Theory and Applications of Categories, Vol. 32, No. 9, 2017, pp. 286–337.

CLASSIFYING TANGENT STRUCTURES USING WEIL ALGEBRAS

POON LEUNG

ABSTRACT. At the heart of differential geometry is the construction of the tangent bundle of a manifold. There are various abstractions of this construction, and of particular interest here is that of Tangent Structures.

Tangent Structure is defined via giving an underlying category $\mathcal{M}$ and a tangent functor T along with a list of natural transformations satisfying a set of axioms, then detailing the behaviour of T in the category $\mathrm{End}(\mathcal{M})$. However, this axiomatic definition at first seems somewhat disjoint from other approaches in differential geometry.

The aim of this paper is to present a perspective that addresses this issue. More specifically, this paper highlights a very explicit relationship between the axiomatic definition of Tangent Structure and the Weil algebras (which have a well established place in differential geometry).

1. Introduction

The starting point for the notion of tangent structure is that given a smooth manifold M, we can construct the *tangent space* TM, which to each point $x \in M$ attaches the vector space T_xM of all tangents to M at x. The functoriality of this construction is used to capture the idea of differentiation of maps between more abstract spaces.

T being a functor (moreover an endofunctor over the category under consideration) allows us to talk about *Tangent Structure*; the ingredients required to give a notion of "tangent space" to an arbitrary category. There is also a more specific, technical meaning of "Tangent Structure" given by Rosický in [Rosický, 1984] and by Cockett and Cruttwell in [Cockett and Cruttwell, 2014].

Weil algebras, on the other hand, have a well established history in the world of differential geometry. For instance, Kolar *et al* give discussion of Weil algebras and Weil functors in [Kolar et al., 2010] (Section 35), and of course there is the work of Weil [Weil, 1953]. Indeed, there are also the ideas of synthetic differential geometry (SDG), which define tangent spaces and related structures through the use of infinitesimals (given as the spectrum of corresponding Weil algebras; for instance, see [Kock, 2006] for more details).

There are, as we shall see, strong connections between these two seemingly different concepts. Furthermore, it will turn out that the tangent functor T is closely related to

Thanks to Steve Lack for his immense support and to the referee for his/her many helpful suggestions.
Received by the editors 2016-09-02 and, in final form, 2017-02-14.
Transmitted by Robert Paré. Published on 2017-02-15.
2010 Mathematics Subject Classification: 18D99,53A99.
Key words and phrases: Tangent Structure, Weil algebra.

a particular Weil algebra in a very meaningful way. We shall begin with a brief look at Tangent Structure, then discuss Weil algebras and some of their properties.

We will then introduce (co)graphs and show that they are a surprisingly useful tool in characterising not only the objects, but also the morphisms of a category $\mathbf{Weil_1}$ (a particular subcategory of **Weil**) we shall be using in our discussion.

More specifically, we will show that each object of $\mathbf{Weil_1}$ corresponds canonically to a particular graph (moreover, what we shall call *piecewise complete* graphs), and further that each morphism $f\colon A \to B$ of such Weil algebras can be described using cliques and independent sets. These observations then provide a language for a methodical process to "construct" any such map using a collection of generating maps.

We will conclude with Theorem 14.1, which states that to give a tangent structure (in the sense of [Cockett and Cruttwell, 2014]) over a category $\mathcal{M}$ is to give a functor

$$F\colon \mathbf{Weil_1} \to [\mathcal{M}, \mathcal{M}]$$

satisfying certain axioms.

One final observation we will make is that we can in fact remove the requirement of the codomain of F needing to be an endofunctor category $[\mathcal{M}, \mathcal{M}]$, and instead replace it with an arbitrary monoidal category $(\mathcal{G}, \Box, I)$. This then more clearly exhibits $\mathbf{Weil_1}$ as what one might call the "initial" tangent structure.

2. Tangent Structure

Tangent Structure is defined by Rosický [Rosický, 1984] using (internal) bundles of abelian groups, but we will be following the more general definition of Cockett-Cruttwell [Cockett and Cruttwell, 2014] using (internal) bundles of commutative monoids. More explicitly, this requires that the tangent bundle TM sitting over a smooth manifold M is a commutative monoid, referred to as an *additive bundle.*

In this section, we shall give said definition of Tangent Structure below in Definition 2.6. However, we first have the following:

2.1. Definition. *Given a category $\mathcal{C}$, a* commutative monoid *in $\mathcal{C}$ consists of*

1. *An object C such that finite powers of C exist (the terminal object we shall call t);*

2. *A pair of maps $\eta\colon t \to C$ and $\mu\colon C \times C \to C$ such that the following diagrams commute*

$$\begin{tikzcd} C \times (C \times C) \arrow[r, "\alpha"] \arrow[d, "1\times\mu"'] & (C \times C) \times C \arrow[r, "\mu\times 1"] & C \times C \arrow[d, "\mu"] \\ C \times C \arrow[rr, "\mu"'] & & C \end{tikzcd} \qquad \begin{tikzcd} t \times C \arrow[r, "\eta\times 1"] \arrow[dr, "\cong"'] & C \times C \arrow[d, "\mu"] & C \times t \arrow[l, "1\times\eta"'] \arrow[dl, "\cong"] \\ & C & \end{tikzcd} \;;$$

where $\alpha\colon C \times (C \times C) \to (C \times C) \to C$ is the obvious associativity map, and μ agrees with the symmetry map

$$s\colon C \times C \to C \times C \ ,$$

so that the diagram

$$\begin{tikzcd} C \times C \arrow[r, "s"] \arrow[dr, "\mu"'] & C \times C \arrow[d, "\mu"] \\ & C \end{tikzcd}$$

also commutes.

2.2. Remark. Often, commutative monoids are considered in categories with all finite products, but we shall not be assuming this.

2.3. Definition. *If A is an object in a category $\mathcal{M}$, then an* additive bundle over A *is a commutative monoid in the slice category $\mathcal{M}/A$. Explicitly, this consists of*

1. *A map $p\colon X \to A$ such that pullback powers of p exist, the n^{th} pullback power denoted by $X^{(n)}$ and projections $\pi_i\colon X^{(n)} \to X$ for $i \in \{1, \ldots, n\}$;*

2. *Maps $+\colon X^{(2)} \to X$ and $\eta\colon A \to X$ with $p \circ + = p \circ \pi_1 = p \circ \pi_2$ and $p \circ \eta = id$ which are associative, commutative, and unital.*

2.4. Remark. We will note here that the notation used in [Cockett and Cruttwell, 2014] for the n^{th} pullback power is instead X_n.

2.5. Definition. *Suppose $p\colon X \to A$ and $q\colon Y \to B$ are additive bundles. An* additive bundle morphism *is a pair of maps $f\colon X \to Y$ and $g\colon A \to B$ such that the following diagrams commute.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ A \arrow[r, "g"'] & B \end{tikzcd} \qquad \begin{tikzcd} X^{(2)} \arrow[r, "f \times f"] \arrow[d, "+"'] & Y^{(2)} \arrow[d, "+"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\eta"'] & B \arrow[d, "\eta'"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

2.6. Definition. *Given a category $\mathcal{M}$, a* tangent structure $\mathbb{T} = (T, p, \eta, +, l, c)$ *consists of*

1. *(**tangent functor**) a functor $T\colon \mathcal{M} \to \mathcal{M}$ and a natural transformation $p\colon T \Rightarrow 1_{\mathcal{M}}$ such that pullback powers $T^{(n)}$ of p exist and the composites T^m of T preserve these pullbacks for all $m \in \mathbb{N}$;*

2. *(**tangent bundle**) natural transformations $+\colon T^{(2)} \Rightarrow T$ and $\eta\colon 1_{\mathcal{M}} \Rightarrow T$ making $p\colon T \Rightarrow 1_{\mathcal{M}}$ into an additive bundle;*

3. *(**vertical lift**) a natural transformation $l\colon T \Rightarrow T^2$ such that*

$$(l,\eta)\colon (p,+,\eta) \to \big(Tp, T+, T\eta\big)$$

is an additive bundle morphism;

4. *(**canonical flip**) a natural transformation $c\colon T^2 \Rightarrow T^2$ such that*

$$(c, id_T)\colon \big(Tp, T+, T\eta\big) \to \big(pT, +T, \eta T\big)$$

is an additive bundle morphism;

where the natural transformations l and c satisfy

1. *(**coherence of** l **and** c) $c^2 = id$, $c \circ l = l$, and the following diagrams commute*

$$\begin{tikzcd} T \arrow[r, "l"] \arrow[d, "l"'] & T^2 \arrow[d, "Tl"] \\ T^2 \arrow[r, "lT"'] & T^3 \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "Tc"] \arrow[d, "cT"'] & T^3 \arrow[r, "cT"] & T^3 \arrow[d, "Tc"] \\ T^3 \arrow[r, "Tc"'] & T^3 \arrow[r, "cT"'] & T^3 \end{tikzcd} \qquad \begin{tikzcd} T^2 \arrow[r, "lT"] \arrow[d, "c"'] & T^3 \arrow[r, "Tc"] & T^3 \arrow[d, "cT"] \\ T^2 \arrow[rr, "Tl"'] & & T^3 \end{tikzcd};$$

2. *(**universality of vertical lift**) the following is an equaliser diagram*

$$\begin{tikzcd} T^{(2)} \arrow[r, "(T+)\circ(l\times_T\eta T)"] & T^2 \arrow[r, shift left, "Tp"] \arrow[r, shift right, "\eta\circ p\circ Tp"'] & T \end{tikzcd},$$

where $(T+)\circ(l \times_T \eta T)$ is the composite

$$\begin{tikzcd} & T \arrow[r, "l"] & T^2 & \\ T^{(2)} \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[rr, dashed] & & TT^{(2)} \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[r, "T+"] & T^2 \\ & T \arrow[r, "\eta T"'] & T^2 & \end{tikzcd}$$

2.7. Remark. We will note here that $l : T \Rightarrow T^2$ and $p : T \Rightarrow 1_{\mathcal{M}}$ do not form a comonad. However, there is a canonical way to make T a monad (detailed in [Cockett and Cruttwell, 2014]).

We may then refer to the pair $(\mathcal{M}, \mathbb{T})$ as a *tangent category*.

3. Weil Algebras

We now introduce Weil algebras. For the purposes of this paper, we will always be using commutative, unital algebras. We shall initially define Weil algebras over a field, but ultimately we are interested in working over a commutative rig; recall a *rig* is a commutative monoid equipped with (a unital) multiplication.

Traditionally, Weil algebras are defined over a field, and we may at first naively use some more general structure (say an arbitrary ring). The problem in doing so is that the notion of "Weil algebra" in complete generality becomes somewhat difficult to define in a consistent and coherent manner.

For the purposes of this paper, however, we will only be interested in Weil algebras with presentations of a particular form (we will describe this in detail in 6). As such, when we restrict to these presentations, we will be able to work unhindered over a rig (rather than a field).

In particular, if we take k to be (the commutative ring) $\mathbb{Z}$, we will ultimately recover the abelian group bundles of [Rosický, 1984], while (the commutative rig) $\mathbb{N}$ corresponds to the additive bundles of [Cockett and Cruttwell, 2014]. Later on in our discussion, we will also be interested in the rig $\mathbb{2}$ (which we shall formally introduce in Definition 7.1).

We shall begin by defining Weil algebras over some given *field* k.

3.1. DEFINITION. *A* Weil algebra B *is an augmented (commutative and unital) algebra with a finite dimensional underlying k-vector space, for which all elements of the augmentation ideal are nilpotent.*

Equivalently, we can say that a Weil algebra is simply a finite dimensional local algebra with residue field k.

3.2. REMARK. The equivalence arises from the fact that the augmentation ideal $\ker(\varepsilon)$ (for augmentation $\varepsilon \colon B \to k$) is the unique maximal ideal of B.

A morphism between Weil algebras B and C is simply an augmented algebra homomorphism, i.e. an algebra map

$$f \colon B \to C$$

that is compatible with the augmentations, i.e. we have a commuting diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\varepsilon_B"'] & C \arrow[dl, "\varepsilon_C"] \\ k & \end{tikzcd}$$

From here onwards, we shall simply refer to these augmented algebra homomorphisms as maps.

3.3. DEFINITION. *Let* **Weil** *be the category with objects the Weil algebras and morphisms the maps described above.*

3.4. Remark. The category **Weil** is a full subcategory of **AugAlg** (= **Alg**/k, the category of augmented algebras).

3.5. Remark. We will (soon) further restrict **Weil** to a full subcategory $\mathbf{Weil_1}$ in order to discuss tangent structure.

It is often convenient to give a Weil algebra B via a presentation

$$B = k[b_1, \ldots, b_m]/Q_B \ ,$$

where we quotient the free algebra $k\,[b_1, \ldots, b_m]$ by the list of terms in Q_B.

3.6. Remark. This is always possible, since each Weil algebra is a finitely generated and commutative algebra, and such algebras always have a presentation of this form.

3.7. Example.

1. $k[x]/x^2$ is the Weil algebra with $\{1, x\}$ as a basis for the underlying k-module and equipped with the obvious multiplication, but with x^2 identified as 0.
2. $k[x]/x^3$ is the Weil algebra with $\{1, x, x^2\}$ as a basis for the underlying k-module and equipped with the obvious multiplication, but with x^3 identified with 0.
3. $k[x, y]/x^2, y^2$ is the Weil algebra with $\{1, x, y, xy\}$ as a basis for the underlying k-module and equipped with the obvious multiplication, but with x^2 and y^2 each identified with 0.

We also note the following:

1. We shall always use presentations for which the augmentation $\varepsilon\colon B \to k$ sends each generator b_i to 0.
2. Recall that for a linear map $h\colon X \to Y$ between vector spaces, it suffices to define how h acts on basis elements of V. Analogously, for an augmented algebra homomorphism $f\colon B \to C$, it suffices to define how f acts on generators (then check that it is suitably compatible with the relations).
3. For Weil algebras $A = k[a_1, \ldots, a_m]/Q_A$ and $B = k[b_1, \ldots, b_n]/Q_B$ and a map $f\colon A \to B$, $f(a_i)$ is a polynomial in the generators $b_1, \ldots, b_n$ with no constant term.

Now that we have defined the category **Weil**, we shall establish some facts about this category. We begin with the following:

1. The category **AugAlg** has all limits and colimits.
2. Coproducts in **AugAlg** are given by $\otimes$.

which are well established and we shall not prove.

3.8. Lemma. *k is a zero object of* **Weil**.

Proof. For each Weil algebra A, the augmentation $\varepsilon_A\colon A \to k$ and the unit $\eta_A\colon k \to A$ are the unique maps to and from k respectively. ∎

3.9. PROPOSITION. *The category* **Weil** *has all finite products.*

PROOF. Since k is a zero object, it is the nullary product. For arbitrary Weil algebras A and B, begin by taking the pullback

$$\begin{tikzcd} A \times_k B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\varepsilon_b"] \\ A \arrow[r, "\varepsilon_A"'] & k \end{tikzcd}$$

(or equivalently, the product) in **AugAlg**. Since both A and B are finitely dimensional and have nilpotent augmentation ideals, then the same is true of $A \times_k B$. Thus it is also a Weil algebra.

Thus **Weil** has all finite products. ∎

3.10. DEFINITION. *Let* **NilAugAlg** *be the full subcategory of* **AugAlg** *containing all augmented algebras whose augmentation ideals are nilpotent.*

3.11. PROPOSITION. *The category* **NilAugAlg** *has all finite limits.*

PROOF. Let $\mathcal{A}$ be a finite category and consider an arbitrary diagram

$$R \colon \mathcal{A} \to \mathbf{NilAugAlg} \ .$$

Since **AugAlg** has all limits, we can form a limiting cone

$$\begin{tikzcd} \mathcal{A} \arrow[r, "R"'] \arrow[rr, bend left, "\Delta X"] & \mathbf{NilAugAlg} \arrow[r, hook] \arrow[u, phantom, "\Downarrow\gamma"] & \mathbf{AugAlg} \end{tikzcd} .$$

But since $\mathcal{A}$ is finite, the (finite) set $\{\gamma_a \mid a \in \mathcal{A}\}$ is jointly monic and each Ra is nilpotent, then the augmentation ideal of X is necessarily nilpotent, and so $X \in$ **NilAugAlg**.

