Theory and Applications of Categories, Vol. 35, No. 13, 2020, pp. 371–416.

HOMOTOPY THEORY WITH MARKED ADDITIVE CATEGORIES

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Abstract. We construct combinatorial model category structures on the categories of (marked) categories and (marked) preadditive categories, and we characterize (marked) additive categories as fibrant objects in a Bousfield localization of preadditive categories. These model category structures are used to present the corresponding ∞-categories obtained by inverting equivalences. We apply these results to explicitly calculate limits and colimits in these ∞-categories. The motivating application is a systematic construction of the equivariant coarse algebraic K-homology with coefficients in an additive category from its non-equivariant version.

Contents

1. Introduction

If $\mathcal{C}$ is a category and W is a set of morphisms in $\mathcal{C}$, then one can consider the localization functor

$$\ell_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}_\infty := \mathcal{C}[W^{-1}]$$

in ∞-categories [Lur, Def. 1.3.4.1] [Cis19, Def. 7.1.2], where we consider $\mathcal{C}$ as an ∞-category given by its nerve (which we will omit in the notation). If the relative category

Received by the editors 2019-03-18 and, in final form, 2020-04-02.
Transmitted by Ieke Moerdijk. Published on 2020-04-08.
2020 Mathematics Subject Classification: 18E05, 18N40.
Key words and phrases: Additive categories, marked categories, model categories.

$(\mathcal{C}, W)$ extends to a simplicial model category in which all objects are cofibrant, then we have an equivalence of ∞-categories

$$\mathcal{C}_\infty \simeq \mathrm{N}^{\mathrm{coh}}(\mathcal{C}^{cf}) \ ,$$

where the right-hand side is the nerve of the simplicial category of cofibrant-fibrant objects of $\mathcal{C}$ [Lur, Def. 1.3.4.15 & Thm. 1.3.4.20]. This explicit description of $\mathcal{C}_\infty$ is sometimes very helpful in order to calculate mapping spaces in $\mathcal{C}_\infty$ or to identify limits or colimits of diagrams in $\mathcal{C}_\infty$.

In the present paper we consider the case where $\mathcal{C}$ belongs to the list

$$\{\mathbf{Cat}, \mathbf{Cat}^+, \mathbf{preAdd}, \mathbf{preAdd}^+\}$$

where $\mathbf{Cat}^{(+)}$ is the category of small (marked) categories (Definition 2.1.3) and $\mathbf{preAdd}^{(+)}$ is the category of small (marked) preadditive categories (Definitions 2.1.4 and 2.1.6). We consider them as relative categories with W the (marking preserving) morphisms (functors or **Ab**-enrichment preserving functors, respectively) which admit inverses up to (marked) isomorphisms (Definition 2.2.1).

To illustrate the role of markings, consider the category of Banach spaces over a complete normed field and bounded linear maps between them. If G is a group, then G-objects in this category are those Banach spaces with an action of G by continuous automorphisms. But often one wants to require that G acts by isometries. This can be ensured by considering the category of Banach spaces as a marked category, where the marked morphisms are the isometric isomorphisms. Taking G-objects in the marked category then yields the category of Banach spaces with an action of G by isometries. A more detailed discussion of this example can be found in Example 3.4.9. Example 3.4.11 discusses a second example appearing in the context of coarse homology theories.

In order to fix set-theoretic issues we choose three Grothendieck universes

$$\mathcal{U} \subset \mathcal{V} \subset \mathcal{W} \ . \tag{1}$$

The objects of $\mathcal{C}$ are categories in $\mathcal{V}$ which are locally $\mathcal{U}$-small, while $\mathcal{C}$ itself belongs to $\mathcal{W}$ and is locally $\mathcal{V}$-small. We will shortly say that the objects of $\mathcal{C}$ are small (as already done above), and correspondingly, that $\mathcal{C}$ itself is large.

Our first main theorem is:

1.0.1. THEOREM. *The pair $(\mathcal{C}, W)$ extends to a combinatorial, simplicial model category structure.*

We refer to Theorem 2.2.2 for a more precise formulation and recall that the adjective *combinatorial* means cofibrantly generated as a model category, and locally presentable as a category. In this model category structure all objects of $\mathcal{C}$ are cofibrant.

The assertion of Theorem 1.0.1 in the case of **Cat** and **preAdd** is well-known or folklore. A discussion of the model structure on **Cat** can be found in [Rez]. The case **preAdd** is also a consequence of [Lur09, Prop. A.3.2.4],[BM13, Thm. 1.9] or [Mur15, Thm. 1.1]; in each case, one considers the category of abelian groups as a monoidal model category in which the isomorphisms are the weak equivalences, and all morphisms are both fibrations and cofibrations. In the proof, which closely follows the standard line of arguments, we therefore put the emphasis on checking that all arguments work in the marked cases as well.

In order to describe the homotopy theory of (marked) additive categories, we show the following.

1.0.2. PROPOSITION. *There exists a Bousfield localization $L\mathbf{preAdd}^{(+)}$ of $\mathbf{preAdd}^{(+)}$ whose fibrant objects are the marked (additive) categories.*

We refer to Proposition 2.3.7 for a more precise statement. Let $W_{\mathbf{Add}^{(+)}}$ denote the weak equivalences in $L\mathbf{preAdd}^{(+)}$. Proposition 1.0.2 then implies that we have an equivalence of ∞-categories

$$\mathbf{Add}_{\infty}^{(+)} := \mathbf{preAdd}^{(+)}[W_{\mathbf{Add}^{(+)}}^{-1}] \simeq \mathrm{N}^{\mathrm{coh}}(\mathbf{Add}^{(+)}) \; , \tag{2}$$

where $\mathbf{Add}^{(+)}$ denotes the category of small (marked) additive categories (see Definitions 2.3.1 and 2.3.3). For example, this allows us to calculate limits in $\mathbf{Add}_{\infty}^{+}$, which is one of the motivating applications of the present paper (see Example 3.4.11).

Since in general an ∞-category modeled by a combinatorial model category is presentable, we get the following (see Corollary 2.4.5).

1.0.3. COROLLARY. *The ∞-categories in the list*

$$\{\mathbf{Cat}_{\infty}, \mathbf{Cat}_{\infty}^{+}, \mathbf{preAdd}_{\infty}, \mathbf{preAdd}_{\infty}^{+}, \mathbf{Add}_{\infty}, \mathbf{Add}_{\infty}^{+}\}$$

are presentable.

Presentability is a very useful property if one wants to show the existence of adjoint functors. For example the inclusion $\mathcal{F}_{\oplus}\colon \mathbf{Add}_{\infty} \to \mathbf{preAdd}_{\infty}$ preserves limits (by inspection) and therefore has a left adjoint, the additive completion functor

$$L_{\oplus}\colon \mathbf{preAdd}_{\infty} \to \mathbf{Add}_{\infty}$$

(see Corollary 2.4.5).

We demonstrate the utility of the model category structures, whose existence is asserted in Theorem 1.0.1, in a variety of examples.

1. In Proposition 2.4.6, we use relation (2) in order to show an equivalence of ∞-categories
$$\mathbf{Add}_{\infty} \simeq \mathrm{N}_2(\mathbf{Add}_{(2,1)}) \; ,$$
where the right-hand side is the 2-categorical nerve of the strict 2-category of small additive categories. This is used in [BEKWb] to extend K-theory functors from $\mathbf{Add}$ to $\mathrm{N}_2(\mathbf{Add}_{(2,1)})$.

2. In Section 3.1 we verify that the localization functor $\ell_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}_\infty$ preserves arbitrary products, where $\mathcal{C}$ belongs to the list
$$\{\mathbf{Cat}, \mathbf{Cat}^+, \mathbf{preAdd}_\infty, \mathbf{preAdd}_\infty^+, \mathbf{Add}_\infty, \mathbf{Add}_\infty^+\}\ ,$$
see Proposition 3.1.1.

3. In Section 3.2 we consider additive categories of modules over rings. For example, we show in Proposition 3.2.1 that
$$L_\oplus(\ell_{\mathbf{preAdd}}(\mathbf{R})) \simeq \ell_{\mathbf{Add}}(\mathbf{Mod}^{\mathrm{fg,free}}(R))\ ,$$
i.e. that the additive completion of a ring (considered as an object $\ell_{\mathbf{preAdd}}(\mathbf{R})$ in $\mathbf{preAdd}_\infty$) is equivalent to the additive category of its finitely generated and free modules (considered in $\mathbf{Add}_\infty$). We also discuss idempotent completion and its relation with the additive category of finitely generated projective modules along the same lines, see Proposition 3.2.7.

4. The main result in Section 3.3, see Theorem 3.3.1, is an explicit formula for the object
$$\operatorname*{colim}_{BG} \ell_{\mathbf{preAdd}^{(+)},BG}(\underline{\mathbf{A}})$$
in $\mathbf{preAdd}^{(+)}$, where $\underline{\mathbf{A}}$ is a (marked) preadditive category with trivial action of a group G and $\ell_{\mathbf{preAdd}^{(+)},BG}$ is induced from $\ell_{\mathbf{preAdd}^{(+)}}$.

5. In Section 3.4 we consider $\mathcal{C}$ in $\{\mathbf{preAdd}_\infty, \mathbf{preAdd}_\infty^+, \mathbf{Add}_\infty, \mathbf{Add}_\infty^+\}$. In Theorem 3.4.3, we provide an explicit formula for the object
$$\lim_{BG} \ell_{\mathcal{C},BG}(\mathbf{A})\ ,$$
where $\mathbf{A}$ is an object of $\mathcal{C}$ with an action of G.

In a parallel paper [Bun19] we consider model category structures on (marked) $*$-categories and a similar application to coarse homology theories including equivariant coarse topological K-homology. The arguments in [Bun19] concerning the model category structures are quite similar to the arguments in the present paper. But while [Bun19] is more concerned with the functional analytic subtleties arising from enrichments in Banach spaces, the emphasis of the present paper is on algebraic aspects and additivity.

Acknowledgements

We thank the referee for their detailed report and numerous suggestions which helped to improve the manuscript.

U. Bunke and A. Engel were supported by the SFB 1085 “Higher Invariants” funded by the Deutsche Forschungsgemeinschaft DFG. C. Winges acknowledges support by the Max

Planck Society and by Wolfgang Lück's ERC Advanced Grant "KL2MG-interactions" (no. 662400). D. Kasprowski and C. Winges are funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - GZ 2047/1, Projekt-ID 390685813.

2. Marked categories

2.1. Categories of marked categories and marked preadditive categories. In this section we introduce categories of marked categories, marked preadditive categories and additive categories. We further describe various relations between these categories given by forgetful functors and their adjoints. We finally describe their enrichments in groupoids and simplicial sets.

Let $\mathbf{C}$ be a category.

2.1.1. Definition. *A* marking on $\mathbf{C}$ *is the choice of a wide subgroupoid* $\mathbf{C}^+$ *of the underlying groupoid of* $\mathbf{C}$.

2.1.2. Example. In this example, we name the two extreme cases of markings. On the one hand, we can consider the minimal marking $\mathbf{C}^+_{min}$ given by the identity morphisms of $\mathbf{C}$. On the other hand, we have the maximal marking $\mathbf{C}^+_{max}$ given by the underlying groupoid of $\mathbf{C}$.

2.1.3. Definition. *A* marked category *is a pair* $(\mathbf{C}, \mathbf{C}^+)$ *of a category and a marking. A morphism between marked categories* $(\mathbf{C}, \mathbf{C}^+) \to (\mathbf{D}, \mathbf{D}^+)$ *is a functor* $\mathbf{C} \to \mathbf{D}$ *which sends* $\mathbf{C}^+$ *to* $\mathbf{D}^+$.

We let $\mathbf{Cat}^+$ denote the category of marked small categories and morphisms between marked categories. Note that $\mathbf{Cat}^+$ is not a marking on $\mathbf{Cat}$. We will not consider markings on $\mathbf{Cat}$ so that no confusion should arise.

We have two functors

$$\mathcal{F}_+ \colon \mathbf{Cat}^+ \to \mathbf{Cat} , \quad (\mathbf{C}, \mathbf{C}^+) \mapsto \mathbf{C} \tag{3}$$

and

$$(-)^+ \colon \mathbf{Cat}^+ \to \mathbf{Groupoids} , \quad (\mathbf{C}, \mathbf{C}^+) \mapsto \mathbf{C}^+ .$$

The functor $\mathcal{F}_+$ (which forgets the markings) fits into adjunctions

$$\mathrm{mi} \colon \mathbf{Cat} \leftrightarrows \mathbf{Cat}^+ \colon \mathcal{F}_+ , \quad \mathcal{F}_+ \colon \mathbf{Cat}^+ \leftrightarrows \mathbf{Cat} \colon \mathrm{ma} ,$$

where the functors mi (mark identities) and ma (mark all isomorphisms) are given (on objects) by

$$\mathrm{mi}(\mathbf{C}) := (\mathbf{C}, \mathbf{C}^+_{min}) , \quad \mathrm{ma}(\mathbf{C}) := (\mathbf{C}, \mathbf{C}^+_{max}) ,$$

and their definition on morphisms as well as the unit and counit of the adjunctions are the obvious ones.

2.1.4. Definition. *A* preadditive category *is a category which is enriched over the category of abelian groups. A* morphism *between preadditive categories is a functor which is compatible with the enrichment.*

We let **preAdd** denote the category of small preadditive categories and functors which are compatible with the enrichment.

The forgetful functor (forgetting the enrichment) is the right adjoint of an adjunction

$$\mathrm{Lin}_{\mathbb{Z}} \colon \mathbf{Cat} \leftrightarrows \mathbf{preAdd} \colon \mathcal{F}_{\mathbb{Z}} \tag{4}$$

whose left adjoint is called the linearization functor. For a preadditive category $\mathbf{A}$ we call $\mathcal{F}_{\mathbb{Z}}(\mathbf{A})$ the underlying category.

2.1.5. Remark. Let $\mathbf{A}$ be a preadditive category. If A and B are two objects of $\mathbf{A}$ such that the product $A \times B$ and the coproduct $A \sqcup B$ exist, then the canonical morphism $A \sqcup B \to A \times B$ induced by the maps $(\mathrm{id}_A, 0) \colon A \to A \times B$ and $(0, \mathrm{id}_B) \colon B \to A \times B$ is an isomorphism. In this case we call the product or coproduct also the sum of A and B and use the notation $A \oplus B$.

2.1.6. Definition. *We define the* category of marked preadditive categories $\mathbf{preAdd}^+$ *as the pullback (in* 1*-categories)*

$$\begin{tikzcd} \mathbf{preAdd}^+ \arrow[r] \arrow[d] & \mathbf{Cat}^+ \arrow[d, "\mathcal{F}_+"] \\ \mathbf{preAdd} \arrow[r, "\mathcal{F}_{\mathbb{Z}}"] & \mathbf{Cat} \end{tikzcd}$$

with the functors $\mathcal{F}_+$ and $\mathcal{F}_{\mathbb{Z}}$ from (3) *and* (4)*.*

Thus a marked preadditive category is a pair $(\mathbf{A}, \mathbf{A}^+)$ of a preadditive category $\mathbf{A}$ and a wide subgroupoid $\mathbf{A}^+$ of the underlying groupoid of $\mathbf{A}$, and a morphism of marked preadditive categories $(\mathbf{A}, \mathbf{A}^+) \to (\mathbf{B}, \mathbf{B}^+)$ is a functor $\mathbf{A} \to \mathbf{B}$ which is compatible with the enrichment and sends $\mathbf{A}^+$ to $\mathbf{B}^+$.

We will denote the vertical arrow forgetting the markings, i.e., taking the underlying preadditive category, also by $\mathcal{F}_+$. We have adjunctions

$$\mathrm{mi} \colon \mathbf{preAdd} \leftrightarrows \mathbf{preAdd}^+ \colon \mathcal{F}_+ \ , \quad \mathcal{F}_+ \colon \mathbf{preAdd}^+ \leftrightarrows \mathbf{preAdd} \colon \mathrm{ma} \ , \tag{5}$$

and

$$\mathrm{Lin}_{\mathbb{Z}} \colon \mathbf{Cat}^+ \leftrightarrows \mathbf{preAdd}^+ \colon \mathcal{F}_{\mathbb{Z}} \ .$$

The unit of adjunction (4) provides an inclusion of categories $\mathbf{C} \to \mathcal{F}_{\mathbb{Z}}(\mathrm{Lin}_{\mathbb{Z}}(\mathbf{C}))$, and the subcategory of marked isomorphisms in $\mathrm{Lin}_{\mathbb{Z}}(\mathbf{C})$ is exactly the image of $\mathbf{C}^+$ under this inclusion.

2.1.7. REMARK. Note that a sum of two addable marked isomorphisms in a marked preadditive category need not be marked. So in general the subcategory of marked isomorphisms of a marked preadditive category is not preadditive.

From now one we will usually shorten the notation and denote marked categories just by one symbol $\mathbf{C}$ instead of $(\mathbf{C}, \mathbf{C}^+)$.

We will now show that the categories $\mathbf{Cat}$, $\mathbf{Cat}^+$ $\mathbf{preAdd}$ and $\mathbf{preAdd}^+$ are enriched over themselves. For categories $\mathbf{A}$ and $\mathbf{B}$ we let $\mathbf{Fun}_{\mathbf{Cat}}(\mathbf{A}, \mathbf{B})$ in $\mathbf{Cat}$ denote the category of functors from $\mathbf{A}$ to $\mathbf{B}$ and natural transformations. Assume now that $\mathbf{A}$ and $\mathbf{B}$ are marked. Then we can consider the functor category $\mathbf{Fun}_{\mathbf{Cat}^+}(\mathbf{A}, \mathbf{B})$ in $\mathbf{Cat}$ of functors preserving the marked subcategories and natural transformations.

2.1.8. DEFINITION. *Define the* marked *functor category* $\mathbf{Fun}^+_{\mathbf{Cat}^+}(\mathbf{A}, \mathbf{B})$ *in* $\mathbf{Cat}^+$ *by marking those natural transformations* $(u_a)_{a \in \mathbf{A}}$ *of* $\mathbf{Fun}_{\mathbf{Cat}^+}(\mathbf{A}, \mathbf{B})$ *for which* u_a *is a marked isomorphism for every* a *in* $\mathbf{A}$.

Similarly, assume that $\mathbf{A}$ and $\mathbf{B}$ are preadditive categories. Then the category of (enrichment preserving) functors $\mathbf{Fun}_{\mathbf{preAdd}}(\mathbf{A}, \mathbf{B})$ and natural transformations is itself naturally enriched in abelian groups, and hence is an object of $\mathbf{preAdd}$. If $\mathbf{A}$ and $\mathbf{B}$ are marked preadditive categories, then the same applies to the category $\mathbf{Fun}_{\mathbf{preAdd}^+}(\mathbf{A}, \mathbf{B})$ of functors preserving the enrichment and the marked subcategories.

