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A CHARACTERIZATION OF FINAL FUNCTORS BETWEEN INTERNAL GROUPOIDS IN EXACT CATEGORIES

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Abstract. This paper provides three characterizations of final functors between internal groupoids in an exact category (in the sense of Barr). In particular, it is proved that a functor between internal groupoids is final if and only if it is internally full and essentially surjective.

1. Introduction

The *comprehensive factorization* of a functor was introduced by Street and Walters in [Street–Walters, 1973], where they showed that any functor $F\colon \mathbb{C} \to \mathbb{D}$ between arbitrary categories is the composite of an initial functor followed by a discrete opfibration, and these two classes of functors form an orthogonal factorization system. By duality, a similar factorization can be obtained by means of a final functor followed by a discrete fibration, and the latter too is known under the name of comprehensive factorization.

Let us recall that a functor $F\colon \mathbb{C} \to \mathbb{D}$ is called *final* if, for any object y in $\mathbb{D}$, the comma category $(y \downarrow F)$ is non-empty and connected. On the other hand, a *discrete fibration* is a functor $F\colon \mathbb{C} \to \mathbb{D}$ such that, for any object x in $\mathbb{C}$ and any arrow $g\colon y \to F(x)$ in $\mathbb{D}$, there exists a unique arrow f in $\mathbb{C}$ with codomain x and such that $F(f) = g$. Initial functors and discrete opfibrations are the obvious dual of the previous concepts.

It is an easy observation that if we restrict our attention to groupoids, then initial and final functors coincide, and the same is true for (discrete) fibrations and opfibrations. Moreover the following elementary and well-known result holds.

1.1. Proposition. *A functor between groupoids is final if and only if it is full and essentially surjective.*

An internal version of the comprehensive factorization system for functors between groupoids in an exact category (in the sense of [Barr, 1971]) has been developed by Bourn in [Bourn, 1987], where he provided an explicit description of the above factorization by means of the *décalage* functor. In that paper, Bourn considers internal discrete fibrations as a basic notion and defines final functors as their orthogonal class. As far as we know, the characterization of Proposition 1.1 has no internal counterpart in the literature. It is the aim of the present work to fill this gap.

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A study of the comprehensive factorization system in the context of semi-abelian categories has been carried out in [Cigoli–Mantovani–Metere, 2014], where final functors were shown to correspond to push forwards of crossed modules. As a consequence, a characterization of final functors was given in terms of the homotopy invariants π_0 and π_1. Namely, a functor F is final if and only if $\pi_0(F)$ is an isomorphism and $\pi_1(F)$ is a regular epimorphism. Our Corollary 4.3 provides a generalization of this result which holds in any exact category.

2. Internal groupoids

We fix here some notation and recall a few basic facts about (internal) groupoids. A(n internal) category $\mathbb{H}$ in a finitely complete category $\mathcal{C}$ is given by a diagram

$$H_1 \times_{(d,c)} H_1 \underset{p_2}{\overset{p_1}{\underset{m_H}{\rightrightarrows}}} H_1 \underset{d_H}{\overset{c_H}{\underset{\xleftarrow{e_H}}{\rightrightarrows}}} H_0$$

satisfying the usual axioms. $\mathbb{H}$ is a groupoid when, in addition, it comes equipped with an "inversion of arrows" morphism

$$i_H \colon H_1 \to H_1$$

satisfying the well-known axioms. We will avoid the use of subscripts as far as no confusion arises.

A(n internal) functor, denoted by $F\colon \mathbb{H} \to \mathbb{G}$, is given by a pair of morphisms (f_0, f_1) in $\mathcal{C}$ such that $f_0 d = d f_1$, $f_0 c = c f_1$, $e f_0 = f_1 e$ and $m_G(f_1 \times_{f_0} f_1) = f_1 m_H$:

$$\begin{array}{ccc} H_1 & \xrightarrow{f_1} & G_1 \\ d \downarrow \uparrow e \downarrow c & & d \downarrow \uparrow e \downarrow c \\ H_0 & \xrightarrow[f_0]{} & G_0 . \end{array}$$

We denote by $\mathsf{Gpd}(\mathcal{C})$ the category of groupoids in $\mathcal{C}$ and functors between them.