Thus **NilAugAlg** has all finite limits. ∎

3.12. DEFINITION. *For each $n \in \mathbb{N}$, let W_n be the Weil algebra $k[x]/x^{n+1}$.*

3.13. PROPOSITION. *The set $\{W_n \mid n \in \mathbb{N}\}$ forms a strong generator for* **NilAugAlg**.

PROOF. We want to show that the set of functors

$$\mathbf{NilAugAlg}(W_n, \_)\colon \mathbf{NilAugAlg} \to \mathbf{Set}$$

for all $n \in \mathbb{N}$ jointly reflect isomorphisms.

Let $f \colon A \to B$ be an arbitrary map of **NilAugAlg** for which

$$\mathbf{NilAugAlg}(W_n, f)\colon \mathbf{NilAugAlg}(W_n, A) \to \mathbf{NilAugAlg}(W_n, B)$$

is an isomorphism for all $n \in \mathbb{N}$.

Let α be an element of A with $f(\alpha) = 0$. In particular, α is an element of the augmentation ideal $\ker(\varepsilon_A)$. Since this is nilpotent, then we can define

$$r = \max\{s \in \mathbb{N} \mid \alpha^s \neq 0\} .$$

Note also that $\alpha^{r+1} = 0$. As such, we may define a map $g\colon W_r \to A$ given as $g(x) = \alpha$. Further, let $z\colon W_r \to A$ be the zero map (i.e. $z(x) = 0$).

Now, we have $g, z \in \mathbf{NilAugAlg}(W_r, A)$. Moreover, we clearly have $f \circ g = f \circ z$. But since $\mathbf{NilAugAlg}(W_r, f)$ is an isomorphism, then we must have $g = z$, i.e. $\alpha = 0$.

$\therefore$ $\ker(f) = \{0\}$.

Now, let β be an arbitrary element of $\ker(\varepsilon_B)$. Since $\ker(\varepsilon_B)$ is nilpotent, then we can define

$$\rho = \max\{\sigma \in \mathbb{N} \mid \beta^\sigma \neq 0\} .$$

Note also that $\beta^{\rho+1} = 0$. As such, we may define a map $\gamma\colon W_\rho \to B$ given as $\gamma(x) = \beta$.

But now we have $\gamma \in \mathbf{NilAugAlg}(W_\rho, B)$, and since $\mathbf{NilAugAlg}(W_\rho, f)$ is an isomorphism, then there is a unique map $h\colon W_\rho \to A$ such that

$$\begin{tikzcd} W_\rho \arrow[r, "h"] \arrow[dr, "\gamma"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes. This shows that f is surjective on elements. But this means that f is an isomorphism in **Vect**.

Thus f is an isomorphism in **NilAugAlg**. Since **NilAugAlg** has all equalisers, then the set $\{W_n \mid n \in \mathbb{N}\}$ forms a strong generator for **NilAugAlg**. ∎

In particular, since each $W_n \in \mathbf{Weil}$, this then says that the inclusion $I\colon \mathbf{Weil} \hookrightarrow \mathbf{NilAugAlg}$ preserves and reflects any existing (finite) limits.

3.14. PROPOSITION. *For an arbitrary $A \in \mathbf{Weil}$, the functor $A \otimes \_\colon \mathbf{Weil} \to \mathbf{Weil}$ preserves finite connected limits.*

PROOF. Consider the diagram

$$\begin{tikzcd} \mathbf{Weil} \arrow[r, "A\otimes\_"] \arrow[d, hook] & \mathbf{Weil} \arrow[d, hook] \\ \mathbf{NilAugAlg} \arrow[r, "A\otimes\_"] \arrow[d, hook] & \mathbf{NilAugAlg} \arrow[d, hook] \\ \mathbf{AugAlg} \arrow[r, "A\otimes\_"'] & \mathbf{AugAlg} . \end{tikzcd}$$

The inclusions all preserve and reflect (finite) limits, and $A \otimes \_\colon \mathbf{AugAlg} \to \mathbf{AugAlg}$ preserves connected limits. ∎

3.15. PROPOSITION. *The category* **Weil** *has all finite coproducts, and moreover, coproduct is given by* $\otimes$*.*

PROOF. (Finite) coproducts in **AugAlg** are given by $\otimes$, and since **Weil** is a full subcategory of **AugAlg**, it remains only to show that **Weil** is closed under (finite) $\otimes$.

Further, as k is a zero object, then it is the nullary coproduct. Now, since Weil algebras are finitely dimensional, then any finite coproduct of them must also be finitely dimensional. The nilpotency of the augmentation ideal is immediate. ∎

3.16. LEMMA. *Let A and B be Weil algebras with presentations*

$$\begin{aligned} A &= k[a_1, \ldots, a_m]/Q_A \\ B &= k[b_1, \ldots, b_n]/Q_B. \end{aligned}$$

Then:

- *The product $A \times B$ has presentation*

$$A \times B = k[a_1, \ldots, a_m, b_1, \ldots, b_n]/Q_A \cup Q_B \cup \{a_i b_j | \forall i, j\} \ ;$$

- *The coproduct $A \otimes B$ has presentation*

$$A \otimes B = k[a_1, \ldots, a_m, b_1, \ldots, b_n]/Q_A \cup Q_B \ .$$

PROOF. The proof is immediate. ∎

Finally, let us define W to be the Weil algebra $k[x]/x^2$. Then, the nth power and copower of W, denoted W^n and nW respectively, have presentations

$$\begin{aligned} W^n &= k[x_1, \ldots, x_n]/\{x_i x_j | \ \forall i \leqslant j\} \\ nW &= k[x_1, \ldots, x_n]/\{{x_i}^2 | \ \forall i\} \ . \end{aligned}$$

3.17. DEFINITION. *For Weil algebras A, B and C, the pullback*

$$\begin{tikzcd} A \otimes (B \times C) \arrow[r, "A\otimes\pi_B"] \arrow[d, "A\otimes\pi_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes B \arrow[d, "A\otimes\varepsilon_B"] \\ A \otimes C \arrow[r, "A\otimes\varepsilon_C"'] & A \end{tikzcd}$$

is a foundational pullback.

3.18. Remark. Foundational pullbacks are a direct application of Proposition 3.14 to Proposition 3.9, with products regarded as pullbacks over the zero object k.

The facts established above assume k is a field. However, we are more interested in $k = \mathbb{N}$, $\mathbb{Z}$ and $\mathbb{2}$ (which, again, we define in Definition 7.1). The notion of Weil algebras in this slightly higher level of generality then becomes somewhat muddled. However, the subcategory $\mathbf{Weil_1}$ of **Weil** we will use in our discussion will always consist of objects having a presentation of the form

$$k[x_1, \dots, x_n]/\left\{c_i c_j \mid \forall\ c_i \sim c_j\right\}\ ,$$

for a symmetric, reflexive relation $\sim$ (although not all such presentations will yield an object of $\mathbf{Weil_1}$), and we will still refer to these as Weil algebras. In particular, such Weil algebras all have finitely generated and free underlying k-modules.

In particular, Proposition 3.14 still holds when restricting to $\mathbf{Weil_1}$ for these more general k using the same arguments.

As we stated at the beginning of this section, the more general k is needed in order to make our comparison with the definitions of [Rosický, 1984] and [Cockett and Cruttwell, 2014]. To reiterate, taking $k = \mathbb{Z}$ (as a ring) will ultimately return the abelian group bundles of [Rosický, 1984]. However, we are more interested in taking $k = \mathbb{N}$ to ultimately obtain the commutative monoid bundles of [Cockett and Cruttwell, 2014]. We will also consider $k = \mathbb{2}$ (Definition 7.1), as this shall provide a convenient tool for our calculations.

4. Tangent Structure and Weil algebras

The tangent functor T is closely related to the Weil algebra $W = k[x]/x^2$. For instance, the tangent functor in synthetic differential geometry (see [Kock, 2006]) is the representable functor $(\_)^D$, where $D = \mathrm{Spec}(W)$.

Here, we will begin to describe a different relationship between **Weil** and tangent structure. Regard coproduct $\otimes$ as a monoidal operation on **Weil** (with unit k).

4.1. Proposition. *The (endo)functor*

$$W \otimes \_ \colon \mathbf{Weil} \to \mathbf{Weil}$$

can be used to define a Tangent Structure on **Weil**.

Proof. With $T = W \otimes \_$, we first give the natural transformations required in order to have a tangent structure on **Weil**. The names for the morphisms used below will be deliberately chosen to coincide with those of tangent structure.

	Natural transformation	Explanation
Projection	$\varepsilon_W \otimes \_\colon T \Rightarrow id_{\mathbf{Weil}}$	$\varepsilon_W\colon W \to k$ is the augmentation for W
Addition	$+\otimes \_\colon T^{(2)} \Rightarrow T$	$T^{(2)}$ is the functor $W^2 \otimes \_$, $+\colon W^2 \to W;\ x_1, x_2 \mapsto x$
Unit	$\eta_W \otimes \_\colon id_{\mathbf{Weil}} \Rightarrow T$	$\eta_W\colon k \to W$ is the (multiplicative) unit for W
Vertical lift	$l \otimes \_\colon T \Rightarrow T^2$	$T^2 = T \circ T$ is the functor $2W \otimes \_$ $l\colon W \to 2W;\ x \mapsto x_1x_2$
Canonical flip	$c \otimes \_\colon T^2 \Rightarrow T^2$	$c\colon 2W \to 2W;\ x_i \mapsto x_{3-i}$, for $i = 1, 2$

With these choices of natural transformations as well as the facts established in Section 3 (so that $(W \otimes \_)^n = (nW \otimes \_)$ preserves the required pullbacks), it is a very routine exercise to verify that this does in fact define a Tangent Structure on **Weil**.

We will also note that the following

$$W^2 \xrightarrow{(W\otimes+)\circ(l\times_W(\eta_W\otimes W))} 2W \underset{\eta_W\circ(\varepsilon_W\otimes\varepsilon_W)}{\overset{W\otimes\varepsilon_W}{\rightrightarrows}} W$$

is an equaliser in **Weil** (the universality of vertical lift equaliser in Definition 2.6).

Note that the map $(W \otimes +) \circ (l \times_W (\eta_W \otimes W))$, which we will denote as v, is given as

$$\begin{aligned} k[x_1, x_2]/x_1^2, x_2^2, x_1x_2 &\to k[y_1, y_2]/y_1^2, y_2^2 \\ x_1 &\mapsto y_1y_2 \\ x_2 &\mapsto y_2\ . \end{aligned}$$

The map $W \otimes \varepsilon_W : k[y_1, y_2]/y_1^2, y_2^2 \to k[z]/z^2$ sends y_1 to z and y_2 to 0, and $\eta_W \circ (\varepsilon_W \otimes \varepsilon_W) : k[y_1, y_2]/y_1^2, y_2^2 \to k[z]/z^2$ sends both y_1 and y_2 to 0. ■

This Tangent Structure on **Weil** relies on the object W, its (finite product) powers W^n and tensors of these. With this in mind, it makes sense to give the following definition:

4.2. Definition. *Let* $\mathbf{Weil_1}$ *be the category consisting of:*

1. *Objects: The closure of the set* $\{W^n \mid n \in \mathbb{N}\}$ *under finite* $\otimes$.
2. *Morphisms: All algebra homomorphisms compatible with units and augmentations.*

4.3. Remark. This definition is valid for $k = \mathbb{N}$, $\mathbb{Z}$ or $\mathbb{2}$ as well.

4.4. Remark. If we wish to use a particular k when discussing $\mathbf{Weil_1}$, we shall use "k-" as a prefix, e.g. $\mathbb{N}$-$\mathbf{Weil_1}$.

Recall that as a consequence of Lemma 3.16, the (finite product) power W^n would have presentation

$$k[x_1, \ldots, x_n]/\{x_ix_j | \forall i \leq j\}\ ,$$

and that the presentation for a tensor $A \otimes B$ took a particular form. As such, a tensor $\bigotimes_{i=1}^{m} W^{n_i}$ of powers of W would have a certain presentation that we will not try to describe explicitly right now (we shall see this in 6).

In general, however, such objects will have a presentation

$$k[x_1, \ldots, x_n]/\{x_i x_j | \forall x_i \sim x_j\}$$

for some symmetric, reflexive relation $\sim$ (although not all symmetric, reflexive relations will yield an object of $\mathbf{Weil_1}$). Since we will always require $x_i^2 = 0$ in these presentations, there is no loss of information if we omit the corresponding relation $x_i \sim x_i$ and take $\sim$ to merely be symmetric (and in fact, anti-reflexive).

However, such symmetric relations can be thought of as graphs.

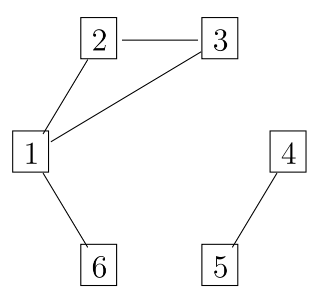
4.5. Remark. We treat the relations as anti-reflexive so that the corresponding graph will not have loops.

5. Graphs

We begin by defining some basic concepts relating to graphs that we will need to use. These are all, for the most part, standard definitions that can be found in any introductory graph theory textbook (for example, see [Bondy and Murty, 1991]). The notation, however, seems to vary depending on the text.

5.1. Definition. *A* graph *G is a pair of sets (V, E), with V a finite set of "vertices" of G, and E a set of unordered pairs of distinct vertices, called the "edges" of G.*

5.2. Example. $G = \Big(\{1, 2, 3, 4, 5, 6\}, \{(1, 2), (1, 3), (1, 6), (2, 3), (4, 5)\}\Big)$ is the graph



5.3. Remark. In more formal graph theory terms, we are actually describing simple (undirected edges, no loops and at most one edge between any pair of vertices) finite graphs.

5.4. Definition. *For graphs $G = (V, E)$ and $G' = (V', E')$, a* graph homomorphism *$h\colon G \to G'$ is a function $h\colon V \to V'$ such that for distinct $u, v \in V$,*

$$(u, v) \in E \Rightarrow (f(u), f(v)) \in E' \text{ or } f(u) = f(v).$$

5.5. Definition. *Let* **Gph** *be the category of graphs and graph homomorphisms.*

5.6. Definition. *For a non-empty graph $G = (V, E)$, we will say it is connected if for any two distinct vertices u and v, there exist vertices $v_1, \ldots, v_s \in V$ with $(v_i, v_{i+1}) \in E$ for each i, with $v_1 = u$ and $v_s = v$.*

5.7. Definition. *Given a graph $G = (V, E)$, the* complement *of G is the graph $G^c = (V, E^c)$, where for any two distinct $u, v \in V$,*

$$(u, v) \in E \Leftrightarrow (u, v) \notin E^c.$$

We now define two important binary operations on graphs. Let graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be given.

5.8. Definition. *The* disjoint union *of G_1 and G_2, denoted as $G_1 \otimes G_2$, is the graph*

$$G_1 \otimes G_2 = (V_1 \sqcup V_2, E_1 \sqcup E_2) \ ;$$

where $\sqcup$ denotes disjoint union.

Or, put simply, it is the graph given by simply placing G_1 adjacent to G_2 without adding or removing any edges.

5.9. Definition. *The* graph join *of G_1 and G_2, denoted $G_1 \times G_2$, is the graph*

$$G_1 \times G_2 = (V_1 \sqcup V_2, \widetilde{E})$$

where $\widetilde{E} = E_1 \sqcup E_2 \sqcup (V_1 \times V_2)$.

Or, put simply, it is the graph given by taking $G_1 \otimes G_2$, then adding in an edge from each vertex in G_1 to each vertex in G_2. Equivalently, it can be defined as

$$G_1 \times G_2 = (G_1^c \otimes G_2^c)^c$$

5.10. Remark. The notation $G_1 \times G_2$ is in no way intended to suggest the product of G_1 and G_2 in the category **Gph** of graphs.

5.11. Remark. The use of $\otimes$ and $\times$ to denote the operations of disjoint union and graph join respectively do not coincide with the notation used in graph theory. Graph union is often denoted as $G_1 \cup G_2$ or $G_1 + G_2$. Further, the graph join, sometimes called "graph sum", is denoted $G_1 \vee G_2$, (to add to the confusion, some texts denote this as $G_1 + G_2$; moreover the meaning of "graph sum" can also vary depending on the literature). However, the notation $\{\otimes, \times\}$ was chosen in place of $\{\cup, \vee\}$ to correspond with the notation for coproduct and product of Weil algebras.

5.12. Definition. *A graph G is said to be* complete *if every pair (u, v) of distinct vertices has an edge joining them (i.e. $(u.v) \in E$ for all $u \neq v$).*

Equivalently, G is the graph join of an appropriate number of instances of the single point graph.

