

## KZ-MONADIC CATEGORIES AND THEIR LOGIC

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ABSTRACT. Given an order-enriched category, it is known that all its KZ-monadic subcategories can be described by Kan-injectivity with respect to a collection of morphisms. We prove the analogous result for Kan-injectivity with respect to a collection  $\mathcal{H}$  of commutative squares. A square is called a Kan-injective consequence of  $\mathcal{H}$  if by adding it to  $\mathcal{H}$  Kan-injectivity is not changed.

We present a sound logic for Kan-injectivity consequences and prove that in “reasonable” categories (such as  $\mathbf{Pos}$  or  $\mathbf{Top}_0$ ) it is also complete for every set  $\mathcal{H}$  of squares.

### 1. Introduction

Scott characterized continuous lattices in [15], 3.8 and following, as those  $T_0$  spaces that are, in modern terminology, Kan-injective with respect to all embeddings. The Kan-injectivity of an object  $X$  of an order-enriched category with respect to a morphism  $h : A \rightarrow B$  was defined by Escardó [8] as follows: for every morphism  $f : A \rightarrow X$  we have a commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ f \downarrow & \swarrow f/h & \\ X & & \end{array}$$

where  $f/h$  is the left Kan-extension of  $f$  along  $h$ . That is, if  $g : B \rightarrow X$  fulfils  $f \leq gh$ , then  $f/h \leq g$ .

Later Carvalho and Sousa [6] extended the above concept from objects to morphisms: a morphism  $u : X \rightarrow X'$  is Kan-injective with respect to  $h$  when  $X$  and  $X'$  are Kan-injective objects and for every morphism  $f : A \rightarrow X$  the following triangle

$$\begin{array}{ccc} B & \xrightarrow{f/h} & X \\ (uf)/h \downarrow & \swarrow u & \\ X' & & \end{array}$$

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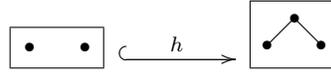
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commutes. Example: the morphisms of  $\mathbf{Top}_0$  Kan-injective with respect to all embeddings are precisely the continuous maps between continuous lattices preserving all infima. A trivial example: in  $\mathbf{Pos}$  Kan-injectivity with respect to



defines the subcategory of join-semilattices and their homomorphisms.

Moreover, in locally ranked categories, e.g., in  $\mathbf{Top}_0$  or  $\mathbf{Pos}$ , Kan-injectivity yields a characterization of KZ-monadic categories in the following sense: given a class  $\mathcal{H}$  of morphisms, let  $\mathbf{LInj}(\mathcal{H})$  be the subcategory of all objects and all morphisms Kan-injective with respect to all members of  $\mathcal{H}$ . Then

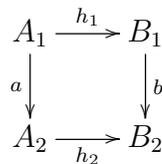
- (1) every KZ-monadic subcategory has the form  $\mathbf{LInj}(\mathcal{H})$  for some class  $\mathcal{H}$  of morphisms, see [6],

and

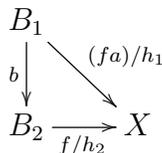
- (2) for every set  $\mathcal{H}$  of morphisms the subcategory  $\mathbf{LInj}(\mathcal{H})$  is KZ-monadic, see [4].

The topic of our paper is the logic of Kan-injectivity, generalizing the logic of orthogonality studied in [1] and [2]. Observe first that given an ordinary category with its trivial order-enrichment (that is, equality), then  $\mathbf{LInj}(\mathcal{H})$  is nothing else than the full subcategory  $\mathcal{H}^\perp$  on objects  $X$  orthogonal to every member  $h : A \rightarrow B$  of  $\mathcal{H}$  (that is, each  $f : A \rightarrow X$  has a unique factorization through  $h$ ). The logic of orthogonality aims to characterize those morphisms  $h$  for which orthogonality to  $\mathcal{H}$  implies that to  $h$ , i.e. with  $\mathcal{H}^\perp = (\mathcal{H} \cup \{h\})^\perp$ . See Section 4 where the simple logic presented in [1] is recalled.

Analogously, we hoped to present a logic that would characterize, in order-enriched categories, those morphisms for which Kan-injectivity with respect to  $\mathcal{H}$  implies that with respect to  $h$ . But we have failed so far. What saved our effort was the idea to "enrich" our language by considering, instead of Kan-injectivity with respect to morphisms, Kan-injectivity with respect to commutative squares  $S$ :



An object  $X$  is *Kan-injective with respect to  $S$*  if it is Kan-injective with respect to  $h_1$  and  $h_2$  and for every morphism  $f : A_2 \rightarrow X$  the following triangle



commutes. And a morphism is Kan-injective with respect to  $S$  iff it is Kan-injective with respect to  $h_1$  and  $h_2$ .

For every class  $\mathcal{H}$  of commutative squares we thus obtain a (non-full) subcategory  $\text{LInj}(\mathcal{H})$  analogously to above. These categories characterize again KZ-monadicity: we prove in Section 3 below that the above statements (1) and (2) remain valid. In other words, this richer language does *not* lead to more examples! However, it enables a formulation of a sound logic (see Section 4) which for sets of squares is, under mild size conditions, also complete (see Section 5). It is our present impression that this enrichment of the structure from morphisms to commutative squares is probably necessary: we suspect that no logic for Kan-injectivity with respect to just morphisms is sound and complete.

## 2. KZ-monadic subcategories

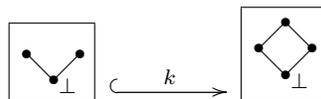
2.1. ASSUMPTION. *Throughout the paper  $\mathcal{X}$  is a category enriched over  $\mathbf{Pos}$ . All squares in our paper are commutative, so when stating that something is a square, we mean a commutative one.*

We introduced Kan-injectivity with respect to a morphism and a square above. Observe that the latter is a generalization of the former: for every morphism  $h : A \rightarrow B$  let  $S(h)$  be the following square

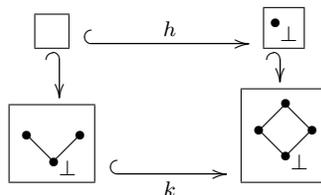
$$S(h) = \begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

Then Kan-injectivity with respect to  $h$  and  $S(h)$  is the same concept for objects and morphisms.

2.2. EXAMPLE. In  $\mathbf{Pos}$  consider Kan-injectivity with respect to  $h : \mathbf{0} \hookrightarrow \mathbf{1}$ , the empty map into the terminal poset. This means the existence (and preservation) of the least element,  $\perp$ . Kan-injectivity with respect to the embedding



characterizes existence (and preservation) of joins of pairs having a lower bound. Combining those two in a square as follows



yields by Kan-injectivity join-semilattices with  $\perp$  and their homomorphisms.

We have already remarked that using squares does not lead to new examples. Indeed, join-semilattices with  $\perp$  are also given by Kan-injectivity with respect to  $h$  and the following embedding



**2.3. EXAMPLE.** Which posets are Kan-injective with respect to all order-embeddings (i.e., regular monomorphisms) in  $\mathbf{Pos}$ ?

As shown in [4], these are precisely the complete lattices, and a monotone map is Kan-injective with respect to order-embeddings iff it preserves joins. We now characterize all squares  $S$  that are Kan-injectivity consequences of order-embeddings. That is, such that every complete lattice and every join-preserving map between complete lattices is Kan-injective with respect to  $S$ .

Let us denote by  $\Omega_0$  the contravariant endofunctor of  $\mathbf{Pos}$  assigning to every posed  $X$  the posed  $\Omega_0 X$  of all  $\downarrow$ -sets of  $X$ , and to every monotone function  $f : X \rightarrow Y$  the function  $\Omega_0 f : \Omega_0 Y \rightarrow \Omega_0 X$  forming preimages. Observe that  $\Omega_0 f$  has a left adjoint

$$(\Omega_0 f)^* : \Omega_0 X \rightarrow \Omega_0 Y, \quad U \mapsto \downarrow f[U] \quad (\text{for all } U \in \Omega_0 X).$$

**2.4. PROPOSITION.** *For every square*

$$S = \begin{array}{ccc} A' & \xrightarrow{h} & B' \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{k} & B \end{array}$$

in  $\mathbf{Pos}$  the following conditions are equivalent:

- (1)  $S$  is a Kan-injectivity consequence of order-embeddings;
- (2)  $h$  and  $k$  are order-embeddings such that whenever  $k(z) \leq b(y)$ , there exists  $x \in A'$  with  $h(x) \leq y$  and  $z \leq a(x)$ ;

and

- (3)  $h$  and  $k$  are order-embeddings yielding a (commutative) square

$$\begin{array}{ccc} \Omega_0 B' & \xrightarrow{\Omega_0 h} & \Omega_0 A' \\ (\Omega_0 b)^* \downarrow & & \downarrow (\Omega_0 a)^* \\ \Omega_0 B & \xrightarrow{\Omega_0 k} & \Omega_0 A \end{array}$$

PROOF. The equivalence of (1) and (2) was proved in [16].

3 ⇒ 2 The inequality  $k(z) \leq b(y)$  states precisely that

$$z \in \Omega_0 k \cdot (\Omega_0 b)^*(\downarrow y) = (\Omega_0 a)^* \cdot \Omega_0 h(\downarrow y),$$

that is,  $z$  lies in  $\downarrow a [\Omega_0 h(\downarrow y)]$ . This means that an  $x$  as in (2) exists.

2 ⇒ 3 It is easy to verify that the inequality

$$(\Omega_0 a)^* \cdot \Omega_0 h \leq (\Omega_0 k) \cdot (\Omega_0 b)^*$$

holds for every square  $S$ . Thus we need to show the opposite inequality only. Let  $V$  be a  $\downarrow$ -set of  $B'$  and  $z$  be an element of  $(\Omega_0 k)(\Omega_0 b)^*(V)$ , i.e.,  $z \in k^{-1}[\downarrow b[V]]$ . Then for some  $y \in V$  we have  $k(z) \leq b(y)$ . Given  $x$  as in (2) we conclude  $z \in (\Omega_0 a)^* \Omega_0 h(V)$ , establishing the required inequality. ■

2.5. EXAMPLE. The category  $\mathbf{Top}_0$  of topological  $T_0$ -spaces is order enriched via the opposite of the specialization order. Recall that the specialization order is given by  $x \sqsubseteq y$  iff  $x \in \overline{\{y\}}$ . Thus, for continuous functions  $f, g : X \rightarrow Y$  we define

$$f \leq g \quad \text{iff} \quad g(x) \sqsubseteq f(x) \quad \text{for all } x \in X.$$

Escardó and Flagg proved that Kan-injective objects with respect to topological embeddings (= regular monomorphisms) are precisely Scott’s continuous lattices, see [9]. Recall that a  $T_0$ -space is a continuous lattice iff for the specialization order its topology is the Scott’s one, and it is a complete lattice with

$$y = \bigsqcup_{U \in \text{nbh}(y)} (\cap U) \quad \text{for all } y \in X.$$

The morphisms Kan-injective with respect to topological embeddings are precisely the continuous functions preserving meets, see [6]. We characterize the squares which are injectivity consequences of topological embeddings. Let  $\Omega : \mathbf{Top}_0 \rightarrow \mathbf{Pos}^{\text{op}}$  be the functor assigning to a space  $X$  the poset  $\Omega X$  of open sets and to a continuous function  $f : X \rightarrow Y$  the preimage function  $\Omega f : \Omega Y \rightarrow \Omega X$ . By  $(\Omega f)_*$  we denote the right adjoint of  $\Omega f$ : to an open set  $U$  it assigns the union of all open  $V \subseteq Y$  with  $f^{-1}(V) \subseteq U$ .

2.6. PROPOSITION. A square

$$S = \begin{array}{ccc} A' & \xrightarrow{h} & B' \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{k} & B \end{array}$$

in  $\mathbf{Top}_0$  is a Kan-injectivity consequence of topological embeddings iff  $h$  and  $k$  are topological embeddings yielding the following (commutative) square

$$\begin{array}{ccc} \Omega A & \xrightarrow{(\Omega k)_*} & \Omega B \\ \Omega a \downarrow & & \downarrow \Omega b \\ \Omega A' & \xrightarrow{(\Omega h)_*} & \Omega B' \end{array}$$

PROOF. (a) Sufficiency. For every continuous lattice  $X$  and every embedding  $f : A \rightarrow X$  the Kan-extension  $f/k$  is, as proved in [9], given by

$$f/k(z) = \sqcup\{\sqcap U; U \in \Omega X \text{ and } z \in (\Omega k)_*[\Omega f(U)]\}, \tag{1}$$

and analogously for  $(fa)/h$ . We are to prove

$$(f/k) \cdot b = (fa)/h.$$

Indeed, given  $b \in B'$ , then  $b(y) \in (\Omega k)_*[\Omega f(U)]$  means that  $y \in (\Omega b)(\Omega k)_*[\Omega f(U)]$ , and, by the commutativity of the last square, this means that  $y \in (\Omega h)_*[\Omega(fa)(U)]$ . Consequently, taken into account the definitions of  $(f/k)(b(y))$  and  $((fa)/h)(y)$  given by (1), we conclude the desired equality.