2.1. Some relevant classes of functors. A *discrete fibration* in $\mathcal{C}$ is a functor $F\colon \mathbb{H} \to \mathbb{G}$ between categories for which the square

$$\begin{array}{ccc} H_1 & \xrightarrow{f_1} & G_1 \\ d \,\vdots\, \downarrow c & & d \,\vdots\, \downarrow c \\ H_0 & \xrightarrow[f_0]{} & G_1 \end{array}$$

of solid arrows is a pullback. In the case of groupoids, thanks to the inversion morphisms, the commutative square with dashed downward arrows is a pullback too, hence F is also a discrete opfibration. It is easy to prove that discrete (op)fibrations are pullback stable.

Given a groupoid $\mathbb{G}$ and a morphism $f\colon X \to G_0$ in $\mathcal{C}$, the groupoid structure on $\mathbb{G}$ induces a groupoid structure over X, whose object of arrows is the object P in the pullback square

$$\begin{array}{ccc} P & \xrightarrow{\;v\;} & G_1 \\ {\scriptstyle (s,t)}\big\downarrow & & \big\downarrow{\scriptstyle (d,c)} \\ X \times X & \xrightarrow[f\times f]{} & G_0 \times G_0 . \end{array}$$

Moreover, we get a functor

$$\begin{array}{ccc} P & \xrightarrow{\;v\;} & G_1 \\ {\scriptstyle s}\downarrow\;{\scriptstyle u}\uparrow\;\downarrow{\scriptstyle t} & & {\scriptstyle d}\downarrow\;{\scriptstyle e}\uparrow\;\downarrow{\scriptstyle c} \\ X & \xrightarrow[f]{} & G_0 \end{array}$$

where $u = ((1,1),e)$. The pair (f,v) provides a cartesian lifting of f at $\mathbb{G}$, with respect to the functor $()_0\colon \mathsf{Gpd}(\mathcal{C}) \to \mathcal{C}$ sending any groupoid in $\mathcal{C}$ to its object of objects. This proves that $()_0$ is indeed a fibration (see also [Bourn, 2010]), so we are allowed to denote by $f^*\mathbb{G}$ the domain of the functor (f,v). The square above is also called the *joint pullback* of (d,c) along f.

In particular, given a morphism $F\colon \mathbb{H} \to \mathbb{G}$ in $\mathsf{Gpd}(\mathcal{C})$, one can factor F through $f_0^*\mathbb{G}$, as in the following diagram:

$$\begin{array}{ccccc} & & \overset{f_1}{\frown} & & \\ H_1 & \xrightarrow[\phi_F]{} & P & \xrightarrow[v]{} & G_1 \\ {\scriptstyle d}\downarrow\;\downarrow{\scriptstyle c} & & {\scriptstyle s}\downarrow\;\downarrow{\scriptstyle t} & & {\scriptstyle d}\downarrow\;\downarrow{\scriptstyle c} \\ H_0 & = & H_0 & \xrightarrow[f_0]{} & G_0 . \end{array}$$

In fact, the above procedure yields a factorization system for $\mathsf{Gpd}(\mathcal{C})$ given by bijective on objects and fully faithful functors, the latter being defined as those functors for which ϕ_F is an isomorphism. A functor F is said to be *full* if ϕ_F is a regular epimorphism and *faithful* if ϕ_F is a monomorphism.