5.13. Definition. *A graph G is said to be* discrete *if the edge set E is empty.*

Equivalently, G is the disjoint union of an appropriate number of instances of the single point graph.

Equivalently again, G is discrete iff its complement G^c is complete.

5.14. Remark. In graph theory literature, sometimes discrete graphs are also called "edgeless graphs" or "null graphs".

5.15. Definition. *We will give an iterative definition of* cograph *(complement-reducible graph) as follows:*

1. *The empty graph (empty vertex set) and one point graph are cographs.*
2. *If G_1 and G_2 are cographs, so are $G_1 \times G_2$ and $G_1 \otimes G_2$.*

5.16. Remark. Cographs are not in any way a dual notion to graphs. The prefix "co-" is an abbreviation of "complement reducible".

In fact, cographs have been studied extensively by graph theorists, and there are various equivalent characterisations of them (for instance, see [Corneil et al., 1981]).

5.17. Remark. For example, given a graph G, the following are equivalent:

1. G is a cograph;
2. G does not contain the graph P_4 (the path graph with four vertices) as a full subgraph (We shall not define P_4 explicitly, but instead simply note that the definition can be found in any introductory graph theory text).

.

6. Graphs and Weil algebras

In 4, we defined the category $\mathbf{Weil_1}$ (Definition 4.2, and noted that each object of this category can be regarded as a graph.

Let us formalise this by first giving the following definition:

6.1. Definition. *The functor*

$$\kappa\colon \mathbf{Gph} \to \mathbf{Weil}$$

is defined as follows:

1. *On objects: For a graph $G = (V, E)$, $\kappa(G)$ is the Weil algebra $k[v_1, \ldots, v_m]/Q_E$, where $V = \{v_1, \ldots, v_m\}$, $v_i^2 \in Q_E$ for all i and for $i \neq j$, $v_iv_j \in Q_E \Leftrightarrow (v_i, v_j) \in E$.*
2. *On morphisms: For a graph homomorphism $h\colon G \to G'$, $\kappa h\colon \kappa(G) \to \kappa(G')$ is given as*
$$(\kappa h)(v_i) = h(v_i) \text{ for all } i \;;$$
where we use the underlying function $h\colon V \to V'$ on the vertex sets.

6.2. Remark. We shall leave as an exercise to the reader to verify that κh is indeed a valid morphism of Weil algebras, and that this definition of κ is functorial, i.e. that it preserves identities and composition.

Conversely, we have the following:

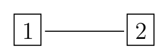
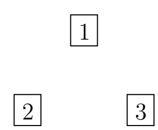
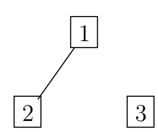
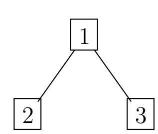
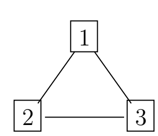
6.3. Definition. *Given a Weil algebra X with presentation of the form*

$$X = k[x_1, \ldots, x_n]/\{x_i x_j \mid \forall\ x_1 \sim x_j\}\ ,$$

let Γ_X denote the graph induced by $\sim$; namely the graph with vertices the generators $x_1, \ldots, x_n$ and an edge between x_i and x_j (for $i \neq j$) whenever $x_i \sim x_j$.

6.4. Remark. With this convention, for a Weil algebra X with presentation as described above, it is easy to see that $\kappa(\Gamma_X) = X$, and for a graph G, we have $\Gamma_{\kappa(G)} = G$.

For example, we have

Weil algebra	**Presentation**	**Graph**
k	$k[\ ]$	
W	$k[x]/x^2$	
$2W$	$k[x_1, x_2]/x_1^2, x_2^2$	
W^2	$k[x_1, x_2]/x_1^2, x_2^2, x_1x_2$	
$3W$	$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2$	
$W^2 \otimes W$	$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2, x_1x_2$	
$W \times 2W$	$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3$	
W^3	$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3$	

6.5. Remark. The object $W \times 2W$ is not contained in the category $\mathbf{Weil_1}$, but we shall include it in the table anyway.

6.6. Proposition. *For graphs G and G', we have:*

1. $\kappa(G) \otimes \kappa(G) = \kappa(G \otimes G')$;
2. $\kappa(G) \times \kappa(G) = \kappa(G \times G')$.

Proof. This is a direct consequence of Lemma 3.16, Definition 5.8 and Definition 5.9. ∎

To require precisely those Weil algebras given as the closure of $\{W^n\}_{n\in\mathbb{N}}$ under $\otimes$ is thus to ask for those that correspond to disjoint unions of complete graphs.

6.7. Definition. *We shall refer to such graphs as being* piecewise complete *(p.c. graphs). Note that p.c. graphs are a subset of the cographs (as defined in* Definition 5.15*).*

6.8. Remark. Although we are in this chapter interested in p.c. graphs, we shall often speak in greater generality by discussing cographs.

Now that we have a description for the objects of our subcategory $\mathbf{Weil_1}$, we may now revisit the idea mentioned towards the end of Section 3; namely that k need not be a field and give formal discussion of the morphisms.

7. The morphisms of $\mathbf{Weil_1}$

We introduced the category $\mathbf{Weil_1}$ in Definition 4.2, and noted that it was worded in such a way that k need not be a field. We noted towards the end of Section 3 (and at a few other points) that we have a particular interest in the case where $k = \mathbb{N}$.

However, we shall for now take $k = \mathbb{2}$ as a tool to help deal with our immediate calculations.

7.1. Definition. *Let $\mathbb{2}$ be the rig $\{0, 1\}$, with the usual multiplication, and addition given by* max*; in particular $1 + 1 = 1$.*

We shall begin by showing that using the maps $\{\varepsilon_W, +, \eta, l, c\}$, composition, $\otimes$ and the universal property of foundational pullbacks (as given in Definition 3.17), we can construct (in some appropriate sense) any map of $\mathbb{2}$-$\mathbf{Weil_1}$.

7.2. Remark. We will not need the universal property of $\otimes$ (the coproduct), but rather we shall consider $\mathbb{2}$-$\mathbf{Weil_1}$ as a monoidal category with respect to $\otimes$ (with k as the unit).

We will need some extra constructions of graphs before we begin.

7.3. Definition. *A* clique *U of G is a (possibly empty) subset of V for which any two distinct vertices in U have an edge between them (or equivalently, the full subgraph of G induced by U is complete).*

7.4. Definition. *Conversely, an* independent set U *of* G *is a (possibly empty) subset of* V *for which no two distinct vertices in* U *have an edge between them (or equivalently, the full subgraph of* G *induced by* U *is discrete).*

7.5. Remark. Given a graph G, an independent set U of G is also a clique of G^c.

We can actually use these notions of cliques and independent sets to form new graphs from existing ones.

7.6. Definition. *Given a graph* $G = (V, E)$*, define* $\mathrm{Ind}(G)$ *to be the graph given by:*

1. *Vertices: the independent sets of* G*;*
2. *Edges: given any two distinct independent sets* U_1 *and* U_2 *of* G*, there is an edge between them in* $\mathrm{Ind}(G)$ *when there exist* $x \in U_1$ *and* $y \in U_2$ *such that either there is an edge between* x *and* y *in* G *or* $x = y$ *(i.e.* $U_1 \cap U_2 \neq \phi$*).*

7.7. Definition. *Given a graph* $G = (V, E)$*, define* $\mathrm{Cl}(G)$ *to be the graph given by:*

1. *Vertices: the cliques of* G*;*
2. *Edges: given any two distinct cliques* U_1 *and* U_2 *of* G*, there is an edge between them in* $\mathrm{Cl}(G)$ *whenever their union* $U_1 \cup U_2$ *is also a clique of* G *(note that there is no requirement for* U_1 *and* U_2 *to be disjoint).*

7.8. Remark. In defining the graph $\mathrm{Cl}(G)$, there is often the additional requirement that cliques U_1 and U_2 are disjoint for there to be an edge between them. If that were the case, then we would have

$$\mathrm{Ind}(G) = (\mathrm{Cl}(G^c))^c$$

7.9. Remark. As defined here, $\mathrm{Cl}\colon \mathbf{Gph} \to \mathbf{Gph}$ is functorial and moreover can be made into a monad. We shall not be needing this fact, so we shall not prove it.

7.10. Definition. *Given a graph* $G = (V, E)$*, define* $\mathrm{Ind}_+(G)$ *to be the full subgraph of* $\mathrm{Ind}(G)$ *where the vertices are the non-empty independent sets of* G*.*

These tools now allow us to canonically express morphisms in $\mathbb{2}$-$\mathbf{Weil_1}$ in a pictorial manner. Recall that to define a map between (Weil) algebras, it suffices to define how the map acts on each of the generators. So, let a map $f\colon A \to B$ in $\mathbb{2}$-$\mathbf{Weil_1}$ be given, where A and B have presentations

$$A = \mathbb{2}[a_1, ..., a_m]/Q_A \text{ and } B = \mathbb{2}[b_1, ..., b_n]/Q_B.$$

Then, for each generator a_i of A, we can express $f(a_i)$ (uniquely) as a sum

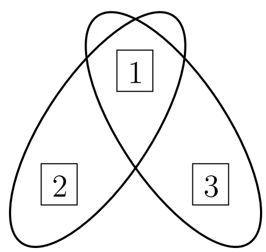
$$f(a_i) = \sum_{\underline{b} \in B} \alpha_{\underline{b}}^{(i)} \underline{b} \ ;$$

the sum being across all non-zero monomials $\underline{b}$ of B in the generators $\{b_1, \ldots, b_n\}$, and $\alpha^{(i)}_{\underline{b}} \in \mathbb{2}$ is a constant (taking value 0 or 1).

In fact, since we are using a presentation for which $\varepsilon_A(a_i) = 0$ for all i, the sum can in fact skip the trivial monomial (i.e. the constant).

We may also try to express f pictorially.

7.11. Example. Consider the map $f\colon W \to 3W$ given by $x \mapsto y_1y_2 + y_1y_3$. We can represent this in graph form:



where each term of $f(x)$ is represented by circling the vertices that generate the term (so the term y_1y_2 is represented by the ellipse encompassing the vertices 1 and 2). Note in particular that $\{1, 2\}$ and $\{1, 3\}$ are independent sets of Γ_{3W}.

We also note that we label the vertices 1, 2 and 3 instead of y_1, y_2 and y_3 for convenience.

This suggests that we can express the map f using the language of graphs.

In the following discussion, we shall not necessarily restrict the discussion to only the p.c. graphs, but rather implicitly refer to all graphs.

7.12. Proposition. *For the Weil algebra $B = \mathbb{2}[b_1, ..., b_n]/Q_B$ with corresponding (p.c.) graph Γ_B, the set of non-zero monomials $\underline{b}$ of B in the generators $\{b_1, \ldots, b_n\}$ are (canonically) in bijection with the independent sets of Γ_B.*

Proof. Since each generator b_i of B squares to zero, then each non-zero monomial $\underline{b}$ can be expressed (uniquely) as

$$\prod_{i \in I} b_i \ ;$$

for some appropriate subset $I \subseteq \{b_1, \ldots, b_n\}$. Since $\underline{b} \neq 0$, then for distinct $i, j \in I$, we must have $b_ib_j \neq 0$, i.e. $b_ib_j \notin Q_B$. This equivalently means there is no edge between the vertices b_i and b_j in Γ_B. I is thus a (possibly empty) independent set of Γ_B.

The reverse direction for the bijection is then obvious. ■

7.13. Remark. Using Proposition 7.12, we can equivalently say that to give a non-constant monomial $\underline{b}$ is to give a vertex of $\mathrm{Ind}_+(\Gamma_B)$.

As such, we may now express $f(a_i)$ (uniquely) as

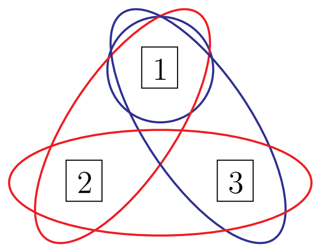
$$f(a_i) = \sum_{U \in \mathrm{Ind}_+(\Gamma_B)} \alpha^{(i)}_U b_U$$

over the non-empty independent sets U of Γ_B

7.14. Notation. For a graph G, let a *circle* U of G simply mean an independent set of G, but regarded pictorially as some shape encompassing the relevant vertices.

We may use this idea to express $f\colon A \to B$ pictorially: start by taking the generator a_1. Then take the graph Γ_B for B, and for each U with $\alpha_U^1 = 1$, we add onto Γ_B a circle corresponding to U, and we do this for all U with $\alpha_U^1 = 1$. Then repeat this process for each generator a_i, but (say) using a different colour for each different generator.

7.15. Example. The map $f\colon 2W \to 3W$ given by $x_1 \mapsto y_1y_2 + y_2y_3$ and $x_2 \mapsto y_1 + y_1y_3$ may be represented as



where $f(x_1)$ is represented in red and $f(x_2)$ is represented in blue.

7.16. Notation. For a map $f\colon A \to B$, let $\{U\}_f$ denote the graph Γ_B together with a set $\{(U, i) \mid \forall\ \alpha_U^i = 1\}$, all of this regarded pictorially as a set of coloured circles on Γ_B.

7.17. Remark. For a map $f\colon W \to B$, we will simply refer to a circle (U, i) of $\{U\}_f$ as U (i.e. we omit the index i).

So, to any map f we can associate a graph with coloured circles. However, not all sets of circles on the graph Γ_B are permissible.

In order to investigate this idea further, we begin with the following:

7.18. Proposition. *Consider maps of the form $f\colon W \to B$. To give such an f is to give a clique of* $\mathrm{Ind}_+(\Gamma_B)$.

Proof. Let x be the generator of W. Recall from Proposition 7.12 that each summand (monomial) of $f(x)$ is a (non-empty) independent set of Γ_B, i.e. a vertex of $\mathrm{Ind}_+(\Gamma_B)$. We may thus regard $f(x)$ as some subset X_f of the vertices of $\mathrm{Ind}_+(\Gamma_B)$.

Let distinct $U_1, U_2 \in X_f$ be given (i.e. two distinct monomials of $f(x)$). Then, since $x^2 = 0$, either

1. $U_1 \cap U_2 \neq \phi$ (so that they have a common vertex which becomes squared in the product $b_{U_1}b_{U_2}$), or

2. there exists $b_i \in U_1$ and $b_j \in U_2$ (with $i \neq j$) such that $(b_jb_{j'})$ is an edge of Γ_B.

In either case, each of the above conditions is equivalent to the independent sets U_1 and U_2 having an edge joining them in $\mathrm{Ind}_+(\Gamma_B)$. X is thus a clique of $\mathrm{Ind}_+(\Gamma_B)$. In particular, $f(x)$ corresponds to a vertex of $\mathrm{Cl}(\mathrm{Ind}_+(\Gamma_B))$.

Conversely, given a clique X of $\mathrm{Ind}_+(\Gamma_B)$, there is the obvious polynomial $p_X(b_1, \ldots, b_n)$ corresponding to X, and it is routine to check that $f_X(x) = p_X(b_1, \ldots, b_n)$ defines a valid morphism $f_X \colon W \to B$. ∎

7.19. Notation. For convenience, we shall let $\chi(\_)$ denote $\mathrm{Cl}(\mathrm{Ind}_+(\_))$.

We can take this one step further:

7.20. Proposition. *To give a map $f\colon A \to B$ is to give a graph homomorphism $\tilde{f}\colon \Gamma_A \to \chi(\Gamma_B)$.*

Proof. We know from Proposition 7.18 that each $f(a_i)$ corresponds to a vertex of $\chi(\Gamma_B)$ (we may view this as pre-composition with $\theta_i \colon W \to A$, with $\theta_i(x) = a_i$).

This gives us a function from the set $\{a_1, \ldots, a_m\}$ of vertices of Γ_A to the set of vertices of $\chi(\Gamma_B)$. We now verify that this function yields a valid graph homomorphism.

Suppose a_i and a_j are two distinct vertices of Γ_A with an edge joining them.

$$\begin{aligned} & (a_i, a_j) \text{ is an edge of } \Gamma_A \\ \Rightarrow\ & a_i a_j = 0 \text{ in } A \\ \Rightarrow\ & f(a_i) f(a_j) = 0 \text{ in } B\ . \end{aligned}$$

This tells us that if $\underline{b}_i$ and $\underline{b}_j$ are each a monomial from $f(a_i)$ and $f(a_j)$ respectively, then $\underline{b}_i \underline{b}_j = 0$. Using the same idea as the proof for Proposition 7.18, this says that there is an edge joining $\underline{b}_i$ and $\underline{b}_j$ in $\mathrm{Ind}_+(\Gamma_B)$.

