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SHIFTED DOUBLE LIE–RINEHART ALGEBRAS

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ABSTRACT. We generalize the notions of shifted double Poisson and shifted double Lie–Rinehart structures, defined by Van den Bergh in [31, 32], to monoids in a symmetric monoidal abelian category. The main result is that an n-shifted double Lie–Rinehart structure on a pair (A, M) is equivalent to a non-shifted double Lie–Rinehart structure on the pair $(A, M[-n])$.

Contents

1. Introduction

1.1. NONCOMMUTATIVE GEOMETRY. In algebraic geometry, a commutative algebra C over a field k corresponds to an affine scheme $\mathrm{Spec}(C)$, via the functor of points. The scheme $\mathrm{Spec}(C)$ is the geometric object associated to its algebra of functions C. Working in noncommutative geometry, a natural question arises: for a noncommutative algebra A, which is viewed as an algebra of noncommutative functions, what is the geometric object associated to A? Recently, Kontsevich and Rosenberg have proposed a new approach to answer this question (see [18]). They consider the family of schemes $\left\{\mathrm{Rep}_V(A)//\mathrm{GL}(V)\right\}_V$, *the moduli space of representations of* A, as successive approximations of a hypothetical noncommutative affine scheme "NCSpec(A)". The scheme $\mathrm{Rep}_V(A)$ is affine, i.e. there exists a commutative k-algebra, denoted by A_V, such that $\mathrm{Rep}_V(A) = \mathrm{Spec}(A_V)$. The quotient $\mathrm{Rep}_V(A)//\mathrm{GL}(V)$ corresponds to taking $A^{\mathrm{GL}(V)}$, the invariant part of the algebra A_V for the action by conjugation of $\mathrm{GL}(V)$.

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Kontsevich and Rosenberg assert that any (noncommutative) property of the non commutative scheme "NCSpec(A)" associated to A should induce its commutative analogue on $\mathrm{Rep}_V(A)/\!/\mathrm{GL}(V)$, for all V: this is the *Kontsevich–Rosenberg principle*. Following this principle, many authors have developed noncommutative structures; the reader can refer to [15] for such constructions in noncommutative geometry.

1.2. Noncommutative Poisson brackets. It is natural to ask what a good definition of a noncommutative Poisson structure. Recall that a Poisson bracket on an associative commutative k-algebra B is a Lie bracket $\{-,-\}\colon B\otimes B\to B$ which satisfies the Leibniz rule $\{ab,c\} = a\{b,c\} + \{a,c\}b$ for all a,b and c in B. For noncommutative algebras, this definition is too restrictive, as shown by [13, Th. 1.2]: for A, an associative algebra with a noncommutative domain, i.e. $[A,A]\neq 0$, a Poisson bracket is the commutator, up to a multiplicative constant. In [10], Crawley-Boevey gives the minimal structure on an associative algebra A which induces a Poisson bracket on $A_V^{\mathrm{GL}(V)}$ for all V, which he calls an H_0-Poisson structure. An H_0-Poisson structure on A is a Lie bracket $\langle -,-\rangle$ on $A_\natural := A/[A,A]$ such that, for all $a\in A$ (with class $\bar{a}$ in $A_\natural$), the application $\langle \bar{a},-\rangle\colon A_\natural\to A_\natural$ is induced by a derivation $d_a\colon A\to A$. Crawley-Boevey shows that, if A is a H_0-Poisson algebra, then there exists a unique Poisson structure on $A_V^{\mathrm{GL}(V)}$ that is compatible with the trace morphism for all V (see [10, Th. 1.6]).

However, there are few examples of H_0-Poisson structures which do not arise from a richer structure. A good example of such a structure is a double Poisson bracket, defined by Van den Bergh in [31]. A double Poisson bracket on an associative algebra A is a morphism

$$\begin{array}{cccc} \{\!\{-,-\}\!\}\colon & A\otimes A & \longrightarrow & A\otimes A \\ & a\otimes b & \longmapsto & \{\!\{a,b\}\!\}'\otimes\{\!\{a,b\}\!\}'' \end{array}$$

(using Sweedler's notation) which is antisymmetric, i.e. for all elements a and b in A, $\{\!\{a,b\}\!\} = -\{\!\{b,a\}\!\}''\otimes\{\!\{b,a\}\!\}'$, which is a derivation in its second variable and satisfies *the double Jacobi relation* (see definition 3.10). There are lots of examples. In [30], Van de Weyer studies double Poisson brackets on semi-simple algebras of finite dimension. However, that double Poisson structures are best suited to the noncommutative world: for example, in [26], Powell shows that any double Poisson bracket on a free commutative algebra with at least two generators is trivial. Van den Bergh shows that (see [31, Lem. 2.6.2]) a double Poisson bracket $\big(A,\{\!\{-,-\}\!\}\big)$ induces a H_0-Poisson structure on A, where the Lie bracket $\{-,-\}_\natural$ is induced by $\{-,-\}_\natural := \mu\circ\{\!\{-,-\}\!\}$, with μ the associative product on A. Double Poisson brackets are connected with many mathematical areas, as we'll now see.

In symplectic geometry, one can associate to an exact symplectic manifold M its Fukaya category $\mathsf{Fuk}(M)$ (see [5]). For an exact symplectic $2d$-dimensional manifold, with vanishing first Chern class, Chen *et al.* show in [7, Th.17] that the linear dual of the reduced bar construction of $\mathsf{Fuk}(M)$ has a naturally defined $(2-d)$-double Poisson bracket. This implies that the cyclic cohomology $\mathrm{HC}^\bullet(\mathsf{Fuk}(M))$ has a $(2-d)$-Lie bracket, an analogue of the Chas–Sullivan bracket in string topology (see [7, Cor. 19]).

To a finite quiver Q (see [9]), Van den Bergh showed that the algebra $k\bar{Q}$ of the double quiver $\bar{Q}$, has a natural double Poisson bracket (see [31]) which induces the Kontsevich Lie bracket on $(k\bar{Q})_\natural$.

Other examples are related to loop spaces of manifolds with boundary (see [23], [24]), the Kashiwara-Vergne problem (see [1]), and noncommutative integrable systems (see [11, 12, 4, 3]).

1.3. In this article. This paper is in two parts. In the first, we extend the definition of a shifted double Poisson algebra to monoids in an additive symmetric monoidal category $(\mathsf{C}, \otimes)$. For Σ an element of the Picard group of C and A an associative monoid in C, a Σ-double Poisson bracket on A is a morphism

$$\{\!\{-,-\}\!\}\colon \Sigma A \otimes \Sigma A \longrightarrow \Sigma A \otimes A$$

where $\Sigma A \coloneqq \Sigma \otimes A$, which satisfies the antisymmetry and derivation properties (see definition 3.3) and the double Jacobi identity (see definition 3.10).

In the second part, we study a particular type of double Poisson algebras called *linear double Poisson algebras*. They correspond to double Lie–Rinehart algebras (called double Lie algebroids by Van den Bergh in [31, Sect. 3.2]), which are a noncommutative version of Lie–Rinehart algebras (see [2, 14]). The principal result of this paper is the shifting property of double Lie–Rinehart algebras:

1.4. Theorem. [cf. Theorem 5.2] *Let C be an additive symmetric monoidal category C, with unit $\mathbb{1}$, a monoid A, an A-bimodule M and Σ an invertible object in C. The following assertions are equivalent:*

1. $(A, M, \rho_M, \{\!\{-,-\}\!\}_M)$ *is a Σ-double Lie–Rinehart algebra;*
2. $(A, \Sigma M, \rho_{\Sigma M}, \{\!\{-,-\}\!\}_{\Sigma M})$ *is a $\mathbb{1}$-double Lie–Rinehart algebra.*

There is an equivalence of categories

$$\Sigma\text{-}\mathsf{DLR}_A \cong \mathbb{1}\text{-}\mathsf{DLR}_A,$$

with Σ-DRL_A, the category of Σ-double Lie–Rinehart algebras over the associative algebra A.

This theorem is a first step to understand *properadically* what is a shifted double Poisson algebra. (see subsection 5.1)

An example of a double Lie–Rinehart algebra is given by Van den Bergh in [31, App. A]: the Koszul double bracket. We extend this example to the general case of a monoid in an additive symmetric monoidal category (without shifting):

1.5. Theorem. [cf. Theorem 5.9] *Let A be a Σ-double Poisson algebra in an additive symmetric monoidal category $(\mathsf{C}, \otimes, \tau)$ with enought coequalizers. The free A-algebra $\mathrm{T}_A\Omega_A$ is a linear Σ-double Poisson algebra (see proposition 5.6 for the definition of Ω_A).*

2. Notation and algebraic background

2.1. Symmetric monoidal category. We recall some classical material about monoidal categories: see [6, section 6] for more details.

We consider $(\mathsf{C}, \otimes)$, an additive category with monoidal structure $\otimes$, unit $\mathbb{1}$ and such that the bifunctor $- \otimes -\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is additive in each entry. We assume that $(\mathsf{C}, \otimes)$ is symmetric for the natural transformation τ i.e., for all objects A and B in C, we have the isomorphism $\tau_{A,B} : A \otimes B \to B \otimes A$ which satisfies $\tau_{B,A}\tau_{A,B} = \mathrm{id}_{A\otimes B}$. We say that the category C is *closed*, if for every object C in the category C, there exists a functor $\underline{\mathrm{hom}}(C, -)$ which is the right adjoint of $- \otimes C$:

$$- \otimes C\colon \mathsf{C} \rightleftarrows \mathsf{C}\colon \underline{\mathrm{hom}}(C, -)\ .$$

Throughout this paper, we fix Σ, an invertible object for the tensor product in C, with inverse Σ^- and an isomorphism $\rho\colon \Sigma^- \otimes \Sigma \to \mathbb{1}$. By the symmetry of C, we also have the isomorphism $\rho\tau_{\Sigma,\Sigma^-} : \Sigma \otimes \Sigma^- \cong \mathbb{1}$. In particular, Σ induces the functor $\Sigma \otimes -\colon \mathsf{C} \to \mathsf{C}$ which is an equivalence of categories. For all objects C in the category C, we denote by

$$\Sigma C \coloneqq \Sigma \otimes C$$

its image by this functor. We denote by $\mathsf{As}(\mathsf{C})$ the category of associative monoids in C : objects in $\mathsf{As}(\mathsf{C})$ are objects $A \in \mathsf{C}$ with an associative product $\mu : A \otimes A \to A$, and, for two monoids A and B, $\mathrm{Hom}_{\mathsf{As}(\mathsf{C})}(A, B)$ is the set of monoid morphisms.

To illustrate these notions, we recall the following classical example.

2.2. Example. Take C to be the category Ch_k of $\mathbb{Z}$-graded chain complexes over a field k. One can equip it with the monoidal structure $\otimes$ given by the tensor product of complexes and the symmetry given, for homogeneous elements $a \in A$ and $b \in B$, by $\tau_{A,B}(a \otimes b) = (-1)^{|a||b|} b \otimes a$, where $|a|$ is the degree of A. Take Σ to be the chain complex k, concentrated in degree r, for r a fixed integer, so that $(\Sigma A)_n = A_{n-r} =: A[-r]$. The monoidal category $(\mathsf{Ch}_k, \otimes_k)$ is closed (see [34] for details). The category $\mathsf{As}(\mathsf{Ch}_k)$ is the category of differential $\mathbb{Z}$-graded algebras, denoted DGA_k.

2.3. A-bimodule structures on $A \otimes A$. Fix A and B, two associative monoids in C. The monoidal structure on A and B induce a monoidal structure on $A \otimes B$. We define A°, the opposite monoid of A, to be given by the same object but with the product $\mu_{A^\circ} \coloneqq \mu_A \circ \tau_{A,A}$. We have the usual notion of left (respectively right) modules over A: we denote by $A\text{-}\mathsf{Mod}_{\mathsf{C}}$ (resp. $\mathsf{Mod}_{\mathsf{C}}\text{-}A$) the category of left A-modules (resp. right A-modules) in the category C. There is an equivalence of categories $\mathsf{Mod}_{\mathsf{C}}\text{-}A \cong A^\circ\text{-}\mathsf{Mod}_{\mathsf{C}}$.

We denote by $(A, B)\text{-}\mathsf{Bimod}_{\mathsf{C}} \coloneqq A\text{-}\mathsf{Mod}_{\mathsf{C}}\text{-}B$, the category of (A, B)-bimodules in C, which is equivalent to $(A \otimes B^\circ)\text{-}\mathsf{Mod}_{\mathsf{C}}$. For M an (A, B)-bimodule and X and Y two objects in C, the product $X \otimes M \otimes Y$ is also an (A, B)-bimodule by the symmetry of C. Fix two (A, B)-bimodules M and N: the product $M \otimes N$ has a structure of (A, B)-bimodule, called the *external* one, given by the left A-action on M and the right B-action on N. $M\otimes N$ also has an *internal* (A,B)-bimodule structure, given by by the left A-action on N and the right B-action on M.

