*Theory and Applications of Categories,* Vol. 33, No. 17, 2018, pp. 476–491.

# DOUBLE POWER MONAD PRESERVING ADJUNCTIONS ARE FROBENIUS

CHRISTOPHER TOWNSEND

ABSTRACT. We give a direct proof that between two toposes, $\mathcal{F}$ and $\mathcal{E}$, bounded over a base topos $\mathcal{S}$, adjunctions $L \dashv R : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}}$ over $\mathbf{Loc}_{\mathcal{S}}$ are Frobenius if and only if $R$ commutes with the double power locale monad and finite coproducts. The proof uses only certain categorical properties of the category of locales, **Loc**. This implies that between categories axiomatized to behave like categories of locales, it does not make a difference whether maps are defined as structure preserving adjunctions (i.e. those that commute with the double power monads) or Frobenius adjunctions.

## 1. Introduction

Let $\mathcal{C}$ be a cartesian category and $S$ some distinguished object of $\mathcal{C}$. The exponential $S^X$ may not exist in $\mathcal{C}$ for another object $X$ but the presheaf $\mathcal{C}(\_ \times X, S) : \mathcal{C}^{op} \longrightarrow Set$ can always be considered (and it is representable if and only if $S^X$ exists). The object $S$ is said to be *double exponentiable* if the exponential $\mathcal{C}(\_, S)^{\mathcal{C}(\_\times X,S)}$ exists in the presheaf category $[\mathcal{C}^{op}, Set]$ and is representable for any object $X$ of $\mathcal{C}$. The Sierpiński locale $\mathbb{S}$ in the category of locales, **Loc**, provides a non-trivial example for which $\mathbb{S}^X$ does not always exists (not all locales are locally compact) but double exponentiation does. This determines a monad structure on **Loc**, the *double power locale monad* and indeed for any double exponentiable object $S$ a monad structure $(\mathbb{P} : \mathcal{C} \longrightarrow \mathcal{C}, \eta : Id \longrightarrow \mathbb{P}, \mu : \mathbb{P}\mathbb{P} \longrightarrow \mathbb{P})$ is determined by the universal properties of double exponentiation. Its functor part is given by defining $\mathbb{P}X$ to be the object that represents $\mathcal{C}(\_, S)^{\mathcal{C}(\_\times X,S)}$. The unit $\eta$ is the double exponential transpose of the identity natural transformation $S^X \longrightarrow S^X$. Note that we write $S^X$ as shorthand for $\mathcal{C}(\_ \times X, S)$.

Say now that we have two cartesian categories, $\mathcal{D}$ and $\mathcal{C}$, each with a distinguished object, say $S_{\mathcal{D}}$ and $S_{\mathcal{C}}$ respectively, both double exponentiable. It can be shown that if $L \dashv R : \mathcal{D} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{C}$ is an adjunction that satisfies Frobenius reciprocity[1] and there is an isomorphism $S_{\mathcal{D}} \cong RS_{\mathcal{C}}$, then $R$ commutes with the double exponentiation monads. That is, there is a natural isomorphism $\phi : R\mathbb{P}_{\mathcal{C}} \xrightarrow{\cong} \mathbb{P}_{\mathcal{D}}R$ that commutes with the monad structure in the obvious manner (explicitly, $\phi(R\eta^{\mathcal{C}}) = \eta^{\mathcal{D}}_R$ and $\phi(R\mu^{\mathcal{C}}) = \mu^{\mathcal{D}}_R(\mathbb{P}_{\mathcal{D}}\phi)\phi_{\mathbb{P}_{\mathcal{C}}}$).

Received by the editors 2017-11-27 and, in final form, 2018-05-24.
Transmitted by Robert Paré. Published on 2018-05-28.
2010 Mathematics Subject Classification: 06D22, 18D35, 18B40, 22A22.
Key words and phrases: topos, locale, geometric morphism, Frobenius reciprocity, power monad.


[1] i.e. each map $(L\pi_1, \epsilon_X L\pi_2) : L(W \times RX) \longrightarrow LW \times X$ is an isomorphism, where $\epsilon$ is the counit of $L \dashv R$.

The purpose of this paper is to provide a partial converse.

At first sight this may seem odd. By taking $S_{\mathcal{C}} = 1$, the terminal object of $\mathcal{C}$, it is easy to construct a monad isomorphism for any adjunction. So certainly we will have to place some restrictions on the distinguished object $S$ to get a converse. In order to show that there are some meaningful examples where we can hope to get a converse consider the main results of [T10b] and [T13]. Let $\mathcal{F}$ and $\mathcal{E}$ be two toposes and let $A$ be the collection of isomorphism classes of geometric morphisms $f : \mathcal{F} \longrightarrow \mathcal{E}$. The paper [T10b] establishes a bijection between $A$ and $B$ where $B$ is defined to be the collection of isomorphism classes of order enriched adjunctions

$$\{L \dashv R : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}} | L \dashv R \text{ Frobenius, } R \text{ preserves Sierpiński}\}.$$

On the other hand, the paper [T13] establishes a bijection between $A$ and $C$, where $C$ is defined to be the collection of isomorphism classes of order enriched adjunctions

$$\{L \dashv R : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}} | R\mathbb{P}_{\mathcal{E}} \cong \mathbb{P}_{\mathcal{F}}R \text{ and } R \text{ preserves finitary coproduct}\}.$$

It is natural to then ask whether a direct proof is available which shows that Frobenius adjunctions between categories of locales are the same thing as adjunctions that commute with the double power locale monad. In other words can we prove that $B$ is in bijection with $C$ directly, referring only to properties of **Loc**? Or is this relationship something particular to geometric morphisms? The question is relevant to investigating the category of locales axiomatically. A key viewpoint provided by the [T13] result (i.e. $A \cong C$) is that geometric morphisms can be seen as structure preserving maps, this view being justified by the fact that the double power locale monad can be axiomatized as a double exponential and this axiomatization used to give structure to the theory of locales. But in other work, particularly [T17], we see that to get a good localic theory of geometric morphisms it is the Frobenius reciprocity condition that appears to be key. By exploiting Frobenius reciprocity a number of results about geometric morphisms can be shown via their representation as Frobenius adjunctions between categories of locales (e.g. pullback stable hyperconnected-localic factorization, results on boundedness etc). So it is quite natural to ask the following question: What is the right notion of continuous map when working localically? Is it those adjunctions that satisfy Frobenius reciprocity or should we be looking at the double power monad preserving adjunctions? What we show here is that, provided we restrict to the bounded case (in the sense of corresponding to bounded geometric morphisms), it makes no difference.

## 2. Summary contents

The paper is structured as follows. The following section covers a number of categorical preliminaries. It clarifies that our context is order enriched cartesian categories and provides a number of categorical characterisations of the connected components adjunction of any internal groupoid.

In the next section we recall the definition of a category of spaces; that is, a category axiomatized to behave like the category of locales. We then recall the definition of triquotient surjection, and provide criteria for when certain forks (i.e. diagrams $\cdot \rightrightarrows \cdot \longrightarrow \cdot$) are coequalizer diagrams, stable under product.

After that we define a notion of morphism between categories of spaces which we call Sierpiński morphism. We show how each geometric morphism determines, uniquely up to isomorphism, a Sierpiński morphism. Sierpiński morphisms are certain double power monad preserving adjunctions, and we recall how being double power monad preserving is the same thing as having an extension to Kleisli categories. Since the morphisms of the Kleisli categories of the double power monad are natural transformations $\mathbb{S}^X \longrightarrow \mathbb{S}^Y$, being double power monad preserving is the same thing as having a contravariant extension to natural transformations. We recall how if an adjunction between categories of spaces satisfies Frobenius reciprocity it then necessarily extends to natural transformations. This is key to showing that Frobenius adjunctions are examples of Sierpiński morphisms between spaces.

Next we define when a Sierpiński morphism is bounded and show that if a geometric morphism is bounded its corresponding Sierpiński morphism is bounded. We also explain, using our preliminary categorical results characterising connected component adjunctions, that bounded Sierpiński morphisms necessarily arise from groupoids; the definition of boundedness implies that the domain category of spaces is a category of $\mathbb{G}$-objects for a groupoid $\mathbb{G}$ internal to the codomain category of spaces. Applying this observation we see that a bounded Sierpiński morphism between categories of locales necessarily arises from a bounded geometric morphism. We then go on to our main result which is that bounded Sierpiński morphisms are Frobenius and show that indeed they are stably Frobenius as bounded Sierpiński morphisms are slice stable.

