Theory and Applications of Categories, Vol. 34, No. 31, 2019, pp. 993–1035.

ADJUNCTION UP TO AUTOMORPHISM

D. TAMBARA

ABSTRACT. We say a set-valued functor on a category is nearly representable if it is a quotient of a representable functor by a group of automorphisms. A distributor is a set-valued functor in two arguments, contravariant in one argument and covariant in the other. We say a distributor is slicewise nearly representable if it is nearly representable in either of the arguments whenever the other argument is fixed. We consider such a distributor a weak analogue of adjunction. Under a finiteness assumption on the domain categories, we show that every slicewise nearly representable functor is a composite of two distributors, each of which may be considered as a weak analogue of (co-)reflective adjunction.

1. Introduction

One of several equivalent presentations of adjunction between categories $\mathcal{B}$ and $\mathcal{C}$ is to give a functor $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ whose slices $L(x, -)$ for every $x \in \mathcal{B}$ and $L(-, y)$ for every $y \in \mathcal{C}$ are representable. Indeed, given such L, we take isomorphisms $L(x, -) \cong \mathrm{Hom}_{\mathcal{C}}(F(x), -)$ and $L(-, y) \cong \mathrm{Hom}_{\mathcal{B}}(-, G(y))$; then we have $\mathrm{Hom}_{\mathcal{C}}(F(x), y) \cong L(x, y) \cong \mathrm{Hom}_{\mathcal{B}}(x, G(y))$, hence a pair of adjoint functors $F: \mathcal{B} \to \mathcal{C}$ and $G: \mathcal{C} \to \mathcal{B}$. A set-valued functor on $\mathcal{B}^{\mathrm{op}} \times \mathcal{C}$ is called a distributor between $\mathcal{B}$ and $\mathcal{C}$. The object of the paper is to study a distributor satisfying the slice condition with representability replaced by a weaker property called near representability. We say a set-valued functor is *nearly representable* if it is a quotient of a representable functor by a group of automorphisms. We say a distributor $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is *slicewise nearly representable* if $L(x, -)$ for every $x \in \mathcal{B}$ is nearly representable and $L(-, y)$ for every $y \in \mathcal{C}$ is nearly representable. In [Tull, 2019] an instance of near representability is considered, the notion named "phased coproduct", which seems to arise from a construction in quantum theory. As for slicewise nearly representable distributors we do not know at present natural occurrences, but we intend here to develop a theory for them analogous to the theory of adjunction.

Our main result is that under a certain finiteness assumption on $\mathcal{B}$ or $\mathcal{C}$ (fulfilled when $\mathcal{B}$ or $\mathcal{C}$ is finite), every slicewise nearly representable distributor $\mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is a composite of two distributors of special kind, each of which may be viewed as an analogue of adjunction for a (co-)reflective subcategory.

We thank Ross Street for pointing out Tull's recent article. We thank the anonymous referee for a number of useful suggestions on presentation of the manuscript.

Received by the editors 2019-03-25 and, in final form, 2019-09-30.

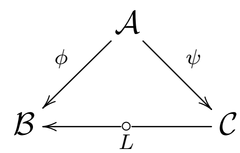
Transmitted by Ross Street. Published on 2019-10-03.

2010 Mathematics Subject Classification: 18A40.

Key words and phrases: distributor, adjoint, nearly representable.

To state the result precisely we define conditions (RH) and (LH). Let $\phi: \mathcal{C} \to \mathcal{D}$ be a functor. Put $G_x = \mathrm{Ker}(\phi: \mathrm{Aut}(x) \to \mathrm{Aut}(\phi(x)))$ for $x \in \mathcal{C}$. Condition (RH) for ϕ is stated as: For every $y \in \mathcal{D}$ there exists $x \in \mathcal{C}$ such that $\phi(x) = y$ and for every $x' \in \mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(x', x)/G_x \to \mathrm{Hom}_{\mathcal{D}}(\phi(x'), y)$ induced by ϕ is bijective. When $G_x = 1$ for all x, the condition reduces to saying that there exists a functor $\mathcal{D} \to \mathcal{C}$ which is a right adjoint and right inverse of ϕ. Dually condition (LH) is defined.

We also need some language of distributor. Given functors $\phi: \mathcal{A} \to \mathcal{B}$ and $\psi: \mathcal{A} \to \mathcal{C}$, we have the induced distributors $(1 \times \phi)^*\mathrm{Hom}_{\mathcal{B}}: \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Set}$ and $(\psi \times 1)^*\mathrm{Hom}_{\mathcal{C}}: \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$: the former takes (x, z) to $\mathrm{Hom}_{\mathcal{B}}(x, \phi(z))$ and the latter (z, y) to $\mathrm{Hom}_{\mathcal{C}}(\psi(z), y)$. By composition we then have the distributor $(1 \times \phi)^*\mathrm{Hom}_{\mathcal{B}} \otimes_{\mathcal{A}} (\psi \times 1)^*\mathrm{Hom}_{\mathcal{C}}: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$. For an arbitrary distributor $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ we say L is *tabulated* by (ϕ, ψ) if L is isomorphic to $(1 \times \phi)^*\mathrm{Hom}_{\mathcal{B}} \otimes_{\mathcal{A}} (\psi \times 1)^*\mathrm{Hom}_{\mathcal{C}}$. This terminology is suggested by the referee, based on a usage in [Freyd and Scedrov, 1990, p.37]. A picture of the tabulation may be a diagram



in Borceux's notation.

Suppose that $\mathcal{C}$ does not have an infinite sequence $(g_i)_{i \geq 0}$ of morphisms $g_i: y_{i+1} \to y_i$ which are split epimorphisms but not isomorphisms. Our theorem states that a distributor $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is slicewise nearly representable if and only if L is tabulated by some pair (λ, μ) of a functor $\lambda: \mathcal{G} \to \mathcal{B}$ satisfying (LH) and a functor $\mu: \mathcal{G} \to \mathcal{C}$ satisfying (RH). We admit however that nature of functors satisfying (RH) is not yet fully understood.

The paper is organized as follows. In Section 2 we review some standard facts about distributors. In Section 3 we collect basic properties of nearly representable functors and slicewise nearly representable distributors. In Section 4 we introduce condition (RG) for a functor $\phi: \mathcal{C} \to \mathcal{D}$, which assures that $\mathrm{Hom}_{\mathcal{D}}(\phi(-), y)$ is nearly representable for every $y \in \mathcal{D}$. It roughly means that the hom-sets of $\mathcal{D}$ are quotients of the hom-sets of $\mathcal{C}$ by groups. In Section 5 we discuss condition (RH) for a functor stated above. Condition (RH) is weaker than (RG). In Section 6, with a distributor $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ we associate certain categories of triples (x, y, a) for $x \in \mathcal{B}$, $y \in \mathcal{C}$, and $a \in L(x, y)$. They are used in later constructions. In Section 7, given a slicewise nearly representable distributor $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$, we construct morphisms η_x in $\mathcal{B}$ and ϵ_y in $\mathcal{C}$, which are analogous to unit and counit for adjunction. Under the finiteness assumption stated above, we show that certain η_x and ϵ_y are isomorphisms.

The proof of the main result is given in Sections 8–10. Let $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ be a slicewise nearly representable distributor. We construct in Section 8 certain subcategories $\mathcal{B}_0 \subset \mathcal{B}$, $\mathcal{C}_0 \subset \mathcal{C}$, and quotient categories $\bar{\mathcal{B}}_0$, $\bar{\mathcal{C}}_0$. We then define three distributors

$$M: \mathcal{B}^{\mathrm{op}} \times \bar{\mathcal{B}}_0 \to \mathbf{Set},\ K: \bar{\mathcal{B}}_0^{\mathrm{op}} \times \bar{\mathcal{C}}_0 \to \mathbf{Set},\ N: \bar{\mathcal{C}}_0^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set},$$

and show that K yields an equivalence $\bar{\mathcal{B}}_0 \simeq \bar{\mathcal{C}}_0$. In Section 9, under the finiteness assumption we show that L is the composite of the three distributors:

$$L \cong M \otimes_{\bar{\mathcal{B}}_0} K \otimes_{\bar{\mathcal{C}}_0} N.$$

In Section 10 we show that N is tabulated by a pair of a functor satisfying (LH) and a functor satisfying (RG). Dually we have a similar tabulation of M. Combining these, we obtain a desired tabulation of L

$$L \cong (1 \times \lambda)^* \mathrm{Hom}_{\mathcal{B}} \otimes_{\mathcal{G}} (\mu \times 1)^* \mathrm{Hom}_{\mathcal{C}},$$

where $\lambda \colon \mathcal{G} \to \mathcal{B}$ is a functor satisfying (LH) and $\mu \colon \mathcal{G} \to \mathcal{C}$ is a functor satisfying (RH).

A set-valued functor F is said to be *familially representable* if F is a sum of representable functors [Carboni and Johnstone, 1995]. As an obvious generalization we have the notion of a familially nearly representable functor and also that of a slicewise familially nearly representable distributor. In Section 11 we show that every slicewise familially nearly representable distributor is a composite of three distributors: a distributor coming from a discrete fibration, a slicewise nearly representable distributor, and a distributor coming from a discrete cofibration. Thus the structure of a slicewise familially nearly representable distributor can be understood to some extent from that of a slicewise nearly representable distributor.

2. Preliminaries

We review here some formal operations on functors and standard facts about distributors.

The category of sets is denoted by **Set**. All categories written in script letters such as $\mathcal{C}$ are small. For a category $\mathcal{C}$ we write $\mathrm{Hom}_{\mathcal{C}}(x, y) = \mathcal{C}(x, y)$. The category of functors $\mathcal{C} \to \mathbf{Set}$ is denoted by $[\mathcal{C}, \mathbf{Set}]$. When $F \colon \mathcal{C} \to \mathbf{Set}$ is a functor, the map $F(f) \colon F(x) \to F(x')$ for a morphism $f \colon x \to x'$ is abbreviated as f_*. When $G \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is a functor, the map $G(f) \colon G(x') \to G(x)$ for a morphism $f \colon x \to x'$ is abbreviated as f^*.

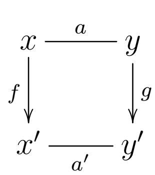
Let $L \colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ be a functor. Such a functor is called a distributor [Borceux, 1994]. For a morphism $f \colon x \to x'$ of $\mathcal{B}$ and an object $y \in \mathcal{C}$ we have the map

$$L(f, 1_y) \colon L(x', y) \to L(x, y).$$

We abbreviate this map as f^*. Similarly for a morphism $g \colon y \to y'$ of $\mathcal{C}$ and an object $x \in \mathcal{B}$ we have the map

$$L(1_x, g) \colon L(x, y) \to L(x, y'),$$

which we abbreviated as g_*. For $a \in L(x, y)$, $a' \in L(x', y')$ and morphisms $f \colon x \to x'$, $g \colon y \to y'$, the equality $f^*(a') = g_*(a)$ in $L(x, y')$ may be pictured as the diagram



For a category $\mathcal{C}$ we have the distributor $\mathrm{Hom}_{\mathcal{C}}: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ taking (x, y) to $\mathrm{Hom}_{\mathcal{C}}(x, y)$.

Let $\phi: \mathcal{C} \to \mathcal{D}$ be a functor. For a functor $G: \mathcal{D} \to \mathbf{Set}$ the composite functor $G \circ \phi: \mathcal{C} \to \mathbf{Set}$ is also denoted by $\phi^* G$. The assignment $G \mapsto \phi^* G$ defines the functor $\phi^*: [\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$ between functor categories. This has a left adjoint functor $[\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$, denoted by $\phi_!$. It operates on a hom-functor as

$$\phi_!(\mathcal{C}(x, -)) \cong \mathcal{D}(\phi(x), -).$$

These notations are used for contravariant functors and distributors as well. For example, given $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$, $\phi: \mathcal{C} \to \mathcal{D}$ and $\psi: \mathcal{A} \to \mathcal{B}$, one has $(1 \times \phi)_! L: \mathcal{B}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ and $(\psi \times 1)^* L: \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$.

For functors $F: \mathcal{C} \to \mathbf{Set}$ and $G: \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ the so-called coend construction [Mac Lane, 1978] yields the set

$$\int^{x \in \mathcal{C}} F(x) \times G(x),$$

which we denote by $F \otimes_{\mathcal{C}} G$. For a functor $F: \mathcal{B} \to \mathbf{Set}$ and a distributor $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ one has the functor $F \otimes_{\mathcal{B}} L: \mathcal{C} \to \mathbf{Set}$ defined by

$$(F \otimes_{\mathcal{B}} L)(y) = F \otimes_{\mathcal{B}} L(-, y).$$

For distributors $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ and $M: \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$, the composition distributor $L \otimes_{\mathcal{C}} M: \mathcal{B}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ is defined by

$$(L \otimes_{\mathcal{C}} M)(x, z) = L(x, -) \otimes_{\mathcal{C}} M(-, z)$$

(denoted $L \circ M$ in [Borceux, 1994]).

The following two propositions are well-known.

2.1. Proposition. *For $F: \mathcal{C} \to \mathbf{Set}$ we have a natural isomorphism*

$$F \otimes_{\mathcal{C}} \mathcal{C}(-, x) \cong F(x).$$

2.2. Proposition. *Let $\phi: \mathcal{C} \to \mathcal{D}$ be a functor. We have natural isomorphisms*

$$\phi_! F \cong F \otimes_{\mathcal{C}} (\phi \times 1)^* \mathrm{Hom}_{\mathcal{D}}$$

for $F: \mathcal{C} \to \mathbf{Set}$, and

$$\phi^* G \cong G \otimes_{\mathcal{D}} (1 \times \phi)^* \mathrm{Hom}_{\mathcal{D}}$$

for $G: \mathcal{D} \to \mathbf{Set}$.

2.3. Proposition. *Let $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C}_2 \to \mathbf{Set}$, $M: {\mathcal{C}_1}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$, and $\gamma: \mathcal{C}_1 \to \mathcal{C}_2$ be functors. We have a natural isomorphism*

$$(1 \times \gamma)^* L \otimes_{\mathcal{C}_1} M \cong L \otimes_{\mathcal{C}_2} (\gamma \times 1)_! M.$$

PROOF. Using the isomorphisms of the preceding proposition and the associativity of composition, we proceed as

$$\begin{aligned}(1\times\gamma)^*L\otimes_{\mathcal{C}_1}M &\cong (L\otimes_{\mathcal{C}_2}(1\times\gamma)^*\mathrm{Hom}_{\mathcal{C}_2})\otimes_{\mathcal{C}_1}M\\ &\cong L\otimes_{\mathcal{C}_2}((1\times\gamma)^*\mathrm{Hom}_{\mathcal{C}_2}\otimes_{\mathcal{C}_1}M)\\ &\cong L\otimes_{\mathcal{C}_2}(\gamma\times 1)_!M\end{aligned}$$

to obtain the asserted isomorphism.

■

2.4. PROPOSITION. *Let $\phi\colon\mathcal{C}\to\mathcal{D}$ be a functor. Then we have a natural isomorphism*

$$(1\times\phi)_!\mathrm{Hom}_{\mathcal{C}}\cong(\phi\times 1)^*\mathrm{Hom}_{\mathcal{D}}$$

of functors on $\mathcal{C}^{\mathrm{op}}\times\mathcal{D}$, and a natural isomorphism

$$(\phi\times 1)_!\mathrm{Hom}_{\mathcal{C}}\cong(1\times\phi)^*\mathrm{Hom}_{\mathcal{D}}$$

of functors on $\mathcal{D}^{\mathrm{op}}\times\mathcal{C}$.

PROOF. For any $x\in\mathcal{C}$ we have

$$((1\times\phi)_!\mathrm{Hom}_{\mathcal{C}})(x,-)=\phi_!(\mathcal{C}(x,-))\cong\mathcal{D}(\phi(x),-)=((\phi\times 1)^*\mathrm{Hom}_{\mathcal{D}})(x,-).$$

Hence

$$(1\times\phi)_!\mathrm{Hom}_{\mathcal{C}}\cong(\phi\times 1)^*\mathrm{Hom}_{\mathcal{D}}.$$

■

2.5. PROPOSITION. *For functors $\lambda\colon\mathcal{A}\to\mathcal{B}$ and $\mu\colon\mathcal{A}\to\mathcal{C}$ we have a natural isomorphism*

$$(\lambda\times\mu)_!\mathrm{Hom}_{\mathcal{A}}\cong(1\times\lambda)^*\mathrm{Hom}_{\mathcal{B}}\otimes_{\mathcal{A}}(\mu\times 1)^*\mathrm{Hom}_{\mathcal{C}}$$

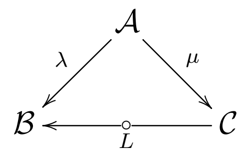
of functors on $\mathcal{B}^{\mathrm{op}}\times\mathcal{C}$.