When $A=B$, we denote by $A\text{-}\mathsf{Bimod}_{\mathsf{C}}$, the category of (A,A)-bimodules $(A,A)\text{-}\mathsf{Bimod}_{\mathsf{C}}$. We denote by A^e, the monoid $A\otimes A^\circ$, so that the category $A\text{-}\mathsf{Bimod}_{\mathsf{C}}$ is equivalent to $A^e\text{-}\mathsf{Mod}_{\mathsf{C}}$. The symmetry of the category C gives us the isomorphism of monoids $\tau_{A,A}: A^e \cong (A^e)^\circ$. The monoid structure of $A\otimes A^\circ$ gives a canonical structure of A^e-bimodule on $\Sigma_1 A\otimes \Sigma_2 A$ for all Σ_1,Σ_2 in C, i.e. two A-bimodule structures (we implicitly use the isomorphism $(A^e)^\circ \cong A^e$):

1. *the external structure* given by

$$\begin{aligned} \mu^e_A &:= (\Sigma_1 A\otimes \Sigma_2\mu) : (\Sigma_1 A\otimes\Sigma_2 A)\otimes A \to \Sigma_1 A\otimes \Sigma_2 A,\\ {}_A\mu^e &:= (\Sigma_1\mu\otimes\Sigma_2 A)(\tau_{A,\Sigma_1}\otimes A\otimes\Sigma_2 A) : A\otimes(\Sigma_1 A\otimes\Sigma_2 A)\to \Sigma A\otimes\Sigma A; \end{aligned}$$

2. *the internal structure* given by

$$\begin{aligned} \mu^i_A &:= (\Sigma_1\mu\otimes\Sigma_2 A)(\Sigma_1 A\otimes \tau_{\Sigma_2 A,A})\colon (\Sigma_1 A\otimes\Sigma_2 A)\otimes A\to\Sigma_1 A\otimes\Sigma_2 A,\\ {}_A\mu^i &:= (\Sigma_1 A\otimes\Sigma_2\mu)(\tau_{A,\Sigma_1 A\Sigma_2}\otimes A) : A\otimes(\Sigma_1 A\otimes\Sigma_2 A)\to\Sigma_1 A\otimes\Sigma_2 A. \end{aligned}$$

2.4. Remark. Let Σ an invertible object in C with the isomorphism $\rho:\Sigma^-\otimes\Sigma\to\mathbb{1}$ and A be a monoid in C. The functor $\Sigma\otimes - : \mathsf{C}\to\mathsf{C}$ induces an equivalence of categories $\Sigma\otimes - : A\text{-}\mathsf{Mod}_{\mathsf{C}} \to A\text{-}\mathsf{Mod}_{\mathsf{C}}$.

2.5. Example. As in example 2.2, we consider the category of chain complexes Ch_k. Let (A,μ) be an associative monoid in Ch_k, i.e. a differential graded algebra. Fix $r\in\mathbb{Z}$, we note s the generator of the chain complex equal to k concentrated in degree r (so $|s|=r$); we note sa an element of $A[r]$.

1. The external A-bimodule structure on $A\otimes A$ is given by the following two morphisms:

$$\begin{aligned} \mu^e_A\colon A[r]\otimes A[r]\otimes A &\longrightarrow A[r]\otimes A[r],\\ {}_A\mu^e\colon A\otimes A[r]\otimes A[r] &\longrightarrow A[r]\otimes A[r], \end{aligned}$$

   where, for homogeneous elements a,b and $c\in A$,

$$\mu^e_A(sa\otimes sb\otimes c) = sa\otimes s\mu(b,c) \text{ and } {}_A\mu^e(c\otimes sa\otimes sb) = (-1)^{d|c|}s\mu(c,a)\otimes sb.$$

2. The internal A-bimodule structure on $A\otimes A$ is given by the following two morphisms:

$$\begin{aligned} \mu^i_A\colon A[r]\otimes A[r]\otimes A &\longrightarrow A[r]\otimes A[r],\\ {}_A\mu^i\colon A\otimes A[r]\otimes A[r] &\longrightarrow A[r]\otimes A[r], \end{aligned}$$

   where, for homogeneous elements a,b and $c\in A$,

$$\mu^i_A(sa\otimes sb\otimes c) = (-1)^{|c|(|b|+r)}s\mu(a,c)\otimes sb \text{ and } {}_A\mu^i(c\otimes sa\otimes sb) = (-1)^{|c||a|}sa\otimes s\mu(c,b).$$

3. Σ-double Poisson algebras

In this section, we extend constructions given by Van den Bergh in [31] to a general categorical framework. As in section 2, we consider $(\mathsf{C}, \otimes)$ a symmetric monoidal additive category and we fix Σ an invertible object in C with the isomorphism $\rho\colon \Sigma^- \otimes \Sigma \to \mathbb{1}$ and (A, μ) a monoid in C.

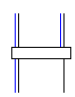
3.1. Σ-DOUBLE BRACKET. We recall that a morphism $\phi\colon A \to M$ in C between an algebra (A, μ) and an A-bimodule $(M, \mu_A, {}_A\mu)$ is a *derivation* if $\phi\mu = \mu_A(\phi \otimes A) + {}_A\mu(A \otimes \phi)$. We denote by $\mathrm{Der}(A, M)$ (resp. $\mathrm{Der}(A)$) the abelian group of derivations between A and M (resp. between A and itself).

3.2. REMARK. If the category C is closed, we can internalize the notion of derivation, so that $\mathrm{Der}(A, M)$ is an object of C. For example, for a differential graded algebra A and an A-bimodule M in Ch_k, the group $\mathrm{Der}(A, M)$ extends to a chain complex.

3.3. DEFINITION. [Σ-shifted double bracket] Let (A, μ) be a monoid of $(\mathsf{C}, \otimes)$. A Σ-*shifted double bracket* or Σ-*double bracket* on A is a morphism

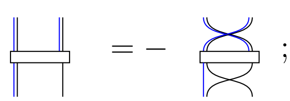
$$f \coloneqq \{\!\{-,-\}\!\}\colon \Sigma A \otimes \Sigma A \longrightarrow \Sigma A \otimes A,$$

represented by the directed coloured graph



where the direction is from top to bottom and where blue edges represent the suspension Σ. The Σ-double bracket f

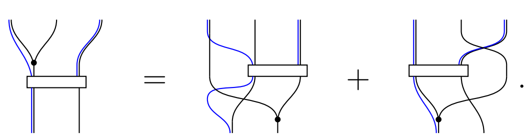
- is *antisymmetric* if $\{\!\{-,-\}\!\} = -\Sigma\tau_{A,A}\{\!\{-,-\}\!\}\tau_{\Sigma A,\Sigma A}$, i.e. in terms of directed graphs:



- is *a left derivation* if the double bracket f is a derivation in its first variable for the internal A-bimodule structure of $\Sigma A \otimes A$, i.e.

$$f(\Sigma\mu \otimes \Sigma A) = {}_A\mu^i(A \otimes f)(\tau_{\Sigma,A} \otimes A \otimes \Sigma A) + \mu^i_A(f \otimes A)(\Sigma A \otimes \tau_{A,\Sigma A}).$$

  This property can be described in terms of directed graphs:



If $\Sigma = \mathbb{1}$, a Σ-shifted double bracket is just called a double bracket.

3.4. Definition. [Compatible morphism] Let $(L, \{\!\{-,-\}\!\}_L)$ and $(H, \{\!\{-,-\}\!\}_H)$ be two objects of the category C, each equipped with a Σ-double bracket. A morphism $\phi\colon L \to H$ of C is said to be *compatible* with the Σ-double brackets if the following diagram commutes:

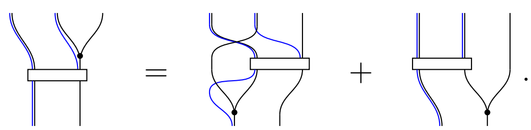
$$\begin{tikzcd} \Sigma L \otimes \Sigma L \arrow[r, "\Sigma\phi\otimes\Sigma\phi"] \arrow[d, "\{\!\{-,-\}\!\}_L"'] \arrow[dr, phantom, "\circlearrowleft"] & \Sigma H \otimes \Sigma H \arrow[d, "\{\!\{-,-\}\!\}_H"] \\ \Sigma L \otimes L \arrow[r, "\Sigma\phi\otimes\phi"'] & \Sigma H \otimes H \end{tikzcd}.$$

3.5. Remark. Suppose that (A, μ, ι) is a monoid with unit $\iota\colon \mathbb{1} \to A$ in the category C, with a Σ-double bracket f which is a left derivation. Then, as the following diagram commutes

$$\begin{tikzcd} \mathbb{1} \otimes \mathbb{1} \arrow[r, "\iota\otimes\iota"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "\mu"] \\ \mathbb{1} \arrow[r, "\iota"'] & A \end{tikzcd},$$

the morphism $\mathbb{1} \otimes A \xrightarrow{f(\iota\otimes 1)} A \otimes A$ is trivial.

3.6. Proposition. *Let (A, μ) be a monoid of the category C, with an antisymmetric Σ-double bracket f which is a left derivation. Then, the Σ-double bracket f is also a* right derivation, *i.e. it is a derivation in its second variable for the external A-bimodule structure of $\Sigma A \otimes A$. In terms of directed graphs:*



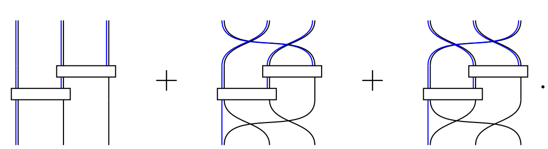
3.7. Definition. [Double Jacobiator] Let A be an object of the category C, with a Σ-double bracket $f \coloneqq \{\!\{-,-\}\!\}\colon \Sigma A \otimes \Sigma A \to \Sigma A \otimes A$. The *double Jacobiator* associated to f is the morphism

$$\mathrm{DJ}_f \coloneqq \{\!\{-,-,-\}\!\}\colon \Sigma A \otimes \Sigma A \otimes \Sigma A \to \Sigma A \otimes A \otimes A$$

defined by

$$\{\!\{-,-,-\}\!\}_l + \Sigma\tau_{A,A\otimes A}\{\!\{-,-,-\}\!\}_l\tau_{\Sigma A\otimes\Sigma A,\Sigma A} + \Sigma\tau_{A\otimes A,}\{\!\{-,-,-\}\!\}_l\tau_{\Sigma A,\Sigma A\otimes\Sigma A}$$

where $\{\!\{-,-,-\}\!\}_l = (f \otimes A)(\Sigma A \otimes f)$; we can describe the double Jacobiator diagrammatically by the following sum of directed graphs:



3.8. Remark. The double Jacobiator is stable under the diagonal action of $\mathbb{Z}/3\mathbb{Z}$, i.e. $\mathrm{DJ}_f = \Sigma\tau_{A,A\otimes A} \circ \mathrm{DJ}_f \circ \tau_{\Sigma A\otimes\Sigma A,\Sigma A}$.

3.9. Definition. [Σ-double Lie algebra] Let L be an object of the category C, with an antisymmetric Σ-double bracket $f := \{\!\{-,-\}\!\}\colon \Sigma L\otimes\Sigma L \to \Sigma L\otimes L$. The double bracket f is a Σ-double Lie bracket if the associated double Jacobiator vanishes, i.e. $\mathrm{DJ}_f = 0$. In this case, we say that L is a Σ-*double Lie* algebra. The category of Σ-double Lie algebras in C is denoted by $\Sigma\text{-}\mathsf{DLie}_{\mathsf{C}}$; its morphisms are the morphisms of the category C which are compatible with the Σ-double brackets (in the sense of definition 3.4).

3.10. Definition. [The category $\Sigma\text{-}\mathsf{DPoiss}_{\mathsf{C}}$] Let A be an object of the category C. A *double Poisson structure on* A is the data of a monoidal product $\mu\colon A\otimes A\to A$ and a Σ-double Lie bracket $f := \{\!\{-,-\}\!\}\colon \Sigma A\otimes\Sigma A \longrightarrow \Sigma A\otimes A$, which also satisfies the left derivation property: such a Σ-double bracket is called a Σ-*double Poisson bracket* and A is a Σ-*double Poisson algebra* in the category C.