Finally we provide an omnibus theorem the highlight of which is to show that between categories of spaces, bounded over some base $\mathcal{C}$, an adjunction is a Sierpiński morphism if and only if it is Frobenius.

In order to keep this paper to a reasonable length we have had to refer to a number of other papers for proofs. None of the proofs involves anything but known topos theory ([J02]) or basic categorical reasoning relative to cartesian categories.

## 3. Categorical preliminaries

In this section we gather together some results about cartesian categories. They are effectively taken from [T17], which contains detailed proofs. Our categories are *order enriched*, which means that universal properties establish order isomorphisms between (partially ordered) homsets rather than just bijections. We state our results for order enriched categories but may refer to proofs that are only given for ordinary categories; this is because the extension of the universal property from being a bijection to an order isomorphism can be seen to be trivial.

An order enriched adjunction $L \dashv R : \mathcal{D} \rightleftarrows \mathcal{C}$ between order enriched cartesian categories is *Frobenius* if for each object $X$ of $\mathcal{C}$ and $W$ of $\mathcal{D}$ the map $(L\pi_1, \epsilon_X L\pi_2)$ : $L(W \times RX) \longrightarrow LW \times X$ is an isomorphism where $\epsilon$ is the counit of the adjunction. Being Frobenius is two conditions away from being an equivalence:

3.1. LEMMA. *Let $L \dashv R : \mathcal{D} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{C}$ be an order enriched Frobenius adjunction between order enriched cartesian categories. If $L1 \cong 1$ and $\eta$, the unit, is a regular monomorphism, then $L = R^{-1}$ (i.e. the adjunction is an equivalence).*

For the short proof of this lemma consult [T17] (or Lemma 3.4 of [T14], where it originally appears).

Any adjunction $L \dashv R : \mathcal{D} \rightleftarrows \mathcal{C}$ can be sliced at any object $X$ of $\mathcal{C}$. The sliced adjunction $L_X \dashv R_X : \mathcal{D}/RX \rightleftarrows \mathcal{C}/X$ is given by $L_X(f) =$ 'the adjoint transpose of $f$' and $R_X(g) = Rg$. An order enriched adjunction is *stably* Frobenius if $L_X \dashv R_X$ is Frobenius for every $X$. If $f : X \longrightarrow Y$ is a morphism of a cartesian order enriched category $\mathcal{C}$ then there is a stably Frobenius order enriched adjunction $\Sigma_f \dashv f^* : \mathcal{C}/X \rightleftarrows \mathcal{C}/Y$, where $f^*$ is pullback. This is the *pullback adjunction* of $f$. We will write $Z_g$ for a typical object of $\mathcal{C}/X$; i.e. for $g : Z \longrightarrow X$. So, for example, $\Sigma_f(Z_g) = Z_{fg}$. We write $Z_X$ for the projection $\pi_1 : X \times Z \longrightarrow X$, an object of $\mathcal{C}/X$, and $\Sigma_X \dashv X^*$ for the pullback adjunction $\mathcal{C}/X \rightleftarrows \mathcal{C}$ of $! : X \longrightarrow 1$. An adjunction of this form is known as a *slice*. All pullback adjunctions can be seen to be slices because for any $f : X \longrightarrow Y$, $\mathcal{C}/X$ is isomorphic to $(\mathcal{C}/Y)/X_f$. We observe that a monomorphism of $\mathcal{C}/X$ which is split in $\mathcal{C}$ is necessarily a regular monomorphism. To prove this observation say $n : Y_f \longrightarrow Z_g$ is split in $\mathcal{C}$ by $k : Z \longrightarrow Y$ (i.e. $kn = Id_Y$), then it is readily checked that $n$ is the equalizer of $Z_g \xrightarrow{(g,k)} Y_X \xrightarrow{Id_X \times n} Z_X$ and $Z_g \xrightarrow{(g,Id_Z)} Z_X$. Note that the unit of $\Sigma_X \dashv X^*$ is split in $\mathcal{C}$ since, at $Y_f$, it is given by $Y \xrightarrow{(f,Id)} X \times Y$ which is split by $\pi_2$. The existence of this split is key to proving that any stably Frobenius adjunction, if its domain is a slice of some base category, is itself a slice:

3.2. LEMMA. *Let $\mathcal{C}$ and $\mathcal{D}$ be two order enriched cartesian categories, $X$ an object of $\mathcal{C}$ and say $\mathcal{D}$ comes equipped with an order enriched adjunction $\Sigma_{\mathcal{D}} \dashv \mathcal{D}^* : \mathcal{D} \rightleftarrows \mathcal{C}$ back to $\mathcal{C}$. Then any stably Frobenius adjunction $L \dashv R : \mathcal{C}/X \rightleftarrows \mathcal{D}$ over $\mathcal{C}$ is equivalent to the slice of $\mathcal{D}$ at $L1$.*

By 'over $\mathcal{C}$' we mean that $L \dashv R$ comes equipped with a natural isomorphism $X^* \cong R\mathcal{D}^*$.

PROOF. The adjunction $L \dashv R$ can be factored as

$$\mathcal{C}/X \underset{\eta_1^*}{\overset{\Sigma_{\eta_1}}{\rightleftarrows}} (\mathcal{C}/X)/RL1 \underset{R_{L1}}{\overset{L_{L1}}{\rightleftarrows}} \mathcal{D}/L1 \underset{L1^*}{\overset{\Sigma_{L1}}{\rightleftarrows}} \mathcal{D}$$

where $\eta$ is the unit of the adjunction $L \dashv R$. The adjunction $L_{L1}\Sigma_{\eta_1} \dashv \eta^* R_{L1}$ must be Frobenius; this is because the composition of two Frobenius adjunctions is Frobenius and $L_{L1} \dashv R_{L1}$ is Frobenius by assumption that $L \dashv R$ is stably Frobenius. Since it is easy to see that $L_{L1}\Sigma_{\eta_1}$ preserves 1 by our first lemma all that remains is to check that the unit of $L_{L1}\Sigma_{\eta_1} \dashv \eta^* R_{L1}$ is a regular monomorphism. But the unit of this adjunction is (up to isomorphism) a factor of the unit of the adjunction $\Sigma_X \dashv X^*$ and so is split in $\mathcal{C}$ because the unit of $\Sigma_X \dashv X^*$ is split in $\mathcal{C}$. ∎

We now recall how the Frobenius reciprocity condition can be used to characterise the connected component adjunctions of internal groupoids. Let $\mathbb{G} = (G_1 \overset{d_0}{\underset{d_2}{\rightrightarrows}} G_0, ...)$ be a groupoid internal to an order enriched cartesian category $\mathcal{C}$. Writing $[\mathbb{G}, \mathcal{C}]$ for the category of $\mathbb{G}$-objects and $\mathbb{G}$-homomorphisms, there is an order enriched functor $\mathbb{G}^* : \mathcal{C} \longrightarrow [\mathbb{G}, \mathcal{C}]$ which sends any object $X$ to the $\mathbb{G}$-object $(X_{G_0}, d_1 \times Id : G_1 \times X \longrightarrow G_0 \times X)$; i.e. $\pi_1 : G_0 \times X \longrightarrow G_0$ with trivial action. Its left adjoint, if it exists, is written $\Sigma_{\mathbb{G}}$, and the adjunction $\Sigma_{\mathbb{G}} \dashv \mathbb{G}^*$ is the *connected components* adjunction of $\mathbb{G}$.