PROOF.

$$\begin{aligned}(1\times\lambda)^*\mathrm{Hom}_{\mathcal{B}}\otimes_{\mathcal{A}}(\mu\times 1)^*\mathrm{Hom}_{\mathcal{C}} &\cong(\lambda\times 1)_!\mathrm{Hom}_{\mathcal{A}}\otimes_{\mathcal{A}}(1\times\mu)_!\mathrm{Hom}_{\mathcal{A}}\\ &\cong(\lambda\times\mu)_!(\mathrm{Hom}_{\mathcal{A}}\otimes_{\mathcal{A}}\mathrm{Hom}_{\mathcal{A}})\\ &\cong(\lambda\times\mu)_!\mathrm{Hom}_{\mathcal{A}}.\end{aligned}$$

■

If a distributor $L\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is isomorphic to the distributor $(\lambda \times \mu)_! \mathrm{Hom}_{\mathcal{A}}$ of the proposition, we say L is *tabulated* by (λ, μ). This may be pictured as a diagram



The word "tabulation" was originally used for binary relations on sets and for morphisms in allegories [Freyd and Scedrov, 1990].

Let $F\colon \mathcal{C} \to \mathbf{Set}$ be a functor. We recall the definition of the category of elements of F, which we denote by $\mathbf{E}(F)$. An object of $\mathbf{E}(F)$ is a pair (x, a) composed of $x \in \mathcal{C}$ and $a \in F(x)$. A morphism $(x, a) \to (x', a')$ in $\mathbf{E}(F)$ is a morphism $f\colon x \to x'$ in $\mathcal{C}$ such that $f_*(a) = a'$. The composition in $\mathbf{E}(F)$ is given by the composition in $\mathcal{C}$. The projection functor $\pi\colon \mathbf{E}(F) \to \mathcal{C}$ is given by $(x, a) \mapsto x$.

The following is well-known.

2.6. Proposition. *For any functor $M\colon \mathbf{E}(F) \to \mathbf{Set}$ and $x \in \mathcal{C}$ we have a natural bijection*

$$(\pi_! M)(x) \cong \coprod_{a \in F(x)} M(x, a).$$

The construction of the category of elements is adapted for a distributor: Given a distributor $L\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$, the category $\mathbb{E}(L)$ is defined as follows.

• An object of $\mathbb{E}(L)$ is a triple (x, y, a) composed of $x \in \mathcal{B}$, $y \in \mathcal{C}$, $a \in L(x, y)$.

• For objects (x, y, a) and (x_1, y_1, a_1), a morphism $(x, y, a) \to (x_1, y_1, a_1)$ is a pair (f, g) composed of $f \in \mathcal{B}(x, x_1)$ and $g \in \mathcal{C}(y, y_1)$ such that $f^*(a_1) = g_*(a)$.

• The composition in $\mathbb{E}(L)$ is defined componentwise.

• The identity morphism of an object (x, y, a) is $(1_x, 1_y)$.

We have the projection functors $\pi_1\colon \mathbb{E}(L) \to \mathcal{B}$ and $\pi_2\colon \mathbb{E}(L) \to \mathcal{C}$:

$$\begin{aligned} \pi_1\colon \quad & (x, y, a) \mapsto x, \\ \pi_2\colon \quad & (x, y, a) \mapsto y. \end{aligned}$$

By the definition of morphisms of $\mathbb{E}(L)$ we have a pullback diagram

$$\begin{array}{ccc} \mathbb{E}(L)((x, y, a), (x_1, y_1, a_1)) & \xrightarrow{\pi_2} & \mathcal{C}(y, y_1) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ \mathcal{B}(x, x_1) & \longrightarrow & L(x, y_1) \end{array}$$

where the right vertical arrow is the map $g \mapsto g_*(a)$, the lower horizontal arrow is the map $f \mapsto f^*(a_1)$.

The following fact is well-known but we include the proof.

2.7. Proposition. *For every distributor L we have an isomorphism*

$$(\pi_1 \times \pi_2)_! \mathrm{Hom}_{\mathbb{E}(L)} \cong L.$$

Thus every distributor has a canonical tabulation.

Proof. We shall establish a natural bijection

$$\mathrm{Hom}(L, M) \cong \mathrm{Hom}(\mathrm{Hom}_{\mathbb{E}(L)}, (\pi_1 \times \pi_2)^* M)$$

for any $M\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$. The asserted isomorphism will then follow by the adjunction between $(\pi_1 \times \pi_2)_!$ and $(\pi_1 \times \pi_2)^*$.

Firstly we have the natural bijection

$$\mathrm{Hom}(\mathrm{Hom}_{\mathbb{E}(L)}, (\pi_1 \times \pi_2)^* M) \cong \int_{\mathbb{E}(L)} (\pi_1 \times \pi_2)^* M$$

where the right-hand side denotes the end of the distributor $(\pi_1 \times \pi_2)^* M$.

An element of $\int_{\mathbb{E}(L)} (\pi_1 \times \pi_2)^* M$ is a family $\lambda = (\lambda_z)_{z \in \mathbb{E}(L)}$ composed of elements $\lambda_z \in ((\pi_1 \times \pi_2)^* M)(z, z)$ for $z \in \mathbb{E}(L)$ satisfying the condition that

$$h_*(\lambda_z) = h^*(\lambda_{z_1})$$

for every morphism $h\colon z \to z_1$ in $\mathbb{E}(L)$.

Write $z = (x, y, a)$, $z_1 = (x_1, y_1, a_1)$, $h = (f, g)$. Then

$$((\pi_1 \times \pi_2)^* M)(z, z) = M(x, y), \quad \lambda_z \in M(x, y),$$

and

$$\begin{aligned} h_*(\lambda_z) &= g_*(\lambda_{(x,y,a)}), \\ h^*(\lambda_{z_1}) &= f^*(\lambda_{(x_1,y_1,a_1)}). \end{aligned}$$

Therefore an element of $\int_{\mathbb{E}(L)} (\pi_1 \times \pi_2)^* M$ is a family $\lambda = (\lambda_{(x,y,a)})_{(x,y,a) \in \mathbb{E}(L)}$ composed of elements $\lambda_{(x,y,a)} \in M(x, y)$ for $(x, y, a) \in \mathbb{E}(L)$ satisfying the condition that

$$g_*(\lambda_{(x,y,a)}) = f^*(\lambda_{(x_1,y_1,a_1)})$$

for every morphism $(f, g)\colon (x, y, a) \to (x_1, y_1, a_1)$ in $\mathbb{E}(L)$.

As every morphism $(f, g)\colon (x, y, a) \to (x_1, y_1, a_1)$ is the composite of

$$(1, g)\colon (x, y, a) \to (x, y_1, g_*(a)) = (x, y_1, f^*(a_1))$$

and

$$(f, 1)\colon (x, y_1, f^*(a_1)) \to (x_1, y_1, a_1),$$

the above condition for $(\lambda_{(x,y,a)})$ is equivalent to the condition that

$$\begin{aligned} g_*(\lambda_{(x,y,a)}) &= \lambda_{(x,y_1,g_*(a))}, \\ f^*(\lambda_{(x_1,y_1,a_1)}) &= \lambda_{(x,y_1,f^*(a_1))} \end{aligned}$$

for every $g : y \to y_1$ and $f\colon x \to x_1$. This means that the family of the maps $t_{x,y}\colon L(x,y) \to M(x,y)$ given by $t_{x,y}(a) = \lambda_{(x,y,a)}$ defines a morphism $t\colon L \to M$. Thus we have a bijection

$$\int_{\mathbb{E}(L)} (\pi_1 \times \pi_2)^* M \cong \mathrm{Hom}(L, M),$$

which completes the proof. ∎

3. Nearly representable functors

In this section we review the definition of a nearly representable functor [Tambara, 2015] and give the definition of a slicewise nearly representable distributor.

Let $\mathcal{C}$ be a category and $F\colon \mathcal{C} \to \mathbf{Set}$ a functor.

Recall that F is said to be representable if there exist an object $v \in \mathcal{C}$ and an isomorphism $F \cong \mathcal{C}(v, -)$. Such an isomorphism is given by an element $a \in F(v)$. A pair (v, a) is then said to be *universal* for F.

When a group G acts on a set X, X/G denotes the quotient set (regardless of the side of the action). When a group G acts on a functor $F\colon \mathcal{C} \to \mathbf{Set}$, that is, when a homomorphism $G \to \mathrm{Aut}(F)$ or $G^{\mathrm{op}} \to \mathrm{Aut}(F)$ is given, F/G denotes the functor $\mathcal{C} \to \mathbf{Set}$ given by $(F/G)(x) = F(x)/G$.

3.1. Definition. *We say F is nearly representable if there exist an object $v \in \mathcal{C}$, a subgroup G of* $\mathrm{Aut}(v)$*, and an isomorphism $F \cong \mathcal{C}(v, -)/G$.*

3.2. Definition. *Let $v \in \mathcal{C}$ and $a \in F(v)$. We say (v, a) is nearly universal for F if there exists a subgroup G of* $\mathrm{Aut}(v)$ *such that G fixes a and the morphism $\mathcal{C}(v, -)/G \to F$ induced by a is an isomorphism. Namely (v, a, G) is required to satisfy the following:*

(1) $f_*(a) = a$ for every $f \in G$.

(2) For every $x \in \mathcal{C}$ and $b \in F(x)$ there exists $f\colon v \to x$ such that $b = f_*(a)$.

(3) For every $x \in \mathcal{C}$ and $f, f'\colon v \to x$, if $f_*(a) = f'_*(a)$, then there exists $g \in G$ such that $f = f'g$.

We note that (1) and (3) imply the following:

(4) $G = \mathrm{Aut}(v, a) = \mathrm{End}(v, a)$.

Here $\mathrm{Aut}(v, a)$ denotes the group $\{f \in \mathrm{Aut}(v) \mid f_*(a) = a\}$, and $\mathrm{End}(v, a)$ the monoid $\{f \in \mathrm{End}(v) \mid f_*(a) = a\}$. Indeed, let $f\colon v \to v$ and suppose $f_*(a) = a$. By (3) applied to $f' = 1_v$, there exists $g \in G$ such that $f = 1_v g$, whence $f \in G$.

The terminology is used for contravariant functors as well.

3.3. Proposition. *If (v, a) and (v', a') are both nearly universal for F, then there exists an isomorphism $h\colon v \to v'$ such that $a = h_*(a')$.*

Proof. Suppose that (v,a) and (v',a') are nearly universal for F. Put $G = \mathrm{Aut}(v,a)$ and $G' = \mathrm{Aut}(v',a')$. As (v,a) is nearly universal for F and $a' \in F(v')$, there exists $h: v \to v'$ such that $a' = h_*(a)$. As (v',a') is nearly universal for F and $a \in F(v)$, there exists $h': v' \to v$ such that $a = {h'}_*(a')$. Then $a = {h'}_* h_*(a) = (h'h)_*(a)$. As $G = \mathrm{End}(v,a)$, we have $h'h \in G$. Similarly $hh' \in G'$. Thus $h'h$ and hh' are both isomorphisms. Hence h is an isomorphism. ∎

3.4. Proposition. *Suppose that $F: \mathcal{C} \to \mathbf{Set}$ is a nearly representable functor. Let K be a subgroup of $\mathrm{Aut}(F)$. Then the quotient functor F/K is nearly representable.*

Proof. Let $F = \mathcal{C}(v,-)/G$ with $v \in \mathcal{C}$ and G a subgroup of $\mathrm{Aut}(v)$. Let N be the normalizer of G in $\mathrm{Aut}(v)$. Then by the Yoneda lemma one has a surjective homomorphism $N \to \mathrm{Aut}(F)$ (See [Tambara, 2015, Prop. 2.1] for details). Let $\tilde{K}$ be the inverse image of K under this map. Then $F/K = \mathcal{C}(v,-)/\tilde{K}$. Thus F/K is nearly representable. ∎

3.5. Proposition. *Let $\phi: \mathcal{C} \to \mathcal{C}'$ be a functor. If $F: \mathcal{C} \to \mathbf{Set}$ is nearly representable, then so is $\phi_! F: \mathcal{C}' \to \mathbf{Set}$.*

Proof. Suppose $F \cong \mathcal{C}(v,-)/G$. As $\phi_!$ preserves colimits and hom-functors, we have

$$\phi_! F \cong (\phi_! \mathcal{C}(v,-))/G \cong \mathcal{C}'(\phi(v),-)/\phi(G).$$

Here $\phi(G)$ is the image of G under $\phi: \mathrm{Aut}(v) \to \mathrm{Aut}(\phi(v))$. ∎

Let $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ be a distributor. Following the terminology "slicing" in [Eilenberg and Mac Lane, 1945, p.245], we call the functor $L(x,-): \mathcal{C} \to \mathbf{Set}$ for $x \in \mathcal{B}$ a *slice* of L, and similar for the functor $L(-,y): \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ for $y \in \mathcal{C}$.

3.6. Definition. *We say L is slicewise nearly representable if for every $x \in \mathcal{B}$ the functor $L(x,-): \mathcal{C} \to \mathbf{Set}$ is nearly representable and for every $y \in \mathcal{C}$ the functor $L(-,y): \mathcal{B}^{\mathrm{op}} \to$ $\mathbf{Set}$ is nearly representable.*

Let $u \in \mathcal{B}$, $v \in \mathcal{C}$, and $a \in L(u,v)$. Then we may use the phrase "(v,a) is nearly universal for $L(u,-)$" or "(u,a) is nearly universal for $L(-,v)$". The former means that there exists a subgroup G of $\mathrm{Aut}(v)$ such that G fixes a and the morphism $\mathcal{C}(v,-)/G \to L(u,-)$ induced by a is an isomorphism. The condition required for (u,v,a,G) amounts to the following:

(1) $\sigma_*(a) = a$ for every $\sigma \in G$.

(2) For every $y \in \mathcal{C}$ and $b \in L(u,y)$ there exists $g: v \to y$ such that $g_*(a) = b$.

(3) For every $y \in \mathcal{C}$ and $g, g': v \to y$, if $g_*(a) = g'_*(a)$, then there exists $\sigma \in G$ such that $g = g'\sigma$.

As a consequence of (1) and (3) we have $G = \mathrm{Aut}(v,a) = \mathrm{End}(v,a)$. Here $\mathrm{Aut}(v,a)$ denotes the group $\{\sigma \in \mathrm{Aut}(v) \mid \sigma_*(a) = a\}$.

That (u,a) is nearly universal for $L(-,v)$ means that there exists a subgroup G of $\mathrm{Aut}(u)$ such that G fixes a and the morphism $\mathcal{B}(-,u)/G \to L(-,v)$ induced by a is an isomorphism. The condition required for (u,v,a,G) amounts to the following:

(1) $\sigma^*(a) = a$ for every $\sigma \in G$.

(2) For every $x \in \mathcal{B}$ and $b \in L(x, v)$ there exists $f: x \to u$ such that $f^*(a) = b$.

(3) For every $x \in \mathcal{B}$ and $f, f': x \to u$, if $f^*(a) = f'^*(a)$, then there exists $\sigma \in G$ such that $f = \sigma f'$.

As a consequence of (1) and (3) we have $G = \mathrm{Aut}(u, a) = \mathrm{End}(u, a)$.

The following is immediate from the definition.

3.7. Proposition. *(i) If (v, a) is nearly universal for $L(u, -)$ and $f: u' \to u$ is an isomorphism, then $(v, f^*(a))$ is nearly universal for $L(u', -)$.*

(ii) If (v, a) is nearly universal for $L(u, -)$ and $h: v \to v'$ is an isomorphism, then $(v', h_*(a))$ is nearly universal for $L(u, -)$.

3.8. Proposition. *Let $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$, $M: \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ be distributors. If L and M are slicewise nearly representable, then so is $L \otimes_{\mathcal{C}} M$.*

Proof. Let $x \in \mathcal{B}$. Take an isomorphism $L(x, -) \cong \mathcal{C}(y, -)/G$ with $y \in \mathcal{C}$ and $G \subset \mathrm{Aut}(y)$. Then

$$\begin{aligned}(L \otimes_{\mathcal{C}} M)(x, -) &\cong L(x, -) \otimes_{\mathcal{C}} M \\ &\cong \mathcal{C}(y, -)/G \otimes_{\mathcal{C}} M \\ &\cong (\mathcal{C}(y, -) \otimes_{\mathcal{C}} M)/G \\ &\cong M(y, -)/G.\end{aligned}$$

Now $M(y, -)$ is nearly representable by assumption. As a quotient of a nearly representable functor, $M(y, -)/G$ is also nearly representable. Thus $(L \otimes_{\mathcal{C}} M)(x, -)$ is nearly representable.

By a similar argument we see that $(L \otimes_{\mathcal{C}} M)(-, z)$ is nearly representable for any $z \in \mathcal{D}$. ∎

4. Condition (G)

We introduce condition (RG) for a functor $\phi: \mathcal{C} \to \mathcal{D}$, which assures that $\mathrm{Hom}_{\mathcal{D}}(\phi(-), y): \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ for every $y \in \mathcal{D}$ is nearly representable. The condition roughly means that $\mathcal{D}$ is obtained by taking quotients by groups of automorphisms of objects of $\mathcal{C}$. A natural example of such a functor is found in group theory.

4.1. Definition. *Let $\phi: \mathcal{C} \to \mathcal{D}$ be a functor. For $x \in \mathcal{C}$ put $G_x = \mathrm{Ker}(\mathrm{Aut}(x) \to \mathrm{Aut}(\phi(x)))$. Condition (RG) for ϕ consists of the following:*

(1) ϕ is surjective on objects.

