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SMALL CATEGORIES OF HOMOLOGICAL DIMENSION ONE

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Abstract. We derive three equivalent necessary conditions for a small category to have homological dimension one, generalizing a result of Novikov. As a consequence, any small cancellative category of homological dimension one is embeddable in a groupoid.

Introduction

Throughout $\mathbb{C}$ will be a small category and R a ring with identity. We will denote the category of left R-modules by $\mathcal{M}$ and the category of covariant functors $\mathbb{C} \to \mathcal{M}$ by $\mathcal{M}^{\mathbb{C}}$. The additive category (or ringoid) $R\mathbb{C}$ has objects those of $\mathbb{C}$ and morphisms from p to q the free module on $\mathbb{C}(p,q)$ such that the composition is bilinear. It is not hard to see that the category $\mathbf{Ab}^{\mathbf{R}\mathbb{C}}$ of additive functors (or $R\mathbb{C}$-modules) $R\mathbb{C} \to \mathbf{Ab}$ is isomorphic to $\mathcal{M}^{\mathbb{C}}$. If $M \in \mathbf{Ab}^{\mathbf{R}\mathbb{C}}$, $\alpha \in R\mathbb{C}(p,q)$ and $x \in M(p)$ then we write αx for $M(\alpha)(x)$.

Let $\mathbb{N}$ denote the set of non-negative integers. The *R-homological dimension* of $\mathbb{C}$ is defined by

$$\mathrm{hd}_R\mathbb{C} = \sup\{k : \mathrm{colim}^k \neq 0\}$$

where colim^k is the kth left derived functor of the colimit functor $\mathrm{colim} : \mathcal{M}^{\mathbb{C}^{op}} \to \mathcal{M}$ and the supremum is taken in the set $\{-1\} \cup \mathbb{N} \cup \{\infty\}$. Since

$$\mathrm{colim}\ M \cong M \bigotimes_{R\mathbb{C}} \Delta R$$

where $\Delta R : \mathbb{C} \to \mathcal{M}$ is the constant R-valued functor, and the isomorphism is natural in $M \in \mathcal{M}^{\mathbb{C}^{op}}$, we have

$$\mathrm{hd}_R\mathbb{C} = \mathrm{fd}\,\Delta\mathrm{R}$$

where fd denotes the flat, or weak dimension. In case $R = \mathbb{Z}$, we shall write $\mathrm{hd}\,\mathbb{C}$ for $\mathrm{hd}_R\mathbb{C}$ and refer to it as the *homological dimension* of $\mathbb{C}$.

Isbell and Mitchell [6, Remark, P. 296] have described all small categories of R-homological dimension zero if R is commutative. (Actually, their description is true even if R is not commutative.) For any ring R, there exist descriptions of R-homological dimension one in case $\mathbb{C}$ is a poset [2], or an abelian monoid [3].

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In this paper, we obtain the following three equivalent necessary Conditions (I), (II) and (III) for a small category $\mathbb{C}$ to have R-homological dimension at most one.

Condition (I) The augmentation module $\mathrm{I}\mathbb{C}$ is full,

Condition (II) All n-crowns, $n > 1$, are supported,

Condition (III) $\mathbb{C}$ is strongly L_∞.

All of these will be defined below. It is worth noting that Condition (I) is a statement in homological algebra, Condition (II) in category theory and Condition (III) in semigroup theory. When $\mathbb{C}$ is cancellative Condition (III) implies that $\mathbb{C}$ is embeddable in a groupoid.

In Section 1 we prove that $\mathrm{hd_R}\mathbb{C} \leq 1$ implies condition (I) in Theorem A. In Section 2, we prove that conditions (I) and (II) are equivalent in Theorem B and in Section 3, we prove that conditions (II) and (III) are equivalent in Theorem C. When $\mathbb{C}$ is cancellative with one object, these Theorems give a slightly more general result than that of Novikov [9]. In Section 4, we prove that if $\mathbb{C}$ is cancellative and $\mathrm{hd_R}\mathbb{C} \leq 1$ then $\mathbb{C}$ is embeddable in a groupoid.

The results in this paper are extracted from the dissertation of K. Sweet [11].

1. $\mathrm{hd_R}\mathbb{C} \leq 1 \implies$ Condition (I)

Let $M \in \mathcal{M}^{\mathbb{C}}$. If $\beta_j, j = 1, \ldots, n$, are morphisms of $\mathbb{C}$ with a common codomain q, it is easy to see that

$$\sum \mathrm{Im\,M}(\alpha) \subseteq \mathrm{Im\,M}(\beta_1) \cap \sum_{\mathrm{j=2}}^{\mathrm{n}} \mathrm{Im\,M}(\beta_\mathrm{j}) \tag{1}$$

where the first sum is indexed by all morphisms α in $\mathbb{C}$ factored through β_1 and β_j for some $j > 1$. If the above is always an equality then we say that M is *full*.

Consider the exact sequence

$$0 \longrightarrow \mathrm{I}\mathbb{C} \longrightarrow E = \bigoplus_{p \in |\mathbb{C}|} R\mathbb{C}(p, \quad) \xrightarrow{\ \varepsilon\ } \Delta R \longrightarrow 0 \tag{2}$$

in $\mathcal{M}^{\mathbb{C}}$. Here ΔR is the constant R-valued functor and ε is defined by $\varepsilon_q(x) = \sum_\alpha r_\alpha$ for $x = \sum_\alpha r_\alpha \alpha$ where $r_\alpha \in R$, and α are morphisms in $\mathbb{C}$ with $\mathrm{cod}\alpha = \mathrm{q}$. Since $E(q)$ is the free R-module on all morphisms with codomain q, $\mathrm{I}\mathbb{C}(q)$ consists of elements $\sum r_\alpha \alpha$ where $r_\alpha \in R, \alpha$ is a morphism in $\mathbb{C}$ with $\mathrm{cod}\alpha = \mathrm{q}$, and $\sum r_\alpha = 0$. From (2) we see that $\mathrm{I}\mathbb{C}$ is flat if and only if $\mathrm{hd}_R\mathbb{C} \leq 1$.

The following is a generalization of the well-known Cohn's Criteria for flatness [10, Theorem 3.2(4)] specialized to $R\mathbb{C}$-modules.

1.1. Proposition. *A left $R\mathbb{C}$-module M is flat if and only if, whenever $\sum_j \beta_j x_j = 0$, where $\beta_j \in R\mathbb{C}(p_j, q)$ and $x_j \in M(p_j)$ for $j = 1, \ldots, n$, there exist objects q_i in $\mathbb{C}$,*

$w_i \in M(q_i)$, and $\alpha_{ij} \in R\mathbb{C}(q_i, p_j)$, for all $i = 1, \ldots, m$ and all j, such that

$$x_j = \sum_i \alpha_{ij} w_i \quad \textit{for all } j$$

and

$$\sum_j \beta_j \alpha_{ij} = 0 \quad \textit{for all } i.$$

1.2. Proposition. *If $M \in \mathcal{M}^{\mathbb{C}}$ is flat then it is full.*

Proof. Let x be a nonzero element of the right side of (1). Then $x = \beta_1 x_1 = \sum_{j=2}^{n} \beta_j x_j$, where $x_j \in M(p_j)$ and $\beta_j \in \mathbb{C}(p_j, q)$ for $j = 1, \ldots, n$. So

$$(-\beta_1)x_1 + \sum_{j=2}^{n} \beta_j x_j = 0.$$

By Proposition 1.1, there exist objects q_i in $\mathbb{C}$, $\alpha_{ij} \in R\mathbb{C}(q_i, p_j)$ and elements $w_i \in M(q_i)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ such that

$$x_j = \sum_{i=1}^{m} \alpha_{ij} w_i, \text{ for all } j \tag{3}$$

and

$$(-\beta_1)\alpha_{i1} + \sum_{j=2}^{n} \beta_j \alpha_{ij} = 0 \text{ for all } i. \tag{4}$$