The following lemma provides characterizations of the connected components adjunction for any internal groupoid:

3.3. Lemma. *Let $\mathcal{D}$ and $\mathcal{C}$ be order enriched cartesian categories and $\Sigma_{\mathcal{D}} \dashv \mathcal{D}^* : \mathcal{D} \rightleftarrows \mathcal{C}$ an order enriched adjunction. The following are equivalent:*

*(i) There exists an internal groupoid $\mathbb{G}$ in $\mathcal{C}$ and an equivalence $\Theta : [\mathbb{G}, \mathcal{C}] \longrightarrow \mathcal{D}$ over $\mathcal{C}$.*

*(ii) There exists an object $W$ of $\mathcal{D}$ such that $! : W \longrightarrow 1$ is an effective descent morphism and $(\Sigma_{\mathcal{D}})_W : \mathcal{D}/W \longrightarrow \mathcal{C}/\Sigma_{\mathcal{D}} W$ is an equivalence.*

*(iii) There exists an object $G_0$ of $\mathcal{C}$ and a stably Frobenius adjunction $\mathbb{T} \dashv U : \mathcal{C}/G_0 \rightleftarrows \mathcal{D}$ over $\mathcal{D}$ with $U$ monadic.*

Recall that a morphism $f : X \longrightarrow Y$ is of *effective descent* if the pullback functor $f^* : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$ is monadic.

Proof. A more detailed proof is given in [T17]. The (i) implies (iii) step follows from easily verified properties of $[\mathbb{G}, \mathcal{C}]$. Send any object $X_f$ of $\mathcal{C}/G_0$ to $(G_1 \times_{G_0} X, m \times Id : G_1 \times_{G_0} G_1 \times_{G_0} X \longrightarrow G_1 \times_{G_0} X)$ to define $\mathbb{T}$ where $m$ is the groupoid multiplication. This determines a free functor left adjoint to the forgetful functor, $U : [\mathbb{G}, \mathcal{C}] \longrightarrow \mathcal{C}/G_0$. The algebras of the monad induced on $\mathcal{C}/G_0$ by $\mathbb{T} \dashv U$ are exactly the $\mathbb{G}$-objects and so $U$ is monadic. Given a $\mathbb{G}$-homomorphism $\phi : (Y_g, b : G_1 \times_{G_0} Y \longrightarrow Y) \longrightarrow (Z_h, c : G_1 \times_{G_0} Z \longrightarrow Z)$ and $X_f$ an object of $\mathcal{C}/G_0$ it is clear how to construct a $\mathbb{G}$-homeomorphism $\mathbb{T}(X \times_Z Y) \cong \mathbb{T}(X_f) \times_{(Z_h, c)} (Y_g, b)$ ($(g, x, y) \mapsto (g, x, gy)$ in one direction and $(g', x', y') \mapsto (g', x', (g')^{-1} y')$ in the other). This is the key step in showing that $\mathbb{T} \dashv U$ is stably Frobenius.

That (iii) implies (ii) follows from the previous lemma, applied to the adjunction $\mathbb{T} \dashv U : \mathcal{C}/G_0 \rightleftarrows \mathcal{D}$. This shows that $\mathcal{C}/G_0$ is equivalent to $\mathcal{D}/W$ over $\mathcal{C}$ where $W$ is taken to be $\mathbb{T}1$. So there must be an equivalence $\phi : \mathcal{C}/G_0 \longrightarrow \mathcal{D}/W$ and as it is over $\mathcal{D}$ there is a natural isomorphism $\Sigma_{\mathcal{D}} \Sigma_W \phi \cong \Sigma_{G_0}$ which by evaluating at 1 shows that $G_0 \cong \Sigma_{\mathcal{D}} W$. But we must also then have that $\Sigma_{G_0} \phi^{-1} \cong \Sigma_{\mathcal{D}} \Sigma_W$ and so $\phi^{-1}$ can be taken to be $(\Sigma_{\mathcal{D}})_W$ (recall that all objects of $\mathcal{D}/W$ can also be seen to be morphisms of $\mathcal{D}/W$, with 1 as codomain). To complete the (iii) implies (ii) implication note that since $U$ is monadic, $! : W \longrightarrow 1$ must be of effective descent.

For (ii) implies (i) apply $\Sigma_{\mathcal{D}}$ to the groupoid ($W \times W \overset{\pi_2}{\underset{\pi_1}{\rightrightarrows}} W, ...$) in order to obtain a groupoid $\mathbb{G}$ in $\mathcal{C}$; i.e. $G_0 = \Sigma_{\mathcal{D}} W$, $G_1 = \Sigma_{\mathcal{D}}(W \times W)$ etc. As $! : W \longrightarrow 1$ is an effective descent morphism, $\mathcal{D}$ is the category of algebras of the pullback adjunction determined by $! : W \longrightarrow 1$. By construction of $\mathbb{G}$ the category of $\mathbb{G}$-objects is the same thing, up to equivalence, as this category of algebras, given that $\mathcal{C}/G_0 \simeq \mathcal{D}/W$. ∎

Finally for this preliminary section we recall some notation and results about lattices internal to order enriched cartesian categories. If $\mathcal{C}$ is an order enriched cartesian category then an *order internal distributive lattice* is an internal distributive lattice such that binary and nullary joins(meets) are left (right) adjoint to finitary diagonals. Being an order internal distributive lattice is therefore a property of the object and not additional structure on the object. If $\mathbb{S}$ is an order internal distributive lattice in an order enriched cartesian category $\mathcal{C}$ then we use $\mathcal{C}_{\mathbb{P}}^{op}$ for the full subcategory of $[\mathcal{C}^{op}, Set]$ consisting of all objects of the form $\mathbb{S}^X$; it is an order enriched category as its morphisms are natural transformations that can be ordered pointwise. If further $\mathcal{C}$ has finitary coproducts and product distributes over them then $\mathcal{C}_{\mathbb{P}}^{op}$ can be seen to have finite products (e.g. $\mathbb{S}^X \times \mathbb{S}^Y$ is given by $\mathbb{S}^{X+Y}$; the diagonal on $\mathbb{S}^X$ is given by $\mathbb{S}^{\nabla}$ where $\nabla : X + X \longrightarrow X$ is the codiagonal). Every object of $\mathcal{C}_{\mathbb{P}}^{op}$ is an order internal distributive lattice. The notation $\mathcal{C}_{P_L}^{op}$ ($\mathcal{C}_{P_U}^{op}$) is used for the categories that are the same as $\mathcal{C}_{\mathbb{P}}^{op}$ but with morphisms only those natural transformations that are internal join (meet) homomorphisms.

## 4. Categories of spaces

We are now in a position to define our categorical context.

4.1. Definition. *A category $\mathcal{C}$ is a* category of spaces *provided:*

*Axiom 1. It is an order enriched cartesian category with finite coproducts.*

*Axiom 2. For any morphism $f : X \longrightarrow Y$ the functor $f^* : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$ preserves finite coproducts.*

*Axiom 3. $\mathcal{C}$ has a distinguished order internal distributive lattice $\mathbb{S} = (\mathbb{S}, \sqcup_{\mathbb{S}} : \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S}, \sqcap_{\mathbb{S}} : \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S}, 0_{\mathbb{S}} : 1 \longrightarrow \mathbb{S}, 1_{\mathbb{S}} : 1 \longrightarrow \mathbb{S})$ such that for any object $X$ the pullback $i^* : \mathcal{C}(X, \mathbb{S}) \longrightarrow Sub(X)$ is an injection for both $i = 0_{\mathbb{S}}$ and $i = 1_{\mathbb{S}}$.*

*Axiom 4. $\mathbb{S}$ is double exponentiable.*

*Axiom 5. Any natural transformation $\alpha : \mathbb{S}^X \longrightarrow \mathbb{S}^Y$ which is also a distributive lattice homomorphism is of the form $\mathbb{S}^f$ for some unique $f : Y \longrightarrow X$.*

*Axiom 6. Inflationary (deflationary) idempotents split in $\mathcal{C}_{P_L}$ ($\mathcal{C}_{P_U}$).*

*Axiom 7. For any equalizer diagram $E \xrightarrow{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in $\mathcal{C}$ the diagram*

$$\mathbb{S}^X \times \mathbb{S}^X \times \mathbb{S}^Y \overset{\sqcap(Id\times\sqcup)(Id\times Id\times\mathbb{S}^f)}{\underset{\sqcap(Id\times\sqcup)(Id\times Id\times\mathbb{S}^g)}{\rightrightarrows}} \mathbb{S}^X \xrightarrow{\mathbb{S}^e} \mathbb{S}^E$$

*is a coequalizer in $\mathcal{C}_{\mathbb{P}}^{op}$.*

Of course, following the notation of the introduction, we use $(\mathbb{P}, \eta : Id \longrightarrow \mathbb{P}, \mu : \mathbb{P}\mathbb{P} \longrightarrow \mathbb{P})$ for the monad induced on $\mathcal{C}$ by the assumption that $\mathbb{S}$ is double exponentiable and will refer to this as the *double power monad* of $\mathcal{C}$. The object $\mathbb{S}$ is called the *Sierpiński object* of $\mathcal{C}$.