2.2. PROPOSITION. *Let $\mathcal{C}$ be a regular category and consider the pullback*

$$\begin{array}{ccc} \mathbb{H} \times_{\mathbb{G}} \mathbb{H}' & \xrightarrow{\;\overline{F}\;} & \mathbb{H}' \\ {\scriptstyle \overline{F}'}\big\downarrow & & \big\downarrow{\scriptstyle F'} \\ \mathbb{H} & \xrightarrow[F]{} & \mathbb{G} \end{array}$$

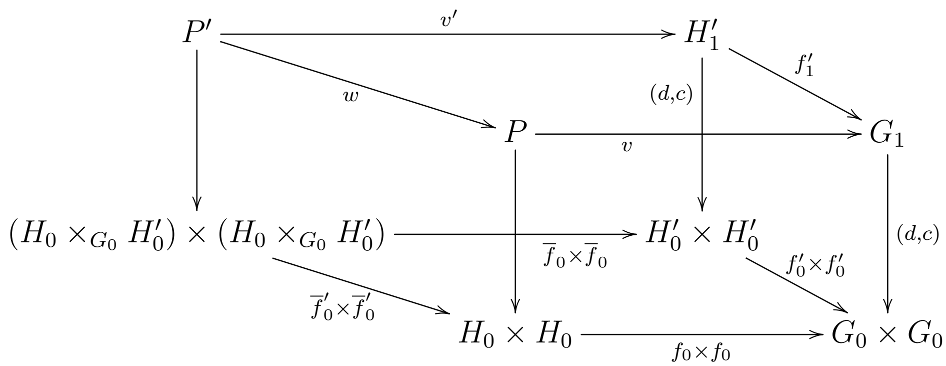
in $\mathsf{Gpd}(\mathcal{C})$.

1. *If F is full, $\overline{F}$ is full;*

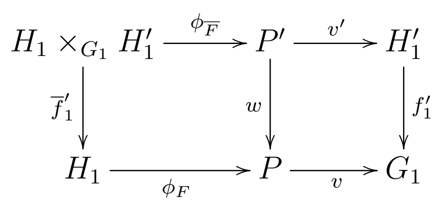
2. If F is faithful, $\overline{F}$ is faithful.

If moreover the arrow component f'_1 of F' is a regular epimorphism, then the converse of 1. and 2., respectively, is true.

Proof. Let us consider the following commutative diagram

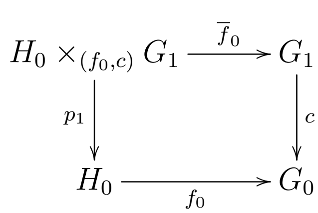


where the front and back faces of the cube are the pullbacks yielding the fully faithful liftings of f_0 and $\overline{f}_0$ at $\mathbb{G}$ and $\mathbb{H}'$ respectively. Since the bottom face is a pullback, then so is the top face. As a consequence, in the diagram below, since the whole rectangle and the right hand square are pullbacks, so is the square on the left hand side.



Now 1. and 2. follow by definition of full and faithful functor and from the fact that monomorphisms and regular epimorphisms are pullback stable in $\mathcal{C}$. If f'_1 is a regular epimorphism, so is w, and the converse of 1. follows trivially. For the converse of 2., let us observe that since in the left hand side pullback w and $\overline{f}'_1$ are regular epimorphisms, by elementary descent theory, if $\phi_{\overline{F}}$ is a monomorphism, so is ϕ_F. The latter argument is often referred to as part of the Barr-Kock Theorem (see [Bourn–Gran, 2004], for instance). ∎

Finally, given a functor $F\colon \mathbb{H} \to \mathbb{G}$, consider the pullback



in $\mathcal{C}$. F is said to be *essentially surjective* if the composite $d\overline{f}_0\colon H_0 \times_{(f_0,c)} G_1 \to G_0$ is a regular epimorphism.