(b) Necessity. Since the Sierpiński space  $\mathbf{S}$  is Kan-injective with respect to  $S$ ,  $k$  and  $h$  are topological embeddings (see 4.3 and 4.4 of [6]). We verify the above square.

Every  $V \in \Omega A$  defines the corresponding characteristic function  $f_V : A \rightarrow \mathbf{S}$  and, using the formula (1) above, we get (by setting  $V = \{1\}$ )

$$(f_V/k)(z) = 1 \text{ iff } z \in (\Omega k)_*(V).$$

Since  $f_V \cdot a = f_{\Omega a(V)}$ , the characteristic function of  $a^{-1}(V)$ , we analogously get

$$(f_V \cdot a)/h(x) = 1 \text{ iff } x \in (\Omega h)_*(\Omega a(V)).$$

Thus, our formula  $(f_V \cdot a)/h = (f_V/k) \cdot b$  reads:

$$x \in (\Omega h)_*(\Omega a(V)) \text{ iff } b(x) \in (\Omega k)_*(V)$$

for all  $x \in B'$ . That is,

$$(\Omega h)_*(\Omega a(V)) = \Omega b((\Omega k)_*(V)),$$

as desired. ■

2.7. REMARK. A slight modification: Scott continuous domains are, as proved by Escardó [8], precisely the  $T_0$ -spaces Kan-injective with respect to dense embeddings, see [10], Exercise 3.19. And the morphisms Kan-injective with respect to dense embeddings are the continuous functions preserving nonempty meets (see [6]). The above corollary holds analogously, just  $h$  and  $k$  are required to be dense embeddings.

2.8. EXAMPLE. Let  $\mathbf{Loc}$  be the category of locales and localic maps. Thus, the objects of  $\mathbf{Loc}$  are complete lattices with the infinite distributive law

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\},$$

and the morphisms are the monotone maps  $f$  which preserve all infima and whose left adjoint  $f^*$  preserves finite meets. We recall that a localic map  $h$  is an embedding provided that  $h^*h = \text{id}$ . Johnstone characterized the stably locally compact locales as the locales injective with respect to flat embeddings, i.e. those preserving finite joins, see [11]. Moreover, with convenient morphisms, stably locally compact locales are precisely  $\mathbf{Llnj}(\mathcal{H})$  for  $\mathcal{H} = \text{flat embeddings}$  (see [11] and [7]).

2.9. PROPOSITION. *A square*

$$\begin{array}{ccc} A' & \xrightarrow{h} & B' \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{k} & B \end{array}$$

in  $\mathbf{Loc}$  is a Kan-injectivity consequence of flat embeddings iff  $h$  and  $k$  are flat embeddings yielding the following (commutative) square

$$\begin{array}{ccc} A & \xrightarrow{k} & B \\ a^* \downarrow & & \downarrow b^* \\ A' & \xrightarrow{h} & B' \end{array}$$

PROOF. Let  $F_0$ ,  $F_1$  and  $F_2$  be the free frames generated by the empty set,  $1 = \{0\}$  and  $2 = \{0, 1\}$ , respectively, and let  $f_i : F_i \rightarrow F_1$ ,  $i = 0, 2$ , be the localic maps determined by  $f_0(\perp) = 0$ ,  $f_2(0 \vee 1) = 0$  and  $f_2(x) = \perp$  for  $x \neq \perp, 0 \vee 1$ . In [7] flat embeddings were characterized as precisely those morphisms with respect to which both  $f_0$  and  $f_2$  are Kan-injective. Furthermore, it was shown there that for every finitely generated free frame  $F$ , in particular for every  $F_i$  ( $i = 0, 1, 2$ ), given a flat embedding  $h : A \rightarrow B$  and a morphism  $f : A \rightarrow F$ , the map  $(hf^*)_*$  is localic and

$$f/h = (hf^*)_* \tag{2}$$

(a) Necessity: suppose the given square is a Kan injectivity consequence of flat embeddings. Since we already know that flat embeddings are characterized by means of  $f_0$  and  $f_2$ , we only need to prove  $ha^* = b^*k$ .

Given  $x \in A$ , we want to show that  $b^*k(x) = ha^*(x)$ . Let  $g : F_1 \rightarrow A$  be the frame homomorphism sending  $0$  to  $x$ . By hypothesis, the localic map  $g_* : A \rightarrow F_1$  satisfies  $(g_*/k) \cdot b = (g_*a)/h$ , that is, by (2),  $(kg)_*b = h(g_*a)^* = ha^*g$ . Consequently, by applying the operator  $-^*$  to the localic maps  $(kg)_*$ ,  $b$  and  $ha^*g$ , we obtain  $b^*kg(0) = ha^*g(0)$ , i.e.,  $b^*k(x) = ha^*(x)$ .

(b) Sufficiency: if the lower square commutes, we prove that, given  $f : A \rightarrow F_i$ ,  $(fa)/h = (f/k)b$ . Indeed, from (2), deduce that

$$(fa)/h = [(h(fa)^*)]_* = (ha^*f^*)_* = (b^*kf^*)_* = (kf^*)_*b = (f/k) \cdot b.$$

■

2.10. REMARK. (a) Given an order-enriched category  $\mathfrak{X}$ , by a *right adjoint retraction* we mean a morphism  $r : X \rightarrow Y$  with  $r^* : Y \rightarrow X$  satisfying  $rr^* = \text{id}$  and  $r^*r \geq \text{id}$ .

(b) Recall that a *KZ-monad* over  $\mathfrak{X}$  is a monad  $\mathbf{T}$  for which

$$T\eta \leq \eta T.$$

Then  $\mathbf{T}$ -algebras are precisely the right adjoint retractions of  $\eta_A : A \rightarrow TA$ . Thus,  $\mathcal{X}^{\mathbf{T}}$  is a (non-full) subcategory of  $\mathcal{X}$ .

(c) In [4] we proved that KZ-monadic subcategories are precisely the subcategories  $\mathcal{K}$  which are

- (1) reflective, i.e., the embedding has a left adjoint,
- (2) *inserter-ideal*, i.e., contain the inserter  $\text{ins}(u, v)$  of every parallel pair  $(u, v)$  with  $u$  in  $\text{Mor}(\mathcal{K})$ ,

and

- (3) closed under right adjoint retractions.

That is, with every morphism  $p : X \rightarrow Y$  the subcategory contains all morphisms  $\bar{p} : \bar{X} \rightarrow \bar{Y}$  for which there exists a square with right adjoint retractions  $x$  and  $y$  as follows

$$\begin{array}{ccc}
 X & \xrightarrow{p} & Y \\
 x \downarrow & & \downarrow y \\
 \bar{X} & \xrightarrow{\bar{p}} & \bar{Y}
 \end{array} \tag{3}$$

(d) Every KZ-monadic subcategory  $\mathcal{K}$  has the form  $\mathcal{K} = \text{LInj}(\mathcal{H})$  for some class  $\mathcal{H}$  of morphisms of  $\mathcal{X}$  ([6]). And we proved that, conversely, for “reasonable” categories  $\mathcal{X}$  (such as  $\text{Top}_0$  and  $\text{Pos}$ ), every set  $\mathcal{H}$  of morphisms defines a KZ-monadic subcategory  $\text{LInj}(\mathcal{H})$ . (The latter is not true for proper classes  $\mathcal{H}$  in  $\text{Top}_0$  as demonstrated by an example in [4]).

2.11. LEMMA. *For every class  $\mathcal{H}$  of squares the category  $\text{LInj}(\mathcal{H})$  is inserter-ideal and closed under right adjoint retractions.*

PROOF. A. Given a pair of morphisms  $u, v : X \rightarrow Y$  and a square

$$\begin{array}{ccc}
 A' & \xrightarrow{h'} & B' \\
 a \downarrow & & \downarrow b \\
 A & \xrightarrow{h} & B
 \end{array} \tag{4}$$

with respect to which  $u$  is Kan-injective, we prove that so is the inserter  $i = \text{ins}(u, v) : I \rightarrow X$ . We have already proved in [4] that the fact that  $u$  is Kan-injective with respect to  $h$  and  $h'$  implies that so is the morphism  $i$ . It remains to show that, for every  $f : A \rightarrow I$ ,  $(f/h)b = (fa)/h'$ , or equivalently, since  $i$  is mono,  $i(f/h)b = i(fa)/h'$ . But the last equality follows immediately from the fact that  $i$  is Kan-injective with respect to both  $h$  and  $h'$  and  $X$  is Kan-injective w.r.t. the above square.

B. Given a square (3) with the left adjoints  $x^*$  and  $y^*$ , respectively, and such that the morphism  $p : X \rightarrow Y$  is Kan-injective with respect to the square (4), we prove that also  $\bar{p}$  is Kan-injective with respect to this square.

- (a)  $\bar{X}$  is Kan-injective with respect to the above square, and for every morphism  $f : A \rightarrow \bar{X}$  we have:  $f/h = x([x^*f]/h)$ .

$$\begin{array}{ccc}
 A' & \xrightarrow{h'} & B' \\
 a \downarrow & & \downarrow b \\
 A & \xrightarrow{h} & B \\
 f \downarrow & & \downarrow (x^*f)/h \\
 \bar{X} & \xrightleftharpoons[x^*]{x} & X
 \end{array}$$

This last formula was proved in [6]. Analogously one proves  $(fa)/h' = x([x^*fa]/h')$ . Consequently, since the Kan-injectivity of  $X$  yields

$$[x^*fa]/h' = ([x^*f]/h)b$$

we conclude  $(fa)/h' = (f/h)b$ .

- (b)  $\bar{Y}$  is Kan-injective. This is completely analogous.  
 (c)  $\bar{p}$  is Kan-injective with respect to  $h$  and  $h'$ . This was also proved in [6].

■

### 3. The reflection chain

Throughout the rest of the paper  $\mathcal{X}$  denotes an order-enriched category with weighted colimits. Given a set  $\mathcal{H}$  of squares, we associate with every object  $X$  a transfinite chain starting in  $X$ . For “reasonable” categories we then prove that there exists a connecting morphism from  $X$  in our chain which is the reflection of  $X$  in the subcategory  $\mathbf{Llnj}(\mathcal{H})$ .

**3.1. DEFINITION.** *Let  $\mathcal{H}$  be a set of squares. For every object  $X$  define a transfinite chain  $X_i$  ( $i \in \mathbf{Ord}$ ) with connecting morphisms called  $x_{ij}$  or simply  $X_i \dashrightarrow X_j$  (for  $i \leq j$ ). We proceed by transfinite recursion. In the isolated steps, given  $i$  we define  $X_i \dashrightarrow X_{i+1} \dashrightarrow X_{i+2}$ , therefore, we can assume that  $i$  is an even ordinal (that is,  $i = 2n$  or  $i = i_0 + 2n$  for  $n < \omega$  and  $i_0$  a limit ordinal).*

- (a) *Initial step.*  $X_0 = X$ .
- (b) *Limit steps.* If  $i$  is a limit ordinal then

$$X_i = \operatorname{colim}_{j < i} X_j$$

*is the colimit of the previous chain with  $X_j \dashrightarrow X_i$  forming the colimit cocone.*

(c) Isolated step  $i \mapsto i + 1$  ( $i$  even). Consider all pairs  $(S, f)$  where

$$S = \begin{array}{ccc} A_1 & \xrightarrow{h_1} & B_1 \\ a \downarrow & & \downarrow b \\ A_2 & \xrightarrow{h_2} & B_2 \end{array} \quad (5)$$

is a member of  $\mathcal{H}$  and  $f : A_r \longrightarrow X_i$  a morphism with  $r = 1$  or  $2$ . Form the pushout of  $h_r$  along  $f$ :

$$\begin{array}{ccc} A_r & \xrightarrow{h_r} & B_r \\ f \downarrow & & \downarrow \bar{f} \\ X_i & \xrightarrow{\bar{h}_r} & Q_f \end{array} \quad (6)$$

Define  $X_i \dashrightarrow X_{i+1}$  as the wide pushout of all the morphisms  $\bar{h}_r$ :

$$\begin{array}{ccc} X_i & \xrightarrow{\bar{h}_r} & Q_f \\ \downarrow & \swarrow q_f & \\ X_{i+1} & & \end{array} \quad (7)$$

ranging over all the above pairs  $(S, f)$ .

