TOPOI OF PARAMETRIZED OBJECTS

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ABSTRACT. We give necessary and sufficient conditions on a presentable ∞-category $\mathcal{C}$ so that families of objects of $\mathcal{C}$ form an ∞-topos. In particular, we prove a conjecture of Joyal that this is the case whenever $\mathcal{C}$ is stable.

Let $\mathfrak{X}$ be an ∞-topos and let $\mathcal{C}$ be a sheaf of ∞-categories on $\mathfrak{X}$. We denote by

$$\int_{\mathfrak{X}} \mathcal{C} \to \mathfrak{X}$$

the cartesian fibration classified by $\mathcal{C}$. An object in $\int_{\mathfrak{X}} \mathcal{C}$ is thus a pair (U, c) with $U \in \mathfrak{X}$ and $c \in \mathcal{C}(U)$. The question we are interested in is the following:

1.1. QUESTION. *When is $\int_{\mathfrak{X}} \mathcal{C}$ an ∞-topos?*

If $\mathcal{C}$ is a presentable ∞-category, we will also denote by $\int_{\mathfrak{X}} \mathcal{C} \to \mathfrak{X}$ the cartesian fibration classified by the sheaf

$$U \mapsto \mathrm{Shv}_{\mathcal{C}}(\mathfrak{X}_{/U}) \simeq \mathcal{C} \otimes \mathfrak{X}_{/U}.$$

Joyal calls $\mathcal{C}$ an ∞-*locus* if $\int_{\mathcal{S}} \mathcal{C}$ is an ∞-topos,[1] and he conjectures that any presentable stable ∞-category is an ∞-locus [2]. The motivating example, due to Biedermann and Rezk, is the ∞-category Sp of spectra: there is an equivalence

$$\int_{\mathfrak{X}} \mathrm{Sp} \simeq \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathfrak{X}),$$

and the right-hand side is an ∞-topos [3, Remark 6.1.1.11]. More generally, for any small ∞-category $\mathcal{A}$ with finite colimits and a final object,

$$\int_{\mathfrak{X}} \mathrm{Exc}_*^n(\mathcal{A}, \mathcal{S}) \simeq \mathrm{Exc}^n(\mathcal{A}, \mathfrak{X})$$

is an ∞-topos.

This paper gives a partial answer to Question 1.1 in Theorem 1.2. As a corollary, we obtain a characterization of ∞-loci (see Corollary 1.5), similar to Rezk's characterization of ∞-topoi, which easily implies Joyal's conjecture (see Example 1.7). The latter also follows more directly from two observations, applicable to any $\mathcal{Pr}^{\mathrm{R}}$-valued sheaf $\mathcal{C}$ on an ∞-topos $\mathfrak{X}$:

[1]Joyal also requires an ∞-locus to be pointed, but it will be convenient to omit this condition.

- If $L\colon \mathcal{C} \to \mathcal{C}$ is objectwise an accessible left exact localization functor, then the inclusion $\int_{\mathfrak{X}} L\mathcal{C} \subset \int_{\mathfrak{X}} \mathcal{C}$ is accessible and has a left exact left adjoint. This is clear once we know that $\int_{\mathfrak{X}} \mathcal{C}$ is presentable [1, Theorem 1.3].

- If $\mathcal{E}$ is a small ∞-category, there is a pullback square in $\mathcal{P}r^{\mathrm{L,R}}$:

$$
\begin{array}{ccc}
\int_{\mathfrak{X}} \mathrm{Fun}(\mathcal{E}, \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\mathcal{E}, \int_{\mathfrak{X}} \mathcal{C}) \\
\downarrow & & \downarrow \\
\mathfrak{X} & \longrightarrow & \mathrm{Fun}(\mathcal{E}, \mathfrak{X}).
\end{array}
$$

These observations show that the class of ∞-loci is closed under accessible left exact localizations and under the formation of functor ∞-categories $\mathrm{Fun}(\mathcal{E}, -)$. Since any presentable stable ∞-category is a left exact localization of $\mathrm{Fun}(\mathcal{E}, \mathrm{Sp})$ for some $\mathcal{E}$ [3, Proposition 1.4.4.9], Joyal's conjecture holds.