2.3. Support and connected components. We shall suppose from now on that $\mathcal{C}$ is exact in the sense of [Barr, 1971]. Recall from [Bourn, 1987] that, for any groupoid $\mathbb{H}$ in $\mathcal{C}$, the pair (d, c) factors through an equivalence relation, denoted by $\operatorname{Supp}\mathbb{H}$, on H_0:

$$\Sigma H_1 \underset{r_2}{\overset{r_1}{\rightrightarrows}} H_0 \quad (s_0 \colon H_0 \to \Sigma H_1)\ ,$$

where ΣH_1 is the regular image of (d, c):

$$H_1 \xrightarrow{\sigma_H} \Sigma H_1 \xrightarrow{(r_1, r_2)} H_0 \times H_0\ , \qquad (d,c) = (r_1,r_2)\cdot\sigma_H .$$

Since $\mathcal{C}$ is exact, $\operatorname{Supp}\mathbb{H}$ is effective and we denote by $q_H \colon H_0 \to \pi_0(\mathbb{H})$ its quotient, which is also the coequalizer of (d, c).

In fact, the above procedure defines two functors:

$$\operatorname{Supp} \colon \mathsf{Gpd}(\mathcal{C}) \to \mathsf{Gpd}(\mathcal{C})\,; \qquad \pi_0 \colon \mathsf{Gpd}(\mathcal{C}) \to \mathcal{C}\,.$$

The first one, the *support* functor, sends each groupoid $\mathbb{H}$ to its associated equivalence relation, also called the support of $\mathbb{H}$. The second one, the *connected components* functor, sends each groupoid $\mathbb{H}$ to its object of connected components $\pi_0(\mathbb{H})$. Let us notice that $\pi_0 \cdot \operatorname{Supp} = \pi_0$.

The next two results are based on Proposition 1.1 in [Bourn, 2003] and will be useful afterwards.

2.4. Proposition. *A functor $F \colon \mathbb{R} \to \mathbb{S}$ between internal equivalence relations in an exact category is fully faithful if and only if $\pi_0(F)$ is monomorphic.*

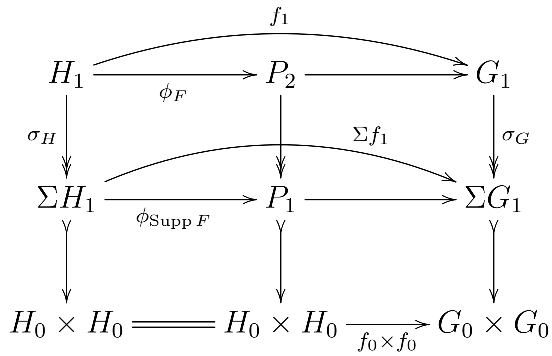
Proof. Let us draw the components of F vertically, and compute $\pi_0(F)$ as the induced arrow between the quotient objects of the domain and codomain:

$$\begin{array}{ccccc}
R_1 & \underset{r_2}{\overset{r_1}{\rightrightarrows}} & R_0 & \xrightarrow{q_R} & \pi_0(\mathbb{R}) \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \pi_0(F)} \\
S_1 & \underset{r_2}{\overset{r_1}{\rightrightarrows}} & S_0 & \xrightarrow[q_S]{} & \pi_0(\mathbb{S})\,.
\end{array}$$

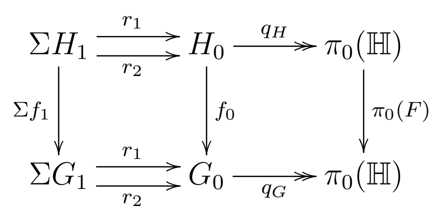
Since the two rows in the above diagram are exact forks (a regular epimorphism with its kernel pair), the thesis follows by Proposition 1.1 in [Bourn, 2003]. ∎

2.5. Corollary. *If a functor $F\colon \mathbb{H} \to \mathbb{G}$ between internal groupoids in an exact category is full, then $\pi_0(F)$ is monomorphic. The converse is true if $\mathbb{G}$ is an equivalence relation.*