4.2. Example. For any topos $\mathcal{E}$ the category of locales over that topos, $\mathbf{Loc}_{\mathcal{E}}$, is a category of spaces. See [T05] and [T10a].

4.3. Example. If $\mathbb{G}$ is a localic groupoid that is not étale complete, then $[\mathbb{G}, \mathbf{Loc}]$ is a category of spaces that is not of the form $\mathbf{Loc}_{\mathcal{E}}$ for any topos $\mathcal{E}$, [T16]. So the class of all categories of spaces is strictly larger than the class determined by the previous example.

To show the axioms in action, let us start with a straightforward lemma:

4.4. Lemma. *If $\mathcal{C}$ is a category of spaces then $\mathbb{P}$ reflects isomorphisms.*

Proof. We are given $f : X \longrightarrow Y$ such that $\mathbb{P}f$ is an isomorphism; that is, there exists $g : \mathbb{P}Y \longrightarrow \mathbb{P}X$ such that $g\mathbb{P}f = Id_{\mathbb{P}X}$ and $(\mathbb{P}f)g = Id_{\mathbb{P}Y}$. Clearly we have to only show that $\mathbb{S}^f$ is an isomorphism since then its inverse must be a distributive lattice homomorphism and so must be of the form $\mathbb{S}^h$ for some $h$ by application of Axiom 5; then by application of the uniqueness part of Axiom 5 we see that $h$ must be $f^{-1}$. It is an exercise in the universal properties of double exponentiation to show that $\mathbb{S}^{\eta_Y}\alpha_g$ is an inverse for $\mathbb{S}^f$, where $\alpha_g : \mathbb{S}^X \longrightarrow \mathbb{S}^{\mathbb{P}Y}$ is the double exponential transpose of $g$. ∎

We now recall the definition of triquotient surjection. These maps were originally introduced by Plewe for locales ([P97]) and the paper [T10a] contains a detailed discussion on how they work using our axiomatic approach. A key aspect of the class of triquotient surjections is that it is a common generalisation of both open and proper surjections.

4.5. Definition. *Given a morphism $p : Z \longrightarrow Y$ in a category of spaces, a* triquotient assignment *on $p$ is a natural transformation $p_\# : \mathbb{S}^Z \longrightarrow \mathbb{S}^Y$ satisfying*

*(i) $\sqcap_{\mathbb{S}^Y}(p_\# \times Id_{\mathbb{S}^Y}) \sqsubseteq p_\# \sqcap_{\mathbb{S}^Z}(Id_{\mathbb{S}^Z} \times \mathbb{S}^p)$ and*

*(ii) $p_\# \sqcup_{\mathbb{S}^Z}(Id_{\mathbb{S}^Z} \times \mathbb{S}^p) \sqsubseteq \sqcup_{\mathbb{S}^Y}(p_\# \times Id_{\mathbb{S}^Y})$.*

*Further $p$ is a* triquotient surjection *if it has a triquotient assignment $p_\#$ such that $p_\#\mathbb{S}^p = Id_{\mathbb{S}^Y}$.*

The usual 'Beck-Chevalley for pullback squares' result holds relative to any category of spaces: if $p_\#$ is a triquotient assignment on $p : Z \longrightarrow Y$ then for any $f : X \longrightarrow Y$ there is a triquotient assignment $(\pi_1)_\#$ on $\pi_1 : X \times_Y Z \longrightarrow X$ such that $(\pi_1)_\#\mathbb{S}^{\pi_2} = \mathbb{S}^f p_\#$. Notice that if $p : Z \longrightarrow Y$ is a triquotient surjection witnessed by the triquotient assignment $p_\# : \mathbb{S}^Z \longrightarrow \mathbb{S}^Y$, then $p_\#(1) = 1$ and $p_\#(0) = 0$. Conversely if $p : Z \longrightarrow Y$ has a triquotient assignment $p_\#$ with $p_\#(1) = 1$ and $p_\#(0) = 0$ then $p_\#(\mathbb{S}^p(b)) = p_\#(0 \sqcup \mathbb{S}^p(b)) \sqsubseteq p_\#(0) \sqcup b = b$ and order dually $b \sqsubseteq p_\#(\mathbb{S}^p(b))$ and so $p$ is a triquotient surjection. Using this characterization of triquotient surjection it is clear from Beck-Chevalley for pullback squares that triquotient surjections are pullback stable. Note that having a triquotient assignment is stable under composition and, in particular, triquotient surjections are closed under composition.

Next we will provide a lemma about triquotient assignments, relating the existence of a $\mathbb{S}^{(\text{-})}$-splitting to showing that certain forks are coequalizers. The lemma will be used to show that bounded Sierpiński morphisms are Frobenius. To state the lemma we need a definition: a fork $X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Q$ (i.e. with $qf = qg$) is $\mathbb{S}^{(\text{-})}$*-split* if there exist natural transformations $q_\# : \mathbb{S}^Y \longrightarrow \mathbb{S}^Q$ and $\alpha : \mathbb{S}^X \longrightarrow \mathbb{S}^Y$ such that (a) $q_\#$ is a triquotient assignment on $q$, (b) $q_\# \mathbb{S}^q = Id_{\mathbb{S}^Q}$, (c) $\mathbb{S}^q q_\# = \alpha \mathbb{S}^f$ and (d) $\alpha \mathbb{S}^g = Id_{\mathbb{S}^Y}$. For example the kernel pair of any triquotient surjection is $\mathbb{S}^{(\text{-})}$-split: take $\alpha$ to be $(\pi_1)_\#$.

4.6. Lemma. *If* $X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Q$ *is a* $\mathbb{S}^{(\text{-})}$*-split fork then for any object* $Z$*,* $X \times Z \underset{g\times Id}{\overset{f\times Id}{\rightrightarrows}} Y \times Z \xrightarrow{q\times Id} Q \times Z$ *is a coequalizer in* $\mathcal{C}$*.*

Proof. Firstly we can reduce to the case $Z = 1$. If $q_\#$ and $\alpha$ are splittings for the diagram, then $q_\#^Z$ and $\alpha^Z$ are splittings for $X \times Z \underset{g\times Id}{\overset{f\times Id}{\rightrightarrows}} Y \times Z \xrightarrow{q\times Id} Q \times Z$. The exponential $\alpha^Z : \mathbb{S}^{X\times Z} \longrightarrow \mathbb{S}^{Y\times Z}$ is defined by $\alpha^Z_V = \alpha_{Z\times V}$ for any object $V$; similarly for $q_\#$. It is straightforward to verify that $q_\#^Z$ is a triquotient assignment on $q \times Id : Y \times Z \longrightarrow Q \times Z$ (recall that the natural transformation $\sqcap_{\mathbb{S}^X}$ is really just 'post compose with $\sqcap_{\mathbb{S}} : \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S}$' for any $X$).