This is true for all such pairs of monomials, and so $f(a_i)$ and $f(a_j)$, viewed as cliques in $\mathrm{Ind}_+(\Gamma_B)$, together (i.e. taking the union of the two cliques) give a clique. As such, when viewed as vertices of $\chi(\Gamma_B)$, there is an edge joining $f(a_i)$ and $f(a_j)$.

Thus $f\colon A \to B$ yields a unique graph homomorphism $\tilde{f}\colon \Gamma_A \to \chi(\Gamma_B)$.

The reverse direction is then obvious. ∎

These ideas actually allow us to prove an interesting fact about χ.

7.21. Proposition. *χ defines an endofunctor on the category* **Gph**, *and moreover, χ is canonically a monad.*

Proof. We first exhibit χ as an endofunctor. It is already well defined on objects. Let $G = (V, E)$ and $G' = (V', E')$ be arbitrary graphs and $h\colon G \to G'$ some chosen graph homomorphism.

Define $\chi(h)\colon \chi(G) \to \chi(G')$ as follows:

1. For a vertex $v \in V$, regarded as the singleton clique of the singleton independent set (so that it is a vertex of $\chi(G)$), define $(\chi h)(v) = h(v)$ (where $h(v) \in V'$ is regarded as a vertex of $\chi(G')$ in the same way).

2. For a non-empty independent set U of G (hence a vertex of $\mathrm{Ind}_+(G)$, and thus a singleton clique) viewed as a vertex of $\chi(G)$, define $(\chi h)(U)$ as

$$\begin{cases} \bigcup_{v\in U} h(v) & ; \text{ if the function } h \text{ restricted to domain } U \text{ is injective,} \\ & \text{ and this defines an independent set of } G' \\ \text{The empty clique} & ; \text{ otherwise} \end{cases} ;$$

if $\bigcup_{v\in U} h(v)$ does indeed define an independent set of G', we again regard it as a singleton clique of $\mathrm{Ind}_+(G')$, hence a vertex in $\chi(G')$.

3. For a clique C of $\mathrm{Ind}_+(G)$, define $\chi(C)$ as the clique of $\mathrm{Ind}_+(G')$ consisting of all $(\chi h)(U)$ not the empty clique, for all (non-empty) independent sets $U \in C$.

We leave as an exercise to the reader to show that this will preserve identities and composition, so that χ is functorial.

To show χ is a monad, we first give the unit $\eta\colon 1_{\mathbf{Gph}} \Rightarrow \chi$ by its components; $\eta_G\colon G \to \chi(G)$ sends each vertex $v \in V$ to the singleton clique of the singleton independent set $\{\{v\}\}$.

Using Proposition 7.20, it is easy to see that each $\eta_G\colon G \to \chi(G)$ corresponds to the identity $id_{\kappa(G)}\colon \kappa(G) \to \kappa(G)$.

The multiplication $\mu\colon \chi^2 \Rightarrow \chi$ has components $\mu_G\colon \chi^2(G) \to \chi(G)$ given as follows: Recall that

1. Vertices of G correspond to generators of $\kappa(G)$ (Definition 6.1);

2. Non-empty independent sets U of G correspond to non-constant, non-zero monomials of $\kappa(G)$ (Proposition 7.12), and an edge in $\mathrm{Ind}_+(G)$ is equivalent to the corresponding monomials multiply to zero;

3. Cliques of such independent sets are polynomials squaring to zero (Proposition 7.18), and an edge in $\chi(G)$ means that the product of the two corresponding polynomials yields zero.

Using Definition 7.7 and Definition 7.10, we can then see that

1. A non-empty independent set of such a clique (i.e. a vertex of $\mathrm{Ind}_+(\chi(G))$) then corresponds to a set X of polynomials for which the product of all polynomials in this set X is not zero, or X contains only the zero polynomial itself (taking the empty clique as a singleton). An edge between X and Y in this graph corresponds to there being polynomials $p \in X$ and $q \in Y$ such that $pq = 0$ in $\kappa(G)$;

2. A (possibly empty) clique of such an independent set (i.e. a vertex of $\chi^2(G)$) is a family ϱ of such sets of polynomials such that for any two distinct sets X and Y of this family, there are polynomials $p \in X$ and $q \in Y$ such that $pq = 0$ in $\kappa(G)$, and an edge between vertices ϱ and σ says that the union of the two families is also such a family.

Then, to give $\mu_G \colon \chi^2(G) \to \chi(G)$ is to associate each family of sets of polynomials to a polynomial squaring to zero. Let ϱ be one such family. Let $X \in \varrho$, and suppose $X = \{p_1, \ldots, p_r\}$, where each p_i is a polynomial of $\kappa(G)$ squaring to zero.

With this notation, we define $\mu_G(\varrho)$ to be the polynomial

$$\sum_{X \in \varrho} \left(\prod_{p_i \in X} p_i \right) .$$

Explicitly, for each set $X \in \varrho$, multiply together all the polynomials in this set (recall that unless X contains only the zero polynomial, then this product is non-zero). Then add up all such resultant polynomials across all $X \in \varrho$.

Now, each polynomial p_i squares to zero, so each product

$$\prod_{p_i \in X} p_i$$

squares to zero. Since ϱ is a clique of $\mathrm{Ind}_+ \chi(G)$, then any two sets $X, Y \in \varrho$ therefore are joined by an edge. As such, there exists $p \in X$ and $q \in Y$ with $pq = 0$ in $\kappa(G)$. As such, the product

$$\left(\prod_{p_i \in X} p_i \right) \left(\prod_{q_j \in Y} q_j \right)$$

must be zero. This is true for all pairs $X, Y \in \varrho$.

Thus, the polynomial $\mu_G(\varrho)$ squares to zero (hence is a vertex of $\chi(G)$).

Finally, suppose σ is another vertex such that (ϱ, σ) is an edge of $\chi^2(G)$. This means that $\varrho \cup \sigma$ is another family. As such, $\mu_G(\varrho \cup \sigma)$ is well defined and moreover squares to zero. In particular, this means that $\mu_G(\varrho)\mu_G(\sigma) = 0$ in $\kappa(G)$.

Therefore there must be an edge between $\mu_G(\varrho)$ and $\mu_G(\sigma)$.

We shall leave verifying the axioms of the monad as an exercise for the reader. ∎

Since χ is a monad, we can then consider the Kleisli category $\mathbf{Gph}_\chi$. Moreover, we can then define $\mathbf{Gph}'_\chi$ as the full subcategory whose objects are precisely the p.c. graphs.

7.22. PROPOSITION. *There exists an equivalence of categories*

$$F \colon \mathbf{Gph}'_\chi \to \mathbb{2}\text{-}\mathbf{Weil_1} .$$

PROOF. The functor F is defined as:

1. On objects: $F(G) = \kappa(G)$
2. On morphisms: A map $h \colon G \nrightarrow G'$ of $\mathbf{Gph}'_\chi$ is a graph homomorphism $h' \colon G \to \chi(G')$, and this corresponds to a unique map $\tilde{h} \colon \kappa(G) \to \kappa(G')$ of $\mathbb{2}$-$\mathbf{Weil_1}$ (Proposition 7.20). Thus, take $F(h) = \tilde{h}$.

Using Proposition 7.20, F is clearly full and faithful. Using Definition 6.1, Definition 6.3, and the fact that $\Gamma_{\kappa(G)} = G$, then F is essentially surjective. ∎

8. Construction of maps

We shall show in this section that using the set $\{\varepsilon_W, +, \eta_W, l, c\}$ (as defined in Section 4), composition, $\otimes$ and the universal property of foundational pullbacks (as given in Definition 3.17) of $\mathbb{2}$-$\mathbf{Weil_1}$, we are able to "construct" (in some appropriate sense) any map $f\colon A \to B$ of $\mathbb{2}$-$\mathbf{Weil_1}$. We begin by expressing the maps $\{\varepsilon_W, +, \eta_W, l, c\}$ in the form $\{U\}_f$ in Table 1 below:

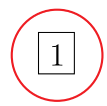
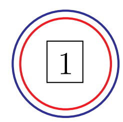
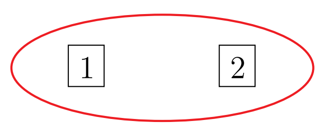
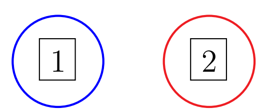
Map	Action on Generators	Graph
$\varepsilon_W\colon W \to \mathbb{2}$	$x_1 \mapsto 0$	(k corresponds to the empty graph)
$id_W\colon W \to W$	$x \mapsto x$	
$+\colon W^2 \to W$	$x_1 \mapsto x,\ x_2 \mapsto x$	
$\eta_W\colon \mathbb{2} \to W$	($\mathbb{2}$ has no generators)	$\boxed{1}$
$l\colon W \to 2W$	$x \mapsto x_1x_2$	
$c\colon 2W \to 2W$	$x_1 \mapsto x_2,\ x_2 \mapsto x_1$	

Table 1:

Pictorially, given $\{U\}_f$ for some map $f\colon A \to B$, we can naively interpret 'post-composition' with the above maps as follows:

- ε_W corresponds to deleting a particular vertex in Γ_B as well as any circles that go through that vertex.

- $+$ corresponds to taking two vertices in Γ_B joined by an edge and collapsing them to a single vertex. Circles that had contained either vertex (but not both) now contain the collapsed vertex instead.
- η_W corresponds to adding a new vertex to Γ_B, but has no effect on any of the existing circles.
- l corresponds to taking a single vertex of Γ_B and splitting it into two vertices without an edge joining them, and any circle U that contained the original vertex now contain both of the new vertices
- c corresponds to switching labels of (unjoined) vertices, and does nothing to the circles themselves.

These ideas will become clearer in subsequent discussion. We shall now precisely define what it means to say that a map $f\colon A \to B$ is "constructible".

8.1. Definition. *Let Ξ be a set given iteratively as follows:*

1. *The maps $\varepsilon_W, +, \eta_W, l, c$ are contained in Ξ.*
2. *Ξ contains all identities.*
3. *For all $n \in \mathbb{N}$, each projection $\pi_i\colon W^n \to W$ is contained in Ξ.*
4. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are both in Ξ, then their composite $g \circ f\colon X \to Z$ is also in Ξ. Equivalently, Ξ is closed under composition.*
5. *If $f\colon X \to Y$ and $g\colon A \to B$ are both in Ξ, then their tensor $f \otimes g\colon X \otimes A \to Y \otimes B$ is also in Ξ. Equivalently, Ξ is closed under tensor.*
6. *For a foundational pullback*
$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$
in $\mathbb{2}$-$\mathbf{Weil_1}$, and for a commuting square
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$
with $X \in \mathbb{2}$-$\mathbf{Weil_1}$ and $f, g \in \Xi$, then the uniquely induced map $h\colon X \to A$ is also in Ξ.

8.2. Definition. *For a map $f\colon A \to B$ of $\mathbb{2}$-$\mathbf{Weil_1}$, we shall say that f is* constructible *if $f \in \Xi$.*

8.3. Lemma. *For any Weil algebra $A \in \mathbb{2}$-$\mathbf{Weil_1}$, the unit η_A and augmentation ε_A are both constructible.*

Proof. The lemma is by definition true for $A = W$. We then simply note that $\varepsilon_{W^n} = \varepsilon_W \circ \pi_i$ (for any i) and η_{W^n} (induced using η_W and product diagrams regarded as foundational pullbacks) are both constructible, and for $X, Y \in \mathbb{2}$-$\mathbf{Weil_1}$ with $\eta_X, \eta_Y, \varepsilon_X, \varepsilon_Y$ constructible, then $\eta_{X\otimes Y} = \eta_X \otimes \eta_Y$, $\varepsilon_{X\otimes Y} = \varepsilon_X \otimes \varepsilon_Y$ are also constructible. ∎

8.4. Corollary. *Any zero map $z\colon A \to B$ is constructible, for all $A, B \in \mathbb{2}$-$\mathbf{Weil_1}$.*

Proof. For given $A, B \in \mathbb{2}$-$\mathbf{Weil_1}$, the zero map $z\colon A \to B$ is the composite

$$A \xrightarrow{\varepsilon_A} k \xrightarrow{\eta_B} B\ .$$

∎

8.5. Lemma. *The only (non-trivial) products in $\mathbb{2}$-$\mathbf{Weil_1}$ are the product powers W^n*

Proof. For arbitrary $X, Y \in \mathbb{2}$-$\mathbf{Weil_1}$, the graph $\Gamma_{X\times Y}$ for their product would need to be connected. The only connected p.c. graphs are the complete ones, and so $X \times Y = W^n$ for some n. ∎

8.6. Lemma. *For arbitrary $0 < n' < n$ in $\mathbb{N}$, all projections*

$$\pi'\colon W^n \to W^{n'}$$

are constructible.

Proof. Let $\pi'\colon W^n \to W^{n'}$ be a given projection. Without loss of generality, suppose π' preserves the first n' generators of W^n. Since each product can be regarded as a foundational pullback (Definition 3.17), π' is then constructed as $id_{W^{n'}} \times \varepsilon_{W^{n-n'}}$. ∎

8.7. Corollary. *Let*

$$\begin{tikzcd}
A \otimes W^{m+n} \arrow[r, "A\otimes\pi_1"] \arrow[d, "A\otimes\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes W^m \arrow[d, "A\otimes\varepsilon_{W^m}"] \\
A \otimes W^n \arrow[r, "A\otimes\varepsilon_{W^n}"'] & A
\end{tikzcd}$$

be an arbitrary foundational pullback (recall from Lemma 8.5 *that the only products in $\mathbb{2}$-$\mathbf{Weil_1}$ are product powers).*

Then, each of the four maps in this pullback diagram are constructible.

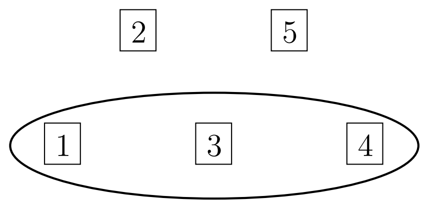
Proof. This is an immediate consequence of Definition 8.1, Lemma 8.3 and Lemma 8.6. ∎

Clearly, any map that is constructible by definition must live in $\mathbb{2}$-$\mathbf{Weil_1}$. We shall now sequentially build up in a different manner the maps of Ξ and show that in fact all maps $f\colon A \to B$ of $\mathbb{2}$-$\mathbf{Weil_1}$ are constructible.

8.8. Lemma. *Any map $f\colon W \to nW$ with precisely one circle is constructible.*

Let us begin with an example.

8.9. Example. The map $f\colon W \to 5W$ given by $x \mapsto x_1x_3x_4$ may be represented as

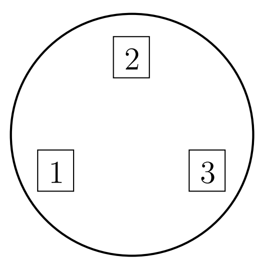


Define a map $\tilde{f}$ as the composite

$$\begin{array}{ccccc} W & \xrightarrow{l} & 2W & \xrightarrow{W\otimes l} & 3W \\ x & \longmapsto & x_1x_2 & \longmapsto & x_1x_2x_3\ . \end{array}$$

Clearly $\tilde{f}$ is constructible.

Then $\{U\}_{\tilde{f}}$ is



i.e. the single circle includes all 3 vertices.

Now define a map g as the map

$$\begin{aligned} W \otimes \eta_W \otimes W \otimes W \otimes \eta_W \colon 3W &\to 5W \\ x_1 &\mapsto y_1 \\ x_2 &\mapsto y_3 \\ x_3 &\mapsto y_4\ . \end{aligned}$$

Clearly, g is constructible.

Then the composite $g \circ \tilde{f}$ is precisely the original map f. Thus f is constructible.

We generalise this idea to prove Lemma 8.8.

Proof. Let $f\colon W \to nW$ with precisely one circle U be given. Let $r = |U|$. Define $\tilde{f}$ as the composite

$$W \xrightarrow{l} 2W \xrightarrow{W\otimes l} \dots \xrightarrow{(r-1)W\otimes l} rW$$

Clearly, $\tilde{f}$ is constructible.

In an analogous manner to Example 8.9, define a constructible map $g\colon rW \to nW$ with $g \circ \tilde{f} = f$. Thus f is constructible. ∎

8.10. Lemma. *All maps $f\colon W \to nW$ are constructible.*

Proof. If there are no circles in $\{U\}_f$ (i.e. $x \mapsto 0$), then the f is given by (say) the composite

$$W \xrightarrow{\varepsilon} k \xrightarrow{\eta} W \xrightarrow{W\otimes\eta} \dots \xrightarrow{(n-1)W\otimes\eta} nW$$

i.e. the zero map, hence f is constructible.