Proof. Let us consider the commutative diagram



where the right hand side squares are pullbacks (i.e. P_1 and P_2 yield the full and faithful liftings of f_0 at ΣG_1 and G_1 respectively). If F is full, ϕ_F is a regular epimorphism by definition, hence $\phi_{\mathrm{Supp}\,F}$ is a regular epimorphism and a monomorphism at the same time, so it is an isomorphism. As a consequence, $\mathrm{Supp}\,F$, i.e. the functor on the left hand side of diagram

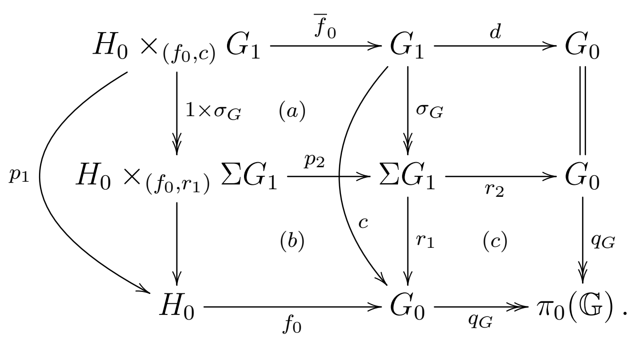


is full and faithful. Hence $\pi_0(F)$ is monomorphic by Proposition 2.4.

Conversely, if $\pi_0(F)$ is a monomorphism then $\mathrm{Supp}\,F$ is fully faithful, i.e. $\phi_{\mathrm{Supp}\,F}$ is an isomorphism. If in addition $\mathbb{G}$ is an equivalence relation, then σ_G is an isomorphism, hence $P_2 \cong P_1$ and ϕ_F is a regular epimorphism, being isomorphic to σ_H. By definition F is then full. ∎

2.6. Proposition. *A functor $F\colon \mathbb{H} \to \mathbb{G}$ between internal groupoids in an exact category is essentially surjective if and only if $\pi_0(F)$ is a regular epimorphism.*

Proof. It suffices to focus on the following commutative diagram:



If F is essentially surjective, then $d\overline{f}_0$ is a regular epimorphism, and so is $q_G d\overline{f}_0 = q_G f_0 p_1 = \pi_0(F) q_H p_1$, hence $\pi_0(F)$ is also a regular epimorphism.

Conversely, if $\pi_0(F)$ is a regular epimorphism, then so is $q_G f_0 = \pi_0(F) q_H$. Now let us observe that (c), (b), $(a)+(b)$, and hence (a) are all pullbacks. So $r_2 p_2$ is a regular epimorphism as a pullback of $q_G f_0$, and $1 \times \sigma_G$ is a regular epimorphism as a pullback of σ_G. Then their composite $d\overline{f}_0$ is a regular epimorphism and F is essentially surjective. ■

3. The comprehensive factorization

We borrow from [Bourn, 1987] the definition and some needed results about the *décalage* functor, first introduced in [Illusie, 1972] for simplicial objects. We actually focus our attention on its restriction $\mathrm{Dec}\colon \mathsf{Gpd}(\mathcal{C}) \to \mathsf{Gpd}(\mathcal{C})$.

We define here Dec as the functor associating with any groupoid $\mathbb{H}$ in $\mathcal{C}$ the groupoid

$$H_1 \times_{(d,c)} H_1 \underset{p_2}{\overset{m}{\underset{\xleftarrow{(ec,1)}}{\rightrightarrows}}} H_1 \ ,$$

which is, in fact, an equivalence relation (being a kernel pair of d), and we denote by $\epsilon\,\mathbb{H}\colon \mathrm{Dec}\,\mathbb{H} \to \mathbb{H}$ the discrete fibration

$$\begin{array}{ccc} H_1 \times_{(d,c)} H_1 & \xrightarrow{p_1} & H_1 \\ {\scriptstyle p_2}\downarrow \uparrow{\scriptstyle (ec,1)} \downarrow{\scriptstyle m} & & {\scriptstyle d}\downarrow \uparrow{\scriptstyle e} \downarrow{\scriptstyle c} \\ H_1 & \xrightarrow[c]{} & H_0 \,. \end{array}$$