Notation: put

$$f // h_r : B_r \xrightarrow{\bar{f}} Q_f \xrightarrow{q_f} X_{i+1} \quad (8)$$

This “approximates” the desired Kan extension  $f/h_r$  in the following sense: we get a (commutative) square

$$\begin{array}{ccc} A_r & \xrightarrow{h_r} & B_r \\ f \downarrow & & \downarrow f // h_r \\ X_i & \dashrightarrow & X_{i+1} \end{array} \quad (9)$$

(d) Isolated step  $i + 1 \mapsto i + 2$  ( $i$  even). Consider all triples  $(S, f, g)$  consisting of  $S \in \mathcal{H}$ , as above, and two morphisms forming an inequality as follows:

$$\begin{array}{ccc} A_r & \xrightarrow{h_r} & B_r \\ f \downarrow & \leq & \downarrow g \\ X_i & \dashrightarrow & X_{i+1} \end{array} \quad (r = 1 \text{ or } 2) \quad (10)$$

For every decomposition of  $f$  as follows

$$f \equiv A_r \xrightarrow{f'} X_j \dashrightarrow X_i \quad (j \leq i \text{ even}) \quad (11)$$

form the coinserter of  $A_r \xrightarrow{f'//h_r} X_{j+1} \dashrightarrow X_{i+1}$  and  $g$ :

$$c_{f,g} = \text{coins}(x_{j+1,i+1}[f'//h_r], g) : X_{i+1} \longrightarrow C_{f,g}. \tag{12}$$

And in case  $r = 1$  for every decomposition of  $f$  as follows

$$f \equiv A_1 \xrightarrow{a} A_2 \xrightarrow{f'} X_i \tag{13}$$

form the coinserter of  $B_1 \xrightarrow{b} B_2 \xrightarrow{f'//h_2} X_{i+1}$  and  $g$ :

$$c_{f,g} = \text{coins}([f'//h_2].b, g). \tag{14}$$

Define  $X_{i+1} \dashrightarrow X_{i+2}$  as the wide pushout of all  $c_{f,g}$  above:

$$\begin{array}{ccc} X_{i+1} & \xrightarrow{c_{f,g}} & C_{f,g} \\ \downarrow & \searrow t_{f,g} & \\ X_{i+2} & & \end{array} \tag{15}$$

**3.2. LEMMA.** *Given a morphism  $p_0 : X_0 \longrightarrow P$  where  $P$  is Kan-injective with respect to  $\mathcal{H}$ , there exists a unique cocone  $p_i : X_i \longrightarrow P$  ( $i \in \text{Ord}$ ) of the reflection chain such that for all pairs  $(S, f)$  in step  $i \mapsto i + 1$  the following triangle*

$$\begin{array}{ccc} A_r & & \\ f//h_r \downarrow & \searrow (p_i f)/h_r & \\ X_{i+1} & \xrightarrow{p_{i+1}} & P \end{array} \tag{16}$$

*commutes.*

**PROOF.** We only need to prove the isolated step of the transfinite induction. Given  $p_i, i$  even, we obtain  $p_{i+1}$  as follows. From the following square

$$\begin{array}{ccc} A_r & \xrightarrow{h_r} & B_r \\ f \downarrow & & \downarrow (p_i f)/h_r \\ X_i & \xrightarrow{p_i} & P \end{array}$$

we conclude that the pushout (6) yields a unique factorization morphism  $p_f : Q_f \longrightarrow P$ . These morphisms  $p_f$  form a cocone of the wide pushout (7). Define  $p_{i+1}$  as the unique factorization morphism

$$p_{i+1} \cdot q_f = p_f.$$

It fulfils  $p_{i+1} \cdot (f//h_r) = p_{i+1} \cdot q_f \cdot \bar{f} = p_f \cdot \bar{f} = (p_i f)//h_r$ . Conversely, whenever the above triangle commutes, then  $p_{i+1}$  is a factorization map of the wide pushout (7).

Next we define  $p_{i+2}$ : since  $X_{i+1} \dashrightarrow X_{i+2}$  is a wide pushout of epimorphisms,  $p_{i+2}$  is unique. And for the proof of existence we only need to verify that  $p_{i+1}$  factorizes through each  $c_{f,g}$ . That is:

(a) In case of (11) we want to verify the inequality

$$\begin{array}{ccc}
 B_r & \xrightarrow{g} & X_{i+1} \\
 f' // h_r \downarrow & & \downarrow p_{i+1} \\
 X_{j+1} & \leq & \\
 \vdots \downarrow & & \\
 X_{i+1} & \xrightarrow{p_{i+1}} & P
 \end{array}$$

The lower passage is, due to  $p_{j+1} = p_{i+1} \cdot x_{j+1,i+1}$  and (16), equal to  $(p_j \cdot f')/h_r$ . By composing (10) with  $p_{i+1}$  we get

$$p_j f' \leq p_{i+1} g h_r$$

hence,

$$(p_j f')/h_r \leq p_{i+1} g$$

(b) In case of (13) we want to verify the inequality

$$\begin{array}{ccc}
 B_1 & \xrightarrow{g} & X_{i+1} \\
 b \downarrow & & \downarrow p_{i+1} \\
 B & \leq & \\
 f' // h_2 \downarrow & & \\
 X_{i+1} & \xrightarrow{p_{i+1}} & P
 \end{array}$$

Indeed, the lower passage is, due to (16) and injectivity of  $P$  with respect to  $S$ , equal to

$$[(p_i f')/h_2]b = (p_i f' a)/h_1 = (p_i f)/h_1.$$

By composing (10) with  $p_{i+1}$  we get  $p_i f \leq p_{i+1} \cdot g \cdot h_1$ . This proves  $(p_i f)/h_1 \leq p_{i+1} \cdot g$  as desired. ■

Recall that in a category with a factorization system  $(\mathcal{E}, \mathcal{M})$  an object is said to have *rank*  $\lambda$  (a regular cardinal) if its hom-functor preserves  $\lambda$ -directed colimits of morphisms in  $\mathcal{M}$ . We use the following concept introduced in [4]:

**3.3. DEFINITION.** *Let  $\mathcal{X}$  be an order-enriched category with a factorization system  $(\mathcal{E}, \mathcal{M})$  such that  $\mathcal{E} \subseteq \text{Epi}$  and  $\mathcal{M} \subseteq \text{Order-Mono}$  (i.e., given  $m \in \mathcal{M}$  then  $mu \leq mv$  implies  $u \leq v$ ). We call  $\mathcal{X}$  locally ranked if*

(i) *it has weighted colimits,*

(ii) it is  $\mathcal{E}$ -cowellpowered,

and

(iii) every object has a rank.

3.4. EXAMPLE.  $\mathbf{Pos}$  is locally ranked w.r.t.  $\mathcal{E} = \mathbf{Epi}$  and  $\mathcal{M} = \mathbf{Order-Embedding}$ .  $\mathbf{Top}_0$  is locally ranked w.r.t.  $\mathcal{E} = \mathbf{surjective\ morphisms}$  and  $\mathcal{M} = \mathbf{subspace\ embeddings}$ .

3.5. REMARK. Let  $\mathcal{X}$  be locally ranked and let  $(X_i)$  be a transfinite chain with connecting morphisms  $X_i \dashrightarrow X_j$  ( $i \leq j$ ). By Proposition 4.1 of [12] there exists a chain  $(Y_i)$  of monomorphisms in  $\mathcal{M}$ , a join-preserving function  $\varphi : \mathbf{Ord} \rightarrow \mathbf{Ord}$  and natural transformations

$$X_i \xrightarrow{\gamma_i} Y_i \xrightarrow{\beta_i} X_{\varphi(i+1)} \quad (i \in \mathbf{Ord})$$

such that for all  $i$ :

- (1)  $\beta_i \gamma_i$  is the connecting morphism  $X_i \dashrightarrow X_{\varphi(i+1)}$ ;
- (2) the composite  $Y_i \xrightarrow{\beta_i} X_{\varphi(i+1)} \dashrightarrow X_j$  lies in  $\mathcal{M}$  for all  $j \geq \varphi(i+1)$ ;

and

- (3) if  $i$  is a limit ordinal, then a colimit of the chain  $(Y_j)_{j < i}$  is given by the following cocone

$$Y_j \xrightarrow{\beta_j} X_{\varphi(j+1)} \dashrightarrow X_{\varphi(i)}$$

Moreover, given such a function  $\varphi$ , every join-preserving function  $\varphi' \geq \varphi$  works too. Consequently, we can clearly choose  $\varphi$  so that  $\varphi(i)$  is even for all ordinals  $i$ .

3.6. THEOREM. Let  $\mathcal{X}$  be a locally ranked order-enriched category. For every set  $\mathcal{H}$  of squares the subcategory  $\mathbf{Llnj}(\mathcal{H})$  is reflective: the reflection of every object  $X$  is given by  $X \dashrightarrow X_k$  (in Definition 3.1) where  $k$  is a suitable cardinal.

PROOF. Apply the above remark to the chain of Definition 3.1, using notation  $\hat{i} = \varphi(i+1)$ . Since  $\mathcal{H}$  is a set, there exists a cardinal  $\lambda$  such that for every square in  $\mathcal{H}$  all the four objects involved have rank  $\lambda$ . The cardinal  $k$  of our theorem is chosen to be

$$k = \varphi(\lambda).$$

(1) We first prove that the object  $X_k$  is Kan-injective with respect to every square in  $\mathcal{H}$ :

$$S = \begin{array}{ccc} A_1 & \xrightarrow{h_1} & A_2 \\ a \downarrow & & \downarrow b \\ B_1 & \xrightarrow{h_2} & B_2 \end{array}$$

(a) Kan extensions modulo  $h_1$  and  $h_2$  exist. Indeed, every morphism  $f_0 : A_r \rightarrow X_k$  ( $r = 1, 2$ ) has, since  $A_r$  has rank  $\lambda$ , a factorization  $f$  through some colimit injection of the colimit  $X_k = \text{colim}_{j < \lambda} Y_j$  of Remark 3.5(3):

$$\begin{array}{ccccc}
 & & & & A_r \\
 & & & & \downarrow f_0 \\
 & & f & & \\
 & & \swarrow & & \\
 Y_j & \xrightarrow{\beta_j} & X_j & \dashrightarrow & X_k \\
 & & \nwarrow f' & & 
 \end{array}$$

Put  $f' = \beta_j f$ . We use Notation (8) and prove that the desired Kan extension is

$$f_0/h_r \equiv A_r \xrightarrow{f'/h_r} X_{j+1} \dashrightarrow X_k . \tag{17}$$

Indeed, from (9) we get  $f_0 = (f_0/h_r) \cdot h_r$  via  $f' = (f'/h_r) \cdot h_r$ :

$$(f_0/h) \cdot h_r \equiv A_r \xrightarrow{f'} X_{j+1} \dashrightarrow X_k$$

Next, let  $g_0 = B_r \rightarrow X_k$  fulfil

$$f_0 \leq g_0 h_r .$$

Since  $B_r$  has rank  $\lambda$ , we can find an analogous factorization:

$$\begin{array}{ccccc}
 & & & & B_r \\
 & & & & \downarrow g_0 \\
 & & g & & \\
 & & \swarrow & & \\
 Y_i & \xrightarrow{\beta_i} & X_i & \dashrightarrow & X_k \\
 & & \nwarrow g' & & 
 \end{array}$$

for some ordinal  $i < k$ . Without loss of generality,  $j \leq i$ . The above inequality yields the following one:

$$\begin{array}{ccccc}
 A_r & \xrightarrow{h_r} & B_r & \xrightarrow{g} & Y_i \\
 \downarrow f & & & & \downarrow \beta_i \\
 & & & \leq & X_i \\
 & & & & \vdots \\
 Y_j & \xrightarrow{\beta_j} & X_j & \dashrightarrow & X_k \\
 \downarrow y_{ji} & & \downarrow & & \\
 & & X_i & & \\
 \downarrow & \nearrow \beta_i & & & \\
 Y_i & & & & 
 \end{array}$$

Since by Remark 3.5 the colimit morphisms lie in  $\mathcal{M}$ , (thus, they are order-monomorphisms) this proves

$$y_{ji} \cdot f \leq g \cdot h_r$$

which by composition with  $\beta_i$  yields the inequality

$$\begin{array}{ccc} A_r & \xrightarrow{h_r} & B_r \\ f' \downarrow & & \downarrow g' \\ X_{\hat{j}} & \leq & X_{\hat{i}} \\ \vdots & & \vdots \\ X_{\hat{i}} & \dashrightarrow & X_{\hat{i}+1} \end{array}$$

This is an instance of (10) and (11) with respect to  $x_{\hat{i}, \hat{i}+1} \cdot g'$ . Let  $c_{f,g'}$  be the corresponding coinsertion (12). Since the map  $X_{\hat{i}+1} \dashrightarrow X_{\hat{i}+2}$  factorizes through  $c_{f,g'}$ , we obtain the following inequality

$$\begin{array}{ccc} B_r & \xrightarrow{g'} & X_{\hat{i}} \dashrightarrow X_{\hat{i}+1} \\ f' // h_r \downarrow & & \vdots \\ X_{\hat{j}+1} & \leq & \vdots \\ \vdots & & \vdots \\ X_{\hat{i}+1} & \dashrightarrow & X_{\hat{i}+2} \end{array}$$