If f has one circle, we apply Lemma 8.8.

If f has more than one circle, then we prove this by induction. Let $S(m)$ be the statement "All maps $f\colon W \to nW$ with m circles or fewer are constructible, for all $n \in \mathbb{N}$".

We know $S(1)$ is true. Suppose that $S(r)$ is true for some $r \in \mathbb{N}$.

Let a map $f\colon W \to nW$ with precisely $r+1$ circles be given. Explicitly, this means that $f(x)$ is a polynomial in the generators of nW (which we shall call $y_1, \dots, y_n$) with precisely $r+1$ monomial summands.

Recall that for f to be a valid map, since the codomain is nW (or equivalently, the corresponding graph Γ_{nW} is discrete), then any two distinct summands of $f(x)$ must have (at least) one generator y_i in common. Let t and t' be distinct summands, and without loss of generality, suppose y_n is a common generator.

Now define a map

$$f'\colon W \to (n-1)W \otimes W^2$$

where $W^2 = \mathbb{2}[y_n, \widetilde{y_n}]/y_n^2, \widetilde{y_n}^2, y_n\widetilde{y_n}$, with $f'(x)$ having the same expression as $f(x)$, except that the y_n in term t' is replaced with $\widetilde{y_n}$. It is a routine task to check that this is a valid map. Furthermore, the composite

$$W \xrightarrow{f'} (n-1)W \otimes W^2 \xrightarrow{(n-1)W\otimes +} nW$$

will return the original map f. Clearly, the map $(n-1)W \otimes +$ is constructible, so it suffices to show that f' is constructible.

But the codomain of f', $(n-1)W \otimes W^2$, is the pullback

$$\begin{tikzcd}
(n-1)W \otimes W^2 \arrow[r, "(n-1)W\otimes\pi_1"] \arrow[d, "(n-1)W\otimes\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & nW \arrow[d, "(n-1)W\otimes\varepsilon_W"] \\
nW \arrow[r, "(n-1)W\otimes\varepsilon_W"'] & (n-1)W
\end{tikzcd},$$

and moreover, this is a foundational pullback.

Thus, to prove that f' is constructible, it suffices to prove that each of the composites

$$\big((n-1)W \otimes \pi_i\big) \circ f' \colon W \to nW;\ i \in \{1,2\}$$

is constructible. But each of these composites have a number of circles strictly fewer than $r+1$. Since we assumed that $S(r)$ was true, then both these composites are constructible, hence f is constructible.

Thus $S(r+1)$ is true.

As such, all maps $f \colon W \to nW$ are constructible. ■

We can actually prove Lemma 8.10 more directly. Suppose we have an arbitrary map $f \colon W \to nW$ with $\{U_f\}$ given. For each $i \in \{1, \dots, n\}$, let m_i be the number of circles containing vertex i (or equivalently, the number of terms of $f(x)$ containing the generator y_i). Then, in a similar manner as before, we can define a map

$$f' \colon W \to W^{m_1} \otimes \cdots \otimes W^{m_n}$$

in such a way that $(+_{m_1} \otimes \cdots \otimes +_{m_n}) \circ f' = f$. Here, since $+$ is an associative and commutative operation, then $+_m \colon W^m \to W$ is well defined, and $+_0$ is the nullary sum η_W.

Clearly, the map $+_{m_1} \otimes \cdots \otimes +_{m_n}$ is constructible. As for $f' \colon W \to W^{m_1} \otimes \cdots \otimes W^{m_n}$, we have the following

8.11. Proposition. *For each $n \in \mathbb{N}$, the object $W^{m_1} \otimes \cdots \otimes W^{m_n}$ is a limit of a (canonical) connected diagram of sW's.*

Proof. This is clearly true for $n = 1$ (as W^m is the m-fold pullback of ε_W).

Suppose this is true for $n = r$ for some $r \in \mathbb{N}$, i.e. for a given $V = W^{m_1} \otimes \cdots \otimes W^{m_r}$, there is a (canonical) connected diagram

$$\mathcal{D} \xrightarrow{D} \mathbf{2}\text{-}\mathbf{Weil_1}$$

with $Dd = s_d W$ for all $d \in \mathcal{D}$ and $\lim D = V$ (the limit calculated using iterations of foundational pullbacks).

Consider $V \otimes W^{m_{r+1}}$ for some $m_{r+1} \in \mathbb{N}_{>0}$. Let

$$\mathcal{C} \xrightarrow{C} \mathbf{2}\text{-}\mathbf{Weil_1}$$

be the m_{r+1}-fold connected diagram of ε_W's (i.e. $\lim C = W^{m_{r+1}}$).

We know from Proposition 3.14 that $V \otimes \_$ preserves $W^{m_{r+1}}$ as a limit, i.e.

$$\lim \left(\mathcal{C} \xrightarrow{G} \mathbf{2}\text{-}\mathbf{Weil_1} \xrightarrow{V \otimes \_} \mathbf{2}\text{-}\mathbf{Weil_1} \right) = V \otimes W^{m_{r+1}} .$$

For each $c \in \mathcal{C}$, by again using Proposition 3.14 as well as the symmetry of $\otimes$, we have

$$\lim \left(\mathcal{D} \xrightarrow{H} \mathbf{2}\text{-}\mathbf{Weil_1} \xrightarrow{\_ \otimes Cc} \mathbf{2}\text{-}\mathbf{Weil_1} \right) = V \otimes Cc ,$$

and note that each $Dd \otimes Cc$ is of the form sW for some $s \in \mathbb{N}$. ■

8.12. Lemma. *Let an arbitrary object $A = \mathbb{2}[a_1, \ldots, a_n]/Q_A$ of $\mathbb{2}$-$\mathbf{Weil_1}$ be given. Then any map $f \colon A \to W$ given as $f(a_i) = x$ for some fixed i and $f(a_j) = 0$ for all $j \neq i$ is constructible.*

Proof. Since $A \in \mathbb{2}$-$\mathbf{Weil_1}$, then Γ_A is a p.c. graph, or more generally, a cograph. We then show that f is constructible recursively as follows:

1) If $\Gamma_A = \{\bullet\}$ (the one point graph), then f is the identity and is thus constructible.
2) If $\Gamma_A = G \otimes H$ with $a_i \in H$, then f is the composite

$$A = \kappa(G) \otimes \kappa(H) \xrightarrow{\varepsilon_{\kappa(G)\otimes\kappa(H)}} \kappa(H) \xrightarrow{f'} W \ ;$$

   for a unique map f', and it thus suffices to show that f' is constructible.
3) If $\Gamma_A = G \times H$ with $a_i \in H$, then f is the composite

$$A = \kappa(G) \times \kappa(H) \xrightarrow{\pi_{\kappa(H)}} \kappa(H) \xrightarrow{f'} W \ ;$$

   for a unique map f', and it thus suffices to show that f' is constructible.

∎

8.13. Lemma. *Every map $f \colon A \to nW$ with no intersecting circles is constructible.*

Proof. Let $\mathcal{A}$ be the full subcategory of $\mathbb{2}$-$\mathbf{Weil_1}$ consisting of all objects A with the property that any map $A \to nW$ with no intersecting circles is constructible.

By Lemma 8.3, we have $\mathbb{2} \in \mathcal{A}$ (since $\mathbb{2}$ is a zero object, the only map to any A is the unit η_A), and by Lemma 8.10, we have $W \in \mathcal{A}$.

For arbitrary $m \in \mathbb{N}_{\geq 2}$, let an arbitrary map $f \colon W^n \to nW$ with no intersecting circles be given. If f is the zero map, then by Corollary 8.4, it is constructible. Suppose then that a is a generator of W_n for which $f(a) \neq 0$. Let a' be any other generator of W^m. Now, since $aa' = 0$ by construction, then $f(aa') = f(a)f(a') = 0$.

But since the codomain of f is nW and f has no intersecting circles, then we must have $f(a') = 0$. This is true for all generators of W^m (other than a, of course). But this means that f factors through the appropriate projection $\pi \colon W^m \to W$ preserving a (the other map being one of the form described in Lemma 8.8), thus f is constructible. Thus $W^m \in \mathcal{A}$ for all $m \in \mathbb{N}$.

Now suppose that A_1 and A_2 are arbitrary objects of $\mathcal{A}$. Let an arbitrary map $f \colon A_1 \otimes A_2 \to nW$ with no intersecting circles is given. Then, with some appropriate post-composition with c's, we can write $f = f_1 \otimes f_2$, for an appropriate pair $f_1 \colon A_1 \to rW$ and $f_2 \colon A_2 \to (n-r)W$ neither of which have intersecting circles. Thus f is constructible. Thus we have $A_1 \otimes A_2 \in \mathcal{A}$.

Now, since $\mathcal{A}$ is a full subcategory of $\mathbb{2}$-$\mathbf{Weil_1}$ containing $W^m \ \forall\, m \in \mathbb{N}$ and is closed under $\otimes$, then $\mathcal{A}$ is just $\mathbb{2}$-$\mathbf{Weil_1}$ itself. Thus any map $A \to nW$ with no intersecting circles is constructible. ∎

8.14. Lemma. *Every map $f\colon A \to nW$ is constructible.*

Proof. Let an arbitrary map $f\colon A \to nW$ be given. Using an analogous idea to that described in the proof of Lemma 8.10, we can construct a map

$$f'\colon A \to W^{m_1} \otimes \cdots \otimes W^{m_n}$$

as follows:

1) For each generator a_i of A, take the polynomial $f(a_i)$ in the generators $z_1, \dots, z_n$ of nW
2) Let m_j be the total number of terms across all the polynomials $f(a_1)$ containing z_j for $j = 1, \dots, n$
3) Define the map $f'\colon A \to W^{m_1} \otimes \cdots \otimes W^{m_n}$ By specifying each $f'(a_i)$ to be $f(a_i)$, but in such a way that each generator of $W^{m_1} \otimes \cdots \otimes W^{m_n}$ is used exactly once (in a similar fashion to the proof for Lemma 8.10)

8.15. Example. Consider the map $f\colon 2W \to 3W$ given as

$$\begin{aligned} x_1 &\mapsto y_1y_2 + y_1y_3 \\ x_2 &\mapsto y_2y_3 \ . \end{aligned}$$

Noting that each generator y_i appears in exactly two monomials, then we have the map $f'\colon 2W \to W^2 \otimes W^2 \otimes W^2$ given as

$$\begin{aligned} x_1 &\mapsto y_1y_2 + y_1'y_3 \\ x_2 &\mapsto y_2'y_3' \end{aligned}$$

Then f is the composite

$$A \xrightarrow{f'} W^{m_1} \otimes \cdots \otimes W^{m_n} \xrightarrow{+_{m_1} \otimes \cdots \otimes +_{m_n}} nW \ ,$$

and so it suffices to show f' is constructible. But now, for each projection

$$\pi = \pi_{i_1} \otimes \cdots \otimes \pi_{i_n}\colon W^{m_1} \otimes \cdots \otimes W^{m_n} \to nW,$$

the composite $\pi \circ f'\colon A \to nW$ has no intersecting circles, and is thus constructible using Lemma 8.13, and we use a series of foundational pullbacks to recover f'. ■

Before we introduce Theorem 8.17, we shall also require the following lemma:

8.16. Lemma. *Let G be a cograph (recall that each p.c. graph is also a cograph) with at least one edge (and hence at least two vertices). Then G can be expressed as $(G_1 \times G_2) \otimes H$, where G_1 and G_2 are non-empty cographs (H may be empty).*

Proof. Let e be a chosen edge of G. Let G' be the connected component of G containing the edge e. Clearly, we can express G as a disjoint union $G' \otimes H$ (with H possibly empty).

Now, since G' contains an edge, it cannot be the one point graph. Since it is connected, it cannot be expressed (non-trivially) as $G_1 \otimes G_2$. Since it is a cograph, then by Definition 5.15, it can be expressed non-trivially as $G_1 \times G_2$. ■

We now have the following:

8.17. THEOREM. *Every map $f\colon A \to B$ in $\mathbb{2}$-$\mathbf{Weil_1}$ is constructible.*

PROOF. Consider the Weil algebra B. If Γ_B has any edges then, using Lemma 8.16, it can be expressed (non-trivially) as $(G_1 \times G_2) \otimes H$ (with H possibly being the empty graph). Correspondingly, $B = (\kappa(G_1) \times \kappa(G_2)) \otimes \kappa(H)$ and we thus have the foundational pullback

$$\begin{tikzcd} B \arrow[r, "\pi_1 \otimes \kappa(H)"] \arrow[d, "\pi_2 \otimes \kappa(H)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \kappa(G_1) \otimes \kappa(H) \arrow[d, "\varepsilon_1 \otimes k[H]"] \\ \kappa(G_2) \otimes \kappa(H) \arrow[r, "\varepsilon_2 \otimes \kappa(H)"'] & \kappa(H) \end{tikzcd}$$

and so $f\colon A \to B$ is uniquely induced by the pair $(\pi_i \otimes \kappa(H)) \circ f$; $i = 1, 2$. As such, it suffices to show that each of these is constructible.

Note now that the graphs $G_i \otimes H$ for the codomains each have strictly fewer edges than Γ_B. As such, we repeat this process until the codomains are all of the form nW, then directly apply Lemma 8.14. ∎

9. Obtaining coefficients outside $\mathbb{2}$

We gave Theorem 8.17, which said that every map $f\colon A \to B$ is constructible. However, this was for the case of $\mathbb{2}$-$\mathbf{Weil_1}$, and so we limit the permissible maps by restricting the coefficients to being either 0 or 1.

Consider k-$\mathbf{Weil_1}$ for an arbitrary rig k. For each $t \in k$, define $\hat{g}_t\colon W \to W$ to be the map given as $\hat{g}_t(x) = tx$. Note that $\hat{g}_0$ is the zero map and $\hat{g}_1$ is the identity id_W.

Define Ξ_k in the same way as Definition 8.1, with the added condition that $\hat{g}_t$ is contained in Ξ_k for all $t \in k$. Define the notion of a *constructible* morphism in the same way.

9.1. PROPOSITION. *Every map $g\colon A \to B$ of k-$\mathbf{Weil_1}$ is constructible.*

PROOF. *(Sketch)* Consider first (the analogue of) Lemma 8.8. Suppose we had a map $f\colon W \to nW$ with $f(x)$ given by a single monomial (with some arbitrary coefficient $r \in k$). Let $f'\colon W \to nW$ be the map with $f'(x)$ being the same monomial, but with coefficient one. Clearly, f' is constructible.

Then, the composite

$$W \xrightarrow{\hat{g}_r} W \xrightarrow{f'} W \quad ,$$

yields f, and so f is constructible.

From there, the proofs for (the analogues of) Lemma 8.10 through to Lemma 8.14 as well as Theorem 8.17 are identical. ∎

Recall, however, that we are ultimately interested in $\mathbb{N}$-$\mathbf{Weil_1}$. We begin with the following:

9.2. Proposition. *Let $\psi^\dagger\colon \mathbb{N} \to \mathbb{2}$ be the rig morphism*

$$\psi^\dagger(n) = \begin{cases} 0 & ;\ n = 0 \\ 1 & ;\ \textit{otherwise} \end{cases} .$$

The canonical functor

$$\psi\colon \mathbb{N}\text{-}\mathbf{Weil_1} \to \mathbb{2}\text{-}\mathbf{Weil_1}$$

induced by the rig morphism above is bijective on objects and full.

Here, ψ sends each object $\mathbb{N}[x_1, \ldots, x_r]/Q$ of $\mathbb{N}$-$\mathbf{Weil_1}$ to its counterpart $\mathbb{2}[x_1, \ldots, x_r]/Q$ in $\mathbb{2}$-$\mathbf{Weil_1}$. There is analogous action of ψ on morphisms.

Proof. Bijectivity on objects follows immediately from the fact that Definition 4.2 defines the objects of k-$\mathbf{Weil_1}$ independently from the choice of k.

For any morphism $f\colon A \to B$ of $\mathbb{2}$-$\mathbf{Weil_1}$, there is a corresponding map $g\colon A \to B$ in $\mathbb{N}$-$\mathbf{Weil_1}$ given by the same action on generators as f. Clearly, we then have $\psi g = f$. ∎

Let us now work with $\mathbb{N}$-$\mathbf{Weil_1}$. Let Ξ be defined as in Definition 8.1 (i.e. we do not explicitly include in Ξ the maps $\hat{g}_t$ for all $t \in \mathbb{N}$). We first have the following:

9.3. Lemma. *For each $t \in \mathbb{N}$, the map $\hat{g}_t$ is constructible.*

Proof. First we note again that $\hat{g}_0$ is the zero map and $\hat{g}_1$ is the identity, and thus both are constructible.

