

CHARACTERIZATION OF LEFT COEXTENSIVE VARIETIES OF UNIVERSAL ALGEBRAS

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ABSTRACT. An extensive category can be defined as a category \mathcal{C} with finite coproducts such that for each pair X, Y of objects in \mathcal{C} , the canonical functor $+: \mathcal{C}/X \times \mathcal{C}/Y \longrightarrow \mathcal{C}/(X + Y)$ is an equivalence. We say that a category \mathcal{C} with finite products is left coextensive if the dual canonical functor $\times: X/\mathcal{C} \times Y/\mathcal{C} \longrightarrow (X \times Y)/\mathcal{C}$ is fully faithful. We then give a syntactical characterization of left coextensive varieties of universal algebras.

An extensive category can be defined as a category \mathcal{C} with finite coproducts such that for each pair X, Y of objects in \mathcal{C} , the canonical functor $+: \mathcal{C}/X \times \mathcal{C}/Y \longrightarrow \mathcal{C}/(X + Y)$ is an equivalence. A category that satisfies the dual condition is called coextensive. According to [1], the term “extensive category” was first used by W. F. Lawvere and S. Schanuel, although “categories with disjoint and universal coproducts” were considered by A. Grothendieck a long time ago, and there are related papers of various authors. Examples of extensive categories include the category **Cat** of all small categories and the category **Top** of topological spaces. An example of a coextensive category is the category **CRing** of commutative rings.

1. **DEFINITION.** A category \mathcal{C} with finite products is called left coextensive if for each pair X, Y of objects in \mathcal{C} , the canonical functor $\times: X/\mathcal{C} \times Y/\mathcal{C} \longrightarrow (X \times Y)/\mathcal{C}$ is fully faithful. Equivalently let L be the left adjoint of \times and let ε be the counit of this adjunction, then $L: (X \times Y)/\mathcal{C} \longrightarrow X/\mathcal{C} \times Y/\mathcal{C}$ sends $v: (X \times Y) \longrightarrow Z$ to the pair of canonical maps $(i_1, i_2): (X, Y) \longrightarrow (X +_{X \times Y} Z, Y +_{X \times Y} Z)$ and \mathcal{C} is left coextensive if and only if ε is a natural isomorphism.

The use of the counit ε in this definition is the motivation for the name left coextensive. Furthermore, as follows from the proof of Proposition 2.2 of [1], \mathcal{C} is left coextensive if and only if for any $A, B \in \mathcal{C}$ and any pair of morphisms $f \times g: X \times Y \longrightarrow A \times B$ the following diagram is a pushout.

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{\pi_1} & X \\
 \downarrow f \times g & & \downarrow f \\
 A \times B & \xrightarrow{p_1} & A
 \end{array}$$

Received by the editors 2019-06-04 and, in final form, 2019-10-15.

Transmitted by Tim Van der Linden. Published on 2019-10-18.

2010 Mathematics Subject Classification: 18A30, 08B05.

Key words and phrases: Coextensivity, Universal Algebra, Syntactic Characterization.

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Let \mathcal{C} be a variety of universal algebras and let $F(X)$ denote the free algebra in \mathcal{C} on the set X . In particular we shall use $F(\{x\})$ which is the algebra consisting of terms of at most one variable. We shall also use $F(\emptyset)$ which is the algebra consisting of all constant terms and is the initial object in \mathcal{C} . When \mathcal{C} is left coextensive we have that $F(\emptyset)$ is non-empty which can be seen by taking $X = Y = B = F(\emptyset)$ in the above diagram. We shall fix a constant $0 \in F(\emptyset)$.

2. PROPOSITION. *Let \mathcal{C} be a variety, the following statements are equivalent:*

1. \mathcal{C} is left coextensive.
2. For all X, Y in \mathcal{C} and any $x \in X$ and $y \in Y$, $((x, y), (x, 0)) \in C$ where C is the congruence on $X \times Y$ generated by the relation $R = \{((a, b), (a, c)) \in F(\emptyset)^4 \mid b = 0 \text{ or } c = 0\}$.
3. There exists a natural number n such that $((x, x), (x, 0)) \in Q^n$, where Q is the reflexive homomorphic relation on $F(\{x\}) \times F(\{x\})$ generated by R .