Composed with  $X_{\hat{i}+2} \dashrightarrow X_k$  this states that

$$x_{\hat{j}+1, k} \cdot (f' // h_r) \leq x_{\hat{i}k} \cdot g'$$

which is the desired inequality

$$f_0/h_r \leq g_0.$$

(b) It remains to prove that for every  $f_0 : A_2 \rightarrow X_k$  we have

$$(f_0 a)/h_1 = (f_0/h_2)b.$$

Put  $\hat{f}_0 = f_0 a$  and factorize it analogously as  $f_0$  above, assuming (without loss of generality) that the same colimit injection can be applied:

$$\begin{array}{ccccc} & & & & A_r \\ & & & & \downarrow \hat{f}_0 \\ & & & & X_k \\ & & & \dashrightarrow & \\ Y_j & \xleftarrow{\beta_j} & X_{\hat{j}} & \dashrightarrow & X_k \end{array}$$

$\begin{array}{ccc} & & A_r \\ & \nearrow f & \downarrow \hat{f}_0 \\ & & X_k \\ & \nwarrow \hat{f}' & \\ & & X_{\hat{j}} \end{array}$

Thus the equality to be proved is, due to (16), the following square

$$\begin{array}{ccc}
 B_1 & \xrightarrow{\hat{f}'//h_1} & X_{\hat{j}+1} \\
 b \downarrow & & \vdots \\
 B_2 & & \\
 f'//h_2 \downarrow & & \downarrow \\
 X_{\hat{j}+1} & \dashrightarrow & X_k
 \end{array} \tag{18}$$

Since  $x_{\hat{j}k} \cdot \beta_j$  is monic by Remark 3.5, the equality  $\hat{f}_0 = f_0 a$  clearly implies  $\hat{f} = f \cdot a$  which multiplied by  $\beta_j$  yields  $\hat{f}' = f' \cdot a$ . From (9) we thus get the square

$$\begin{array}{ccc}
 A_1 & \xrightarrow{h_1} & B_1 \\
 a \downarrow & & \downarrow \hat{f}'//h_1 \\
 A_2 & & \\
 f' \downarrow & & \downarrow \\
 X_{\hat{j}} & \dashrightarrow & X_{\hat{j}+1}
 \end{array}$$

This is an instance of (10) and (13). Let  $c_{f,g}$  be the corresponding coinsertor. Since the map  $X_{\hat{j}+1} \dashrightarrow X_{\hat{j}+2}$  factorizes through it, we obtain the following inequality

$$\begin{array}{ccc}
 B_1 & \xrightarrow{\hat{f}'//h_1} & X_{\hat{j}+1} \\
 b \downarrow & & \vdots \\
 B_2 & \leq & \\
 f'//h_2 \downarrow & & \downarrow \\
 X_{\hat{j}+1} & \dashrightarrow & X_{\hat{j}+2}
 \end{array}$$

which multiplied by  $X_{\hat{j}+2} \dashrightarrow X_k$  yields "almost" the desired square (18): indeed, the opposite inequality  $(f_0/h_2) \cdot b \geq (f_0 a)/h_1$  is trivial.

(2) For every morphism  $p : X_0 \rightarrow P$  where  $P$  lies in  $\text{Llnj}(\mathcal{H})$  we prove that the morphism  $p_k : X_k \rightarrow P$  in Lemma 3.2 is Kan-injective with respect to  $\mathcal{H}$ . The proof is entirely analogous to that of Part (2) of the proof of Theorem 6.10 of [4]. The proof that the extension of the morphism  $p$  via  $X \dashrightarrow X_k$  is unique in  $\text{Llnj}(\mathcal{H})$  is entirely analogous to Part (3) of the proof mentioned above. ■

3.7. REMARK. (a) The ordinal  $k$  above depends on the choice of the object  $X$ . However, given two objects  $X$  and  $\tilde{X}$ , we can find an ordinal  $k$  such that both of the reflection chains for  $X$  and  $\tilde{X}$  yield reflections in  $\text{Llnj}(\mathcal{H})$  after  $k$  steps. This follows from the choice

$k = \varphi(\lambda)$  made in the above proof, since the same function  $\varphi$  can be used for both chains. (This can be deduced from the fact that we can always use any function  $\varphi' \geq \varphi$  preserving joins.)

(b) Denote by

$$R : \mathcal{X} \longrightarrow \mathbf{LInj}(\mathcal{H})$$

the reflector, i.e., the left adjoint of the embedding. And by  $\eta_X : X \longrightarrow RX$  the reflection morphisms. We have just proved that for all objects  $X$  we have

$$\eta_X \equiv X \dashrightarrow X_k$$

How is  $R$  characterized on morphisms  $u : X \longrightarrow \tilde{X}$ ?

We prove below that

$$Ru = u_k : X_k \longrightarrow \tilde{X}_k$$

for the following natural transformation  $u_i : X_i \longrightarrow \tilde{X}_i$  ( $i \in \mathbf{Ord}$ ). Here  $\tilde{X}_i$  denotes the reflection chain for  $\tilde{X}$  (and we use the obvious notation  $\tilde{Q}_f, \tilde{C}_{f,g}$ , etc.).

**3.8. NOTATION.** Let  $u : X \longrightarrow \tilde{X}$  be a morphism of  $\mathcal{X}$ . We define a natural transformation  $u_i : X_i \longrightarrow \tilde{X}_i$  ( $i \in \mathbf{Ord}$ ) of the reflection chains of  $X$  and  $\tilde{X}$  by the following transfinite induction:

Initial step:  $u_0 = u$ .

Limit step: this is automatic from naturality.

Isolated step  $i \mapsto i + 1$  ( $i$  even): every pair  $(S, f)$  with respect to  $X_i$  defines a pair  $(S, u_i \cdot f)$  with respect to  $\tilde{X}_i$ . For the corresponding pushouts  $Q_f$  and  $\tilde{Q}_{u_i f}$ , see (6), we get a unique factorization  $f^*$  as follows:

$$\begin{array}{ccc}
 A_r & \xrightarrow{h_r} & B_r = B_r \\
 f \downarrow & & \bar{f} \downarrow \\
 X_i & \xrightarrow{\bar{h}_r} & Q_f \\
 u_i \downarrow & & \searrow f^* \\
 \tilde{X}_i & \xrightarrow{\tilde{h}_r} & \tilde{Q}_{u_i f}
 \end{array}
 \quad (19)$$

Then  $(\tilde{q}_{u_i f} \cdot f^*)\bar{h}_r = \tilde{x}_{i,i+1} \cdot u_i$  is independent of  $f$ . Therefore we can define  $u_{i+1}$  via the following squares (for all  $(S, f)$ ):

$$\begin{array}{ccc}
 Q_f & \xrightarrow{f^*} & \tilde{Q}_{u_i f} \\
 q_f \downarrow & & \downarrow \tilde{q}_{u_i f} \\
 X_{i+1} & \xrightarrow{u_{i+1}} & \tilde{X}_{i+1}
 \end{array}
 \quad (20)$$

Observe also that  $(u_i f) // h_r$  is the composite of  $f // h_r$  and  $u_{i+1}$ :

$$\left( \begin{array}{ccc} B_r & \xlongequal{\quad} & B_r \\ \downarrow \bar{f} & & \downarrow \widetilde{u_i f} \\ Q_f & \xrightarrow{f^*} & \widetilde{Q}_{u_i f} \\ \downarrow q_f & & \downarrow \widetilde{q}_{u_i f} \\ X_{i+1} & \xrightarrow{-u_{i+1}} & \widetilde{X}_{i+1} \end{array} \right) \begin{array}{l} f // h_r \\ \\ \\ \\ (u_i f) // h_r \end{array} \quad (21)$$

Isolated step  $i + 1 \mapsto i + 2$  ( $i$  even). Every triple  $(S, f, g)$  in the definition of  $X_{i+2}$  yields a triple  $(S, \widetilde{f}, \widetilde{g})$  with respect to  $\widetilde{X}_{i+2}$  as follows:

$$\widetilde{f} \equiv A_r \xrightarrow{f} X_i \xrightarrow{u_i} \widetilde{X}_i \quad \text{and} \quad \widetilde{g} \equiv B_r \xrightarrow{g} X_{i+1} \xrightarrow{u_{i+1}} \widetilde{X}_{i+1}$$

The naturality of  $(u_j)$  guarantees that every factorization (11) of  $f$  yields the corresponding factorization

$$\widetilde{f} \equiv A_r \xrightarrow{f'} X_j \xrightarrow{u_j} \widetilde{X}_j \dashrightarrow \widetilde{X}_i$$

of  $\widetilde{f}$ . And we prove below that this leads to a unique morphism  $d_{f,g}$  forming the following square

$$\begin{array}{ccc} X_{i+1} & \xrightarrow{c_{f,g}} & C_{f,g} \\ u_{i+1} \downarrow & & \downarrow d_{f,g} \\ \widetilde{X}_{i+1} & \xrightarrow{\widetilde{c}_{\widetilde{f},\widetilde{g}}} & \widetilde{C}_{\widetilde{f},\widetilde{g}} \end{array} \quad (22)$$

Analogously, every factorization (13) of  $f$  yields one for  $\widetilde{f}$  and we again obtain a square (22). We define  $u_{i+2}$  via the following squares (for all  $(S, f, g)$ ):

$$\begin{array}{ccc} C_{f,g} & \xrightarrow{d_{f,g}} & \widetilde{C}_{f,g} \\ t_{f,g} \downarrow & & \downarrow \widetilde{t}_{\widetilde{f},\widetilde{g}} \\ X_{i+2} & \xrightarrow{u_{i+2}} & \widetilde{X}_{i+2} \end{array} \quad (23)$$

**3.9. PROPOSITION.** *The above natural transformation  $u_i : X_i \longrightarrow \widetilde{X}_i$  ( $i \in \text{Ord}$ ) is well defined, and  $u_k = Ru$  for every morphism  $u : X \longrightarrow \widetilde{X}$ .*

**PROOF.** (1) Firstly, given an even ordinal  $i$ , the morphisms  $q_{u_i f} \cdot f^*$  form a cocone of the wide pushout (7) defining  $X_i \dashrightarrow X_{i+1}$ , hence, in (21) the morphism  $u_{i+1}$  is unique. We

need to verify the naturality square. For that observe that the following diagram

$$\begin{array}{ccc}
 X_i & \xrightarrow{u_i} & \tilde{X}_i \\
 \downarrow \text{---} & \searrow^{h_r} & \swarrow_{\tilde{h}_r} \\
 & Q_f & \xrightarrow{f^*} & \tilde{Q}_{u_i f} \\
 & \swarrow_{q_f} & & \searrow_{\tilde{q}_{u_i f}} \\
 X_{i+1} & \xrightarrow{u_{i+1}} & \tilde{X}_{i+1}
 \end{array}$$

commutes.