(2) For every $x, x' \in \mathcal{C}$ the map

$$\mathcal{C}(x', x)/G_x \to \mathcal{D}(\phi(x'), \phi(x))$$

induced by ϕ is bijective.

(2) is phrased as the natural morphism

$$\mathcal{C}(-,x)/G_x \to \phi^*(\mathcal{D}(-,\phi(x)))$$

in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is an isomorphism for every $x \in \mathcal{C}$. (1) and (2) imply that $\phi^*(\mathcal{D}(-,y))$ is nearly representable for every $y \in \mathcal{D}$.

4.2. Definition. *A functor $\phi\colon \mathcal{C} \to \mathcal{D}$ is called a surjective equivalence if ϕ is fully faithful and surjective on objects.*

Thus ϕ is a surjective equivalence if and only if ϕ satisfies (RG) and the groups G_x are trivial for all x.

The following is immediate from the definition.

4.3. Proposition. *The functors satisfying (RG) are closed under composition.*

Here is a construction of a functor satisfying (RG). Let $\mathcal{C}$ be a category. Suppose that for each object x in $\mathcal{C}$ a subgroup G_x of $\mathrm{Aut}(x)$ is given so that the following condition is satisfied.

($\star$) For every morphism $f\colon x' \to x$ in $\mathcal{C}$ and $v \in G_{x'}$, there exists $u \in G_x$ such that $fv = uf$.

This amounts to saying the action of $G_{x'}$ on $\mathcal{C}(x',x)/G_x$ is trivial for every $x, x' \in \mathcal{C}$.

We then define a category $\mathcal{D}$ and a functor $\phi\colon \mathcal{C} \to \mathcal{D}$ as follows:

- $\mathrm{Obj}(\mathcal{D}) = \mathrm{Obj}(\mathcal{C})$.
- $\mathcal{D}(x',x) = \mathcal{C}(x',x)/G_x$ for objects x, x'.
- The composition

$$\mathcal{C}(x'',x') \times \mathcal{C}(x',x) \to \mathcal{C}(x'',x)$$

in $\mathcal{C}$ induces a map

$$\mathcal{C}(x'',x') \times \mathcal{C}(x',x)/G_x \to \mathcal{C}(x'',x)/G_x,$$

which in turn induces

$$\mathcal{C}(x'',x')/G_{x'} \times \mathcal{C}(x',x)/G_x \to \mathcal{C}(x'',x)/G_x$$

owing to the triviality of the action of $G_{x'}$ on $\mathcal{C}(x',x)/G_x$. Define the composition

$$\mathcal{D}(x'',x') \times \mathcal{D}(x',x) \to \mathcal{D}(x'',x)$$

in $\mathcal{D}$ to be the above map.

Thus the category $\mathcal{D}$ is defined. The functor $\phi\colon \mathcal{C} \to \mathcal{D}$ is defined as follows:

- ϕ is identical on objects.
- $\phi : \mathcal{C}(x',x) \to \mathcal{D}(x',x)$ is the natural surjection $\mathcal{C}(x',x) \to \mathcal{C}(x',x)/G_x$.

One sees readily that ϕ satisfies (RG).

4.4. Remark. In [Puig, 2009, p.12] the above construction of $\mathcal{D}$ from $\mathcal{C}$ is called the exterior quotient and utilized in his theory of Frobenius categories. Here is a classical example. Let $\mathcal{C}$ be the category of groups. For each group x let G_x be the inner automorphism group of x. The assignment $x \mapsto G_x$ satisfies the above condition $(\star)$. Morphisms of the resulting quotient category $\mathcal{D}$ are group homomorphisms modulo inner automorphisms. In [Tull, 2019] the term "choice of trivial isomorphisms" is used for a collection of subgroups G_x satisfying $(\star)$, and some examples of quotient categories are provided from projective geometry and quantum theory.

The left-sided version of (RG) is named (LG):

4.5. Definition. *Let $\phi: \mathcal{C} \to \mathcal{D}$ be a functor. Put $G_x = \mathrm{Ker}(\mathrm{Aut}(x) \to \mathrm{Aut}(\phi(x)))$. Condition (LG) for ϕ consists of the following:*

(1) ϕ is surjective on objects.

(2) For every $x, x' \in \mathcal{C}$ the map

$$\mathcal{C}(x, x')/G_x \to \mathcal{D}(\phi(x), \phi(x'))$$

induced by ϕ is bijective.

(2) amounts to saying that

$$\mathcal{C}(x, -)/G_x \to \phi^*(\mathcal{D}(\phi(x), -))$$

in $[\mathcal{C}, \mathbf{Set}]$ is an isomorphism for every $x \in \mathcal{C}$. (1) and (2) imply that $\phi^*(\mathcal{D}(y, -))$ is nearly representable for every $y \in \mathcal{D}$.

We have the left-sided version of the above quotient construction. Let $\mathcal{C}$ be a category. Suppose that for each object x in $\mathcal{C}$ a subgroup G_x of $\mathrm{Aut}(x)$ is given so that the following condition is satisfied.

$(\star)$ For every morphism $f: x \to x'$ in $\mathcal{C}$ and $v \in G_{x'}$, there exists $u \in G_x$ such that $vf = fu$.

This is equivalent to saying the action of $G_{x'}$ on $\mathcal{C}(x, x')/G_x$ is trivial for every x, x'.

We then define a category $\mathcal{D}$ and a functor $\phi: \mathcal{C} \to \mathcal{D}$ as follows:

- $\mathrm{Obj}(\mathcal{D}) = \mathrm{Obj}(\mathcal{C})$.
- $\mathcal{D}(x, x') = \mathcal{C}(x, x')/G_x$ for objects x, x'.

The composition in $\mathcal{D}$ is induced from the composition in $\mathcal{C}$.

The identity on objects and the natural surjections $\mathcal{C}(x, x') \to \mathcal{D}(x, x')$ give a functor $\phi: \mathcal{C} \to \mathcal{D}$, which satisfies (LG).

4.6. Proposition. *Suppose that $\phi: \mathcal{C} \to \mathcal{D}$ satisfies (RG). Put $G_x = \mathrm{Ker}(\mathrm{Aut}(x) \to \mathrm{Aut}(\phi(x))$ for $x \in \mathcal{C}$. For any functor $F: \mathcal{C} \to \mathbf{Set}$ we have a natural isomorphism*

$$(\phi_! F)(\phi(x)) \cong F(x)/G_x$$

for every $x \in \mathcal{C}$.

Proof. For any $x \in \mathcal{C}$ we have an isomorphism $\mathcal{C}(-,x)/G_x \cong \mathcal{D}(\phi(-),\phi(x))$ as functors on $\mathcal{C}$. Also we have by Proposition 2.2 a natural isomorphism $\phi_! F \cong F \otimes_{\mathcal{C}} (\phi \times 1)^* \mathrm{Hom}_{\mathcal{D}}$, hence $(\phi_! F)(y) \cong F \otimes_{\mathcal{C}} \mathcal{D}(\phi(-), y)$ for $y \in \mathcal{D}$. Let $y = \phi(x)$ for $x \in \mathcal{C}$. Then

$$\begin{aligned}(\phi_! F)(\phi(x)) &\cong F \otimes_{\mathcal{C}} \mathcal{D}(\phi(-),\phi(x)) \cong F \otimes_{\mathcal{C}} (\mathcal{C}(-,x)/G_x) \\ &\cong (F \otimes_{\mathcal{C}} \mathcal{C}(-,x))/G_x \cong F(x)/G_x.\end{aligned}$$

Thus $(\phi_! F)(\phi(x)) \cong F(x)/G_x$. ∎

4.7. Proposition. *Let*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\xi} & \mathcal{C} \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathcal{D}' & \xrightarrow[\eta]{} & \mathcal{D} \end{array}$$

be a fiber square of categories and suppose that ϕ satisfies (RG). Then we have the following:

(i) ϕ' satisfies (RG).

*(ii) For any functor $F\colon \mathcal{C} \to$ **Set** the natural morphism*

$$\phi'_! \xi^* F \to \eta^* \phi_! F$$

is an isomorphism.

Proof. (i) Since the square

$$\begin{array}{ccc} \mathrm{Obj}(\mathcal{C}') & \longrightarrow & \mathrm{Obj}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Obj}(\mathcal{D}') & \longrightarrow & \mathrm{Obj}(\mathcal{D}) \end{array}$$

is a pullback and $\phi\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ is a surjection, $\phi'\colon \mathrm{Obj}(\mathcal{C}') \to \mathrm{Obj}(\mathcal{D}')$ is a surjection.

Let $x' \in \mathcal{C}'$ and $x = \xi(x')$. Put

$$\begin{aligned} G_x &= \mathrm{Ker}(\phi\colon \mathrm{Aut}(x) \to \mathrm{Aut}(\phi(x))), \\ G_{x'} &= \mathrm{Ker}(\phi'\colon \mathrm{Aut}(x') \to \mathrm{Aut}(\phi'(x'))). \end{aligned}$$

Since the square

$$\begin{array}{ccc} \mathrm{Aut}(x') & \longrightarrow & \mathrm{Aut}(x) \\ \downarrow & & \downarrow \\ \mathrm{Aut}(\phi'(x')) & \longrightarrow & \mathrm{Aut}(\phi(x)) \end{array}$$

is a pullback, ξ induces an isomorphism $G_{x'} \cong G_x$.

Let $x_1' \in \mathcal{C}'$, $x_1 = \xi(x_1')$. The square

$$\begin{array}{ccc} \mathcal{C}'(x_1', x') & \longrightarrow & \mathcal{C}(x_1, x) \\ \downarrow & & \downarrow \\ \mathcal{D}'(\phi'(x_1'), \phi'(x')) & \longrightarrow & \mathcal{D}(\phi(x_1), \phi(x)) \end{array}$$

is a pullback and the right vertical arrow is the quotient map by G_x, hence the left vertical arrow is the quotient map by $G_{x'}$, namely

$$\mathcal{C}'(x_1', x')/G_{x'} \cong \mathcal{D}'(\phi'(x_1'), \phi'(x')).$$

Thus ϕ' satisfies (RG).

(ii) Let $F: \mathcal{C} \to \mathbf{Set}$. For $x' \in \mathcal{C}'$ put $x = \xi(x')$. Using the isomorphism of Proposition 4.6, we have

$$(\eta^* \phi_! F)(\phi'(x')) = (\phi_! F)(\eta\phi'(x')) = (\phi_! F)(\phi(x)) \cong F(x)/G_x,$$

$$(\phi'_! \xi^* F)(\phi'(x')) \cong (\xi^* F)(x')/G_{x'} = F(x)/G_x.$$

Thus

$$(\eta^* \phi_! F)(\phi'(x')) \cong (\phi'_! \xi^* F)(\phi'(x')).$$

As ϕ' is surjective on objects, we conclude $\eta^* \phi_! F \cong \phi'_! \xi^* F$. ∎

Pullbacks need not preserve equivalences, but they do preserve surjective equivalences:

4.8. Proposition. *Let*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\xi} & \mathcal{C} \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathcal{D} & \xrightarrow[\eta]{} & \mathcal{D} \end{array}$$

be a fiber square of categories and suppose that ϕ is a surjective equivalence. Then ϕ' is a surjective equivalence.

5. Condition (H)

Here we introduce condition (RH) for a functor $\phi: \mathcal{C} \to \mathcal{D}$, which is weaker than condition (RG) of the preceding section. This condition still assures that the functor $\mathrm{Hom}_{\mathcal{D}}(\phi(-), y)$ for every $y \in \mathcal{D}$ is nearly representable, but does not require that ϕ induces a bijection of isomorphism classes. We may say that a functor satisfying (RH) admits a right adjoint inverse modulo a functor satisfying (RG).

5.1. Definition. *Let* $\phi\colon\mathcal{C}\to\mathcal{D}$ *be a functor. For* $x\in\mathcal{C}$ *put* $G_x=\mathrm{Ker}(\mathrm{Aut}(x)\to\mathrm{Aut}(\phi(x)))$*. Condition (RH) for* ϕ *is stated as: For every* $y\in\mathcal{D}$ *there exists* $x\in\mathcal{C}$ *such that* $\phi(x)=y$ *and for every* $x'\in\mathcal{C}$ *the map*

$$\mathcal{C}(x',x)/G_x\to\mathcal{D}(\phi(x'),y)$$

induced by ϕ *is bijective.*

When (RH) holds, the functor $\mathcal{D}(\phi(-),y)$ is nearly representable for every $y\in\mathcal{D}$, hence $(\phi\times 1)^*\mathrm{Hom}_{\mathcal{D}}$ is slicewise nearly representable. Obviously (RG) implies (RH).

The following is immediate from the definition.

5.2. Proposition. *The functors satisfying (RH) are closed under composition.*

5.3. Proposition. *Let* $\phi\colon\mathcal{C}\to\mathcal{D}$ *be a functor. The following are equivalent.*

(i) ϕ *satisfies (RH).*

(ii) There exist a category $\mathcal{B}$ *and a functor* $\tau\colon\mathcal{B}\to\mathcal{C}$ *such that* $\psi=\phi\tau$ *satisfies (RG) and the morphism*

$$(1\times\psi)_!(1\times\tau)^*\mathrm{Hom}_{\mathcal{C}}\to(1\times\phi)_!\mathrm{Hom}_{\mathcal{C}}$$

induced by the adjunction $\tau_!\tau^*\to 1$ *is an isomorphism.*

Proof. Put $G_x=\mathrm{Ker}(\mathrm{Aut}(x)\to\mathrm{Aut}(\phi(x)))$ for $x\in\mathcal{C}$. Let $\tau\colon\mathcal{B}\to\mathcal{C}$ be a functor such that $\psi=\phi\tau$ satisfies (RG). Put $F_u=\mathrm{Ker}(\mathrm{Aut}(u)\to\mathrm{Aut}(\psi(u)))$ for $u\in\mathcal{B}$. As ψ satisfies (RG), applying Proposition 4.6 to the functor $\mathcal{C}(x',\tau(-))$, we have

$$((1\times\psi)_!(1\times\tau)^*\mathrm{Hom}_{\mathcal{C}})(x',\psi(u))\cong\mathcal{C}(x',\tau(u))/F_u$$

for $x'\in\mathcal{C}$ and $u\in\mathcal{B}$. Also by the general isomorphism

$$(1\times\phi)_!\mathrm{Hom}_{\mathcal{C}}\cong(\phi\times 1)^*\mathrm{Hom}_{\mathcal{D}}$$

we have

$$((1\times\phi)_!\mathrm{Hom}_{\mathcal{C}})(x',\psi(u))\cong\mathrm{Hom}_{\mathcal{D}}(\phi(x'),\psi(u)).$$

In view of these isomorphisms the morphism

$$(1\times\psi)_!(1\times\tau)^*\mathrm{Hom}_{\mathcal{C}}\to(1\times\phi)_!\mathrm{Hom}_{\mathcal{C}}$$

in (ii), evaluated at $(x',\psi(u))$, is regarded as the map

$$\mathcal{C}(x',\tau(u))/F_u\to\mathcal{D}(\phi(x'),\psi(u))$$

induced by ϕ.

Now suppose $(1\times\psi)_!(1\times\tau)^*\mathrm{Hom}_{\mathcal{C}}\cong(1\times\phi)_!\mathrm{Hom}_{\mathcal{C}}$. Let $y\in\mathcal{D}$. Take $u\in\mathcal{B}$ such that $\psi(u)=y$. By the above observation we have

$$\mathcal{C}(-,\tau(u))/F_u\cong\mathcal{D}(\phi(-),y).$$

This implies that the group $\tau(F_u)$ coincides with $G_{\tau(u)}$ and ϕ satisfies (RH).

Suppose conversely that ϕ satisfies (RH). Let $\mathcal{B}$ be the full subcategory of $\mathcal{C}$ consisting of $x \in \mathcal{C}$ such that the morphism

$$\mathcal{C}(-, x)/G_x \to \mathcal{D}(\phi(-), \phi(x))$$

induced by ϕ is isomorphic. Let $\tau : \mathcal{B} \to \mathcal{C}$ be the inclusion and $\psi = \phi\tau$.

Clearly ψ satisfies (RG) and

$$\mathcal{C}(-, \tau(u))/G_{\tau(u)} \cong \mathcal{D}(\phi(-), \psi(u))$$

for every $u \in \mathcal{B}$. By the earlier observation we see that

$$(1 \times \psi)_!(1 \times \tau)^*\mathrm{Hom}_{\mathcal{C}} \cong (1 \times \phi)_!\mathrm{Hom}_{\mathcal{C}}.$$

Thus (ii) holds. ∎

The dual version of (RH) is named (LH):

5.4. Definition. *Condition (LH) for $\phi: \mathcal{C} \to \mathcal{D}$ is stated as: For every $y \in \mathcal{D}$ there exists $x \in \mathcal{C}$ such that $\phi(x) = y$ and for every $x' \in \mathcal{C}$ the map*

$$\mathcal{C}(x, x')/G_x \to \mathcal{D}(y, \phi(x'))$$

induced by ϕ is bijective.