By (3) $x = \beta_1 x_1 = \sum_{i=1}^{m} \beta_1 \alpha_{i1} w_i$. Since x is nonzero, $\beta_1 \alpha_{i1} \neq 0$ for at least one i. For each such i, $\alpha_{i1} = \sum_k r_{ik} \gamma_{ik}$, where $\gamma_{ik} \in \mathbb{C}(q_i, p_1)$ and $r_{ik} \in R$ for all k. Hence, for each such i, we have, by (4),

$$\sum_{j=2}^{n} \beta_j \alpha_{ij} = \beta_1 \alpha_{i1} = \sum_k \beta_1 (r_{ik} \gamma_{ik}) = \sum_k r_{ik} (\beta_1 \gamma_{ik}).$$

Since the above element is nonzero, without loss of generality, we may assume that each $r_{ik} \neq 0$ and $\beta_1 \gamma_{ik}$ are pairwise distinct in $\mathbb{C}$. Thus each $\beta_1 \gamma_{ik}$ factors through β_j for some $j \geq 2$. Therefore

$$x = \beta_1 x_1 = \sum_i \beta_1 \alpha_{i1} w_i = \sum_i \sum_k \beta_1 (r_{ik} \gamma_{ik}) w_i = \sum_i \sum_k (\beta_1 \gamma_{ik})(r_{ik} w_i)$$

which is contained in the left side of (1). ∎

The following theorem states that condition (I) is a necessary condition for $\mathrm{hd_R}\mathbb{C} \leq 1$.

1.3. Theorem. [Theorem A] *If* $\mathrm{hd_R}\mathbb{C} \leq 1$ *then* $\mathrm{I}\mathbb{C}$ *is full.*

Proof. Since $\mathrm{I}\mathbb{C}$ is flat, it is full by the above Proposition. ∎

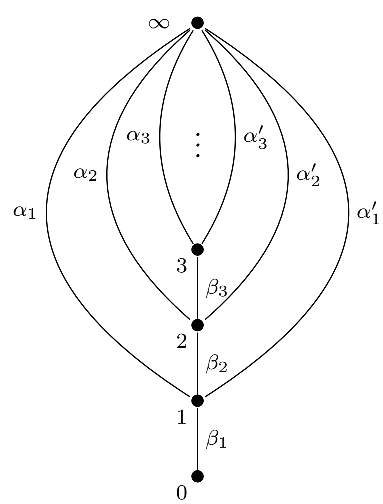
The *R-cohomological dimension* of $\mathbb{C}$ can be defined by

$$\mathrm{cd}_R\mathbb{C} = \mathrm{pd}\,\Delta R.$$

Hence if $\mathrm{cd_R}\mathbb{C} \leq 1$ then $\mathrm{I}\mathbb{C}$ is projective and, therefore, is flat implying that $\mathrm{hd_R}\mathbb{C} \leq 1$. Thus we have deduced from Theorem A the following.

1.4. Corollary. *If* $\mathrm{cd_R}\mathbb{C} \leq 1$ *then* $\mathrm{I}\mathbb{C}$ *is full.*

1.5. Remark. *When* $\mathbb{C}$ *is a poset, the Converse of Theorem A is true by Theorem B (below) and [2, Corollary 11]. However, it is not true in general. Consider the following graph* $\mathbb{G}$ *(all arrows downward) where the vertices are* $0, 1, \ldots, \infty$ *and arrows from* ∞ *to* i *are* α_i *and* α'_i*, from* i *to* $i-1$ *is* β_i *for any positive integers* i*.*



Let $\mathbb{G}_0$ *be the free category on* $\mathbb{G}$ *and let* $\sim$ *be the least congruence relation on* $\mathbb{G}_0$ *containing* $\{(\beta_1\alpha_1, \beta_1\alpha'_1)\} \cup \{(\beta_{i+1}\alpha_{i+1}, \alpha_i), (\beta_{i+1}\alpha'_{i+1}, \alpha'_i) \,|\, i \geq 1 \in \mathbb{N}\}$ *. Define* $\mathbb{C} = \mathbb{G}_0/\sim$*. We will show that* $\mathrm{I}\mathbb{C}$ *is full, but* $\mathrm{hd_R}\mathbb{C} \geq 2$*.*

Suppose

$$x \in \mathrm{Im}\ \mathrm{I}\mathbb{C}(\gamma_1) \cap \sum_{i=2}^{n} \mathrm{Im}\ \mathrm{I}\mathbb{C}(\gamma_i),$$

where $x \neq 0$ and $\mathrm{cod}\gamma_\mathrm{i} = \mathrm{q}$. If $q = \infty$ then $\gamma_i = 1$ for all i. So certainly $x \in \sum_\alpha \mathrm{Im}\ \mathrm{I}\mathbb{C}(\alpha)$ where $\mathrm{cod}\alpha = \mathrm{q}$. Assume $q \neq \infty$ and let $\gamma_i \in \mathbb{C}(p_i, j)$. Since $\mathrm{I}\mathbb{C}(\infty) = 0$, we may assume each p_i is a positive integer greater than or equal to j. Thus,

$$\gamma_i = \beta_{j+1} \cdots \beta_{p_i},$$

for each i. Let S be the set of morphisms in $\mathbb{C}$ that factor through γ_1 and γ_i for some $i > 1$. If $p_1 \geq p_i$ for some $i > 1$, then γ_1 factors through γ_1 and γ_i, and so

$$x \in \operatorname{Im} \mathrm{I}\mathbb{C}(\gamma_1) \subseteq \sum_{y \in S} \operatorname{Im} \mathrm{I}\mathbb{C}(y).$$

If $p_1 < p_i$ for all $i > 1$, then each γ_i factors through γ_1, so

$$x \in \sum_{i=2}^{n} \operatorname{Im} \mathrm{I}\mathbb{C}(\gamma_i) \subseteq \sum_{y \in S} \operatorname{Im} \mathrm{I}\mathbb{C}(y).$$

Next we will show that $\mathrm{hd}_{\mathrm{R}}\mathbb{C} > 1$. Suppose not. Then $\mathrm{I}\mathbb{C}$ is flat. Since $\beta_1\alpha_1 = \beta_1\alpha_1'$,

$$\beta_1(\alpha_1 - \alpha_1') = 0.$$

By Proposition 1.1,

$$\alpha_1 - \alpha_1' = \sum_i \gamma_i w_i,$$

for some $w_i \in \mathrm{I}\mathbb{C}(q_i)$, and $\gamma_i \in R\mathbb{C}(q_i, 1)$, $1 \leq i \leq m$, and

$$\beta_1\gamma_i = 0 \quad \text{for all } i. \tag{5}$$

Since $\mathrm{I}\mathbb{C}(\infty) = 0$, we may assume each q_i is a positive integer, so $\gamma_i = r_i\beta_2 \cdots \beta_{q_i}$ for some $r_i \in R$. By (5), we have $r_i = 0$, and consequently $\gamma_i = 0$, for all i, contradicting the fact that $\alpha_1 \neq \alpha_1'$. Therefore, $\mathrm{hd}_R\mathbb{C} \geq 2$.