2.1.9. DEFINITION. *Define the* marked *functor category* $\mathbf{Fun}^+_{\mathbf{preAdd}^+}(\mathbf{A}, \mathbf{B})$ *in* $\mathbf{preAdd}^+$ *by marking those natural transformations* $(u_a)_{a \in \mathbf{A}}$ *of* $\mathbf{Fun}_{\mathbf{preAdd}^+}(\mathbf{A}, \mathbf{B})$ *for which* u_a *is marked for every* a *in* $\mathbf{A}$.

2.1.10. REMARK. This is a remark about notation. For $\mathcal{C} = \mathbf{Cat}$ or $\mathcal{C} = \mathbf{preAdd}$ and $\mathbf{A}, \mathbf{B}$ in $\mathcal{C}^+$ we can consider the functor category $\mathbf{Fun}_{\mathcal{C}^+}(\mathbf{A}, \mathbf{B})$ in $\mathcal{C}$. The +-sign indicates that we only consider functors which preserve marked isomorphisms. In general we have a full inclusion of categories $\mathbf{Fun}_{\mathcal{C}^+}(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Fun}_{\mathcal{C}}(\mathcal{F}_+(\mathbf{A}), \mathcal{F}_+(\mathbf{B}))$. The upper index + in $\mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{A}, \mathbf{B})$ indicates that we consider the functor category as a marked category, i.e., as an object of $\mathcal{C}^+$. The symbol $\mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{A}, \mathbf{B})^+$ denotes the subcategory of marked isomorphisms. In our longer pair notation for marked objects we thus have

$$\mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{A}, \mathbf{B}) = (\mathbf{Fun}_{\mathcal{C}^+}(\mathbf{A}, \mathbf{B}), \mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{A}, \mathbf{B})^+) \ .$$

We now introduce enrichments of the categories over simplicial sets using the nerve functor

$$\mathrm{N} \colon \mathbf{Cat} \to \mathbf{sSet} \ .$$

2.1.11. REMARK. The usual enrichment of $\mathbf{Cat}$ over simplicial sets is given by setting

$$\mathrm{Map}^{\mathrm{standard}}_{\mathbf{Cat}}(\mathbf{A}, \mathbf{B}) := \mathrm{N}(\mathbf{Fun}_{\mathbf{Cat}}(\mathbf{A}, \mathbf{B})) \ .$$

In the present paper we will consider a different enrichment which only takes the invertible natural transformations between functors into account.

For the rest of this section $\mathcal{C}$ serves as a placeholder for either $\mathbf{Cat}$ or $\mathbf{preAdd}$.

We start with marked categories $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}^+$.

2.1.12. Definition. *We define*

$$\mathrm{Map}_{\mathcal{C}^+}(\mathbf{A},\mathbf{B}) := \mathrm{N}(\mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{A},\mathbf{B})^+)\ .$$

In other words, $\mathrm{Map}_{\mathcal{C}^+}(\mathbf{A},\mathbf{B})$ is the nerve of the groupoid of marked isomorphisms in $\mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{A},\mathbf{B})$.

Let now $\mathbf{A}$ and $\mathbf{B}$ be categories in $\mathcal{C}$.

2.1.13. Definition. *We define*

$$\mathrm{Map}_{\mathcal{C}}(\mathbf{A},\mathbf{B}) := \mathrm{N}(\mathbf{Fun}^+_{\mathcal{C}^+}(\mathrm{ma}(\mathbf{A}),\mathrm{ma}(\mathbf{B}))^+)\ .$$

In other words, $\mathrm{Map}_{\mathcal{C}}(\mathbf{A},\mathbf{B})$ is the nerve of the groupoid of isomorphisms in $\mathbf{Fun}_{\mathcal{C}}(\mathbf{A},\mathbf{B})$.

The composition of functors and natural transformations naturally induces the composition law for these mapping spaces. In this way we have turned the categories $\mathbf{Cat}$, $\mathbf{Cat}^+$, $\mathbf{preAdd}$ and $\mathbf{preAdd}^+$ into simplicially enriched categories.

2.1.14. Remark. Since the mapping spaces are nerves of groupoids they are Kan complexes. Therefore these simplicial categories are fibrant in Bergner's model structure on simplicial categories [Ber07].

2.2. The model categories $\mathbf{preAdd}^+$ and $\mathbf{Cat}^+$.

In this section we describe the model category structures on the categories $\mathbf{Cat}$, $\mathbf{Cat}^+$, $\mathbf{preAdd}$ and $\mathbf{preAdd}^+$, see Definition 2.2.1. The main result is Theorem 2.2.2.

As before, $\mathcal{C}$ serves as a placeholder for either $\mathbf{Cat}$ or $\mathbf{preAdd}$. We first introduce the data for the model category structure on $\mathcal{C}$ or $\mathcal{C}^+$.

2.2.1. Definition.

1. *A morphism $f\colon \mathbf{A}\to\mathbf{B}$ in $\mathcal{C}$ (or $\mathcal{C}^+$) is a weak equivalence if it admits an inverse $g\colon \mathbf{B}\to\mathbf{A}$ up to isomorphisms (or marked isomorphisms).*
2. *A morphism in $\mathcal{C}$ (or $\mathcal{C}^+$) is called a cofibration if it is injective on objects.*
3. *A morphism in $\mathcal{C}$ (or $\mathcal{C}^+$) is called a fibration, if it has the right lifting property for trivial cofibrations.*

In the marked case, we call the weak equivalences also marked equivalences.

The following is the main theorem of the present section.

2.2.2. Theorem. *The simplicial category $\mathcal{C}$ (or $\mathcal{C}^+$) with the weak equivalences, cofibrations and fibrations as in Definition 2.2.1 is a simplicial and combinatorial model category.*

PROOF. We refer to [Hov99, Def. 1.1.3 and Def. 1.1.4] or [Hir03, Def. 7.1.3] for the axioms (M1)-(M5) for a model category and [Hir03, Def. 9.1.6] for the additional axioms (M6) and (M7) for a simplicial model category. For the Definition of cofibrant generation we refer to [Hov99, Def. 2.1.17] or [Hir03, Def. 11.1.2]. Finally, a model category is called combinatorial if it is cofibrantly generated and locally presentable [Dug01], [Lur09, Def. A.2.6.1].

1. In Proposition 2.2.4 we verify completeness and cocompleteness (M1).
2. Weak equivalences have the two-out-of-three property (M2) by Lemma 2.2.14.
3. Weak equivalences, cofibrations and fibrations are closed under retracts (M3) by Proposition 2.2.15.
4. Lifting along trivial cofibrations holds by definition. Lifting along trivial fibrations (M4) holds by Proposition 2.2.13.
5. Existence of factorizations (M5) follows from Lemma 2.2.26 and Lemma 2.2.28.
6. For (M6), the simplicial enrichment was already discussed. Tensors and cotensors with simplicial sets are treated by Corollary 2.2.22. The pushout-product axiom (M7) is verified in Proposition 2.2.24.
7. The category is cofibrantly generated by Corollary 2.2.34.
8. It is locally presentable by Proposition 2.2.35.

■

2.2.3. REMARK. The case of **Cat** is well-known. In the following, in order to avoid case distinctions, we will only consider the marked case in full detail. In fact, the functor $\mathrm{ma}\colon \mathcal{C} \to \mathcal{C}^+$ is the inclusion of a full simplicial subcategory and the model category structure is inherited. We will indicate the necessary modifications (e.g, list the generating (trivial) cofibrations or the generators of the category in the unmarked case) in remarks at the appropriate places.

Completeness and cocompleteness in the following means admitting limits and colimits with indexing categories in the universe $\mathcal{U}$, see (1).

2.2.4. PROPOSITION. *The category $\mathcal{C}^+$ is complete and cocomplete.*

PROOF. We will deduce the marked case from the unmarked one and use as a known fact that $\mathcal{C}$ is complete and cocomplete, see [Bor94a, Prop. 5.1.7] for cocompleteness for $\mathcal{C} = \mathbf{Cat}$.

Let I be a category in $\mathcal{U}$ (see (1)) and $X\colon I \to \mathcal{C}^+$ be a diagram. We form the object $\operatorname{colim}_I \mathcal{F}_+(X)$ of $\mathcal{C}$. We have a canonical morphism $\mathcal{F}_+(X) \to \underline{\operatorname{colim}_I \mathcal{F}_+(X)}$, where $\underline{-}$ denotes the constant I-object. We define the marked subcategory of $\operatorname{colim}_I \mathcal{F}_+(X)$ as the subcategory generated by the images of marked isomorphisms under the canonical functors $\mathcal{F}_+(X(i)) \to \operatorname{colim}_I \mathcal{F}_+(X)$ for all i in I and denote the resulting object of $\mathcal{C}^+$ by Y. We claim that the resulting morphism $X \to \underline{Y}$ represents the colimit of the diagram X. If $Y \to T$ is a morphism in $\mathcal{C}^+$, then the induced functor $\mathcal{F}_+(X) \to \underline{\mathcal{F}_+(Y)} \to \underline{\mathcal{F}_+(T)}$ preserves marked isomorphisms, i.e., refines to a morphism in $(\mathcal{C}^+)^I$. Vice versa, if $X \to \underline{T}$ is a morphism in $(\mathcal{C}^+)^I$, then we get an induced morphism $\mathcal{F}_+(Y) \to \underline{\mathcal{F}_+(T)}$. It preserves marked isomorphisms and therefore refines to a morphism in $\mathcal{C}^+$. This shows that $\mathcal{C}^+$ is cocomplete.

Let $X \colon I \to \mathcal{C}^+$ again be a diagram. We form the object $\lim_I \mathcal{F}_+(X)$ of $\mathcal{C}$. We have a canonical morphism $\underline{\lim_I \mathcal{F}_+(X)} \to \mathcal{F}_+(X)$. We mark all isomorphisms in $\lim_I \mathcal{F}_+(X)$ whose evaluations at every i in I are marked isomorphisms in $X(i)$. In this way we define an object Y of $\mathcal{C}^+$. We claim that the resulting morphism $\underline{Y} \to X$ represents the limit of the diagram X.

If $T \to Y$ is a morphism in $\mathcal{C}^+$, then the induced $\underline{\mathcal{F}_+(T)} \to \underline{\mathcal{F}_+(Y)} \to \mathcal{F}_+(X)$ refines to a morphism in $(\mathcal{C}^+)^I$. Vice versa, if $\underline{T} \to X$ is a morphism in $(\mathcal{C}^+)^I$, then we get an induced morphism $\underline{\mathcal{F}_+(T)} \to \mathcal{F}_+(Y)$ which again refines to a morphism in $\mathcal{C}^+$. This shows that $\mathcal{C}^+$ is complete. ■

We let

$$\mathcal{F}_{\mathbf{All}} \colon \mathcal{C}^+ \to \mathbf{Cat}$$

denote the functor which takes the underlying category, i.e., which forgets markings and enrichments (in the case of $\mathbf{preAdd}^+$). Recall further that we have the functor

$$(-)^+ \colon \mathcal{C}^+ \to \mathbf{Groupoids}$$

taking the groupoid of marked isomorphisms.

Let $f \colon \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathcal{C}^+$.

2.2.5. Lemma. *The following are equivalent.*

1. *f is a weak equivalence.*
2. *$\mathcal{F}_{\mathbf{All}}(f)$ and f^+ are equivalences in* $\mathbf{Cat}$ *and* $\mathbf{Groupoids}$*, respectively.*

Proof. If f is a weak equivalence, then by Definition 2.2.1 there exists an inverse g up to marked isomorphism. Then $\mathcal{F}_{\mathbf{All}}(g)$ and g^+ are the required inverse equivalences of $\mathcal{F}_{\mathbf{All}}(f)$ and f^+.

We now show the converse. We can choose an inverse equivalence $g^+ \colon \mathbf{B}^+ \to \mathbf{A}^+$ of f^+ and a natural isomorphism $u \colon \mathrm{id}_{\mathbf{B}^+} \xrightarrow{\cong} f^+g^+$. We then define a functor $g \colon \mathbf{B} \to \mathbf{A}$ as follows.

1. On objects: For an object B of $\mathbf{B}$ we set $g(B) := g^+(B)$.

2. On morphisms: On the set of morphisms $\mathrm{Hom}_{\mathbf{B}}(B, B')$, we define g as the composition

$$\mathrm{Hom}_{\mathbf{B}}(B,B') \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{B}}(fg(B), fg(B')) \xleftarrow{\cong} \mathrm{Hom}_{\mathbf{A}}(g(B), g(B')) \ .$$

Here the first isomorphism sends b to $u_{B'}bu_B^{-1}$ and the second isomorphism employs the fact that $\mathcal{F}_{\mathbf{All}}(f)$ is an equivalence. In the case $\mathcal{C} = \mathbf{preAdd}$ it is clear from this description that the isomorphism is compatible with abelian group structures on the morphism sets. Since u is given by marked isomorphisms and f induces a bijection on marked isomorphisms, this map also preserves marked isomorphisms.

Then g is the required inverse of f up to marked isomorphism. The natural transformations are u and $v\colon \mathrm{id}_{\mathbf{A}} \to gf$ determined by $f(v_A) = u_{f(A)}$. Note that both are by marked isomorphisms since f is a bijection on marked isomorphisms. ∎

Note that a weak equivalence not only preserves marked isomorphisms, but also detects them.

Let $\mathbf{C}$ and $\mathbf{D}$ be two objects of $\mathcal{C}^+$ and $a\colon \mathbf{C} \to \mathbf{D}$ be a morphism.

2.2.6. DEFINITION. *The morphism a is called a* marked isofibration, *if for every object d of $\mathbf{D}$, every object c of $\mathbf{C}$ and every marked isomorphism $u\colon a(c) \to d$ in $\mathbf{D}$ there exists a marked isomorphism $v\colon c \to c'$ in $\mathbf{C}$ such that $a(v) = u$.*

Note that this is equivalent to $a^+\colon \mathbf{C}^+ \to \mathbf{D}^+$ being an isofibration in the usual sense.

2.2.7. EXAMPLE. The object classifier in $\mathbf{Cat}$ is the category $\Delta^0_{\mathbf{Cat}}$ with one object $*$ and one morphism id_*. The object classifier in $\mathbf{Cat}^+$ is given by $\Delta^0_{\mathbf{Cat}^+} := \mathrm{mi}(\Delta^0_{\mathbf{Cat}})$. Furthermore, the object classifiers in $\mathbf{preAdd}$ and $\mathbf{preAdd}^+$ are given by $\Delta^0_{\mathbf{preAdd}} := \mathrm{Lin}_{\mathbb{Z}}(\Delta^0_{\mathbf{Cat}})$ and $\Delta^0_{\mathbf{preAdd}^+} := \mathrm{Lin}_{\mathbb{Z}}(\Delta^0_{\mathbf{Cat}^+})$, respectively.

The morphism classifier in $\mathbf{Cat}$ is the category $\Delta^1_{\mathbf{Cat}}$ with two objects 0 and 1, and one non-identity morphism $0 \to 1$. The morphism classifier in $\mathbf{Cat}^+$ is given by $\Delta^1_{\mathbf{Cat}^+} := \mathrm{mi}(\Delta^1_{\mathbf{Cat}})$. Furthermore, the morphism classifiers in $\mathbf{preAdd}$ and $\mathbf{preAdd}^+$ are given by $\Delta^1_{\mathbf{preAdd}} := \mathrm{Lin}_{\mathbb{Z}}(\Delta^1_{\mathbf{Cat}})$ and $\Delta^1_{\mathbf{preAdd}^+} := \mathrm{Lin}_{\mathbb{Z}}(\Delta^1_{\mathbf{Cat}^+})$, respectively.

The invertible morphism classifier in $\mathbf{Cat}$ is the category $\mathbb{I}_{\mathbf{Cat}}$ with two objects 0 and 1, and non-identity morphisms $0 \to 1$ and its inverse $1 \to 0$. The invertible morphism classifier in $\mathbf{Cat}^+$ is given by $\mathbb{I}_{\mathbf{Cat}^+} := \mathrm{mi}(\mathbb{I}_{\mathbf{Cat}})$. Furthermore, the invertible morphism classifiers in $\mathbf{preAdd}$ and $\mathbf{preAdd}^+$ are given by $\mathbb{I}_{\mathbf{preAdd}} := \mathrm{Lin}_{\mathbb{Z}}(\mathbb{I}_{\mathbf{Cat}})$ and $\mathbb{I}_{\mathbf{preAdd}^+} := \mathrm{Lin}_{\mathbb{Z}}(\mathbb{I}_{\mathbf{Cat}^+})$, respectively.

Finally, the marked isomorphism classifier in $\mathbf{Cat}^+$ is given by $\mathbb{I}^+_{\mathbf{Cat}^+} := \mathrm{ma}(\mathbb{I}_{\mathbf{Cat}})$, and the one in $\mathbf{preAdd}^+$ is given by $\mathbb{I}^+_{\mathbf{preAdd}^+} := \mathrm{Lin}_{\mathbb{Z}}(\mathbb{I}^+_{\mathbf{Cat}^+})$.

We have the following statement about morphisms in $\mathcal{C}^+$.

2.2.8. LEMMA.

1. *Trivial fibrations are surjective on objects.*

2. *Weak equivalences which are surjective on objects have the right lifting property with respect to all cofibrations.*

In particular, a weak equivalence is a trivial fibration if and only if it is surjective on objects.

PROOF. Let $f\colon \mathbf{C} \to \mathbf{D}$ be a trivial fibration and let D in $\mathbf{D}$ be an object. Since f is a weak equivalence, there exists an object C in $\mathbf{C}$ and an isomorphism $d\colon f(C) \xrightarrow{\cong} D$. Consider the commutative diagram

$$\begin{tikzcd} \Delta^0_{\mathcal{C}^+} \arrow[r, "C"] \arrow[d] & \mathbf{C} \arrow[d, "f"] \\ \mathbb{I}_{\mathcal{C}^+} \arrow[r, "d"] & \mathbf{D} \end{tikzcd}$$

Since $\Delta^0_{\mathcal{C}^+} \to \mathbb{I}_{\mathcal{C}^+}$ is a trivial cofibration, d admits a lift c to $\mathbf{C}$ whose codomain is a preimage of D.

Let now $f\colon \mathbf{C} \to \mathbf{D}$ be a weak equivalence which is surjective on objects. Consider a commutative diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\alpha"] \arrow[d, "i"'] & \mathbf{C} \arrow[d, "f"] \\ \mathbf{B} \arrow[r, "\beta"] & \mathbf{D} \end{tikzcd}$$

in which i is a cofibration.

We first define the lift γ of β on objects. If B in $\mathbf{B}$ lies in the image of i, there exists a unique object A in $\mathbf{A}$ with $i(A) = B$, and we set $\gamma(B) = \alpha(A)$. Otherwise, pick any C in $\mathbf{C}$ such that $f(C) = \beta(B)$ and set $\gamma(B) = C$. For a morphism b in $\mathbf{B}$, define $\gamma(b)$ as the unique preimage of $\beta(b)$ under f. Since the preimage $\gamma(b)$ is unique (subject to the choices made on objects), γ is a functor. Then $f \circ \gamma = \beta$ holds by definition, and $\gamma \circ i = \alpha$ also follows easily from the fact that f is faithful. ■

2.2.9. LEMMA. *A morphism in $\mathcal{C}^+$ is a marked isofibration if and only if it has the right lifting property with respect to the morphism*

$$\Delta^0_{\mathcal{C}^+} \xrightarrow{0} \mathbb{I}^+_{\mathcal{C}^+}$$

classifying the object 0 *of* $\mathbb{I}^+_{\mathcal{C}^+}$.