The dual of Proposition 5.3 is the following:

5.5. Proposition. *Let $\phi: \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent.*

(i) ϕ satisfies (LH).

(ii) There exist a category $\mathcal{B}$ and a functor $\tau: \mathcal{B} \to \mathcal{C}$ such that $\psi = \phi\tau$ satisfies (LG) and the morphism

$$(\psi \times 1)_!(\tau \times 1)^*\mathrm{Hom}_{\mathcal{C}} \to (\phi \times 1)_!\mathrm{Hom}_{\mathcal{C}}$$

induced by the adjunction $\tau_!\tau^* \to 1$ is an isomorphism.

5.6. Proposition. *Let*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\xi} & \mathcal{C} \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathcal{D}' & \xrightarrow[\eta]{} & \mathcal{D} \end{array}$$

be a fiber square of categories and suppose that ϕ satisfies (RH). Then we have the following:

(i) ϕ' satisfies (RH).

*(ii) For any functor $F: \mathcal{C} \to$ **Set** the natural morphism $\phi'_!\xi^*F \to \eta^*\phi_!F$ is an isomorphism.*

PROOF. (i) For any $x \in \mathcal{C}$ and $x' \in \mathcal{C}'$ put

$$\begin{aligned} G_x &= \operatorname{Ker}(\phi\colon \operatorname{Aut}(x) \to \operatorname{Aut}(\phi(x))), \\ G_{x'} &= \operatorname{Ker}(\phi'\colon \operatorname{Aut}(x') \to \operatorname{Aut}(\phi'(x'))) \end{aligned}$$

as before. Then $G_{x'} \cong G_{\xi(x')}$.

Let $y' \in \mathcal{D}'$. Put $y = \eta(y')$. As ϕ satisfies (RH), we can take $x \in \mathcal{C}$ such that $\phi(x) = y$ and

$$\phi\colon \mathcal{C}(-, x) \to \mathcal{D}(\phi(-), y)$$

is the quotient map by G_x. Take $x' \in \mathcal{C}'$ such that $\phi'(x') = y'$ and $\xi(x') = x$. Then we have a pullback diagram

$$\begin{CD} \mathcal{C}'(-, x') @>{\xi}>> \mathcal{C}(\xi(-), x) \\ @V{\phi'}VV @VV{\phi}V \\ \mathcal{D}'(\phi'(-), y') @>>{\eta}> \mathcal{D}(\phi\xi(-), y) \end{CD}$$

Since the right vertical arrow is quotient by G_x, the left vertical arrow is quotient by $G_{x'}$. Thus ϕ' satisfies (RH).

(ii) Recall that

$$\phi_! F \cong F \otimes_{\mathcal{C}} (\phi \times 1)^* \mathrm{Hom}_{\mathcal{D}}$$

for any $F\colon \mathcal{C} \to \mathbf{Set}$, and

$$\phi'_! F' \cong F' \otimes_{\mathcal{C}'} (\phi' \times 1)^* \mathrm{Hom}_{\mathcal{D}'}$$

for any $F'\colon \mathcal{C}' \to \mathbf{Set}$.

Let $y' \in \mathcal{D}'$. Take x', x, y as in (i). Then

$$\mathcal{D}(\phi(-), y) \cong \mathcal{C}(-, x)/G_x$$

and

$$\mathcal{D}'(\phi'(-), y') \cong \mathcal{C}'(-, x')/G_{x'}.$$

Then

$$(\phi_! F)(y) \cong F \otimes_{\mathcal{C}} \mathcal{D}(\phi(-), y) \cong F \otimes_{\mathcal{C}} \mathcal{C}(-, x)/G_x \cong F(x)/G_x,$$

so

$$(\eta^* \phi_! F)(y') = (\phi_! F)(y) \cong F(x)/G_x.$$

Similarly

$$\begin{aligned} (\phi'_! \xi^* F)(y') &\cong \xi^* F \otimes_{\mathcal{C}'} \mathcal{D}'(\phi'(-), y') \cong \xi^* F \otimes_{\mathcal{C}'} \mathcal{C}'(-, x')/G_{x'} \cong (\xi^* F)(x')/G_{x'} \\ &= F(\xi(x'))/G_{x'} = F(x)/G_x. \end{aligned}$$

Thus

$$(\phi'_! \xi^* F)(y') \cong (\eta^* \phi_! F)(y').$$

This proves (ii). ∎

6. The subcategory ${}_{\mathrm{nu}}\mathbb{E}(L)$ of $\mathbb{E}(L)$

Let $L\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ be a distributor. We defined in Section 2 the category $\mathbb{E}(L)$. Its objects are triples (x, y, a) for $x \in \mathcal{B}$, $y \in \mathcal{C}$, and $a \in L(x, y)$. Here we introduce some subcategories of $\mathbb{E}(L)$ defined by conditions of near universality. They will be used in Sections 8 and 10.

Firstly we define ${}_{\mathrm{nu}}\mathbb{E}(L)$ as a full subcategory of $\mathbb{E}(L)$: An object of ${}_{\mathrm{nu}}\mathbb{E}(L)$ is an object (x, y, a) of $\mathbb{E}(L)$ such that (x, a) is nearly universal for $L(-, y)$.

Likewise we define $\mathbb{E}_{\mathrm{nu}}(L)$ as a full subcategory of $\mathbb{E}(L)$: An object of $\mathbb{E}_{\mathrm{nu}}(L)$ is an object (x, y, a) of $\mathbb{E}(L)$ such that (y, a) is nearly universal for $L(x, -)$.

We define ${}_{\mathrm{nu}}\mathbb{E}_{\mathrm{nu}}(L)$ to be ${}_{\mathrm{nu}}\mathbb{E}(L) \cap \mathbb{E}_{\mathrm{nu}}(L)$.

Using universality in place of near universality, we define ${}_{\mathrm{u}}\mathbb{E}_{\mathrm{u}}(L)$ as a full subcategory of $\mathbb{E}(L)$: An object of ${}_{\mathrm{u}}\mathbb{E}_{\mathrm{u}}(L)$ is an object (x, y, a) of $\mathbb{E}(L)$ such that (x, a) is universal for $L(-, y)$ and (y, a) is universal for $L(x, -)$.

Firstly we consider ${}_{\mathrm{nu}}\mathbb{E}(L)$. Put $\check{\mathcal{C}} = {}_{\mathrm{nu}}\mathbb{E}(L)$. We have the projection functors $\sigma\colon \check{\mathcal{C}} \to \mathcal{B}$ and $\pi\colon \check{\mathcal{C}} \to \mathcal{C}$:

$$\begin{aligned} \sigma\colon &\quad (x, y, a) \mapsto x, \\ \pi\colon &\quad (x, y, a) \mapsto y. \end{aligned}$$

We have a pullback diagram

$$\begin{array}{ccc} \check{\mathcal{C}}((x, y, a), (x_1, y_1, a_1)) & \xrightarrow{\ \pi\ } & \mathcal{C}(y, y_1) \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ \mathcal{B}(x, x_1) & \longrightarrow & L(x, y_1) \end{array}$$

where the right vertical arrow is the map $g \mapsto g_*(a)$, the lower horizontal arrow is the map $f \mapsto f^*(a_1)$.

6.1. Proposition. *Assume that for every $y \in \mathcal{C}$ the functor $L(-, y)\colon \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ is nearly representable. Then we have the following:*

(i) π satisfies (RG).

(ii) The pair (σ, π) tabulates L, that is, $(\sigma \times \pi)_! \mathrm{Hom}_{\check{\mathcal{C}}} \cong L$.

Proof. (i) The assumption implies that π is surjective on objects.

Let (x, y, a), (x_1, y_1, a_1) be objects of $\check{\mathcal{C}}$. Put $K_1 = \mathrm{Aut}(x_1, a_1)$. As (x_1, a_1) is nearly universal for $L(-, y_1)$, the map

$$\mathcal{B}(x, x_1) \to L(x, y_1)\colon f \mapsto f^*(a_1)$$

is quotient by the group K_1. The pullback diagram shows that the map

$$\pi\colon \check{\mathcal{C}}((x, y, a), (x_1, y_1, a_1)) \to \mathcal{C}(y, y_1)$$

is also quotient by K_1. Thus π satisfies (RG).

(ii) In view of the general isomorphism $(\sigma \times 1)_!\mathrm{Hom}_{\check{\mathcal{C}}} \cong (1 \times \sigma)^*\mathrm{Hom}_{\mathcal{B}}$, it is enough to show $(1 \times \pi)_!(1 \times \sigma)^*\mathrm{Hom}_{\mathcal{B}} \cong L$. Let $(x, y, a) \in \check{\mathcal{C}}$. Put $K = \mathrm{Aut}(x, a)$ so that

$$\mathcal{B}(-, x)/K \cong L(-, y).$$

As $\pi\colon \check{\mathcal{C}} \to \mathcal{C}$ satisfies (RG) and

$$K \cong \mathrm{Ker}(\pi\colon \mathrm{Aut}(x, y, a) \to \mathrm{Aut}(y)),$$

we have by Proposition 4.6

$$(\pi_! F)(y) \cong F(x, y, a)/K$$

for any functor $F\colon \check{\mathcal{C}} \to \mathbf{Set}$. Taking $F = \sigma^*\mathcal{B}(x', -)$ for any $x' \in \mathcal{B}$, we have

$$(\pi_!\sigma^*\mathcal{B}(x', -))(y) \cong \mathcal{B}(x', \sigma(x, y, a))/K = \mathcal{B}(x', x)/K \cong L(x', y).$$

Thus

$$(1 \times \pi)_!(1 \times \sigma)^*\mathrm{Hom}_{\mathcal{B}} \cong L.$$

■

We next consider ${}_{\mathrm{nu}}\mathbb{E}_{\mathrm{nu}}(L)$. Put $\mathcal{A} = {}_{\mathrm{nu}}\mathbb{E}_{\mathrm{nu}}(L)$. We have the projection functors $\lambda\colon \mathcal{A} \to \mathcal{B}$ and $\mu\colon \mathcal{A} \to \mathcal{C}$.

6.2. Proposition. *The functors λ and μ are full.*

Proof. Let (x, y, a), $(x_1, y_1, a_1) \in \mathcal{A}$. As in the preceding proof we have a pullback diagram

$$\begin{array}{ccc} \mathcal{A}((x, y, a), (x_1, y_1, a_1)) & \xrightarrow{\mu} & \mathcal{C}(y, y_1) \\ {\scriptstyle\lambda}\downarrow & & \downarrow \\ \mathcal{B}(x, x_1) & \longrightarrow & L(x, y_1) \end{array}$$

As (x_1, a_1) is nearly universal for $L(-, y_1)$, the lower arrow is a quotient map. Hence the upper arrow is also a quotient map and in particular surjective. Thus μ is full. ■

Next we put $\mathcal{D} = {}_{\mathrm{u}}\mathbb{E}_{\mathrm{u}}(L)$. We have the projection functors $\beta\colon \mathcal{D} \to \mathcal{B}$ and $\gamma\colon \mathcal{D} \to \mathcal{C}$.

6.3. Proposition. *The functors β and γ are fully faithful.*

Proof. Let $(x, y, a), (x_1, y_1, a_1) \in \mathcal{D}$. We have again a pullback diagram

$$\begin{array}{ccc} \mathcal{D}((x, y, a), (x_1, y_1, a_1)) & \xrightarrow{\gamma} & \mathcal{C}(y, y_1) \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ \mathcal{B}(x, x_1) & \longrightarrow & L(x, y_1) \end{array}$$

As (x_1, a_1) is universal for $L(-, y_1)$ and (y, a) is universal of $L(x, -)$, the lower arrow and the right arrow are bijections. Hence the other arrows are bijections. Thus β and γ are fully faithful. ■

6.4. Corollary. *Suppose that β and γ are surjective on objects. Then β and γ are surjective equivalences, and we have $L \cong (\beta \times \gamma)_! \mathrm{Hom}_{\mathcal{D}}$.*

Proof. The pullback diagram shows $\mathcal{D}((x, y, a), (x_1, y_1, a_1)) \cong L(x, y_1)$. This means $\mathrm{Hom}_{\mathcal{D}} \cong (\beta \times \gamma)^* L$. As β and γ are equivalences, this implies $(\beta \times \gamma)_! \mathrm{Hom}_{\mathcal{D}} \cong L$. ∎

7. ϵ and η

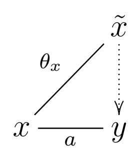
An adjunction gives rise to two natural transformations called unit and counit. In this section we pursue an analogous construction for a slicewise nearly representable distributor. Under a certain finiteness hypothesis we show a theorem about the invertibility of a unit-like morphism, on which our factorization theorem depends.

Let $L \colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ be a distributor. Throughout this section we assume that L is slicewise nearly representable.

For each $x \in \mathcal{B}$ take an object $\tilde{x} \in \mathcal{C}$, a subgroup H_x of $\mathrm{Aut}(\tilde{x})$, and an isomorphism

$$\mathcal{C}(\tilde{x}, -)/H_x \cong L(x, -).$$

Take an element $\theta_x \in L(x, \tilde{x})$ which induces this isomorphism. Thus, for every $y \in \mathcal{C}$ and $a \in L(x, y)$, there exists $g \in \mathcal{C}(\tilde{x}, y)$ such that $g_*(\theta_x) = a$; such g is unique up to the action of H_x. This is pictured as the diagram (Section 2)

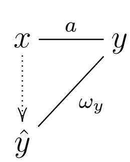


In the language of Section 3 the pair $(\tilde{x}, \theta_x)$ is nearly universal for $L(x, -)$ and $H_x = \mathrm{Aut}(\tilde{x}, \theta_x)$.

Likewise, for each $y \in \mathcal{C}$ take an object $\hat{y} \in \mathcal{B}$, a subgroup K_y of $\mathrm{Aut}(\hat{y})$, and an isomorphism

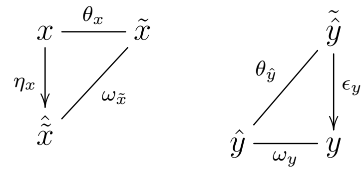
$$\mathcal{B}(-, \hat{y})/K_y \cong L(-, y).$$

Take an element $\omega_y \in L(\hat{y}, y)$ which induces this isomorphism. Thus, for every $x \in \mathcal{B}$ and $a \in L(x, y)$, there exists $f \in \mathcal{B}(x, \hat{y})$ such that $f^*(\omega_y) = a$; such f is unique up to the action of K_y.

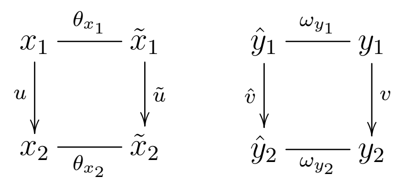


The pair $(\hat{y}, \omega_y)$ is nearly universal for $L(-, y)$ and $K_y = \mathrm{Aut}(\hat{y}, \omega_y)$.