Let (A, f_A) and (B, f_B) be two Σ-double Poisson algebras and $\phi\colon A\to B$ a morphism in C. The morphism ϕ is a Σ-*double Poisson algebra morphism* if ϕ is a monoid morphism and a Σ-double Lie morphism. We denote by $\Sigma\text{-}\mathsf{DPoiss}_{\mathsf{C}}$ the category of Σ-double Poisson algebras in C, so that there are forgetful functors

$$\Sigma\text{-}\mathsf{DPoiss}_{\mathsf{C}} \longrightarrow \Sigma\text{-}\mathsf{DLie}_{\mathsf{C}} \quad\text{and}\quad \Sigma\text{-}\mathsf{DPoiss}_{\mathsf{C}} \longrightarrow \mathsf{As}(\mathsf{C}).$$

3.11. Definition. [Left Σ-Leibniz algebra] Let L be a object in C, Σ an invertible object in C and $f\colon \Sigma L\otimes\Sigma L\to\Sigma L$. The pair (L, f) is a *left Σ-Leibniz algebra* if f satisfies the Leibniz identity:

$$f(\Sigma A\otimes f) = f(f\otimes\Sigma A) + f(\Sigma A\otimes f)(\tau_{\Sigma A,\Sigma A}\otimes\Sigma A).$$

3.12. Proposition. [cf. [31]] *Let (A,μ) be a monoid in C equipped with a Σ-double Poisson bracket f. Then $(\Sigma A, \Sigma\mu f)$ is a Σ-left Leibniz algebra in C.*

3.13. Double Poisson structure on a free monoid. Fix (A,μ) a monoid in C and M an A-bimodule. Recall that an A algebra is a monoid B in the category C equip with a morphism of monoid $A\to B$. We consider the free A-algebra on M:

$$\mathrm{T}_A(M) := A \oplus \bigoplus_{n\in\mathbb{N}^*} M^{\otimes_A n}$$

satisfying the following universal property: for an A-algebra B with an A-bimodule morphism $M\to B$, we have the following canonical extension

$$\begin{tikzcd} M \arrow[r] \arrow[d, hook] & B \\ \mathrm{T}_A(M) \arrow[ur, dotted, "\exists!\phi"'] & \end{tikzcd}$$

with ϕ an A-algebra morphism. There is a canonical inclusion $A\oplus M \hookrightarrow \mathrm{T}_A M$. Then, we have the following result:

3.14. Lemma. *Let A be a monoid and M be an A-bimodule. An antisymmetric Σ-double bracket on $\mathrm{T}_A(M)$ that satisfies the left derivation property is determined by its restrictions to $\Sigma A \otimes \Sigma A$, $\Sigma M \otimes \Sigma A$ and $\Sigma M \otimes \Sigma M \subset \Sigma \mathrm{T}_A(M) \otimes \Sigma \mathrm{T}_A(M)$.*

Let A and M be fixed in C, with A a monoid, M an A-bimodule and let $\{\!\{-,-\}\!\}$ be a Σ-double bracket on $\mathrm{T}_A(M)$. We will define three classes of Poisson double brackets on $\mathrm{T}_A(M)$ using the terminology of [25].

3.14.1. Constant double brackets.

3.15. Definition. [Constant double Poisson bracket] The double bracket $\{\!\{-,-\}\!\}$ is a *constant* Poisson double bracket if its restrictions to $\Sigma A \otimes \Sigma A$ and $\Sigma M \otimes \Sigma A$ vanish and if its restriction to $\Sigma M \otimes \Sigma M$ takes values in $\Sigma A \otimes A$, i.e. $\{\!\{-,-\}\!\}$ is completely defined by the morphism

$$\{\!\{-,-\}\!\} \colon \Sigma M \otimes \Sigma M \longrightarrow \Sigma A \otimes A.$$

3.16. Example. If $A = \mathbb{1}$, a constant Σ-double Poisson bracket on $\mathrm{T}_{\mathbb{1}}(M)$, corresponds to an antisymmetric bilinear form on ΣM.

3.17. Remark. This example should be compared to the commutative case: for every finite-dimensional vector space V of degree d, there is a natural one-to-one correspondence between constant Poisson structures on V and skew-symmetric matrices of size d (see [19, Proposition 6.2]).

3.17.1. Linear double brackets.

3.18. Definition. [Linear double Poisson bracket] The double bracket $\{\!\{-,-\}\!\}$ is a *linear* Poisson double bracket if its restriction to $\Sigma A \otimes \Sigma A$ vanishes and its restrictions to $\Sigma M \otimes \Sigma A$ and $\Sigma M \otimes \Sigma M$ take values respectively in $\Sigma A \otimes A$ and $\Sigma(M \otimes A \oplus A \otimes M)$, that is if $\{\!\{-,-\}\!\}$ is determined by the morphisms

$$\begin{aligned} &\{\!\{-,-\}\!\} \colon \Sigma M \otimes \Sigma A \longrightarrow \Sigma A \otimes A \quad \text{and} \\ &\{\!\{-,-\}\!\} \colon \Sigma M \otimes \Sigma M \longrightarrow \Sigma(M \otimes A \oplus A \otimes M)\ . \end{aligned}$$

We denote by Σ-$\mathsf{DPFree}_A^{\mathrm{lin}}$ the category where objects are free A-algebras with a linear Σ-double Poisson bracket, the morphisms are Σ-double Poisson algebra morphisms induced by an A-bimodule morphism.

3.19. Example. [cf. [25, Sect. 2]] We consider $\{\!\{-,-\}\!\}$, a linear $\mathbb{1}$-double Poisson bracket on $T_{\mathbb{1}}M$, which is determined by morphisms $f : M \otimes M \longrightarrow M \otimes \mathbb{1}$ and $g : M \otimes \mathbb{1} \longrightarrow \mathbb{1} \otimes \mathbb{1}$. By the derivation property of $\{\!\{-,-\}\!\}$, the morphism g vanishes and the double Jacobi identity gives us the identity

$$pr_{M \otimes \mathbb{1} \otimes \mathbb{1}} \{\!\{-,-,-\}\!\}|_{M^{\otimes 3}} = 0;$$

so we have $(f \otimes \mathbb{1})(M \otimes f) - (\tau_{\mathbb{1},M} \otimes \mathbb{1})(\mathbb{1} \otimes f)(f \otimes M) = 0$, which is equivalent to

$$f(M \otimes f) = f(f \otimes M).$$

This corresponds to an associative monoid structure on M (without unit). Futhermore, we have the equivalence of categories

$$\mathbb{1}\text{-}\mathsf{DPFree}_{\mathbb{1}}^{\mathrm{lin}} \cong \mathsf{As}(\mathsf{C}).$$

3.20. Remark. This example should be compared to the commutative case: for every finite-dimensional vector space V, there is a natural one-to-one correspondence between linear Poisson structures on V^* and Lie algebra structure on V (see [19, Proposition 7.3]).

3.20.1. Quadratic double Poisson brackets.

3.21. Definition. [Quadratic double Poisson bracket] We say that $\{\{-,-\}\}$ is a *quadratic* double Poisson bracket if its restrictions to $\Sigma A \otimes \Sigma A$ and $\Sigma M \otimes \Sigma A$ are trivial and its restriction to $\Sigma M \otimes \Sigma M$ takes values in $\Sigma M \otimes M$, i.e. $\{\{-,-\}\}$ is determined by the antisymmetric Σ-double bracket

$$\{\{-,-\}\} \colon \Sigma M \otimes \Sigma M \longrightarrow \Sigma M \otimes M.$$

We denote by $\Sigma\text{-}\mathsf{DPFree}_A^{\mathrm{quad}}$ the subcategory of free A-algebras with a quadratic Σ-double Poisson bracket, where morphisms are induced by A-bimodules morphisms.

3.22. Proposition. *The free associative functor* $\mathrm{T}_{\mathbb{1}}(-)$ *induces an equivalence of categories*

$$\mathrm{T}_{\mathbb{1}}(-) \colon \Sigma\text{-}\mathsf{DLie}_{\mathsf{C}} \xrightarrow{\cong} \Sigma\text{-}\mathsf{DPFree}_{\mathbb{1}}^{\mathrm{quad}}.$$

Proof. We extend the Σ-double Lie bracket by the derivation property. ∎

3.23. Example. In [25, Sect. 2.1], Odesskii et al. give a complete classification of quadratic double Poisson brackets on $\mathbb{C}\langle x, y\rangle$ with $|x| = |y| = 0$.

3.24. Example. In [27], Sokolov gives a complete classification of quadratic double Poisson brackets on $\mathbb{C}\langle x, y, z\rangle$ with $|x| = |y| = |z| = 0$.

4. Double Lie–Rinehart algebras

In this section, we extend the notion of a double Lie–Rinehart algebra, first defined by Van den Bergh in [32], which is a noncommutative version of a Lie–Rinehart algebra.

4.1. Recollections on Lie–Rinehart algebras. For a more complete exposition, the reader is refered to [17], [22, Sect. 13.3.8] and [14, Sect. 5.1.2].

4.2. Definition. [Lie–Rinehart algebra] Let $(\mathsf{C}, \otimes, \tau)$ be an additive symmetric monoidal category and A a commutative monoid in C. A Lie algebra $(\mathfrak{g}, [-,-])$ is a *Lie–Rinehart algebra* over A if $\mathfrak{g}$ is an A-module for ${}_A\mu \colon A \otimes \mathfrak{g} \to \mathfrak{g}$, A is a $\mathfrak{g}$-module for $\rho \colon \mathfrak{g} \otimes A \to A$ (which is called *the anchor*) and these module structures are compatible, i.e. satisfy the following properties.

1. The Lie algebra $\mathfrak{g}$ acts by derivations on A:

$$\rho \circ \mu = \mu(A \otimes \rho)(\tau_{A,\mathfrak{g}} \otimes A) + \mu(\rho \otimes A) \ .$$

2. The bracket and the anchor satisfy the *Leibniz relation*:

$$[-,-](\mathfrak{g} \otimes_A \mu) = [-,-](\rho \otimes \mathfrak{g}) +_A \mu(A \otimes [-,-])(\tau_{A,\mathfrak{g}} \otimes \mathfrak{g}) \ .$$

3. The bracket and the anchor satisfy the compatibility relation in $\mathrm{Hom}(\mathfrak{g}^{\otimes 2} \otimes A, A)$:

$$\rho\big([-,-] \otimes A\big) = \rho(\mathfrak{g} \otimes \rho) - \rho(\mathfrak{g} \otimes \rho)(\tau_{\mathfrak{g},\mathfrak{g}} \otimes A).$$

Let $(M, \rho_M, [-,-]_M)$ and $(N, \rho_N, [-,-]_N)$ be two Lie–Rinehart algebras over A. An A-module morphism $\phi\colon M \to N$ is a *morphism of Lie–Rinehart algebras* if ϕ is a Lie algebra morphism.

4.3. Remark. In the case where C is closed, condition 1 holds if and only if the anchor corresponds to a morphism $\rho^* : \mathfrak{g} \to \mathrm{Der}(A)$. Then, condition 3 holds if and only if ρ^* is a Lie algebra morphism.

For examples, the reader is refered to [17, Ex. 1.3.3] or [14, Chap. 5].

4.4. Proposition. [cf. [22, Prop. 13.3.8]] *Any Lie–Rinehart algebra (A, L) gives rise to a Poisson algebra $P = A \oplus L$, where $A \oplus L$ is the square-zero extension as algebra and the operations μ and $\{-,-\}$ are as follows:*

$$\begin{aligned} A \otimes A \xrightarrow{\mu} A, \quad & A \otimes A \xrightarrow{\{-,-\}} 0, \\ A \otimes L \xrightarrow{\mu} L, \quad & L \otimes A \xrightarrow{\{-,-\}} A, \\ L \otimes L \xrightarrow{\mu} 0, \quad & L \otimes L \xrightarrow{\{-,-\}} L. \end{aligned}$$

Conversely, any Poisson algebra P, whose underlying vector space can be split as $P = A \oplus L$ and such that the two operations take values as indicated above, defines a Lie–Rinehart algebra. The two constructions are inverse to each other.

4.5. Remark. (cf. [33, Prop. 3.6.2]) This result is operadic. In fact, a Lie–Rinehart algebra is an algebra over the two-coloured operad $\mathcal{L}ie\mathcal{R}in$.

4.6. Double Lie–Rinehart algebras. Now, we introduce the noncommutative version of Lie–Rinehart algebras: double Lie–Rinehart algebras (called double Lie algebroids by Van den Bergh in [32]; note that this notion is not related in any manner with those of Mackenzie–Xu).