We shall show that all $\hat{g}_t$'s are constructible by induction. Let $S(t)$ be the statement "$\hat{g}_t$ is constructible". We have established that $S(0)$ and $S(1)$ are true. Suppose $S(r)$ is true.

We then have

$$\begin{tikzcd}
W \arrow[dr, dashed, "h"] \arrow[drr, bend left, "\hat{g}_r"] \arrow[ddr, bend right, "id_W"'] & & \\
& W^2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "\varepsilon_W"] \\
& W \arrow[r, "\varepsilon_W"'] & k
\end{tikzcd} ,$$

so the map h is constructible. Then, the map $\hat{g}_{r+1}$ is clearly the composite

$$W \xrightarrow{h} W^2 \xrightarrow{+} W ,$$

so that $\hat{g}_{r+1}$ is also constructible, so that $S(r+1)$ is true. ∎

9.4. Remark. We may try to give a similar construction in $\mathbb{2}$-$\mathbf{Weil_1}$, but note that for all $t \in \mathbb{N}_{>0}$, we will have $\hat{g}_t = id_W$, since $1+1 = 1$ in $\mathbb{2}$.

With these "coefficient maps" $\hat{g}_t$ being constructible along with Theorem 8.17, we now have the following:

9.5. Proposition. *Every map $g\colon A \to B$ of $\mathbb{N}$-$\mathbf{Weil_1}$ is constructible.*

Proof. This is a direct consequence of Proposition 9.1 and Lemma 9.3. ∎

10. Instructions for assembly

In Lemmas 8.10 and 8.14, there was an element of choice involved; namely given a map $f\colon A \to nW$ (for the case of Lemma 8.14, say), the corresponding map $f'\colon A \to W^{m_1} \otimes \cdots \otimes W^{m_n}$ required a choice as to which circle would correspond to which projection. Ultimately, this choice is inconsequential as different choices are (up to isomorphism) equivalent.

However, for the purposes of what we wish to do, we will assume that for each $f\colon A \to nW$, there is some pre-determined choice that has already been made regarding the corresponding map f'.

This then implicitly equips each map $f\colon A \to B$ of $\mathbb{2}$-$\mathbf{Weil_1}$ (and hence $\mathbb{N}$-$\mathbf{Weil_1}$) with a set of instructions for its construction.

11. The map Ω

We will now describe a certain construction Ω, a map in $\mathbb{N}$-$\mathbf{Weil_1}$, which we shall require in order to prove Proposition 12.13 later.

Let $s \in \mathbb{N}$ be given. For an arbitrary map $g\colon B \to sW$, recall that g decomposes (as described in the proof of Lemma 8.14) as

$$\begin{tikzcd} & W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} \arrow[d, "+_\beta"] \\ B \arrow[ur, "g'"] \arrow[r, "g"'] & sW \end{tikzcd} .$$

By 10, the particular decomposition is fixed (i.e. g' is uniquely determined by g).

One way we can view this decomposition involves the slice category $\mathbb{N}$-$\mathbf{Weil_1}/sW$; the pair $(g, +_\beta)$ (again, g' is uniquely determined by g) can be seen as an object of the arrow category $(\mathbb{N}\text{-}\mathbf{Weil_1}/sW)^2$. We shall now extend this to a functor

$$\tau\colon \mathbb{N}\text{-}\mathbf{Weil_1}/sW \to (\mathbb{N}\text{-}\mathbf{Weil_1}/sW)^2$$

whose composite with the domain functor $d\colon (\mathbb{N}\text{-}\mathbf{Weil_1}/sW)^2 \to \mathbb{N}\text{-}\mathbf{Weil_1}/sW$ is the identity.

This amounts to giving, for each arrow

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & sW \end{tikzcd}$$

of $\mathbb{N}$-$\mathbf{Weil_1}/sW$, a morphism

$$\Omega\colon W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} \to \ W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \ ,$$

such that the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "h'"'] & & B \arrow[d, "g'"] \\ W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[rr, "\Omega"'] \arrow[dr, "+_\delta"'] & & W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} \arrow[dl, "+_\beta"] \\ & sW & \end{tikzcd}$$

commutes, and satisfying the evident functoriality conditions.

11.1. Remark. We are, of course, taking $h\colon A \to sW$ to decompose as

$$\begin{tikzcd} & W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[d, "+_\delta"] \\ A \arrow[ur, "h'"] \arrow[r, "h"'] & sW \ . \end{tikzcd}$$

It now remains to specify Ω.

Since $W^{\beta_1} \otimes \cdots \otimes W^{\beta_s}$ is a limit (Proposition 8.11), it then suffices to define each map $\Omega_{(r_1,\ldots,r_s)}$ as shown below

$$\begin{tikzcd} W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[d, dashed, "\Omega"'] \arrow[dr, "\Omega_{(r_1,\ldots,r_s)}"] & \\ W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} \arrow[r, "(r_1,\ldots,r_s)=\pi_{r_1}\otimes\cdots\otimes\pi_{r_s}"'] & sW \ , \end{tikzcd}$$

where the $\Omega_{(r_1,\ldots,r_s)}$ are suitably compatible.

But to give $\Omega_{(r_1,\ldots,r_s)}$, it suffices to say where each generator of $W^{\delta_1} \otimes \cdots \otimes W^{\delta_s}$ is sent. Let y_1 be a generator of W^{δ_1} (without loss of generality, let $\alpha_1 \geq 1$). We shall refer to the generators of sW as $z_1, \ldots, z_s$. Observe that $+_\beta(y_1) = z_1$.

Recall from 10 the construction of $h'\colon A \to W^{\delta_1} \otimes \cdots \otimes W^{\delta_s}$. There is a unique circle (U_1, a) for some generator a of A with $y_1 \in U_1$ (and correspondingly, a unique circle (U_1, a) of h as well with $z_1 \in U_1$). Recall also that $h = g \circ f$. Let

$$h(a) = U_1 + U_2 + \ldots \ ,$$

where each U_i is a monomial in the generators $z_1, \dots, z_s$. Similarly, let

$$f(a) = V_1 + V_2 + \dots ,$$

where each V_i is a monomial in the generators $\{b_j\}$ of B.

Then (ignoring coefficients), since g preserves addition and multiplication, we can express $(g \circ f)(a)$ as

$$\begin{aligned}(g \circ f)(a) &= g(f(a)) \\ &= g(V_1 + V_2 + \dots) \\ &= g(V_1) + g(V_2) + \dots \\ &= \left[\prod_{b_j \in V_1} g(b_j)\right] + \left[\prod_{b_j \in V_2} g(b_j)\right] + \dots .\end{aligned}$$

But this needs to be equal to $h(a)$. In particular, U_1 must be somewhere in the expression for $(g \circ f)(a)$. Without loss of generality, suppose U_1 is contained in the first term

$$\prod_{b_j \in V_1} g(b_j).$$

Now, for each $b_j \in V_1$, we must be able to choose precisely one circle Q_j in such a way that

$$\bigcup_{b_j \in V_1} Q_j = U_1$$

with the Q_j's pairwise distinct. This is because for each $b_j \in V_1$, $g(b_j)$ is a polynomial in the generators $z_1, \dots, z_n$. Then, if the product of these polynomials (which in turn is another polynomial) is to contain a particular monomial (namely U_1), then this monomial must have arisen as the product of one monomial from each of the factor polynomials.

Moreover, since $z_1 \in U_1$, then we also have $z_1 \in Q_j$ for a unique j. Take $j = 1$ so that Q_1 is one of the terms of the polynomial $g(b_1)$.

$\Rightarrow$ Q_1 is a circle of g corresponding to b_1

$\Rightarrow$ In $g' \colon B \to W^{\beta_1} \otimes \dots \otimes W^{\beta_n}$, $\exists!$ generator v of W^{β_1} corresponding to the circle Q_1

$\Rightarrow$ Define $\Omega_{(r_1,\dots,r_s)}(y_1) = \begin{cases} z_1 & ;\ (r_1, \dots, r_s) \text{ preserves } v \text{ (in particular, } r_1 \text{ preserves } v) \\ 0 & ;\ \text{otherwise} \end{cases}$

and repeat for all generators of $W^{\beta_1} \otimes \dots \otimes W^{\beta_n}$.

In particular, note that since Ω can only assign a generator from any W^{δ_i} to a generator of the corresponding W^{β_i}, then we have $\Omega = \Omega_1 \otimes \dots \otimes \Omega_s$, for appropriate maps $\Omega_i \colon W^{\delta_i} \to W^{\beta_i}$.

11.2. Remark. We shall note here that in full formality, we should use the label $\Omega_{f,g}$ (or something to this effect), but we shall not be doing this.

12. Back to Tangent Structure

We defined the category k-$\mathbf{Weil_1}$ in Definition 4.2 and in Section 8, we defined the notion of a *constructible* morphism (Definition 8.2) and showed that any map of $\mathbb{2}$-$\mathbf{Weil_1}$ was constructible (Theorem 8.17). We then said in Proposition 9.5 that in fact any map of $\mathbb{N}$-$\mathbf{Weil_1}$ was constructible, and moreover in 10 we noted that each map $g\colon A \to B$ was equipped with a set of instructions for its construction.

We shall conclude by linking these ideas about Weil algebras back to Tangent Structures in an explicit manner.

We wish to construct a functor

$$F\colon \mathbb{N}\text{-}\mathbf{Weil_1} \to \mathrm{End}(\mathcal{M})$$

with certain properties (which we shall specify in due course). However, we first need to establish some facts.

Suppose that a given category $\mathcal{M}$ is equipped with a Tangent Structure $\mathbb{T}$ (in the sense of Definition 2.6). Regard $\mathrm{End}(\mathcal{M})$ as a monoidal category with respect to the operation of composition $\circ$, and with unit the identity functor $1_{\mathcal{M}}$.

12.1. Notation. To avoid confusion, when we want to regard composition as a monoidal operation in $\mathrm{End}(\mathcal{M})$, we will use concatenation if the meaning is clear (otherwise we will explicitly use $\otimes$), and save $\circ$ for actual composition. For example, if we have natural transformations $\alpha\colon R \Rightarrow S$ and $\beta\colon S \to U$ in $\mathrm{End}(\mathcal{M})$, then $\beta \circ \alpha$ denotes the composite

$$R \overset{\alpha}{\Longrightarrow} S \overset{\beta}{\Longrightarrow} U \ ;$$

whereas $\beta\alpha$ denotes the natural transformation

$$S \circ R \overset{\beta\alpha}{\Longrightarrow} U \circ S \quad .$$

12.2. Definition. *Let*

$$F_0\colon ob(\mathbb{N}\text{-}\mathbf{Weil_1}) \to ob(\mathrm{End}(\mathcal{M}))$$

be the function given as $F_0(\mathbb{N}) = 1_{\mathcal{M}}$, $F_0(W^m) = T^{(m)}$ *for all* $m \in \mathbb{N}$, *and then recursively, if* $A, B \in \mathbb{N}$-$\mathbf{Weil_1}$ *with* $F_0(A) = R$, $F_0(B) = S$, *then* $F_0(A \otimes B) = R \circ S$.

12.3. Proposition. *For any foundational pullback*

$$\begin{tikzcd} A \otimes (B \times C) \arrow[r, "A\otimes\pi_B"] \arrow[d, "A\otimes\pi_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes B \arrow[d, "A\otimes\varepsilon_B"] \\ A \otimes C \arrow[r, "A\otimes\varepsilon_C"'] & A \end{tikzcd}$$

in $\mathbb{N}$-$\mathbf{Weil_1}$ *(recall from* Lemma 8.5 *that the only products in* $\mathbb{N}$-$\mathbf{Weil_1}$ *are the powers* W^n *of* W*), we have a corresponding pullback*

$$\begin{tikzcd} F_0(A\otimes(B\times C)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_0(A\otimes B) \arrow[d] \\ F_0(A\otimes C) \arrow[r] & F_0(A) \end{tikzcd}$$

in $\mathrm{End}(\mathcal{M})$*, which we may also equivalently express as*

$$\begin{tikzcd} F_0(A)F_0(B\times C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_0(A)F_0(B) \arrow[d] \\ F_0(A)F_0(C) \arrow[r] & F_0(A)\ . \end{tikzcd}$$

We shall also refer to these as foundational pullbacks (in $\mathrm{End}(\mathcal{M})$*).*

Proof. The square in $\mathrm{End}(\mathcal{M})$ being a pullback is a direct consequence of the axioms of $\mathbb{T}$ ■

12.4. Definition. *Let* Ψ *be a collection of pairs* (f,α)*, where* $f\colon X\to Y$ *is a morphism in* $\mathbb{N}$-$\mathbf{Weil_1}$ *and* $\alpha : F_0(X)\Rightarrow F_0(Y)$ *is a morphism in* $\mathrm{End}(\mathcal{M})$ *(i.e. a natural transformation), given as follows:*

We begin with the following pairs:

- *Each element of* $\{\varepsilon_W,\eta_W,+,l,c\}$ *is paired with its obvious counterpart* $\{p,+,\eta,l,c\}$*.*
- *For each object* $A\in\mathbb{N}$-$\mathbf{Weil_1}$*, the pair* $(id_A, id_{F_0(A)})$*.*
- *For any given foundational pullback in* $\mathbb{N}$-$\mathbf{Weil_1}$*, each map in this pullback is paired with its obvious counterpart in the corresponding pullback in* $\mathrm{End}(\mathcal{M})$ *(in the sense of* Proposition 12.3*).*

This gives us a starting point for Ψ*. Recall from* 10 *that any map* $h\colon A\to B$ *of* $\mathbb{N}$-$\mathbf{Weil_1}$ *is equipped with a (finite) sequential set of instructions for its construction. We then iteratively add to* Ψ *as follows:*

Let f,g *and* h *be maps in* $\mathbb{N}$-$\mathbf{Weil_1}$*, and suppose we already have pairs*

$$(f,\alpha),(g,\beta)\in\Psi\ .$$

- *If the final step of the instructions of* h *was to obtain* h *as the composite* $g\circ f$*, then we add to* Ψ *the pair* $(h,\beta\circ\alpha)$*. That is, we close* Ψ *under certain compositions.*
- *If the final step of the instructions of* h *was to obtain* h *as the tensor* $g\otimes f$*, then we add to* Ψ *the pair* $(h,\beta\alpha)$*. That is, we close* Ψ *under certain tensors.*

- *If the final step of the instructions of h was to (uniquely) induce h using f and g as*

$$\begin{tikzcd} A \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\ & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d] \\ & B_2 \arrow[r] & C \end{tikzcd} ;$$

where the pullback square is a foundational one, then we consider the diagram

$$\begin{tikzcd} F_0(A) \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\ & F_0(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_0(B_1) \arrow[d] \\ & F_0(B_2) \arrow[r] & F_0(C) \end{tikzcd}$$

in $\mathrm{End}(\mathcal{M})$ *(where we use the foundational pullback in* $\mathrm{End}(\mathcal{M})$ *corresponding to the one above).*

If the exterior commutes, then by the universal property of the pullback, a unique map $\gamma\colon F_0(A) \to F_0(B)$ will be induced. In that case, add to Ψ the pair (h, γ).

If the exterior does not commute, then we will say that "h does not have a pairing in Ψ".

12.5. Definition. *Let Φ be the collection of all maps $h\colon A \to B$ in $\mathbb{N}$-$\mathbf{Weil_1}$ which do not have a pairing in Ψ.*

12.6. Lemma. *Each coefficient map $\hat{g}_t$ is paired with some (unique) natural transformation λ_t in Ψ.*

Proof. We know that λ_0 is the composite

$$T \xRightarrow{p} 1_{\mathcal{M}} \xRightarrow{\eta} T$$

(since this was how $\hat{g}_0$ was constructed) and λ_1 is the identity id_T (since $\hat{g}_1$ was the identity).

Since each $\hat{g}_t$ is then constructed recursively as

$$\begin{tikzcd} & W & \\ W \arrow[ur, "\hat{g}_{t-1}"] \arrow[r, dashed] \arrow[dr, "id_W"'] & W^2 \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[r, "+"] & W \\ & W & \end{tikzcd} ,$$

then each λ_t is constructed recursively in the same way. ∎

Clearly, Ψ and Φ are mutually exclusive and exhaustive collections. For now, let us focus on Ψ.