PROOF. In view of the universal properties of $\Delta^0_{\mathcal{C}^+}$ and $\mathbb{I}^+_{\mathcal{C}^+}$ this is just a reformulation of Definition 2.2.6. ■

Since $\Delta^0_{\mathcal{C}^+} \xrightarrow{0} \mathbb{I}^+_{\mathcal{C}^+}$ is a trivial cofibration we conclude that fibrations are marked isofibrations.

2.2.10. PROPOSITION. *The marked isofibrations in $\mathcal{C}^+$ have the right lifting property with respect to trivial cofibrations.*

PROOF. We consider a diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\alpha"] \arrow[d, "i"'] & \mathbf{C} \arrow[d, "f"] \\ \mathbf{B} \arrow[r, "\beta"] \arrow[ur, dotted, "\ell"] & \mathbf{D} \end{tikzcd}$$

in $\mathcal{C}^+$, where f is a marked isofibration and i is a trivial cofibration. In the usual construction of an inverse $j \colon \mathbf{B} \to \mathbf{A}$ to i, we can choose the preimages of the objects in the essential image of $\mathbf{A}$ such that $j \circ i = \mathrm{id}_{\mathbf{A}}$ since i is injective on objects. Moreover, we can choose a marked isomorphism $u \colon i \circ j \to \mathrm{id}_{\mathbf{B}}$ which in addition satisfies $u \circ i = \mathrm{id}_i$.

On objects we define ℓ as follows: For every object B of $\mathbf{B}$ we get a marked isomorphism

$$\beta(u_B) \colon f(\alpha(j(B)) = \beta(i(j(B))) \to \beta(B) \ .$$

Using that f is a marked isofibration we choose a marked isomorphism $v_B \colon \alpha(j(B)) \to C$ such that $f(v_B) = \beta(u_B)$. If B is in the image of i, we can and will choose v_B to be the identity. We then set $\ell(B) := C$. This makes both triangles commute.

We now define the lift ℓ on a morphism $\phi \colon B \to B'$ by

$$\ell(\phi) := v_{B'} \circ \alpha(j(\phi)) \circ v_B^{-1} \ .$$

One can check that then both triangles commute and that this really defines a functor. One further checks that ℓ is a morphism of marked categories (and preserves the enrichment in the case of preadditive categories). Here we use that i detects marked isomorphisms. ■

2.2.11. COROLLARY. *The notions of marked isofibration and fibration in $\mathcal{C}^+$ coincide.*

2.2.12. REMARK. We note that all objects in $\mathcal{C}^+$ are fibrant and cofibrant. Consequently, the model category $\mathcal{C}^+$ is proper by [Hir03, Cor. 13.1.3]

2.2.13. PROPOSITION. *The cofibrations in $\mathcal{C}^+$ have the left lifting property with respect to trivial fibrations.*

PROOF. We consider a diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\alpha"] \arrow[d, "i"'] & \mathbf{C} \arrow[d, "f"] \\ \mathbf{B} \arrow[r, "\beta"] \arrow[ur, dotted, "\ell"] & \mathbf{D} \end{tikzcd}$$

in $\mathcal{C}^+$, where f is a trivial fibration and i is a cofibration.

Since the map i is injective on objects and the morphism f is surjective on objects by Lemma 2.2.8, we can find a lift ℓ on the level of objects. Let now B, B' be objects in $\mathbf{B}$. Since f is fully faithful we have a bijection

$$\mathrm{Hom}_{\mathbf{C}}(\ell(B), \ell(B')) \xrightarrow[\cong]{f} \mathrm{Hom}_{\mathbf{D}}(\beta(B), \beta(B')) \ .$$

We can therefore define ℓ on $\mathrm{Hom}_{\mathbf{B}}(B, B')$ by

$$\mathrm{Hom}_{\mathbf{B}}(B, B') \xrightarrow{\beta} \mathrm{Hom}_{\mathbf{D}}(\beta(B), \beta(B')) \cong \mathrm{Hom}_{\mathbf{C}}(\ell(B), \ell(B')) \ .$$

Since f detects marked isomorphisms, ℓ preserves them. The lower triangle commutes by construction. One can furthermore check that the upper triangle commutes. Finally one checks that this really defines a functor. ∎

2.2.14. LEMMA. *The weak equivalences in $\mathcal{C}^+$ satisfy the two-out-of-three axiom.*

PROOF. This follows from Lemma 2.2.5. ∎

2.2.15. PROPOSITION. *The cofibrations, fibrations, and weak equivalences in $\mathcal{C}^+$ are closed under retracts.*

PROOF. Since fibrations are characterized by a right lifting property they are closed under retracts. Cofibrations are closed under retracts since a retract diagram of marked categories induces a retract diagram on the level of sets of objects, and injectivity of maps between sets is closed under retracts. It remains to consider weak equivalences. We consider a diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "i"] \arrow[d, "f"] & \mathbf{A}' \arrow[r, "p"] \arrow[d, "f'"] & \mathbf{A} \arrow[d, "f"] \\ \mathbf{B} \arrow[r, "j"] & \mathbf{B}' \arrow[r, "q"] & \mathbf{B} \end{tikzcd}$$

in $\mathcal{C}^+$ with $p \circ i = \mathrm{id}_{\mathbf{A}}$ and $q \circ j = \mathrm{id}_{\mathbf{B}}$, and where f' is a weak equivalence. Let $g' \colon \mathbf{B}' \to \mathbf{A}'$ be an inverse of f' up to marked isomorphism. Then $p \circ g' \circ j \colon \mathbf{B} \to \mathbf{A}$ is an inverse of f up to marked isomorphism. ∎

We have finished the verification of the basic model category axioms except the existence of factorizations. This follows from considerations about the simplicial structure which we do now.

We define a functor

$$Q \colon \mathbf{Groupoids} \to \mathcal{C}^{(+)} \tag{6}$$

as follows. Let $i \colon \mathbf{Groupoids} \to \mathbf{Cat}$ be the inclusion.

1. In the case $\mathbf{Cat}$, we define $Q := i$.
2. In the case $\mathbf{Cat}^+$, we define $Q := \mathrm{ma} \circ i$.
3. In the case $\mathbf{preAdd}$, we define $Q := \mathrm{Lin}_{\mathbb{Z}} \circ i$.
4. In the case $\mathbf{preAdd}^+$, we define $Q := \mathrm{Lin}_{\mathbb{Z}} \circ \mathrm{ma} \circ i$.

2.2.16. LEMMA. *We have an adjunction*

$$Q(-) \colon \mathbf{Groupoids} \leftrightarrows \mathcal{C}^+ \colon (-)^+$$

between groupoid-enriched categories.

PROOF. It is straightforward to check that the obvious isomorphism $Q(G)^+ \cong G$ for G in **Groupoids** and the morphism $Q(\mathbf{A}^+) \to \mathbf{A}$ of marked categories induced by the inclusion $\mathbf{A}^+ \to \mathbf{A}$ for $\mathbf{A}$ in $\mathcal{C}^+$ give rise to an isomorphism

$$\mathbf{Fun}(G, \mathbf{A}^+) \cong \mathbf{Fun}^+_{\mathcal{C}^+}(Q(G), \mathbf{A})^+ \ .$$

■

For (marked) preadditive categories we need a further symmetric monoidal product structure $\otimes$ (which differs from the cartesian structure) on $\mathbf{preAdd}^{(+)}$ given as follows:

1. (objects) The objects of $\mathbf{A} \otimes \mathbf{B}$ are pairs (A, B) of objects A in $\mathbf{A}$ and B in $\mathbf{B}$.
2. (morphisms) The abelian group of morphisms between (A, B) and (A', B') is given by
$$\mathrm{Hom}_{\mathbf{A}\otimes\mathbf{B}}((A, B), (A', B')) := \mathrm{Hom}_{\mathbf{A}}(A, A') \otimes \mathrm{Hom}_{\mathbf{B}}(B, B') \ .$$
The composition is defined in the obvious way.
3. (marking) We mark tensor products of marked isomorphisms.

We refrain from writing out the remaining data (unit, unit- and associativity constraints) explicitly.

In order to define a tensor structure of $\mathcal{C}^{(+)}$ over simplicial sets, we start with a tensor structure over groupoids.

2.2.17. DEFINITION. *In the case* $\mathbf{Cat}^{(+)}$ *we define the functor*

$$-\sharp-\colon \mathbf{Cat}^{(+)} \times \mathbf{Groupoids} \to \mathbf{Cat}^{(+)} \ , \quad (\mathbf{A}, G) \mapsto \mathbf{A}\sharp G := \mathbf{A} \times Q(G).$$

In the case $\mathbf{preAdd}^{(+)}$ *we define the functor*

$$-\sharp-\colon \mathbf{preAdd}^{(+)} \times \mathbf{Groupoids} \to \mathbf{preAdd}^{(+)} \ , \quad (\mathbf{A}, G) \mapsto \mathbf{A}\sharp G := \mathbf{A} \otimes Q(G).$$

Let $\mathbf{B}$ be in $\mathcal{C}^+$. In the following lemma, we will write $\otimes$ for the product in $\mathbf{Cat}^+$, to avoid distinguishing between $\mathbf{Cat}^+$ and $\mathbf{preAdd}^+$.

2.2.18. LEMMA. *We have an adjunction*

$$- \otimes \mathbf{B}\colon \mathcal{C}^+ \leftrightarrows \mathcal{C}^+\colon \mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{B}, -) \ ,$$

where we view $\mathcal{C}^+$ *as enriched over* $\mathcal{C}^+$.

PROOF. We provide an explicit description of the unit and the counit of the adjunction. For $\mathbf{A}$ in $\mathcal{C}^+$ they are given by morphisms

$$\eta_{\mathbf{A}}\colon \mathbf{A} \to \mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{B}, \mathbf{A} \otimes \mathbf{B}) \quad \text{and} \quad \epsilon_{\mathbf{A}}\colon \mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{B}, \mathbf{A}) \otimes \mathbf{B} \to \mathbf{A}$$

defined as follows:

1. The morphism $\eta_{\mathbf{A}}$ takes an object A in $\mathbf{A}$ to the functor sending an object B in $\mathbf{B}$ to (A, B) and a morphism b in $\mathbf{B}$ to $\mathrm{id}_A \otimes b$. A morphism $a\colon A \to A'$ is sent by $\eta_{\mathbf{A}}$ to the natural transformation $\{a \otimes \mathrm{id}_B\colon (A, B) \to (A', B)\}_{B \in \mathbf{B}}$.

2. The morphism ϵ_A is induced by evaluation of functors.

One checks that η and ϵ are enriched natural transformations. One furthermore checks the triangle identities by explicit calculations. ■

Recall that for $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}^+$ the category $\mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{A}, \mathbf{B})^+$ is a groupoid. Let G be a groupoid. From Lemmas 2.2.16 and 2.2.18 we get natural isomorphisms

$$\mathbf{Fun}_{\mathbf{Groupoids}}(G, \mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{A}, \mathbf{B})^+) \cong \mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{A}\sharp G, \mathbf{B})^+ \cong \mathbf{Fun}^+_{\mathcal{C}^+}(\mathbf{A}, \mathbf{Fun}^+_{\mathcal{C}^+}(Q(G), \mathbf{B}))^+ \tag{7}$$

In order to define the tensor structure of $\mathcal{C}^+$ with simplicial sets we consider the fundamental groupoid functor.

2.2.19. DEFINITION. *The* fundamental groupoid *functor* Π *is defined as the left adjoint of the adjunction*

$$\Pi\colon \mathbf{sSet} \leftrightarrows \mathbf{Groupoids}\colon \mathrm{N}\ ,$$

where N *takes the nerve of a groupoid.*

Explicitly, the fundamental groupoid $\Pi(K)$ of a simplicial set K is the groupoid freely generated by the path category $P(K)$ of K. The category $P(K)$ in turn is given as follows:

1. The objects of $P(K)$ are the 0-simplices.

2. The morphisms of $P(K)$ are generated by the 1-simplices of K subject to the relation $g \circ f \sim h$ if there exists a 2-simplex σ in K with $d_2\sigma = f$, $d_0\sigma = g$ and $d_1\sigma = h$.

2.2.20. LEMMA. *We have an adjunction*

$$\Pi\colon \mathbf{sSet} \leftrightarrows \mathbf{Groupoids}\colon \mathrm{N}\ ,$$

of simplicially enriched categories.

PROOF. Since Π commutes with finite products, this is a consequence of the unenriched adjunction:

$$\begin{aligned}\mathrm{Map}_{\mathbf{sSet}}(K, \mathrm{N}(G))_n &\cong \mathrm{Hom}_{\mathbf{sSet}}(K \times \Delta^n, \mathrm{N}(G)) \\ &\cong \mathrm{Hom}_{\mathbf{Groupoids}}(\Pi(K \times \Delta^n), G) \\ &\cong \mathrm{Hom}_{\mathbf{Groupoids}}(\Pi(K) \times \Pi(\Delta^n), G) \\ &\cong \mathrm{Hom}_{\mathbf{Groupoids}}(\Pi(\Delta^n), \mathbf{Fun}(\Pi(K), G)) \\ &\cong \mathrm{Map}_{\mathbf{Groupoids}}(\Pi(K), G)_n \ .\end{aligned}$$

■

Using the tensor and cotensor structure with groupoids we define the corresponding structures with simplicial sets by precomposition with the fundamental groupoid functor. Recall the definition (6) of Q.

2.2.21. DEFINITION. *We define* tensor and cotensor structures *on $\mathcal{C}^+$ with simplicial sets by*

$$\mathcal{C}^+ \times \mathbf{sSet} \to \mathcal{C}^+ \ , \quad (\mathbf{A}, K) \mapsto \mathbf{A}\sharp\Pi(K) \ .$$

$$\mathbf{sSet}^{op} \times \mathcal{C}^+ \to \mathcal{C}^+ \ , \quad (K, \mathbf{B}) \mapsto \mathbf{Fun}^+_{\mathcal{C}^+}(Q(\Pi(K)), \mathbf{B}) \ .$$

In order to simplify notation, we will usually write $\mathbf{A}\sharp K$ instead of $\mathbf{A}\sharp\Pi(K)$ and $\mathbf{B}^K$ instead of $\mathbf{Fun}^+_{\mathcal{C}^+}(Q(\Pi(K)), \mathbf{B})$.

Recall Definition 2.1.12 of the simplicial mapping sets in $\mathcal{C}^+$. Applying the nerve functor to (7) and using Lemma 2.2.20, we obtain the following corollary.

2.2.22. COROLLARY. *For K in $\mathbf{sSet}$ and $\mathbf{A}$, $\mathbf{B}$ in $\mathcal{C}^+$ we have natural isomorphisms of simplicial sets*

$$\mathrm{Map}_{\mathbf{sSet}}(K, \mathrm{Map}_{\mathcal{C}^+}(\mathbf{A}, \mathbf{B})) \cong \mathrm{Map}_{\mathcal{C}^+}(\mathbf{A}\sharp K, \mathbf{B}) \cong \mathrm{Map}_{\mathcal{C}^+}(\mathbf{A}, \mathbf{B}^K) \ .$$

We consider a commutative square

$$\begin{tikzcd} \mathbf{A} \arrow[r, "i"] \arrow[d, "f"'] & \mathbf{B} \arrow[d, "g"] \\ \mathbf{C} \arrow[r, "j"] & \mathbf{D} \end{tikzcd} \qquad (8)$$

in $\mathcal{C}^+$.

2.2.23. LEMMA.

1. *If (8) is a pushout and i is a trivial cofibration, then j is a trivial cofibration.*
2. *If (8) is a pullback and g is a trivial fibration, then f is a trivial fibration.*

Proof. We show Assertion 1. Because i is a trivial cofibration, there exists a morphism $i'\colon \mathbf{B} \to \mathbf{A}$ such that $i' \circ i = \mathrm{id}_{\mathbf{A}}$ and a marked isomorphism $u\colon i \circ i' \to \mathrm{id}_{\mathbf{B}}$ satisfying $u \circ i = \mathrm{id}_i$. By the universal property of the pushout, the morphism $f \circ i'\colon \mathbf{B} \to \mathbf{C}$ induces a morphism $j'\colon \mathbf{D} \to \mathbf{C}$ such that $j' \circ j = \mathrm{id}_{\mathbf{C}}$. In particular, j is a cofibration and it remains to show that it is a weak equivalence. Moreover, $g \circ u$ provides a marked isomorphism $j \circ f \circ i' = g \circ i \circ i' \to g$.

The functor $-\sharp\mathbb{I}_{\mathbf{Cat}}\colon \mathcal{C}^+ \to \mathcal{C}^+$ (see Example 2.2.7 for $\mathbb{I}_{\mathbf{Cat}}$ in **Groupoids**) is a left adjoint by Lemma 2.2.18. Therefore it preserves pushouts. Using the first isomorphism in (7) and the fact that $\mathbb{I}_{\mathbf{Cat}}$ is the morphism classifier in **Groupoids**, we consider the natural transformation $g \circ u$ as a functor $\mathbf{B}\sharp\mathbb{I}_{\mathbf{Cat}} \to \mathbf{D}$. Together with the functor $\mathbf{C}\sharp\mathbb{I}_{\mathbf{Cat}} \to \mathbf{D}$ corresponding to the identity natural transformation of j, by the universal property of the pushout diagram (8)$\sharp\mathbb{I}_{\mathbf{Cat}}$ we obtain an induced functor $\mathbf{D}\sharp\mathbb{I}_{\mathbf{Cat}} \to \mathbf{D}$ which provides, by a converse application of the first isomorphism in (7), a marked isomorphism $j \circ j' \to \mathrm{id}_{\mathbf{D}}$. This proves that j is a weak equivalence.

The proof of Assertion 2 can be obtained by dualizing the proof above. ∎

The following proposition verifies the pushout-product axiom (M7).

2.2.24. Proposition. *Let $a\colon \mathbf{A} \to \mathbf{B}$ be a cofibration in $\mathcal{C}^+$ and $i\colon X \to Y$ be a cofibration in* **sSet**. *Then*

$$(\mathbf{A}\sharp Y) \sqcup_{\mathbf{A}\sharp X} (\mathbf{B}\sharp X) \to (\mathbf{B}\sharp Y) \tag{9}$$

is a cofibration. Moreover, if i or a is in addition a weak equivalence, then (9) is a weak equivalence.

In the proof of this proposition we use the following lemma.

