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CHARACTERIZATION OF LEFT COEXTENSIVE VARIETIES OF UNIVERSAL ALGEBRAS

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Abstract. An extensive category can be defined as a category $\mathcal{C}$ with finite coproducts such that for each pair X, Y of objects in $\mathcal{C}$, the canonical functor $+\colon \mathcal{C}/X \times \mathcal{C}/Y \longrightarrow \mathcal{C}/(X+Y)$ is an equivalence. We say that a category $\mathcal{C}$ with finite products is left coextensive if the dual canonical functor $\times\colon X/\mathcal{C} \times Y/\mathcal{C} \longrightarrow (X \times Y)/\mathcal{C}$ is fully faithful. We then give a syntactical characterization of left coextensive varieties of universal algebras.

An extensive category can be defined as a category $\mathcal{C}$ with finite coproducts such that for each pair X, Y of objects in $\mathcal{C}$, the canonical functor $+\colon \mathcal{C}/X \times \mathcal{C}/Y \longrightarrow \mathcal{C}/(X+Y)$ is an equivalence. A category that satisfies the dual condition is called coextensive. According to [1], the term "extensive category" was first used by W. F. Lawvere and S. Schanuel, although "categories with disjoint and universal coproducts" were considered by A. Grothendieck a long time ago, and there are related papers of various authors. Examples of extensive categories include the category **Cat** of all small categories and the category **Top** of topological spaces. An example of a coextensive category is the category **CRing** of commutative rings.

1. Definition. *A category $\mathcal{C}$ with finite products is called left coextensive if for each pair X, Y of objects in $\mathcal{C}$, the canonical functor $\times\colon X/\mathcal{C} \times Y/\mathcal{C} \longrightarrow (X \times Y)/\mathcal{C}$ is fully faithful. Equivalently let L be the left adjoint of $\times$ and let ε be the counit of this adjunction, then $L\colon (X \times Y)/\mathcal{C} \longrightarrow X/\mathcal{C} \times Y/\mathcal{C}$ sends $v\colon (X \times Y) \longrightarrow Z$ to the pair of canonical maps $(i_1, i_2)\colon (X, Y) \longrightarrow (X +_{X\times Y} Z, Y +_{X \times Y} Z)$ and $\mathcal{C}$ is left coextensive if and only if ε is a natural isomorphism.*

The use of the counit ε in this definition is the motivation for the name left coextensive. Furthermore, as follows from the proof of Proposition 2.2 of [1], $\mathcal{C}$ is left coextensive if and only if for any $A, B \in \mathcal{C}$ and any pair of morphisms $f \times g\colon X \times Y \longrightarrow A \times B$ the following diagram is a pushout.

$$\begin{tikzcd}
X \times Y \arrow[r, "\pi_1"] \arrow[d, "f\times g"'] & X \arrow[d, "f"] \\
A \times B \arrow[r, "p_1"'] & A
\end{tikzcd}$$

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Let $\mathcal{C}$ be a variety of universal algebras and let $F(X)$ denote the free algebra in $\mathcal{C}$ on the set X. In particular we shall use $F(\{x\})$ which is the algebra consisting of terms of at most one variable. We shall also use $F(\emptyset)$ which is the algebra consisting of all constant terms and is the initial object in $\mathcal{C}$. When $\mathcal{C}$ is left coextensive we have that $F(\emptyset)$ is non-empty which can be seen by taking $X = Y = B = F(\emptyset)$ in the above diagram. We shall fix a constant $0 \in F(\emptyset)$.

2. Proposition. *Let $\mathcal{C}$ be a variety, the following statements are equivalent:*

1. *$\mathcal{C}$ is left coextensive.*
2. *For all X, Y in $\mathcal{C}$ and any $x \in X$ and $y \in Y$, $((x, y), (x, 0)) \in C$ where C is the congruence on $X \times Y$ generated by the relation $R = \{((a, b), (a, c)) \in F(\emptyset)^4 \mid b = 0 \text{ or } c = 0\}$.*
3. *There exists a natural number n such that $((x, x), (x, 0)) \in Q^n$, where Q is the reflexive homomorphic relation on $F(\{x\}) \times F(\{x\})$ generated by R.*