The remainder of the proof essentially follows the proof of Proposition 6.2 of [T16]. We have a diagram

$$\begin{tikzcd} \mathbb{S}^Q \arrow[r, shift left, "\mathbb{S}^q"] & \mathbb{S}^Y \arrow[l, shift left, "q_\#"] \arrow[r, shift left=2, "\mathbb{S}^f"] \arrow[r, shift right, "\mathbb{S}^g"'] & \mathbb{S}^X \arrow[l, shift left=3, "\alpha"] \end{tikzcd}$$

which is a split fork in $\mathcal{C}_{\mathbb{P}}^{op}$ and so certainly $\mathbb{S}^q$ is the equalizer of $\mathbb{S}^f$ and $\mathbb{S}^g$. So for any $p : Y \longrightarrow W$ with $pf = pg$ we therefore have that $\mathbb{S}^p$ factors (uniquely) as $\mathbb{S}^p\beta$ for some natural transformation $\beta$ (it is given by $q_\#\mathbb{S}^p$). By Axiom 5 it therefore only remains to check that $\beta$ is a distributive lattice homomorphism. Since we have already observed $q_\#$ preserves 0 and 1 we just need to show that $\beta$ preserves binary meet and join, and for this it is sufficient to check $q_\#\mathbb{S}^p(c_1) \sqcap q_\#\mathbb{S}^p(c_2) \sqsubseteq q_\#\mathbb{S}^p(c_1 \sqcap c_2)$ and $q_\#\mathbb{S}^p(c_1 \sqcup c_2) \sqsubseteq q_\#\mathbb{S}^p(c_1) \sqcup q_\#\mathbb{S}^p(c_2)$. But

$$\begin{aligned} q_\#\mathbb{S}^p(c_1) \sqcap q_\#\mathbb{S}^p(c_2) &\sqsubseteq q_\#(\mathbb{S}^p c_1 \sqcap \mathbb{S}^q q_\# \mathbb{S}^p c_2) \\ &= q_\#(\mathbb{S}^p c_1 \sqcap \alpha \mathbb{S}^f \mathbb{S}^p c_2) \\ &= q_\#(\mathbb{S}^p c_1 \sqcap \alpha \mathbb{S}^g \mathbb{S}^p c_2) \text{ (since } pf = pg) \\ &= q_\#(\mathbb{S}^p c_1 \sqcap \mathbb{S}^p c_2) \\ &= q_\#\mathbb{S}^p(c_1 \sqcap c_2) \end{aligned}$$

and $q_\#\mathbb{S}^p(c_1 \sqcup c_2) \sqsubseteq q_\#\mathbb{S}^p(c_1) \sqcup q_\#\mathbb{S}^p(c_2)$ follows an order dual proof and so we are done. ∎

One can probably state this lemma more generally to cover full pullback stability of the regular epimorphisms induced by $\mathbb{S}^{(\_)}$-splittings, but we do not need that in applications investigated here. The lemma shows that if $! : W \longrightarrow 1$ is a triquotient surjection then $\pi_1 : X \times W \longrightarrow X$ is a regular epimorphism for any $X$; this is key to showing that $W \longrightarrow 1$ is an effective descent morphism. Originally observed for locales by Plewe, the general result is:

4.7. PROPOSITION. *Any triquotient surjection is an effective descent morphism.*

PROOF. A full proof is in [T04]. ∎

## 5. Sierpiński morphisms

We now define our notion of morphism between categories of spaces.

5.1. DEFINITION. *If $\mathcal{C}$ and $\mathcal{D}$ are two categories of spaces, then a* Sierpiński morphism *$f : \mathcal{D} \longrightarrow \mathcal{C}$ consists of an order enriched adjunction $\Sigma_f \dashv f^* : \mathcal{D} \rightleftarrows \mathcal{C}$ together with a monad morphism $(f^*, \phi)$ such that $\phi : f^*\mathbb{P}_{\mathcal{C}} \longrightarrow \mathbb{P}_{\mathcal{D}} f^*$ is an isomorphism and $f^*$ preserves finitary coproducts.*

The Sierpiński object $\mathbb{S}$ is isomorphic to $\mathbb{P}0$; so the right adjoint of any Sierpiński morphism necessarily preserves the Sierpiński object.

5.2. EXAMPLE. If $f : \mathcal{F} \longrightarrow \mathcal{E}$ is a geometric morphism then there is an order enriched adjunction $\Sigma_f \dashv f^* : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}}$; $f^*$ is given by pullback in the category of toposes ($\Sigma_f \dashv f^*$ is the *pullback adjunction* of the geometric morphism $f$). The paper [T13] describes how this adjunction necessarily commutes with the double power locale monad and so is a Sierpiński morphism as we have defined here.

Our next lemma provides another class of examples, showing how Frobenius adjunctions give rise to Sierpiński morphisms. To prove it we will need to recall how lifting adjunctions to Kleisli categories corresponds to the right adjoint commuting with monads. An adjunction is said to be a lifting to Kleisli categories if it commutes with the left adjoints $\mathbb{P} : \mathcal{C} \longrightarrow \mathcal{C}_{\mathbb{P}}$. Lemma 3.3 of [T13] shows that such a lifting exists if and only if the right adjoint commutes with the monads. In the context of double power monads giving a lifting for $L \dashv R : \mathcal{D} \rightleftarrows \mathcal{D}$ is the same thing as having an order enriched adjunction $\bar{R}^{op} \dashv \bar{L}^{op} : \mathcal{C}^{op}_{\mathbb{P}_{\mathcal{C}}} \rightleftarrows \mathcal{D}^{op}_{\mathbb{P}_{\mathcal{D}}}$ such that (i) for any morphism $f : X \longrightarrow Y$ of $\mathcal{C}$, $\bar{R}^{op}\mathbb{S}^f_{\mathcal{C}} = \mathbb{S}^{Rf}_{\mathcal{D}}$, (ii) for any morphism $g : W \longrightarrow V$ of $\mathcal{D}$, $\bar{L}^{op}\mathbb{S}^g_{\mathcal{D}} = \mathbb{S}^{Lg}_{\mathcal{C}}$, and (iii) $\bar{\eta} = \mathbb{S}^{\epsilon}_{\mathcal{C}}$, $\bar{\epsilon} = \mathbb{S}^{\eta}_{\mathcal{D}}$ where $\bar{\eta}(\bar{\epsilon})$ is the unit(counit) of $\bar{R}^{op} \dashv \bar{L}^{op}$. In summary, therefore, a right adjoint commutes with the double power monad if and only if the adjunction extends to natural transformations in the obvious manner.

5.3. LEMMA. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories of spaces and $L \dashv R : \mathcal{D} \rightleftarrows \mathcal{C}$ a Frobenius order enriched adjunction for which there is an isomorphism $\mathbb{S}_{\mathcal{D}} \xrightarrow{\cong} R\mathbb{S}_{\mathcal{C}}$. Then $L \dashv R$ is the adjunction of a Sierpiński morphism $\mathcal{D} \longrightarrow \mathcal{C}$.*

This lemma is effectively the content of Proposition 5.1 of [T10b].

Proof. If $\alpha : \mathbb{S}_{\mathcal{C}}^X \longrightarrow \mathbb{S}_{\mathcal{C}}^Y$ is a natural transformation then define $\bar{R}^{op}(\alpha)$ by setting $[\bar{R}^{op}(\alpha)]_W$ to

$$\begin{aligned}\mathcal{D}(W \times RX, \mathbb{S}_{\mathcal{D}}) &\cong \mathcal{D}(W \times RX, R\mathbb{S}_{\mathcal{C}}) \cong \mathcal{C}(L(W \times RX), \mathbb{S}_{\mathcal{C}})\\ &\cong \mathcal{C}(LW \times X, \mathbb{S}_{\mathcal{C}}) \xrightarrow{\alpha_{LW}} \mathcal{C}(LW \times Y, \mathbb{S}_{\mathcal{C}}) \cong \mathcal{D}(W \times RY, \mathbb{S}_{\mathcal{D}})\end{aligned}$$

for each object $W$ of $\mathcal{D}$. If $\beta : \mathbb{S}_{\mathcal{D}}^W \longrightarrow \mathbb{S}_{\mathcal{D}}^V$ is a natural transformation then define $\bar{L}^{op}(\beta)$ by setting $[\bar{L}^{op}(\beta)]_X$ to

$$\begin{aligned}\mathcal{C}(X \times LW, \mathbb{S}_{\mathcal{C}}) &\cong \mathcal{C}(L(RX \times W), \mathbb{S}_{\mathcal{C}}) \cong \mathcal{D}(RX \times W, R\mathbb{S}_{\mathcal{C}})\\ &\cong \mathcal{D}(RX \times W, \mathbb{S}_{\mathcal{D}}) \xrightarrow{\beta_{RX}} \mathcal{D}(RX \times V, \mathbb{S}_{\mathcal{D}}) \cong \mathcal{C}(X \times LV, \mathbb{S}_{\mathcal{C}})\end{aligned}$$