The next theorem gives more intrinsic conditions on $\mathcal{C}$ implying that $\int_{\mathfrak{X}} \mathcal{C}$ is an ∞-topos, and it leads to a second proof of Joyal's conjecture. Recall that a colimit in an ∞-category $\mathcal{C}$ with pullbacks is *van Kampen* if it is preserved by the functor $\mathcal{C} \to \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{op}}$, $c \mapsto \mathcal{C}_{/c}$, and recall that an ∞-category $\mathfrak{X}$ is an ∞-topos iff it is presentable and all small colimits in $\mathfrak{X}$ are van Kampen.

1.2. THEOREM. *Let $\mathfrak{X}$ be an ∞-topos and $\mathcal{C}$ a sheaf of ∞-categories on $\mathfrak{X}$. Suppose that:*

1. for every $U \in \mathfrak{X}$:

   (a) $\mathcal{C}(U)$ is accessible and admits pushouts;

   (b) pushouts in $\mathcal{C}(U)$ are van Kampen;[2]

2. for every $f\colon V \to U$ in $\mathfrak{X}$:

   (a) $f^*\colon \mathcal{C}(U) \to \mathcal{C}(V)$ preserves pushouts;

   (b) f^* has a left adjoint $f_!$;

   (c) for every $c, 0 \in \mathcal{C}(V)$ with 0 initial, the square

$$
\begin{array}{ccc}
\mathcal{C}(U)_{/f_!(c)} & \longrightarrow & \mathcal{C}(U)_{/f_!(0)} \\
{\scriptstyle \eta^* \circ f^*}\downarrow & & \downarrow{\scriptstyle \eta^* \circ f^*} \\
\mathcal{C}(V)_{/c} & \longrightarrow & \mathcal{C}(V)_{/0}
\end{array}
$$

   is cartesian, where $\eta\colon \mathrm{id} \to f^* f_!$ is the unit of the adjunction;

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[2]The existence of pullbacks in $\mathcal{C}(U)$ follows from the other assumptions, as the proof will show.

3. for every cartesian square

$$\begin{array}{ccc} V' & \xrightarrow{g} & V \\ q\downarrow & & \downarrow p \\ U' & \xrightarrow{f} & U \end{array}$$

in $\mathfrak{X}$, the canonical transformation $q_! g^* \to f^* p_! \colon \mathcal{C}(V) \to \mathcal{C}(U')$ is an equivalence.

Then $\mathcal{C}$ is a $\mathrm{Pr}^{\mathrm{L,R}}$-valued sheaf and $\int_{\mathfrak{X}} \mathcal{C}$ is an ∞-topos.

PROOF. It is clear that the coproduct of a family (U_α, c_α) in $\int_{\mathfrak{X}} \mathcal{C}$ is given by $(\coprod_\alpha U_\alpha, c)$, where $c = (c_\alpha)_\alpha \in \prod_\alpha \mathcal{C}(U_\alpha) \simeq \mathcal{C}(\coprod_\alpha U_\alpha)$. Given a span $(U, c) \leftarrow (W, e) \to (V, d)$, (1a) and (2b) imply that it has a pushout given by $(U \amalg_W V, u_!(c) \coprod_{w_!(e)} v_!(d))$, where u, v, and w are the canonical maps to $U \amalg_W V$. Hence, $\int_{\mathfrak{X}} \mathcal{C}$ has small colimits, and they are preserved by the projection $\int_{\mathfrak{X}} \mathcal{C} \to \mathfrak{X}$. By [4, Lemma 5.4.5.5], this implies that $\mathcal{C}(U)$ has weakly contractible colimits, being the pullback $\{U\} \times_{\mathfrak{X}} \int_{\mathfrak{X}} \mathcal{C}$. By (2b), $\mathcal{C}(U)$ also has an initial object, namely $i_!(0)$ where $i \colon \emptyset \to U$ and 0 is the unique object of $\mathcal{C}(\emptyset) \simeq *$. Hence, by (1a), $\mathcal{C}(U)$ is presentable. By (2b), we deduce that $\int_{\mathfrak{X}} \mathcal{C}$ has pullbacks that are computed in a similar manner to pushouts. Note that $f^* \colon \mathcal{C}(U) \to \mathcal{C}(V)$ preserves the initial object by condition (3). To show that f^* preserves weakly contractible colimits, it suffices to show that pullback along $(V, *) \to (U, *)$ in $\int_{\mathfrak{X}} \mathcal{C}$ does, since the canonical functors $\mathcal{C}(U) \to \int_{\mathfrak{X}} \mathcal{C}$ are conservative and preserve weakly contractible colimits. Once this is done, we will know that $\mathcal{C}$ is $\mathrm{Pr}^{\mathrm{L,R}}$-valued and that $\int_{\mathfrak{X}} \mathcal{C}$ is presentable, by [1, Theorem 1.3].