2.2.25. Lemma. *Let $\mathbf{A}$ be in $\mathcal{C}^+$ and let K be a simplicial set. The functors*

$$\mathbf{A}\sharp-\colon \mathbf{sSet} \to \mathcal{C}^+$$

and

$$-\sharp K\colon \mathcal{C}^+ \to \mathcal{C}^+$$

preserve (trivial) cofibrations.

Proof. Observe that $\mathbf{A} \otimes -$ preserves both cofibrations and marked equivalences. For $\mathbf{A} = Q(\Pi(K))$ this directly implies that $-\sharp K$ preserves (trivial) cofibrations. To see that also $\mathbf{A}\sharp-$ preserves (trivial) cofibrations, it suffices to show that $Q \circ \Pi$ preserves cofibrations and sends weak equivalences to marked equivalences. For cofibrations, this is clear since the set of objects of $Q(\Pi(K))$ is given by the vertices of K for every simplicial set K. Moreover, Π sends weak equivalences to equivalences of groupoids. Since Q marks all isomorphisms, any equivalence of groupoids induces a marked equivalence. ∎

Proof Proof of Proposition 2.2.24. Consider the diagram

$$\begin{tikzcd}
\mathbf{A}\sharp X \arrow[rr, "\mathbf{A}\sharp i"'] \arrow[dd, "a\sharp X"'] & & \mathbf{A}\sharp Y \arrow[dl, "b"] \arrow[dd, "a\sharp Y"] \\
& (\mathbf{A}\sharp Y) \sqcup_{\mathbf{A}\sharp X} (\mathbf{B}\sharp X) \arrow[dr, "?"] & \\
\mathbf{B}\sharp X \arrow[ur] \arrow[rr, "\mathbf{B}\sharp i"] & & \mathbf{B}\sharp Y
\end{tikzcd}$$

The set of objects of the pushout is equal to the pushout of the object sets. Hence it is easy to check that ? is injective on objects and thus a cofibration.

Assume that a is a weak equivalence. By Lemma 2.2.25 the maps $a\sharp X$ and $a\sharp Y$ are trivial cofibrations. Since b is a pushout of a trivial cofibration, it is a trivial cofibration by Lemma 2.2.23. It follows from the two-out-of-three property, see Lemma 2.2.14, that the morphism ? is a weak equivalence.

The case that i is a weak equivalence is similar. ■

2.2.26. Lemma. *Every morphism in $\mathcal{C}^+$ can be functorially factored into a cofibration followed by a trivial fibration.*

Proof. Let $a\colon \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathcal{C}^+$. Denote by $i_1\colon \mathbf{A} \cong \mathbf{A}\sharp\Delta^0 \to \mathbf{A}\sharp\partial\Delta^1$ the morphism induced by the map classifying the vertex 1, and let $j\colon \mathbf{A}\sharp\partial\Delta^1 \to \mathbf{A}\sharp\Delta^1$ be the morphism induced by the inclusion $\partial\Delta^1 \to \Delta^1$. Consider the diagram

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "i_1"] \arrow[d, "a"'] & \mathbf{A}\sharp\partial\Delta^1 \arrow[r, "j"] \arrow[d, "\mathrm{id}_{\mathbf{A}} \sqcup a"] & \mathbf{A}\sharp\Delta^1 \arrow[d] \\
\mathbf{B} \arrow[r, "e_{\mathbf{B}}"] & \mathbf{A} \sqcup \mathbf{B} \arrow[r, "b"] & Z(a)
\end{tikzcd}$$

in which $e_{\mathbf{B}}$ is the canonical morphism, and in which the right square is defined to be a pushout. Since $\mathbf{A}\sharp\partial\Delta^1 \cong \mathbf{A} \sqcup \mathbf{A}$, it is easy to see that the left square is also a pushout. Hence, the outer square is also a pushout. By the universal property of the pushout, the composed morphism

$$\mathbf{A}\sharp\Delta^1 \xrightarrow{a\sharp\Delta^1} \mathbf{B}\sharp\Delta^1 \xrightarrow{\mathrm{pr}_{\mathbf{B}}} \mathbf{B}$$

and the identity on $\mathbf{B}$ induce a morphism $q\colon Z(a) \to \mathbf{B}$ such that $q \circ b \circ e_{\mathbf{B}} = \mathrm{id}_{\mathbf{B}}$. In particular, q is surjective on objects. Moreover, $b \circ e_{\mathbf{B}}$ is a trivial cofibration by Lemma 2.2.25 and Lemma 2.2.23.1. The two-out-of-three property (Lemma 2.2.14) implies that q is a weak equivalence, and hence a trivial fibration by Lemma 2.2.8.

Since the structure morphism $e_{\mathbf{A}}\colon \mathbf{A} \to \mathbf{A} \sqcup \mathbf{B}$ is a cofibration, the morphism $b \circ e_{\mathbf{A}}\colon \mathbf{A} \to Z(a)$ is also a cofibration. Considering the left-hand pushout square, it is easy to see that $q \circ b = a + \mathrm{id}_{\mathbf{B}}$. Regarding $Z(a)$ as the pushout of the right square, it follows from the universal property that $q \circ (b \circ e_{\mathbf{A}}) = a$, and thus provides the required factorization. Note that this construction is functorial in a. ■

Let $\mathbf{A}$ be an object of $\mathcal{C}^+$. Recall the notation $\mathbf{A}^K$ for a simplicial set K from Definition 2.2.21.

2.2.27. LEMMA. *The functor*

$$\mathbf{A}^{(-)}\colon \mathbf{sSet}^{op} \to \mathcal{C}^+$$

sends (trivial) cofibrations to (trivial) fibrations.

PROOF. This follows from Lemma 2.2.18 and Lemma 2.2.25 by explicitly checking lifting properties. ∎

2.2.28. LEMMA. *Every morphism in $\mathcal{C}^+$ can be functorially factored into a trivial cofibration followed by a fibration.*

PROOF. Let $a\colon \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathcal{C}^+$. Denote by $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathbf{B}^{\Delta^1} \to \mathbf{B}^{\partial\Delta^1} \cong \mathbf{B}\times\mathbf{B}$ the morphism induced by the canonical inclusion $\partial\Delta^1 \to \Delta^1$. Let $p_1\colon \mathbf{B}^{\partial\Delta^1} \cong \mathbf{B}\times\mathbf{B} \to \mathbf{B}$ denote the projection on the second factor (which corresponds to the vertex 1), and let $p_{\mathbf{A}}\colon \mathbf{A}\times\mathbf{B} \to \mathbf{A}$ be the projection. Consider the diagram

$$\begin{tikzcd}
P(a) \arrow[r, "q"] \arrow[d] & \mathbf{A}\times\mathbf{B} \arrow[r, "p_{\mathbf{A}}"] \arrow[d, "a\times\mathrm{id}_{\mathbf{B}}"] & \mathbf{A} \arrow[d, "a"] \\
\mathbf{B}^{\Delta^1} \arrow[r, "(\mathrm{ev}_0,\mathrm{ev}_1)"] & \mathbf{B}\times\mathbf{B} \arrow[r, "p_1"] & \mathbf{B}
\end{tikzcd}$$

in which the left square is defined to be a pullback. Since the right square is also a pullback, the outer square is a pullback, too. By the universal property of the pullback, the composed morphism

$$\mathbf{A} \xrightarrow{a} \mathbf{B} \xrightarrow{\mathrm{const}} \mathbf{B}^{\Delta^1}$$

and the identity on $\mathbf{A}$ induce a morphism $i\colon \mathbf{A} \to P(a)$ such that $p_{\mathbf{A}} \circ q \circ i = \mathrm{id}_{\mathbf{A}}$. In particular, i is a cofibration. Since $\mathrm{ev}_1 = p_1 \circ (\mathrm{ev}_0, \mathrm{ev}_1)$ is a trivial fibration by Lemma 2.2.27, it follows from Lemma 2.2.23.2 that $p_{\mathbf{A}} \circ q$ is a trivial fibration. The two-out-of-three property (Lemma 2.2.14) implies that i is a weak equivalence, and thus a trivial cofibration.

Note that q is a fibration since it is the pullback of a fibration (use again Lemma 2.2.27 and Lemma 2.2.23.2). Since the structure morphism $p_{\mathbf{B}}\colon \mathbf{A}\times\mathbf{B} \to \mathbf{B}$ is a fibration, the morphism $p_{\mathbf{B}} \circ q\colon P(a) \to \mathbf{B}$ is also a fibration. Regarding $P(a)$ as the pullback of the left square, it follows from the universal property that $(p_{\mathbf{B}} \circ q) \circ i = a$, and thus provides the required factorization. Note that this construction is functorial in a. ∎

We thus have finished the verification of the model category axioms (M1) to (M7).

2.2.29. REMARK. By considering the full embedding $\mathrm{ma}\colon \mathcal{C} \to \mathcal{C}^+$, we obtain a verification of the axioms in the unmarked case.

We next describe the generating cofibrations and the generating trivial cofibrations.

Recall that by Lemma 2.2.9 and Corollary 2.2.11 we can take

$$J := \{\Delta^0_{\mathcal{C}^+} \to \mathbb{I}^+_{\mathcal{C}^+}\}$$

as the generating trivial cofibrations for $\mathcal{C}^+$.

2.2.30. Remark. The set of generating trivial cofibrations for $\mathcal{C}$ is given by

$$J := \{\Delta^0_{\mathcal{C}} \to \mathbb{I}_{\mathcal{C}}\}\ .$$

We furthermore define

$$I := \{U, V, V^+, W, W^+\}$$

where U, V, V^+, W, W^+ are cofibrations defined as follows (see Example 2.2.7):

1. $U\colon \emptyset \to \Delta^0_{\mathcal{C}^+}$.
2. We let $V\colon \Delta^0_{\mathcal{C}^+} \sqcup \Delta^0_{\mathcal{C}^+} \to \Delta^1_{\mathcal{C}^+}$ classify the pair of objects $(0,1)$ of $\Delta^1_{\mathcal{C}^+}$.
3. We let $V^+\colon \Delta^0_{\mathcal{C}^+} \sqcup \Delta^0_{\mathcal{C}^+} \to \mathbb{I}^+_{\mathcal{C}^+}$ classify the pair of objects $(0,1)$ of $\mathbb{I}^+_{\mathcal{C}^+}$.
4. We define P as the pushout

$$\begin{tikzcd} \Delta^0_{\mathcal{C}^+} \sqcup \Delta^0_{\mathcal{C}^+} \arrow[r, "V"] \arrow[d, "V"] & \Delta^1_{\mathcal{C}^+} \arrow[d] \\ \Delta^1_{\mathcal{C}^+} \arrow[r] & P \end{tikzcd}$$

   and let $W\colon P \to \Delta^1_{\mathcal{C}^+}$ be the obvious map induced by $\mathrm{id}_{\Delta^1_{\mathcal{C}^+}}$.

5. We define P^+ as the pushout

$$\begin{tikzcd} \Delta^0_{\mathcal{C}^+} \sqcup \Delta^0_{\mathcal{C}^+} \arrow[r, "V^+"] \arrow[d, "V^+"] & \mathbb{I}^+_{\mathcal{C}^+} \arrow[d] \\ \mathbb{I}^+_{\mathcal{C}^+} \arrow[r] & P^+ \end{tikzcd}$$

   and let $W^+\colon P^+ \to \mathbb{I}^+_{\mathcal{C}^+}$ be the obvious map induced by $\mathrm{id}_{\mathbb{I}^+_{\mathcal{C}^+}}$.

2.2.31. Lemma. *The trivial fibrations in $\mathcal{C}^+$ are exactly the morphisms which have the right lifting property with respect to I.*

Proof. A trivial fibration is a weak equivalence which is in addition surjective on objects by Lemma 2.2.8.

We first observe that lifting with respect to U exactly corresponds to the surjectivity on objects.

We now use the characterization of weak equivalences given in Lemma 2.2.5. Lifting with respect to V and W corresponds to surjectivity and injectivity on morphisms, and lifting with respect to V^+ and W^+ corresponds to surjectivity and injectivity on marked isomorphisms. ∎

2.2.32. Lemma. *The objects $\emptyset$, $\Delta^0_{\mathcal{C}^+}$, $\Delta^1_{\mathcal{C}^+}$, $\mathbb{I}^+_{\mathcal{C}}$, P and P^+ are compact.*

Proof. (Additive) functors out of these categories are determined by their values on finitely many morphisms. ∎

2.2.33. Remark. In the unmarked case, we can take the set of generating cofibrations

$$I := \{U, V, W\}$$

defined analogous to the marked case.

The objects $\emptyset$, $\Delta^0_{\mathcal{C}}$, $\Delta^1_{\mathcal{C}}$, $\mathbb{I}_{\mathcal{C}}$ and P involved in their definition are compact.

2.2.34. Corollary. *The model category $\mathcal{C}^+$ is cofibrantly generated by finite sets of generating cofibrations and trivial cofibrations between compact objects.*

2.2.35. Proposition. *The category $\mathcal{C}^+$ is locally presentable.*

Proof. Since we have already shown that $\mathcal{C}^+$ is cocomplete, by [AR94, Thm 1.20] it suffices to show that $\mathcal{C}^+$ has a strong generator consisting of compact objects. For this it suffices to show that there exists a set of compact objects such that every other object of $\mathcal{C}$ is isomorphic to a colimit of a diagram with values in this set, see [Bun19, Lem. 8.4]. We will call such a set strongly generating.

We will first show that $\mathbf{Cat}^+$ is strongly generated by a finite set of compact objects. We consider the category $\mathbf{DirGraph}^+$ of marked directed graphs. It consists of directed graphs with distinguished subsets of edges called marked edges. Morphisms in $\mathbf{DirGraph}^+$ must preserve marked edges. The category $\mathbf{DirGraph}^+$ is locally presentable by [AR94, Thm 1.20]. Indeed, it is cocomplete and strongly generated by the objects in the list

$$\{\bullet\ , \bullet \to \bullet, \bullet \xrightarrow{+} \bullet\}\ .$$

We have a forgetful functor from $\mathbf{Cat}^+$ to marked directed graphs which fits into an adjunction

$$\mathrm{Free}_{\mathbf{Cat}^+} \colon \mathbf{DirGraph}^+ \leftrightarrows \mathbf{Cat}^+ \colon \mathcal{F}_\circ\ .$$

The left adjoint takes the free category on the marked directed graph[1] and localizes at the marked morphisms. The marking is then extended to inverses of marked morphisms, identities and compositions of those. The counit of the adjunction provides a canonical morphism

$$v_{\mathbf{A}} \colon \mathbf{F}(\mathbf{A}) := \mathrm{Free}_{\mathbf{Cat}^+}(\mathcal{F}_\circ(\mathbf{A})) \to \mathbf{A}$$

of marked categories.

Consider the pullback

$$\begin{tikzcd}
\mathbf{F}(\mathbf{A}) \times_{\mathbf{A}} \mathbf{F}(\mathbf{A}) \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathbf{F}(\mathbf{A}) \arrow[d, "v_{\mathbf{A}}"] \\
\mathbf{F}(\mathbf{A}) \arrow[r, "v_{\mathbf{A}}"] & \mathbf{A}
\end{tikzcd}$$

[1] Note that this includes freely adjoining identities for all the nodes in the graph. Even if the given graph underlies a category, $\mathrm{Free}_{\mathbf{Cat}^+}$ adjoins new identity morphisms.

We claim that the diagram

$$\mathbf{F}(\mathbf{A}) \times_{\mathbf{A}} \mathbf{F}(\mathbf{A}) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathbf{F}(\mathbf{A}) \xrightarrow{v_{\mathbf{A}}} \mathbf{A}$$

is a coequalizer. We have $v_{\mathbf{A}} \circ p_1 = v_{\mathbf{A}} \circ p_2$ by definition. That every morphism $f \colon \mathbf{F}(\mathbf{A}) \to \mathbf{B}$ with $f \circ p_1 = f \circ p_2$ factors uniquely through $v_{\mathbf{A}}$ follows from the fact that $v_{\mathbf{A}}$ is surjective on objects and full.

We know that $\mathbf{F}(\mathbf{A})$ is isomorphic to a colimit of a small diagram involving the list of finite categories

$$\{\mathrm{Free}_{\mathbf{Cat}^+}(\bullet)\ , \mathrm{Free}_{\mathbf{Cat}^+}(\bullet \to \bullet), \mathrm{Free}_{\mathbf{Cat}^+}(\bullet \xrightarrow{+} \bullet)\}\ .$$

The fiber product over $\mathbf{A}$ is not a colimit. But we have a surjection

$$v'_{\mathbf{A}} = v_{\mathbf{F}(\mathbf{A}) \times_{\mathbf{A}} \mathbf{F}(\mathbf{A})} \colon \mathbf{F}(\mathbf{F}(\mathbf{A}) \times_{\mathbf{A}} \mathbf{F}(\mathbf{A})) \to \mathbf{F}(\mathbf{A}) \times_{\mathbf{A}} \mathbf{F}(\mathbf{A})$$

and therefore a coequalizer diagram

$$\mathbf{F}(\mathbf{F}(\mathbf{A}) \times_{\mathbf{A}} \mathbf{F}(\mathbf{A})) \underset{p_2 \circ v'_{\mathbf{A}}}{\overset{p_1 \circ v'_{\mathbf{A}}}{\rightrightarrows}} \mathbf{F}(\mathbf{A}) \xrightarrow{v_{\mathbf{A}}} \mathbf{A}.$$

The marked category $\mathbf{F}(\mathbf{F}(\mathbf{A}) \times_{\mathbf{A}} \mathbf{F}(\mathbf{A}))$ is again a colimit of a diagram involving the generators in the list above. Hence $\mathbf{A}$ itself is a colimit of a diagram built from this list.

A similar argument applies in the case $\mathbf{preAdd}^+$. In this case we must replace $\mathcal{F}_\circ$ by $\mathcal{F}_\circ \circ \mathcal{F}_{\mathbb{Z}}$ and $\mathrm{Free}_{\mathbf{Cat}^+}$ by $\mathrm{Lin}_{\mathbb{Z}} \circ \mathrm{Free}_{\mathbf{Cat}^+}$. The list of generators is

$$\{\mathrm{Lin}_{\mathbb{Z}}(\mathrm{Free}_{\mathbf{Cat}}(\bullet))\ , \mathrm{Lin}_{\mathbb{Z}}(\mathrm{Free}_{\mathbf{Cat}}(\bullet \to \bullet))\ , \mathrm{Lin}_{\mathbb{Z}}(\mathrm{Free}_{\mathbf{Cat}}(\bullet \xrightarrow{+} \bullet))\}\ .$$

These categories are again compact since they have finitely many objects and their morphism groups are finitely generated. ∎

2.2.36. Remark. In order to show that **Cat** and **preAdd** are locally presentable one argues similarly using the category of directed graphs **DirGraph** and the adjunctions

$$\mathrm{Free}_{\mathbf{Cat}} \colon \mathbf{DirGraph} \leftrightarrows \mathbf{Cat} \colon \mathcal{F}_\circ\ , \quad \mathrm{Lin}_{\mathbb{Z}} \circ \mathrm{Free}_{\mathbf{Cat}} \colon \mathbf{DirGraph} \leftrightarrows \mathbf{preAdd} \colon \mathcal{F}_{\mathbb{Z}} \circ \mathcal{F}_\circ\ .$$

2.3. (Marked) additive categories as fibrant objects. In Theorem 2.2.2 we have shown that the simplicial categories **preAdd** and $\mathbf{preAdd}^+$ are locally presentable and have a simplicial, cofibrantly generated model category structures. In the present section we introduce Bousfield localizations of these categories whose categories of fibrant objects are exactly the additive categories or marked additive categories.