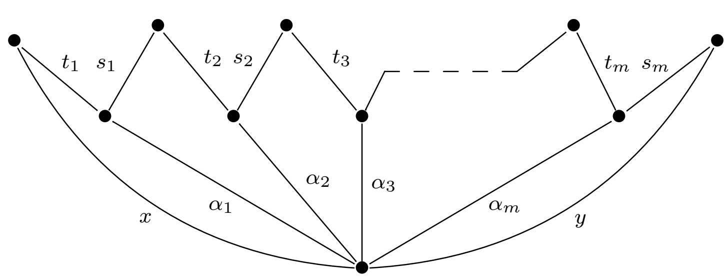
1.6. Remark. *It should be pointed out that the description of* $\mathbb{C}$ *with* $\mathrm{hd}_{\mathrm{R}}\mathbb{C} = 0$ *in [6] is true for all noncommutative rings* R *as well.*

2. Condition (I) $\iff$ Condition (II)

2.1. Lemma. *Suppose* A *is a set of morphisms of* $\mathbb{C}$ *with codomain* q *and suppose* x, y *are distinct morphisms in* A. *Then*

$$x - y \in \sum_{\alpha \in A} \operatorname{Im} \mathrm{I}\mathbb{C}(\alpha)$$

if and only if there exists a commutative diagram



(6)

where $\alpha_i \in A$.

Proof. ($\Leftarrow$) Since $(x-\alpha_1 t_1)+(\alpha_1 s_1-\alpha_2 t_2)+\cdots+(\alpha_m s_m - y)=0$, we have

$$\begin{aligned} x-y &= (\alpha_1 t_1-\alpha_1 s_1)+(\alpha_2 t_2-\alpha_2 s_2)+\cdots+(\alpha_m t_m-\alpha_m s_m) \\ &= \alpha_1(t_1-s_1)+\alpha_2(t_2-s_2)+\cdots+\alpha_m(t_m-s_m) \end{aligned}$$

and the result follows.

($\Rightarrow$) By assumption, $x-y=\sum_i \alpha_i x_i$ where $\alpha_i \in A$ and $x_i \in I\mathbb{C}(\mathrm{dom}\alpha_\mathrm{i})$ for each $i \in I$. Hence $x_i = \sum_j r_{ij}\beta_{ij}$ for some $r_{ij} \in R$, and morphisms β_{ij} in $\mathbb{C}$ with $\mathrm{cod}\beta_\mathrm{ij} = \mathrm{dom}\alpha_\mathrm{i}$, where $\sum_j r_{ij} = 0$. Therefore

$$x-y=\sum_i \alpha_i \sum_j r_{ij}\beta_{ij} = \sum_i \sum_j r_{ij}\alpha_i\beta_{ij}. \tag{7}$$

For each i, let B_i be the set of all morphisms β_{ij}. We say a morphism z in $\mathbb{C}$ is *reachable* from α_k via β if there exists a sequence of equalities

$$\begin{aligned} z &= \alpha_{i_1}c_{i_1} \\ \alpha_{i_1}b_{i_1} &= \alpha_{i_2}c_{i_2} \\ &\cdots \\ \alpha_{i_m}b_{i_m} &= \alpha_k\beta \end{aligned} \tag{8}$$

for some $m \geq 0$ where $b_i, c_i \in B_i$ and $\beta \in B_k$. By adding the equality $\alpha_k\beta_{kj} = \alpha_k\beta_{kj}$ to (8), we see that z is reachable from α_k via β if and only if z is reachable from α_k via any $\beta_{kj} \in B_k$. We say that z is *reachable* from α_k if it is reachable from α_k via some $\beta \in B_k$.

Let I_1 be the set of all i such that x is reachable from α_i and I_2 those i for which y is reachable from α_i. Since $x \neq y$, x and y must appear in the right side of (7), so I_1 and I_2 are both nonempty.

If $I_1 \cap I_2 \neq \emptyset$, then both x and y are reachable from some α_k and we have (6).

Suppose $I_1 \cap I_2 = \emptyset$. Let $I_3 = I \setminus (I_1 \cup I_2)$, where I is the set of all i. Then I_1, I_2, I_3 are pairwise disjoint. Let $S_k = \{\alpha_i\beta_i \,|\, i \in I_k, \beta_i \in B_i\}$ for $k=1,2,3$. Suppose $\alpha_i\beta_i = \alpha_j\beta_j, \beta_i \in B_i, \beta_j \in B_j$. If $i \in I_1$ then $j \in I_1$. Hence $S_1 \cap S_2 = \phi$ and $S_1 \cap S_3 = \phi$. Similarly $S_2 \cap S_3 = \phi$. Thus S_1, S_2, S_3 are pairwise disjoint. Since I_1, I_2, I_3 are pairwise disjoint, we obtain from (7)

$$x-y=\sum_{i\in I_1}\sum_j r_{ij}\alpha_i\beta_{ij} + \sum_{i\in I_2}\sum_j r_{ij}\alpha_i\beta_{ij} + \sum_{i\in I_3}\sum_j r_{ij}\alpha_i\beta_{ij}. \tag{9}$$

Since both sides are elements of $E(q)$, a free R-module, and S_1, S_2, S_3 are disjoint, using the fact that $x \in S_1$ and $y \in S_2$, we have

$$\sum_{i\in I_1}\sum_j r_{ij}\alpha_i\beta_{ij} = x.$$

So $\sum_{i\in I_1}\sum_j r_{ij}=1$. But

$$\sum_{i\in I_1}\sum_j r_{ij}=\sum_{i\in I_1}(\sum_j r_{ij})=0,$$

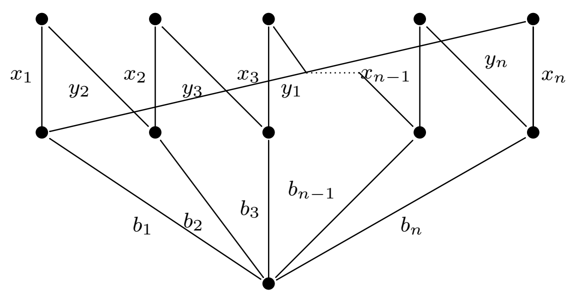
a contradiction. ∎

An *n-crown* C_n, $n\geq 1$, in $\mathbb{C}$ is a subcategory of $\mathbb{C}$ generated by morphisms b_i, x_i and $y_i, i=1,\dots,n$, satisfying the following

$$\begin{aligned} b_1x_1 &= b_2y_2\\ b_2x_2 &= b_3y_3\\ &\dots\\ b_nx_n &= b_1y_1. \end{aligned} \tag{10}$$

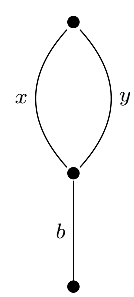
We sometimes will follow semigroup theory and call it a *cyclic system* in $\mathbb{C}$. It gives the following commutative diagram in $\mathbb{C}$. (Here and elsewhere all arrows are downwards.)