Proof. First note that by Proposition 4.1 of [1] we need only consider $X = Y = F(\emptyset)$ in the above diagram. Then the diagram is a pushout if and only if for any $m\colon A \times B \longrightarrow C$ and $n\colon X \longrightarrow C$ with $m(f \times g) = n\pi_1$ there exists a unique $\phi\colon A \longrightarrow C$ with $\phi p_1 = m$ and $\phi f = n$. However, for $X = Y = F(\emptyset)$ to give such a pair (n, m) is simply to give m such that $m(a, b) = m(a, 0)$ for all $(a, b) \in F(\emptyset)^2$. Such a ϕ exists exactly when $m(a, b) = m(a, 0)$ for all $(a, b) \in A \times B$. Therefore $\mathcal{C}$ is left coextensive if and only if for any A, B and $m\colon A \times B \longrightarrow C$ we have that $m(a, b) = m(a, 0)$ for all $(a, b) \in F(\emptyset)^2$ implies that $m(a, b) = m(a, 0)$ for all $(a, b) \in A \times B$. Written in terms of congruences this is simply statement 2 of the proposition. To attain statement 3 from statement 2 note that it is sufficient to consider $A = B = F(\{x\})$ and that the congruence generated by some symmetric relation R is simply the transitive closure of the reflexive homomorphic relation generated by R. ∎

3. Theorem. *A variety of universal algebras $\mathcal{C}$ is left coextensive if and only if there exist $(n+m)$-ary terms $u_0, \ldots, u_k$, unary terms $t_0, \ldots, t_m, t'_0, \ldots, t'_m \in F(\{x\})$, and constants $e_0, \ldots, e_n, e'_0, \ldots, e'_n, e''_0, \ldots, e''_n \in F(\emptyset)$ such that $u_0 = x, u_k = 0$ and for all $0 \le i < k$ the following identities hold:*

$$u_i(t_1, t_2, \ldots, t_m, e_1, e_2, \ldots, e_n) = x$$
$$u_i(t'_1, t'_2, \ldots, t'_m, e'_1, e'_2, \ldots, e'_n) = u_{i+1}(t'_1, t'_2, \ldots, t'_m, e''_1, e''_2, \ldots, e''_n)$$

Proof. By Proposition 2, $\mathcal{C}$ is left coextensive if and only if $((x, x), (x, 0)) \in Q^n$ for some natural n, which is true if and only if there exist $a_0, \ldots, a_k \in F(\{x\})$ such that $a_0 = x$, $a_k = 0$ and $((x, a_i), (x, a_{i+1})) \in Q$ for $i < k$. But $((x, a_i), (x, a_{i+1})) \in Q$ if and only if for some term u_i, terms $t_0, \ldots, t_{m_i}, t'_0, \ldots, t'_{m_i} \in F(\{x\})$, and constants $e_0, \ldots, e_{n_i}, e'_0, \ldots, e'_{n_i}, e''_0, \ldots, e''_{n_i} \in F(\emptyset)$ we have the following equalities:

$$(x, a_i) = u_i((t_1, t'_1), \ldots, (t_{m_i}, t'_{m_i}), (e_1, e''_1), \ldots, (e_{n_i}, e''_{n_i}))$$
$$(x, a_{i+1}) = u_i((t_1, t'_1), \ldots, (t_{m_i}, t'_{m_i}), (e_1, e'_1), \ldots, (e_{n_i}, e'_{n_i}))$$

Furthermore we can assume that the t, t', e, e', e'' terms are the same for each i. Using this simplified notation gives, for all $0 \leq i < k$:

$$u_i(t_1, \ldots, t_m, e_1, \ldots, e_n) = x$$
$$u_i(t'_1, \ldots, t'_m, e'_1, \ldots, e'_n) = a_{i+1} = u_{i+1}(t'_1, \ldots, t'_m, e''_1, \ldots, e''_n)$$

as required. ■

4. Example. Let $\mathcal{C}$ be a variety of universal algebras and suppose that the algebraic theory of $\mathcal{C}$ has constants 0 and 1 and a binary term u with $u(x, 0) = 0$ and $u(x, 1) = x$. Then let $t_1 = t'_1 = x, e_1 = e''_1 = 1, e'_1 = 0, u_0 = x, u_1 = u, u_2 = 0$. Then we have that the required syntactic condition holds and so $\mathcal{C}$ is left coextensive. Clearly this is true when $\mathcal{C}$ is the variety of rings. However, unlike the variety of rings, $\mathcal{C}$ is not co-extensive in general, i.e., $\times\colon X/\mathcal{C} \times Y/\mathcal{C} \longrightarrow (X \times Y)/\mathcal{C}$ is fully faithful but not necessarily an equivalence. Therefore left coextensivity is different to coextensivity.

5. Remark. It should be noted that this characterization is similar in a sense to [2] in spite of the fact that the varieties which satisfy the conditions characterized in each paper are completely different. It should also be noted that the paper [3] gives a syntactic characterization of coextensive varieties, however the meaning of syntactic characterization in [3] is different from the meaning in this paper.

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