for each object $X$ of $\mathcal{C}$, where we are repressing the isomorphisms induced by twist isomorphisms such as $\tau : RX \times W \longrightarrow W \times RX$ in order to ease the presentation. By checking that the triangular identities hold (since they hold for $L \dashv R$) it can be established that $\bar{R}^{op} \dashv \bar{L}^{op}$. That an extension is defined is routine from construction. To complete the proof we therefore only need to confirm that $R$ preserves finitary coproduct. That $R$ preserves nullary coproduct (i.e. 0) follows because of our assumption that $R$ preserves the Sierpiński object; there are isomorphisms $\mathbb{P}_{\mathcal{D}}0_{\mathcal{D}} \cong R\mathbb{P}_{\mathcal{C}}0_{\mathcal{C}} \cong \mathbb{P}_{\mathcal{D}}R0_{\mathcal{C}}$ and so $0_{\mathcal{D}} \cong R0_{\mathcal{C}}$ by Lemma 4.4. For binary coproduct, let $X$ and $Y$ be objects of $\mathcal{C}$ and $W$ an object of $\mathcal{D}$; then

$$\begin{aligned}\mathcal{D}(W \times R(X+Y), \mathbb{S}_{\mathcal{D}}) &\cong \mathcal{C}(LW \times (X+Y), \mathbb{S}_{\mathcal{C}}) \cong \mathcal{C}(LW \times X, \mathbb{S}_{\mathcal{C}}) \times \mathcal{C}(LW \times Y, \mathbb{S}_{\mathcal{C}})\\ &\cong \mathcal{D}(W \times RX, \mathbb{S}_{\mathcal{D}}) \times \mathcal{D}(W \times RY, \mathbb{S}_{\mathcal{D}}) \cong \mathcal{D}(W \times (RX + RY), \mathbb{S}_{\mathcal{D}}).\end{aligned}$$

Therefore $\mathbb{S}_{\mathcal{D}}^{R(X+Y)} \cong \mathbb{S}_{\mathcal{D}}^{RX+RY}$ and so $R(X+Y) \cong RX + RY$ by Lemma 4.4. ∎

So Frobenius adjunctions give rise to Sierpiński morphisms categorically; this paper's primary aim is to shown that the implication can reverse. As a step towards achieving that aim we now show how the adjunction of a Sierpiński morphism, extended to natural transformations, necessarily preserves triquotient assignments in both directions:

5.4. Lemma. *If $f : \mathcal{D} \longrightarrow \mathcal{C}$ is a Sierpiński morphism then the extended adjunction $\bar{\Sigma}_f \dashv \bar{f}^* : \mathcal{D}_{\mathbb{P}_{\mathcal{D}}} \rightleftarrows \mathcal{C}_{\mathbb{P}_{\mathcal{C}}}$ preserves triquotient assignments in both directions. That is, (a) if $p_\# : \mathbb{S}_{\mathcal{C}}^Z \longrightarrow \mathbb{S}_{\mathcal{C}}^Y$ is a triquotient assignment on $p : Z \longrightarrow Y$, a morphism of $\mathcal{C}$, then ${\bar{f}^*}{}^{op} p_\#$ is a triquotient assignment on $f^*p$ and (b) if $q_\# : \mathbb{S}_{\mathcal{D}}^W \longrightarrow \mathbb{S}_{\mathcal{D}}^V$ is a triquotient assignment on $q : W \longrightarrow V$, a morphism of $\mathcal{D}$, then ${\bar{\Sigma}_f}{}^{op} q_\#$ is a triquotient assignment on $\Sigma_f q$. In particular both $f^*$ and $\Sigma_f$ preserve triquotient surjections.*

Proof. For any object $X$ in a category of spaces the join map $\sqcup_{\mathbb{S}^X} : \mathbb{S}^X \times \mathbb{S}^X \longrightarrow \mathbb{S}^X$ is the left adjoint of $\mathbb{S}^X \xrightarrow{\mathbb{S}^\nabla} \mathbb{S}^{X+X} \cong \mathbb{S}^X \times \mathbb{S}^X$ where $\nabla : X + X \longrightarrow X$ is the codiagonal. Any order enriched functor necessarily preserves the property of being a left adjoint in the order enrichment and so will preserve the join map if it preserves $\mathbb{S}^\nabla$; but if an order enriched functor preserves binary coproduct then it must preserve the codiagonal and so its extension to natural transformations must preserve the join $\sqcup_{\mathbb{S}^X}$. This applies to both $\Sigma_f$ and $f^*$ as both preserve binary coproduct ($\Sigma_f$ since it is a left adjoint, $f^*$ by definition). An order dual argument shows that extensions to natural transformations also preserve binary meet on $\mathbb{S}^X$. This is enough to show that triquotient assignments are preserved in both directions because triquotient assignments are defined in terms of certain inequalities involving binary meet and join on $\mathbb{S}^X$. ∎

We now define boundedness for Sierpiński morphisms, which is the restriction that we need in order to get our converse for Lemma 5.3.

5.5. Definition. *A Sierpiński morphism $p : \mathcal{D} \longrightarrow \mathcal{C}$ is bounded provided there exists an object $W$ of $\mathcal{D}$ such that the map $! : W \longrightarrow 1$ is a triquotient surjection and $(\Sigma_p)_W : \mathcal{D}/W \longrightarrow \mathcal{C}/\Sigma_p W$ is an equivalence of categories.*

Note that Lemma 3.3 can be applied to this definition because triquotient surjections are of effective descent. Therefore Sierpiński morphisms are the connected component adjunctions of groupoids internal to the codomain. This agrees with our well known intuition about bounded geometric morphisms, e.g. [JT84], which is that they can be represented using localic groupoids. Let us make the relationship precise:

5.6. Example. Let $p : \mathcal{E} \longrightarrow \mathcal{S}$ be a bounded geometric morphism. If $B$ is a bound for $p$ then the unique map $! : [\mathbb{N} \twoheadrightarrow B] \longrightarrow 1$ is an open surjection and so is a triquotient surjection. Here $[\mathbb{N} \twoheadrightarrow B]$ is the locale of surjections from the naturals onto $B$. The paper [T14] recalls this fact and shows that $(\Sigma_p)_{[\mathbb{N} \twoheadrightarrow B]} : \mathbf{Loc}_{\mathcal{E}}/[\mathbb{N} \twoheadrightarrow B] \longrightarrow \mathbf{Loc}_{\mathcal{S}}/\Sigma_p[\mathbb{N} \twoheadrightarrow B]$ is an equivalence. Therefore $\Sigma_p \dashv p^*$ corresponds to a bounded Sierpiński morphism.

On the other hand, say we have a bounded Sierpiński morphism $p : \mathbf{Loc}_{\mathcal{E}} \longrightarrow \mathbf{Loc}_{\mathcal{S}}$. Then we know from [T13] that there exists a geometric morphism $p : \mathcal{E} \longrightarrow \mathcal{S}$, unique up to isomorphism, such that $\Sigma_p \dashv p^*$ is the pullback adjunction of $p$. But is the geometric morphism $p$ necessarily bounded? By (ii) of Lemma 3.3 there is an equivalence $\mathbf{Loc}_{\mathcal{E}} \simeq [\mathbb{G}, \mathbf{Loc}_{\mathcal{S}}]$ over $\mathbf{Loc}_{\mathcal{S}}$ and by restricting to discrete locales one sees that there is an equivalence $\mathcal{E} \simeq B\mathbb{G}$ over $\mathcal{S}$, where $B\mathbb{G}$ is the topos of $\mathbb{G}$-equivariant sheaves which is well known (e.g. B3.1.14(b) of [J02]) to be bounded over $\mathcal{S}$. This observation, together with the example just given, justifies our use of the term 'bounded' in the context of Sierpiński morphisms. We can now state our main result:

5.7. Theorem. *If $p : \mathcal{D} \longrightarrow \mathcal{C}$ is a bounded Sierpiński morphism then $\Sigma_p \dashv p^*$ is Frobenius.*