4.7. Notation. Let A be a monoid in an additive symmetric monoidal category $(\mathsf{C}, \otimes, \tau)$, M and N two A-bimodules and $\phi : M \to N$ a morphism of A-bimodules. We canonically extend the morphism ϕ to an A-bimodule morphism $\tilde{\phi}\colon A\otimes M \oplus M\otimes A \longrightarrow A\otimes N\oplus N\otimes A$ where the structures of A-bimodule are induced by those of M and N and such that the restrictions to $A \otimes M$ and $M \otimes A$ are given by

$$\tilde{\phi}|_{A\otimes M} = A \otimes \phi \quad \text{ and } \quad \tilde{\phi}|_{M\otimes A} = \phi \otimes A.$$

Hereafter, we do not distinguish between ϕ and $\tilde{\phi}$.

4.8. Definition. [Σ-Double Lie–Rinehart algebra] Let (A, μ) be a monoid in an additive symmetric monoidal category $(\mathsf{C}, \otimes, \tau)$ and $(M, {}_A\mu, \mu_A)$ an A-bimodule. We say that M is a Σ-*double Lie–Rinehart algebra over* A if M is equipped with:

1. an A-bimodule morphism (called *the anchor*)

$$\rho\colon \Sigma M \otimes \Sigma A \longrightarrow \Sigma A \otimes A$$

(where, for the left term, the A-bimodule structure induced by that of M and, for the right term, the internal structure) which is a derivation in the second input for the external A-bimodule structure on the codomain;

2. a morphism

$$\{\!\{-,-\}\!\}^M : \Sigma M \otimes \Sigma M \longrightarrow \Sigma(M \otimes A \oplus A \otimes M)$$

with components $\{\!\{-,-\}\!\}^M_l := pr_{\Sigma M\otimes A} \circ \{\!\{-,-\}\!\}^M$ and $\{\!\{-,-\}\!\}^M_r := pr_{\Sigma A\otimes M} \circ \{\!\{-,-\}\!\}^M$; which satisfy the following conditions:

(Antisymmetry):

$$\{\!\{-,-\}\!\}^M \tau_{\Sigma M,\Sigma M} = -\Sigma(\tau_{M,A}, \tau_{A,M})\{\!\{-,-\}\!\}^M \ ;$$

The derivation property (Derivation): the first compatibility with the anchor: we have the following commutative diagram

$$\begin{tikzcd}
\Sigma M \otimes \Sigma A \otimes M \arrow[r, "\Sigma M\otimes \Sigma_A\mu"] \arrow[dr, bend right, "\phi^l"'] & \Sigma M \otimes \Sigma M \arrow[d, "\{\!\{-,-\}\!\}^M"] & \Sigma M \otimes \Sigma M \otimes A \arrow[l, "\Sigma M\otimes \Sigma\mu_A"'] \arrow[dl, bend left, "\phi^r"] \\
 & \Sigma(M \otimes A \oplus A \otimes M), &
\end{tikzcd}$$

where

$$\begin{aligned}
\phi^l :=& (\Sigma A \otimes_A \mu)(\rho \otimes M) \\
&+ ({}_A\mu^{\Sigma M} \otimes A, {}_A\mu^{\Sigma A} \otimes M)(A \otimes \{\!\{-,-\}\!\}^M)(\tau_{\Sigma M \Sigma, A} \otimes M) \\
\phi^r :=& (\Sigma M \otimes \mu, \Sigma A \otimes \mu_A)(\{\!\{-,-\}\!\}^M \otimes A) \\
&+ (\Sigma\mu_A \otimes A)(\tau_{M,\Sigma} \otimes A \otimes A)(M \otimes \rho)(\tau_{\Sigma M \Sigma, M} \otimes A) \ ;
\end{aligned}$$

The anchor relation (Anchor): the second compatibility with the anchor: we have the following relation in $\mathrm{Hom}(\Sigma A \otimes (\Sigma M)^{\otimes 2}, \Sigma A^{\otimes 3})$

$$\begin{aligned}
&(\rho_\tau \otimes A)(\Sigma A \otimes \{\!\{-,-\}\!\}_l^M) \\
&\quad + \quad \tau_{\Sigma A\otimes\Sigma A,\Sigma A}(\rho \otimes A)(\Sigma M \otimes \rho)\tau_{\Sigma A,\Sigma M\otimes\Sigma M} \\
&\quad + \quad \tau_{\Sigma A,\Sigma A\otimes\Sigma A}(\rho \otimes A)(\Sigma M \otimes \rho_\tau)\tau_{\Sigma A\otimes\Sigma M,\Sigma M} = 0
\end{aligned}$$

where $\rho_\tau \coloneqq -(\Sigma\tau_{A,A})\rho\tau_{\Sigma A,\Sigma M} : \Sigma A \otimes \Sigma M \to \Sigma A \otimes A$;

The double Jacobi identity (Double Jacobi): which is the following relation in $\mathrm{Hom}((\Sigma M)^{\otimes 3}, \Sigma M \otimes A^{\otimes 2})$:

$$\begin{aligned}
&(\{\!\{-,-\}\!\}_l^M \otimes A)(\Sigma M \otimes \{\!\{-,-\}\!\}_l^M) \\
&\quad + \quad \Sigma\tau_{A,M\otimes A}(\{\!\{-,-\}\!\}_r^M \otimes A)(\Sigma M \otimes \{\!\{-,-\}\!\}_l^M)\tau_{\Sigma M\otimes\Sigma M,\Sigma M} \\
&\quad + \quad \Sigma\tau_{A\otimes A,M}(\rho \otimes A)(\Sigma M \otimes \{\!\{-,-\}\!\}_r^M)\tau_{\Sigma M,\Sigma M\otimes\Sigma M} = 0.
\end{aligned}$$

Let $(M, \{\!\{-,-\}\!\}^M)$ and $(N, \{\!\{-,-\}\!\}^N)$ be two Σ-double Lie–Rinehart algebras over A. A Σ-*double Lie–Rinehart algebra morphism* ϕ is an A-bimodule morphism such that:

$$\phi\big(\{\!\{-,-\}\!\}^M\big) = \{\!\{\phi(-), \phi(-)\}\!\}^N.$$

We denote by $\Sigma\text{-}\mathsf{DLR}_A$ the category of Σ-double Lie–Rinehart algebras over A.

4.9. Remark. In the case where C is a *closed* symmetric monoidal category (for example, $\mathsf{C} = \mathsf{Ch}_k$), then, by adjunction, the anchor of a Σ-double Lie–Rinehart algebra A is equivalent to the A-bimodule morphism $\rho^* : \Sigma M \to \mathrm{Der}(\Sigma A, \Sigma A \otimes A)$.

4.10. Remark. When the category C is the category Ch_k and when A is a finitely generated differential graded associative algebra, the condition ((Anchor)) can be expressed using the Schouten–Nijenhuis double bracket (introduced in section 5.11 below), as:

$$\rho^*\big(\{\!\{-,-\}\!\}^M\big) = \{\!\{\rho^*, \rho^*\}\!\}^{\mathrm{SN}}$$

4.11. Example.

1. For a finitely-generated differential graded associative algebra A, the A-bimodule $\mathbb{D}\mathrm{er}(A)$ of biderivations (cf. section 5.11 for the definition) with the Schouten–Nijenhuis double Poisson bracket and the identity plays for the anchor, is a 0-double Lie–Rinehart algebra over A.

2. In [14, Sect. 5.5], the noncommutative version of the Atiyah algebra is defined as follows. Let A be an associative k-algebra and M a finitely-presented A-bimodule. We denote $\mathbb{E}\mathrm{nd}(M)$, the A-bimodule $\mathrm{Hom}_k(M, M \otimes A \oplus A \otimes M)$, and, for ϕ in $\mathbb{E}\mathrm{nd}(M)$, we denote $\phi_l \coloneqq pr_{M\otimes A} \circ \phi$ and $\phi_r \coloneqq pr_{A\otimes M} \circ \phi$, the compositions with the projections. The *Atiyah double algebra on* M, denoted $\mathbb{A}\mathrm{t}(M)$, is the set of pairs (d, ϕ) with $d \in \mathbb{D}\mathrm{er}(A)$ and $\phi \in \mathbb{E}\mathrm{nd}(M)$, with compatibilities analogous to

the commutative case. *The Atiyah double bracket* is defined as follow: for (d^1, ϕ^1) and (d^2, ϕ^2) in $\mathbb{A}\mathrm{t}(M)$

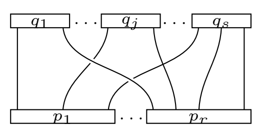
$$\{\!\{(d^1,\phi^1),(d^2,\phi^2)\}\!\}^{\mathbb{A}\mathrm{t}} := \left(\{\!\{d^1,d^2\}\!\}^{\mathrm{SN}}, \{\!\{(d^1,\phi^1),(d^2,\phi^2)\}\!\}^{\mathbb{E}}\right),$$

where $\{\!\{-,-\}\!\}^{\mathrm{SN}}$ is the Schouten–Nijenhuis double Poisson bracket (cf. section 5.11) and $\{\!\{-,-\}\!\}^{\mathbb{E}}$ is described in [14]. By [14, Prop. 5.5.3], $\mathbb{A}\mathrm{t}(M)$ equipped with the Atiyah double bracket and the anchor morphism given by

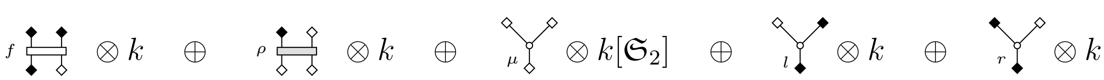
$$\begin{array}{rccc} \rho: & \mathbb{A}\mathrm{t}(M) & \longrightarrow & \mathbb{D}\mathrm{er}(A) \\ & (d,\phi) & \longmapsto & d \end{array}$$

is a 0-double Lie–Rinehart algebra over A and ρ is a morphism of double Lie–Rinehart algebras.

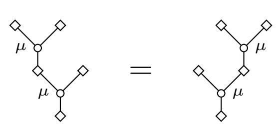
4.12. Remark. A double Lie–Rinehart algebra is an algebra over a (coloured) properad. Properads encode algebraic structures which have several inputs and outputs: they generalize operads, which encode algebraic structures with several inputs and one output (see [22]). Formally, a properad is a $\mathfrak{S}$-bimodule, i.e. a family of $\mathfrak{S}_n \times \mathfrak{S}_m^{op}$-module with m and n in $\mathbb{N}^*$, which is a monoid for the connected product $\boxtimes_c$ defined by Vallette in [28, 29]. This product is controlled by connected graphs: for two $\mathfrak{S}$-bimodules P and Q, elements in $P \boxtimes_c Q$ can be described as a sum of graphs of the following form:



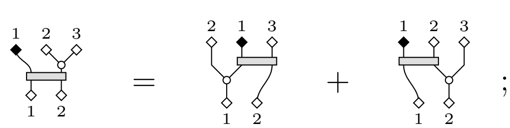
where $p_1, \ldots, p_r$ are elements in P and $q_1, \ldots, q_s$ are in Q. There is a notion of free properad (see [29, Sect. 2.7]), so we can talk about properads presented by generators and relations. As for operads, we have the notion of coloured properads (for the definition, the reader can refer to [16]). The 2-coloured properad $\mathcal{DL}ie\mathcal{R}in$ which encodes double Lie–Rinehart algebras (cf. definition 4.8), is generated by



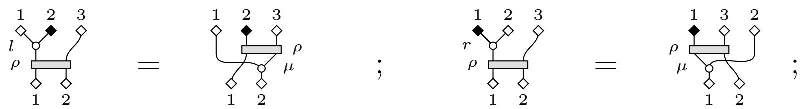
these satisfy the following relations:



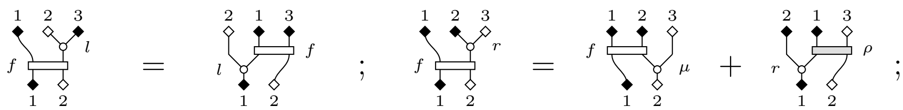
(associativity of μ)



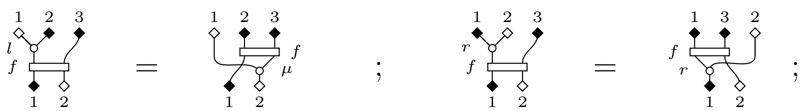
((1) - derivation)



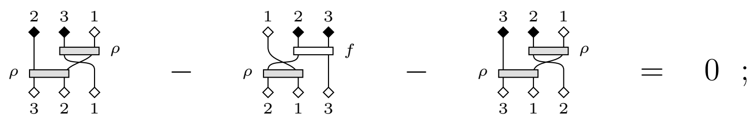
((1) - bimodule)



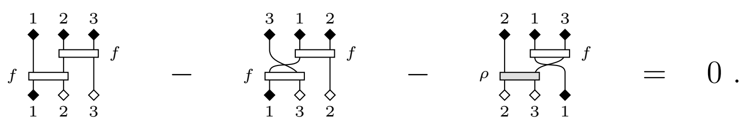
(right derivation)



(bimodule)



((Anchor))



((Double Jacobi))

In the next proposition, we establish the noncommutative version of the correspondence between Lie–Rinehart algebras and a class of Poisson algebras stated in 4.4. Namely, we explain the correspondence between Σ-double Lie–Rinehart algebras and linear Σ-double Poisson algebras.