(11)



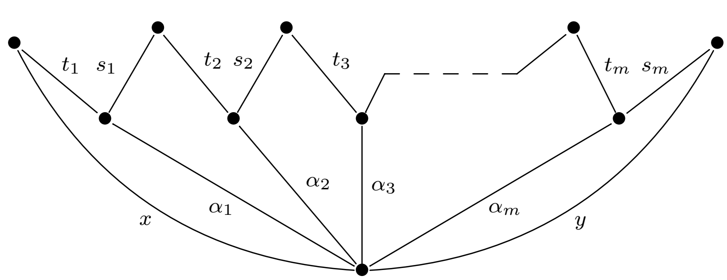
Therefore a 1-crown is simply a cyclic system $bx=by$, or a diagram

(12)



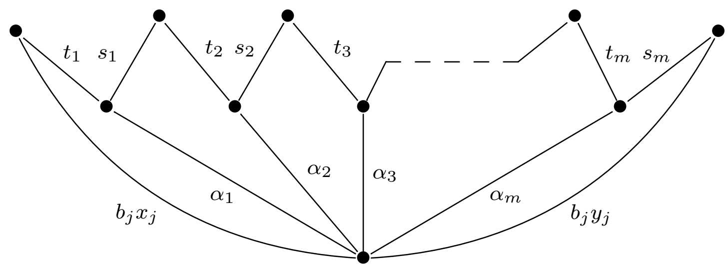
It is *supported* if there is a commutative diagram

(13)



where if $m = 0$ then $x = y$.

If $n \geq 2$ we say that C_n is *supported between* x_j *and* y_j if there is a commutative diagram



(14)

in $\mathbb{C}$ where $m \geq 0$ and each α_i factors through b_j and b_k for some $k \neq j$. (When $m = 0$, $b_j x_j = b_j y_j$.) If $n \geq 2$ we say that C_n is *supported* if C_n is supported between x_j and y_j for all $j = 1, \ldots, n$.

Suppose F is a free R-module with basis B. So every nonzero element x of F is of the unique form $\sum r_b b, r_b \in R, b \in B$. If $r_b \neq 0$ then b is said to be a *term* of x. If $y \in F$ and every term of y is a term of x then y is a *subsum* of x. Consider the exact sequence

$$0 \to K \to F \xrightarrow{\epsilon} R \to 0$$

where $\epsilon(\sum r_b b) = \sum r_b$. So every element of K is of form $\sum r_b b$ where $\sum r_b = 0$.

2.2. Lemma. [Lemma 5, [2]] *If* $\sum_{i=1}^{n} x_i = 0$ *where each* $x_i \in K$ *is nonzero then, for each* i, *there exists a subsum* y_i *of* x_i, *not all zero, such that* $\sum_{i=1}^{n} y_i = 0$. *Moreover, after rearranging subscripts, there exists an integer* k, $2 \leq k \leq n$ *such that*

$$y_i = \begin{cases} b_i - b_{i+1}, & 1 \leq i \leq k \\ 0, & i > k \end{cases} \tag{15}$$

where each $b_i \in B$ *and* $b_{k+1} = b_1$.

Proof. Since $x_1 \neq 0$, there exist distinct terms b_1, b_2 of x_1. Since $\sum_{i=1}^{n} x_i = 0$, b_2 is a term of some $x_i \neq x_1$, say, x_2. Since $x_2 \neq 0$, it has a term $b_3 \neq b_2$. If b_3 is a term of x_1 then define $y_1 = b_1' - b_2, y_2 = b_2 - b_3$ where $b_1' = b_3$ and $y_i = 0$ for $i > 2$ and the result follows. If b_3 is not a term of x_1 then it must be a term of some x_3 distinct from x_1 and x_2. Since $x_3 \neq 0$, it must have a term $b_4 \neq b_3$. Repeating this process, there exists s, such that, for $i \leq s$, b_i, b_{i+1} are distinct terms of x_i, where x_i are distinct. Furthermore b_{j+1} is not a term of $x_1, \ldots, x_{j-1}$ for $j < s$ but b_{s+1} is a term of x_t for some $t < s$. In this case we define $y_t = b_t' - b_{t+1}, y_{t+1} = b_{t+1} - b_{t+2}, \cdots, y_s = b_s - b_{s+1}$ where $b_t' = b_{s+1}$ and all other $y_i = 0$. Then the result follows. ∎

In the following we will apply the above Lemma to the case where $F = E(q)$ and $K = \mathrm{I}\mathbb{C}(q)$ in (2) to prove that conditions (I) and (II) are equivalent.

2.3. THEOREM. [Theorem B] *$I\mathbb{C}$ is full if and only if every n-crown in $\mathbb{C}$, $n \geq 2$, is supported.*

PROOF. ($\Rightarrow$) Suppose $\mathbb{C}$ contains an n-crown C_n of form (11), with $n \geq 2$. We will prove that it is supported between x_j and y_j. This is clear if $b_jx_j = b_jy_j$. Otherwise, since

$$b_1(x_1 - y_1) + b_2(x_2 - y_2) + \cdots + b_n(x_n - y_n) = 0,$$

$b_j(x_j - y_j) \in \mathrm{Im}\,\mathrm{I}\mathbb{C}(\mathrm{b_j}) \cap \sum_{\mathrm{k}\neq\mathrm{j}} \mathrm{Im}\,\mathrm{I}\mathbb{C}(\mathrm{b_k})$. Since $\mathrm{I}\mathbb{C}$ is full,

$$b_j(x_j - y_j) \in \sum \mathrm{Im}\,\mathrm{I}\mathbb{C}(\alpha)$$

where the sum is indexed by α that factors through b_j and b_k for some $k \neq j$. Since $b_j(x_j - y_j) \neq 0$, by Lemma 2.1, C_n is supported between x_j and y_j, and hence, C_n is supported in $\mathbb{C}$.

($\Leftarrow$) Suppose $\mathrm{I}\mathbb{C}$ is not full. Then there is an element x_1 in the right side of (1) but not the left side. Therefore, $-x_1 = \sum_{i=2}^{n} x_i$ where $x_i \in \mathrm{Im}\,\mathrm{I}\mathbb{C}(\beta_\mathrm{i})$, so $\sum_i x_i = 0$. Consider the exact sequence

$$0 \to I\mathbb{C}(q) \to E(q) = \bigoplus_p R\mathbb{C}(p, q) \xrightarrow{\epsilon_q} R \to 0$$

where $\epsilon(\sum r_\alpha \alpha) = \sum r_\alpha$, $r_\alpha \in R$ and $\alpha \in \mathbb{C}(p, q)$. Because the middle term of the exact sequence is a free R-module on morphisms with codomain q, we may assume that $\sum_i \#\mathrm{Terms}(x_i)$ is a minimum. By Lemma 2.2, for each i, there is a subsum y_i of x_i of form (15) with $\sum_i y_i = 0$. Let j be the smallest index such that $y_j \neq 0$ and let α be the term of y_j with coefficient 1. If r is the coefficient of α in x_j, then $\sum_i (x_i - ry_i) = 0$, and

$$\sum_i \#\ \mathrm{Terms}(x_i - ry_i) < \sum_i \#\mathrm{Terms}(x_i).$$

If $j > 1$, then $y_1 = 0$ so $-x_1 = \sum_{i=2}^{n} (x_i - ry_i)$, contradicting the minimality assumption. Thus, $j = 1$, so $y_1 \neq 0$. Since x_1 is not on the left side of (1), the same is true for either $x_1 - ry_1$ or ry_1. In the former case, if $x_1 - ry_1 \neq 0$, as $\#\mathrm{Terms}(x_1 - ry_1) < \#\mathrm{Terms}(x_1)$ and $\#\mathrm{Terms}(x_i - ry_i) \leq \#\mathrm{Terms}(x_i)$ for $i > 1$, it contradicts the minimality assumption. Otherwise ry_1, and hence, y_1 is not on the left side of (1). Since $\sum_i y_i = 0$, y_1 is on the right side of (1). After relabeling the indices beyond 1, we have $\sum_{i=1}^{k} y_i = 0$ for some $k > 1$ where