Let $\mathbf{A}$ be a preadditive category.

2.3.1. Definition. *We say that* $\mathbf{A}$ *is* additive *if* $\mathbf{A}$ *has a zero object and the sum, see Remark 2.1.5, of any two objects of* $\mathbf{A}$ *exists.*

We let **Add** denote the full subcategory of **preAdd** of additive categories.

2.3.2. Remark. In contrast to being a preadditive category, being an additive category is a property of a category. In the following we describe the conditions for an additive category just in terms of category language. First of all we require the existence of a zero object which by definition is an object which is both initial and final. Furthermore we require the existence of finite products and coproducts, and that the natural transformation

$$- \sqcup - \to - \times -$$

of bifunctors (its definition uses the zero object) is an isomorphism. This leads naturally to an enrichment over commutative monoids. Finally we require that the morphism monoids are in fact abelian groups.

A morphism between additive categories can be characterized as a functor which preserves finite products. It then automatically preserves finite sums, zero objects, and the enrichment. Here one can also interchange the roles of sums and products.

Therefore **Add** can be considered as a (non-full) subcategory of **Cat**.

Let $(\mathbf{A}, \mathbf{A}^+)$ be a marked preadditive category.

2.3.3. Definition. $(\mathbf{A}, \mathbf{A}^+)$ *is a* marked additive category *if the following conditions are satisfied:*

1. *The underlying category* $\mathbf{A}$ *is additive.*
2. $\mathbf{A}^+$ *is closed under sums.*

In detail, Condition 2 means that for every two morphisms $a\colon A \to A'$ and $b\colon B \to B'$ in $\mathbf{A}^+$ the induced isomorphism $a \oplus b\colon A \oplus B \to A' \oplus B'$ (for any choice of objects and structure maps representing the sums) also belongs to $\mathbf{A}^+$.

We denote by $\mathbf{Add}^+$ the full subcategory of $\mathbf{preAdd}^+$ spanned by the marked additive categories.

In Example 3.4.9 below we will discuss a natural example of a marked preadditive category in which the Condition 2 is violated.

2.3.4. Example. A category $\mathbf{C}$ with cartesian products can be refined to a symmetric monoidal category with the cartesian symmetric monoidal structure [Bor94b, Sec. 6.1], [Lur, Sec. 2.4.1]. In particular we have a functor (uniquely defined up to unique isomorphism)

$$- \times -\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}\ .$$

This applies to an additive category $\mathbf{A}$ where the cartesian product is denoted by $\oplus$. We therefore have a sum functor

$$- \oplus -\colon \mathbf{A} \times \mathbf{A} \to \mathbf{A}\ .$$

Note that $\mathbf{A} \times \mathbf{A}$ (the product is taken in **preAdd**) is naturally an additive category again, and that the sum functor is a morphism of additive categories.

If $(\mathbf{A}, \mathbf{A}^+)$ is now a marked additive category, then $(\mathbf{A}, \mathbf{A}^+) \times (\mathbf{A}, \mathbf{A}^+)$ (the product is taken in $\mathbf{preAdd}^+$) is marked again, and Condition 2.3.3.2 implies that we also have a functor

$$- \oplus -\colon (\mathbf{A}, \mathbf{A}^+) \times (\mathbf{A}, \mathbf{A}^+) \to (\mathbf{A}, \mathbf{A}^+)$$

between marked additive categories.

We want to reformulate the characterization of (marked) additive categories from Definition 2.3.1 and Definition 2.3.3 as a right lifting property. To this end we introduce the preadditive categories $\mathbf{S}_{\mathbf{preAdd}}$ and $\emptyset_{\mathbf{preAdd}}$ in $\mathbf{preAdd}$ given as follows:

1. The preadditive category $\mathbf{S}_{\mathbf{preAdd}}$ has three objects 1, 2, and S and the morphisms are generated by the morphisms

$$\{1 \xrightarrow{i_1} S, 2 \xrightarrow{i_2} S, S \xrightarrow{p_1} 1, S \xrightarrow{p_2} 2\} \ .$$

   subject to the following relations:

$$p_1 \circ i_1 = \mathrm{id}_1 \ , \quad p_2 \circ i_2 = \mathrm{id}_2 \ , \quad i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_S \ .$$

2. $\emptyset_{\mathbf{preAdd}}$ has one object 0 and $\mathrm{Hom}_{\emptyset_{\mathbf{preAdd}}}(0,0) = \{\mathrm{id}_0\}$. Note that id_0 is the zero morphism.

We further define the marked versions

$$\mathbf{S}_{\mathbf{preAdd}^+} := \mathrm{mi}(\mathbf{S}_{\mathbf{preAdd}}) \ , \quad \emptyset_{\mathbf{preAdd}^+} := \mathrm{mi}(\emptyset_{\mathbf{preAdd}})$$

in $\mathbf{preAdd}^+$ by marking the identities.

In the following let $\mathcal{C}$ be a placeholder for $\mathbf{preAdd}$ or $\mathbf{preAdd}^+$.

2.3.5. Remark. We consider the object $\mathbf{S}_{\mathcal{C}}$ of $\mathcal{C}$. Note that the relations $p_1 \circ i_2 = 0$ and $p_2 \circ i_1 = 0$ are implied. The morphisms p_1, p_2 present S as the product of 1 and 2, and the morphisms i_1 and i_2 present S as a coproduct of 1 and 2. Consequently, S is the sum of the objects 1 and 2, see Remark 2.1.5.

If $\mathbf{A}$ belongs to $\mathcal{C}$ and $f\colon \mathbf{S}_{\mathcal{C}} \to \mathbf{A}$ is a morphism, then the morphisms $f(p_1)$, $f(p_2)$ present $f(S)$ as the product of $f(1)$ and $f(2)$, and the morphisms $f(i_1)$, $f(i_2)$ present $f(S)$ as a coproduct of $f(1)$ and $f(2)$. Hence again, $f(S)$ is the sum of the objects $f(1)$ and $f(2)$.

A functor $\mathbf{S}_{\mathcal{C}} \to \mathbf{A}$ is the same as the choice of two objects A, B in $\mathbf{A}$ together with a representative of the sum $A \oplus B$ and the corresponding structure maps.

2.3.6. Remark. The object 0 of $\emptyset_{\mathcal{C}}$ is a zero object. If $\mathbf{A}$ belongs to $\mathcal{C}$ and $f\colon \emptyset_{\mathcal{C}} \to \mathbf{A}$ is a morphism, then $f(0)$ is an object satisfying $\mathrm{id}_{f(0)} = 0$. Since $\mathbf{A}$ is enriched over abelian groups, every object in $\mathbf{A}$ admits a morphism to $f(0)$ and a morphism from $f(0)$, both of which are necessarily unique. Hence $f(0)$ is a zero object of $\mathbf{A}$. In fact, $\emptyset_{\mathcal{C}}$ is the zero-object classifier in $\mathcal{C}$.

Recall the notation introduced in Example 2.2.7. We let

$$w\colon \Delta^0_{\mathcal{C}} \sqcup \Delta^0_{\mathcal{C}} \to \mathbf{S}_{\mathcal{C}} \tag{10}$$

be the morphism which classifies the two objects 1 and 2. We furthermore let

$$v\colon \emptyset \to \emptyset_{\mathcal{C}} \tag{11}$$

be the canonical morphism from the initial object of $\mathcal{C}$.

We now use that $\mathcal{C}$ is a left-proper (see Remark 2.2.12), combinatorial simplicial model category (see Theorem 2.2.2). By [Lur09, Prop. A.3.7.3], for every set $\mathcal{S}$ of cofibrations in $\mathcal{C}$ the left Bousfield localization $L_{\mathcal{S}}\mathcal{C}$ (see [Hir03, Def. 3.3.1] or [Lur09, Sec. A.3.7] for a definition) exists and is again a combinatorial simplicial model category. We will consider the set $\mathcal{S} := \{v, w\}$ consisting of the cofibrations (10) and (11).

2.3.7. Proposition. *The fibrant objects in $L_{\{v,w\}}\mathcal{C}$ are exactly the (marked) additive categories.*

Proof. The fibrant objects in $L_{\{v,w\}}\mathcal{C}$ are the fibrant objects $\mathbf{A}$ in $\mathcal{C}$ which are local for $\{v, w\}$, i.e., for which the maps of simplicial sets $\mathrm{Map}_{\mathcal{C}}(v, \mathbf{A})$ and $\mathrm{Map}_{\mathcal{C}}(w, \mathbf{A})$ are trivial Kan fibrations, see [Lur09, Prop. A.3.7.3(3)].

Let $\mathbf{A}$ be in $\mathcal{C}$ and consider the lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(\mathbf{S}_{\mathcal{C}}, \mathbf{A}) \arrow[d, "\mathrm{Map}_{\mathcal{C}}(w,\mathbf{A})"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathrm{Map}_{\mathcal{C}}(\Delta^0_{\mathcal{C}} \sqcup \Delta^0_{\mathcal{C}}, \mathbf{A}) \end{tikzcd} . \tag{12}$$

Since the mapping spaces in $\mathcal{C}$ are nerves of groupoids they are 2-coskeletal. Hence the lifting problem is uniquely solvable for all $n \geq 3$ without any condition on $\mathbf{A}$. It therefore suffices to consider the cases $n = 0, 1, 2$.

n=0 The outer part of the diagram reflects the choice of two objects in $\mathbf{A}$, and a lift corresponds to a choice of a sum of these objects together with the corresponding structure maps. Therefore the lifting problem is solvable if and only if $\mathbf{A}$ admits sums of pairs of objects.

n=1 The outer part of the diagram reflects the choice of (marked) isomorphisms $A \to A'$ and $B \to B'$ in $\mathbf{A}$ and choices of objects $A \oplus A'$ and $B \oplus B'$ together with structure maps (inclusions and projections) representing the sums. The lift then corresponds to the choice of a (marked) isomorphism $A \oplus A' \to B \oplus B'$ which is compatible with the structure maps. In fact such an isomorphism exists (and is actually uniquely determined). In the marked case the fact that the isomorphism is marked is equivalent to the compatibility condition between the sums and the marking required for a marked additive category.

n=2 The outer part reflects the choice of data: six objects A, A', A'' and B, B', B''; objects representing the sums $A \oplus B$, $A' \oplus B'$ and $A'' \oplus B''$ together with structure maps; (marked) isomorphisms $a\colon A \to A'$, $a'\colon A' \to A''$, $a''\colon A \to A''$, and $b\colon B \to B'$, $b'\colon B' \to B''$, $b''\colon B \to B''$; and (marked) isomorphisms $a \oplus b\colon A \oplus B \to A' \oplus B'$, $a' \oplus b'\colon A' \oplus B' \to A'' \oplus B''$, $a'' \oplus b''\colon A \oplus B \to A'' \oplus B''$ which are compatible with the structure maps and hence uniquely determined. Thereby we have the relations $a'' = a' \circ a$ and $b'' = b' \circ b$. A lift corresponds to a witness of the fact that $a'' \oplus b'' = (a' \oplus b') \circ (a \oplus b)$. Hence the lift exists and is unique by the universal properties of the sums.

We have

$$\mathrm{Map}_{\mathcal{C}}(v, \mathbf{A})\colon\ \mathrm{Map}_{\mathcal{C}}(\emptyset_{\mathcal{C}}, \mathbf{A}) \to * \ .$$

The domain of this map is the space of zero objects in $\mathbf{A}$ which is either empty or a contractible Kan complex. Consequently, $\mathrm{Map}_{\mathcal{C}}(v, \mathbf{A})$ is a trivial Kan fibration exactly if $\mathbf{A}$ admits a zero object. ∎

2.4. ∞-categories of (marked) preadditive and additive categories. In the present paper we use the language of ∞-categories as developed in [Joy], [Lur09] and [Cis19]. Let $\mathbf{C}$ be a simplicial model category. By [Lur, Thm. 1.3.4.20], we have an equivalence of ∞-categories

$$\mathrm{N}^{\mathrm{coh}}(\mathbf{C}^{cf}) \simeq \mathbf{C}^{c}[W^{-1}] \ , \tag{13}$$

where $\mathrm{N}^{\mathrm{coh}}(\mathbf{C}^{cf})$ is the coherent nerve of the simplicial category of cofibrant-fibrant objects in $\mathbf{C}$, and $\mathbf{C}^{c}[W^{-1}]$ is the ∞-category obtained from (the nerve of) $\mathbf{C}^{c}$ by inverting the weak equivalences of the model category structure, where $\mathbf{C}^{c}$ denotes the ordinary category of cofibrant objects of $\mathbf{C}$. If $\mathbf{C}$ is in addition combinatorial, then $\mathbf{C}^{c}[W^{-1}]$ is a presentable ∞-category [Lur, Prop. 1.3.4.22].

For the following we assume that $\mathbf{C}$ is a combinatorial simplicial model category. Let $L_{\mathcal{S}}\mathbf{C}$ be the Bousfield localization of the model category structure on $\mathbf{C}$ at a set $\mathcal{S}$ of morphisms in $\mathbf{C}^{cf}$ and let $\mathrm{N}^{\mathrm{coh}}(\mathbf{C}^{cf}) \to L_{\mathcal{S}}\mathrm{N}^{\mathrm{coh}}(\mathbf{C}^{cf})$ be the localization at the same set of morphisms in the sense of [Lur09, Def. 5.2.7.2]. Using [Lur, Rem. 1.3.4.27] we get an adjunction

$$\mathrm{N}^{\mathrm{coh}}(\mathbf{C}^{cf}) \simeq \mathbf{C}^{c}[W^{-1}] \leftrightarrows (L_{\mathcal{S}}\mathbf{C})^{c}[W^{-1}] \simeq \mathrm{N}^{\mathrm{coh}}((L_{\mathcal{S}}\mathbf{C})^{cf})$$

which presents $\mathrm{N}^{\mathrm{coh}}((L_{\mathcal{S}}\mathbf{C})^{cf})$ as a Bousfield localization at $\mathcal{S}$. Hence there is an equivalence of ∞-categories

$$L_{\mathcal{S}}\mathrm{N}^{\mathrm{coh}}(\mathbf{C}^{cf}) \simeq \mathrm{N}^{\mathrm{coh}}((L_{\mathcal{S}}\mathbf{C})^{cf}) \ .$$

We let $W_{\mathbf{preAdd}^{(+)}}$ denote the weak equivalences in $\mathbf{preAdd}^{(+)}$. Note that in $\mathbf{preAdd}^{(+)}$ all objects are cofibrant and fibrant.

2.4.1. Definition. *We define the* ∞-category of (marked) pre-additive *categories by*

$$\mathbf{preAdd}_{\infty}^{(+)} := \mathbf{preAdd}^{(+)}[W_{\mathbf{preAdd}^{(+)}}^{-1}] \ .$$

By a specialization of (13) we have an equivalence of ∞-categories

$$\mathrm{N}^{\mathrm{coh}}(\mathbf{preAdd}^{(+)}) \simeq \mathbf{preAdd}_{\infty}^{(+)} \ . \tag{14}$$

A weak equivalence between fibrant objects in a Bousfield localization is a weak equivalence in the original model category. Consequently, a morphism between (marked) additive categories is a weak equivalence in $L_{\{v,w\}}\mathcal{C}$ if and only if it is a weak equivalence in (marked) preadditive categories. We let $W_{\mathbf{Add}^{(+)}}$ denote the weak equivalences in the Bousfield localization $L_{\{v,w\}}\mathbf{preAdd}^{(+)}$.

2.4.2. Definition. *We define the* ∞-category of (marked) additive categories *by*

$$\mathbf{Add}_{\infty}^{(+)} := \mathbf{preAdd}^{(+)}[W_{\mathbf{Add}^{(+)}}^{-1}] \ .$$

By specialization of (13), we then have an equivalence of ∞-categories

$$\mathrm{N}^{\mathrm{coh}}(\mathbf{Add}^{(+)}) \simeq \mathbf{Add}_{\infty}^{(+)} \ . \tag{15}$$

2.4.3. Remark. The equivalences (14) and (15) can be shown directly using Prop. 1.3.4.7 in [Lur], since fibrant replacement in the marked model structure corresponds to localization at the marked edges (apply [Lur09, Thm. 3.1.5.1 and Prop. 3.1.4.1] with $S = \Delta^0$). Indeed, the categories $\mathbf{preAdd}^{(+)}$ and $\mathbf{Add}^{(+)}$ are enriched in groupoids and therefore fibrant simplicial categories. The interval object of $\mathbf{A}$ is given by $\mathbf{A}\sharp\Delta^1$. In the case of (marked) additive categories we must observe that $\mathbf{A}\sharp\Delta^1$ is again (marked) additive.

Recall that $\mathcal{F}_+$ is the functor which forgets the markings, and that its right-adjoint ma is the functor which marks all equivalences. Since these functors preserve equivalences, the adjunction descends to the ∞-categories.

2.4.4. Corollary. *We have an adjunction*

$$\mathcal{F}_+ \colon \mathbf{preAdd}_{\infty}^{+} \leftrightarrows \mathbf{preAdd}_{\infty} \colon \mathrm{ma} \ .$$

2.4.5. Corollary.

1. *The* ∞*-categories* $\mathbf{preAdd}_{\infty}^{(+)}$ *and* $\mathbf{Add}_{\infty}^{(+)}$ *are presentable.*
2. *We have an adjunction*

$$L_{\oplus} \colon \mathbf{preAdd}_{\infty}^{(+)} \leftrightarrows \mathbf{Add}_{\infty}^{(+)} \colon \mathcal{F}_{\oplus} \ , \tag{16}$$

where $\mathcal{F}_{\oplus}$ *is the inclusion of a full subcategory.*

The functor $L_{\oplus}$ is the additive completion functor.

In the following $\mathcal{C}$ is a placeholder for $\mathbf{Cat}^{(+)}$, $\mathbf{Add}^{(+)}$ or $\mathbf{preAdd}^{(+)}$.