For every $x \in \mathcal{B}$, using the near universality of $(\hat{\tilde{x}}, \omega_{\tilde{x}})$, we take a morphism $\eta_x \in \mathcal{B}(x, \hat{\tilde{x}})$ such that $\theta_x = \eta_x^*(\omega_{\tilde{x}})$. For every $y \in \mathcal{C}$, using the near universality of $(\tilde{\hat{y}}, \theta_{\hat{y}})$, we take a morphism $\epsilon_y \in \mathcal{C}(\tilde{\hat{y}}, y)$ such that $\epsilon_{y*}(\theta_{\hat{y}}) = \omega_y$. These are pictured as the diagrams

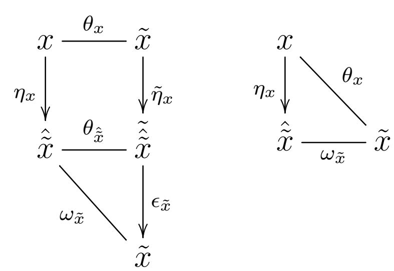


For $u \in \mathcal{B}(x_1, x_2)$ take $\tilde{u} \in \mathcal{C}(\tilde{x}_1, \tilde{x}_2)$ such that $u^*(\theta_{x_2}) = \tilde{u}_*(\theta_{x_1})$; such $\tilde{u}$ is unique up to the action of H_{x_1}. For $v \in \mathcal{C}(y_1, y_2)$ take $\hat{v} \in \mathcal{B}(\hat{y}_1, \hat{y}_2)$ such that $v_*(\omega_{y_1}) = \hat{v}^*(\omega_{y_2})$; such $\hat{v}$ is unique up to the action of K_{y_2}. Thus

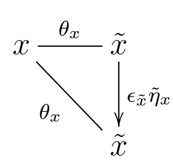


7.1. Proposition. *For $x \in \mathcal{B}$ we have $\epsilon_{\tilde{x}}\tilde{\eta}_x \in H_x$.*

Proof. We have the diagrams



Hence



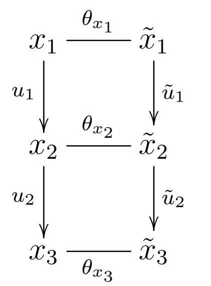
By the uniqueness modulo H_x we see $\epsilon_{\tilde{x}}\tilde{\eta}_x \equiv 1_{\tilde{x}} \bmod H_x$, that is, $\epsilon_{\tilde{x}}\tilde{\eta}_x \in H_x$ as required. ∎

Dually we have

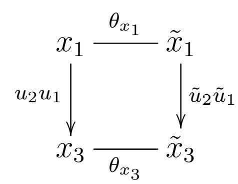
7.2. Proposition. *For $y \in \mathcal{C}$ we have $\hat{\epsilon}_y \eta_{\hat{y}} \in K_y$.*

7.3. Proposition. *For $u_1 \in \mathcal{B}(x_1, x_2)$ and $u_2 \in \mathcal{B}(x_2, x_3)$ we have $\widetilde{u_2 u_1} \equiv \widetilde{u_2}\,\widetilde{u_1} \bmod H_{x_1}$.*

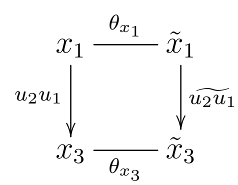
Proof. We have the diagram



Hence



Also we have the diagram



It follows that $\tilde{u}_2\tilde{u}_1 \equiv \widetilde{u_2u_1} \bmod H_{x_1}$. ∎

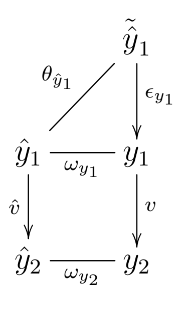
Dually we have

7.4. Proposition. *For $v_1 \in \mathcal{C}(y_1, y_2)$ and $v_2 \in \mathcal{C}(y_2, y_3)$ we have $\widehat{v_2v_1} \equiv \widehat{v}_2\widehat{v}_1 \bmod K_{y_3}$.*

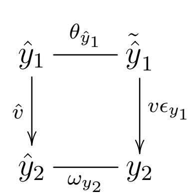
7.5. Corollary. *If u is an isomorphism in $\mathcal{B}$, then $\tilde{u}$ is an isomorphism in $\mathcal{C}$. If v is an isomorphism in $\mathcal{C}$, then $\hat{v}$ is an isomorphism in $\mathcal{B}$.*

7.6. Proposition. *For $v \in \mathcal{C}(y_1, y_2)$ we have $v\epsilon_{y_1} \equiv \epsilon_{y_2}\tilde{\hat{v}} \bmod H_{\hat{y}_1}$.*

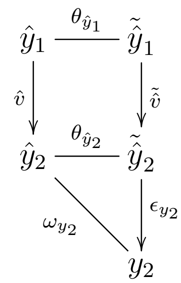
Proof. We have the diagram



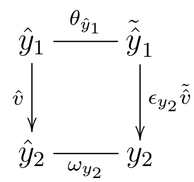
hence



Also we have



hence



Owing to the isomorphism $\mathcal{C}(\tilde{\hat{y}}_1, -)/H_{\hat{y}_1} \cong L(\hat{y}_1, -)$, we conclude from the two squares above that $v\epsilon_{y_1} \equiv \epsilon_{y_2}\tilde{\hat{v}}$ mod $H_{\hat{y}_1}$. ∎

7.7. Proposition. *Let $x \in \mathcal{B}$. If $\eta_{\hat{\tilde{x}}}$ is an isomorphism, then so is $\epsilon_{\tilde{x}}$.*

Proof. Put $x_1 = \hat{\tilde{x}}$, $v_1 = \epsilon_{\tilde{x}}$ so that

$$\epsilon_{\tilde{x}}\colon \tilde{\hat{\tilde{x}}} \to \tilde{x}$$

is written as

$$v_1\colon \tilde{x}_1 \to \tilde{x}.$$

Assume that $\eta_{x_1}\colon x_1 \to \hat{\tilde{x}}_1$ is an isomorphism. By Proposition 7.2 for $\tilde{x}$ we have $\hat{\epsilon}_{\tilde{x}}\eta_{\hat{\tilde{x}}} \in K_{\tilde{x}}$, so this is an isomorphism. Namely $\hat{v}_1\eta_{x_1}$ is an isomorphism. As η_{x_1} is an isomorphism, it follows that $\hat{v}_1$ is also an isomorphism.

The morphism

$$\eta_x\colon x \to \hat{\tilde{x}}$$

gives rise to the morphism

$$\tilde{\eta}_x\colon \tilde{x} \to \tilde{\hat{\tilde{x}}}.$$

Denote this by v_2 so that

$$v_2\colon \tilde{x} \to \tilde{x}_1.$$

Proposition 7.1 says $\epsilon_{\tilde{x}}\tilde{\eta}_x \in H_x$, namely $v_1v_2 \in H_x$. In particular v_1v_2 is an isomorphism. Then $\widehat{v_1v_2}$ is an isomorphism, and

$$\hat{v}_1\hat{v}_2 \equiv \widehat{v_1v_2} \text{ mod } K_{\tilde{x}}.$$

Therefore $\hat{v}_1\hat{v}_2$ is an isomorphism. As $\hat{v}_1$ is an isomorphism, so is $\hat{v}_2$.

Next we have

$$\widehat{v_2v_1} \equiv \hat{v}_2\hat{v}_1 \text{ mod } K_{\tilde{x}_1},$$

so $\widehat{v_2v_1}$ is an isomorphism. Hence $\widetilde{\widehat{v_2v_1}}$ is an isomorphism. Proposition 7.1 for x_1 says $\epsilon_{\tilde{x}_1}\tilde{\eta}_{x_1} \in H_{x_1}$. As η_{x_1} is an isomorphism, it follows that $\epsilon_{\tilde{x}_1}$ is an isomorphism. And Proposition 7.6 for $v_2v_1\colon \tilde{x}_1 \to \tilde{x}_1$ says

$$(v_2v_1)\epsilon_{\tilde{x}_1} \equiv \epsilon_{\tilde{x}_1}\widetilde{\widehat{v_2v_1}} \bmod H_{\hat{\tilde{x}}_1}.$$

As $\epsilon_{\tilde{x}_1}$ and $\widetilde{\widehat{v_2v_1}}$ are isomorphisms, it follows that v_2v_1 is an isomorphism.

As the both v_1v_2 and v_2v_1 are isomorphisms, v_1 and v_2 are isomorphisms, that is, $\epsilon_{\tilde{x}}$ and $\tilde{\eta}_x$ are isomorphisms.

■

The following is similarly proved.

7.8. Proposition. *Let $y \in \mathcal{C}$. If $\epsilon_{\tilde{\hat{y}}}$ is an isomorphism, then so is $\eta_{\hat{y}}$.*

7.9. Theorem. *Suppose that $\mathcal{C}$ satisfies the following condition: If*

$$\cdots \xrightarrow{g_2} y_2 \xrightarrow{g_1} y_1 \xrightarrow{g_0} y_0$$

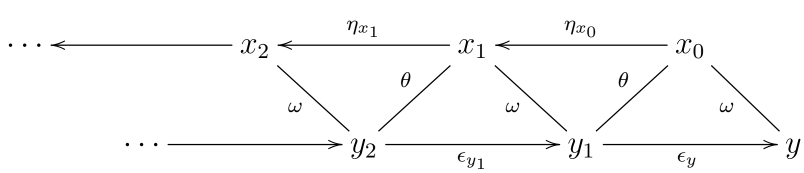
is a sequence of morphisms in $\mathcal{C}$ and all g_i have right inverses, then g_n for large n are isomorphisms.

Then $\epsilon_{\tilde{x}}$ is an isomorphism for every $x \in \mathcal{B}$, and $\eta_{\hat{y}}$ is an isomorphism for every $y \in \mathcal{C}$.

Proof. Let $y \in \mathcal{C}$. Put

$$y_0 = y, \quad x_n = \hat{y}_n \text{ for } n \geq 0, \quad y_n = \tilde{x}_{n-1} \text{ for } n > 0.$$

We have a diagram



By Propositions 7.7 and 7.8 we have implications

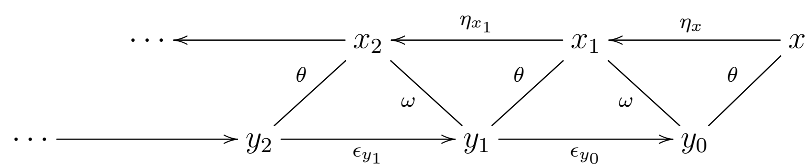
$$\begin{aligned} \eta_{x_n} \text{ is an isomorphism} &\implies \epsilon_{y_n} \text{ is an isomorphism} \quad (n = 1, 2, \ldots), \\ \epsilon_{y_n} \text{ is an isomorphism} &\implies \eta_{x_{n-1}} \text{ is an isomorphism} \quad (n = 1, 2, \ldots). \end{aligned}$$

For every $n \geq 1$, Proposition 7.1 for x_{n-1} says $\epsilon_{y_n}\tilde{\eta}_{x_{n-1}} \in H_{x_{n-1}}$. Hence ϵ_{y_n} has a right inverse. By assumption ϵ_{y_n} for a large n is an isomorphism. Then it follows that η_{x_0} is an isomorphism, that is, $\eta_{\hat{y}}$ is an isomorphism.

Let $x \in \mathcal{B}$. Put

$$x_0 = x, \quad y_n = \tilde{x}_n \text{ for } n \geq 0, \quad x_n = \hat{y}_{n-1} \text{ for } n > 0.$$

We have a diagram



By Propositions 7.7 and 7.8

$$\begin{aligned} \eta_{x_n} \text{ is an isomorphism} &\implies \epsilon_{y_{n-1}} \text{ is an isomorphism} \quad (n = 1, 2, \ldots), \\ \epsilon_{y_n} \text{ is an isomorphism} &\implies \eta_{x_n} \text{ is an isomorphism} \quad (n = 1, 2, \ldots). \end{aligned}$$

By assumption ϵ_{y_n} for a large n is an isomorphism. Then ϵ_{y_0} is an isomorphism, that is, $\epsilon_{\tilde{x}}$ is an isomorphism.

■

The same conclusion holds when $\mathcal{B}$ satisfies the dual condition: if

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} \cdots$$

is a sequence of morphisms in $\mathcal{B}$ and all f_i have left inverses, then f_n for large n are isomorphisms.

8. Equivalence $\bar{\mathcal{B}}_0 \simeq \bar{\mathcal{C}}_0$

Let $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ be a slicewise nearly representable distributor. In this section we construct from L subcategories $\mathcal{B}_0$ of $\mathcal{B}$, $\mathcal{C}_0$ of $\mathcal{C}$, and quotient categories $\bar{\mathcal{B}}_0$ of $\mathcal{B}_0$, $\bar{\mathcal{C}}_0$ of $\mathcal{C}_0$. We then construct distributors $K: \bar{\mathcal{B}}_0^{\mathrm{op}} \times \bar{\mathcal{C}}_0 \to \mathbf{Set}$, $M: \mathcal{B}^{\mathrm{op}} \times \bar{\mathcal{B}}_0 \to \mathbf{Set}$, and $N: \bar{\mathcal{C}}_0^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$. We show that K gives an equivalence $\bar{\mathcal{B}}_0 \simeq \bar{\mathcal{C}}_0$.

We first make the category ${}_{\mathrm{nu}}\mathbb{E}_{\mathrm{nu}}(L)$ from L (Section 6). Put $\mathcal{A} = {}_{\mathrm{nu}}\mathbb{E}_{\mathrm{nu}}(L)$. Recall that an object of $\mathcal{A}$ is an object (x, y, a) of $\mathbb{E}(L)$ such that (x, a) is nearly universal for $L(-, y)$ and (y, a) is nearly universal for $L(x, -)$. We have the projection functors $\lambda: \mathcal{A} \to \mathcal{B}$, $\mu: \mathcal{A} \to \mathcal{C}$, which are known to be full (Proposition 6.2). Define $\mathcal{B}_0 = \mathrm{Im}\lambda$: This is a full subcategory of $\mathcal{B}$; an object of $\mathcal{B}_0$ is an object x of $\mathcal{B}$ such that $(x, y, a) \in \mathcal{A}$ for some y, a. Define $\mathcal{C}_0 = \mathrm{Im}\mu$: This a full subcategory of $\mathcal{C}$; an object of $\mathcal{C}_0$ is an object y of $\mathcal{C}$ such that $(x, y, a) \in \mathcal{A}$ for some x, a.

8.1. Proposition. *Let $x \in \mathcal{B}_0$. Take $y \in \mathcal{C}$ and $a \in L(x, y)$ such that $(x, y, a) \in \mathcal{A}$. Then the subgroup* $\mathrm{Aut}(x, a)$ *of* $\mathrm{Aut}(x)$ *does not depend on the choice of* y, a.

Proof. Suppose $(x, y, a), (x, y', a') \in \mathcal{A}$. As (y, a) and (y', a') are both nearly universal for $L(x, -)$, there exists an isomorphism $h: y \to y'$ such that $a' = h_*(a)$ by Proposition 3.3. Then $\mathrm{Aut}(x, a) = \mathrm{Aut}(x, a')$. ■

Owing to this proposition, we can define for every $x \in \mathcal{B}_0$ the group $\Delta_x = \mathrm{Aut}(x, a)$ by taking $(x, y, a) \in \mathcal{A}$. As (x, a) is nearly universal for $L(-, y)$, a induces

$$L(-, y) \cong \mathcal{B}(-, x)/\Delta_x.$$

Similarly

8.2. Proposition. *Let $y \in \mathcal{C}_0$. Take $x \in \mathcal{B}$ and $a \in L(x, y)$ such that $(x, y, a) \in \mathcal{A}$. Then the subgroup $\mathrm{Aut}(y, a)$ of $\mathrm{Aut}(y)$ does not depend on the choice of x, a.*

We define for every $y \in \mathcal{C}_0$ the group $\Gamma_y = \mathrm{Aut}(y, a)$ by taking $(x, y, a) \in \mathcal{A}$. The element a induces

$$L(x, -) \cong \mathcal{C}(y, -)/\Gamma_y.$$

8.3. Proposition. *For every $x \in \mathcal{B}_0$ and $y' \in \mathcal{C}$, the action of Δ_x on $L(x, y')$ is trivial.*

Proof. Take $(x, y, a) \in \mathcal{A}$. Then $\Delta_x = \mathrm{Aut}(x, a)$. For any $y' \in \mathcal{C}$ and $a' \in L(x, y')$ take $g \colon y \to y'$ such that $a' = g_*(a)$. As Δ_x fixes a and g_* commutes with the action of $\mathrm{Aut}(x)$, Δ_x fixes a'. ∎

Similarly we have

8.4. Proposition. *For every $y \in \mathcal{C}_0$ and $x' \in \mathcal{B}$, the action of Γ_y on $L(x', y)$ is trivial.*

8.5. Proposition. *For every $y, y' \in \mathcal{C}_0$ the action of $\Gamma_{y'}$ on $\mathcal{C}(y, y')/\Gamma_y$ is trivial.*

Proof. Let $y, y' \in \mathcal{C}_0$. Take $(x, y, a) \in \mathcal{A}$. The element a gives

$$L(x, -) \cong \mathcal{C}(y, -)/\Gamma_y,$$

hence

$$L(x, y') \cong \mathcal{C}(y, y')/\Gamma_y$$

as $\mathrm{Aut}(y')$-sets. On the other hand, as $y' \in \mathcal{C}_0$, the action of $\Gamma_{y'}$ on $L(x, y')$ is trivial (Proposition 8.4). It follows that the action of $\Gamma_{y'}$ on $\mathcal{C}(y, y')/\Gamma_y$ is trivial. ∎

Similarly we have

8.6. Proposition. *For every $x, x' \in \mathcal{B}_0$ the action of Δ_x on $\mathcal{B}(x, x')/\Delta_{x'}$ is trivial.*

Let $y, y' \in \mathcal{C}_0$ and $y'' \in \mathcal{C}$. The composition in $\mathcal{C}$ induces a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(y, y') \times \mathcal{C}(y', y'') & \longrightarrow & \mathcal{C}(y, y'') \\ \downarrow & & \downarrow \\ \mathcal{C}(y, y')/\Gamma_y \times \mathcal{C}(y', y'')/\Gamma_{y'} & \longrightarrow & \mathcal{C}(y, y'')/\Gamma_y \end{array}$$

because $\Gamma_{y'}$ acts trivially on $\mathcal{C}(y, y')/\Gamma_y$.

The construction in Section 4 then gives us a quotient category $\bar{\mathcal{C}}_0$ and a functor $q\colon \mathcal{C}_0 \to \bar{\mathcal{C}}_0$: The category $\bar{\mathcal{C}}_0$ has the same objects as $\mathcal{C}_0$; its hom-sets are given by

$$\bar{\mathcal{C}}_0(y, y') = \mathcal{C}(y, y')/\Gamma_y.$$

The functor $q\colon \mathcal{C}_0 \to \bar{\mathcal{C}}_0$ is identical on objects and the natural surjections on hom-sets. We know q satisfies (LG).