$$y_i = \beta_i w_i - \beta_i z_i,$$

for some morphisms w_i, z_i in $\mathbb{C}$, where $\beta_i z_i = \beta_{i+1} w_{i+1}$ for all $1 \le i \le k$ with subscripts modulo k. Hence $\mathbb{C}$ contains the k-crown

$$\begin{aligned} \beta_1 z_1 &= \beta_2 w_2 \\ \beta_2 z_2 &= \beta_3 w_3 \\ &\vdots \\ \beta_k z_k &= \beta_1 w_1 \end{aligned}$$

for some $k > 1$. Since all crowns in $\mathbb{C}$ are supported and $k \ge 2$, there is a commutative diagram (14) with $b_j x_j, b_j y_j$ replaced by $\beta_1 z_1, \beta_1 w_1$, and each α_j factors through β_1 and β_j for some $j \ne 1$. If $\beta_1 z_1 = \beta_1 w_1$ then $y_1 = 0$, a contradiction. Otherwise,

$$y_1 = \beta_1 w_1 - \beta_1 z_1 = \sum_{j=1}^{m} \alpha_j (t_j - s_j),$$

contradicting the fact that y_1 is not on the left side of (1). ■

2.4. Corollary. *If $hd_R\mathbb{C} \le 1$ then all n-crowns in $\mathbb{C}$, $n > 1$, are supported in $\mathbb{C}$.*

Actually the above is true with $n = 1$ as well.

2.5. Proposition. *If $hd_R\mathbb{C} \le 1$, then every 1-crown in $\mathbb{C}$ is supported in $\mathbb{C}$.*

Proof. Suppose $\mathbb{C}$ contains a 1-crown of the form (12) and q is the common codomain of x and y. (For convenience we suppress the subscripts.) Since $\mathrm{hd}_R\mathbb{C} \le 1$, $\mathrm{I}\mathbb{C}$ is flat, so by Proposition 1.1, $b(x - y) = 0$ implies that there exist objects q_i in $\mathbb{C}$, $w_i \in \mathrm{I}\mathbb{C}(q_i)$, and $\alpha_i \in R\mathbb{C}(q_i, q)$, for all $i = 1, \ldots, m$ such

$$x - y = \sum_i \alpha_i w_i \quad ,$$

and

$$b\alpha_i = 0 \quad \text{for all } i.$$

For each i, $\alpha_i = \sum_j r_{ij}\alpha_{ij}$, where $r_{ij} \in R$ and α_{ij} are morphisms $\mathbb{C}$, so

$$x - y = \sum_i \sum_j r_{ij}\alpha_{ij} w_i = \sum_i \sum_j \alpha_{ij}(r_{ij} w_i) \in \sum_i \sum_j \mathrm{Im}\, \mathrm{I}\mathbb{C}(\alpha_{\mathrm{ij}}).$$

The result now follows from Lemma 2.1. ■

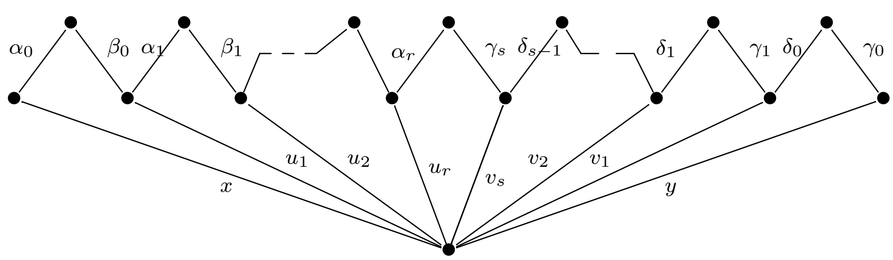
2.6. Remark. *From Corollary 6 and Proposition 7, we see that* $\mathrm{hd_R}\mathbb{C} \leq 1$ *implies all* n*-crowns are supported. But the converse is not true. For, in the example of Remark 1 of the previous section, it is easy to see that every* 1*-crown is supported and that every* n*-crown,* $n > 1$*, is supported by Theorem 6 and Remark 1. But* $\mathrm{hd_R}\mathbb{C} > 1$*. However, it is true if* $\mathbb{C}$ *is a DCC category.*

2.7. Corollary. *Suppose* $\mathbb{C}$ *is a DCC category. Then* $\mathrm{hd_R}\mathbb{C} \leq 1$ *if and only if all crowns are supported.*

Proof. Corollary 6 and Proposition 7 give the necessary direction. For the other direction, we have $\mathrm{cd_R}\mathbb{C} \leq 1$ by [2]. Since $\mathrm{cd_R}\mathbb{C} = \mathrm{pd}\,\Delta\mathrm{R}$, $\mathrm{I}\mathbb{C}$ is projective. But projectives are flat, so the result follows. ∎

3. Condition (II) $\iff$ Condition (III)

If U and V are subsets of the morphisms of $\mathbb{C}$, then $x, y \in \mathbb{C}$ are said to be *connected via* U *and* V if there exists a commutative diagram



(16)

in $\mathbb{C}$, where $u_i \in U$, and $v_j \in V$. They are *strongly connected via* U *and* V if, in addition, α_0 and γ_0 are identities. When $U = V$ we simply say that it is *via* U.

For each z in $\mathbb{C}$, we shall denote by $z\mathbb{C}$ the set of all morphisms of the form zw, where w is a morphism in $\mathbb{C}$. A small category $\mathbb{C}$ is (*strongly*) L_∞ if for every cyclic system (10) in $\mathbb{C}$, there exists an integer j, $1 \leq j \leq n$, such that $b_j x_j$ and $b_{j+1}x_{j+1}$ are (strongly) connected via $b_j\mathbb{C} \cap b_{j+1}\mathbb{C}$ and $b_{j+1}\mathbb{C} \cap b_{j+2}\mathbb{C}$ (where indices are taken modulo n). It is clear that if $\mathbb{C}$ is strongly L_∞ then $\mathbb{C}$ is L_∞. Note that C_n of form (11) is supported between x_i and y_i if and only if $b_i x_i$ and $b_i y_i$ are strongly connected via $b_i\mathbb{C} \cap U$ where U is the set of all morphisms that factor through some $b_j, j \neq i$.

A cyclic system (10) is said to be *reducible* if $b_i\mathbb{C} \cap b_j\mathbb{C} \neq \emptyset$ for some $i \neq j$ where $i \not\equiv j \pm 1 \pmod{n}$.