The category $\mathcal{C}$ can be considered as a category enriched in groupoids and therefore as a strict $(2,1)$-category which will be denoted by $\mathcal{C}_{(2,1)}$. A strict $(2,1)$-category gives rise to an ∞-category as follows. We first apply the usual nerve functor to the morphism categories of $\mathcal{C}_{(2,1)}$ and obtain a category enriched in Kan complexes. Then we apply the coherent nerve functor and get a quasi-category which we will denote by $\mathrm{N}_2(\mathcal{C}_{(2,1)})$. The obvious functor $\mathrm{N}(\mathcal{C}_{(1,1)}) \to \mathrm{N}_2(\mathcal{C}_{(2,1)})$ (where $\mathcal{C}_{(1,1)}$ denotes the underlying ordinary category of $\mathcal{C}$) sends equivalences to equivalences and therefore descends to a functor

$$\mathcal{C}_\infty \to \mathrm{N}_2(\mathcal{C}_{(2,1)})\ . \tag{17}$$

2.4.6. Proposition. *The functor* (17) *is an equivalence.*

Proof. Note that $\mathrm{N}_2(\mathcal{C}_{(2,1)})$ and $\mathrm{N}^{\mathrm{coh}}(\mathcal{C})$ are isomorphic by the definition of the simplicial enrichment of $\mathcal{C}$.

We consider the following commuting diagram of quasi-categories

$$\begin{tikzcd} & \mathrm{N}(\mathcal{C}_{(1,1)}) \arrow[dl, "\ell_{\mathcal{C}}"'] \arrow[d, "!"] \arrow[dr] & \\ \mathcal{C}_\infty \arrow[r, "!!", "\simeq"'] & \mathrm{N}^{\mathrm{coh}}(\mathcal{C}) \arrow[r, "\cong"'] & \mathrm{N}_2(\mathcal{C}_{(2,1)}) \end{tikzcd} .$$

The left triangle commutes since the morphism marked by ! is an explicit model of the localization morphism, where we use (14) (or (15), depending on the case) for the equivalence marked by !!. The composition of the two horizontal arrows is an explicit model of (17). ∎

3. Applications

3.1. Localization preserves products. We show that the localizations

$$\ell_{\mathcal{C}^{(+)}} \colon \mathcal{C}^{(+)} \to \mathcal{C}^{(+)}_\infty$$

for $\mathcal{C}$ in $\{\mathbf{Cat}\ , \mathbf{preAdd}\ , \mathbf{Add}\}$ preserve products. Here and in the following we view categories as ∞-categories using the nerve. However we will not write the nerve explicitly.

Let I be a set. Then we consider the functor

$$\ell_{I,\mathcal{C}} \colon \mathcal{C}^I \to \mathcal{C}^I_\infty$$

defined by postcomposition with $\ell_{\mathcal{C}}$. For every category $\mathbf{C}$ with products we have a functor $\prod_I \colon \mathbf{C}^I \to \mathbf{C}$. We apply this to $\mathbf{C} = \mathcal{C}$ and $\mathbf{C} = \mathcal{C}_\infty$.

3.1.1. Proposition. *We have an equivalence of functors*

$$\ell_{\mathcal{C}} \circ \prod_I \xrightarrow{\simeq} \prod_I \ell_{I,\mathcal{C}} \colon \mathcal{C}^I \to \mathcal{C}_\infty\ .$$

PROOF. We start with the case $\mathcal{C} = \mathbf{preAdd}^{(+)}$ or $\mathcal{C} = \mathbf{Cat}^{(+)}$. We use that $\mathcal{C}$ has a combinatorial model category structure in which all objects are cofibrant and fibrant. It is a general fact, that in this case the localization $\ell\colon \mathcal{C} \to \mathcal{C}_\infty$ preserves products. Here is the (probably much too complicated) argument. We can consider the injective model category structure on the diagram category $\mathcal{C}^I$. Since I is discrete one easily observes that all objects in this diagram category are fibrant again. So we can take the identity as a fibrant replacement functor for $\mathcal{C}^I$. This gives the equivalence

$$\ell_{\mathcal{C}} \circ \prod_I \xrightarrow{\simeq} \prod_I \ell_{I,\mathcal{C}} \ ,$$

(e.g. by specializing [Bun19, Prop. 13.6]).

In order to deduce the assertion for additive categories we consider the inclusion functor $\mathcal{F}_{\oplus,1}\colon \mathbf{Add}^{(+)} \to \mathbf{preAdd}^{(+)}$. This functor preserves weak equivalences and therefore descends essentially uniquely to the functor $\mathcal{F}_\oplus$ in (16) such that

$$\mathcal{F}_\oplus \circ \ell_{\mathbf{Add}^{(+)}} \simeq \ell_{\mathbf{preAdd}^{(+)}} \circ \mathcal{F}_{\oplus,1} \ .$$

The functor $\mathcal{F}_\oplus$ is a right adjoint which preserves and detects limits. We do not claim that $\mathcal{F}_{\oplus,1}$ is a right adjoint, but it clearly preserves products by inspection. We let $\mathcal{F}_{I,\oplus,1}$ and $\mathcal{F}_{I,\oplus}$ be the factorwise application of $\mathcal{F}_{\oplus,1}$ and $\mathcal{F}_\oplus$. With this notation we have an equivalence

$$\mathcal{F}_{\oplus,1} \circ \prod_I \cong \prod_I \circ \mathcal{F}_{I,\oplus,1} \ .$$

The assertion in the case $\mathcal{C} = \mathbf{Add}^{(+)}$ now follows from the chain of equivalences

$$\begin{aligned}
\mathcal{F}_\oplus \circ \ell_{\mathbf{Add}^{(+)}} \circ \prod_I &\simeq \ell_{\mathbf{preAdd}^{(+)}} \circ \mathcal{F}_{\oplus,1} \circ \prod_I \\
&\simeq \ell_{\mathbf{preAdd}^{(+)}} \circ \prod_I \circ \mathcal{F}_{I,\oplus,1} \\
&\simeq \prod_I \circ \ell_{I,\mathbf{preAdd}^{(+)}} \circ \mathcal{F}_{I,\oplus,1} \\
&\simeq \prod_I \circ \mathcal{F}_{I,\oplus} \circ \ell_{I,\mathbf{Add}^{(+)}} \\
&\simeq \mathcal{F}_\oplus \circ \prod_I \circ \ell_{I,\mathbf{Add}^{(+)}}
\end{aligned}$$

by removing $\mathcal{F}_\oplus$. ■

3.2. RINGS AND MODULES. A unital ring R can be considered as a preadditive category $\mathbf{R}$ with one object $*$ and ring of endomorphisms $\mathrm{Hom}_{\mathbf{R}}(*,*) := R$. The category of finitely generated free R-modules $\mathbf{Mod}^{\mathrm{fg,free}}(R)$ is an additive category. We have a canonical functor $\mathbf{R} \to \mathbf{Mod}^{\mathrm{fg,free}}(R)$ sending $*$ to R which presents $\mathbf{Mod}^{\mathrm{fg,free}}(R)$ as the additive completion of $\mathbf{R}$. This fact is well-known, see e.g. [DL98, Sec. 2]. In the following we provide a precise formulation using the language of ∞-categories.

Recall the sum-completion functor $L_\oplus$ from Corollary 2.4.5.

3.2.1. PROPOSITION. *The morphism of preadditive categories* $\mathbf{R} \to \mathbf{Mod}^{\mathrm{fg,free}}(R)$ *induces an equivalence*

$$L_{\oplus}(\ell_{\mathbf{preAdd}}(\mathbf{R})) \simeq \ell_{\mathbf{Add}}(\mathbf{Mod}^{\mathrm{fg,free}}(R)) \ .$$

PROOF. We must show that

$$\mathrm{Map}_{\mathbf{preAdd}_\infty}(\ell_{\mathbf{preAdd}}(\mathbf{Mod}^{\mathrm{fg,free}}(R)), \ell_{\mathbf{preAdd}}(\mathbf{B})) \to \mathrm{Map}_{\mathbf{preAdd}_\infty}(\ell_{\mathbf{preAdd}}(\mathbf{R}), \ell_{\mathbf{preAdd}}(\mathbf{B}))$$

is an equivalence for every additive category $\mathbf{B}$. In view of (14), this is equivalent to the fact that

$$\mathrm{Map}_{\mathbf{preAdd}}(\mathbf{Mod}^{\mathrm{fg,free}}(R), \mathbf{B}) \to \mathrm{Map}_{\mathbf{preAdd}}(\mathbf{R}, \mathbf{B})$$

is a trivial Kan fibration. Here we use that by (14) the mapping spaces in $\mathbf{preAdd}_\infty$ are represented by the simplicial mapping spaces in $\mathbf{preAdd}$, see [Lur09, Sec. 2.2.2]. The proof is very similar to the proof of Proposition 2.3.7. We must check the lifting property against the inclusions $\partial\Delta^n \to \Delta^n$. Again we must only consider the case $n \leq 2$.

n=0 A functor $\mathbf{R} \to \mathbf{B}$ (sending $*$ to an R-module B) determines a functor

$$\mathbf{Mod}^{\mathrm{fg,free}}(R) \to \mathbf{B}$$

which sends R^k to $B^{\oplus k}$.

n=1 An isomorphism of functors $\mathbf{R} \to \mathbf{B}$ is an isomorphism of objects $f\colon B \to B'$ which is compatible with the R-module structures. It induces an isomorphism of induced functors $\mathbf{Mod}^{\mathrm{fg,free}}(R) \to \mathbf{B}$ which on R^k is given by $\oplus_k f\colon B^{\oplus k} \to B'^{,\oplus k}$.

n=2 The existence of the lift expresses the naturality of the isomorphisms obtained in the case $n = 1$.

■

To understand the category of finitely generated projective modules $\mathbf{Mod}^{\mathrm{fg,proj}}(R)$ and the morphism $\mathbf{R} \to \mathbf{Mod}^{\mathrm{fg,proj}}(R)$ in a similar manner we must consider idempotent complete additive categories. Recall that a projection in an additive category $\mathbf{A}$ is an endomorphism $e\colon A \to A$ such that $e^2 = e$.

Let $\mathbf{A}$ be an additive category.

3.2.2. DEFINITION. *The category* $\mathbf{A}$ *is* idempotent complete *if for every object* A *in* $\mathbf{A}$ *and projection* e *in* $\mathrm{End}_{\mathbf{A}}(A)$ *there exists an isomorphism* $A \cong e(A) \oplus e(A)^\perp$ *such that* $e(A)$ *and* $e(A)^\perp$ *are images of* e *and* $\mathrm{id}_A - e$.

The last part of this definition more precisely means that there exist morphisms $e(A) \to A$ and $e(A)^\perp \to A$ such that the diagrams

$$\begin{tikzcd} A \arrow[d, "e"] & e(A) \oplus e(A)^\perp \arrow[l, "\cong"'] \arrow[d, "\mathrm{pr}_{e(A)}"] \\ A & e(A) \oplus e(A)^\perp \arrow[l, "\cong"'] \end{tikzcd} \qquad \begin{tikzcd} e(A) \oplus e(A)^\perp \arrow[r, "\cong"] \arrow[d, "\mathrm{pr}_{e(A)^\perp}"] & A \arrow[d, "\mathrm{id}_A - e"] \\ e(A) \oplus e(A)^\perp \arrow[r, "\cong"] & A \end{tikzcd}$$

commute.

Let $\mathbf{A}$ be a marked additive category.

3.2.3. Definition. *The marked additive category* $\mathbf{A}$ *is* idempotent complete *if the underlying additive category* $\mathcal{F}_+(\mathbf{A})$ *is idempotent complete (Definition 3.2.2), and if in addition for every* A *in* $\mathbf{A}$*, every projection* e *on* A*, and every marked isomorphism* $f\colon A \to A'$ *the induced isomorphism* $e(A) \to e'(A')$ *is marked, where* $e' := f \circ e \circ f^{-1}$*.*

We let $\mathbf{Add}^{(+),\mathrm{idem}}$ be the full subcategory of $\mathbf{Add}^{(+)}$ of idempotent complete small (marked) additive categories.

We can characterize idempotent completeness of a marked additive category as a lifting property. To this end we consider the following preadditive category $\mathbf{E}_{\mathbf{preAdd}}$:

1. $\mathbf{E}_{\mathbf{preAdd}}$ has the object $*$.
2. The morphisms of $\mathbf{E}_{\mathbf{preAdd}}$ are generated by id_* and e subject to the relation $e^2 = e$.

We then consider the functor

$$u\colon \mathbf{E}_{\mathbf{preAdd}} \to \mathbf{S}_{\mathbf{preAdd}} \tag{18}$$

(see Section 2.3 for $\mathbf{S}_{\mathbf{preAdd}}$) which sends $*$ to S and e to $i_1 \circ p_1$. In the marked case we consider

$$u\colon \mathbf{E}_{\mathbf{preAdd}^+} \to \mathbf{S}_{\mathbf{preAdd}^+}$$

obtained from (18) by applying the functor mi marking the identities. Then one checks:

3.2.4. Lemma. *A (marked) additive category* $\mathbf{A}$ *is idempotent complete if and only if it is local with respect to the map* u*.*

Proof. The proof is similar to the proof of Proposition 2.3.7. The extra condition in Definition 3.2.3 in the marked case arises from the lifting problem for $n = 1$. ∎

3.2.5. Corollary. *The fibrant objects in the Bousfield localization* $L_{\{u,v,w\}}\mathbf{preAdd}^{(+)}$ *are exactly the idempotent complete small (marked) additive categories.*

Consider the equivalences $W_{\mathbf{Add}^{(+),\mathrm{idem}}}$ in the Bousfield localization $L_{\{u,v,w\}}\mathbf{preAdd}^{(+)}$ and the ∞-category

$$\mathbf{Add}_\infty^{(+),\mathrm{idem}} := \mathbf{preAdd}^{(+)}[W^{-1}_{\mathbf{Add}^{(+),\mathrm{idem}}}]\ .$$

Using (13), we have an equivalence

$$\mathrm{N}^{\mathrm{coh}}(\mathbf{Add}^{(+),\mathrm{idem}}) \simeq \mathbf{Add}_\infty^{(+),\mathrm{idem}}\ . \tag{19}$$

We obtain the analog of Corollary 2.4.5.

3.2.6. Corollary.

1. *The ∞-category $\mathbf{Add}_\infty^{(+),\mathrm{idem}}$ is presentable.*
2. *We have an adjunction*

$$L_{\mathrm{idem}}\colon \mathbf{Add}_\infty^{(+)} \leftrightarrows \mathbf{Add}_\infty^{(+),\mathrm{idem}} \colon \mathcal{F}_{\mathrm{idem}}$$

   where $\mathcal{F}_{\mathrm{idem}}$ is the inclusion and L_{idem} is the idempotent completion functor.
3. *We have an adjunction*

$$L_{\oplus,\mathrm{idem}}\colon \mathbf{preAdd}_\infty^{(+)} \leftrightarrows \mathbf{Add}_\infty^{(+),\mathrm{idem}} \colon \mathcal{F}_{\oplus,\mathrm{idem}}$$

   where $\mathcal{F}_{\oplus,\mathrm{idem}} \simeq \mathcal{F}_\oplus \circ \mathcal{F}_{\mathrm{idem}}$ and $L_{\oplus,\mathrm{idem}} \simeq L_{\mathrm{idem}} \circ L_\oplus$.

3.2.7. Proposition. *The morphism of preadditive categories $\mathbf{R} \to \mathbf{Mod}^{\mathrm{fg,proj}}(R)$ induces an equivalence $L_{\oplus,\mathrm{idem}}(\ell_{\mathbf{preAdd}}(\mathbf{R})) \simeq \ell_{\mathbf{Add}^{\mathrm{idem}}}(\mathbf{Mod}^{\mathrm{fg,proj}}(R))$.*

Proof. The proof is similar to Proposition 3.2.1. ∎

The following is a precise version of the assertion that $\mathbf{Mod}^{\mathrm{fg,proj}}(R)$ is the idempotent completion of $\mathbf{Mod}^{\mathrm{fg,free}}(R)$.

3.2.8. Corollary. *The morphism of additive categories $\mathbf{Mod}^{\mathrm{fg,free}}(R) \to \mathbf{Mod}^{\mathrm{fg,proj}}(R)$ induces an equivalence*

$$\ell_{\mathbf{Add}^{\mathrm{idem}}}(\mathbf{Mod}^{\mathrm{fg,proj}}(R)) \simeq L_{\mathrm{idem}}(\ell_{\mathbf{Add}}(\mathbf{Mod}^{\mathrm{fg,free}}(R))) \ .$$

3.3. G-coinvariants. Let G be a group. In this subsection we want to calculate explicitly the homotopy G-orbits of preadditive categories with trivial G-action. The precise formulation of the result is Theorem 3.3.1. We then discuss applications to group rings.

By BG we denote the groupoid with one object $*$ and group of automorphisms G. The functor category $\mathbf{Fun}(BG, \mathbf{C})$ is the category of objects in $\mathbf{C}$ with G-action and equivariant morphisms. The underlying object or morphism of an object or morphism in $\mathbf{Fun}(BG, \mathbf{C})$ is the evaluation of the functor or morphism at $*$.

If $\mathbf{I}$ is a category and $F\colon \mathbf{C} \to \mathbf{D}$ is a functor, then we will use the notation

$$F_{\mathbf{I}}\colon \mathbf{Fun}(\mathbf{I}, \mathbf{C}) \to \mathbf{Fun}(\mathbf{I}, \mathbf{D}) \tag{20}$$

for the functor defined by postcomposition with F.

We consider a (marked) preadditive category $\mathbf{A}$. It gives rise to a constant functor $\underline{\mathbf{A}}$ in $\mathbf{Fun}(BG, \mathbf{preAdd}^{(+)})$ and hence to an object $\ell_{\mathbf{preAdd}^{(+)},BG}(\underline{\mathbf{A}})$ in $\mathbf{Fun}(BG, \mathbf{preAdd}_\infty^{(+)})$.

Since the ∞-category $\mathbf{preAdd}_\infty^{(+)}$ is presentable, it is cocomplete and the colimit in the following theorem exists. Recall the functor $-\sharp-$ from Definition 2.2.17.

3.3.1. Theorem. *We have a natural equivalence*

$$\operatorname*{colim}_{BG} \ell_{\mathbf{preAdd}^{(+)},BG}(\underline{\mathbf{A}}) \simeq \ell_{\mathbf{preAdd}^{(+)}}(\mathbf{A}\sharp BG) \ .$$

3.3.2. Remark. Note that the order of taking the colimit and the localization is relevant. Indeed, we have $\mathrm{colim}_{BG}\,\underline{\mathbf{A}} \cong \mathbf{A}$ and therefore $\ell_{\mathbf{preAdd}^{(+)}}(\mathrm{colim}_{BG}\,\underline{\mathbf{A}}) \simeq \ell_{\mathbf{preAdd}^{(+)}}(\mathbf{A})$.

3.3.3. Remark. Note that the unmarked version of Theorem 3.3.1 can be deduced from the marked version using the functor ma introduced in (5) using Corollary 2.4.4.

In order to avoid case distinctions, we will formulate the details of the proof in the marked case. The unmarked case can be shown similarly, or alternatively deduced formally from the marked case as noted in Remark 3.3.3.