Likewise, let $x \in \mathcal{B}$ and $x', x'' \in \mathcal{B}_0$. The composition in $\mathcal{B}$ induces a commutative diagram

$$\begin{array}{ccc} \mathcal{B}(x, x') \times \mathcal{B}(x', x'') & \longrightarrow & \mathcal{B}(x, x'') \\ \downarrow & & \downarrow \\ \mathcal{B}(x, x')/\Delta_{x'} \times \mathcal{B}(x', x'')/\Delta_{x''} & \longrightarrow & \mathcal{B}(x, x'')/\Delta_{x''} \end{array}$$

because $\Delta_{x'}$ acts trivially on $\mathcal{B}(x', x'')/\Delta_{x''}$.

The construction in Section 4 gives us a quotient category $\bar{\mathcal{B}}_0$ and a functor $p : \mathcal{B}_0 \to \bar{\mathcal{B}}_0$: $\bar{\mathcal{B}}_0$ has the same objects as $\mathcal{B}_0$; its hom-sets are

$$\bar{\mathcal{B}}_0(x, x') = \mathcal{B}(x, x')/\Delta_{x'}.$$

The functor $p : \mathcal{B}_0 \to \bar{\mathcal{B}}_0$ is identical on objects and the natural surjections on hom-sets. We know p satisfies (RG).

Let $i\colon \mathcal{B}_0 \to \mathcal{B}$ and $j\colon \mathcal{C}_0 \to \mathcal{C}$ be the inclusion functors. For $x \in \mathcal{B}$, $y \in \mathcal{C}_0$, $y \in \mathcal{C}$ the map

$$L(x, y) \times \mathcal{C}(y, y') \to L(x, y')$$

induces

$$L(x, y) \times \mathcal{C}(y, y')/\Gamma_y \to L(x, y')$$

because Γ_y acts trivially on $L(x, y)$. If $y' \in \mathcal{C}_0$, we then have a map

$$L(x, y) \times \bar{\mathcal{C}}_0(y, y') \to L(x, y').$$

These maps for all $x \in \mathcal{B}$, $y, y' \in \mathcal{C}_0$ define a functor $\mathcal{B}^{\mathrm{op}} \times \bar{\mathcal{C}}_0 \to \mathbf{Set}$, which is denoted by L''. The restriction of L to $\mathcal{B}^{\mathrm{op}} \times \mathcal{C}_0$ is denoted by L', so that

$$L' = (1 \times j)^* L, \quad L' = (1 \times q)^* L''.$$

Thus we have a commutative diagram

$$\begin{array}{ccc} \mathcal{B}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{L} & \mathbf{Set} \\ \uparrow & \nearrow_{L'} & \\ \mathcal{B}^{\mathrm{op}} \times \mathcal{C}_0 & & \\ \downarrow & \nearrow_{L''} & \\ \mathcal{B}^{\mathrm{op}} \times \bar{\mathcal{C}}_0 & & \end{array}$$

Likewise, for $x \in \mathcal{B}$, $x' \in \mathcal{B}_0$, $y \in \mathcal{C}$ the map

$$\mathcal{B}(x,x') \times L(x',y) \to L(x,y)$$

induces a map

$$\mathcal{B}(x,x')/\Delta_{x'} \times L(x',y) \to L(x,y)$$

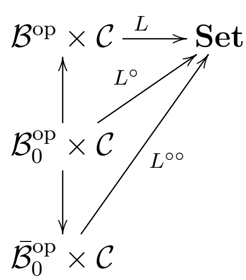
because $\Delta_{x'}$ acts trivially on $L(x',y)$. If $x \in \mathcal{B}_0$, we then have a map

$$\bar{\mathcal{B}}_0(x,x') \times L(x',y) \to L(x,y).$$

These maps for all $x, x' \in \mathcal{B}_0$ and $y \in \mathcal{C}$ define a functor $\bar{\mathcal{B}}_0^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$, which is denoted by $L^{\circ\circ}$. The restriction of L to $\mathcal{B}_0^{\mathrm{op}} \times \mathcal{C}$ is denoted by L°, so that

$$L^\circ = (i \times 1)^* L, \quad L^\circ = (p \times 1)^* L^{\circ\circ}.$$

Thus we have a commutative diagram



In particular we have a functor $K: \bar{\mathcal{B}}_0^{\mathrm{op}} \times \bar{\mathcal{C}}_0 \to \mathbf{Set}$, so that

$$(i \times j)^* L = (p \times q)^* K.$$

We make the category ${}_{\mathrm{u}}\mathbb{E}_{\mathrm{u}}(K)$ from the distributor $K: \bar{\mathcal{B}}_0^{\mathrm{op}} \times \bar{\mathcal{C}}_0 \to \mathbf{Set}$ (Section 6). We put $\mathcal{D} = {}_{\mathrm{u}}\mathbb{E}_{\mathrm{u}}(K)$. Recall that an object of $\mathcal{D}$ is a triple (x, y, a) composed of $x \in \bar{\mathcal{B}}_0$, $y \in \bar{\mathcal{C}}_0$, and $a \in K(x,y)$ such that (x,a) is universal for $K(-,y)$ and (y,a) is universal for $K(x,-)$. We have the projection functors $\beta: \mathcal{D} \to \bar{\mathcal{B}}_0$ and $\gamma: \mathcal{D} \to \bar{\mathcal{C}}_0$.

8.7. Proposition. *If $(x,y,a) \in \mathcal{A}$, then $(p(x), q(y), a) \in \mathcal{D}$.*

Proof. Let $(x,y,a) \in \mathcal{A}$. Then $y \in \mathcal{C}_0$ and $L(x,-) \cong \mathcal{C}(y,-)/\Gamma_y$ on $\mathcal{C}$, hence on $\mathcal{C}_0$. Now $L(x,-) = K(p(x), q(-))$ on $\mathcal{C}_0$ and $\mathcal{C}(y,-)/\Gamma_y = \bar{\mathcal{C}}_0(q(y), q(-))$ on $\mathcal{C}_0$. Hence $K(p(x), q(-)) \cong \bar{\mathcal{C}}_0(q(y), q(-))$ on $\mathcal{C}_0$. It follows that $K(p(x), -) \cong \bar{\mathcal{C}}_0(q(y), -)$ on $\bar{\mathcal{C}}_0$. This isomorphism is induced by the element $a \in K(p(x), q(y))$. Thus $(p(x), a)$ is universal for $K(-, q(y))$.

Similarly $(q(y), a)$ is universal for $K(p(x), -)$.

This proves that $(p(x), q(y), a) \in \mathcal{D}$.

∎

8.8. PROPOSITION. *The functors β and γ are surjective equivalences.*

PROOF. We know by Proposition 6.3 that β and γ are fully faithful. It remains to show that they are surjective on objects. As $\lambda\colon \mathcal{A} \to \mathcal{B}$ has the image $\mathcal{B}_0$ and $\rho\colon \mathcal{A} \to \mathcal{C}$ has the image $\mathcal{C}_0$, it follows by the preceding proposition that $\beta\colon \mathcal{D} \to \bar{\mathcal{B}}_0$ has the image $\bar{\mathcal{B}}_0$ and $\gamma\colon \mathcal{D} \to \bar{\mathcal{C}}_0$ has the image $\bar{\mathcal{C}}_0$.

■

Therefore $\bar{\mathcal{B}}_0$ and $\bar{\mathcal{C}}_0$ are equivalent.

We next define distributors $N\colon \bar{\mathcal{C}}_0^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ and $M\colon \mathcal{B}^{\mathrm{op}} \times \bar{\mathcal{B}}_0 \to \mathbf{Set}$.

For $y \in \mathcal{C}_0$ and $y' \in \mathcal{C}$ set

$$N(y, y') = \mathcal{C}(y, y')/\Gamma_y.$$

As seen before, for $y, y' \in \mathcal{C}_0$ and $y'' \in \mathcal{C}$ the composition

$$\mathcal{C}(y, y') \times \mathcal{C}(y', y'') \to \mathcal{C}(y, y'')$$

induces a map

$$\mathcal{C}(y, y')/\Gamma_y \times \mathcal{C}(y', y'')/\Gamma_{y'} \to \mathcal{C}(y, y'')/\Gamma_y,$$

that is,

$$\bar{\mathcal{C}}_0(y, y') \times N(y', y'') \to N(y, y'').$$

This makes N a distributor $\bar{\mathcal{C}}_0^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$.

Likewise, for $x \in \mathcal{B}$ and $x' \in \mathcal{B}_0$ set

$$M(x, x') = \mathcal{B}(x, x')/\Delta_{x'}.$$

For $x \in \mathcal{B}$, $x', x'' \in \mathcal{B}_0$ the composition

$$\mathcal{B}(x, x') \times \mathcal{B}(x', x'') \to \mathcal{B}(x, x'')$$

induces a map

$$\mathcal{B}(x, x')/\Delta_{x'} \times \mathcal{B}(x', x'')/\Delta_{x''} \to \mathcal{B}(x, x'')/\Delta_{x''},$$

that is,

$$M(x, x') \times \bar{\mathcal{B}}_0(x', x'') \to M(x, x'').$$

This makes M a distributor $\mathcal{B}^{\mathrm{op}} \times \bar{\mathcal{B}}_0 \to \mathbf{Set}$.

Thus we have obtained distributors

$$\begin{aligned} M\colon &\quad \mathcal{B}^{\mathrm{op}} \times \bar{\mathcal{B}}_0 \to \mathbf{Set}, \\ K\colon &\quad \bar{\mathcal{B}}_0^{\mathrm{op}} \times \bar{\mathcal{C}}_0 \to \mathbf{Set}, \\ N\colon &\quad \bar{\mathcal{C}}_0^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}. \end{aligned}$$

Recall that $p\colon \mathcal{B}_0 \to \bar{\mathcal{B}}_0$, $q\colon \mathcal{C}_0 \to \bar{\mathcal{C}}_0$ denote the projections and $i\colon \mathcal{B}_0 \to \mathcal{B}$, $j\colon \mathcal{C}_0 \to \mathcal{C}$ denote the inclusions. The natural maps $\mathcal{B}(x, x_0) \to M(x, p(x_0))$ for all $x \in \mathcal{B}$ and $x_0 \in \mathcal{B}_0$ yield a morphism $(1 \times i)^* \mathrm{Hom}_{\mathcal{B}} \to (1 \times p)^* M$, which by adjunction induces a morphism $(1 \times p)_! (1 \times i)^* \mathrm{Hom}_{\mathcal{B}} \to M$ or by Proposition 2.4 a morphism $(i \times p)_! \mathrm{Hom}_{\mathcal{B}_0} \to M$. This is shown to be an isomorphism:

8.9. LEMMA. $(i \times p)_! \mathrm{Hom}_{\mathcal{B}_0} \cong M$.

PROOF. We shall show $(1 \times p)_!(1 \times i)^* \mathrm{Hom}_{\mathcal{B}} \cong M$. The functor $p\colon \mathcal{B}_0 \to \bar{\mathcal{B}}_0$ satisfies (RG) and has the kernel

$$\mathrm{Ker}(\mathrm{Aut}(x_0) \to \mathrm{Aut}(p(x_0))) = \Delta_{x_0}$$

for $x_0 \in \mathcal{B}_0$. Proposition 4.6 then tells us that

$$(p_! F)(p(x_0)) \cong F(x_0)/\Delta_{x_0}$$

for any functor $F\colon \mathcal{B}_0 \to \mathbf{Set}$. Applying this to $F = i^*\mathcal{B}(x, -)\colon \mathcal{B}_0 \to \mathbf{Set}$ for $x \in \mathcal{B}$, we have

$$(p_! i^* \mathcal{B}(x, -))(p(x_0)) \cong \mathcal{B}(x, x_0)/\Delta_{x_0} = M(x, p(x_0)).$$

Hence

$$(1 \times p)_!(1 \times i)^* \mathrm{Hom}_{\mathcal{B}} \cong M.$$

This proves the proposition. ∎

Similarly we have

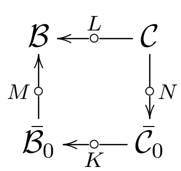
8.10. LEMMA. $(q \times j)_! \mathrm{Hom}_{\mathcal{C}_0} \cong N$.

9. Factorization: the first step

Let $L\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ be a slicewise nearly representable distributor. From now on we assume that $\mathcal{C}$ satisfies the assumption of Theorem 7.9, that is, that $\mathcal{C}$ does not have an infinite chain of non-isomorphic split epimorphisms. In this section we show that L is the composite of the three distributors

$$M\colon \mathcal{B}^{\mathrm{op}} \times \bar{\mathcal{B}}_0 \to \mathbf{Set},\ K\colon \bar{\mathcal{B}}_0^{\mathrm{op}} \times \bar{\mathcal{C}}_0 \to \mathbf{Set},\ N\colon \bar{\mathcal{C}}_0^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$$

defined in Section 8, and that M and N are slicewise nearly representable. A picture in Borceux's notation:



Exactly as in Section 7, for each $x \in \mathcal{B}$ take $\tilde{x} \in \mathcal{C}$, a subgroup H_x of $\mathrm{Aut}(\tilde{x})$, and an isomorphism $\mathcal{C}(\tilde{x}, -)/H_x \cong L(x, -)$. Take $\theta_x \in L(x, \tilde{x})$ which induces this isomorphism. Then $(\tilde{x}, \theta_x)$ is nearly universal for $L(x, -)$. For each morphism u in $\mathcal{B}$ take a morphism $\tilde{u}$ in $\mathcal{C}$ as in Section 7.

For each $y \in \mathcal{C}$ take $\hat{y} \in \mathcal{B}$, a subgroup K_y of $\mathrm{Aut}(\hat{y})$, and an isomorphism $\mathcal{B}(-, \hat{y})/K_y \cong L(-, y)$. Take $\omega_y \in L(\hat{y}, y)$ which induces this isomorphism. Then $(\hat{y}, \omega_y)$ is nearly universal for $L(-, y)$. For each morphism v in $\mathcal{C}$ take a morphism $\hat{v}$ in $\mathcal{B}$ as in Section 7.

For each $x \in \mathcal{B}$ take a morphism $\eta_x: x \to \hat{\tilde{x}}$, and for each $y \in \mathcal{C}$ take a morphism $\epsilon_y: \tilde{\hat{y}} \to y$ as in Section 7.

Let

$$\begin{aligned} \mathcal{B}_0' &= \{x \in \mathcal{B} \mid \eta_x \text{ is an isomorphism}\}, \\ \mathcal{C}_0' &= \{y \in \mathcal{C} \mid \epsilon_y \text{ is an isomorphism}\}. \end{aligned}$$

We regard these as full subcategories of $\mathcal{B}$ and $\mathcal{C}$, respectively. We shall show that $\mathcal{B}_0' = \mathcal{B}_0$, $\mathcal{C}_0' = \mathcal{C}_0$.

We restate Theorem 7.9:

9.1. Proposition. *(i) If $x \in \mathcal{B}$, then $\tilde{x} \in \mathcal{C}_0'$.*
(ii) If $y \in \mathcal{C}$, then $\hat{y} \in \mathcal{B}_0'$.

9.2. Proposition. *(i) If $x \in \mathcal{B}_0'$, then (x, θ_x) is nearly universal for $L(-, \tilde{x})$.*
(ii) If $y \in \mathcal{C}_0'$, then (y, ω_y) is nearly universal for $L(\hat{y}, -)$.

Proof. (i) Let $x \in \mathcal{B}_0'$. Then $\eta_x: x \to \hat{\tilde{x}}$ is an isomorphism. As $(\hat{\tilde{x}}, \omega_{\tilde{x}})$ is nearly universal for $L(-, \tilde{x})$ and $\theta_x = \eta_x^*(\omega_{\tilde{x}})$, it follows by Proposition 3.7 that (x, θ_x) is nearly universal for $L(-, \tilde{x})$. (ii) is similarly proved. ■

9.3. Proposition. *(i) If $x \in \mathcal{B}_0'$, then $(x, \tilde{x}, \theta_x) \in \mathcal{A}$.*
(ii) If $y \in \mathcal{C}_0'$, then $(\hat{y}, y, \omega_y) \in \mathcal{A}$.

Proof. (i) Let $x \in \mathcal{B}_0'$. The pair $(\tilde{x}, \theta_x)$ is nearly universal for $L(x, -)$ by definition, while the pair (x, θ_x) is nearly universal for $L(-, \tilde{x})$ by Proposition 9.2. Thus $(x, \tilde{x}, \theta_x) \in \mathcal{A}$. (ii) is similarly proved. ■

9.4. Proposition. *If $(x, y, a) \in \mathcal{A}$, then $x \in \mathcal{B}_0'$ and $y \in \mathcal{C}_0'$.*

Proof. Let $(x, y, a) \in \mathcal{A}$. As (x, a) and $(\hat{y}, \omega_y)$ are both nearly universal for $L(-, y)$, there exists an isomorphism $f: x \to \hat{y}$ such that $a = f^*(\omega_y)$ by Proposition 3.3. As (y, a) and $(\tilde{x}, \theta_x)$ are both nearly universal for $L(x, -)$, there exists likewise an isomorphism $g: \tilde{x} \to y$ such that $a = g_*(\theta_x)$. We have $\eta_x^*(\omega_{\tilde{x}}) = \theta_x$ and $\hat{g}^*(\omega_y) = g_*(\omega_{\tilde{x}})$. So $(\hat{g}\eta_x)^*(\omega_y) = g_*(\theta_x)$, hence $(\hat{g}\eta_x)^*(\omega_y) = a$. Comparing this with $f^*(\omega_y) = a$, we have by the near universality of $(\hat{y}, \omega_y)$ that $\hat{g}\eta_x \equiv f \bmod K_y$. As $\hat{g}$ and f are isomorphisms, so is η_x. Thus $x \in \mathcal{B}_0'$.