Proof. Say $W \longrightarrow 1$ is the triquotient surjection such that there is an equivalence $\mathcal{D}/W \simeq \mathcal{C}/\Sigma_p W$ over $\mathcal{C}$. Since the pullback adjunction $\Sigma_{\Sigma_p W} \dashv (\Sigma_p W)^*$ is Frobenius we know that the composite adjunction

$$\mathcal{D}/W \underset{W^*}{\overset{\Sigma_W}{\rightleftarrows}} \mathcal{D} \underset{p^*}{\overset{\Sigma_p}{\rightleftarrows}} \mathcal{C} \quad (*)$$

is Frobenius. Now for any object $V$ of $\mathcal{D}$ the fork

$$W \times W \times V \underset{\pi_2 \times Id}{\overset{\pi_1 \times Id}{\rightrightarrows}} W \times V \xrightarrow{\pi_2} V$$

is $\mathbb{S}_{\mathcal{D}}^{(\text{-})}$-split (it is the kernel pair of the triquotient surjection $\pi_2 : W \times V \longrightarrow V$). By Lemma 4.6 we therefore know that it is a coequalizer diagram in $\mathcal{D}$, stable under product. In particular for any object $X$ of $\mathcal{C}$,

$$W \times W \times V \times p^*X \underset{\pi_2 \times Id \times Id}{\overset{\pi_1 \times Id \times Id}{\rightrightarrows}} W \times V \times p^*X \xrightarrow{\pi_2 \times Id} V \times p^*X$$

is a coequalizer diagram in $\mathcal{D}$ and so since the left adjoint $\Sigma_p$ preserves coequalizers we get that

$$\Sigma_p(W \times W \times V \times p^*X) \underset{\Sigma_p(\pi_2 \times Id \times Id)}{\overset{\Sigma_p(\pi_1 \times Id \times Id)}{\rightrightarrows}} \Sigma_p(W \times V \times p^*X) \xrightarrow{\Sigma_p(\pi_2 \times Id)} \Sigma_p(V \times p^*X)$$

is a coequalizer diagram in $\mathcal{C}$.

Now $\bar{\Sigma_p}^{op}$ preserves triquotient assignments and so the fork

$$\Sigma_p(W \times W \times V) \underset{\Sigma_p(\pi_2 \times Id)}{\overset{\Sigma_p(\pi_1 \times Id)}{\rightrightarrows}} \Sigma_p(W \times V) \xrightarrow{\Sigma_p \pi_2} \Sigma_p V$$

is $\mathbb{S}_{\mathcal{C}}^{(\text{-})}$-split and is therefore a coequalizer diagram in $\mathcal{C}$, stable under product. In particular

$$\Sigma_p(W \times W \times V) \times X \underset{\Sigma_p(\pi_2 \times Id) \times Id}{\overset{\Sigma_p(\pi_1 \times Id) \times Id}{\rightrightarrows}} \Sigma_p(W \times V) \times X \xrightarrow{\Sigma_p \pi_2 \times Id} \Sigma_p V \times X$$

is a coequalizer diagram in $\mathcal{C}$. This effectively completes the proof because as the composite adjunction (*) is Frobenius we can see that the two pairs of arrows that determine $\Sigma_p(V \times p^*X)$ and $\Sigma_p V \times X$ (i.e. $(\Sigma_p(\pi_1 \times Id \times Id), \Sigma_p(\pi_2 \times Id \times Id))$ and $(\Sigma_p(\pi_1 \times Id) \times Id, \Sigma_p(\pi_2 \times Id) \times Id))$ are isomorphic (recall $W \times V = \Sigma_W W^* V$ etc). ∎

Now that we know that bounded Sierpiński morphisms are Frobenius we can get results about them by using known results about Frobenius adjunctions:

5.8. Proposition. *The composition of two bounded Sierpiński morphisms is bounded.*

Proof. The proof is effectively taken from Proposition 14.1 of [T17]. Say $p_0 : \mathcal{D}_0 \longrightarrow \mathcal{D}$ and $p : \mathcal{D} \longrightarrow \mathcal{C}$ are two bounded Sierpiński morphisms with objects $W_0$ and $W$ of $\mathcal{D}_0$ and $\mathcal{D}$ respectively such that $W_0 \longrightarrow 1$ and $W \longrightarrow 1$ are both triquotient surjections and there are equivalences $(\Sigma_{p_0})_{W_0} : \mathcal{D}_0/W_0 \xrightarrow{\simeq} \mathcal{D}/\Sigma_{p_0} W_0$ and $(\Sigma_p)_W : \mathcal{D}/W \xrightarrow{\simeq} \mathcal{C}/\Sigma_p W$. By slicing the equivalence $(\Sigma_{p_0})_{W_0}$ at $\pi_1 : W_0 \times p_0^* W \longrightarrow W_0$ we obtain an equivalence $\mathcal{D}_0/W_0 \times p_0^* W \simeq \mathcal{D}/\Sigma_{p_0}(W_0 \times p_0^* W)$ and since $\Sigma_{p_0} \dashv p_0^*$ is Frobenius there is therefore an equivalence $\mathcal{D}_0/W_0 \times p_0^* W \simeq \mathcal{D}/\Sigma_{p_0} W_0 \times W$. But $\mathcal{D}/\Sigma_{p_0} W_0 \times W$ is a slice of $\mathcal{D}/W$ which is itself a slice $\mathcal{C}$ and therefore $\mathcal{D}/\Sigma_{p_0} W_0 \times W$ is a slice of $\mathcal{C}$. So to complete the proof we just need to check that $! : W_0 \times p_0^* W \longrightarrow 1$ is a triquotient surjection. $W_0 \longrightarrow 1$ is a triquotient surjection and triquotient surjections are closed under composition and so it remains to check that $\pi_1 : W_0 \times p^* W \longrightarrow W_0$ is a triquotient surjection which will follow by the pullback stability of triquotient surjections provided we can show that $p_0^* W \longrightarrow 1$ is a triquotient surjection. But we know that $p_0^* W \longrightarrow 1$ is a triquotient surjection because $p_0^*$ preserves triquotient surjections. ∎

However to make further progress and so get a good theory of geometric morphisms using their localic representations, it is clear from [T17] that we need the adjunctions to be *stably* Frobenius and not just Frobenius. In fact the hard work on this has already been done in [T13]:

5.9. PROPOSITION. *If $p : \mathcal{D} \longrightarrow \mathcal{C}$ is a Sierpiński morphism and $X$ is an object of $\mathcal{C}$ then the sliced adjunction $(\Sigma_p)_X \dashv p_X^* : \mathcal{D}/p^*X \rightleftarrows \mathcal{C}/X$ determines a Sierpiński morphism $p_X : \mathcal{D}/p^*X \longrightarrow \mathcal{C}/X$. Further if $p$ is bounded, so is $p_X$.*

Recall from [T12] that if $\mathcal{C}$ is a category of spaces then so is $\mathcal{C}/X$; for example $\mathbb{S}_X$ is a Sierpiński object of the slice. The pullback adjunction $\Sigma_X \dashv X^*$ determines a Sierpiński morphism $\mathcal{C}/X \longrightarrow \mathcal{C}$. In particular note that this Sierpiński morphism is bounded (take $W = Id_X$; the identity map is a triquotient surjection).

PROOF. Consult Theorem 6.13 of [T13]. Whilst Theorem 6.13 is stated and proved as a property of adjunctions between categories of locales, care was taken in the proof to ensure that it only used properties of locales that could be proved using our axiomatic approach.