Since $\mathbf{preAdd}^+$ has a cofibrantly generated model category structure, the projective model category structure on $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$ exists [Hir03, Thm. 11.6.1]. For every cofibrant replacement functor $l\colon L \to \mathrm{id}_{\mathbf{Fun}(BG,\mathbf{preAdd}^+)}$ for this projective model category structure we have an equivalence

$$\ell_{\mathbf{preAdd}^+} \circ \operatorname*{colim}_{BG} \circ L \simeq \operatorname*{colim}_{BG} \circ \ell_{\mathbf{preAdd}^+,BG} \tag{21}$$

of functors from $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$ to $\mathbf{preAdd}^+_\infty$, see e.g. [Bun19, Prop. 14.3] for an argument.

We derive the formula asserted in Theorem 3.3.1 by considering a particular choice of a cofibrant replacement functor.

3.3.4. Definition. *Let $\widetilde{G}$ in $\mathbf{Fun}(BG, \mathbf{Groupoids})$ be the groupoid with G-action given as follows:*

1. *The objects of $\widetilde{G}$ are the elements of G.*
2. *For every pair of objects g, g' there is a unique morphism $g \to g'$.*
3. *The group G acts on $\widetilde{G}$ by left-multiplication.*

The G-groupoid $\widetilde{G}$ is often called the transport groupoid of G.

We now define the functor

$$L := -\sharp\widetilde{G}\colon \mathbf{Fun}(BG, \mathbf{preAdd}^+) \to \mathbf{Fun}(BG, \mathbf{preAdd}^+)$$

(more precisely $L(\mathbf{D})$ is the G-object obtained from the $G\times G$-object $\mathbf{D}\sharp\widetilde{G}$ in $\mathbf{preAdd}^+$ by restriction of the action along the diagonal $G \to G\times G$). We have a natural transformation $L \to \mathrm{id}$ induced by the morphism of G-groupoids $\widetilde{G} \to \underline{\Delta^0_{\mathbf{Cat}}}$, where we use the canonical isomorphism $\mathbf{D}\sharp\underline{\Delta^0_{\mathbf{Cat}}} \cong \mathbf{D}$.

3.3.5. Lemma. *The functor L together with the transformation $L \to \mathrm{id}$ is a cofibrant replacement functor for the projective model category structure on $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$.*

PROOF. Since $\mathrm{Res}^G_{\{1\}}(\widetilde{G}) \to \Delta^0_{\mathbf{Cat}}$ is an (non-equivariant) equivalence of groupoids and for every object $\mathbf{A}$ in $\mathbf{preAdd}^+$ the functor $\mathbf{A}\sharp-\colon \mathbf{Groupoids} \to \mathbf{preAdd}^+$ preserves equivalences (see the proof of Lemma 2.2.25), the morphism $\mathbf{D}\sharp\widetilde{G} \to \mathbf{D}$ is a weak equivalence in the projective model category structure on $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$ for every $\mathbf{D}$ in $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$.

We must show that $L(\mathbf{D})$ is cofibrant. To this end we consider the lifting problem

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathbf{A} \arrow[d, "f"] \\ \mathbf{D}\sharp\widetilde{G} \arrow[r, "u"] \arrow[ur, dashed, "c"] & \mathbf{B} \end{tikzcd}$$

where f is a trivial fibration in $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$. By Lemma 2.2.8, f is a marked equivalence and surjective on objects. Pick for each object B in $\mathbf{B}$ a preimage $s(B)$. Then define c on objects by setting $c(D, g) := gs(u(g^{-1}D, 1))$. Note that $f(c(D, g)) = u(D, g)$ since both f and u are G-equivariant. For $(D, g), (E, h)$ in $\mathbf{D}\sharp\widetilde{G}$, define c on morphisms as

$$\mathrm{Hom}_{\mathbf{D}\sharp\widetilde{G}}((D,g),(E,h)) \xrightarrow{u} \mathrm{Hom}_{\mathbf{B}}(u(D,g),u(E,h)) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{A}}(c(D,g),c(E,h)) \ ,$$

where the second map is the inverse of the map induced by f. Since both u and f are G-equivariant functors, this defines a G-equivariant functor $c\colon \mathbf{D}\sharp\widetilde{G} \to \mathbf{A}$. The equality $fc = u$ also holds by definition. ∎

PROOF PROOF OF THEOREM 3.3.1. According to (21) and Lemma 3.3.5, we must calculate the object

$$\operatorname*{colim}_{BG} L(\underline{\mathbf{A}}) \cong \operatorname*{colim}_{BG}(\underline{\mathbf{A}}\sharp\widetilde{G})$$

for an object $\mathbf{A}$ of $\mathbf{preAdd}^+$. To this end, we note that for a fixed marked preadditive category $\mathbf{D}$, we have by (7) an adjunction

$$\mathbf{D}\sharp-\colon \mathbf{Groupoids} \leftrightarrows \mathbf{preAdd}^+ \colon \mathbf{Fun}^+_{\mathbf{preAdd}^+}(\mathbf{D}, -) \ .$$

Since $\mathbf{D}\sharp-$ is a left adjoint, it commutes with colimits. Consequently, we get

$$\operatorname*{colim}_{BG}(\underline{\mathbf{A}}\sharp\widetilde{G}) \simeq \mathbf{A}\sharp \operatorname*{colim}_{BG} \widetilde{G} \ . \tag{22}$$

The assertion of Theorem 3.3.1 now follows from a combination of the relations (22) and (21) together with $\mathrm{colim}_{BG}\,\widetilde{G} \cong BG$. ∎

3.3.6. REMARK. Theorem 3.3.1 admits a generalization to preadditive categories with arbitrary G-action. Let us consider the unmarked case for simplicity.

For $\mathbf{A}$ in $\mathbf{Fun}(BG, \mathbf{preAdd})$, Bartels and Reich have defined a preadditive category $\mathbf{A} *_G G/G$ whose objects are precisely the objects of $\mathbf{A}$ and whose morphisms are given by families $(\phi_\gamma\colon A \to \gamma B)_{\gamma \in G}$ such that $\phi_\gamma = 0$ for all but finitely many γ in G (see [BR07, Def. 2.1]). Consider the functor

$$S\colon \mathbf{A}\sharp\widetilde{G} \to \underline{\mathbf{A} *_G G/G}$$

which sends an object (A, g) to $g^{-1}A$ and a morphism $(f, g \to h)\colon (A, g) \to (B, h)$ to the family $S(f)$ given by

$$S(f)_\gamma := \begin{cases} g^{-1}f & \gamma = g^{-1}h, \\ 0 & \gamma \neq g^{-1}h. \end{cases}$$

Then one can check directly that S exhibits $\mathbf{A}*_G G/G$ as the colimit of $\mathbf{A}\sharp\widetilde{G}$. So $\mathbf{A}*_G G/G$ always models the homotopy G-orbits of $\mathbf{A}$.

We apply Theorem 3.3.1 to identify module categories over a group ring as homotopy orbits of the category of modules over the base ring. This allows us to describe functors on the orbit category of a discrete group G analogous to the ones defined by Davis and Lück in [DL98, Sec. 2] in terms of universal constructions.

Let R be a unital ring. By $R[G]$ we denote the group ring of G with coefficients in R. Recall from Section 3.2 that we can consider unital rings as preadditive categories which will be denoted by the corresponding bold-face letters.

3.3.7. Lemma. *We have an equivalence*

$$\operatorname*{colim}_{BG} \ell_{\mathbf{preAdd},BG}(\underline{\mathbf{R}}) \simeq \ell_{\mathbf{preAdd}}(\mathbf{R}[\mathbf{G}]) \ .$$

Proof. By Theorem 3.3.1, we have an equivalence

$$\operatorname*{colim}_{BG} \ell_{\mathbf{preAdd},BG}(\underline{\mathbf{R}}) \simeq \ell_{\mathbf{preAdd}}(\mathbf{R}\sharp BG) \ .$$

Unfolding the definitions (see e.g. Definition 2.2.17) we observe that $\mathbf{R}\sharp BG$ has one object, and its ring of endomorphisms is given by $R \otimes_{\mathbb{Z}} \mathbb{Z}[G] \cong R[G]$. ∎

3.3.8. Proposition. *We have equivalences*

$$\operatorname*{colim}_{BG} \ell_{\mathbf{preAdd},BG}(\underline{\mathbf{Mod}^{\mathrm{fg,free}}(R)}) \simeq \ell_{\mathbf{preAdd}}(\mathbf{Mod}^{\mathrm{fg,free}}(R[G]))$$

and

$$\operatorname*{colim}_{BG} \ell_{\mathbf{preAdd},BG}(\underline{\mathbf{Mod}^{\mathrm{fg,proj}}(R)}) \simeq \ell_{\mathbf{preAdd}}(\mathbf{Mod}^{\mathrm{fg,proj}}(R[G]))$$

Proof. By Proposition 3.2.1, we have an equivalence

$$\operatorname*{colim}_{BG} \ell_{\mathbf{Add},BG}(\underline{\mathbf{Mod}^{\mathrm{fg,free}}(R)}) \simeq \operatorname*{colim}_{BG} L_{\oplus,BG}(\ell_{\mathbf{preAdd},BG}(\underline{\mathbf{R}})) \ .$$

Since $L_\oplus$ is a left adjoint, it commutes with colimits. Therefore,

$$\operatorname*{colim}_{BG} L_{\oplus,BG}(\ell_{\mathbf{preAdd},BG}(\underline{\mathbf{R}})) \simeq L_\oplus(\operatorname*{colim}_{BG} \ell_{\mathbf{preAdd},BG}(\underline{\mathbf{R}})) \ .$$

By Lemma 3.3.7, we have the equivalence

$$L_\oplus(\operatorname*{colim}_{BG} \ell_{\mathbf{preAdd},BG}(\underline{\mathbf{R}})) \simeq L_\oplus(\ell_{\mathbf{preAdd}}(\mathbf{R}[\mathbf{G}])) \ .$$

Finally, by Proposition 3.2.1 again

$$L_\oplus(\ell_{\mathbf{preAdd}}(\mathbf{R}[\mathbf{G}])) \simeq \ell_{\mathbf{preAdd}}(\mathbf{Mod}^{\mathrm{fg,free}}(R[G])) \ .$$

The second equivalence is shown similarly, using Proposition 3.2.7 and $L_{\oplus,\mathrm{idem}}$ instead of Proposition 3.2.1 and $L_\oplus$. ∎

3.3.9. Example. A unital ring R gives rise to two canonical marked preadditive categories $\mathrm{mi}(\mathbf{R})$ (only the identity is marked) and $\mathrm{ma}(\mathbf{R})$ (all units are marked). Then

$$\operatorname*{colim}_{BG} \ell_{\mathbf{preAdd}^+,BG}(\mathrm{mi}(\mathbf{R})) \simeq \ell_{\mathbf{preAdd}^+}(\mathbf{R}[\mathbf{G}]^{\mathrm{can}_G}) \ ,$$

where the marked isomorphisms in $\mathbf{R}[\mathbf{G}]^{\mathrm{can}_G}$ are the elements of G (canonically considered as elements in $R[G]$). In contrast,

$$\operatorname*{colim}_{BG} \ell_{\mathbf{preAdd}^+,BG}(\mathrm{ma}(\mathbf{R})) = \ell_{\mathbf{preAdd}^+}(\mathbf{R}[\mathbf{G}]^{\mathrm{can}}) \ ,$$

where the marked isomorphisms in $\mathbf{R}[\mathbf{G}]^{\mathrm{can}}$ are the canonical units in $R[G]$, i.e., the elements of the form ug for a unit u of R and an element g of G.

Let us now use the general machine in order to construct interesting functors on the orbit category $G\mathbf{Orb}$ of G. Recall that $G\mathbf{Orb}$ is the category of transitive G-sets and equivariant maps. The group G with the left action is an object of $G\mathbf{Orb}$. Since the right action of G on itself implements an isomorphism $\mathrm{End}_{G\mathbf{Orb}}(G) \cong G$, we get a fully faithful functor

$$i\colon BG \to G\mathbf{Orb} \ . \tag{23}$$

If $\mathbf{C}$ is a presentable ∞-category, then we have an adjunction

$$i_!\colon \mathbf{Fun}(BG,\mathbf{C}) \leftrightarrows \mathbf{Fun}(G\mathbf{Orb},\mathbf{C})\colon i^* \ . \tag{24}$$

The functor $i_!$ is the left Kan extension functor along i. We now consider the composition

$$\begin{aligned} \mathbf{preAdd} \quad &\xrightarrow{\underline{(-)}} \quad \mathbf{Fun}(BG,\mathbf{preAdd}) \\ &\xrightarrow{\ell_{\mathbf{preAdd},BG}} \quad \mathbf{Fun}(BG,\mathbf{preAdd}_\infty) \\ &\xrightarrow{i_!} \quad \mathbf{Fun}(G\mathbf{Orb},\mathbf{preAdd}_\infty) \end{aligned} \tag{25}$$

which we denote by J^G.

We are interested in the calculation of the value $J^G(\mathbf{A})(G/H)$ for a subgroup H.

Let $\mathbf{A}$ be a preadditive category.

3.3.10. PROPOSITION. *We have an equivalence*

$$J^G(\mathbf{A})(G/H) \simeq \ell_{\mathbf{preAdd}}(\mathbf{A}\sharp BH)\ .$$

PROOF. The functor $S \mapsto (G\times_H S \to G/H)$ induces an equivalence of categories $H\mathbf{Orb} \xrightarrow{\simeq} G\mathbf{Orb}/(G/H)$ which restricts to an equivalence

$$BH \simeq i/(G/H)\ , \tag{26}$$

where $i/(G/H)$ denotes the slice of $i\colon BG \to G\mathbf{Orb}$ over G/H. Using the pointwise formula [Lur09, Def. 4.3.2.2] for the left Kan extension functor $i_!$ at the equivalence marked by ! we get

$$\begin{aligned} J^G(\mathbf{A})(G/H) &\simeq i_!(\ell_{\mathbf{preAdd},BG}(\underline{\mathbf{A}}))(G/H) \\ &\overset{!}{\simeq} \operatorname*{colim}_{(i(*)\to G/H)\in i/(G/H)} \ell_{\mathbf{preAdd},BG}(\underline{\mathbf{A}})(*) \\ &\overset{!!}{\simeq} \operatorname*{colim}_{BH} \ell_{\mathbf{preAdd},BH}(\underline{\mathbf{A}}) \\ &\overset{Theorem\ 3.3.1}{\simeq} \ell_{\mathbf{preAdd}}(\mathbf{A}\sharp BH)\ , \end{aligned}$$

where at !! we use (26) and that the argument of the colimit is a constant functor. ■

Applying Proposition 3.3.10 in the case $\mathbf{A} := \mathbf{R}$ for a ring R leads to a functor

$$J^G(\mathbf{R})\colon G\mathbf{Orb} \to \mathbf{preAdd}_\infty$$

whose value at G/H is given by $J^G(\mathbf{R})(G/H) \simeq \ell_{\mathbf{preAdd}}(\mathbf{R}[\mathbf{H}])$. If we postcompose by $L_\oplus$ and use Proposition 3.2.1, then we get a functor

$$L_{\oplus,G\mathbf{Orb}} \circ J^G(\mathbf{R})\colon G\mathbf{Orb} \to \mathbf{Add}_\infty$$

with values $L_{\oplus,G\mathbf{Orb}} \circ J^G(\mathbf{R})(G/H) \simeq \ell_{\mathbf{Add}}(\mathbf{Mod}^{\mathrm{fg,free}} R[H])$. The composition

$$K \circ L_{\oplus,G\mathbf{Orb}} \circ J^G(\mathbf{R})\colon G\mathbf{Orb} \to \mathbf{Sp}$$

therefore has the same values as the functor representing the equivariant K-homology with R-coefficients constructed by Davis and Lück [DL98, Sec. 2 and 4]. In ∞-categorical language, equivariant K-homology is given by the left Kan extension of this functor along the Yoneda embedding $G\mathbf{Orb} \to \mathbf{PSh}(G\mathbf{Orb})$.

3.4. G-INVARIANTS. Let G be a group. In this section we calculate the homotopy G-invariants of marked preadditive categories with G-action. The precise formulation is Theorem 3.4.3. Ignoring enrichments, these categories have previously been considered by Merling [Mer17].

Let $\mathbf{A}$ be an object of $\mathbf{Fun}(BG, \mathbf{preAdd}^{(+)})$, i.e. a (marked) preadditive category with G-action.

3.4.1. DEFINITION. *We define a (marked) preadditive category* $\widehat{\mathbf{A}}^G$ *as follows:*

1. *The objects of* $\widehat{\mathbf{A}}^G$ *are pairs* (A,ρ) *of an object* A *of* $\mathbf{A}$ *and a collection* $\rho := (\rho(g))_{g\in G}$*, where* $\rho(g)\colon A\to g(A)$ *is a (marked) isomorphism in* $\mathbf{A}$ *and the equality*
$$g(\rho(h))\circ\rho(g)=\rho(hg)$$
holds true for all pairs g,h *in* G.
2. *The morphisms* $(A,\rho)\to(A',\rho')$ *in* $\widehat{\mathbf{A}}^G$ *are morphisms* $a\colon A\to A'$ *in* $\mathbf{A}$ *such that the equality* $g(a)\circ\rho(g)=\rho'(g)\circ a$ *holds true for all* g *in* G.
3. *The enrichment of* $\widehat{\mathbf{A}}^G$ *over abelian groups is inherited from the enrichment of* $\mathbf{A}$.
4. *(in the marked case) The marked isomorphisms in* $\widehat{\mathbf{A}}^G$ *are those morphisms which are marked isomorphisms in* $\mathbf{A}$.

3.4.2. EXAMPLE. If $\mathbf{A}$ is an object of $\mathbf{preAdd}^{(+)}$, then we will shorten the notation and write $\widehat{\mathbf{A}}^G$ for $\widehat{\underline{\mathbf{A}}}^G$, where $\underline{\mathbf{A}}$ is $\mathbf{A}$ with the trivial G-action.

In this case $\widehat{\mathbf{A}}^G$ is the category of objects of $\mathbf{A}$ with an action of G by (marked) isomorphisms, and equivariant morphisms. In the marked case, the marked isomorphisms in $\widehat{\mathbf{A}}^G$ are those which are marked in $\mathbf{A}$.

Recall Notation (20). We will identify $\widehat{\mathbf{A}}^G$ with $\lim_{BG} R(\mathbf{A})$ for a fibrant replacement $R(\mathbf{A})$ of $\mathbf{A}$ to obtain the following statement.