PROOF. First note that by Proposition 4.1 of [1] we need only consider $X = Y = F(\emptyset)$ in the above diagram. Then the diagram is a pushout if and only if for any $m: A \times B \rightarrow C$ and $n: X \rightarrow C$ with $m(f \times g) = n\pi_1$ there exists a unique $\phi: A \rightarrow C$ with $\phi p_1 = m$ and $\phi f = n$. However, for $X = Y = F(\emptyset)$ to give such a pair (n, m) is simply to give m such that $m(a, b) = m(a, 0)$ for all $(a, b) \in F(\emptyset)^2$. Such a ϕ exists exactly when $m(a, b) = m(a, 0)$ for all $(a, b) \in A \times B$. Therefore \mathcal{C} is left coextensive if and only if for any A, B and $m: A \times B \rightarrow C$ we have that $m(a, b) = m(a, 0)$ for all $(a, b) \in F(\emptyset)^2$ implies that $m(a, b) = m(a, 0)$ for all $(a, b) \in A \times B$. Written in terms of congruences this is simply statement 2 of the proposition. To attain statement 3 from statement 2 note that it is sufficient to consider $A = B = F(\{x\})$ and that the congruence generated by some symmetric relation R is simply the transitive closure of the reflexive homomorphic relation generated by R . ■

3. THEOREM. *A variety of universal algebras \mathcal{C} is left coextensive if and only if there exist $(n + m)$ -ary terms u_0, \dots, u_k , unary terms $t_0, \dots, t_m, t'_0, \dots, t'_m \in F(\{x\})$, and constants $e_0, \dots, e_n, e'_0, \dots, e'_n, e''_0, \dots, e''_n \in F(\emptyset)$ such that $u_0 = x, u_k = 0$ and for all $0 \leq i < k$ the following identities hold:*

$$u_i(t_1, t_2, \dots, t_m, e_1, e_2, \dots, e_n) = x$$

$$u_i(t'_1, t'_2, \dots, t'_m, e'_1, e'_2, \dots, e'_n) = u_{i+1}(t'_1, t'_2, \dots, t'_m, e''_1, e''_2, \dots, e''_n)$$

PROOF. By Proposition 2, \mathcal{C} is left coextensive if and only if $((x, x), (x, 0)) \in Q^n$ for some natural n , which is true if and only if there exist $a_0, \dots, a_k \in F(\{x\})$ such that $a_0 = x, a_k = 0$ and $((x, a_i), (x, a_{i+1})) \in Q$ for $i < k$. But $((x, a_i), (x, a_{i+1})) \in Q$ if and only if for some term u_i , terms $t_0, \dots, t_{m_i}, t'_0, \dots, t'_{m_i} \in F(\{x\})$, and constants $e_0, \dots, e_{n_i}, e'_0, \dots, e'_{n_i}, e''_0, \dots, e''_{n_i} \in F(\emptyset)$ we have the following equalities:

$$\begin{aligned}(x, a_i) &= u_i((t_1, t'_1), \dots, (t_{m_i}, t'_{m_i}), (e_1, e''_1), \dots, (e_{n_i}, e''_{n_i})) \\ (x, a_{i+1}) &= u_i((t_1, t'_1), \dots, (t_{m_i}, t'_{m_i}), (e_1, e'_1), \dots, (e_{n_i}, e'_{n_i}))\end{aligned}$$

Furthermore we can assume that the t, t', e, e', e'' terms are the same for each i . Using this simplified notation gives, for all $0 \leq i < k$:

$$\begin{aligned}u_i(t_1, \dots, t_m, e_1, \dots, e_n) &= x \\ u_i(t'_1, \dots, t'_m, e'_1, \dots, e'_n) &= a_{i+1} = u_{i+1}(t'_1, \dots, t'_m, e''_1, \dots, e''_n)\end{aligned}$$

as required. ■

4. **EXAMPLE.** Let \mathcal{C} be a variety of universal algebras and suppose that the algebraic theory of \mathcal{C} has constants 0 and 1 and a binary term u with $u(x, 0) = 0$ and $u(x, 1) = x$. Then let $t_1 = t'_1 = x, e_1 = e''_1 = 1, e'_1 = 0, u_0 = x, u_1 = u, u_2 = 0$. Then we have that the required syntactic condition holds and so \mathcal{C} is left coextensive. Clearly this is true when \mathcal{C} is the variety of rings. However, unlike the variety of rings, \mathcal{C} is not co-extensive in general, i.e., $\times: X/\mathcal{C} \times Y/\mathcal{C} \longrightarrow (X \times Y)/\mathcal{C}$ is fully faithful but not necessarily an equivalence. Therefore left coextensivity is different to coextensivity.

5. **REMARK.** It should be noted that this characterization is similar in a sense to [2] in spite of the fact that the varieties which satisfy the conditions characterized in each paper are completely different. It should also be noted that the paper [3] gives a syntactic characterization of coextensive varieties, however the meaning of syntactic characterization in [3] is different from the meaning in this paper.

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