Similarly we have $y \in \mathcal{C}_0'$.

■

The preceding two propositions give the following:

9.5. Proposition. *The categories $\mathcal{B}_0'$ and $\mathcal{C}_0'$ respectively coincide with $\mathcal{B}_0$ and $\mathcal{C}_0$ defined in Section 8: $\mathcal{B}_0 = \mathcal{B}_0'$, $\mathcal{C}_0 = \mathcal{C}_0'$.*

Recall that $i: \mathcal{B}_0 \to \mathcal{B}$ and $j: \mathcal{C}_0 \to \mathcal{C}$ denote the inclusions and $p: \mathcal{B}_0 \to \bar{\mathcal{B}}_0$ and $q: \mathcal{C}_0 \to \bar{\mathcal{C}}_0$ the projections.

9.6. Lemma. *(i) $L \cong (1 \times j)_!(1 \times j)^* L$.*
(ii) $L \cong (i \times 1)_!(i \times 1)^* L$.

Proof. (i) Let $x \in \mathcal{B}$. We have

$$L(x,-) \cong \mathcal{C}(\tilde{x},-)/H_x$$

on $\mathcal{C}$. We have $\tilde{x} \in \mathcal{C}_0$ by Proposition 9.1. Hence

$$j^*(L(x,-)) \cong \mathcal{C}_0(\tilde{x},-)/H_x.$$

Then

$$j_!j^*(L(x,-)) \cong j_!(\mathcal{C}_0(\tilde{x},-)/H_x) \cong \mathcal{C}(\tilde{x},-)/H_x.$$

Thus

$$j_!j^*(L(x,-)) \cong L(x,-).$$

This proves (i). ∎

9.7. Lemma. *$L \cong (i\times j)_!(i\times j)^*L$.*

Proof.

$$\begin{aligned} L &\cong (1\times j)_!(1\times j)^*L \\ &\cong (1\times j)_!(1\times j)^*(i\times 1)_!(i\times 1)^*L \\ &\cong (1\times j)_!(i\times 1)_!(1\times j)^*(i\times 1)^*L \\ &\cong (i\times j)_!(i\times j)^*L. \end{aligned}$$

∎

9.8. Proposition. *We have an isomorphism $L \cong M \otimes_{\bar{\mathcal{B}}_0} K \otimes_{\bar{\mathcal{C}}_0} N$.*

Proof. We know (Section 8)

$$\begin{aligned} (i\times j)^*L &= (p\times q)^*K, \\ (i\times p)_!\mathrm{Hom}_{\mathcal{B}_0} &\cong M, \\ (q\times j)_!\mathrm{Hom}_{\mathcal{C}_0} &\cong N. \end{aligned}$$

Then we proceed as

$$\begin{aligned} L &\cong (i\times j)_!(i\times j)^*L \\ &\cong (i\times j)_!(p\times q)^*K \\ &\cong (i\times j)_!\,[(1\times p)_!\mathrm{Hom}_{\mathcal{B}_0} \otimes_{\bar{\mathcal{B}}_0} K \otimes_{\bar{\mathcal{C}}_0} (q\times 1)_!\mathrm{Hom}_{\mathcal{C}_0}] \\ &\qquad\qquad \text{(by Propositions 2.2 and 2.4)} \\ &\cong (i\times 1)_!(1\times p)_!\mathrm{Hom}_{\mathcal{B}_0} \otimes_{\bar{\mathcal{B}}_0} K \otimes_{\bar{\mathcal{C}}_0} (1\times j)_!(q\times 1)_!\mathrm{Hom}_{\mathcal{C}_0} \\ &\cong M \otimes_{\bar{\mathcal{B}}_0} K \otimes_{\bar{\mathcal{C}}_0} N. \end{aligned}$$

This proves the proposition. ∎

Recall that $L^\circ = (i \times 1)^* L$, $L^\circ = (p \times 1)^* L^{\circ\circ}$ (Section 8).

9.9. Proposition. *The distributors L° and $L^{\circ\circ}$ are slicewise nearly representable.*

Proof. For any $y \in \mathcal{C}$ we have $L(-,y) \cong \mathcal{B}(-,\hat{y})/K_y$ on $\mathcal{B}$. As $\hat{y} \in \mathcal{B}_0$, $L(-,y)$ is nearly representable on $\mathcal{B}_0$. For any $x \in \mathcal{B}$, $L(x,-)$ is nearly representable on $\mathcal{C}$, hence also for any $x \in \mathcal{B}_0$. Thus $L^\circ\colon \mathcal{B}_0^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is slicewise nearly representable.

For any $x \in \mathcal{B}_0$ we have $L^{\circ\circ}(p(x),-) = L^\circ(x,-)$ on $\mathcal{C}$, which is nearly representable. For any $y \in \mathcal{C}$ we have $L^\circ(-,y) = p^* L^{\circ\circ}(-,y)$. As p is full and surjective on objects, we have $p_! p^* \cong 1$, so $p_! L^\circ(-,y) \cong L^{\circ\circ}(-,y)$. As $L^\circ(-,y)$ is nearly representable, so is $p_! L^\circ(-,y)$. Hence $L^{\circ\circ}(-,y)$ is nearly representable. Thus $L^{\circ\circ}$ is slicewise nearly representable. ■

Recall that $N\colon \bar{\mathcal{C}}_0^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is defined as $N(y,y') = \mathcal{C}(y,y')/\Gamma_y$.

9.10. Lemma. $L^{\circ\circ} \cong K \otimes_{\bar{\mathcal{C}}_0} N$.

Proof. We know

$$\begin{aligned} L &\cong (1 \times j)_! (1 \times j)^* L, \\ (i \times j)^* L &= (p \times q)^* K, \\ N &\cong (q \times j)_! \mathrm{Hom}_{\mathcal{C}_0}. \end{aligned}$$

Using these, we proceed as

$$\begin{aligned} L^\circ &= (i \times 1)^* L \\ &\cong (i \times 1)^* (1 \times j)_! (1 \times j)^* L \\ &\cong (1 \times j)_! (i \times j)^* L \\ &= (1 \times j)_! (p \times q)^* K \\ &\cong (p \times q)^* K \otimes_{\mathcal{C}_0} (1 \times j)_! \mathrm{Hom}_{\mathcal{C}_0} \text{ (by Propositions 2.2 and 2.4)} \\ &\cong (p \times 1)^* [(1 \times q)^* K \otimes_{\mathcal{C}_0} (1 \times j)_! \mathrm{Hom}_{\mathcal{C}_0}] \\ &\cong (p \times 1)^* [K \otimes_{\bar{\mathcal{C}}_0} (q \times j)_! \mathrm{Hom}_{\mathcal{C}_0}] \text{ (by Proposition 2.3)} \\ &\cong (p \times 1)^* (K \otimes_{\bar{\mathcal{C}}_0} N), \end{aligned}$$

hence

$$L^\circ \cong (p \times 1)^* (K \otimes_{\bar{\mathcal{C}}_0} N).$$

As $L^\circ = (p \times 1)^* L^{\circ\circ}$ and $p_! p^* \cong 1$, we conclude

$$L^{\circ\circ} \cong K \otimes_{\bar{\mathcal{C}}_0} N.$$

■

9.11. Proposition. *The distributor N is slicewise nearly representable.*

Proof. The distributor $K\colon \bar{\mathcal{B}}_0^{\mathrm{op}} \times \bar{\mathcal{C}}_0 \to \mathbf{Set}$ gives an equivalence between $\bar{\mathcal{B}}_0$ and $\bar{\mathcal{C}}_0$. The distributor $L^{\circ\circ}$ is slicewise nearly representable and isomorphic to $K \otimes_{\bar{\mathcal{C}}_0} N$. It then follows that N is also slicewise nearly representable. ■

Similarly

9.12. Proposition. *The distributor M is slicewise nearly representable.*

10. Factorization: the second step

We keep the assumption of the preceding section. Here we prove that the distributor N is tabulated by a pair of a functor satisfying (RG) and a functor satisfying (LH). A corresponding fact holds also for the distributor M. Then we prove the main theorem that L is tabulated by a pair of a functor satisfying (LH) and a functor satisfying (RH).

From the distributor $N\colon \bar{\mathcal{C}}_0^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ we make the category ${}_{\mathrm{nu}}\mathbb{E}(N)$ (Section 6). Put $\check{\mathcal{C}} = {}_{\mathrm{nu}}\mathbb{E}(N)$. An object of $\check{\mathcal{C}}$ is a triple (x, y, a) composed of $x \in \bar{\mathcal{C}}_0$, $y \in \mathcal{C}$, $a \in N(x, y)$ such that (x, a) is nearly universal of $N(-, y)$.

We have the projection functors

$$\begin{aligned} \sigma\colon\quad & \check{\mathcal{C}} \to \bar{\mathcal{C}}_0\colon (x, y, a) \mapsto x, \\ \pi\colon\quad & \check{\mathcal{C}} \to \mathcal{C}\colon (x, y, a) \mapsto y. \end{aligned}$$

As $N(-, y)$ is nearly representable for every $y \in \mathcal{C}$ (Proposition 9.11), we see by Proposition 6.1 the following:

10.1. Proposition. *The functor π satisfies (RG). The pair (σ, π) tabulates N, that is, $N \cong (\sigma \times \pi)_! \mathrm{Hom}_{\check{\mathcal{C}}}$.*

Define a functor $\tau\colon \mathcal{C}_0 \to \check{\mathcal{C}}$ as follows. For $y \in \mathcal{C}_0$ we set

$$\tau(y) = (q(y), y, 1_{q(y)}).$$

Note that $N(q(y'), y) = \bar{\mathcal{C}}_0(q(y'), q(y))$ for $y, y' \in \mathcal{C}_0$. So $(q(y), 1_{q(y)})$ is universal for $N(-, y)$, hence $(q(y), y, 1_{q(y)}) \in \check{\mathcal{C}}$. For a morphism $f\colon y \to y_1$ of $\mathcal{C}_0$ we set

$$\tau(f) = (q(f), f).$$

Note that

$$q(f)^*(1_{q(y_1)}) = q(f) = f_*(1_{q(y)}).$$

Hence

$$(q(f), f)\colon (q(y), y, 1_{q(y)}) \to (q(y_1), y_1, 1_{q(y_1)})$$

is really a morphism.

Thus τ is a functor and $\sigma\tau = q$, $\pi\tau = j$.

10.2. Proposition. *For $y \in \mathcal{C}_0$ and $(x_1, y_1, a_1) \in \check{\mathcal{C}}$ the map*

$$\check{\mathcal{C}}(\tau(y), (x_1, y_1, a_1))/\Gamma_y \to \bar{\mathcal{C}}_0(q(y), x_1)$$

induced by σ is bijective.

Proof. Let $(x, y, a), (x_1, y_1, a_1) \in \check{\mathcal{C}}$. We have a pullback diagram

$$\begin{array}{ccc} \check{\mathcal{C}}((x,y,a),(x_1,y_1,a_1)) & \xrightarrow{\pi} & \mathcal{C}(y,y_1) \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ \bar{\mathcal{C}}_0(x,x_1) & \longrightarrow & N(x,y_1) \end{array}$$

where the right vertical arrow is the map $g \mapsto g_*(a)$ and the lower horizontal arrow is the map $f \mapsto f^*(a_1)$.

Now let $y \in \mathcal{C}_0$. Set $(x, y, a) = (q(y), y, 1_{q(y)})$. The diagram becomes

$$\begin{array}{ccc} \check{\mathcal{C}}((q(y),y,1_{q(y)}),(x_1,y_1,a_1)) & \longrightarrow & \mathcal{C}(y,y_1) \\ \downarrow & & \downarrow \\ \bar{\mathcal{C}}_0(q(y),x_1) & \longrightarrow & N(q(y),y_1) \end{array}$$

Note

$$N(q(y), y_1) = \mathcal{C}(y, y_1)/\Gamma_y,$$
$$g_*(1_{q(y)}) = q(g) \in \mathcal{C}(y, y_1)/\Gamma_y$$

for $g \in \mathcal{C}(y, y_1)$. So the right vertical arrow is the quotient map by Γ_y.

Since the diagram is a pullback, it follows that the left vertical arrow is also the quotient map by Γ_y, that is,

$$\check{\mathcal{C}}((q(y), y, 1_{q(y)}), (x_1, y_1, a_1))/\Gamma_y \cong \bar{\mathcal{C}}_0(q(y), x_1).$$

This proves the proposition. ∎

10.3. Proposition. *The functor $\sigma: \check{\mathcal{C}} \to \bar{\mathcal{C}}_0$ satisfies (LH).*

Proof. The bijection of the preceding proposition gives an isomorphism

$$\check{\mathcal{C}}(\tau(y), -)/\Gamma_y \cong \bar{\mathcal{C}}_0(q(y), \sigma(-))$$

of functors on $\check{\mathcal{C}}$ for every $y \in \mathcal{C}_0$. Then $\tau(\Gamma_y)$ coincides with $\mathrm{Ker}(\sigma: \mathrm{Aut}(\tau(y)) \to \mathrm{Aut}(q(y)))$. As q is surjective on objects, it follows that σ satisfies (LH). ∎

The above proof, compared with the proof of Proposition 5.3, shows that τ satisfies condition (ii) of Proposition 5.5: $q = \sigma\tau$ satisfies (LG) and

$$(q \times 1)_!(\tau \times 1)^*\mathrm{Hom}_{\check{\mathcal{C}}} \cong (\sigma \times 1)_!\mathrm{Hom}_{\check{\mathcal{C}}}.$$

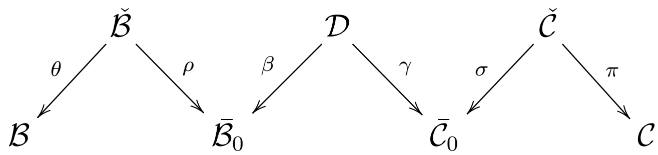
Dually we make the category $\mathbb{E}_{\mathrm{nu}}(M)$ from $M\colon \mathcal{B}^{\mathrm{op}} \times \bar{\mathcal{B}}_0 \to \mathbf{Set}$. We put $\check{\mathcal{B}} = \mathbb{E}_{\mathrm{nu}}(M)$. An object of $\check{\mathcal{B}}$ is a triple (x, y, a) composed of $x \in \mathcal{B}$, $y \in \bar{\mathcal{B}}_0$, $a \in M(x, y)$ such that (y, a) is nearly universal for $M(x, -)$.

Define $\theta\colon \check{\mathcal{B}} \to \mathcal{B}$ and $\rho\colon \check{\mathcal{B}} \to \bar{\mathcal{B}}_0$ as the projections.

10.4. Proposition. *The functor θ satisfies (LG). The pair (θ, ρ) tabulates M, that is, $M \cong (\theta \times \rho)_!\mathrm{Hom}_{\check{\mathcal{B}}}$.*

10.5. Proposition. *The functor ρ satisfies (RH).*

Now we deduce the final factorization of L. We have so far constructed the functors



and the distributors

$$\begin{aligned} M&\colon \quad \mathcal{B}^{\mathrm{op}} \times \bar{\mathcal{B}}_0 \to \mathbf{Set}, \\ K&\colon \quad \bar{\mathcal{B}}_0^{\mathrm{op}} \times \bar{\mathcal{C}}_0 \to \mathbf{Set}, \\ N&\colon \quad \bar{\mathcal{C}}_0^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}. \end{aligned}$$

We know the factorization

$$L \cong M \otimes_{\bar{\mathcal{B}}_0} K \otimes_{\bar{\mathcal{C}}_0} N. \tag{1}$$

We know π satisfies (RG), σ satisfies (LH), and

$$N \cong (\sigma \times \pi)_!\mathrm{Hom}_{\check{\mathcal{C}}}. \tag{2}$$

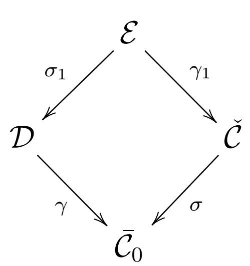
We know θ satisfies (LG), ρ satisfies (RH), and

$$M \cong (\theta \times \rho)_!\mathrm{Hom}_{\check{\mathcal{B}}}. \tag{3}$$

Also β and γ are surjective equivalences, and

$$K \cong (\beta \times \gamma)_!\mathrm{Hom}_{\mathcal{D}}. \tag{4}$$

Form the pullback of categories

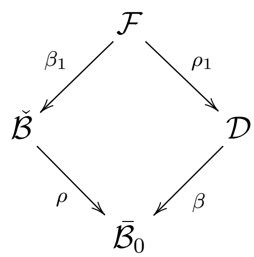


Then we find that σ_1 satisfies (LH), γ_1 is a surjective equivalence, and

$$\sigma^*\gamma_!F \cong \gamma_{1!}{\sigma_1}^*F \tag{5}$$

for any functor $F\colon \mathcal{D} \to \mathbf{Set}$ (because γ is an equivalence).