3.4.3. THEOREM. *We have an equivalence*
$$\lim_{BG}\ell_{\mathbf{preAdd}^{(+)},BG}(\mathbf{A})\simeq\ell_{\mathbf{preAdd}^{(+)}}(\widehat{\mathbf{A}}^G)\ .$$

3.4.4. REMARK. The forgetful functor $\mathcal{F}_+\colon\mathbf{preAdd}^+\to\mathbf{preAdd}$ descends to a functor $\mathcal{F}_+\colon\mathbf{preAdd}^+_\infty\to\mathbf{preAdd}_\infty$, see Corollary 2.4.4. If $\mathbf{A}$ is a preadditive category with G-action, then the unmarked version of Theorem 3.4.3 can be obtained from the marked versions by
$$\begin{aligned}\lim_{BG}\ell_{\mathbf{preAdd}^{(+)},BG}(\mathbf{A}) &\simeq \mathcal{F}_+(\mathrm{ma}(\lim_{BG}\ell_{\mathbf{preAdd}^{(+)},BG}\mathbf{A})))\\ &\simeq \mathcal{F}_+(\lim_{BG}\ell_{\mathbf{preAdd}^{(+)},BG}(\mathrm{ma}_{BG}(\mathbf{A})))\\ &\simeq \ell_{\mathbf{preAdd}^{(+)}}(\mathcal{F}_+(\widehat{\mathrm{ma}_{BG}(\mathbf{A})}^G))\end{aligned}$$
using that ma (as a right adjoint, see (5)) preserves limits. Note that
$$\mathcal{F}_+(\widehat{\mathrm{ma}_{BG}(\mathbf{A})}^G)=\widehat{\mathbf{A}}^G\ ,$$
where on the left-hand side we use Definition 3.4.1 in the marked case, and on the right-hand side we use it in the unmarked case.

3.4.5. Remark. The order of taking the limit $\lim_{BG}$ and the localization $\ell_{\dots}$ matters. For example, consider the additive category $\mathbf{Mod}(\mathbb{Z})$ with the trivial G-action. Then

$$\lim_{BG} \mathbf{Mod}(\mathbb{Z}) \cong \mathbf{Mod}(\mathbb{Z}) \ .$$

On the other hand, $\widehat{\mathbf{Mod}(\mathbb{Z})}^{G}$ is the category of representations of G on $\mathbb{Z}$-modules. If G is non-trivial, then it is not equivalent to $\mathbf{Mod}(\mathbb{Z})$.

For simplicity (and in view of Remark 3.4.4), we formulate the proof in the marked case, only. Since the category $\mathbf{preAdd}^+$ has a combinatorial model category structure the injective model category structure in $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$ exists. The proof of this fact involves Smith's theorem, see e.g. [Bek00, Thm. 1.7], [Lur09, Sec. A.2.6]. A textbook reference of the fact stated precisely in the form we need is [Lur09, Prop. A.2.8.2].

For every fibrant replacement functor $r\colon \mathrm{id} \to R$ in the injective model category structure on $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$ we have an equivalence

$$\ell_{\mathbf{preAdd}^+} \circ \lim_{BG} \circ R \simeq \lim_{BG} \circ \ell_{\mathbf{preAdd}^+, BG} \tag{27}$$

of functors from $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$ to $\mathbf{preAdd}^+_\infty$ (see e.g. [Bun19, Prop. 13.6] for an argument). In the following we use the notation introduced in Definition 2.1.9 and before Lemma 2.2.18. Furthermore, we consider the G-groupoid $\widetilde{G}$ defined in Definition 3.3.4. We define the functor

$$R := \mathbf{Fun}^+_{\mathbf{preAdd}^+}(Q(\widetilde{G}), -)\colon \mathbf{Fun}(BG, \mathbf{preAdd}^+) \to \mathbf{Fun}(BG, \mathbf{preAdd}^+) \tag{28}$$

together with the natural transformation $r\colon \mathrm{id} \to R$ induced by $\widetilde{G} \to \Delta^0_{\mathbf{Cat}}$ using the canonical isomorphism $\mathbf{Fun}^+_{\mathbf{preAdd}^+}(Q(\Delta^0_{\mathbf{Cat}}), -) \cong \mathrm{id}$.

3.4.6. Lemma. *The functor* (28) *together with the natural transformation* r *is a fibrant replacement functor.*

Proof. The morphism $\widetilde{G} \to \Delta^0_{\mathbf{Cat}}$ is a non-equivariant equivalence of groupoids. An inverse equivalence is given by any map $\Delta^0_{\mathbf{Cat}} \to \widetilde{G}$ classifying some object of $\widetilde{G}$. Since this functor is injective on objects, we conclude as in Lemma 2.2.27 that the induced (non-equivariant) morphism $p\colon R(\mathbf{A}) \to \mathbf{A}$ is a weak equivalence. Since $p \circ r = \mathrm{id}$ we conclude that $r\colon \mathbf{A} \to R(\mathbf{A})$ is a (non-equivariant) weak equivalence, too. Hence $r\colon \mathbf{A} \to R(\mathbf{A})$, now considered as a morphism in $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$, is an equivalence in the injective model category structure.

In order to finish the proof we must show that $R(\mathbf{A})$ is fibrant. To this end we consider the following square in $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$, where $c\colon \mathbf{C} \to \mathbf{D}$ is a trivial cofibration in $\mathbf{Fun}(BG, \mathbf{preAdd}^+)$:

$$\begin{tikzcd} \mathbf{C} \arrow[r] \arrow[d, "c"] & R(\mathbf{A}) \arrow[d] \\ \mathbf{D} \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

We must show the existence of the diagonal lift.

We use the identification $\mathbf{Fun}^+_{\mathbf{preAdd}^+}(Q(\widetilde{G}),*) \simeq *$ and the adjunction of Lemma 2.2.18 in order to rewrite the lifting problem as follows.

$$\begin{tikzcd} \mathbf{C}\sharp\widetilde{G} \arrow[r, "\phi"] \arrow[d] & \mathbf{A} \arrow[d] \\ \mathbf{D}\sharp\widetilde{G} \arrow[r] \arrow[u, dotted, bend left, "\widetilde{d}"] \arrow[ur, dotted] & * \end{tikzcd}$$

Since, after forgetting the G-action, the morphism of $c\colon \mathbf{C} \to \mathbf{D}$ is a trivial cofibration it is injective on objects. We can therefore choose an inverse equivalence $d\colon \mathbf{D} \to \mathbf{C}$ (not necessarily G-equivariant) up to marked isomorphism with $d \circ c = \mathrm{id}_{\mathbf{C}}$. We can extend the composition

$$\mathbf{D} \xrightarrow{d} \mathbf{C} \to \mathbf{C} \times \{1\} \to \mathbf{C}\sharp\widetilde{G}$$

uniquely to a G-equivariant morphism

$$\widetilde{d}\colon \mathbf{D}\sharp\widetilde{G} \to \mathbf{C}\sharp\widetilde{G}$$

by setting

$$\widetilde{d}(D,g) := (gd(g^{-1}D), g)\ , \quad \widetilde{d}(f\colon D \to D', g \to h) := gd(g^{-1}f)\sharp(g \to h)\ .$$

The desired lift can now be obtained as the composition $\phi \circ \widetilde{d}$. ∎

Proof Proof of Theorem 3.4.3. By (27) and Lemma 3.4.6, we have an equivalence

$$\lim_{BG} \ell_{\mathbf{preAdd}^+,BG}(\mathbf{A}) \simeq \ell_{\mathbf{preAdd}^+}(\lim_{BG} R(\mathbf{A}))\ .$$

In order to finish the proof of Theorem 3.4.3, it remains to show that

$$\lim_{BG} R(\mathbf{A}) \cong \widehat{\mathbf{A}}^G\ .$$

We define a functor

$$\Psi\colon\ \lim_{BG} R(\mathbf{A}) = \lim_{BG} \mathbf{Fun}^+_{\mathbf{preAdd}^+}(Q(\widetilde{G}), \mathbf{A}) \to \widehat{\mathbf{A}}^G$$

as follows.

1. on objects:
$$\Psi(\phi) := (\phi(1), (\phi(1 \to g))_{g\in G})\ .$$
Note that $\phi(g) = g\phi(1)$ by G-equivariance of ϕ. Functoriality of ϕ guarantees that Definition 3.4.1.1 is fulfilled.

2. on morphisms:
$$\Psi((a_h)_{h\in\widetilde{G}}\colon \phi\to\psi) := a_1\colon \phi(1)\to\psi(1)\ .$$
One easily checks Definition 3.4.1.2 using that ϕ and ψ are G-equivariant and that $(a_h)_{h\in\widetilde{G}}$ is a natural transformation.

3. We observe that Ψ preserves marked isomorphisms.

Finally we check that the functor Ψ is an isomorphism of categories. This finishes the proof of Theorem 3.4.3. ∎

Theorem 3.4.3 implies an analogous statement for additive categories.

Let $\mathbf{A}$ be in $\mathbf{Fun}(BG, \mathbf{preAdd}^{(+)})$.

3.4.7. Lemma. *If $\mathbf{A}$ belongs to the subcategory $\mathbf{Fun}(BG, \mathbf{Add}^{(+)})$, then $\widehat{\mathbf{A}}^G$ is a (marked) additive category.*

Proof. We must show that $\widehat{\mathbf{A}}^G$ admits finite coproducts. If (M,ρ) and (M',ρ') are two objects, then $(M\oplus M', \rho\oplus\rho')$ together with the canonical inclusions represents the coproduct of (M,ρ) and (M',ρ'). In the marked case, one furthermore checks by inspection condition 2 from Definition 2.3.3 for $\mathbf{A}$ implies this condition for $\widehat{\mathbf{A}}^G$. This condition also implies that $\rho\oplus\rho'$ acts by marked isomorphisms as required in the marked case. ∎

Let $\mathbf{A}$ be in $\mathbf{Fun}(BG, \mathbf{Add}^{(+)})$.

3.4.8. Corollary. *We have an equivalence*
$$\lim_{BG} \ell_{\mathbf{Add}^{(+)},BG}(\mathbf{A}) \simeq \ell_{\mathbf{Add}^{(+)}}(\widehat{\mathbf{A}}^G)\ .$$

Proof. The functor $\mathcal{F}_\oplus\colon \mathbf{Add}^{(+)}_\infty \to \mathbf{preAdd}^{(+)}_\infty$ is a right adjoint and hence preserves limits. Using Theorem 3.4.3, we obtain equivalences
$$\begin{aligned}\mathcal{F}_\oplus(\lim_{BG}(\ell_{\mathbf{Add}^{(+)},BG}(\mathbf{A}))) &\simeq \lim_{BG} \ell_{\mathbf{preAdd}^{(+)},BG}(\mathcal{F}_{\oplus,BG}(\mathbf{A}))\\ &\simeq \ell_{\mathbf{preAdd}^{(+)}}(\widehat{\mathcal{F}_{\oplus,BG}(\mathbf{A})}^G)\\ &\simeq \mathcal{F}_\oplus(\ell_{\mathbf{Add}^{(+)}}(\widehat{\mathbf{A}}^G))\end{aligned}$$
Since $\widehat{\mathbf{A}}^G$ is additive by Lemma 3.4.7, this implies the assertion by omitting $\mathcal{F}_\oplus$ on both sides. ∎

3.4.9. Example. Let k be a complete normed field and let $\mathbf{Ban}$ denote the category of Banach spaces over k and bounded linear maps. This category is additive. Note that only the equivalence class of the norm on an object of $\mathbf{Ban}$ is an invariant of the isomorphism class of the object. We use the norms in order to define a marked preadditive category $\mathbf{Ban}^+$ by marking isometries.

It is first interesting to observe that $\mathbf{Ban}^+$ is not a marked additive category. In fact, the Condition 2.3.3.2 is violated since only the equivalence class of the norm on a direct sum is fixed by the norms on the summands.

We can now calculate the G-invariants: By Corollary 3.4.8,

$$\lim_{BG} \ell_{\mathbf{Add},BG}(\underline{\mathbf{Ban}}) \simeq \ell_{\mathbf{Add}}(\widehat{\mathbf{Ban}}^G) .$$

By Example 3.4.2, $\widehat{\mathbf{Ban}}^G$ is the category of Banach-spaces over k with an action by G and equivariant bounded linear maps. On the other hand, by Theorem 3.4.3

$$\lim_{BG} \ell_{\mathbf{preAdd}^+,BG}(\underline{\mathbf{Ban}^+}) \simeq \ell_{\mathbf{preAdd}^+}(\widehat{\mathbf{Ban}^+}^G) .$$

By Example 3.4.2, $\widehat{\mathbf{Ban}^+}^G$ is the category of Banach-spaces over k with an *isometric* action by G and equivariant bounded linear maps which are marked if they are isometric. Hence $\mathcal{F}_+(\widehat{\mathbf{Ban}^+}^G)$ is contained properly in $\widehat{\mathbf{Ban}}^G$.

This shows that even if we forget the marking at the end, the marking matters when we form limits.

3.4.10. Example. Let R be a unital ring. We consider the additive categories $\mathbf{Mod}^?(R)$ and $\mathbf{Mod}^?(R)$, where the decoration ? is a condition like *free*, *projective*, *finitely generated* or some combination of these. By Corollary 3.4.8 and Example 3.4.2, we get

$$\lim_{BG} \ell_{\mathbf{Add},BG}(\underline{\mathbf{Mod}^?(R)}) \simeq \ell_{\mathbf{Add}}(\mathbf{Fun}(BG, \mathbf{Mod}^?(R))) .$$

Note the difference between limits and colimits: By Proposition 3.3.8 we have an equivalence

$$\operatorname*{colim}_{BG} \ell_{\mathbf{Add},BG}(\underline{\mathbf{Mod}^?(R)}) \simeq \ell_{\mathbf{Add}}(\mathbf{Mod}^?(R[G]))$$

for $? = (\mathrm{fg}, \mathrm{proj}), (\mathrm{fg}, \mathrm{free})$. If G is infinite, then the interpretation of ? on the right-hand side leads to different categories (e.g. finitely generated free $R[G]$-modules are in general not finitely generated free R-modules with a G-action).

3.4.11. Example. For the following example we assume familiarity with equivariant coarse homology theories and the example of equivariant coarse algebraic K-homology, see for example [BEKWa, Sec. 2, 3 and 8]. In particular, recall the definition of the functor $\mathbf{V_A} \colon \mathbf{BornCoarse} \to \mathbf{Add}$ of X-controlled $\mathbf{A}$-objects for a bornological coarse space X and an additive category $\mathbf{A}$ from [BEKWa, Sec. 8.2]. We define the functor

$$\mathbf{V}^+_{\mathbf{A}} \colon \mathbf{BornCoarse} \to \mathbf{Add}^+$$

by considering $\mathbf{V_A}$ and marking the $\mathrm{diag}(X)$-controlled isomorphisms.

Let X be a G-bornological coarse space and let $\mathbf{A}$ be an additive category with a G-action. By functoriality the marked additive category $\mathbf{V}^+_{\mathbf{A}}(X)$ then has an action of $G \times G$. We consider $\mathbf{V}^+_{\mathbf{A}}(X)$ as a marked additive category with G-action by restricting the $G \times G$ action along the diagonal. As in Definition 3.4.1 we can form the category $\widehat{\mathbf{V}^+_{\mathbf{A}}}^G$.

We define the functor

$$\mathbf{V}^G_{\mathbf{A}} := \mathcal{F}_+ \circ \widehat{\mathbf{V}^+_{\mathbf{A}}}^G : G\mathbf{BornCoarse} \to \mathbf{Add} \ .$$

One checks that this definition agrees with the definition of $\mathbf{V}^G_{\mathbf{A}}$ from [BEKWa, Sec. 8.2]. By definition, equivariant coarse algebraic K-homology is the functor $K\mathbf{A}\mathcal{X}^G := K \circ \mathbf{V}^G_{\mathbf{A}}$.

The functor $\mathcal{F}_+ \colon \mathbf{Add}^+ \to \mathbf{Add}$ descends to a functor $\mathcal{F}_+ \colon \mathbf{Add}^+_\infty \to \mathbf{Add}_\infty$. Using Corollary 3.4.8, we now obtain

$$\begin{aligned} K\mathbf{A}\mathcal{X}^G = K \circ \mathbf{V}^G_{\mathbf{A}} &= K \circ \mathcal{F}_+ \circ \widehat{\mathbf{V}^+_{\mathbf{A}}}^G \\ &\simeq K_\infty \circ \ell_{\mathbf{Add}} \circ \mathcal{F}_+ \circ \widehat{\mathbf{V}^+_{\mathbf{A}}}^G \\ &\simeq K_\infty \circ \mathcal{F}_+ \circ \ell_{\mathbf{Add}^+} \circ \widehat{\mathbf{V}^+_{\mathbf{A}}}^G \\ &\simeq K_\infty \circ \mathcal{F}_+ \circ \lim_{BG} \circ \ell_{\mathbf{Add}^+,BG} \circ \mathbf{V}^+_{\mathbf{A}} \ . \end{aligned}$$

This shows that equivariant coarse algebraic K-homology can be computed from the non-equivariant version by taking G-invariants in marked additive categories.

In addition to the adjunction (24), for a presentable ∞-category $\mathbf{C}$ we also have an adjunction

$$i^{op,*} \colon \mathbf{Fun}(G\mathbf{Orb}^{op}, \mathbf{C}) \leftrightarrows \mathbf{Fun}(BG^{op}, \mathbf{C}) \colon i^{op}_* \ . \tag{29}$$

In analogy to (25) we consider the functor C^G defined as the composition

$$\mathbf{Fun}(BG^{op}, \mathbf{preAdd}^{(+)}) \xrightarrow{\ell_{\mathbf{preAdd}^{(+)},BG}} \mathbf{Fun}(BG^{op}, \mathbf{preAdd}^{(+)}_\infty) \xrightarrow{i^{op}_*} \mathbf{Fun}(G\mathbf{Orb}^{op}, \mathbf{preAdd}^{(+)}_\infty)$$

For a (marked) preadditive category with G-action $\mathbf{A}$ we are interested in the values $C^G(\mathbf{A})(G/H)$ for subgroups H of G.

3.4.12. Lemma. *We have an equivalence*

$$C^G(\mathbf{A})(G/H) \simeq \ell_{\mathbf{preAdd}^{(+)}}(\widehat{\mathrm{Res}^G_H(\mathbf{A})}^H) \ .$$

Proof. The argument is very similar to the proof of Proposition 3.3.10. We use that the induction $S \mapsto G \times_H S$ induces an equivalence

$$BH^{op} \simeq (G/H)/i^{op} \ ,$$

where $(G/H)/i^{op}$ denotes the slice of $i^{op} \colon BG^{op} \to G\mathbf{Orb}^{op}$ under G/H. Further employing the pointwise formula for the right Kan extension functor i^{op}_* and the equivalence $BH \simeq BH^{op}$ given by inversion, we get

$$\begin{aligned} C^G(\mathbf{A})(G/H) &\simeq i^{op}_*(\ell_{\mathbf{preAdd}^{(+)},BG^{op}}(\mathbf{A}))(G/H) \\ &\simeq \lim_{(G/H \to i^{op}(*)) \in (G/H)/i^{op}} \ell_{\mathbf{preAdd}^{(+)},BG^{op}}(\mathbf{A})(*) \\ &\simeq \lim_{BH^{op}} \ell_{\mathbf{preAdd}^{(+)},BH^{op}}(\mathrm{Res}^G_H(\mathbf{A})) \\ &\simeq \ell_{\mathbf{preAdd}^{(+)}}(\widehat{\mathrm{Res}^G_H(\mathbf{A})}^H) \end{aligned}$$

■

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