The following is an exact fork in $\mathsf{Gpd}(\mathcal{C})$:

$$\mathrm{Dec}^2\,\mathbb{H} \underset{\mathrm{Dec}\,\epsilon\,\mathbb{H}}{\overset{\epsilon\,\mathrm{Dec}\,\mathbb{H}}{\rightrightarrows}} \mathrm{Dec}\,\mathbb{H} \xrightarrow{\epsilon\,\mathbb{H}} \mathbb{H} \ .$$

Following [Bourn–Rodelo, 2012], we give here a description of the comprehensive factorization in $\mathsf{Gpd}(\mathcal{C})$. Further details can be found also in [Bourn, 1987]. Recall that our base category $\mathcal{C}$ is now assumed to be exact, while the context considered in [Bourn–Rodelo, 2012] is a bit more general, and conditions are given for the existence of the comprehensive factorization.

Let $F\colon \mathbb{H} \to \mathbb{G}$ be a morphism in $\mathsf{Gpd}(\mathcal{C})$, then the pair $(\mathrm{Dec}\,F, \mathrm{Dec}^2\,F)$ gives rise to a functor between equivalence relations in $\mathsf{Gpd}(\mathcal{C})$:

$$\begin{array}{ccc} \mathrm{Dec}^2\,\mathbb{H} & \xrightarrow{\mathrm{Dec}^2 F} & \mathrm{Dec}^2\,\mathbb{G} \\ {\scriptstyle \mathrm{Dec}\,\epsilon\,\mathbb{H}}\downarrow\downarrow{\scriptstyle \epsilon\,\mathrm{Dec}\,\mathbb{H}} & & {\scriptstyle \mathrm{Dec}\,\epsilon\,\mathbb{G}}\downarrow\downarrow{\scriptstyle \epsilon\,\mathrm{Dec}\,\mathbb{G}} \\ \mathrm{Dec}\,\mathbb{H} & \xrightarrow[\mathrm{Dec}\,F]{} & \mathrm{Dec}\,\mathbb{G} \,. \end{array}$$

We consider the following factorization of the above functor, where all the right hand side squares are pullbacks:

$$\begin{array}{ccccc}
\mathrm{Dec}^2\,\mathbb{H} & \longrightarrow & \mathbb{H}\times_{\mathbb{G}}\mathrm{Dec}^2\,\mathbb{G} & \longrightarrow & \mathrm{Dec}^2\,\mathbb{G} \\
{\scriptstyle \mathrm{Dec}\,\epsilon\,\mathbb{H}}\downarrow\downarrow{\scriptstyle \epsilon\,\mathrm{Dec}\,\mathbb{H}} & & \downarrow\downarrow & & {\scriptstyle \mathrm{Dec}\,\epsilon\,\mathbb{G}}\downarrow\downarrow{\scriptstyle \epsilon\,\mathrm{Dec}\,\mathbb{G}} \\
\mathrm{Dec}\,\mathbb{H} & \longrightarrow & \mathbb{H}\times_{\mathbb{G}}\mathrm{Dec}\,\mathbb{G} & \xrightarrow{\overline{F}} & \mathrm{Dec}\,\mathbb{G} \\
{\scriptstyle \epsilon\,\mathbb{H}}\downarrow & & {\scriptstyle U}\downarrow & (*) & \downarrow{\scriptstyle \epsilon\,\mathbb{G}} \\
\mathbb{H} & = & \mathbb{H} & \xrightarrow[F]{} & \mathbb{G}\,.
\end{array} \tag{1}$$

Finally, applying the functor π_0 to the upper rectangle, we get the factorization

$$\begin{array}{ccccc}
H_1 & \xrightarrow{j_1} & T_1 & \xrightarrow{k_1} & G_1 \\
{\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\
H_0 & \xrightarrow[j_0]{} & T_0 & \xrightarrow[k_0]{} & G_0
\end{array} \tag{2}$$

of F into a final functor $J = (j_0, j_1)$ followed by a discrete fibration $K = (k_0, k_1)$.