12.7. Notation. When describing pairs in Ψ, if f is a map in $\mathbb{N}$-$\mathbf{Weil_1}$, then we will use $\widetilde{f}$ to denote the corresponding natural transformation in $\mathrm{End}(\mathcal{M})$, i.e. we have $(f,\widetilde{f})\in\Psi$, as we shall see below.

We will now sequentially show that the pairings of Ψ "preserve" (arbitrary) composition, i.e. if we have arbitrary pairings $(f,\widetilde{f}),(g,\widetilde{g}),(h,\widetilde{h})\in\Psi$ such that $h=g\circ f$ in $\mathbb{N}$-$\mathbf{Weil_1}$, then we have $\widetilde{h}=\widetilde{g}\circ\widetilde{f}$ in $\mathrm{End}(\mathcal{M})$.

Explicitly, suppose we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "h=g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

in $\mathbb{N}$-$\mathbf{Weil_1}$. We wish to show that

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[r, "\widetilde{g}"] & F_0C \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$, for all $(f,\widetilde{f}),(g,\widetilde{g}),(h=g\circ f,\widetilde{h})\in\Psi$.

As with Section 8, we shall begin with the most basic case for "preservation" of composition by the pairings in Ψ, and then sequentially build our way up to the general case.

12.8. Proposition. *For all $f\colon qW\to rW$ and $g\colon rW\to sW$, neither of which having intersecting circles, the diagram*

$$\begin{tikzcd} T^q \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & T^r \arrow[r, "\widetilde{g}"] & T^s \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$.

Proof. First, since f and g have no intersecting circles, then h also has no intersecting circles.

Since f has domain qW and has no intersecting circles, then it can be expressed in the form (modulo some appropriate post-composition with $c's$)

$$f=f_1\otimes\cdots\otimes f_q\otimes\eta_{q'_W}\colon W\otimes\cdots\otimes W\otimes k\to\xi_1W\otimes\cdots\otimes\xi_qW\otimes q'W$$

(and f is constructed as such); where each f_i is has a single circle and is either given as ε_W if $\xi_i=0$, or constructed as the composite

$$W\xrightarrow{\hat{g}_{a_i}}W\xrightarrow{l}\ldots\xrightarrow{(\xi_i-1)W\otimes l}\xi_iW$$

(as described in the proof of Lemma 8.8), for an appropriate coefficient map $\hat{g}_{a_i}$.

12.9. Remark. Note that we also have

$$\left(\sum_{i=1}^{q} \xi_i\right) + q' = m \ .$$

An analogous fact is true for g and h. The natural transformations $\widetilde{f}, \widetilde{g}$ and $\widetilde{h}$ are then constructed in a corresponding manner.

Now, it can be shown that for all $c, d \in \mathbb{N}$, the diagram

$$\begin{tikzcd}
T \arrow[r, "\lambda_c"] \arrow[d, "\lambda_{cd}"'] & T \arrow[r, "l"] & T^2 \arrow[d, "\lambda_d T"] \\
T \arrow[rr, "l"'] & & T^2
\end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$ (recall that each λ_t is paired with the coefficient map $\hat{g}_t$ in Ψ). Together with the fact that the diagram

$$\begin{tikzcd}
T \arrow[r, "l"] \arrow[d, "l"'] & T^2 \arrow[d, "lT"] \\
T^2 \arrow[r, "Tl"'] & T^3
\end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$ (an axiom of $\mathbb{T}$), then we have $\widetilde{g} \circ \widetilde{f} = \widetilde{h}$.

Thus $\widetilde{h} = \widetilde{g} \circ \widetilde{f}$ ∎

12.10. Proposition. *For all $f\colon A \to rW$ and $g\colon rW \to sW$, neither of which having intersecting circles, the diagram*

$$\begin{tikzcd}
F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & T^r \arrow[r, "\widetilde{g}"] & T^s
\end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$.

Proof. First, we note that if f and g do not have intersecting circles, then neither does h.

Consider $f\colon A \to rW$. Using the arguments from the proof of Lemma 8.13, since f has no intersecting circles, it must factor through some particular projection $\pi\colon A \to qW$ of A (as the final step in its construction), and the same is true for h.

Correspondingly, α and γ both factor through the corresponding projection $\pi\colon F_0A \to T^q$.

As such, it suffices to assume $A = qW$, and so $F_0A = T^q$. Then, we can apply Proposition 12.8 directly.

$\therefore\ \widetilde{h} = \widetilde{g} \circ \widetilde{f}$ ∎

12.11. Proposition. *For all arbitrary $f\colon A \to rW$, and $g\colon rW \to sW$ with no intersecting circles, the diagram*

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & T^r \arrow[r, "\widetilde{g}"] & T^s \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$.

Proof. Let $\{y_1, \dots, y_r\}$ denote the generators of rW and $\{z_1 \dots, z_s\}$ denote the generators of sW.

By the same argument as used in the proof for Proposition 12.8, then modulo appropriate post-composition with c's, we can express g as

$$g = g_1 \otimes \cdots \otimes g_r \otimes \eta_{r'_W} \colon W \otimes \cdots \otimes W \otimes k \to \nu_1 W \otimes \cdots \otimes \nu_q W \otimes r'W$$

(and g is constructed as such); where each $g_i\colon W \to \nu_i W$ has a single circle.

Without loss of generality, we shall assume that $r' = 0$ and $\nu_i > 0$ for all i. This amounts to asking that no generator y_i of rW is sent by g to zero, and that each generator z_j of sW belongs to exactly one of the r circles of $\{U\}_g$.

This then defines a surjective function

$$\psi\colon \{z_1, \dots, z_n\} \to \{y_1, \dots, y_m\} \; .$$

Without loss of generality, suppose that $\psi(z_1) = y_1$.

Suppose the maps h and f factorise as the composites

$$\begin{tikzcd} & W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[d, "+_\delta"] \\ A \arrow[ur, "h'"] \arrow[r, "h"'] & sW \end{tikzcd} \qquad \begin{tikzcd} & W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} \arrow[d, "+_\vartheta"] \\ A \arrow[ur, "f'"] \arrow[r, "f"'] & rW \; . \end{tikzcd}$$

(and are constructed as such, recall Lemma 8.14).

Note that correspondingly, $\widetilde{f}$ and $\widetilde{h}$ are given as composites

$$\begin{tikzcd} & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[d, "+_\delta"] \\ F_0A \arrow[ur, "\widetilde{h}'"] \arrow[r, "\widetilde{h}"'] & T^s \end{tikzcd} \qquad \begin{tikzcd} & T^{(\vartheta_1)} \otimes \cdots \otimes T^{(\vartheta_r)} \arrow[d, "+_\vartheta"] \\ F_0A \arrow[ur, "\widetilde{f}'"] \arrow[r, "\widetilde{f}"'] & T^r \; , \end{tikzcd}$$

noting that since $+\colon T^{(2)} \to T$ is an associative, commutative and unital map, then there is a well defined map

$$+_{(\delta_i)}\colon T^{(\delta_i)} \to T$$

for each i, and finally, we define

$$+_\delta \colon = +_{(\delta_1)} \otimes \cdots \otimes +_{(\delta_s)}$$

in $\mathrm{End}(\mathcal{M})$. The same is true for $+_\vartheta$.

Firstly, this means that for the map h, there are precisely δ_1 circles (say $U_1, \dots, U_{\delta_1}$) containing the generator z_1. But given what we've established about g, and noting that $h = g \circ f$, then the z_1 term in each of these U_i must arise as a result of the generator y_1 (since $\psi(z_1) = y_1$). More explicitly, to each circle U_i of h containing z_1 we can associate a unique circle of f containing y_1.

Conversely, for each circle V_j of f containing y_1, we have $g(V_j) \neq 0$ (moreover, $g(V_j)$ is a single circle) and $z_1 \in g(V_j)$. Therefore the number of circles of f containing y_1 (namely ϑ_1) is the same as the number of circles of h containing z_1 (namely α_1). Thus, if $\psi(z_i) = y_j$, then $\delta_i = \vartheta_j$.

We then define a map

$$\Lambda\colon W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} \to W^{\delta_1} \otimes \cdots \otimes W^{\delta_s}$$

induced using

$$\begin{tikzcd}
 & rW \arrow[rr, "g"] & & sW \\
W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} \arrow[ur, "t"] \arrow[rr, dashed, "\exists!\ \Lambda"'] & & W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[ur, "\overline{a}"'] &
\end{tikzcd} \tag{1}$$

where, for each fixed projection $\overline{a} = (a_1, \dots, a_s)$, t is determined as follows:

- Consider $\overline{a} \circ h'\colon A \to sW$. If this is the zero map, then t is also the zero map.
- If not, this means that there is at least one circle U of h (and hence h') with each of its generators preserved by $\overline{a}$. Moreover, if h has multiple circles, then they must be disjoint and each corresponds to a different generator of A (see proof of Lemma 8.14).

  Without loss of generality, assume there is only one such circle U. Regard U as a subset of $\{z_1, \dots, z_s\}$. Then we know $\psi(U)$ (the image of U under ψ) is the unique circle of f corresponding to U. Choose t (in the unique way) so that this circle $\psi(U)$ of f is preserved, but sends any $y_j \notin \psi(U)$ to 0.

We shall also note that there is a corresponding natural transformation $\widetilde{t}$ (i.e. $\big(t, \widetilde{t}\big) \in \Psi$, we shall not prove this).

Now, it is fairly routine (albeit tedious) to show that Λ is paired with some unique natural transformation $\widetilde{\Lambda}$ in Ψ (i.e. that it exists). It can also be shown that the diagram

$$\begin{tikzcd}
W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} \arrow[r, "\Lambda"] \arrow[d, "+_\vartheta"'] & W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[d, "+_\delta"] \\
rW \arrow[r, "g"'] & sW
\end{tikzcd}$$

commutes in $\mathbb{N}$-**Weil$_1$** (and that the corresponding diagram commutes in $\mathrm{End}(\mathcal{M})$). We now have the following diagram

$$\begin{tikzcd}
& T^r \arrow[ddr, bend left, "\widetilde{g}"] & \\
& T^{(\vartheta_1)} \otimes \cdots \otimes T^{(\vartheta_r)} \arrow[u, "+_\vartheta"'] \arrow[d, "\widetilde{\Lambda}"] & \\
F_0A \arrow[uur, bend left, "\widetilde{f}"] \arrow[ur, "\widetilde{f'}"] \arrow[r, "\widetilde{h'}"] \arrow[rr, bend right, "\widetilde{h}"'] & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[r, "+_\delta"] & T^n ,
\end{tikzcd}$$

and to show the exterior commutes, it suffices to show that

$$\begin{tikzcd}
& T^{(\vartheta_1)} \otimes \cdots \otimes T^{(\vartheta_r)} \arrow[d, "\widetilde{\Lambda}"] \\
FA \arrow[ur, "\widetilde{f'}"] \arrow[r, "\widetilde{h'}"'] & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)}
\end{tikzcd}$$

commutes.

Since $T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)}$ is a limit with projections $(a_1, \ldots, a_s)$, and since Λ (and hence $\widetilde{\Lambda}$) was given using Diagram (1), then it suffices to show that

$$\overline{a} \circ \widetilde{\Lambda} \circ \widetilde{f'} = \overline{a} \circ \widetilde{h'}$$

for all projections $\overline{a} = (a_1, \ldots, a_s)$.

Consider the diagram

$$\begin{tikzcd}
F_0A \arrow[rr, "\widetilde{f'}"] \arrow[dr, "\widetilde{t \circ f'}"] \arrow[ddrr, bend right, "\widetilde{\overline{a} \circ h'}"'] \arrow[dd, "\widetilde{h'}"'] & & T^{(\vartheta_1)} \otimes \cdots \otimes T^{(\vartheta_r)} \arrow[dl, "\widetilde{t}"'] \arrow[d, "\widetilde{\Lambda}"] \\
& T^r \arrow[dr, "\widetilde{g}"] & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[d, "\overline{a}"] \\
T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[rr, "\overline{a}"'] & & T^s
\end{tikzcd}$$

(again, we leave as an exercise to the reader to verify that all the necessary pairs in Ψ exist and are well defined).

To show the commutativity of the exterior, we first note that the lower left triangle and right square commute by construction, and further that it is routine to check that the top triangle commutes. So all that remains is to verify the commutativity of the innermost triangle.

But since $t \circ f'$ by definition has no intersecting circles, then we can apply Proposition 12.10 directly.

$\therefore\ \widetilde{h} = \widetilde{g} \circ \widetilde{f}$ ∎

12.12. PROPOSITION. *For all arbitrary $f: A \to B$, and $g: B \to sW$ with no intersecting circles, the diagram*

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[r, "\widetilde{g}"] & T^s \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$.

PROOF. Using Lemma 8.16 and Proposition 6.6, we can see that if the graph Γ_B contains any edges, then B is part of a (foundational) pullback

$$\begin{tikzcd} B = B' \otimes (B_1 \times B_2) \arrow[r, "B'\otimes\pi_1"] \arrow[d, "B'\otimes\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & B' \otimes B_1 \arrow[d, "B'\otimes\varepsilon_{B_1}"] \\ B' \otimes B_2 \arrow[r, "B'\otimes\varepsilon_{B_2}"'] & B' \end{tikzcd}.$$

Further, recall the proof of Lemma 8.13. Since $g: B \to sW$ has no intersecting circles, then if Γ_B has any edges, we know that g must then factorise through one of B's (foundational) projections, say as

$$B \xrightarrow{B'\otimes\pi_1} B' \otimes B_1 \xrightarrow{\gamma} sW$$

(and moreover, this would be in the instructions for its construction). The same is thus true of $\widetilde{g}$. We shall simply denote the projection as π_1 for convenience.

We now have

$$\begin{tikzcd} & & F_0B'F_0B_1 \arrow[d, "\widetilde{\gamma}"] \\ F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[ur, "\widetilde{\pi_1}"] \arrow[r, "\widetilde{g}"] & T^s \end{tikzcd},$$

and note that the map $\widetilde{\pi_1}: F_0B \to F_0B'F_0B_1$ is part of a foundational pullback in $\mathrm{End}(\mathcal{M})$ (Proposition 12.3).

As such, this tells us that $\widetilde{\pi_1} \circ \widetilde{f} = \widetilde{\pi_1 \circ f}$ (Definition 12.4). It thus suffices to show the commutativity of

$$\begin{tikzcd} & F_0B'F_0B_1 \arrow[d, "\widetilde{\gamma}"] \\ F_0A \arrow[ur, "\widetilde{\pi_1\circ f}"] \arrow[r, "\widetilde{h}"'] & sW \end{tikzcd}.$$

But we know that since g has no intersecting circles, then neither does γ, and we can thus repeat this iteratively until there are no more edges in B, i.e. we have $B = rW$ and apply Proposition 12.11 directly. ∎

12.13. Proposition. *For all arbitrary $f\colon A \to B$ and $g\colon B \to sW$, the diagram*

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[r, "\widetilde{g}"] & T^s \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$.

Proof. Recall from the proof of Lemma 8.14 that g factorises as the composite

$$\begin{tikzcd} & W^{\beta_1} \otimes \cdots \otimes W^{\beta_n} \arrow[d, "+_\beta"] \\ B \arrow[ur, "g'"] \arrow[r, "g"'] & sW \end{tikzcd}$$

(and is constructed as such). Thus, $\widetilde{g}$ is constructed as the corresponding composite

$$\begin{tikzcd} & T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)} \arrow[d, "+_\beta"] \\ F_0B \arrow[ur, "\widetilde{g'}"] \arrow[r, "\widetilde{g}"'] & T^n \end{tikzcd}$$

Recall that we also said $\widetilde{h}$ was constructed as the composite

$$\begin{tikzcd} & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[d, "+_\delta"] \\ F_0A \arrow[ur, "\widetilde{h'}"] \arrow[r, "\widetilde{h}"'] & T^s\ . \end{tikzcd}$$

We now have the following diagram

$$\begin{tikzcd} & F_0B \arrow[d, "\widetilde{g'}"] \arrow[ddr, bend left, "\widetilde{g}"] & \\ & T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)} \arrow[dr, "+_\beta"] & \\ F_0A \arrow[uur, bend left, "\widetilde{f}"] \arrow[ur, "\widetilde{g' \circ f}"] \arrow[r, "\widetilde{h'}"] \arrow[rr, bend right, "\widetilde{h}"'] & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[r, "+_\delta"] & T^s\ ; \end{tikzcd} \tag{2}$$

for which we wish to show the commutativity of the exterior.