4.13. Proposition. [cf. [31, (3.4-1)-(3.4-8)] – [32, Sect. 3.2]] *Let A be a monoid in C, M an A-bimodule and Σ an invertible object in C. The following are equivalent:*

1. *M is a Σ-double Lie–Rinehart algebra over A ;*
2. *$\mathrm{T}_A(M)$ is a linear Σ-double Poisson algebra.*

We have the equivalence of categories

$$\Sigma\text{-}\mathsf{DPFree}_A^{\mathrm{lin}} \cong \Sigma\text{-}\mathsf{DLR}_A.$$

Proof. The anchor ρ of a Σ-double Lie–Rinehart algebra M over A is a morphism $\rho : \Sigma M \otimes \Sigma A \to \Sigma A \otimes A$ which we extend, by antisymmetry, to a double bracket $\{\!\{-,-\}\!\}^A \colon \Sigma M \otimes \Sigma A \oplus \Sigma A \otimes \Sigma M \to \Sigma A \otimes A$.

The condition (Derivation) of definition 4.8 corresponds to the derivation properties of the restriction $\Sigma M \otimes \Sigma A \oplus \Sigma A \otimes \Sigma M$ of a linear Σ-double bracket on $\mathrm{T}_A M$. Hence, a linear Σ-double bracket on $\mathrm{T}_A M$ corresponds to an anchor $\rho \colon \Sigma M \otimes \Sigma A \to \Sigma A \otimes A$ and a morphism $f \colon \Sigma M \otimes \Sigma M \to \Sigma(M \otimes A \oplus A \otimes M)$ such that conditions (Antisymmetry) and (Derivation) of definition 4.8 are satisfied.

We will check that the conditions (Anchor) and (Double Jacobi) of definition 4.8 exactly correspond to the double Jacobi identity for the associated linear Σ-double bracket on $\mathrm{T}_A M$. The double Jacobiator of $\mathrm{T}_A M$, restricted to $\Sigma A \otimes \Sigma M \otimes \Sigma M$ is given by the following diagram

$$\begin{tikzcd}
\Sigma M \otimes \Sigma A \otimes \Sigma M \arrow[d, "(\rho^*\otimes A)(\Sigma M\otimes \rho_r^*)"] & \Sigma A \otimes \Sigma M \otimes \Sigma M \arrow[l, "\tau_{\Sigma A\otimes \Sigma M, \Sigma M}"'] \arrow[r, "\tau_{\Sigma A, \Sigma M\otimes\Sigma M}"] \arrow[d, "(\rho_r^*\otimes A)(\Sigma A\otimes \{\!\{-,-\}\!\}_l)"] & \Sigma M \otimes \Sigma M \otimes \Sigma A \arrow[d, "(\rho^*\otimes A)(\Sigma A\otimes \rho^*)"] \\
\Sigma(A)^{\otimes 3} \arrow[r, "\Sigma\tau_{A, A\otimes A}"'] & \Sigma(A)^{\otimes 3} & \Sigma(A)^{\otimes 3}, \arrow[l, "\Sigma\tau_{A\otimes A, A}"]
\end{tikzcd}$$

then, the morphisms ρ and $\{\!\{-,-\}\!\}$ satisfy the condition (Anchor) of the Σ-double Lie–Rinehart algebra. The restriction to $(\Sigma M)^{\otimes 3}$ of the double Jacobiator on $\mathrm{T}_A M$ takes values in

$$\Sigma(M \otimes A^{\otimes 2} \oplus A^{\otimes 2} \otimes M \oplus A \otimes M \otimes A).$$

By invariance under the $\mathbb{Z}/3\mathbb{Z}$-action (see remark 3.8), the vanishing of this restriction is equivalent to the vanishing of its projection to $\Sigma M \otimes A^{\otimes 2}$. This projection is given by the sum of the morphisms $(\Sigma M)^{\otimes 3} \to \Sigma M \otimes A^{\otimes 2}$ given in the following commutative diagram

$$\begin{tikzcd}
(\Sigma M)^{\otimes 3} \arrow[d, "(\{\!\{-,-\}\!\}_r^M\otimes A)(\Sigma M\otimes \{\!\{-,-\}\!\}_l^M)"] & (\Sigma M)^{\otimes 3} \arrow[l, "\tau_{\Sigma M\otimes \Sigma M, \Sigma M}"'] \arrow[r, "\tau_{\Sigma M, \Sigma M\otimes\Sigma M}"] \arrow[d, "(\{\!\{-,-\}\!\}_l^M\otimes A)(\Sigma A\otimes \{\!\{-,-\}\!\}_l^M)"] & (\Sigma M)^{\otimes 3} \arrow[d, "(\rho^M\otimes A)(\Sigma M\otimes \{\!\{-,-\}\!\}_r^M)"] \\
\Sigma A \otimes M \otimes A \arrow[r, "\Sigma\tau_{A, M\otimes A}"'] & \Sigma M \otimes A^{\otimes 2} & \Sigma A^{\otimes 2} \otimes M; \arrow[l, "\Sigma\tau_{A\otimes A, M}"]
\end{tikzcd}$$

the vanishing of the restriction of the double Jacobiator to $(\Sigma M)^{\otimes 3}$ is equivalent to the condition (Double Jacobi) of definition 4.8.

Then, if we consider $(\mathrm{T}_A M, f := \{\!\{-,-\}\!\})$ a linear Σ-double Poisson algebra, by taking the following restrictions of the linear Σ-double bracket, the morphisms

$$\rho^M := f|_{\Sigma M\otimes A} \quad \text{and} \quad \{\!\{-,-\}\!\}^M := f|_{\Sigma M\otimes\Sigma M}$$

make M a Σ-double Lie–Rinehart algebra over A. Conversely, consider $(M, \rho, \{\!\{-,-\}\!\})$ a Σ-double Lie–Rinehart algebra over A. By the universal property of $\mathrm{T}_A(M)$ (see 3.13), we extend by derivation the morphism $\rho\colon \Sigma M \otimes A \to \Sigma A \otimes A$ to

$$\widetilde{\rho}\colon \Sigma\mathrm{T}_A(M) \otimes A \to \Sigma\mathrm{T}_A(M) \otimes \mathrm{T}_A(M),$$

which is a left derivation. We extend $\widetilde{\rho}$ to a morphism

$$\{\!\{-,-\}\!\}^A\colon \Sigma\mathrm{T}_A(M) \otimes \Sigma\mathrm{T}_A(M) \to \Sigma\mathrm{T}_A(M) \otimes \mathrm{T}_A(M)$$

by antisymmetry. Similarly, we extend $\{\!\{-,-\}\!\}$ to a double derivation

$$\{\!\{-,-\}\!\}^M\colon \Sigma\mathrm{T}_A(M) \otimes \Sigma\mathrm{T}_A(M) \to \Sigma\mathrm{T}_A(M) \otimes \mathrm{T}_A(M).$$

The sum $\{\!\{-,-\}\!\}^A+\{\!\{-,-\}\!\}^M$ gives a linear Σ-double Poisson bracket on $\mathrm{T}_A(M)$ because we have proved that the double Lie–Rinehart conditions (Antisymmetry), (Derivation), (Anchor) and (Double Jacobi) are equivalent to the the double Poisson conditions.

Let $(M,\{\!\{-,-\}\!\}^M)$ and $(N,\{\!\{-,-\}\!\}^N)$ be two Σ-double Lie–Rinehart algebras and $\phi\colon M\to N$, a double Lie–Rinehart algebra morphism. The morphism ϕ induces a morphism of A-algebras $\phi'\colon \mathrm{T}_AM\to \mathrm{T}_AN$. We have $\phi\big(\{\!\{-,-\}\!\}^M\big)=\{\!\{\phi(-),\phi(-)\}\!\}^N$ and, as ϕ' is an algebra morphism and the double brackets on $\{\!\{-,-\}\!\}^M$ and $\{\!\{-,-\}\!\}^N$ are constructed by extending using the derivation property, then ϕ' is a Σ-double Poisson algebra morphism. Hence, we have defined the functor

$$\mathrm{T}_A(-)\colon \Sigma\text{-}\mathsf{DPFree}_A^{\mathrm{lin}}\to \Sigma\text{-}\mathsf{DLR}_A. \tag{1}$$

Let $\psi\colon \mathrm{T}_AM\to \mathrm{T}_AN$ be a morphism of linear Σ-double Poisson algebras. The morphism $\psi'\colon pr_N\circ\psi\circ i\colon M\to N$ with $i\colon M\hookrightarrow \mathrm{T}_AM$ and $pr_N\colon \mathrm{T}_AN\twoheadrightarrow N$ is an A-bimodule morphism, which commutes with the double brackets. Then, the functor (1) is an equivalence of categories. ■

4.14. Remark. By proposition 3.12, if M is a $\mathbb{1}$-double Lie–Rinehart algebra over A, then the morphism

$$\{-,-\}\coloneqq \mu_A^M\{\!\{-,-\}\!\}_l^M+_A\mu^M\{\!\{-,-\}\!\}_r^M\colon M\otimes M\longrightarrow M$$

yields a left Leibniz algebra structure on M, which is an A-bimodule morphism, where the A-bimodule structure on $M\otimes M$ is given by that of the right factor. The composition of the anchor with the product of A is a derivation in its second input:

$$\widetilde{\rho}\coloneqq \mu\circ\rho\colon M\otimes A\longrightarrow A\ .$$

By proposition 4.13 and [31, Prop. 2.4.2] generalised to the categorical framework, the double-Jacobi identity, restricted to $M\otimes M\otimes A$ implies that $\widetilde{\rho}$ gives A the structure of an antisymmetrical representation of M (for the definition of representations of left Leibniz algebras, the reader can refer to [8, Def. 1.2.1 and 1.2.4]).

5. The shifting property

5.1. The main result. In the case of algebras over an operad, a Σ-shifted structure on an object M is equivalent to a non-shifted structure on ΣM (for more detail, the reader can refer to [22]). However, this is not true for the case of algebras over a properad. For example, for a chain complex A, an r-double Lie structure on A is the datum of a double bracket

$$\{\!\{-,-\}\!\}\colon A[r]\otimes A[r]\longrightarrow (A\otimes A)[r]$$

and a 0-double Lie structure on $A[r]$ is the datum of a double bracket

$$\{\!\{-,-\}\!\}\colon A[r]\otimes A[r]\longrightarrow A[r]\otimes A[r].$$

These morphisms have different degrees. However, this shifting property does hold for double Lie–Rinehart algebras (which are algebras over the coloured properad $\mathcal{DL}ie\mathcal{R}in$). In fact, the equivalence of categories $\Sigma \otimes -\colon A\text{-}\mathsf{Bimod}_{\mathsf{C}} \to A\text{-}\mathsf{Bimod}_{\mathsf{C}}$ induces an equivalence of categories $\Sigma\text{-}\mathsf{DLR}_A \to \mathbb{1}\text{-}\mathsf{DLR}_A$. In the following theorem and its proof, we implicitly use the isomorphism $\Sigma^- \otimes \Sigma \cong \mathbb{1}$.

5.2. Theorem. *The following assertions are equivalent:*

1. $(A, M, \rho_M, \{\!\{-,-\}\!\}_M)$ *is a Σ-double Lie–Rinehart algebra;*
2. $(A, \Sigma M, \rho_{\Sigma M}, \{\!\{-,-\}\!\}_{\Sigma M})$ *is a $\mathbb{1}$-double Lie–Rinehart algebra.*

Under this correspondance, there is an equivalence of categories

$$\Sigma\text{-}\mathsf{DLR}_A \cong \mathbb{1}\text{-}\mathsf{DLR}_A.$$

5.3. Remark. In this theorem, the anchors are related by

$$\rho_{\Sigma M} = (\tau_{\Sigma^-, \Sigma M} \otimes A)(\Sigma^- \otimes \rho_M)$$

and the double brackets by

$$\{\!\{-,-\}\!\}_{\Sigma M} = (\Sigma M \otimes A, \tau_{\Sigma, A} \otimes M) \circ \{\!\{-,-\}\!\}_M.$$

Proof. An A-bimodule structure on M canonically corresponds to an A-bimodule structure on ΣM (see section 2.3). The commutative square

$$\begin{tikzcd}
\Sigma(M \otimes A \oplus A \otimes M) \arrow[r, "\cong"] \arrow[d, "\Sigma\tau_{M,A}"'] & \Sigma M \otimes A \oplus A \otimes \Sigma M \arrow[d, "\tau_{\Sigma M, A}"] \\
\Sigma(M \otimes A \oplus A \otimes M) \arrow[r, "\cong"'] & \Sigma M \otimes A \oplus A \otimes \Sigma M
\end{tikzcd}$$

implies that we have a canonical correspondance between an antisymmetrical Σ-double bracket $\{\!\{-,-\}\!\}_M\colon \Sigma M \otimes \Sigma M \longrightarrow \Sigma(M \otimes A \oplus A \otimes M)$ and an antisymmetrical $\mathbb{1}$-double bracket $\{\!\{-,-\}\!\}_{\Sigma M}\colon \Sigma M \otimes \Sigma M \longrightarrow \Sigma M \otimes A \oplus A \otimes \Sigma M$ given by

$$\{\!\{-,-\}\!\}_{\Sigma M} = (\Sigma M \otimes A, \tau_{\Sigma, A} \otimes M) \circ \{\!\{-,-\}\!\}.$$

Futhermore, $\{\!\{-,-\}\!\}_M$ satisfies the double Jacobi identity if and only if $\{\!\{-,-\}\!\}_{\Sigma M}$ does.