3.1. Proposition. *If every* n*-crown,* $n \geq 2$*, contained in* $\mathbb{C}$ *is supported in* $\mathbb{C}$*, then every cyclic system in* $\mathbb{C}$ *of length* $n > 3$ *is reducible.*

Proof. Suppose to the contrary that the system (10), i.e. C_n, with $n > 3$ is irreducible. Then $b_2\mathbb{C} \cap b_n\mathbb{C} = \emptyset$ as $n > 3$. Since C_n is supported in $\mathbb{C}$, it is so between x_1 and y_1. Therefore we have commutative diagram (14) with $j = 1$ and each α_i factors through β_1 and β_i for some $i \neq 1$. If $m = 0$, then $b_1 x_1 = b_1 y_1$. So $b_2 y_2 = b_1 x_1 = b_1 y_1 =$

b_nx_n contradicting $b_2\mathbb{C} \cap b_n\mathbb{C} = \emptyset$. Therefore, $m > 0$. Since the system is irreducible, $b_1\mathbb{C} \cap b_j\mathbb{C} = \emptyset$ for $j \neq 2, n$. If α_{i-1} factors through b_2 and α_i factors through b_n then $\alpha_{i-1} = b_2\gamma_1$ and $\alpha_i = b_n\gamma_2$ for some morphisms γ_1, γ_2 in $\mathbb{C}$. Hence

$$b_2\gamma_1 s_{i-1} = \alpha_{i-1}s_{i-1} = \alpha_i t_i = b_n\gamma_2 t_i.$$

This contracts $b_2\mathbb{C} \cap b_n\mathbb{C} = \emptyset$. Similarly, if α_{i-1} factors through b_n and α_i factors through b_2 we also get a contradiction. Therefore all α_i factor through b_2, or all α_i factors through b_n. In the first case, $\alpha_m = b_2\gamma$ for some morphism γ in $\mathbb{C}$, so $b_nx_n = b_1y_1 = \alpha_m s_m = b_2\gamma s_m$, a contradiction. In the second case $\alpha_1 = b_n\gamma$ for some morphism γ in $\mathbb{C}$, so, $b_2y_2 = b_1x_1 = \alpha_1 t_1 = b_n\gamma t_1$, another contradiction. Therefore, the system is reducible. ∎

3.2. Corollary. *If every n-crown, $n \geq 2$, contained in $\mathbb{C}$ is supported in $\mathbb{C}$, then for every cyclic system (10) of length $n \geq 3$, there is an index j such that $b_j\mathbb{C} \cap b_{j+2}\mathbb{C} \neq \emptyset$.*

Proof. If $n = 3$, the statement clearly holds. If $n > 3$, by Proposition 3.1, the system is reducible, so there exist indices i, j, $2 \leq |j - i| \leq n - 2$, so that $b_i\mathbb{C} \cap b_j\mathbb{C} \neq \emptyset$. Among all such pairs i, j, choose i and j so that $|j - i|$ is minimal. Without loss of generality, we may assume $i = 1$ and $3 \leq j \leq n - 1$. Let $s, t \in \mathbb{C}$ be such that $b_1s = b_jt$. Then we have the cyclic system

$$\begin{aligned} b_1x_1 &= b_2y_2 \\ b_2x_2 &= b_3y_3 \\ &\cdots \\ b_jt &= b_1s \end{aligned}$$

of length j. By Proposition 3.1, this system is reducible, so there exists indices k, l, where $2 \leq |l - k| \leq j - 2$, such that $b_k\mathbb{C} \cap b_l\mathbb{C} \neq \emptyset$. If $j > 3$, then $|l - k| \leq j - 2 < j - 1 = |j - i|$, contradicting the minimality of $|j - i|$. Therefore $j = 3$, and the proof is complete. ∎

3.3. Proposition. *If every 3-crown contained in $\mathbb{C}$ is supported in $\mathbb{C}$, then for a cyclic system (10) of length three in $\mathbb{C}$, there exist morphisms z_1, z_2, z_3 in $\mathbb{C}$ such that*

$$b_1z_1 = b_2z_2 = b_3z_3.$$

Proof. Since every C_3 is supported in $\mathbb{C}$, it is so between x_1 and y_1. Consequently, there exists a commutative diagram (14) where j=1 and each α_i factors through b_1 and b_k for some $k \neq 1$. If $m = 0$, then $b_1x_1 = b_1y_1$, so $b_2y_2 = b_1x_1 = b_1y_1 = b_3x_3$. Hence we may assume that $m > 0$. Suppose α_{i-1} factors through b_2 and α_i factors through b_3. So $\alpha_{i-1} = b_2\gamma_2$ for some morphism γ_2 in $\mathbb{C}$ and $\alpha_i = b_3\gamma_3$ for some morphism γ_3 in $\mathbb{C}$. By (14), we have

$$b_2\gamma_2 s_i = \alpha_{i-1}s_{i-1} = \alpha_i t_i = b_3\gamma_3 t_i$$

and this morphism factors through b_1 as α_i does. Hence the result follows. Similarly, if α_{i-1} factors through b_3 and α_i factors through b_2 the result also follows. Therefore we may assume either all α_i factor through b_2 or all α_i factor through b_3. In the earlier case, $\alpha_m = b_2\gamma$ for some morphism γ in $\mathbb{C}$, so $b_3x_3 = b_1y_1 = \alpha_m s_m = b_2\gamma s_m$. In the latter case, $\alpha_1 = b_3\gamma$ for some morphism γ in $\mathbb{C}$, so $b_2y_2 = b_1x_1 = \alpha_1 t_1 = b_3\gamma t_1$. ∎

3.4. Proposition. *If every 2-crown contained in $\mathbb{C}$ is supported in $\mathbb{C}$, then, in any cyclic system of form (10) of length two, b_1x and b_2x_2 are strongly connected via $b_1\mathbb{C} \cap b_2\mathbb{C}$.*

Proof. Since C_2 is supported between x_1 and y_1 in $\mathbb{C}$, there exists a commutative diagram (14) where each α_i factors through both b_1 and b_2 with $j = 1$. Thus, b_1x_1 and $b_2x_2 = b_1y_1$ are strongly connected via $b_1\mathbb{C} \cap b_2\mathbb{C}$. ∎

The next Theorem shows that conditions (II) and (III) are equivalent.

3.5. Theorem. [Theorem C] *Every n-crown, $n \geq 2$, in $\mathbb{C}$ is supported if and only if $\mathbb{C}$ is strongly L_∞.*

Proof. ($\Rightarrow$) Consider the cyclic system (10). If $n = 2$, then the result follows from Proposition 3.4. Suppose $n \geq 3$. By Corollary 3.2, there is an index j such that $b_j\mathbb{C} \cap b_{j+2}\mathbb{C} \neq \emptyset$. Without loss of generality, we may assume that $j = 1$. Then there exist morphisms s, t in $\mathbb{C}$ such that $b_1s = b_3t$, so we have the cyclic system

$$\begin{aligned} b_1x_1 &= b_2y_2 \\ b_2x_2 &= b_3y_3 \\ b_3t &= b_1s \end{aligned}$$

in $\mathbb{C}$. We will show that b_1x_1 and b_2x_2 are connected via $b_1\mathbb{C} \cap b_2\mathbb{C}$ and $b_2\mathbb{C} \cap b_3\mathbb{C}$. By Proposition 3.3, there are morphisms z_1, z_2, z_3 in $\mathbb{C}$ such that

$$b_1z_1 = b_2z_2 = b_3z_3.$$

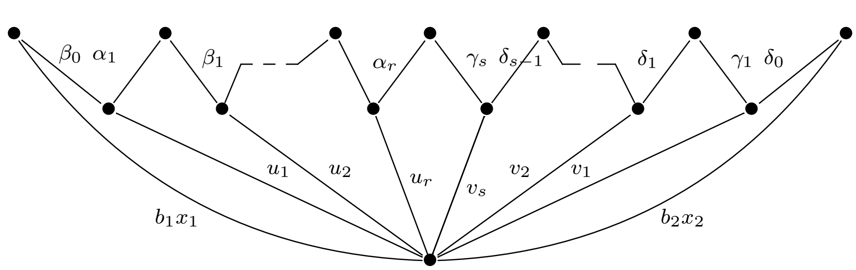
Applying Proposition 3.4 to the systems

$$\begin{aligned} b_1x_1 &= b_2y_2 \\ b_2z_2 &= b_1z_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} b_2z_2 &= b_3z_3 \\ b_3y_3 &= b_2x_2 \end{aligned}$$

we get that b_1x_1 and b_2z_2 are strongly connected via $b_1\mathbb{C} \cap b_2\mathbb{C}$; b_2z_2 and $b_3y_3 = b_2x_2$ are strongly connected via $b_2\mathbb{C} \cap b_3\mathbb{C}$. Thus, b_1x_1 and b_2x_2 are strongly connected via $b_1\mathbb{C} \cap b_2\mathbb{C}$ and $b_2\mathbb{C} \cap b_3\mathbb{C}$. Therefore, $\mathbb{C}$ is strongly L_∞.