It thus remains to show that colimits in $\int_{\mathfrak{X}} \mathcal{C}$ are van Kampen. The statement for coproducts is straightforward, so we only consider pushouts. We will use the factorization system on $\int_{\mathfrak{X}} \mathcal{C}$ induced by the cocartesian fibration $\int_{\mathfrak{X}} \mathcal{C} \to \mathfrak{X}$: any map $(V, d) \to (U, c)$ factors uniquely as $(V, d) \to (U, f_!(d)) \to (U, c)$ where the first map is cocartesian and the second one is vertical.

We must show that the functor

$$\int_{\mathfrak{X}} \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}, \quad (U, c) \mapsto \left(\int_{\mathfrak{X}} \mathcal{C}\right)_{/(U,c)} \simeq \int_{\mathfrak{X}_{/U}} \mathcal{C}_{/c},$$

preserves pushouts. Consider a span $(U, c) \leftarrow (W, e) \to (V, d)$ in $\int_{\mathfrak{X}} \mathcal{C}$. Since the canonical map

$$\int_{\mathfrak{X}_{/U \amalg_W V}} \mathcal{C}_{/u_! c \amalg_{w_! e} v_! d} \longrightarrow \int_{\mathfrak{X}_{/U}} \mathcal{C}_{/c} \times_{\int_{\mathfrak{X}_{/W}} \mathcal{C}_{/e}} \int_{\mathfrak{X}_{/V}} \mathcal{C}_{/d}$$

is a map of cartesian fibrations over $\mathfrak{X}_{/U \amalg_W V} \simeq \mathfrak{X}_{/U} \times_{\mathfrak{X}_{/W}} \mathfrak{X}_{/V}$, it suffices to show that it is a fiberwise equivalence. By (2a) and (3), it suffices to consider the fiber over $U \amalg_W V$, which is

$$\mathcal{C}(U \amalg_W V)_{/u_! c \amalg_{w_! e} v_! d} \longrightarrow \mathcal{C}(U)_{/c} \times_{\mathcal{C}(W)_{/e}} \mathcal{C}(V)_{/d}. \qquad (*)$$

Decomposing a given span in $\int_{\mathfrak{X}} \mathcal{C}$ using the cocartesian factorization system, we see that it suffices to consider two types of spans:

(i) a vertical span $(U, c) \leftarrow (U, e) \rightarrow (U, d)$ in $\mathcal{C}(U)$;

(ii) a span of the form $(U, f_! e) \xleftarrow{f} (W, e) \rightarrow (V, d)$.

In case (i), the map $(*)$ is an equivalence by (1b). In case (ii), we consider the following cube, where $P = U \amalg_W V$ and $v \colon V \rightarrow P$ is the canonical map:

$$
\begin{array}{ccc}
\mathcal{C}(P)_{/0} & \longrightarrow & \mathcal{C}(U)_{/0} \\
\mathcal{C}(P)_{/v_! d} & \longrightarrow \mathcal{C}(U)_{/f_! e} & \\
& \mathcal{C}(V)_{/0} \longrightarrow & \mathcal{C}(W)_{/0} \\
\mathcal{C}(V)_{/d} & \longrightarrow & \mathcal{C}(W)_{/e}
\end{array}
$$

The lateral faces are cartesian by (2c). The back face is cartesian since $\mathcal{C}$ is a sheaf whose restriction maps preserve initial objects. Hence, the front face is cartesian, as desired. ∎