We already know the bottom triangle as well as top right triangle commute by construction. We begin with the innermost square

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{g' \circ f}"] \arrow[d, "\widetilde{h'}"'] & T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)} \arrow[d, "+_\beta"] \\ T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[ur, dashed, "\widetilde{\Omega}"'] \arrow[r, "+_\delta"'] & FB\ , \end{tikzcd} \tag{3}$$

and introduce the map $\widetilde{\Omega}$ (for Ω defined in 11, again, we shall not explicitly show explicitly the existence of $\widetilde{\Omega}$). Recall that

$$\Omega\colon W^{\delta_1}\otimes\cdots\otimes W^{\delta_s}\to W^{\beta_1}\otimes\cdots\otimes W^{\beta_n}$$

assigned each generator of the domain to a particular generator in the codomain, and moreover we have $\Omega=\Omega_1\otimes\cdots\otimes\Omega_s$ for $\Omega_i\colon W^{\delta_i}\to W^{\beta_i}$.

First, it now becomes rather routine to show that

$$+_\delta=+_\beta\circ\widetilde{\Omega}$$

(i.e. the lower triangle in Diagram (3)). To show $\widetilde{g'\circ f}=\widetilde{\Omega}\circ\widetilde{h'}$ in $\mathrm{End}(\mathcal{M})$, note first that $g'\circ f=\Omega\circ h'$ in $\mathbb{N}$-$\mathbf{Weil_1}$ by design. So equivalently, we can show that

$$\begin{tikzcd}
F_0A \arrow[r, "\widetilde{h'}"] \arrow[rr, bend right, "\widetilde{\Omega\circ h'}"'] & T^{(\delta_1)}\otimes\cdots\otimes T^{(\delta_s)} \arrow[r, "\widetilde{\Omega}"] & T^{(\beta_1)}\otimes\cdots\otimes T^{(\beta_s)}
\end{tikzcd}$$

commutes. But recall that $T^{(\beta_1)}\otimes\cdots\otimes T^{(\beta_n)}$ is a limit (constructed as iterations of foundational pullbacks in $\mathrm{End}(\mathcal{M})$) with projections $\overline{r}=(r_1,\ldots,r_s)$. As such, it suffices to show the commutativity of

$$\begin{tikzcd}
F_0A \arrow[r, "\widetilde{h'}"] \arrow[rr, bend right, "\overline{r}\circ\widetilde{\Omega\circ h'}"'] & T^{(\delta_1)}\otimes\cdots\otimes T^{(\delta_s)} \arrow[r, "\overline{r}\circ\widetilde{\Omega}"] & T^s
\end{tikzcd}$$

for each $\overline{r}$.

But noting that $\Omega=\Omega_1\otimes\cdots\otimes\Omega_s$ and $\overline{r}=\pi_{r_1}\otimes\cdots\otimes\pi_{r_s}$, and noting the form of each $\pi_{r_i}\circ\Omega_i\colon T^{(\alpha_1)}\otimes\cdots\otimes T^{(\alpha_n)}\to T$ from 11, then the commutativity of the upper triangle in Diagram (3) above is immediate.

Hence, all that remains is to show the commutativity of the upper left triangle of Diagram (2), namely the commutativity of

$$\begin{tikzcd}
 & F_0B \arrow[d, "\widetilde{g'}"] \\
F_0A \arrow[ur, "\widetilde{f}"] \arrow[r, "\widetilde{g'\circ f}"'] & T^{(\beta_1)}\otimes\cdots\otimes T^{(\beta_s)}
\end{tikzcd}.$$

Again, since $T^{(\beta_1)}\otimes\cdots\otimes T^{(\beta_n)}$ is a limit, it suffices to show the commutativity of

$$\begin{tikzcd}
 & F_0B \arrow[d, "\overline{r}\circ\widetilde{g'}=\widetilde{\overline{r}\circ g'}"] \\
F_0A \arrow[ur, "\widetilde{f}"] \arrow[r, "\widetilde{\overline{r}\circ g'\circ f}"'] & T^s
\end{tikzcd}$$

for each projection $\overline{r}$.

Finally, note that by definition, each map

$$\overline{r} \circ g' \colon B \to nW$$

has no intersecting circles, so we may apply Proposition 12.12 directly. ∎

12.14. Proposition. *For all arbitrary $f \colon A \to B$ and $g \colon B \to C$, the diagram*

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[r, "\widetilde{g}"] & F_0C \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$.

Proof. using the same argument as in the proof of Proposition 12.12, if the graph Γ_C contains any edges, then C is part of a (foundational) pullback

$$\begin{tikzcd} C = C' \otimes (C_1 \times C_2) \arrow[r, "C' \otimes \pi_1"] \arrow[d, "C' \otimes \pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & C' \otimes C_1 \arrow[d, "C' \otimes \varepsilon_{C_1}"] \\ C' \otimes C_2 \arrow[r, "C' \otimes \varepsilon_{C_2}"'] & C' \end{tikzcd} .$$

Correspondingly, F_0C is part of the pullback

$$\begin{tikzcd} F_0C \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & F_0C'F_0C_1 \arrow[d] \\ F_0C'F_0C_2 \arrow[r] & F_0C' \end{tikzcd} .$$

We now have

$$\begin{tikzcd} & & & F_0C_1 \\ F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[r, "\widetilde{g}"] & F_0C \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & & F_0C_2 \end{tikzcd}$$

in $\mathrm{End}(\mathcal{M})$, and using the fact that $\pi_i \circ \widetilde{g} = \widetilde{\pi_i \circ g}$ for $i = 1, 2$ (and a corresponding fact for h), it suffices to show the commutativity of

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{\pi_i \circ h}"'] & F_0B \arrow[r, "\widetilde{\pi_i \circ g}"] & F_0C_i \end{tikzcd}$$

for each i. Using this argument iteratively, it suffices to assume the graph Γ_C has no edges, i.e. $C = sW$ and apply Proposition 12.13 directly. ∎

We have now shown that the pairings of the collection Ψ "preserve" arbitrary compositions. We now need to consider the collection Φ (Definition 12.5).

13. The Problem with Pullbacks

As we mentioned in Definition 12.4, we may have maps $f, g, h \in \mathbb{N}$-$\mathbf{Weil_1}$ for which the final step of the instructions for h is to uniquely induce it using f and g as

$$\begin{tikzcd}
A \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d] \\
& B_2 \arrow[r] & C
\end{tikzcd},$$

but the exterior of

$$\begin{tikzcd}
F_0A \arrow[drr, bend left, "\widetilde{f}"] \arrow[ddr, bend right, "\widetilde{g}"'] & & \\
& F_0B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_0B_1 \arrow[d] \\
& F_0B_2 \arrow[r] & F_0C
\end{tikzcd}$$

does not commute, and so we cannot construct $\widetilde{h}$, and said that $\widetilde{h}$ is *not well defined*. We then defined Φ to be the set of such "undefined" maps (Definition 12.5). We will now show that this set Φ is in fact empty.

13.1. PROPOSITION. *The collection Φ is empty.*

PROOF. Suppose that Φ is non-empty. Then for each $f \in \Phi$ (with $f\colon A \to B$ a map in $\mathbb{N}$-$\mathbf{Weil_1}$), let $n(f)$ be the number of vertices in the graph Γ_B. Finally, let $N(\Phi) = \{n(f) \mid \forall f \in \Phi\}$.

Since $N(\Phi)$ is a non-empty subset of $\mathbb{N}$, then by the well ordering principle, it has a least element. Choose a map $h\colon A \to B$ corresponding to this least element. Further, suppose that the cograph for this codomain has at least one edge (if Γ_B has no edges, i.e. $B = nW$, then we construct $\widetilde{h}$ directly using the methods in the proof of 8.14).

We then have the diagram

$$\begin{tikzcd}
A \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f=\pi_1\circ h"] \arrow[ddr, bend right, "g=\pi_2\circ h"'] & & \\
& B = (B_1 \times B_2) \otimes C \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \otimes C \arrow[d, "\varepsilon\otimes C"] \\
& B_2 \otimes C \arrow[r, "\varepsilon_2\otimes C"'] & C
\end{tikzcd}$$

and noting that since Γ_B has at least one edge, then $\Gamma_{B_1 \otimes C}$ and $\Gamma_{B_2 \otimes C}$ each have strictly fewer vertices in their respective cographs than Γ_B. Thus, $\widetilde{f}$ and $\widetilde{g}$ are both well defined.

We wish to show the commutativity of

$$\begin{tikzcd} F_0 A \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] & F_0 B_1 \otimes F_0 C \arrow[d] \\ F_0 B_2 \otimes C \arrow[r] & F_0 C \end{tikzcd}$$

so that $\widetilde{h}$ can be induced using the foundational pullback in $\mathrm{End}(\mathcal{M})$.

Let $\psi = (\varepsilon_1 \otimes C) \circ f \colon A \to C$ in $\mathbb{N}$-$\mathbf{Weil_1}$, i.e. the composite

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\psi"'] & B_1 \otimes C \arrow[d, "\varepsilon_1 \otimes C"] \\ & C . \end{tikzcd}$$

Since Γ_C has strictly fewer vertices than Γ_B, then $\widetilde{\psi}$ is also well defined. But by Proposition 12.14, each of the triangles in the diagram

$$\begin{tikzcd} F_0 A \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] \arrow[dr, "\widetilde{\psi}"] & F_0 B_1 \otimes F_0 C \arrow[d] \\ F_0 B_2 \otimes C \arrow[r] & F_0 C \end{tikzcd}$$

commute in $\mathrm{End}(\mathcal{M})$, and thus the exterior commutes.

Therefore $\widetilde{h}$ is well defined. Thus the original assumption is incorrect, i.e. Φ is an empty set. ■

What we have shown then is that F_0 and the pairings of Ψ together define precisely a functor.

14. The Functor F and the universality of $\mathbb{N}$-$\mathbf{Weil_1}$

We now have the following:

14.1. Theorem. *Suppose we have a given category $\mathcal{M}$. Regard* $\mathrm{End}(\mathcal{M})$ *as a monoidal category with respect to composition and* $\mathbb{N}$-$\mathbf{Weil_1}$ *as monoidal with respect to coproduct.*

Then to give a Tangent Structure $\mathbb{T}$ to $\mathcal{M}$ is equivalent (up to isomorphism) to giving a strong monoidal functor $F \colon \mathbb{N}$-$\mathbf{Weil_1} \to \mathrm{End}(\mathcal{M})$ satisfying the following conditions:

1) Given a product $A = A_1 \times A_2$ in $\mathbb{N}$-$\mathbf{Weil_1}$, regarded as a pullback of the augmentations, and an arbitrary Weil algebra $B \in \mathbb{N}$-$\mathbf{Weil_1}$, then F preserves the pullback

$$\begin{tikzcd} B \otimes A \arrow[r, "B\otimes\pi_1"] \arrow[d, "B\otimes\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \otimes A_1 \arrow[d, "B\otimes\varepsilon_1"] \\ B \otimes A_2 \arrow[r, "B\otimes\varepsilon_2"'] & B \end{tikzcd}$$

i.e. it preserves all "foundational pullbacks" of $\mathbb{N}$-$\mathbf{Weil_1}$ (as defined in Definition 3.17*).*

2) The equaliser

$$W^2 \xrightarrow{v} 2W \underset{\eta_W\circ(\varepsilon_W\otimes\varepsilon_W)}{\overset{W\otimes\varepsilon_W}{\rightrightarrows}} W$$

as given in 4 is preserved.

Proof. Given such a functor F, the corresponding Tangent Structure is given as

$$\mathbb{T} = (FW, F\varepsilon_W, F\eta_W, F+, Fl, Fc) \ ,$$

and it can be readily verified that this satisfies all the necessary conditions to be a Tangent Structure.

Conversely, suppose we have a Tangent Structure $\mathbb{T}$. Then $F_0\colon ob(\mathbb{N}\text{-}\mathbf{Weil_1}) \to ob(\mathrm{End}(\mathcal{M}))$ and Ψ give us our assignations for objects and morphisms, and Propositions 12.14 and 13.1 together give functoriality.

Moreover, F_0 actually makes F monoidal (see Definition 12.2). F being strong monoidal as well as the preservation of foundational pullbacks is then a direct consequence of the fact that we are using composition as the monoidal structure of $\mathrm{End}(\mathcal{M})$ together with Proposition 12.3.

Finally, preservation of the equaliser

$$W^2 \xrightarrow{v} 2W \underset{\eta_W\circ(\varepsilon_W\otimes\varepsilon_W)}{\overset{W\otimes\varepsilon_W}{\rightrightarrows}} W$$

is trivial, since it is a condition of $\mathbb{T}$ that the corresponding fork in $\mathrm{End}(\mathcal{M})$ is also an equaliser. ∎

We have thus shown that to equip a category $\mathcal{M}$ with a Tangent Structure $\mathbb{T}$ is equivalent to giving (up to a suitable isomorphism) a strong monoidal functor $F\colon \mathbb{N}\text{-}\mathbf{Weil_1} \to \mathrm{End}(\mathcal{M})$ satisfying some extra properties.

As such, $\mathbb{N}$-$\mathbf{Weil_1}$ becomes an initial Tangent Structure in the sense that it characterises any Tangent Structure $\mathbb{T}$ via this functor F.

We also note that this functor F only required that $\mathrm{End}(\mathcal{M})$ was a monoidal category (with respect to composition and with unit $1_{\mathcal{M}}$) and that certain pullbacks were preserved. As a result, we make the following generalisation.

14.2. Definition. *Let* $(\mathcal{G}, \square, I)$ *be a monoidal category. Regard the category* $\mathbb{N}$-$\mathbf{Weil_1}$ *as monoidal with respect to coproduct and having unit* $\mathbb{N}$. *A* Tangent Structure $\mathbb{G}$ internal to $\mathcal{G}$ *is a strong monoidal functor*

$$F\colon (\mathbb{N}\text{-}\mathbf{Weil_1}, \otimes, \mathbb{N}) \to (\mathcal{G}, \square, I)$$

satisfying the following conditions:

1) F preserves foundational pullbacks

2) The equaliser

$$W^2 \xrightarrow{v} 2W \underset{\eta_W \circ (\varepsilon_W \otimes \varepsilon_W)}{\overset{W \otimes \varepsilon_W}{\rightrightarrows}} W$$

is preserved

14.3. Corollary. *A Tangent Structure on* $\mathcal{M}$ *(in the sense of* Theorem 14.1*) is the same as a Tangent Structure internal to* $\mathrm{End}(\mathcal{M})$ *(in the sense of* Definition 14.2*).*

In fact, Definition 14.2 actually gives a universal property of the category $\mathbb{N}$-$\mathbf{Weil_1}$ in relation to Tangent Structures. One way we might express this is that Tangent Structures are simply models of $\mathbb{N}$-$\mathbf{Weil_1}$ (regarded as a theory).

References

[Bondy and Murty, 1991] A.J. Bondy and U.S.R. Murty. *Graph Theory with Applications*, Wiley, 1991.

[Cockett and Cruttwell, 2014] J.R.B. Cockett and G.S.H. Cruttwell. Differential structure, tangent structure, and SDG. *Appl. Categ. Structures*, 22(2):331-417, 2014.

[Corneil et al., 1981] D.G. Corneil, H. Lerchs and L. Stewart Burlingham. Complement reducible graphs. *Discrete Appl. Math.*, 3(3):163-174, 1981.

[Kolar et al., 2010] I. Kolar, P.W. Michor and J. Slovak. *Natural Operations in Differential Geometry*, Springer Berlin Heidelberg, 2010.

[Kock, 2006] A. Kock. *Synthetic Differential Geometry*, Cambridge University Press (2nd ed.), 2006.

[Rosický, 1984] J. Rosický. Abstract tangent functors. *Diagrammes*, 12, Exp. No. 3, 1984.

[Weil, 1953] A. Weil. Théorie des points proches sur les variétés differentielles. *Colloque de topologie et géométrie différentielle*, pages 111-117, Strasbourg, 1953

Department of Mathematics,
Macquarie University,
Sydney, NSW 2109

Email: `poon.leung@mq.edu.au`

This article may be accessed at `http://www.tac.mta.ca/tac/`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.
Full text of the journal is freely available from the journal's server at `http://www.tac.mta.ca/tac/`. It is archived electronically and in printed paper format.

Subscription information Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

Information for authors The typesetting language of the journal is TeX, and LaTeX2e is required. Articles in PDF format may be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at `http://www.tac.mta.ca/tac/`.