4. Final functors

We adopt here the notation from the previous section, so any functor F factors as $F = KJ$, where J is final and K is a discrete fibration. The following lemma provides a first characterization of final functors and is a key step towards Theorem 4.2.

4.1. Lemma. *A functor $F\colon \mathbb{H} \to \mathbb{G}$ between internal groupoids in an exact category is final if and only if the pullback $\overline{F}\colon \mathbb{H}\times_{\mathbb{G}}\mathrm{Dec}\,\mathbb{G} \to \mathrm{Dec}\,\mathbb{G}$ of F along $\epsilon\,\mathbb{G}$ is inverted by π_0.*

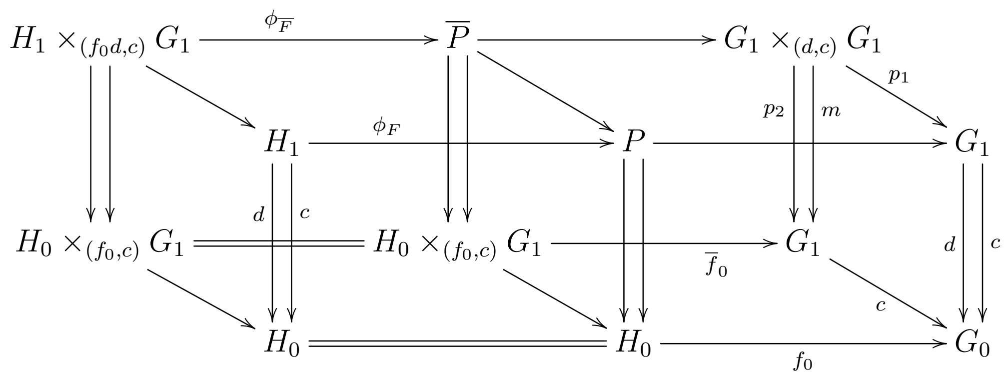
Proof. First of all, notice that $\pi_0(\overline{F})$ is nothing but the arrow k_0 in diagram (2), so saying that $\overline{F}$ is inverted by π_0 means that the arrow k_0 is an isomorphism.

Now, if F is final, K is an isomorphism, and so is k_0. Conversely, since K is a discrete fibration, the right hand side commutative squares in (2) are pullbacks, hence if k_0 is an isomorphism, so is k_1, and F is final. ∎

We are now ready for an internal version of Proposition 1.1.

4.2. Theorem. *A functor $F\colon \mathbb{H} \to \mathbb{G}$ between internal groupoids in an exact category is final if and only if it is full and essentially surjective.*

Proof. Let us consider the vertical expansion of the pullback (∗) in diagram (1) and take the (bijective on objects, fully faithful) factorizations of the functors F and $\overline{F}$:



Thanks to Lemma 4.1, we only have to prove that $\overline{F}$ is inverted by π_0 if and only if F is full and essentially surjective.

Suppose $\pi_0(\overline{F})$ is an isomorphism. Then, by Corollary 2.5, since $\mathrm{Dec}\,\mathbb{G}$ is an equivalence relation, $\overline{F}$ is full, i.e. $\phi_{\overline{F}}$ is a regular epimorphism. But p_1 is a split epimorphism, so F is full by Proposition 2.2. Moreover, in our context, regular epimorphic functors are just levelwise regular epimorphisms. Then $\epsilon\,\mathbb{G}$ is a regular epimorphism, since both c and p_1 are. As a consequence, $\pi_0(\epsilon\,\mathbb{G})$ is a regular epimorphism, and the same holds for $\pi_0(F)\cdot\pi_0(U) = \pi_0(\epsilon\,\mathbb{G})\cdot\pi_0(\overline{F})$. This implies that $\pi_0(F)$ is a regular epimorphism, hence, by Proposition 2.6, F is essentially surjective.