1.3. REMARK. The conditions of Theorem 1.2 are almost necessary. Suppose that $\mathcal{C}(*)$ has initial and final objects that restrict to initial and final objects of $\mathcal{C}(U)$ for every $U \in \mathfrak{X}$. Under this mild assumption, if $\int_{\mathfrak{X}} \mathcal{C}$ is an ∞-topos, then conditions (1) and (2a) hold with "pushouts" replaced by "weakly contractible colimits", and condition (2c) and (3) hold provided (2b) does. Thus, (2b) is the only condition that may not be necessary for $\int_{\mathfrak{X}} \mathcal{C}$ to be an ∞-topos. However, if $\mathcal{C}$ is the sheaf associated with a presentable ∞-category, it is clear that (1a), (2a), (2b), and (3) always hold; in that case, therefore, $\int_{\mathfrak{X}} \mathcal{C}$ is an ∞-topos iff (1b) and (2c) hold.

1.4. EXAMPLE. Let $\mathcal{C}$ be a sheaf on $\mathfrak{X}$ satisfying the assumptions of Theorem 1.2. Then the following sheaves of ∞-categories on $\mathfrak{X}$ also satisfy the assumptions of Theorem 1.2:

1. the subsheaf $L\mathcal{C} \subset \mathcal{C}$ for any $L \colon \mathcal{C} \rightarrow \mathcal{C}$ that is objectwise an accessible left exact localization functor;

2. the sheaf $\mathrm{Fun}(\mathcal{E}, \mathcal{C})$ for any small ∞-category $\mathcal{E}$;

3. the sheaf $\mathrm{Exc}^n(\mathcal{A}, \mathcal{C})$ for any $n \geq 0$ and any small ∞-category $\mathcal{A}$ with finite colimits and a final object (this follows from the previous two cases, noting that $\mathcal{C}$ is objectwise differentiable);

4. the sheaves $\mathcal{C}_{/c}$ and $\mathcal{C}_{c/}$ for any $c \in \mathcal{C}(*)$.

Specializing to sheaves of the form $U \mapsto \mathrm{Shv}_{\mathcal{C}}(\mathfrak{X}_{/U})$, we obtain the following characterization of ∞-loci:

1.5. COROLLARY. *Let $\mathcal{C}$ be a presentable ∞-category. The following assertions are equivalent:*

1. *$\mathcal{C}$ is an ∞-locus.*

2. *For every ∞-topos $\mathcal{X}$, $\int_{\mathcal{X}} \mathcal{C}$ is an ∞-topos.*

3. *Weakly contractible colimits in $\mathcal{C}$ are van Kampen.*

4. *Pushouts in $\mathcal{C}$ are van Kampen, and for every ∞-groupoid A and every functor $F \colon A^{\triangleleft} \to \mathcal{C}$ sending the initial vertex to the initial object of $\mathcal{C}$, the colimit of F is van Kampen.*

PROOF. The equivalence of (1) and (2) follows from

$$\int_{\mathcal{X}} \mathcal{C} \simeq \mathcal{X} \otimes \int_{\mathcal{S}} \mathcal{C}.$$

Note that the functor $\mathcal{C} \to \int_{\mathcal{S}} \mathcal{C}$, $c \mapsto (*, c)$, is fully faithful and preserves limits and weakly contractible colimits. The implication $(1) \Rightarrow (3)$ follows immediately, and $(3) \Rightarrow (4)$ is obvious. To prove $(4) \Rightarrow (1)$, we apply Theorem 1.2. The only nontrivial condition to check is (2c): given a morphism $f \colon V \to U$ in $\mathcal{S}$ and a functor $F \colon V \to \mathcal{C}$, we must show that the square

$$\begin{array}{ccc} \mathrm{Fun}(U, \mathcal{C})_{/f_! F} & \longrightarrow & \mathrm{Fun}(U, \mathcal{C}_{/0}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(V, \mathcal{C})_{/F} & \longrightarrow & \mathrm{Fun}(V, \mathcal{C}_{/0}) \end{array}$$