(2) Next we verify the existence of  $d_{f,g}$  in (22). In case (11) this is equivalent to proving that  $\tilde{c}_{\tilde{f},\tilde{g}} u_{i+1}$  satisfies the following inequality:

$$\begin{array}{ccc}
 A_r & \xrightarrow{g} & X_{i+1} \\
 \downarrow f' // h_r & & \downarrow u_{i+1} \\
 X_{j+1} & \leq & \tilde{X}_{i+1} \\
 \vdots & & \downarrow \tilde{c}_{\tilde{f},\tilde{g}} \\
 X_{i+1} & \xrightarrow{u_{i+1}} \tilde{X}_{i+1} \xrightarrow{\tilde{c}_{\tilde{f},\tilde{g}}} & \tilde{C}_{\tilde{f},\tilde{g}}
 \end{array}$$

Indeed, this follows from the definition of  $\tilde{c}_{\tilde{f},\tilde{g}}$  and (21). And in case (13) we need the inequality

$$\begin{array}{ccc}
 B_1 & \xrightarrow{g} & X_{i+1} \\
 \downarrow b & & \downarrow u_{i+1} \\
 B & \leq & \tilde{X}_{i+1} \\
 \downarrow f' // h_2 & & \downarrow \tilde{c}_{\tilde{f},\tilde{g}} \\
 X_{i+1} & \xrightarrow{u_{i+1}} \tilde{X}_{i+1} \xrightarrow{\tilde{c}_{\tilde{f},\tilde{g}}} & \tilde{C}_{\tilde{f},\tilde{g}}
 \end{array}$$

which also follows from the definition of  $\tilde{c}_{\tilde{f},\tilde{g}}$  and (21). The morphisms

$$C_{f,g} \xrightarrow{d_{f,g}} \tilde{C}_{\tilde{f},\tilde{g}} \xrightarrow{\tilde{t}_{\tilde{f},\tilde{g}}} \tilde{X}_{i+2}$$

form a cocone of the wide pushout (15) due to (22):

$$\tilde{t}_{\tilde{f},\tilde{g}} \cdot d_{f,g} \cdot c_{f,g} = t_{\tilde{f},\tilde{g}} \cdot \tilde{c}_{\tilde{f},\tilde{g}} \cdot u_{i+1} = \tilde{x}_{i+1,i+2} \cdot u_{i+1}.$$

Thus (23) defines  $u_{i+2}$  uniquely. Moreover, the naturality square now follows:

$$\begin{array}{ccc}
 X_{i+1} & \xrightarrow{u_{i+1}} & \tilde{X}_{i+1} \\
 \downarrow c_{f,g} & & \downarrow \tilde{c}_{\tilde{f},\tilde{g}} \\
 & C_{f,g} \xrightarrow{d_{f,g}} \tilde{C}_{\tilde{f},\tilde{g}} & \\
 \downarrow t_{f,g} & & \downarrow \tilde{t}_{\tilde{f},\tilde{g}} \\
 X_{i+1} & \xrightarrow{u_{i+2}} & \tilde{X}_{i+2}
 \end{array}$$

(3) Consequently, for every morphism  $u : X \rightarrow \tilde{X}$  the ordinal  $k$  of Theorem 3.6 provides a square

$$\begin{array}{ccc}
 X & \xrightarrow{\eta_X} & RX \equiv X_k \\
 u \downarrow & & \downarrow u_k \\
 \tilde{X} & \xrightarrow{\eta_{\tilde{X}}} & R\tilde{X} \equiv \tilde{X}_k
 \end{array}$$

To prove  $u_k = Ru$ , we only need to verify that  $u_k$  is a morphism of  $\mathbf{LInj}(\mathcal{H})$ . That is, for every square

$$\begin{array}{ccc}
 A_1 & \xrightarrow{h_1} & B_1 \\
 a \downarrow & & \downarrow b \\
 A_2 & \xrightarrow{h_2} & B_2
 \end{array}$$

in  $\mathcal{H}$  and every morphism  $f_0 : A_r \rightarrow X_k$  the triangle

$$\begin{array}{ccc}
 B_r & \xrightarrow{f_0/h_r} & X_k \\
 \searrow (u_k f_0)/h_r & & \downarrow u_k \\
 & & \tilde{X}_k
 \end{array}$$

commutes. We use (17) for  $f_0/h_r$ , and we assume without loss of generality that the same ordinal  $j$  can be used for  $(u_k f_0)/h_r$ . Then the triangle above commutes due to (21):

$$\begin{array}{ccc}
 A_r & \xrightarrow{f'/h_r} & X_{\hat{j}+1} \dashrightarrow X_k \\
 \searrow (u_j f')/h_r & & \downarrow u_{\hat{j}+1} \\
 & & \tilde{X}_{\hat{j}+1} \dashrightarrow \tilde{X}_k
 \end{array}$$

■

3.10. REMARK. In the following diagram (see (19))

$$\begin{array}{ccc}
 A_r & \xrightarrow{h_r} & B_r \\
 f \downarrow & & \downarrow \bar{f} \\
 X_i & \xrightarrow{\bar{h}_r} & Q_f \\
 u_i \downarrow & & \downarrow f^* \\
 \tilde{X}_i & \xrightarrow{\tilde{h}_r} & \tilde{Q}_{u_i f}
 \end{array}$$

the lower square is a pushout. This follows from the fact that both the composite and the upper square are pushouts.

### 4. Kan-injectivity logic

We now introduce a sound logic for deriving a square  $S$  from a given class  $\mathcal{H}$  of squares of  $\mathcal{X}$ . (Recall that “square” means a commutative one throughout.) Soundness means that every object and every morphism Kan-injective with respect to members of  $\mathcal{H}$  is also Kan-injective with respect to squares derived from  $\mathcal{H}$ . In the subsequent section we prove that our logic is also complete if  $\mathcal{H}$  is small and the base category is locally ranked.

Since for ordinary categories Kan-injectivity w.r.t. a morphism is just the usual orthogonality, it is not surprising that our logic is very close to the orthogonality logic presented in [1]. Let us recall this logic here shortly. Firstly, every isomorphism  $s$  has the property that all objects are orthogonal to  $s$ . Hence that logic has one axiom

AXIOM  $\frac{}{s}$  for  $s$  an isomorphism.

The deduction rules are such that whenever an object is orthogonal to the assumptions (above the horizontal line), then it is orthogonal also to the conclusion. We have the following deduction rules:

COMPOSITION  $\frac{h_1 \quad h_2}{h_2 \cdot h_1}$  for morphisms  $\xrightarrow{h_1} \xrightarrow{h_2}$

PUSHOUT  $\frac{h}{k}$  for a pushout  $\begin{array}{ccc} & \xrightarrow{h} & \\ \downarrow & & \downarrow \\ & \xrightarrow{k} & \end{array}$

WIDE PUSHOUT	$\frac{h_i (i \in I)}{k}$	for a wide pushout	$\begin{array}{ccc} & h_i & \\ & \searrow & \downarrow \bar{h}_i \\ & k & \end{array} \quad (i \in I)$
COEQUALIZER	$\frac{k}{c}$	for a coequalizer	$\begin{array}{ccc} & \overset{k}{\downarrow} & \\ & g_2 & \xrightarrow{c} \\ & \xrightarrow{g_1} & \end{array}$
		and a morphism $k$ with	$g_1 k = g_2 k$
WEAK CANCELLATION	$\frac{h_3 \cdot h_2 \quad h_2 \cdot h_1}{h_1}$	for morphisms	$\xrightarrow{h_1} \xrightarrow{h_2} \xrightarrow{h_3}$

This logic is sound in every category  $\mathcal{X}$  with colimits. And for small sets of morphisms  $\mathcal{H}$  it is complete, i.e., every morphism  $h$  such that  $\mathcal{H}^\perp = (\mathcal{H} \cup \{h\})^\perp$  can be derived from  $\mathcal{H}$  (and the isomorphisms) provided that  $\mathcal{X}$  is *locally ranked*, see [1]. This means that for some factorization system  $(\mathcal{E}, \mathcal{M})$

- (a)  $\mathcal{X}$  is cocomplete and  $\mathcal{E}$ -cowellpowered

and

- (b) every object  $X$  has a *rank*, i.e., an infinite cardinal  $\lambda$  such that  $\mathcal{X}(X, -)$  preserves  $\lambda$ -directed colimits of  $\mathcal{M}$ -monics.

The logic presented below deals with collections  $\mathcal{H}$  of squares and the following concept:

4.1. DEFINITION. A square  $S$  is said to be a Kan-injectivity consequence of a class  $\mathcal{H}$  of squares provided that every object and every morphism Kan-injective with respect to the members of  $\mathcal{H}$  is also Kan-injective with respect to  $S$ . Shortly:

$$\text{LInj}(\mathcal{H}) = \text{LInj}(\mathcal{H} \cup \{S\}).$$

4.2. EXAMPLE. If  $\mathcal{H} = \emptyset$ , we are speaking about squares w.r.t. everything is Kan-injective. Let us call a square

$$\begin{array}{ccc} A_1 & \xrightarrow{h_1} & B_1 \\ a \downarrow & & \downarrow b \\ A_2 & \xrightarrow{h_2} & B_2 \end{array} \tag{24}$$

*split* if  $h_1$  and  $h_2$  are left adjoint sections and the Beck-Chevalley condition holds. That is, we have a commutative square

$$\begin{array}{ccc} A_1 & \xleftarrow{h_1^*} & B_1 \\ a \downarrow & & \downarrow b \\ A_2 & \xleftarrow{h_2^*} & B_2 \end{array} \tag{25}$$

with  $h_i^*h_i = \text{id}$  and  $h_i h_i^* \leq \text{id}$  for  $i = 1, 2$ .

Each such square has the property that every object is Kan-injective with respect to it. Indeed, given  $f : A_r \rightarrow X$  for  $r = 1, 2$  the formula for the Kan extension is easily seen to be

$$f/h_r = f \cdot h_r^*$$

from which this fact is obvious. Moreover, this formula implies also that every morphism is Kan-injective w.r.t.  $S$ .

This explains why in the following deduction system the split squares can serve as axioms.

The following logic has as formulas (commutative) squares in a given category. Recall that every morphism  $h : A \rightarrow B$  is represented by the square

$$S(h) = \begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array} .$$

4.3. DEFINITION. *The Kan-injectivity Logic consists of one*

$$\text{AXIOM} \quad \frac{}{S} \quad \text{for split squares } S$$

and the following deduction rules:

$$\text{COMPOSITION} \quad \frac{S_1, S_2}{S} \quad \text{for a composite } S, \text{ horizontal or vertical, of } S_1 \text{ and } S_2$$

$$\text{PUSHOUT} \quad \frac{\begin{array}{ccc} & \xrightarrow{h_1} & \\ \downarrow & & \downarrow \\ & \xrightarrow{h_2} & \end{array}}{\begin{array}{ccc} & \xrightarrow{h_r} & \\ a \downarrow & \lrcorner & \downarrow \\ & \xrightarrow{\quad} & \end{array}}$$

*for a pushout of  $h_r$ ,  $r = 1$  or  $2$ , along an arbitrary morphism  $a$*

WIDE PUSHOUT	$\frac{\begin{array}{ccc} & \xrightarrow{h} & \\ \parallel & \searrow & \downarrow b_i \ (i \in I) \\ & \xrightarrow{b_i h} & \end{array}}{\begin{array}{ccc} & \xrightarrow{h} & \xrightarrow{b_j} \\ \parallel & \searrow & \downarrow \bar{b}_j \\ & \xrightarrow{h} & \xrightarrow{k} \end{array}}$	<p>for any wide pushout</p> $\begin{array}{ccc} & \xrightarrow{h} & \\ & \searrow & \downarrow b_i \ (i \in I) \\ & \xrightarrow{k} & \end{array}$ <p>and any <math>j \in I</math></p>
COINSERTER	$\frac{S, S_0, \bar{S}}{\begin{array}{ccc} & \xrightarrow{k'} & \\ \parallel & \searrow & \downarrow c \\ & \xrightarrow{k'} & \xrightarrow{c} \end{array}}$	<p>for a coinserter</p> $\begin{array}{ccc} & \xrightarrow{k} & \\ & \searrow & \downarrow c \\ & \xrightarrow{g_1} & \xrightarrow{g_2} \end{array},$ <p>a morphism <math>k</math> with <math>g_1 k \leq g_2 k</math>, and a commutative diagram</p> $\begin{array}{ccccc} & \xrightarrow{k} & \xrightarrow{g_1} & & \\ \downarrow & S & \downarrow & \parallel & \xrightarrow{c} \\ \downarrow & \xrightarrow{h_0} & \downarrow & \xrightarrow{k'} & \\ \underline{\underline{S_0}} & \uparrow & \underline{\underline{\bar{S}}} & \uparrow & \end{array}$
RIGHT CANCELLATION	$\frac{S, S(h), S(k)}{S_0}$	<p>for <math>S = \begin{array}{ccc} &amp; \xrightarrow{h} &amp; \\ \downarrow &amp; S_0 &amp; \downarrow \\ \downarrow &amp; \xrightarrow{k} &amp; \downarrow \end{array}</math></p>
UPPER CANCELLATION	$\frac{S_i, \bar{S}_i \ (i \in I)}{S}$	<p>for <math>\bar{S}_i = \begin{array}{ccc} &amp; \xrightarrow{h_i} &amp; \\ a_i \downarrow &amp; S_i &amp; \downarrow b_i \\ &amp; \xrightarrow{a} &amp; \downarrow b \\ a \downarrow &amp; S &amp; \downarrow b \\ &amp; \xrightarrow{h} &amp; \end{array}</math></p> <p>with <math>(b_i)_{i \in I}</math> collectively epic</p>

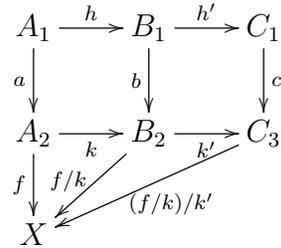
By a deduction of a square  $S$  from a collection  $\mathcal{H}$  of squares is, as usual, meant a sequence of squares obtained by the application of the above rules where  $S$  is the last square and the assumptions are (a) members of  $\mathcal{H}$ , (b) axiom instances or (c) squares already deduced.

4.4. PROPOSITION. (Soundness of the Kan-Injectivity Logic) Let  $\mathcal{X}$  have weighted colimits. Every square deduced from a class of squares is a Kan-injectivity consequence of that class.