($\Leftarrow$) Let C_n be of the form (11). Suppose $n = 2$. Since $\mathbb{C}$ is strongly L_∞, b_1x_1 and b_2x_2 are strongly connected via $b_1\mathbb{C} \cap b_2\mathbb{C}$. Because $b_2x_2 = b_1y_1$, the result follows.

Suppose $n = 3$. Since $\mathbb{C}$ is strongly L_∞, there is an index j such that b_jx_j and $b_{j+1}x_{j+1}$ are strongly connected via $b_j\mathbb{C} \cap b_{j+1}\mathbb{C}$ and $b_{j+1}\mathbb{C} \cap b_{j+2}\mathbb{C}$ (indices modulo n). Without loss of generality, we may assume that $j = 1$. That is, there is a commutative diagram



(17)

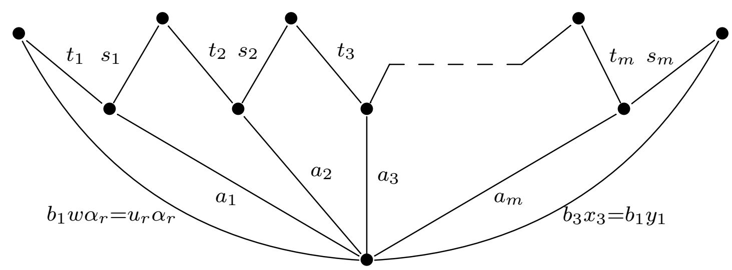
in $\mathbb{C}$, where $u_i \in b_1\mathbb{C} \cap b_2\mathbb{C}$ for all i and $v_l \in b_2\mathbb{C} \cap b_3\mathbb{C}$ for all l. Since $b_2y_2 = b_1x_1$, the crown is supported between x_2 and y_2. Since $u_r = b_1w$ and $v_s = b_3z$ for some morphisms w, z in $\mathbb{C}$, we have

$$b_1w\alpha_r = u_r\alpha_r = v_s\gamma_s = b_3z\gamma_s.$$

So we have the cyclic system

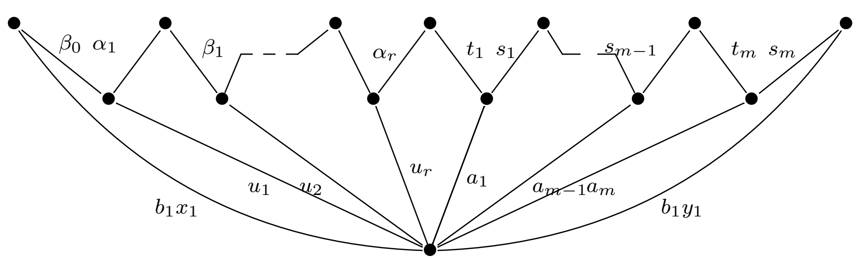
$$\begin{aligned} b_1(w\alpha_r) &= b_3(z\gamma_s) \\ b_3x_3 &= b_1y_1. \end{aligned}$$

Since $\mathbb{C}$ is strongly L_∞, there is a commutative diagram

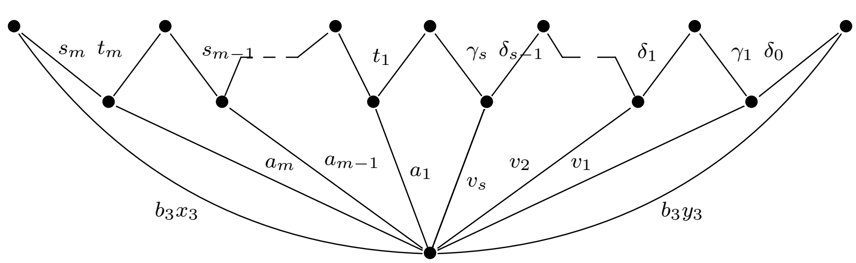


(18)

where each $a_j \in b_1\mathbb{C} \cap b_3\mathbb{C}$. Hence, together with the left half of (17), we obtain the commutative diagram



Since each u_i, a_j factor through b_1 and either b_2 or b_3, C_3 is supported between x_1 and y_1. Similarly, since $b_3y_3 = b_2x_2$ and $v_s\gamma_s = u_r\alpha_r = a_1t_1$, (18) together with the right half of (17) gives the commutative diagram



Since each a_j, v_l factor through b_3 and either b_1 or b_2, C_3 is supported between x_3 and y_3. Therefore, C_n is supported when $n = 3$.

Now suppose the result holds for all $t \leq m$ for some $m \geq 3$, and suppose $n = m + 1$. Since $\mathbb{C}$ is strongly L_∞, without loss of generality, we may assume that b_1x_1 and b_2x_2 are strongly connected via $b_1\mathbb{C} \cap b_2\mathbb{C}$ and $b_2\mathbb{C} \cap b_3\mathbb{C}$. Hence there is a commutative diagram (17) with $u_i \in b_1\mathbb{C} \cap b_2\mathbb{C}$ for all i, and $v_l \in b_2\mathbb{C} \cap b_3\mathbb{C}$ for all l. Therefore $u_r = b_1 w', v_s = b_3 z'$ for some $w', z' \in \mathbb{C}$. Hence $u_r\alpha_r = v_s\gamma_s$ implies $b_1 w'\alpha_r = b_3 z'\gamma_s$. Taking $w = w'\alpha_r$ and $z = z'\gamma_s$, we have $b_1 w = b_3 z$. So we have two cyclic systems

$$\begin{aligned} b_1x_1 &= b_2y_2 \\ b_2x_2 &= b_3y_3 \\ b_3z &= b_1w \end{aligned} \qquad\qquad \begin{aligned} b_3x_3 &= b_4y_4 \\ b_4x_4 &= b_5x_5 \\ &\cdots \\ b_nx_n &= b_1y_1 \\ b_1w &= b_3z \end{aligned} \tag{19}$$

By induction, the 3-crown and $(n-1)$-crown in (19) are supported in $\mathbb{C}$. Hence C_n is supported between x_i and y_i for all i, except perhaps when $i = 1$ or 3. Let U_1 be the set of all morphisms that factor through some $b_i, i \neq 1$. Since the 3-crown on the left is supported in $\mathbb{C}$, b_1x_1 and b_1w are strongly connected via $b_1\mathbb{C} \cap U_1$. Since the $(n-1)$-crown on the right is supported in $\mathbb{C}$, b_1w and b_1y_1 are also strongly connected via $b_1\mathbb{C} \cap U_1$. Thus b_1x_1 and b_1y_1 are strongly connected via $b_1\mathbb{C} \cap U_1$, i.e. C_n is supported between x_1 and y_1. Similarly, let U_3 be the set of all morphisms factoring through some $b_i, i \neq 3$. Since b_3x_3 and b_3z are strongly connected via $b_3\mathbb{C} \cap U_3$, b_3z and b_3y_3 are also strongly connected via $b_3\mathbb{C} \cap U_3$, we see that b_3x_3 and b_3y_3 are also. Hence C_n is supported between x_3 and y_3. ∎

A small category is *cancellative* if $ab = ac$ or $ba = ca \implies b = c$ for all morphisms a, b, c. When $\mathbb{C}$ is cancellative having only one object, using the fact that $\mathrm{hd}_{\mathrm{R}}\mathbb{C} \leq \mathrm{cd}_{\mathrm{R}}\mathbb{C}$ and Theorems A, B and C we derive the following.

