B\times A \arrow[r] \arrow[d, "(\mathrm{pr}_A, B\times f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\Delta C"] \\ A\times B\times C \arrow[r, "f\times \mathrm{pr}_C"'] & C\times C \end{tikzcd}.$$

Consider the following object in $\mathcal{C}_{/B}$

$$E := \sum_{B\times C\to B}\left(\prod_{A\times B\times C\to B\times C} (\mathrm{pr}_A, B\times f)^{(\mathrm{pr}_A, g\times C)}\right)$$

where the displayed internal hom is taken in $\mathcal{C}_{/A\times B\times C}$. Then the following hold.

(i) If we let $f\colon C^B \to C^A$ be the composite $C^B \to 1 \xrightarrow{f} C^A$, there is an equivalence

$$\prod_B E \simeq \sum_{C^B} (f, C^g)^*\left(\Delta(C^A)\right). \tag{6}$$

(ii) The space of global elements of the right-hand side in (6) is equivalent to the space $\mathrm{Ext}(f,g)$ of extensions of f along g. In particular, if $\prod_B E$ is contractible in $\mathcal{C}_{/B}$, then there is a unique dotted extension in

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & C \\ B \arrow[ur, dashed] \end{tikzcd}.$$

Proof. We start by proving the first claim. We have

$$(\mathrm{pr}_A, B\times f)^{(\mathrm{pr}_A, g\times C)} = \prod_{(\mathrm{pr}_A, g\times C)} (\mathrm{pr}_A, g\times C)^*(\mathrm{pr}_A, B\times f).$$

Since $(\mathrm{pr}_A, B\times f) = (f\times \mathrm{pr}_C)^*(\Delta C)$ and $(f\times \mathrm{pr}_C)(\mathrm{pr}_A, g\times C) = f\times C$, we get that $(\mathrm{pr}_A, g\times C)^*(\mathrm{pr}_A, B\times f) = (\mathrm{id}_A, f)\colon A\to A\times C$. If $\mathrm{pr}_{B\times C}\colon A\times B\times C\to B\times C$ is the projection map, we then have

$$\prod_{\mathrm{pr}_{B\times C}} (\mathrm{pr}_A, B\times f)^{(\mathrm{pr}_A, g\times C)} = \prod_{\mathrm{pr}_{B\times C}} \left(\prod_{(\mathrm{pr}_A, g\times C)} (\mathrm{id}_A, f)\right) = \prod_{g\times C} (\mathrm{id}_A, f)$$

Using Lemma 5.3.3, we then get

$$\prod_B E = \prod_B \sum_{B\times C\to B} \prod_{g\times C} (\mathrm{id}_A, f) \simeq \sum_{C^B} \prod_{\mathrm{pr}_{C^B}} (\mathrm{pr}_1, \mathrm{ev})^* \left(\prod_{g\times C} (\mathrm{id}_A, f)\right) =: E'$$

where $\mathrm{pr}_{C^B}\colon B\times C^B\to C^B$ is the projection map. There are pullback squares

$$\begin{tikzcd}
A\times C^B \arrow[r, "{(\mathrm{id}_A,\ \mathrm{ev}(g\times C^B))}"] \arrow[d, "g\times C^B"'] \arrow[dr, phantom, "\lrcorner", very near start] & A\times C \arrow[d, "g\times C"] \\
B\times C^B \arrow[r, "{(\mathrm{pr}_1, \mathrm{ev})}"'] & B\times C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "{(\mathrm{id}_A, f)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\Delta C"] \\
A\times C \arrow[r, "f\times C"'] & C\times C
\end{tikzcd}.$$

Thus, using the Beck–Chevalley condition, we get

$$\begin{aligned}
E' &\simeq \sum_{C^B} \prod_{\mathrm{pr}_{C^B}} \prod_{g\times C^B} \left((f\times C)(\mathrm{id}_A,\ \mathrm{ev}(g\times C^B))\right)^* (\Delta C) \simeq \\
&\simeq \sum_{C^B} \prod_{A\times C^B\to C^B} \left((f\times C)(\mathrm{id}_A,\ \mathrm{ev}(g\times C^B))\right)^* (\Delta C) \simeq \\
&\simeq \sum_{C^B} \prod_{A\times C^B\to C^B} (\mathrm{ev}(A\times (f, C^g)))^* (\Delta C) =: E''
\end{aligned}$$

where the last equivalence is due to the fact that $(f\times C)(\mathrm{id}_A,\ \mathrm{ev}(g\times C^B))$ is equal to the composite map $\mathrm{ev}\circ(A\times(f, C^g))$. Using the Beck–Chevalley condition applied to the pullback square

$$\begin{tikzcd}
A\times C^B \arrow[r, "{A\times(f,C^g)}"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A\times C^A\times C^A \arrow[d, "\mathrm{pr}_2"] \\
C^B \arrow[r, "{(f,C^g)}"'] & C^A\times C^A
\end{tikzcd}$$

we further deduce that

$$E'' = \sum_{C^B} \prod_{A\times C^B\to C^B} (A\times (f, C^g))^*(\mathrm{ev}^*(\Delta C)) \simeq$$

$$\simeq \sum_{C^B}(f, C^g)^* \left(\prod_{\mathrm{pr}_2} \mathrm{ev}^*(\Delta C)\right) \simeq \sum_{C^B}(f, C^g)^*(\Delta(C^A))$$

where the last equivalence is given by Function Extensionality.

For the second part, $P := \sum_{C^B}(f, C^g)^*(\Delta(C^A))$ is the pullback object of C^g along $f\colon 1 \to C^A$ and thus $\mathcal{C}(1, P)$ is the homotopy fiber of $\mathcal{C}(1, C^g)$ at $f \in \mathcal{C}(1, C^A)$. The latter homotopy fiber gives the needed space of extensions. ■

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