PROOF. For the soundness of AXIOM see Example 4.2. Therefore, all we need to prove is that for every deduction rule in Definition 4.3 the deduced square  $S$  is a Kan-injectivity consequence of the assumptions of that rule. To do so we take an object  $X$  Kan-injective with respect to each of the assumptions and verify that  $X$  is Kan-injective with respect to  $S$ . By doing so we actually give a formula for the Kan extensions needed.

We leave out the verification that also morphisms  $u : X \rightarrow X'$  Kan-injective with respect to all assumptions are Kan-injective with respect to  $S$ . Indeed, due to the formula presented for objects this verification is in each case trivial.

(1) COMPOSITION. (a) Horizontal composition:



Let  $X$  be Kan-injective with respect to both of the above squares. We prove for all  $f : A_2 \rightarrow X$  the formula

$$f/(k'k) = (f/k)/k'.$$

Clearly,

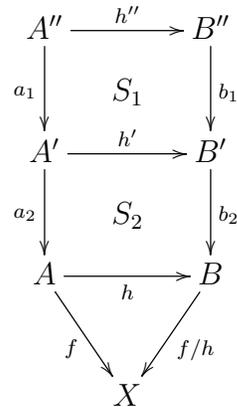
$$[(f/k)/k']k'k = f.$$

Given  $g$  with  $f \leq gk'k$ , we conclude  $f/k \leq gk'$ , hence  $(f/k)/k' \leq g$ . Analogously,

$$\bar{f}/(h'h) = (\bar{f}/h)/h' \quad \text{for all } \bar{f} : A_1 \rightarrow X.$$

The formula  $[fa]/(h'h) = [f/(k'k)] \cdot c$  easily follows.

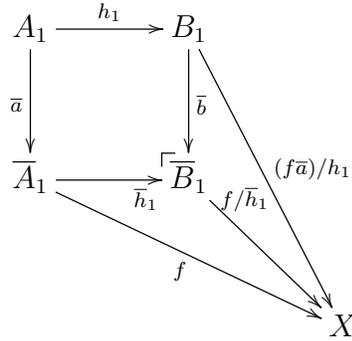
(b) For a vertical composition:



we prove that  $(fa_2a_1)/h'' = (f/h)b_2b_1$  easily from  $(fa_2)/h' = (f/h)b_2$  and  $[(fa_2)a_1]/h'' = [(fa_2)/h'] \cdot b_1$ .

(2) PUSHOUT. (a) Suppose  $X$  is Kan-injective w.r.t.  $\begin{array}{ccc} & \xrightarrow{h_1} & \\ \downarrow & & \downarrow \\ & \xrightarrow{h_2} & \end{array}$ . Let  $r = 1$  and  $f :$

$\bar{A}_1 \longrightarrow X$  be given:



For the square formed by  $f$  and  $(f\bar{a})/h_1$  we have a unique factorization, let us call it  $f/\bar{h}_1$ . This is justified by the lower triangle above together with the implication

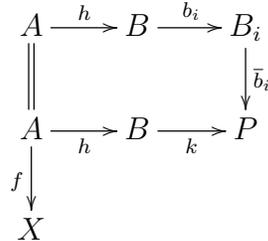
$$f \leq g\bar{h}_1 \text{ implies } f/\bar{h}_1 \leq g$$

which we verify easily. The above pushout is conical so to prove  $f/\bar{h}_1 \leq g$  we only need a verification when precomposed by  $\bar{h}_1$  (this is our assumption) and by  $\bar{b}$ . To prove  $(f/\bar{h}_1)\bar{b} \leq g\bar{b}$ , that is,  $(f\bar{a})h_1 \leq g\bar{b}$ , we just observe that our assumption implies  $f\bar{a} \leq g\bar{b}h_1$  due to  $\bar{b}h_1 = \bar{h}_1\bar{a}$ .

The required rule  $(f/\bar{h}_1)/\bar{b} = (f\bar{a})/h_1$  is the right-hand triangle above.

(b) The proof for  $r = 2$  is completely analogous.

(3) WIDE PUSHOUT. Let  $X$  be Kan-injective w.r.t. each of the squares  $\begin{array}{ccc} & \xrightarrow{h} & \\ \parallel & & \downarrow b_i \\ & \xrightarrow{b_i h} & \end{array}$ . Given a morphism  $f : A \longrightarrow X$



we know that  $f/h$  and  $f/(b_i h)$  exist and fulfil

$$(f/(b_i h))b_i = f/h.$$

Consequently, for the given wide pushout  $P$  there exists a unique morphism

$$\hat{f} : P \longrightarrow X \text{ with } \hat{f}\bar{b}_i = f/(b_i h) \text{ for } i \in I.$$

We prove

$$f/(kh) = \hat{f}.$$

Firstly

$$\hat{f}kh = \hat{f}\bar{b}_i b_i h = [f/(b_i h)] b_i h = f.$$

Next if  $f \leq gkh$ , we prove  $\hat{f} \leq g$ . It suffices to observe that for all  $i$  we have  $\hat{f}\bar{b}_i \leq g\bar{b}_i$ , and use that our wide pushout is conical. Indeed, we have

$$\hat{f}\bar{b}_i = f/(b_ih) \leq g\bar{b}_i$$

or, equivalently,

$$f \leq g\bar{b}_ib_ih$$

by assumption.

The desired formula

$$(f/(kh)) \cdot \bar{b}_j = f/(b_jh)$$

now follows from  $\hat{f}\bar{b}_j = f/(b_jh)$ .

(4) COINSERTER. We are given a diagram

$$A \xrightarrow{k} B \begin{array}{c} \xrightarrow{g_2} \\ \xrightarrow{g_1} \end{array} C \xrightarrow{c} D$$

with  $g_1k \leq g_2k$  and  $c = \text{coins}(g_1, g_2)$ , and a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{k} & B & \xrightarrow{g_1=h'b} & C \\ a \downarrow & S & \downarrow b & & \parallel \\ A_0 & \xrightarrow{h} & B_0 & \xrightarrow{h'} & C \\ h_0 \uparrow & S_0 & \uparrow & \bar{S} & \uparrow k' \\ R & \equiv & R & \equiv & R \end{array}$$

Observe that  $k' = h'hh_0$ . If  $X$  is Kan-injective with respect to  $S$ ,  $S_0$  and  $\bar{S}$ , we prove Kan-injectivity with respect to the following square:

$$\begin{array}{ccccccc} R & \xrightarrow{h_0} & A_0 & \xrightarrow{h} & B_0 & \xrightarrow{h'} & C \\ \parallel & & & & & & \downarrow c \\ R & \xrightarrow{h_0} & A_0 & \xrightarrow{h} & B_0 & \xrightarrow{h'} & C \xrightarrow{c} D \end{array}$$

Kan-injectivity with respect to  $\bar{S}$  yields

$$\hat{f} = f/(h'hh_0) \quad \text{with} \quad \hat{f}h' = f/(hh_0) \tag{26}$$

We prove below that

$$\hat{f}g_1 \leq \hat{f}g_2 \tag{27}$$

which means that  $\hat{f}$  factorizes through  $c = \text{coins}(g_1, g_2)$ . This concludes the proof: the factorization

$$\tilde{f} : C \rightarrow X \quad \text{with} \quad \hat{f} = \tilde{f}c$$

is the desired  $f/(ch'hh_0)$ . Indeed,

$$\tilde{f}ch'hh_0 = \hat{f}(h'hh_0) = f$$

and given  $t$  with  $f \leq tch'hh_0$ , then  $\hat{f} = f/(h'hh_0) \leq tc$ . Thus  $\tilde{f}c \leq tc$  which implies  $\tilde{f} \leq t$ . The desired equality

$$[f/(ch'hh_0)] \cdot c = f/(h'hh_0)$$

in other words

$$\hat{f} = \tilde{f}c = f/(h'hh_0)$$

is the definition of  $\hat{f}$ .

In order to prove (27) first recall that  $X$  is Kan-injective with respect to  $S_0$  and  $S$ , hence,  $f/h_0$  and  $(f/h_0)/h$  exist. From this we easily deduce

$$f/(hh_0) = (f/h_0)/h$$

and then (26) yields

$$\hat{f}h' = (f/h_0)/h. \tag{28}$$

By Kan-injectivity with respect to  $S$  this implies

$$\hat{f}g_1 = \hat{f}h'b = [(f/h_0)/h] \cdot b = [(f/h_0) \cdot a] / k. \tag{29}$$

Next recall that COINSERTER also assumes  $g_1k \leq g_2k$ , thus  $\hat{f}g_1k \leq \hat{f}g_2k$ , and then, using (29), we obtain (27).

(5) RIGHT CANCELLATION. Let  $X$  be Kan-injective with respect to the squares

$$S = \begin{array}{ccccc} A' & \xrightarrow{k'} & B' & \xrightarrow{k} & C' \\ a \downarrow & & \downarrow b & & \downarrow c \\ A & \xrightarrow{h'} & B & \xrightarrow{h} & C \end{array}, \quad S(h) = \begin{array}{ccc} B & \xrightarrow{h} & C \\ \parallel & & \parallel \\ B & \xrightarrow{h} & C \end{array} \quad \text{and} \quad S(k) = \begin{array}{ccc} B' & \xrightarrow{k} & C' \\ \parallel & & \parallel \\ B' & \xrightarrow{k} & C' \end{array}.$$

Given a morphism  $f : A \rightarrow X$ , put

$$f/h' = [f/(hh')] \cdot h.$$

Then  $(f/h') \cdot h' = f$  is clear. And if  $f \leq gh'$ , then, recalling Kan-injectivity with respect to  $S(h)$ , we get  $g = (g/h)h$ , hence

$$f \leq (g/h)(hh')$$

which implies

$$f/(hh') \leq g/h$$

and yields

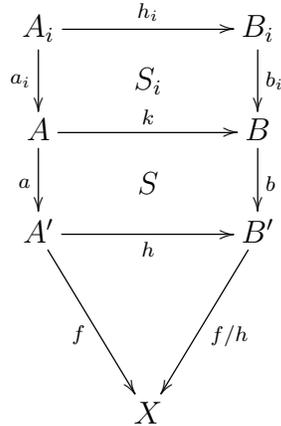
$$[f/(h'h)] \cdot h \leq g$$

as desired. Analogously,  $(fa)/k' = [(fa)/(kk')] \cdot k$  and this yields

$$(f/h') \cdot b = (fa)/k'$$

as required.

(6) UPPER CANCELLATION. Let  $X$  be Kan-injective w.r.t. the squares  $S_i$  and composites of  $S_i$  with  $S$  ( $i \in I$ ). Given a morphism  $f$  as follows



we have  $f/h$  satisfying

$$(f/h)bb_i = (faa_i)/h_i \quad (i \in I).$$

The desired equality

$$(fa)/k = (f/h) \cdot b$$

follows, since  $(b_i)$  is collectively epic, from

$$\begin{aligned}
 [(fa)/k] \cdot b_i &= (faa_i)/h_i && \text{by Kan-injectivity with respect to } S \\
 &= (f/h)(bb_i) && \text{by Kan-injectivity with respect to the composite.}
 \end{aligned}$$

This concludes the proof of soundness. ■

4.5. LEMMA. *The following deduction rules are consequences of the Kan-injectivity Logic:*

$$\text{S-RULE} \quad \frac{\begin{array}{ccc} & \xrightarrow{h_1} & \\ \downarrow & & \downarrow \\ & \xrightarrow{h_2} & \end{array}}{S(h_r)} \quad \text{for } r = 1 \text{ or } 2$$

and

$$\text{TRANSFER} \quad \frac{\begin{array}{ccc} & \xrightarrow{\quad} & \\ \downarrow & & \downarrow \\ & \xrightarrow{h} & \end{array}}{\begin{array}{ccc} & \xrightarrow{\quad} & \\ \downarrow & & \downarrow h \\ & \xrightarrow{h} & \end{array}}$$

PROOF. S-RULE is a special case of PUSHOUT:  $S(h_r)$  is a pushout square. TRANSFER follows by applying RIGHT CANCELLATION as follows:

$$S(h) = \begin{array}{ccc} \parallel & \xrightarrow{h} & \parallel \\ \parallel & \downarrow h & \parallel \\ \parallel & \xrightarrow{h} & \parallel \end{array}$$

■

4.6. LEMMA. *The following deduction rule is a consequence of the Kan-Injectivity Logic:*

$$\text{COEQUALIZER} \quad \frac{\begin{array}{ccc} \xrightarrow{k} & & \xrightarrow{k} \\ a \downarrow & \xrightarrow{h} & \downarrow b_1 \\ & & \xrightarrow{h} \\ & & \downarrow b_2 \end{array} \quad \begin{array}{ccc} \xrightarrow{k} & & \xrightarrow{k} \\ a \downarrow & \xrightarrow{h} & \downarrow b_2 \\ & & \xrightarrow{h} \\ & & \downarrow b_1 \end{array}}{\begin{array}{ccc} \xrightarrow{h} & & \\ \parallel & \downarrow c & \\ \xrightarrow{ch} & & \end{array}} \quad \text{for } c = \text{coeq}(b_1, b_2)$$