3.6. Corollary. [Novikov [9]] *Suppose M is a cancellative monoid such that* $\mathrm{cd}_{\mathrm{R}}\mathbb{C} \leq 1$. *Then M is L_∞.*

4. Embeddability

Since a small category can be thought of as a monoid with a "partial" binary operation, many results in semigroup theory can be generalized to small categories. In this section we shall state two such generalizations and use them to derive another necessary condition for $\mathrm{hd}_{\mathrm{R}}\mathbb{C} \leq 1$. For details, we refer the readers to [5, Chapter 12], [14] and [11].

Let $L_1, \cdots, L_h, L_1^*, \cdots, L_h^*, R_0, \cdots, R_k, R_0^*, \cdots, R_k^*$ be a set of $2(h+k+1)$ symbols. A *Malcev sequence*, or *M-sequence*, denoted $I(h, k, n)$ is a sequence of these symbols, each occurring exactly once, so that

(1) R_0 and R_0^* are respectively the first and last symbols;

(2) the L's and R's occur in their natural order;

(3) L_i^* occurs after L_i and if L_j occurs between L_i and L_i^*, then so does L_j^*; and similarly for R_i^*;

(4) n symbols occur between L_h and L_h^*.

To each M-sequence, we can associate a system of equations called an *M-system*. This is done by associating products of variables with the symbols of $I(h,k,n)$ using the following table where each symbol of the table represents an unknown in the M-system. An adjacent pair of symbols in $I(h,k,n)$ determines an equation as follows: the left (right) member of the equation is the top (bottom) entry of the column in the table associated with the first (second) symbol of the pair. The adjacent pairs of symbols (from left to right) of $I(h,k,n)$ determine an ordered set of $2(h+k)+1$ equations, which we call the M-system associated with $I(h,k,n)$.

L_i	L_i^*	R_i	R_i^*
d_ia_i	c_ib_i	A_iD_i	B_iC_i
c_ia_i	d_ib_i	A_iC_i	B_iD_i

Suppose the M-system of equations associated with an M-sequence $I(h,k,n)$ holds for some set $X = \{a_i, b_i, c_i, d_i, A_j, B_j, C_j, D_j \mid 0 < i \leq h, 0 \leq j \leq k\}$ of morphisms of $\mathbb{C}$. Then the $2(h+k)+1$ equations, with the elements of X substituted in for the variables, form an *M-chain* $I(h,k,n,X)$ in $\mathbb{C}$ with *locked morphisms* A_0, B_0. We say $I(h,k,n,X)$ is a *closed M-chain* if $A_0 = B_0$.

4.1. Theorem. [Bouleau [1]] *If $\mathbb{C}$ is a cancellative L_∞ category, then every M-chain in $\mathbb{C}$ is closed.*

Malcev's Theorem, as proved in [5, Theorem 12.17], can also be generalized.

4.2. Theorem. [Malcev] *Let $\mathbb{C}$ be a small category. Then $\mathbb{C}$ is embeddable in a groupoid if and only if every M-chain in $\mathbb{C}$ is closed.*

Using the above two Theorems, which are proved in [11], together with Theorems A, B and C, we have the following.

4.3. Corollary. *If $\mathbb{C}$ is cancellative and* $\mathrm{hd}_{\mathrm{R}}\mathbb{C} \leq 1$*, then $\mathbb{C}$ is embeddable in a groupoid.*

4.4. Remark. *The converse of the above is not true. For it can easily be shown that any poset is embeddable in a groupoid but a poset can have arbitrarily large homological dimensions. Bouleau has supplied a one-object counterexample in [1].*

4.5. Corollary. *If $\mathbb{C}$ is cancellative connected, and* $\mathrm{cd}\,\mathbb{C} \leq 1$ *then $\mathbb{C}$ is embeddable in a groupoid which is equivalent to a free group.*

Proof. By [4, Corollary 3.3], $\operatorname{cd}\mathbb{C} \leq 1$ implies $\operatorname{cd}\mathrm{S}(\mathbb{C}) \leq 1$ where $S(\mathbb{C})$ is the groupoid reflection of $\mathbb{C}$. Since $\mathbb{C}$ is connected, $S(\mathbb{C})$ is connected and, therefore, is equivalent to a group G. Hence $\operatorname{cd}\mathrm{S}(\mathbb{C}) = \operatorname{cd}\mathrm{G}$. By Stallings [12] and Swan [13], $\operatorname{cd}\mathrm{G} \leq 1$ implies that G is free. Since $\operatorname{hd}_{\mathrm{R}}\mathbb{C} \leq \operatorname{cd}_{\mathrm{R}}\mathbb{C}$ the result follows from Corollary 4.5.

■

References

[1] N. Bouleau, *Classes de semigroupes immersible dans groupes.* Semigroup Forum 6 (1973) 100–112.

[2] C.C. Cheng, *DCC categories of cohomological dimension one.* J. Pure Appl. Algebra 80 (1992) 107–116.

[3] C.C. Cheng, J. Shapiro, *Cohomological dimension of an abelian monoid.* Proc. Amer. Math. Soc. 80 (1980) 547–551.

[4] C.C. Cheng, Y. Wu and B. Mitchell, *Categories of fractions preserve dimension one.* Communications in Algebra 8 (1980) 927-939.

[5] A.H. Clifford, G.B. Preston, *The Algebraic Theory of Semigroups.* II, The American Mathematical Society, 1967

[6] J. Isbell, B. Mitchell, *Exact colimits and fixed points.* Trans. Amer. Math. Soc. 220 (1976) 289–298.

[7] D. Latch, B. Mitchell, *On the difference between homological dimension and cohomological dimension.* J. Pure Appl. Algebra 5 (1974) 333–343.

[8] B. Mitchell, *Rings with several objects.* Advances in Mathematics 8 (1972) 1–161.

[9] B.V. Novikov, *Semigroups of cohomological dimension one.* J. Algebra 2 (1998) 386–393.

[10] U. Oberst, H. Rohrl, *Flat and coherent functors.* J. Algebra 14 (1971) 91–105.

[11] K. Sweet, Small Categories of Homological Dimension One, Dissertation, Oakland University, Rochester, MI, 2018.

[12] J. Stallings, *On torsion free groups with infinitely many ends.* Ann. Math. 88 (1968) 312–334.

[13] R. Swan, *Groups of cohomological dimension one.* J. Algebra 1 (1969) 585–610.

[14] P.G. Trotter, *A class of semigroups embeddable in groups.* Semigroup Forum 5 (1972) 1–13.

[15] P. Webb, *An introduction to the representation and cohomology of categories*. Group representation theory, EPFL Press, Lausanne (2007) 149-173.

[16] R.W. Wong, Some projective free ringoids, Dissertation, Rutgers University, New Brunswick, NJ, 1977.

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