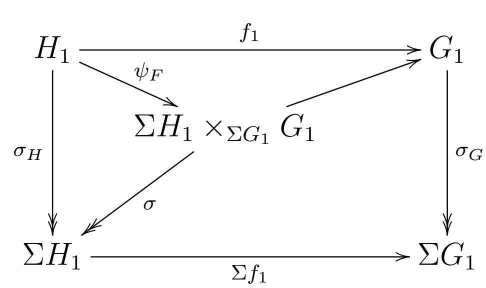
Conversely, if F is full, then $\overline{F}$ is full by Proposition 2.2, hence $\pi_0(\overline{F})$ is a monomorphism by Corollary 2.5. If in addition F is essentially surjective, $d\overline{f}_0 = \pi_0(\overline{F})q_{H\times_G \mathrm{Dec}\,G}$ is a regular epimorphism. So $\pi_0(\overline{F})$ is a regular epimorphism, hence an isomorphism. ■

We end with a further characterization of final functors, which follows from the latter.

4.3. Corollary. *A functor $F\colon \mathbb{H}\to\mathbb{G}$ between internal groupoids in an exact category is final if and only if*

(i) $\pi_0(F)$ is an isomorphism;

(ii) the arrow ψ_F in the following commutative diagram is a regular epimorphism:



(3)

Proof. By Proposition 2.4, $\pi_0(F)$ is a monomorphism if and only if $\operatorname{Supp} F$ is fully faithful. In this case, the arrow ψ_F coincides with the arrow ϕ_F defined in Section 2.1.

Suppose F is final. Then by Theorem 4.2 it is full and essentially surjective, hence, by Proposition 2.6 and Corollary 2.5, $\pi_0(F)$ is an isomorphism and $\phi_F = \psi_F$ is a regular epimorphism.

Conversely, if (i) and (ii) hold, then $\phi_F = \psi_F$ is a regular epimorphism, so F is full, and moreover $\pi_0(F)$ is a regular epimorphism, i.e. F is essentially surjective. ■

4.4. Remark. When the ground category $\mathcal{C}$ is exact and pointed, it is possible to define a functor

$$\pi_1 \colon \mathsf{Gpd}(\mathcal{C}) \to \mathcal{C}\,, \qquad \pi_1(\mathbb{H}) = \operatorname{Ker}(\sigma_H)\,.$$

Then, let us look at the diagram

$$\begin{array}{ccccc}
\pi_1(\mathbb{H}) & \xrightarrow{\pi_1(F)} & \pi_1(\mathbb{G}) & = & \pi_1(\mathbb{G}) \\
\downarrow & & \downarrow & & \downarrow \\
H_1 & \xrightarrow{\psi_F} & \Sigma H_1 \times_{\Sigma G_1} G_1 & \longrightarrow & G_1 \\
{\scriptstyle \sigma_H}\downarrow & & {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma_G} \\
\Sigma H_1 & = & \Sigma H_1 & \xrightarrow[\Sigma f_1]{} & \Sigma G_1
\end{array}$$

obtained from (3) by taking kernels vertically. Since the right lower square is a pullback, $\operatorname{Ker}(\sigma) \cong \pi_1(\mathbb{G})$, so $\pi_1(F)$ is the upper horizontal arrow in the left upper square, which is a pullback. As a consequence, when F is final, by Corollary 4.3, $\pi_1(F)$ is a regular epimorphism, being a pullback of ψ_F.

In fact, in a semi-abelian setting, by the short five lemma, the request that $\pi_1(F)$ is a regular epimorphism is also sufficient to imply that ψ_F is a regular epimorphism. Combined with the request that $\pi_0(F)$ is an isomorphism, this yields the characterization of final functors given by Corollary 5.4 in [Cigoli–Mantovani–Metere, 2014], which can be interpreted accordingly as a very special case of Corollary 4.3 above.

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