PROOF. Form coinserters

$$k_1 = \text{coins}(b_1, b_2) \quad \text{and} \quad k_2 = \text{coins}(b_2, b_1).$$

Then we apply COINSERTER to the following diagrams

$$\begin{array}{ccc} \xrightarrow{k} & \xrightarrow{b_i} & \\ a \downarrow & S \downarrow b_i & \parallel \\ A \xrightarrow{h} & B \xrightarrow{=} & B \\ \parallel & S_0 \uparrow h \bar{S} & \uparrow h \\ A \xrightarrow{=} & A \xrightarrow{=} & A \end{array} \quad (i = 1, 2)$$

where  $S_0$  follows by TRANSFER and  $\bar{S}$  by S-RULE, and we deduce the squares below from the above assumptions:

$$\begin{array}{ccc} A \xrightarrow{h} & B & \\ \parallel & \downarrow k_1 & \\ A \xrightarrow{k_1 h} & K_1 & \end{array} \quad \text{and} \quad \begin{array}{ccc} A \xrightarrow{h} & B & \\ \parallel & \downarrow k_2 & \\ A \xrightarrow{k_2 h} & K_2 & \end{array}$$

The coequalizer  $c$  is clearly a pushout of  $k_1$  and  $k_2$ :

$$\begin{array}{ccc} & B & \\ k_1 \swarrow & & \searrow k_2 \\ K_1 & & K_2 \\ & \downarrow c & \\ & C & \end{array}$$

By applying WIDE PUSHOUT with  $s = h$  we derive

$$\begin{array}{ccc} & \xrightarrow{h} & \\ \parallel & & \downarrow c \\ & \xrightarrow{ch} & \end{array}$$

■

4.7. REMARK. Let  $X_i$  ( $i < \lambda$ ) be an  $\alpha$ -chain with connecting maps  $x_{ij}$ . We can construct its colimit using wide pushouts as follows:

First form a wide pushout of all  $x_{0i}$

$$\begin{array}{ccc} X_0 & \xrightarrow{x_{0i}} & X_i \\ & \searrow p_0 & \downarrow p_i \\ & & P \end{array} \quad (i < \alpha) \tag{30}$$

For all pairs  $i \leq i' < \alpha$  form the following coequalizer

$$X_i \begin{array}{c} \xrightarrow{p_i} \\ \xrightarrow{p_{i'}x_{ii'}} \end{array} P \xrightarrow{c_{ii'}} C_{ii'} \tag{31}$$

Finally, the wide pushout of all these coequalizers is formed

$$\begin{array}{ccc} P & \xrightarrow{c_{ii'}} & C_{ii'} \\ & \searrow g & \downarrow \bar{c}_{ii'} \\ & & K \end{array} \tag{32}$$

Then  $K$  is the colimit of the given diagram with the following colimit cocone:

$$X_i \xrightarrow{p_i} P \xrightarrow{g} K \quad (i < \alpha).$$

Indeed, since  $g$  merges the parallel pair in (31), all  $gp_i$  form a cocone.

Let  $y_i : X_i \rightarrow Y$  ( $i < \alpha$ ) be another cocone. Then the unique  $y : P \rightarrow Y$  with  $y_i = yp_i$  ( $i < \alpha$ ) clearly merges the parallel pair of (31). Hence,  $y$  factorizes through each  $c_{ii'}$ . Consequently,  $y = \bar{y}g$  for a unique  $\bar{y} : K \rightarrow Y$ . This is the desired factorization:

$$y_i = yp_i = \bar{y}(gp_i) \quad \text{for } i < \alpha.$$

The uniqueness of this factorization is easy to verify.

4.8. LEMMA. (*Transfinite Composition*) Let  $(X_i)_{i \in \alpha}$  be an  $\alpha$ -chain with connecting morphisms  $X_i \dashrightarrow X_j$  and a colimit  $k_i : X_i \rightarrow K$  ( $i < \alpha$ ). The following deduction rule follows, for every  $j < \alpha$ , from the Kan-Injectivity Logic.

$$\begin{array}{c}
 \text{TRANSFINITE} \\
 \text{COMPOSITION}
 \end{array}
 \frac{
 \begin{array}{ccc}
 X_0 \dashrightarrow X_j & & \\
 \parallel & \dashrightarrow & \downarrow \\
 X_0 \dashrightarrow X_k & & (j \leq k \leq \alpha)
 \end{array}
 }{
 \begin{array}{ccc}
 X_0 \dashrightarrow X_j & & \\
 \parallel & \downarrow k_j & \\
 X_0 \xrightarrow{k_0} K & & 
 \end{array}
 }$$

PROOF. We can assume  $k_i = gp_i$  for all  $i$ , see the above remark. By applying WIDE PUSHOUT to the premisses of our rule we get, for every  $i < \alpha$ , the deduction of the square

$$\begin{array}{ccc}
 X_0 \dashrightarrow X_i & & \\
 \parallel & \downarrow p_i & \\
 X_0 \xrightarrow{p_0} P & & 
 \end{array} \tag{33}$$

This makes it possible to apply COEQUALIZER to (30) to derive

$$\begin{array}{ccc}
 X_b \xrightarrow{p_0} P & & \\
 \parallel & \downarrow c_{ii'} & \\
 X_0 \xrightarrow{c_{ii'}p_0} C_{ii'} & & 
 \end{array} \tag{34}$$

for all  $i \leq i' < \alpha$ . Next apply WIDE PUSHOUT to (34) with  $h = p_0$  to derive

$$\begin{array}{ccccc}
 X_0 \xrightarrow{p_0} P & \xrightarrow{c_{ii'}} & C_{ii'} & & \\
 \parallel & & \downarrow \bar{c}_{ii'} & & \\
 X_0 \xrightarrow{p_0} P & \xrightarrow{g} & K & & 
 \end{array} .$$

For  $i = i' = j$  we can put  $c_{jj} = \text{id}$  in (31), thus  $g = \bar{c}_{jj}$  in (32). Hence we have derived the following square

$$\begin{array}{ccc}
 X_0 \xrightarrow{p_0} P & & \\
 \parallel & \downarrow g & \\
 X_0 \xrightarrow{p_0} P & \xrightarrow{g} & K
 \end{array}$$

Vertical COMPOSITION with (33) (for  $i = j$ ) yields the desired square:

$$\begin{array}{ccc}
 X_0 \dashrightarrow X_j & & \\
 \parallel & \downarrow h_j = gp_j & \\
 X_0 \xrightarrow{h = gp_0} K & & 
 \end{array}$$

■

### 5. Proof of Completeness

Throughout this section  $\mathcal{H}$  is a set of squares in a locally ranked order-enriched category  $\mathcal{X}$ . We prove that the Kan-Injectivity Logic is complete.

First a preliminary result. Recall the reflection chain from Section 3.

5.1. LEMMA. *For all ordinals  $m \leq i$  the squares*

$$\begin{array}{ccc}
 X_0 & \dashrightarrow & X_m \\
 \parallel & & \downarrow \\
 X_0 & \dashrightarrow & X_i
 \end{array} \tag{35}$$

can be deduced from  $\mathcal{H}$ .

PROOF. We proceed by transfinite induction in  $i$ .

Initial step. Use AXIOM on the split square

$$\begin{array}{ccc}
 X_0 & \equiv & X_0 \\
 \parallel & & \parallel \\
 X_0 & \equiv & X_0
 \end{array}$$

Limit step. This follows from TRANSFINITE COMPOSITION, see Lemma 4.8.

For  $m = i$ ,  $S(X_0 \dashrightarrow X_i)$  is obtained by S-RULE (Lemma 4.5).

Isolated step  $i \mapsto i + 1$  ( $i$  even). Given a square

$$\begin{array}{ccc}
 A_1 & \xrightarrow{h_1} & B_1 \\
 a \downarrow & S & \downarrow b \\
 A_2 & \xrightarrow{h_2} & B_2
 \end{array}$$

in  $\mathcal{H}$  and a morphism  $f : A_r \rightarrow X_i$  ( $r = 1$  or  $2$ ), we have the following deduction:

$$\begin{array}{ccc}
 \frac{S}{\begin{array}{ccc} & \xrightarrow{h_r} & \\ f \downarrow & & \downarrow \bar{f} \\ & \xrightarrow{\bar{h}_r} & \end{array}} & \text{PUSHOUT} & \\
 \frac{\begin{array}{ccc} X_i & \equiv & X_i \\ \parallel & & \downarrow \bar{h}_r \\ X_i & \xrightarrow{\bar{h}_r} & Q_f \end{array}} & \text{TRANSFER} & \\
 & \text{(Lemma 4.5)} & 
 \end{array} \tag{36}$$

$$\begin{array}{ccc}
 \hline & \text{WIDE PUSHOUT} & \\
 X_i \xrightarrow{\bar{h}_r} Q_f & & \\
 \parallel & \searrow q_f & \\
 X_i \dashrightarrow X_{i+1} & & 
 \end{array} \tag{37}$$

$$\begin{array}{ccc}
 \hline & \text{TRANSFER} & \\
 & \text{(Lemma 4.5)} & \\
 X_i \equiv X_i & & \\
 \parallel & \dashrightarrow & \downarrow \\
 X_i \dashrightarrow X_{i+1} & & 
 \end{array} \tag{38}$$

By induction hypothesis, (35) are given. Our task is to verify, for every  $m \leq i + 1$ , the corresponding square. Horizontal COMPOSITION of (38) with (35) for  $m = i$  deduces

$$\begin{array}{ccc}
 X_0 \dashrightarrow X_i & & \\
 \parallel & \dashrightarrow & \downarrow \\
 X_0 \dashrightarrow X_{i+1} & & 
 \end{array}$$

and vertical COMPOSITION of (38) with (35),  $m \leq i$ , yields the desired square

$$\begin{array}{ccc}
 X_0 \dashrightarrow X_k & & \\
 \parallel & \dashrightarrow & \downarrow \\
 X_0 & & X_i \\
 \parallel & \dashrightarrow & \downarrow \\
 X_0 \dashrightarrow X_{i+1} & & 
 \end{array} \tag{39}$$

For  $m = i + 1$ , we deduce the square  $S(X_0 \dashrightarrow X_{i+1})$  by (39) via S-RULE (Lemma 4.5).

Isolated step  $i + 1 \mapsto i + 2$  ( $i$  even). We first observe that the square (9) of Definition 3.1 is deduced from  $\mathcal{H}$  by COMPOSITION applied to (36) and (37):

$$\begin{array}{ccc}
 A_r \xrightarrow{h_r} B_r & & \\
 f \downarrow & & \downarrow \bar{f} \\
 X_i \xrightarrow{\bar{h}_r} Q_f & & \\
 \parallel & \searrow q_f & \\
 X_i \dashrightarrow X_{i+1} & & 
 \end{array} \Bigg) f // h_r \tag{40}$$

For every coinserter (12) we can thus apply COINSERTER to  $k = h_r$  and the following

diagram

$$\begin{array}{ccccccc}
 A_r & \xrightarrow{h_r} & B_r & \longrightarrow & & & \\
 f' \downarrow & & S & & \downarrow f' // h_r & & \parallel \\
 X_j & \dashrightarrow & X_{j+1} & \dashrightarrow & X_{i+1} & \xrightarrow{c_{f,g}} & C_{f,g} \\
 \uparrow & & S_0 & & \bar{S} & & \uparrow \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 X_0 & \equiv & X_0 & \equiv & X_0 & & 
 \end{array}$$

Indeed, the assumptions are  $S$ , by (40),  $S_0$ , which is (35) for  $m = j$  and  $i = j + 1$ , and  $\bar{S}$ , which is (35) with  $m = j + 1$  and  $i + 1$  in the place of  $i$ . Consequently, we deduce the square

$$\begin{array}{ccc}
 X_0 & \dashrightarrow & X_{i+1} \\
 \parallel & & \downarrow c_{f,g} \\
 X_0 & \dashrightarrow & X_{i+1} \xrightarrow{c_{f,g}} C_{f,g}
 \end{array} \tag{41}$$

For every coinserter (14), first observe that the square

$$S = \begin{array}{ccc}
 A_1 & \xrightarrow{h_1} & B_1 \\
 a \downarrow & & S' \downarrow b \\
 A_2 & \xrightarrow{h_2} & B_2 \\
 f' \downarrow & & \downarrow f' // h_2 \\
 X_i & \dashrightarrow & X_{i+1}
 \end{array}$$

is deduced from  $S'$  and (40) by COMPOSITION. We apply COINSERTER to  $k = h_1$  and the following diagram

$$\begin{array}{ccccccc}
 A_1 & \xrightarrow{h_1} & B_1 & \longrightarrow & X_{i+1} & & \\
 \downarrow & & S & & \downarrow & & \parallel \\
 X_i & \dashrightarrow & X_{i+1} & \equiv & X_{i+1} & \xrightarrow{c_{f,g}} & C_{f,g} \\
 \uparrow & & S_0 & & \bar{S} & & \uparrow \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 X_0 & \equiv & X_0 & \equiv & X_0 & & 
 \end{array}$$

where the second and last assumptions of COINSERTER are just instances of (35) which are deduced from  $\mathcal{H}$  by induction hypothesis. Thus we again deduce the square (41).