By remark 2.4, the functor $\Sigma^- \otimes -\colon A\text{-}\mathsf{Bimod}_{\mathsf{C}} \to A\text{-}\mathsf{Bimod}_{\mathsf{C}}$ is an equivalence of categories: the morphism $\rho_M\colon \Sigma M \otimes \Sigma A \to \Sigma A \otimes A$ corresponds to the morphism $\rho_{\Sigma M} \coloneqq (\tau_{\Sigma^-, \Sigma M} \otimes A)(\Sigma^- \otimes \rho_M)\colon \Sigma M \otimes A \to A \otimes A$. Conversely, by the equivalence $\Sigma \otimes -\colon A\text{-}\mathsf{Bimod}_{\mathsf{C}} \to A\text{-}\mathsf{Bimod}_{\mathsf{C}}$, a morphism $\rho_{\Sigma M} : \Sigma M \otimes A \to A \otimes A$ corresponds to $\rho_M : \Sigma M \otimes \Sigma A \to \Sigma A \otimes A$. So ρ_M satisfies the condition ((Anchor)) of definition 4.8 if and only if $\rho_{\Sigma M}$ satisfies the condition ((Anchor)).

It remains to establish the anchors' compatibilities (condition ((Derivation)) of definition 4.8). We need to show that the following diagram

$$
\begin{tikzcd}
 & \Sigma M \otimes A \otimes \Sigma M \arrow[r, "\Sigma M \otimes {}_A\mu"] \arrow[dl, "\cong"'] \arrow[ddrr, dotted, "\psi^l"] & \Sigma M \otimes \Sigma M \arrow[dl, "="'] \arrow[ddr, dotted, "\{\!\{-,-\}\!\}_{\Sigma M}"] & \Sigma M \otimes \Sigma M \otimes A \arrow[l, "\Sigma M \otimes \mu_A"'] \arrow[dl, "="'] \arrow[dd, "\psi^r"] \\
\Sigma M \otimes \Sigma(A \otimes M) \arrow[r, "\Sigma M \otimes \Sigma_A\mu"] \arrow[dr, bend right, "\phi^l"'] & \Sigma M \otimes \Sigma M \arrow[d, "\{\!\{-,-\}\!\}_M"] & \Sigma M \otimes \Sigma(M \otimes A) \arrow[l, "\Sigma M \otimes \Sigma\mu_A"'] \arrow[dl, bend left, "\phi^r"] & \\
 & \Sigma(M \otimes A \oplus A \otimes M) & & \Sigma M \otimes A \oplus A \otimes \Sigma M \arrow[ll, "\cong"']
\end{tikzcd}
$$

commutes, with

$$
\begin{aligned}
\psi^l &\coloneqq (({}_A\mu, \mu) \otimes M)(A \otimes \{\!\{-,-\}\!\}_{\Sigma M})(\tau_{\Sigma M, A} \otimes \Sigma M) + (A \otimes_A \mu)(\rho_{\Sigma M} \otimes \Sigma M) \ ; \\
\psi^r &\coloneqq (\Sigma M \otimes \mu + A \otimes \mu_A)(\{\!\{-,-\}\!\}_{\Sigma M} \otimes A) + (\mu_A \otimes A)(\Sigma M \otimes \rho_{\Sigma M})(\tau_{\Sigma M, \Sigma M} \otimes A) \ ; \\
\phi^l &\coloneqq ({}_A\mu \otimes M)(A \otimes \{\!\{-,-\}\!\}_M)(\tau_{\Sigma M \Sigma, A} \otimes M) + (\Sigma A \otimes_A \mu)(\rho_M \otimes M) \ ; \\
\phi^r &\coloneqq (\Sigma M \otimes \mu + \Sigma A \otimes \mu_A)(\{\!\{-,-\}\!\}_M \otimes A) \\
&\qquad + (\Sigma \mu_A \otimes A)(\tau_{M,\Sigma} \otimes A \otimes A)(M \otimes \rho_M)(\tau_{\Sigma M \Sigma, M} \otimes A).
\end{aligned}
$$

It suffices to show that the following squares

$$
\begin{tikzcd}
\Sigma M \otimes A \otimes \Sigma M \arrow[r, "\psi^l"] \arrow[d, "\Sigma M \otimes \tau_{A,\Sigma} \otimes M"', "\cong"] & \Sigma M \otimes A \oplus A \otimes \Sigma M \arrow[d, "\cong"', "(\mathrm{id}, \tau_{A,\Sigma} \otimes M)"] \\
\Sigma M \otimes \Sigma(A \otimes M) \arrow[r, "\phi^l"'] & \Sigma(M \otimes A \oplus A \otimes M)
\end{tikzcd}
\quad
\begin{tikzcd}
\Sigma M \otimes \Sigma M \otimes A \arrow[r, "\psi^r"] \arrow[d, "="'] & \Sigma M \otimes A \oplus A \otimes \Sigma M \arrow[d, "\cong"', "(\mathrm{id}, \tau_{A,\Sigma} \otimes M)"] \\
\Sigma M \otimes \Sigma(M \otimes A) \arrow[r, "\phi^r"'] & \Sigma(M \otimes A \oplus A \otimes M)
\end{tikzcd}
$$

commute. The following diagrams are commutative:

$$
\begin{tikzcd}
\Sigma M \otimes A \otimes \Sigma M \arrow[r, "\Sigma M \otimes \tau_{A,\Sigma} \otimes M"] \arrow[d, "\rho_{\Sigma M} \otimes \Sigma M"'] & \Sigma M \otimes \Sigma A \otimes M \arrow[d, "\rho_M \otimes M"] \\
A \otimes A \otimes \Sigma M \arrow[d, "A \otimes_A \mu"'] & \Sigma A \otimes A \otimes M \arrow[d, "\Sigma A \otimes_A \mu"] \\
A \otimes \Sigma M \arrow[r, "\tau_{A,\Sigma} \otimes M"'] & \Sigma A \otimes M
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma M \otimes \Sigma M \otimes A \arrow[r, "="] \arrow[d, "\tau_{\Sigma M, \Sigma M} \otimes A"'] & \Sigma M \otimes \Sigma M \otimes A \arrow[d, "\tau_{\Sigma M \Sigma, M} \otimes A"] \\
\Sigma M \otimes \Sigma M \otimes A \arrow[d, "\Sigma M \otimes \rho_{\Sigma M}"'] & M \otimes \Sigma M \otimes \Sigma A \arrow[d, "M \otimes \rho_M"] \\
\Sigma M \otimes A \otimes A \arrow[d, "\mu_A \otimes A"'] & M \otimes \Sigma A \otimes A \arrow[d, "(\Sigma\mu_A \otimes A)(\tau_{M,\Sigma} \otimes A \otimes A)"] \\
\Sigma M \otimes A \arrow[r, "="'] & \Sigma M \otimes A
\end{tikzcd}
\ ;
$$

as $\{\!\{-,-\}\!\}_{\Sigma M} = (\mathrm{id}, \tau_{\Sigma,A} \otimes M) \circ \{\!\{-,-\}\!\}_M$, the following equalities hold:

$$
\begin{aligned}
&(\mathrm{id}, \tau_{A,\Sigma} \otimes M)(({}_A\mu, \mu) \otimes M)(A \otimes \{\!\{-,-\}\!\}_{\Sigma M})(\tau_{\Sigma M, A} \otimes \Sigma M) = \\
&\qquad ({}_A\mu \otimes M)(A \otimes \{\!\{-,-\}\!\}_M)(\tau_{\Sigma M \Sigma, A} \otimes M)(\Sigma M \otimes \tau_{A,\Sigma} \otimes M)
\end{aligned}
$$

and

$$(\mathrm{id}, \tau_{A,\Sigma} \otimes M)(\Sigma M \otimes \mu + A \otimes \mu_A)(\{\!\{-,-\}\!\}_{\Sigma M} \otimes A) = (\Sigma M \otimes \mu + \Sigma A \otimes \mu_A)(\{\!\{-,-\}\!\}_M \otimes A)$$

so, finally, $(\mathrm{id}, \tau_{A,\Sigma} \otimes M) \circ \psi^r = \phi^r$. ■

5.4. Example. When C is the category of chain complexes Ch_k, consider a linear 0-double Poisson bracket on $\mathrm{T}M$, as in example 3.19, which corresponds to a (non-unital) associative product on M. Then, using proposition 4.13, the shifting property implies that $M[1]$ has a (non-unital) associative product of homological degree -1. This recovers the shifting property of algebras over the operad $\mathcal{A}s$ (see [22, Chapter 9] for the definition): a (non-unital) associative product of degree -1 on M corresponds to a non-unital associative algebra structure of degree 0 on $M[-1]$.

5.5. Example: the Koszul double bracket. For this example, we suppose that the category C has coequalizers. We begin by recalling the definition of the A-bimodule of noncommutative one forms associated to a unital monoid A (see [21] for details in the case of $\mathsf{C} = \mathsf{Ch}_k$).

5.6. Proposition. [Noncommutative differential 1-forms] *Let (A, μ, ι) be a unital associative monoid in C. We define the* A-bimodule of noncommutative differential 1-forms *Ω_A as the coequalizer*

$$A^{\otimes 4} \underset{\mu\otimes A^{\otimes 2}+A^{\otimes 2}\otimes\mu}{\overset{A\otimes\mu\otimes A}{\rightrightarrows}} A^{\otimes 3} \xrightarrow{\tilde{d}} \Omega_A$$

in the category of A-bimodules in C, where the $A^{\otimes i}$, for $i = 3, 4$, are equipped with their external A-bimodule structure. We denote by

$$d : \tilde{d}(\iota \otimes A \otimes \iota) \colon A \longrightarrow \Omega_A$$

the universal derivation.

5.7. Proposition. *Let A be a unital associative monoid. The A-bimodule Ω_A satisfies the following universal property: for an A-bimodule M and a derivation $h \colon A \to M$, there exists a unique A-bimodule morphism $i_h \colon \Omega_A \to M$ such that $h = i_h \circ d$. That is, we have the canonical isomorphism*

$$\begin{array}{ccc} \mathrm{Der}(A, M) & \xrightarrow{\cong} & \mathrm{Hom}_{A\text{-}\mathsf{Bimod}}(\Omega_A, M) \\ h & \longmapsto & i_h \end{array},$$

where $\mathrm{Der}(A, M)$ is the subgroup of $\mathrm{Hom}_{\mathsf{C}}(A, M)$ of morphisms satisfying the derivation property, i.e. $\phi : A \to M \in \mathrm{Der}(A, M)$ if $\phi \circ \mu = \mu_A(\phi \otimes A) +_A \mu(A \otimes \phi)$. We also have the following canonical isomorphism of A-bimodules: for an A-bimodule M and X and Y two objects in C

$$\mathrm{Der}(X \otimes A \otimes Y, M) \cong \mathrm{Hom}_{A\text{-}\mathsf{Bimod}}(X \otimes \Omega_A \otimes Y, M).$$