For the 'Further' part, say $W \longrightarrow 1$ is an effective descent morphism and $(\Sigma_p)_W : \mathcal{D}/W \longrightarrow \mathcal{C}/\Sigma_p W$ is an equivalence. The pullback morphism $(p^*X)^* : \mathcal{D} \longrightarrow \mathcal{D}/p^*X$ will preserve triquotient surjections because it is the right adjoint of a Sierpiński morphism. Therefore $! : W_{p^*X} \longrightarrow 1_{\mathcal{D}/p^*X}$ (i.e. $\pi_1 : p^*X \times W \longrightarrow p^*X$) is a triquotient surjection relative to $\mathcal{D}/p^*X$. Finally $(\mathcal{D}/p^*X)/W_{p^*X}$ must be equivalent to $(\mathcal{C}/X)/\Sigma_{p_X}(W_{p^*X})$; this can be seen by slicing the equivalence $\mathcal{D}/W \simeq \mathcal{C}/\Sigma_p W$. ∎

Combining this with our main result (Theorem 5.7) we see that:

5.10. PROPOSITION. *The adjunction of any bounded Sierpiński morphism is stably Frobenius.*

Our final result clarifies the main aim of this paper which is to show that between categories of spaces, bounded over a base, it makes no difference whether we define Sierpiński morphisms as Frobenius adjunctions or double power monad preserving adjunctions:

5.11. THEOREM. *If $p' : \mathcal{D}' \longrightarrow \mathcal{C}$ and $p : \mathcal{D} \longrightarrow \mathcal{C}$ are two bounded Sierpiński morphisms then the following conditions on order enriched adjunctions $L \dashv R : \mathcal{D}' \rightleftarrows \mathcal{D}$ over $\mathcal{C}$ are equivalent:*

*(a) $L \dashv R$ is stably Frobenius.*
*(b) $L \dashv R$ is Frobenius.*
*(c) $L \dashv R$ is the adjunction of a Sierpiński morphism over $\mathcal{C}$.*

*(d) $L \dashv R$ is the adjunction of a bounded Sierpiński morphism over $\mathcal{C}$.*

Proof. That (a) implies (b) is clear because (b) requires less of the adjunction. For (b) implies (c) we can appeal to Lemma 5.3 provided we can show, assuming (b), that $R$ preserves the Sierpiński object. But $L \dashv R$ is over $\mathcal{C}$ and both $p'$ and $p$ preserve the Sierpiński object as they are Sierpiński morphisms.

Showing (c) implies (a) in fact reduces to showing (c) implies (b). To see this assume (c) and let $V$ be any object of $\mathcal{D}$. The adjunction $L_V \dashv R_V$ is an adjunction of a Sierpiński morphism which is over $\mathcal{C}$ via the adjunctions $\Sigma_{p'}\Sigma_{RV} \dashv (RV)^*(p')^*$ and $\Sigma_p\Sigma_V \dashv V^*p^*$, both of which are the adjunctions of bounded Sierpiński morphisms (the composition of bounded Sierpiński morphisms is bounded). So if we have proved that (c) implies (b) we can conclude that $L_V \dashv R_V$ is Frobenius. This is true for all $V$ and so $L \dashv R$ is stably Frobenius.

To prove (c) implies (b) let us now assume that $L \dashv R$ is the adjunction of a Sierpiński morphism. Let $W$ be the object of $\mathcal{D}$ that witnesses that $p$ is bounded; i.e. $W \longrightarrow 1$ is a triquotient surjection and $(\Sigma_p)_W : \mathcal{D}/W \longrightarrow \mathcal{C}/\Sigma_p W$ is an equivalence. Now the composite adjunction

$$\mathcal{D}'/RW \underset{R_W}{\overset{L_W}{\rightleftarrows}} \mathcal{D}/W \underset{[(\Sigma_p)_W]^{-1}}{\overset{(\Sigma_p)_W}{\rightleftarrows}} \mathcal{C}/\Sigma_p W$$

is naturally isomorphic to

$$\mathcal{D}'/RW \underset{(R\eta^{\mathcal{D}}_W)^*}{\overset{\Sigma_{R\eta^{\mathcal{D}}_W}}{\rightleftarrows}} \mathcal{D}'/(p')^*\Sigma_p W \underset{(p')^*_{\Sigma_p W}}{\overset{(\Sigma_{p'})_{\Sigma_p W}}{\rightleftarrows}} \mathcal{C}/\Sigma_p W$$

where $\eta^{\mathcal{D}}$ is the unit of $\Sigma_p \dashv p^*$ (recall that $Rp^* \cong (p')^*$) and so $L_W \dashv R_W$ must be Frobenius because $\Sigma_{p'} \dashv (p')^*$ is stably Frobenius by assumption that $p'$ is bounded. But then the adjunction $L\Sigma_{RW} \dashv (RW)^*R$ must be Frobenius since it can be factored as $\Sigma_W L_W \dashv R_W W^*$ and so we can proceed exactly as in the proof of Theorem 5.7: the adjunction $L \dashv R$ is a factor of a Frobenius adjunction where the first factor is $\Sigma_{RW} \dashv RW^* : \mathcal{D}'/RW \rightleftarrows \mathcal{D}'$ with $RW \longrightarrow 1$ a triquotient surjection (because $R$ preserves triquotient surjections). This completes the proof of (c) implies (b).

Certainly (d) implies (c) as (c) requires less of the Sierpiński morphism.

To complete we prove that (a) implies (d) and from our earlier observations we just need to prove boundedness. Let $W'$ be the object of $\mathcal{D}'$ that witnesses that $p'$ is bounded; i.e. $W' \longrightarrow 1$ is a triquotient surjection and $(\Sigma_{p'})_{W'} : \mathcal{D}'/W' \longrightarrow \mathcal{C}/\Sigma_{p'}W'$ is an equivalence. Then the adjunction

$$\mathcal{D}'/W' \underset{(W')^*}{\overset{\Sigma_{W'}}{\rightleftarrows}} \mathcal{D}' \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

is stably Frobenius, over $\mathcal{C}$ and its domain is equivalent to a slice of $\mathcal{C}$. It follows that Lemma 3.2 can be applied to show that $\mathcal{D}'/W' \simeq \mathcal{D}/LW'$ which proves boundedness. ∎

In particular, we note as a final observation, that if two composable Sierpiński morphisms $f$ and $g$ are such that $fg$ is bounded and $f$ is bounded, then $g$ is bounded.

## 6. Acknowledgement

The author is embarrassed that he is mostly quoting his own work. This has been done on the grounds of efficiency. I would like to take this opportunity to acknowledge the many great written works on topos theory and category theory, without which my research would not have been possible, notably [J02]. I have also benefited from the works of Vickers, Moerdijk, Streicher, Janelidze, Bunge, Taylor, Kock, Joyal and Tierney amongst others. I have freely dipped into all your work over the years and my work would not be possible if you had not put in the immense effort needed to complete your work. Thank you.

## 7. Dedication

This paper is dedicated to my wife, Beth.

## References

[J02] Johnstone, P.T. *Sketches of an elephant: A topos theory compendium.* Vols 1, 2, Oxford Logic Guides **43**, **44**, Oxford Science Publications, 2002.

[JT84] Joyal, A. and Tierney, M. *An Extension of the Galois Theory of Grothendieck*, Memoirs of the American Mathematical Society **309**, 1984.

[P97] Plewe, T. *Localic triquotient maps are effective descent maps*, Math. Proc. Cambridge Philos. Soc. **122** (1997), 17-43.

[T04] Townsend, C.F. *An axiomatic account of weak localic triquotient assignments. Early draft.* www.christophertownsend.org.

[T05] Townsend, C.F. *A categorical account of the Hofmann-Mislove theorem.* Math. Proc. Camb. Phil. Soc. **139** (2005) 441-456.

[T10a] Townsend, C.F. *An axiomatic account of weak triquotient assignments in locale theory* Journal of Pure and Applied Algebra, Volume 214, Issue 6, June 2010, Pages 729-739

[T10b] Townsend, C.F. *A representation theorem for geometric morphisms* Appl. Categ. Struct. 18(6), 573-583 (2010)

[T12] Townsend, C.F. *Aspects of slice stability in Locale Theory* Georgian Mathematical Journal. Vol. 19, Issue 2, 317-374 (2012)

[T13] Townsend, C.F. *Representing geometric morphisms using power locale monads* Appl. Categ. Struct. 21(1), 15-47 (2013)

[T14] Townsend, C.F. *A localic proof of the localic groupoid representation of Grothendieck toposes* Proc. Amer. Math. Soc. 142, 859-866 (2014)

[T16] Townsend, C.F. *Stability of Properties of Locales Under Groups* Appl. Categ. Struct. 25(3), 363-380 (2016)

[T17] Townsend, C.F. *Hilsum-Skandalis maps as Frobenius adjunctions with application to geometric morphisms* Tbilisi Math. J. Volume 10, Issue 3 (2017), 83-120. (A special volume of Tbilisi Mathematical Journal in honor of P.J. Freyd and F.W. Lawvere.)

*8 Gordon Villas, Aylesbury Road, Tring,*
*HERTFORDSHIRE, HP23 4DJ, U.K.*

Email: `info@christophertownsend.org`

This article may be accessed at `http://www.tac.mta.ca/tac/`