is cartesian. But this square is the limit over $u \in U$ of the squares

$$\begin{array}{ccc} \mathcal{C}_{/\operatorname{colim}_{v \in f^{-1}(u)} F(v)} & \longrightarrow & \mathcal{C}_{/0} \\ \downarrow & & \downarrow \\ \lim_{v \in f^{-1}(u)} \mathcal{C}_{/F(v)} & \longrightarrow & \lim_{v \in f^{-1}(u)} \mathcal{C}_{/0}, \end{array}$$

which are cartesian by assumption. ∎

1.6. EXAMPLE. An ∞-category is an ∞-topos iff it is an ∞-locus with a strictly initial object.

1.7. EXAMPLE. If $\mathcal{C}$ is a stable ∞-category, any weakly contractible colimit in $\mathcal{C}$ is van Kampen. This follows easily from the fact that cartesian squares in $\mathcal{C}$ are preserved by colimits. In particular, we obtain another proof that any presentable stable ∞-category is an ∞-locus.

1.8. EXAMPLE. Let $\mathcal{C}' \to \mathcal{C}$ be a conservative functor between presentable ∞-categories that preserves pullbacks and weakly contractible colimits. If $\mathcal{C}$ is an ∞-locus, so is $\mathcal{C}'$. For example:

1. If $\mathcal{Y}$ is an ∞-topos, the ∞-category $\mathcal{Y}_*^{\geq\infty}$ of pointed ∞-connective objects of $\mathcal{Y}$ is an ∞-locus.

2. If $\mathcal{C}$ is an ∞-locus and $\mathcal{A}$ is a small ∞-category with finite colimits and a final object, the ∞-category $\mathrm{Exc}_*^n(\mathcal{A}, \mathcal{C})$ of reduced n-excisive functors from $\mathcal{A}$ to $\mathcal{C}$ is an ∞-locus (use Example 1.4 (3)).

3. If $\mathcal{C}$ is an ∞-locus and $T\colon \mathcal{C} \to \mathcal{C}$ is an accessible pullback-preserving comonad, the ∞-category $\mathrm{coAlg}_T(\mathcal{C})$ of T-coalgebras is an ∞-locus. Similarly, if $T\colon \mathcal{C} \to \mathcal{C}$ is a monad that preserves weakly contractible colimits, then the ∞-category $\mathrm{Alg}_T(\mathcal{C})$ of T-algebras is an ∞-locus.

We conclude this note by considering an amusing family of ∞-topoi. Let $\mathcal{C}$ be an ∞-locus in which every truncated object is contractible, e.g., a presentable stable ∞-category. The projection $\int_{\mathcal{X}} \mathcal{C} \to \mathcal{X}$ admits left and right adjoints given by $U \mapsto (U, \emptyset)$ and $U \mapsto (U, *)$, so that the functor

$$\mathcal{X} \hookrightarrow \int_{\mathcal{X}} \mathcal{C}, \quad U \mapsto (U, *),$$

is an essential geometric embedding. Moreover, it is clear that an object $(U, c) \in \int_{\mathcal{X}} \mathcal{C}$ is hypercomplete if and only if U is hypercomplete and $c \simeq *$; in other words, $\mathcal{X}$ is a cotopological localization of $\int_{\mathcal{X}} \mathcal{C}$ [4, Definition 6.5.2.17]. In particular, $\int_{\mathcal{S}} \mathcal{C}$ is an ∞-topos whose hypercompletion is $\mathcal{S}$ and whose full subcategory of ∞-connective objects is $\mathcal{C}$.

References

[1] D. Gepner, R. Haugseng, and T. Nikolaus, *Lax colimits and free fibrations in ∞-categories*, Doc. Math. **22** (2017), pp. 1225–1266, preprint arXiv:1501.02161v2 [math.CT]

[2] A. Joyal, *New variations on the notion of topos*, talk at IHÉS, 2015, https://www.youtube.com/watch?v=qmCh9KwrQq8

[3] J. Lurie, *Higher Algebra*, September 2017, http://www.math.harvard.edu/~lurie/papers/HA.pdf

[4] ———, *Higher Topos Theory*, April 2017, http://www.math.harvard.edu/~lurie/papers/HTT.pdf

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