Form the pullback

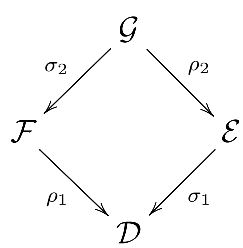


Then we find that ρ_1 satisfies (RH), β_1 is a surjective equivalence, and

$$\beta^*\rho_!F \cong \rho_{1!}{\beta_1}^*F \tag{6}$$

for any functor $F\colon \check{\mathcal{B}} \to \mathbf{Set}$ (because β is an equivalence).

Form the pullback



Then we find that ρ_2 satisfies (RH), σ_2 satisfies (LH), and

$$\sigma_1^*\rho_{1!}F \cong \rho_{2!}\sigma_2^*F \tag{7}$$

for any functor $F\colon \mathcal{F} \to \mathbf{Set}$ (because ρ_1 satisfies (RH)).

For any functor $F\colon \mathcal{B} \to \mathbf{Set}$ we have

$$\begin{aligned} F \otimes_{\mathcal{B}} L &\cong F \otimes_{\mathcal{B}} M \otimes_{\bar{\mathcal{B}}_0} K \otimes_{\bar{\mathcal{C}}_0} N \quad \text{(by (1))} \\ &\cong \pi_!\sigma^*\gamma_!\beta^*\rho_!\theta^*F \quad \text{(by (2), (3), (4)).} \end{aligned}$$

Now

$$\begin{aligned} \pi_!\sigma^*\gamma_!\beta^*\rho_!\theta^* &\cong \pi_!\gamma_{1!}{\sigma_1}^*\rho_{1!}{\beta_1}^*\theta^* \quad \text{(by (5), (6))} \\ &\cong \pi_!\gamma_{1!}\rho_{2!}{\sigma_2}^*{\beta_1}^*\theta^* \quad \text{(by (7))} \\ &\cong (\pi\gamma_1\rho_2)_!(\theta\beta_1\sigma_2)^*. \end{aligned}$$

Put $\mu = \pi\gamma_1\rho_2$ and $\lambda = \theta\beta_1\sigma_2$, so that

$$\mathcal{B} \xleftarrow{\ \lambda\ } \mathcal{G} \xrightarrow{\ \mu\ } \mathcal{C}.$$

We have obtained

$$F \otimes_{\mathcal{B}} L \cong \mu_! \lambda^* F$$

for any F. And canonically

$$\mu_! \lambda^* F \cong F \otimes_{\mathcal{B}} (\lambda \times \mu)_! \mathrm{Hom}_{\mathcal{G}}.$$

Hence

$$L \cong (\lambda \times \mu)_! \mathrm{Hom}_{\mathcal{G}}.$$

As θ satisfies (LG) and σ_2 satisfies (LH), λ satisfies (LH). As π satisfies (RG) and ρ_2 satisfies (RH), μ satisfies (RH).

Thus we obtain

10.6. Theorem. *The functor $\lambda\colon \mathcal{G} \to \mathcal{B}$ satisfies (LH), the functor $\mu\colon \mathcal{G} \to \mathcal{C}$ satisfies (RH), and we have an isomorphism $L \cong (\lambda \times \mu)_! \mathrm{Hom}_{\mathcal{G}}$.*

10.7. Theorem. *Suppose that $\mathcal{C}$ satisfies the assumption of Theorem 7.9. Let $L\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to$ **Set** be a distributor. The following are equivalent.*

(i) L is slicewise nearly representable.

(ii) There exist a category $\mathcal{M}$, a functor $\phi\colon \mathcal{M} \to \mathcal{B}$ satisfying (LH), a functor $\psi\colon \mathcal{M} \to \mathcal{C}$ satisfying (RH), and an isomorphism $L \cong (\phi \times \psi)_! \mathrm{Hom}_{\mathcal{M}}$.

Proof. We have proved that (i) implies (ii). For the converse suppose

$$L \cong (\phi \times \psi)_! \mathrm{Hom}_{\mathcal{M}}$$

with $\phi\colon \mathcal{M} \to \mathcal{B}$ satisfying (LH), $\psi\colon \mathcal{M} \to \mathcal{C}$ satisfying (RH). We have

$$L \cong (1 \times \phi)^* \mathrm{Hom}_{\mathcal{B}} \otimes_{\mathcal{G}} (\psi \times 1)^* \mathrm{Hom}_{\mathcal{C}}.$$

By Definitions 5.1 and 5.4 $(\psi \times 1)^* \mathrm{Hom}_{\mathcal{C}}$ and $(1 \times \phi)^* \mathrm{Hom}_{\mathcal{B}}$ are slicewise nearly representable. Then so is their composite by Proposition 3.8. Thus L is slicewise nearly representable. ∎

Our factorization of a slicewise nearly representable distributor $L\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to$ **Set** relies on the finiteness assumption on $\mathcal{B}$ or $\mathcal{C}$. For a slicewise truly representable distributor $L\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to$ **Set**, namely an adjunction, [Applegate and Tierney, 1970] gives a factorization of L under the completeness assumption on $\mathcal{B}$ or $\mathcal{C}$.

11. Familial condition

Recall that a set-valued functor F is said to be familially representable if F is a sum of representable functors [Carboni and Johnstone, 1995]. Following this terminology we say F is familially nearly representable if F is a sum of nearly representable functors. We say a distributor $L\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to$ **Set** is slicewise familially nearly representable if $L(x, -)$ for every $x \in \mathcal{B}$ is familially nearly representable and $L(-,y)$ for every $y \in \mathcal{C}$ is familially nearly representable.

In this section we show that every slicewise familially nearly representable distributor is a composite of three: a distributor coming from a discrete fibration, a slicewise nearly representable distributor, and a distributor coming from a discrete cofibration.

Let $L\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ be a functor. For each $x \in \mathcal{B}$ let $F(x)$ be the set of connected components of the functor $L(x,-)\colon \mathcal{C} \to \mathbf{Set}$. Each element of $F(x)$ is a connected subfunctor of $L(x,-)$ and $L(x,-)$ is a disjoint union of all elements of $F(x)$:

$$L(x,-) = \bigcup_{U \in F(x)} U \text{ (disjoint union).}$$

For a morphism $f\colon x \to x'$ in $\mathcal{B}$ the induced morphism $f^*\colon L(x',-) \to L(x,-)$ maps each connected component of $L(x',-)$ into a connected component of $L(x,-)$, hence defines a map $F(x') \to F(x)$. Thus F becomes a functor $\mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$. Let $\pi_{x,y}\colon L(x,y) \to F(x)$ for $x \in \mathcal{B}, y \in \mathcal{C}$ denote the natural map: For $U \in F(x)$ we have $\pi_{x,y}^{-1}(\{U\}) = U(y)$.

Likewise, for each $y \in \mathcal{C}$ let $G(y)$ be the set of connected components of the functor $L(-,y)\colon \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$. Then G naturally becomes a functor $\mathcal{C} \to \mathbf{Set}$. Let $\sigma_{x,y}\colon L(x,y) \to G(y)$ denote the natural map: For $V \in G(y)$ we have $\sigma_{x,y}^{-1}(\{V\}) = V(x)$.

Consider the category of elements $\mathbf{E}(F)$ with projection $p\colon \mathbf{E}(F) \to \mathcal{B}$, and the category of elements $\mathbf{E}(G)$ with projection $q\colon \mathbf{E}(G) \to \mathcal{C}$. For $(x,U) \in \mathbf{E}(F)$ and $(y,V) \in \mathbf{E}(G)$ define

$$M((x,U),(y,V)) = \pi_{x,y}^{-1}(\{U\}) \cap \sigma_{x,y}^{-1}(\{V\}) = U(y) \cap V(x).$$

This is a subset of $L(x,y)$ and

$$L(x,y) = \bigcup_{U \in F(x), V \in G(y)} M((x,U),(y,V)) \quad \text{(disjoint union).}$$

Let $f\colon x \to x'$ be a morphism in $\mathcal{B}$, and $g\colon y \to y'$ a morphism in $\mathcal{C}$. For $U' \in F(x')$ let $U = f^*(U')$, and for $V \in G(y)$ let $V' = g_*(V)$. Then we have the morphism $f\colon (x,U) \to (x',U')$ in $\mathbf{E}(F)$ and the morphism $g\colon (y,V) \to (y',V')$ in $\mathbf{E}(G)$.

We have commutative diagrams

$$\begin{array}{ccc} L(x,y) & \xrightarrow{\pi_{x,y}} & F(x) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ L(x',y) & \xrightarrow[\pi_{x',y}]{} & F(x') \end{array} \qquad \begin{array}{ccc} L(x,y) & \xrightarrow{\pi_{x,y}} & F(x) \\ {\scriptstyle g_*}\downarrow & \nearrow{\scriptstyle \pi_{x,y'}} & \\ L(x,y') & & \end{array}$$

$$\begin{array}{ccc} L(x,y) & \xrightarrow{\sigma_{x,y}} & G(y) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ L(x,y') & \xrightarrow[\sigma_{x,y'}]{} & G(y') \end{array} \qquad \begin{array}{ccc} L(x,y) & \xrightarrow{\sigma_{x,y}} & G(y) \\ {\scriptstyle f^*}\uparrow & \nearrow{\scriptstyle \sigma_{x',y}} & \\ L(x',y) & & \end{array}$$

By the first and the forth of the diagrams we see that $f^*: L(x', y) \to L(x, y)$ maps the subset $M((x', U'), (y, V))$ into the subset $M((x, U), (y, V))$. Denote the resulting map

$$M((x', U'), (y, V)) \to M((x, U), (y, V))$$

by f^*, so that the diagram

$$\begin{array}{ccc} L(x, y) & \longleftarrow & M((x, U), (y, V)) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ L(x', y) & \longleftarrow & M((x', U'), (y, V)) \end{array}$$

commutes, where the horizontal arrows are the inclusion maps. By the second and the third of the diagrams we see that $g_*: L(x, y) \to L(x, y')$ maps the subset $M((x, U), (y, V))$ into the subset $M((x, U), (y', V'))$. Denote the resulting map

$$M((x, U), (y, V)) \to M((x, U), (y', V'))$$

by g_*, so that the diagram

$$\begin{array}{ccc} L(x, y) & \longleftarrow & M((x, U), (y, V)) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ L(x, y') & \longleftarrow & M((x, U), (y', V')) \end{array}$$

commutes. The sets $M((x, U), (y, V))$ together with thus defined maps f^* and g_* make a functor $M: \mathbf{E}(F)^{\mathrm{op}} \times \mathbf{E}(G) \to \mathbf{Set}$.

11.1. Proposition. *We have an isomorphism* $(p \times q)_! M \cong L$.

Proof. Use Proposition 2.6 and its dual. ∎

11.2. Proposition. *The distributor L is slicewise familially nearly representable if and only if M is slicewise nearly representable.*

Proof. Let $(x, U) \in \mathbf{E}(F)$. We shall show that $U: \mathcal{C} \to \mathbf{Set}$ is nearly representable if and only if $M((x, U), -): \mathbf{E}(G) \to \mathbf{Set}$ is nearly representable.

As

$$\bigcup_{V \in G(y)} V = L(-, y),$$

we have

$$\bigcup_{V \in G(y)} V(x) = L(x, y).$$

And $U(y) \subset L(x, y)$. Hence

$$\bigcup_{V \in G(y)} (U(y) \cap V(x)) = U(y).$$

By Proposition 2.6

$$(q_!(M((x,U),-)))(y) \cong \coprod_{V\in G(y)} M((x,U),(y,V)) = \coprod_{V\in G(y)} (U(y)\cap V(x)) \cong U(y),$$

hence

$$q_!(M((x,U),-)) \cong U.$$

Let $y \in \mathcal{C}$, $t \in U(y)$. Put $\Gamma = \mathrm{Aut}(y,t)$. The element t gives a morphism

$$\tau : \mathcal{C}(y,-)/\Gamma \to U.$$

Put $K = \sigma_{x,y}(t)$, the image of t under the map $\sigma_{x,y} : L(x,y) \to G(y)$. Then $(y,K) \in \mathbf{E}(G)$ and $t \in K(x)$. Hence $t \in U(y) \cap K(x) = M((x,U),(y,K))$. As Γ stabilizes t, Γ stabilizes K. Thus $\Gamma \subset \mathrm{Aut}((y,K),t)$. The element t gives a morphism

$$\tau' : \mathbf{E}(G)((y,K),-)/\Gamma \to M((x,U),-).$$

Through the isomorphisms

$$q_!(\mathbf{E}(G)((y,K),-)/\Gamma) \cong \mathcal{C}(y,-)/\Gamma$$

and

$$q_!(M((x,U),-)) \cong U,$$

the functor $q_!$ takes τ' to τ. As $q_!$ reflects isomorphisms, we see that τ is an isomorphism if and only if τ' is an isomorphism. This means that (y,t) is nearly universal for U if and only if $((y,K),t)$ is nearly universal for $M((x,U),-)$.

This proves that U is nearly representable if and only if $M((x,U),-)$ is nearly representable.

Therefore $L(x,-)$ is familially nearly representable if and only if $M((x,U),-)$ is nearly representable for all $U \in F(x)$.

Likewise, $L(-,y)$ is familially nearly representable if and only if $M(-,(y,V))$ is nearly representable for all $V \in G(y)$.

This proves the proposition.

■

Under the assumption of Theorem 7.9, if L is slicewise familially nearly representable, the factorization theorem can apply to M, from which a factorization for L results. We refrain from going into details.

11.3. Proposition. *Let $L : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ be a distributor. The following are equivalent.*

(i) L is slicewise familially nearly representable.

(ii) There exist a discrete fibration $p : \mathcal{B}' \to \mathcal{B}$, a discrete cofibration $q : \mathcal{C}' \to \mathcal{C}$, a slicewise nearly representable distributor $L' : \mathcal{B}'^{\mathrm{op}} \times \mathcal{C}' \to \mathbf{Set}$, and an isomorphism $L \cong (p \times q)_! L'$.

Proof. We have shown that (i) implies (ii). Let us show the converse. Let $p: \mathcal{B}' \to \mathcal{B}$ be a discrete fibration, $q: \mathcal{C}' \to \mathcal{C}$ a discrete cofibration, and $L': \mathcal{B}'^{\mathrm{op}} \times \mathcal{C}' \to \mathbf{Set}$ a slicewise nearly representable distributor. We shall show that $(p \times q)_! L'$ is slicewise familially nearly representable.

We may assume that $\mathcal{B}' = \mathbf{E}(H)$ for a functor $H: \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ and $\mathcal{C}' = \mathbf{E}(K)$ for a functor $K: \mathcal{C} \to \mathbf{Set}$, and p, q are the natural projections.

For $x \in \mathcal{B}$ we have by Proposition 2.6 that

$$((p \times q)_! L')(x, -) \cong \coprod_{a \in H(x)} q_!(L'((x, a), -)).$$

By assumption $L'((x, a), -): \mathbf{E}(K) \to \mathbf{Set}$ is nearly representable. By Proposition 3.5 it follows that $q_!(L'((x, a), -)): \mathcal{C} \to \mathbf{Set}$ is nearly representable. Hence $((p \times q)_! L')(x, -)$ is familially nearly representable.

Argue similarly for $((p \times q)_! L')(-, y)$.

■

We have also the notion of a *slicewise familially representable distributor*. By the same argument as above we see that it is exactly the composite of a discrete fibration, a slicewise representable distributor, and a discrete cofibration.

11.4. Remark. A distributor $L: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ whose one-sided slice $L(x, -)$ for every $x \in \mathcal{B}$ is familially representable is the same thing as a familially representable functor $\mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ in the sense of [Leinster, 2004, Appendix C.3].

References

H. Applegate and M. Tierney, Iterated cotriples, Reports of the Midwest Category Seminar IV, Lecture Notes in Mathematics 137, Springer-Verlag, 1970, 56–99.

F. Borceux, "Handbook of Categorical Algebra 1", Cambridge University Press, 1994.

A. Carboni and P. Johnstone, Connected limits, familial representability and Artin glueing, Math. Struct. Comp. Science 5(1995), 441–459.

S. Eilenberg and S. Mac Lane, General theory of natural equivalences, Trans. Amer. Math. Soc. 58 (1945), 231–294.

P. J. Freyd and A. Scedrov, "Categories, Allegories", North-Holland, 1990.

S. Mac Lane, "Categories for the Working Mathematician", second edition, Springer-Verlag, 1978.

T. Leinster,"Higher Operads, Higher Categories", Cambridge University Press, 2004.

L. Puig, "Frobenius Categories versus Brauer Blocks", Birkhäuser, 2009.

D. Tambara, Finite categories with pushouts, Theory and Applications of Categories vol.30 (2015), 1017–1031.

S. Tull, Quotient categories and phases, Theory and Applications of Categories vol.34 (2019), 573–603.

Department of Mathematics and Physics, Hirosaki University
Hirosaki 036-8561, Japan

Email: tambara@hirosaki-u.ac.jp

This article may be accessed at http://www.tac.mta.ca/tac/