In this section, we consider a unital associative monoid A, with a Σ-double Poisson bracket $\{\!\{-,-\}\!\}\colon \Sigma A\otimes\Sigma A\longrightarrow \Sigma A\otimes A$. We associate to this double-bracket, a natural Σ-double Lie–Rinehart structure (over A) on Ω_A, the A-bimodule of noncommutative one forms. By the derivation property of $\{\!\{-,-\}\!\}$, proposition 5.7 implies that we can extend the double bracket to the following A^e-bimodule morphism:

$$\phi\colon \Sigma\Omega_A\otimes\Sigma\Omega_A\longrightarrow\Sigma A\otimes A,$$

where the A^e-bimodule structure is given by the external and internal A-bimodule structures. By composition with d (extended as a derivation to $A\otimes A$), we obtain, in the category C, the *Koszul double bracket* $\{\!\{-,-\}\!\}^\Omega$:

$$\begin{tikzcd}
\Sigma A\otimes\Sigma A \arrow[r, "\{\!\{-,-\}\!\}"] \arrow[d, "\Sigma d\otimes\Sigma d"'] & \Sigma A\otimes A \arrow[d, "d\otimes A+A\otimes d"] \\
\Sigma\Omega_A\otimes\Sigma\Omega_A \arrow[ur, "\phi"] \arrow[r, dotted, "=:\{\!\{-,-\}\!\}^\Omega"'] & \Sigma\big(\Omega_A\otimes A\oplus A\otimes\Omega_A\big).
\end{tikzcd}$$

As in definition 4.8, we denote by:

$$\{\!\{-,-\}\!\}^\Omega_r \coloneqq pr_{\Sigma\Omega_A\otimes A}\circ\{\!\{-,-\}\!\}^\Omega \quad\text{and}\quad \{\!\{-,-\}\!\}^\Omega_l \coloneqq pr_{\Sigma A\otimes\Omega_A}\circ\{\!\{-,-\}\!\}^\Omega$$

the projections of $\{\!\{-,-\}\!\}^\Omega$ to $\Sigma\Omega_A\otimes A$ and $\Sigma A\otimes\Omega_A$. By the derivation property of $\{\!\{-,-\}\!\}$ and proposition 5.7, we extend the double bracket canonically to two A-bimodule morphisms:

- the morphism
$$\rho^\Omega_l\colon \Sigma\Omega_A\otimes\Sigma A\longrightarrow\Sigma A\otimes A$$
with, for the left term, the A-bimodule structure induced by that of Ω_A and, for the right term, the internal structure;
- the morphism
$$\rho^\Omega_r\colon \Sigma A\otimes\Sigma\Omega_A\longrightarrow\Sigma A\otimes A$$
with for the left term, the A-bimodule structure induced by that of Ω_A and the external structure for the right term.

By definition, the following diagrams commute:

$$\begin{tikzcd}
\Sigma A\otimes\Sigma A \arrow[r, "\{\!\{-,-\}\!\}"] \arrow[d, "\Sigma d\otimes\Sigma A"'] & \Sigma A\otimes A \\
\Sigma\Omega_A\otimes\Sigma A \arrow[ur, bend right, "\rho^\Omega_l"'] &
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\Sigma A\otimes\Sigma A \arrow[r, "\{\!\{-,-\}\!\}"] \arrow[d, "\Sigma A\otimes\Sigma d"'] & \Sigma A\otimes A \\
\Sigma A\otimes\Sigma\Omega_A \arrow[ur, bend right, "\rho^\Omega_r"'] &
\end{tikzcd}.$$

As $\{\!\{-,-\}\!\}$ is antisymmetric, we have the following anticommutative diagram:

$$\begin{tikzcd} \Sigma\Omega_A \otimes \Sigma A \arrow[r, "\rho_l^\Omega"] \arrow[d, "\tau_{\Sigma\Omega_A,\Sigma A}"'] \arrow[dr, phantom, "\ominus"] & \Sigma A \otimes A \arrow[d, "\Sigma\tau_{A,A}"] \\ \Sigma A \otimes \Sigma\Omega_A \arrow[r, "\rho_r^\Omega"'] & \Sigma A \otimes A. \end{tikzcd}$$

To simplify, we use the notation $\rho^\Omega \coloneqq \rho_l^\Omega$. In the next proposition, we prove that the morphisms ρ^Ω and $\{\!\{-,-\}\!\}^\Omega$ provide the A-bimodule Ω_A with a Σ-double Lie–Rinehart algebra structure.

5.8. Remark. Van den Bergh gives a similar construction in [31, Prop. A.2.1] but with a weight shifting. Here, give a construction in the general categorical setting, but without the shifting. Section 5 applied to the particular case of chain complexes, allows us to recover Van den Bergh's result.

5.9. Theorem. *Let A be a Σ-double Poisson algebra in an additive symmetric monoidal category $(\mathsf{C},\otimes,\tau)$ with coequalizers. The morphisms ρ^Ω and $\{\!\{-,-\}\!\}^\Omega$ endow the A-bimodule Ω_A with a Σ-double Lie–Rinehart algebra structure.*

5.10. Remark. By proposition 4.13, the morphisms ρ^Ω and $\{\!\{-,-\}\!\}^\Omega$ also endow the free A-algebra $\mathrm{T}_A\Omega_A$ with a linear Σ-double Poisson algebra.

Proof. Let $(A,\{\!\{-,-\}\!\})$ be a Σ-double Poisson algebra, we write$\Omega \coloneqq \Omega_A$ with ${}_A\mu^\Omega$ and μ_A^Ω the morphisms which define the A-bimodule structure of Ω. We'll show that the morphisms ρ^Ω and $\{\!\{-,-\}\!\}^\Omega$ endow Ω with a Σ-double Lie–Rinehart algebra structure. The double bracket $\{\!\{-,-\}\!\}^\Omega$ is antisymmetric: in fact, it is defined by the following commutative diagram

$$\begin{tikzcd} \Sigma A \otimes \Sigma A \arrow[r, "\{\!\{-,-\}\!\}"] \arrow[d, "d\otimes d"'] & \Sigma A \otimes A \arrow[d, "d\otimes A + A\otimes d"] \\ \Sigma\Omega \otimes \Sigma\Omega \arrow[r, "\{\!\{-,-\}\!\}^\Omega"'] & \Sigma\big(\Omega\otimes A \oplus A\otimes\Omega\big) \end{tikzcd}$$

so that $\{\!\{-,-\}\!\}^\Omega$ satisfies the antisymmetry condition ((Antisymmetry)) of definition 4.8 by the antisymmetry of the double bracket of A.

We will show that ρ^Ω and $\{\!\{-,-\}\!\}^\Omega$ satisfy the derivation condition ((Derivation)) of definition 4.8. We write $A_1 = A = A_2$; we have the diagram of figure (1). The front and back faces commute by definition of $\{\!\{-,-\}\!\}_l^\Omega$. The left and right faces commute because d is a derivation and the top face commutes because ρ^Ω is a derivation. Then the bottom face commutes. Similarly, the diagram of figure (2) commutes. Then, the morphisms ρ^Ω and $\{\!\{-,-\}\!\}^\Omega$ satisfy the compatibility condition ((Derivation)) of definition 4.8.

$$
\begin{tikzcd}[column sep=small]
 & \Sigma\Omega\otimes\Sigma A \arrow[rr, "\rho^{\Omega}"] \arrow[dd, "\Sigma\Omega\otimes\Sigma d"] & & \Sigma A\otimes A \arrow[dd, "\Sigma d\otimes A"] \\
\Sigma\Omega\otimes\Sigma A_1\otimes A_2 \arrow[ur, "\Sigma\Omega\otimes\Sigma\mu"] \arrow[rr, "(\rho^{\Omega}\otimes A)+(A\otimes\rho^{\Omega})(\tau_{\Sigma\Omega\otimes\Sigma,A_1}\otimes A_2)"] \arrow[dd, "\Sigma\Omega\otimes\Sigma d\otimes A+\Sigma\Omega\otimes\Sigma A\otimes d)"'] & & \substack{\Sigma A\otimes A\otimes A_2\\ \oplus A_1\otimes\Sigma A\otimes A} \arrow[ur, "(\Sigma A\otimes\mu,(\Sigma\mu\otimes A)(\tau_{A_1,\Sigma}\otimes A^{\otimes 2})"] \arrow[dd, "(\Sigma d\otimes A\otimes A,d\otimes\Sigma A\otimes A+A_1\otimes\Sigma d\otimes A)"] & \\
 & \Sigma\Omega\otimes\Sigma\Omega \arrow[rr, "\{\!\{-,-\}\!\}_l^{\Omega}"'] & & \Sigma\Omega\otimes A \\
\substack{\Sigma\Omega\otimes\Sigma\Omega\otimes A_2\\ \oplus\Sigma\Omega\otimes\Sigma A_1\otimes\Omega} \arrow[ur, "(\Sigma\Omega\otimes\Sigma\mu_A^{\Omega},\Sigma\Omega\otimes\Sigma_A\mu^{\Omega})"] \arrow[rr, "\big(\{\!\{-,-\}\!\}_l^{\Omega}\otimes A_2+\Sigma\Omega\otimes\rho^{\Omega}", "+(A_1\otimes\{\!\{-,-\}\!\}_l^{\Omega})(\tau_{\Sigma\Omega\otimes\Sigma,A_1}\otimes\Omega)\big)"'] & & \substack{\Sigma\Omega\otimes A\otimes A_2\\ \oplus\Omega\otimes\Sigma A\otimes A\\ \oplus A_1\otimes\Sigma\Omega\otimes A} \arrow[ur, "\Phi_l"'] &
\end{tikzcd}
$$

with $\qquad \Phi_l \coloneqq \big(\Sigma\Omega\otimes\mu, (\Sigma\mu_A^{\Omega}\otimes A)(\tau_{\Omega,\Sigma}\otimes A^{\otimes 2}), (\Sigma_A\mu^{\Omega}\otimes A)(\tau_{A,\Sigma}\otimes\Omega\otimes A)\big).$

Figure 1: First diagram of compatibility ((Derivation))

We show that ρ^{Ω} and $\{\!\{-,-\}\!\}^{\Omega}$ satisfy the condition ((Anchor)). We call Ψ_L the morphism $\big\{\!\!\big\{-,\{\!\{-,-\}\!\}\big\}\!\!\big\}_L = \big(\{\!\{-,-\}\!\}\otimes A\big)\big(\Sigma A\otimes\{\!\{-,-\}\!\}\big)$. We have the following diagram with commuting vertical faces:

$$
\begin{tikzcd}[column sep=small]
 & (\Sigma A)^{\otimes 3} \arrow[ddl, bend right, "\Psi_L"'] \arrow[d, "\Sigma d\otimes\Sigma A\otimes\Sigma d"] & & (\Sigma A)^{\otimes 3} \arrow[ll, "\tau_{\Sigma A\otimes\Sigma A,\Sigma A}"'] \arrow[rr, "\tau_{\Sigma A,\Sigma A\otimes\Sigma A}"] \arrow[ddl, bend right, "\Psi_L"'] \arrow[d, "\Sigma A\otimes\Sigma d\otimes\Sigma d"] & & (\Sigma A)^{\otimes 3} \arrow[ddl, bend right, "\Psi_L"'] \arrow[d, "(\Sigma d)^{\otimes 2}\otimes\Sigma A"] \\
 & \Sigma\Omega\otimes\Sigma A\otimes\Sigma\Omega \arrow[dl, "(\rho_l^{\Omega}\otimes A)(\Sigma\Omega\otimes\rho_r^{\Omega})"] & & \Sigma A\otimes\Sigma\Omega\otimes\Sigma\Omega \arrow[ll, "\tau_{\Sigma A\otimes\Sigma\Omega,\Sigma\Omega}"'] \arrow[rr, "\tau_{\Sigma A,\Sigma\Omega\otimes\Sigma\Omega}"] \arrow[dl, "(\rho_r^{\Omega}\otimes A)(\Sigma A\otimes\{\!\{-,-\}\!\}_l^{\Omega})"] & & (\Sigma\Omega)^{\otimes 2}\otimes\Sigma A \arrow[dl, "(\rho_l^{\Omega}\otimes A)(\Sigma\Omega\otimes\rho_l^{\Omega})"] \\
\Sigma(A)^{\otimes 3} \arrow[rr, "\Sigma\tau_{A,A\otimes A}"'] & & \Sigma(A)^{\otimes 3} & & \Sigma(A)^{\otimes 3} \arrow[ll, "\Sigma\tau_{A\otimes A,A}"] & .
\end{tikzcd}
$$

Then, since $\{\!\{-,-\}\!\}$ satisfies the double Jacobi identity, the term

$$
\begin{aligned}
&(\rho_r^{\Omega}\otimes A)(\Sigma A\otimes\{\!\{-,-\}\!\}_l^{\Omega})\\
&\qquad+\tau_{\Sigma A\otimes\Sigma A,\Sigma A}(\rho_l^{\Omega}\otimes A)(\Sigma\Omega\otimes\rho_l^{\Omega})\tau_{\Sigma A,\Sigma\Omega\otimes\Sigma\Omega}\\
&\qquad+\tau_{\Sigma A,\Sigma A\otimes\Sigma A}(\rho_l^{\Omega}\otimes A)(\Sigma\Omega\otimes\rho_r^{\Omega})\tau_{\Sigma A\otimes\Sigma\Omega,\Sigma\Omega}
\end{aligned}
$$

is equal to zero, hence the $\{\!\{-,-\}\!\}^{\Omega}$ and ρ^{Ω} satisfy the condition ((Anchor)) of definition

$$
\begin{tikzcd}
& \Sigma\Omega\otimes\Sigma A \arrow[rr, "\rho^{\Omega}"] \arrow[dd, "\Sigma\Omega\otimes\Sigma d"] & & \Sigma A\otimes A \arrow[dd, "\Sigma A\otimes d"] \\
\Sigma\Omega\otimes\Sigma A_1\otimes A_2 \arrow[ur, "\Sigma\Omega\otimes\Sigma\mu"] \arrow[rr, "(\rho^{\Omega}\otimes A)+(A\otimes\rho^{\Omega})(\tau_{\Sigma\Omega\otimes\Sigma,A_1}\otimes A_2)"] \arrow[dd, "\Sigma\Omega\otimes\Sigma d\otimes A+\Sigma\Omega\otimes\Sigma A\otimes d)"'] & & {\begin{smallmatrix}\Sigma A\otimes A\otimes A_2\\ \oplus A_1\otimes\Sigma A\otimes A\end{smallmatrix}} \arrow[ur, "(\Sigma A\otimes\mu,(\Sigma\mu\otimes A)(\tau_{A_1,\Sigma}\otimes A^{\otimes 2})"] \arrow[dd, "(\Sigma A\otimes d\otimes A+\Sigma A\otimes A\otimes d,A\otimes\Sigma A\otimes d)"] & \\
& \Sigma\Omega\otimes\Sigma\Omega \arrow[rr, "\{\!\{-,-\}\!\}_r^{\Omega}"'] & & \Sigma A\otimes\Omega \\
{\begin{smallmatrix}\Sigma\Omega\otimes\Sigma\Omega\otimes A_2\\ \oplus\Sigma\Omega\otimes\Sigma A_1\otimes\Omega\end{smallmatrix}} \arrow[ur, "(\Sigma\Omega\otimes\Sigma\mu_A^{\Omega},\Sigma\Omega\otimes\Sigma_A\mu^{\Omega})"] \arrow[rr, "\begin{smallmatrix}(\{\!\{-,-\}\!\}_r^{\Omega}\otimes A_2,\rho^{\Omega}\otimes\Omega\\ +(A_1\otimes\{\!\{-,-\}\!\}_r^{\Omega})(\tau_{\Sigma\Omega\otimes\Sigma,A_1}\otimes\Omega))\end{smallmatrix}"] & & {\begin{smallmatrix}\Sigma A\otimes\Omega\otimes A_2\\ \oplus A_1\otimes\Sigma A\otimes\Omega\\ \oplus\Sigma A\otimes A\otimes\Omega\end{smallmatrix}} \arrow[ur, "\Phi_r"'] &
\end{tikzcd}
$$

with $\quad \Phi_r \coloneqq (\Sigma A\otimes\mu_A^{\Omega}, (\Sigma\mu\otimes\Omega)(\tau_{A,\Sigma}\otimes A\otimes\Omega), \Sigma A\otimes_A\mu^{\Omega}).$

Figure 2: Second diagram of compatibility ((Derivation))

4.8. We have the following diagram with commuting vertical faces:

$$
\begin{tikzcd}
& (\Sigma A)^{\otimes 3} \arrow[d, "(\Sigma d)^{\otimes 3}"] \arrow[ddl, bend right, "\Psi_L"'] & & (\Sigma A)^{\otimes 3} \arrow[ll, "\tau_{\Sigma A\otimes\Sigma A,\Sigma A}"'] \arrow[rr, "\tau_{\Sigma A,\Sigma A\otimes\Sigma A}"] \arrow[d, "(\Sigma d)^{\otimes 3}"] \arrow[ddl, bend right, "\Psi_L"'] & & (\Sigma A)^{\otimes 3} \arrow[d, "(\Sigma d)^{\otimes 3}"] \arrow[ddl, bend right, "\Psi_L"'] \\
& (\Sigma\Omega)^{\otimes 3} \arrow[dl, "(\{\!\{-,-\}\!\}_r^{\Omega}\otimes A)(\Sigma\Omega\otimes\{\!\{-,-\}\!\}_l^{\Omega})"] & & (\Sigma\Omega)^{\otimes 3} \arrow[ll, "\tau_{\Sigma\Omega\otimes\Sigma\Omega,\Sigma\Omega}"'] \arrow[rr, "\tau_{\Sigma\Omega,\Sigma\Omega\otimes\Sigma\Omega}"] \arrow[dl, "(\{\!\{-,-\}\!\}_l^{\Omega}\otimes A)(\Sigma\Omega\otimes\{\!\{-,-\}\!\}_l^{\Omega})"] & & (\Sigma\Omega)^{\otimes 3}\ ; \arrow[dl, "(\rho_l^{\Omega}\otimes A)(\Sigma\Omega\otimes\{\!\{-,-\}\!\}_r^{\Omega})"] \\
\Sigma A\otimes\Omega\otimes A \arrow[rr, "\Sigma\tau_{A,\Omega\otimes A}"'] & & \Sigma\Omega\otimes A^{\otimes 2} & & \Sigma A^{\otimes 2}\otimes\Omega \arrow[ll, "\Sigma\tau_{A\otimes A,\Omega}"] &
\end{tikzcd}
$$

as $\{\!\{-,-\}\!\}$ satisfies the double Jacobi identity, the term

$$
\begin{aligned}
&(\{\!\{-,-\}\!\}_l^{\Omega}\otimes A)(\Sigma\Omega\otimes\{\!\{-,-\}\!\}_l^{\Omega}) \\
&\quad+\Sigma\tau_{A,\Omega\otimes A}(\{\!\{-,-\}\!\}_r^{\Omega}\otimes A)(\Sigma\Omega\otimes\{\!\{-,-\}\!\}_l^{\Omega})\tau_{\Sigma\Omega\otimes\Sigma\Omega,\Sigma\Omega} \\
&\quad+\Sigma\tau_{A\otimes A,\Omega}(\rho_l^{\Omega}\otimes A)(\Sigma\Omega\otimes\{\!\{-,-\}\!\}_r^{\Omega})\tau_{\Sigma\Omega,\Sigma\Omega\otimes\Sigma\Omega}
\end{aligned}
$$

is equal to zero. By invariance under the $\mathbb{Z}/3\mathbb{Z}$-action of the double Jacobiator, the double bracket $\{\!\{-,-\}\!\}^{\Omega}$ satisfies the double Jacobi identity. ■

Using the shifting property, we recove the original construction of the Koszul double bracket of Van den Bergh as follows: in [31, Ann. A], Van den Bergh constructs the Koszul double bracket as a Gerstenhaber double bracket, i.e. a Poisson double bracket of degree -1, on $\mathrm{T}_A(\Omega_A[1])$. The shifting property 5.2, applied to our Koszul double bracket construction, recovers the original Koszul double bracket of Van den Bergh.

5.11. Example: the Schouten–Nijenhuis double bracket. For this example, we take $\mathsf{C} = \mathsf{DGA}_k$ with k a field and we consider a differential graded algebra A. We start by the definition of the A-bimodule of biderivations of A; biderivations play the role of derivations in noncommutative geometry.

The *(external) A-bimodule of biderivations of A*, denoted by $\mathbb{D}\mathrm{er}(A)$, is defined by

$$\mathbb{D}\mathrm{er}(A) \coloneqq \mathrm{Der}(A, A \otimes A),$$

where $A \otimes A$ is equipped with its external A-bimodule structure.

We recall the definition of the Schouten Nijenhuis 0-double Poisson bracket following Van den Bergh [31, Sect. 3.2]. We consider A, a *finitely-generated* differential graded algebra: this implies that the A-bimodule Ω_A is finitely-generated. The morphisms $\Phi, \Psi \colon \mathbb{D}\mathrm{er}(A)^{\otimes 2} \otimes A \to A^{\otimes 3}$ defined by

$$\begin{aligned}
\Phi \coloneqq &(A \otimes \tau_{A,A})\Big((ev \otimes A)(\mathbb{D}\mathrm{er}(A) \otimes ev) \\
&\quad - (A \otimes ev)(\tau_{\mathbb{D}\mathrm{er}(A),A} \otimes A)(\mathbb{D}\mathrm{er}(A) \otimes ev)(\tau_{\mathbb{D}\mathrm{er}(A),\mathbb{D}\mathrm{er}(A)} \otimes A)\Big) \\
\Psi \coloneqq &(\tau_{A,A} \otimes A)\Big((A \otimes ev)(\tau_{\mathbb{D}\mathrm{er}(A),A} \otimes A)(\mathbb{D}\mathrm{er}(A) \otimes ev) \\
&\quad - (ev \otimes A)(\mathbb{D}\mathrm{er}(A) \otimes ev)(\tau_{\mathbb{D}\mathrm{er}(A),\mathbb{D}\mathrm{er}(A)} \otimes A)\Big)
\end{aligned}$$

yield, by adjunction, the morphisms

$$\begin{aligned}
\Phi^* &\coloneqq \{\!\{-,-\}\!\}_l^{\mathrm{SN}} \colon \mathbb{D}\mathrm{er}(A) \otimes \mathbb{D}\mathrm{er}(A) \to \mathrm{Hom}(A, A^{\otimes 3}), \\
\Psi^* &\coloneqq \{\!\{-,-\}\!\}_r^{\mathrm{SN}} \colon \mathbb{D}\mathrm{er}(A) \otimes \mathbb{D}\mathrm{er}(A) \to \mathrm{Hom}(A, A^{\otimes 3}).
\end{aligned}$$

As the A-bimodule Ω_A is finitely-generated, the following chain complexes are isomorphic $\mathrm{Der}(A, A^{\otimes 3}) \cong \mathrm{Hom}_{A^e}(\Omega_A, A \otimes A) \otimes A$. So, the morphism $\{\!\{-,-\}\!\}_l^{\mathrm{SN}}$ (resp. $\{\!\{-,-\}\!\}_r^{\mathrm{SN}}$) factorizes through $\mathbb{D}\mathrm{er}(A) \otimes A$ (resp. $A \otimes \mathbb{D}\mathrm{er}(A)$) (see [31, Prop. 3.2.1]). Using the theorem of Van den Bergh [31, Th. 3.2.2], we apply theorem 5.2: the morphisms

$$\begin{aligned}
\{\!\{-,-\}\!\}_l^{\mathrm{SN}} + \{\!\{-,-\}\!\}_r^{\mathrm{SN}} &\colon \mathbb{D}\mathrm{er}(A) \otimes \mathbb{D}\mathrm{er}(A) \to \mathbb{D}\mathrm{er}(A) \otimes A \oplus A \otimes \mathbb{D}\mathrm{er}(A) \\
(ev, -\tau_{A,A} \circ ev \circ \tau_{A,\mathbb{D}\mathrm{er}(A)}) &\colon \mathbb{D}\mathrm{er}(A) \otimes A \oplus A \otimes \mathbb{D}\mathrm{er}(A) \to A \otimes A
\end{aligned} \tag{2}$$

induce an A-linear 0-double Poisson algebra structure on the free A-algebra $\mathrm{T}_A\mathbb{D}\mathrm{er}(A)$ (see [31, Thm. 3.2.2]) called the *Schouten–Nijenhuis double bracket*. Using theorem 4.13, this structure corresponds to a A-double Lie–Rinehart structure on $\mathbb{D}\mathrm{er}(A)$.

Now, consider A a finitely-generated differential graded algebra with a double Poisson bracket. The A-bimodule Ω_A is a double Lie–Rinehart algebra for the Koszul structure $(\rho^\Omega, \{\!\{-,-\}\!\}^\Omega)$ (see section 5.5). As the category Ch_k is closed, the anchor ρ^Ω corresponds to a morphism

$$(\rho^\Omega)^* \colon \Omega \to \mathrm{Hom}(A, A \otimes A)$$

which factorizes through $\mathbb{D}\mathrm{er}(A)$.

5.12. Proposition. [Relation between the Schouten–Nijenhuis and Koszul double brackets] *Let A be a finitely-generated differential graded algebra with a double Poisson bracket. The morphism $(\rho^\Omega)^*$ is an A-double Lie–Rinehart morphism between Ω_A and $\mathbb{D}\mathrm{er}(A)$.*

PROOF. As remarked in 4.10, in this case, the compatibility condition ((Anchor)) can be expressed using the Schouten–Nijenhuis double bracket introduced in (2), as:

$$(\rho^{\Omega})^{*}\big(\{\!\{-,-\}\!\}^{M}\big) = \{\!\{(\rho^{\Omega})^{*},(\rho^{\Omega})^{*}\}\!\}^{\mathrm{SN}}.$$

■

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