Apply WIDE PUSHOUT to (15) with  $h \equiv X_0 \dashrightarrow X_{i+1}$  to deduce all the squares

$$\begin{array}{ccc}
 X_0 & \dashrightarrow & X_{i+1} \xrightarrow{c_{f,g}} C_{f,g} \\
 \parallel & & \downarrow t_{f,g} \\
 X_0 & \dashrightarrow & X_{i+1} \dashrightarrow X_{i+2}
 \end{array} \tag{42}$$

A vertical COMPOSITION with (41) deduces

$$\begin{array}{ccc} X_0 & \dashrightarrow & X_{i+1} \\ \parallel & & \downarrow \\ X_0 & \dashrightarrow & X_{i+2} \end{array}$$

which vertically composed with (39) yields

$$\begin{array}{ccc} X_0 & \dashrightarrow & X_m \\ \parallel & & \downarrow \\ X_0 & \dashrightarrow & X_{i+2} \end{array} \quad \text{for all } m \leq i + 1.$$

The remaining case  $m = i + 2$  is then deduced by S-RULE. ■

5.2. THEOREM. (*Kan-Injectivity Logic is Complete and Sound*) *A square is a Kan-injectivity consequence of a set of squares iff it can be deduced from that set.*

PROOF. For soundness see Proposition 4.4. Let  $\mathcal{H}$  be a set of squares and let the following square

$$S \equiv \begin{array}{ccc} A_1 & \xrightarrow{h_1} & B_1 \\ a \downarrow & & \downarrow b \\ A_2 & \xrightarrow{h_2} & B_2 \end{array}$$

be a Kan-injectivity consequence of  $\mathcal{H}$ . We find a deduction of  $S$  from  $\mathcal{H}$ .

Let  $R : \mathcal{X} \rightarrow \text{LInj}(\mathcal{H})$  denote the reflector,  $RX = X_k$  and  $Ru = u_k$  (see Proposition 3.9).

(1) We first use the fact that  $RA_1$  and  $RA_2$  are Kan-injective with respect to  $S$ , thus,  $\eta_{A_1}/h_1$  and  $\eta_{A_2}/h_2$  exist, and we prove that they form a square as follows:

$$\begin{array}{ccc} B_1 & \xrightarrow{\eta_{A_1}/h_1} & RA_1 \\ b \downarrow & & \downarrow Ra \\ B_2 & \xrightarrow{\eta_{A_2}/h_2} & RA_2 \end{array}$$

Indeed, the morphism  $Ra$  is Kan-injective with respect to  $h_1$ , thus

$$Ra \cdot (\eta_{A_1}/h_1) = (Ra \cdot \eta_{A_1})/h_1 = (\eta_{A_2} \cdot a)/h_1$$

and, since  $RA_2$  is Kan-injective with respect to  $S$ , we have

$$(\eta_{A_2} \cdot a)/h_1 = (\eta_{A_2}/h_2) \cdot b.$$

(2) The morphism  $Rh_1 : RA_1 \rightarrow RB_1$  is a left adjoint section. (This was proved in [6] but we include the short proof for the convenience of the reader.) Indeed, since  $RA_1$  is Kan-injective with respect to  $h_1$  (being Kan-injective with respect to  $S$ ), using the universal property of  $\eta_{A_1}$ , we have  $\eta_{A_1}/h_1 = h_1^* \cdot \eta_{B_1}$

$$\begin{array}{ccc}
 A_1 & \xrightarrow{h_1} & B_1 \\
 \eta_{A_1} \downarrow & \nearrow \eta_{A_1}/h_1 & \downarrow \eta_{B_1} \\
 RA_1 & \xleftarrow{Rh_1} & RB_1 \\
 & \xleftarrow{h_1^*} & 
 \end{array}$$

for a unique  $h_1^*$  in  $\mathbf{LInj}(\mathcal{H})$ . This is the desired morphism with

$$h_1^* \cdot Rh_1 = id \quad \text{and} \quad Rh_1 \cdot h_1^* \leq id.$$

Indeed, both composites lie in  $\mathbf{LInj}(\mathcal{H})$ , thus, it is sufficient to verify (a)  $h_1^* \cdot Rh_1 \cdot \eta_{A_1} = \eta_{A_1}$  - see the above diagram, and (b)  $Rh_1 \cdot h_1^* \cdot \eta_{B_1} \leq \eta_{B_1}$ . We use the trivial inequality  $(\eta_{B_1} \cdot h_1)/h_1 \leq \eta_{B_1}$  and the above diagram to see that (b) holds.

(3) The square

$$R(S) \equiv \begin{array}{ccc}
 RA_1 & \xrightarrow{Rh_1} & RB_1 \\
 Ra \downarrow & & \downarrow Rb \\
 RA_2 & \xrightarrow{h_2} & RB_2
 \end{array}$$

splits. (This is Lemma 3.4 in [16]; here we present a different proof.) Indeed, we have  $h_1^* : RB_1 \rightarrow RA_1$  in (2) and, analogously,  $h_2^* : RB_2 \rightarrow RA_2$ . It remains to verify the following square

$$\begin{array}{ccc}
 RB_1 & \xrightarrow{h_1^*} & RA_1 \\
 Rb \downarrow & & \downarrow Ra \\
 RB_2 & \xrightarrow{h_2^*} & RA_2
 \end{array}$$

It lies in  $\mathbf{LInj}(\mathcal{H})$ , thus, it is sufficient to prove that it commutes when precomposed by  $\eta_{B_1}$ :

$$\begin{array}{ccccc}
 RB_1 & \xrightarrow{h_1^*} & RA_1 & & \\
 \eta_{B_1} \swarrow & & \nearrow \eta_{A_1}/h_1 & & \\
 B_1 & & & & \\
 \downarrow b & & & & \\
 B_2 & & & & \\
 \eta_{B_2} \swarrow & & \nearrow \eta_{A_2}/h_2 & & \\
 RB_2 & \xrightarrow{h_2^*} & RA_2 & & \\
 Rb \downarrow & & \downarrow Ra & & 
 \end{array}$$

Indeed, use the square in (1) above.

(4) In part (5) we are going to prove that every naturality square of  $\eta$ , in particular

$$\begin{array}{ccc} A_1 & \xrightarrow{\eta_{A_1}} & RA_1 \\ a \downarrow & & \downarrow Ra \\ A_2 & \xrightarrow{\eta_{A_2}} & RA_2 \end{array}$$

can be deduced from  $\mathcal{H}$ . Due to AXIOM, the square  $R(S)$  of part (3) is also deducible, and their horizontal composite yields

$$\begin{array}{ccccc} A_1 & \xrightarrow{\eta_{A_1}} & RA_1 & \xrightarrow{Rh_1} & RB_1 \\ a \downarrow & & & & \downarrow Rb \\ A_2 & \xrightarrow{\eta_{A_2}} & RA_2 & \xrightarrow{Rh_2} & RB_2 \end{array}$$

which is the same square as the following composite

$$\begin{array}{ccccc} A_1 & \xrightarrow{h_1} & B_1 & \xrightarrow{\eta_{B_1}} & RB_1 \\ a \downarrow & S & \downarrow b & & \downarrow Rb \\ A_2 & \xrightarrow{h_2} & B_2 & \xrightarrow{\eta_{B_2}} & RB_2 \end{array}$$

Thus,  $S$  is deduced via RIGHT CANCELLATION, since the right-hand square is deducible (being, again, a naturality square of  $\eta$ ). This concludes the proof.

(5) To prove that naturality squares of  $\eta$  are deducible from  $\mathcal{H}$ , we consider the squares

$$\begin{array}{ccc} X & \dashrightarrow & X_i \\ u \downarrow & & \downarrow u_i \\ \tilde{X} & \dashrightarrow & \tilde{X}_i \end{array} \tag{43}$$

for all ordinals  $i$  and prove their deducibility by transfinite induction. By Proposition 3.9 the case  $i = k$  is the desired square.

Initial step: the square

$$\begin{array}{ccc} X & \equiv & X \\ u \downarrow & & \downarrow u \\ \tilde{X} & \equiv & \tilde{X} \end{array}$$

is split, we can apply AXIOM .

Limit step: Given a limit ordinal  $i$ , such that (43) is deducible for every  $m < i$  in place of  $i$ , compose (43) vertically with (35) to get a deducible (outward) square as follows:

$$\begin{array}{ccc}
 X_0 & \dashrightarrow & X_m \\
 \parallel & & \downarrow \\
 X & \dashrightarrow & X_i \\
 u \downarrow & & \downarrow u_i \\
 \tilde{X} & \dashrightarrow & \tilde{X}_i
 \end{array}$$

The upper square is deducible by Lemma 5.1, and all  $X_m \dashrightarrow X_i$  are collectively epic, hence, the desired lower square is deduced by UPPER CANCELLATION .

Isolated step  $i \mapsto i + 1$  ( $i$  even). We are going to derive the square

$$\begin{array}{ccc}
 X_i & \dashrightarrow & X_{i+1} \\
 u_i \downarrow & & \downarrow u_{i+1} \\
 \tilde{X}_i & \dashrightarrow & X_{i+1}
 \end{array}$$

and compose it horizontally with the square assumed by induction hypothesis.

For that take all pairs  $(S, f)$  defining the step  $i \mapsto i + 1$ . Then we apply UPPER CANCELLATION to the following diagram

$$\begin{array}{ccc}
 X_i & \xrightarrow{\bar{h}_r} & Q_f \\
 \parallel & & \downarrow q_f \\
 X_i & \dashrightarrow & X_{i+1} \\
 u_i \downarrow & & \downarrow u_{i+1} \\
 \tilde{X}_i & \dashrightarrow & \tilde{X}_{i+1}
 \end{array}$$

The upper squares are deduced, see (37), and all  $q_f$  are collectively epic, thus, we only need a deduction of the composite square. This is, by the definition of  $u_{i+1}$  in Remark 3.7 the square

$$\begin{array}{ccc}
 X_i & \xrightarrow{\bar{h}_r} & Q_f \\
 \downarrow u_i & & \downarrow f^* \\
 & & \tilde{Q}_{u_i f} \\
 & & \downarrow \tilde{q}_{u_i f} \\
 \tilde{X}_i & \dashrightarrow & \tilde{X}_{i+1}
 \end{array} \tag{44}$$

Now in order to derive (44), recall the pushout

$$\begin{array}{ccc} X_i & \xrightarrow{\bar{h}_r} & Q'_f \\ u_i \downarrow & S_0 & \downarrow f^* \\ \tilde{X}_i & \xrightarrow{\tilde{h}_r} & \tilde{Q}_{u_i f} \end{array}$$

from Notation 3.8 and compose it vertically with the square

$$\begin{array}{ccc} \tilde{X}_i & \xrightarrow{\tilde{h}_r} & \tilde{Q}_{u_i f} \\ \parallel & & \downarrow \tilde{q}_{u_i f} \\ \tilde{X}_i & \dashrightarrow & \tilde{X}_{i+1} \end{array}$$

The former square can be deduced from (36) via PUSHOUT, for the latter one see (37). Thus (44) is deducible.

Isolated step  $i + 1 \mapsto i + 2$  ( $i$  even). Using (42), we can again apply UPPER CANCELLATION :

$$\begin{array}{ccc} X & \dashrightarrow X_{i+1} \xrightarrow{c_{f,g}} C_{f,g} & (45) \\ \parallel & & \downarrow \bar{c}_{f,g} \\ X & \dashrightarrow \dots \dashrightarrow X_{i+2} & \\ u \downarrow & & \downarrow u_{i+2} \\ \tilde{X} & \dashrightarrow \dots \dashrightarrow \tilde{X}_{i+2} & \end{array}$$

We only need to deduce the composite square.

The following composite

$$\begin{array}{ccc} X & \dashrightarrow X_{i+1} \\ u \downarrow & \downarrow u_{i+1} \\ \tilde{X} & \dashrightarrow \tilde{X}_{i+1} \\ \parallel & \vdots \\ \tilde{X} & \dashrightarrow \tilde{X}_{i+2} \end{array}$$

is deducible due to Lemma 5.1, and the right-hand vertical morphism is

$$X_{i+1} \xrightarrow{c_{f,g}} C_{f,g} \xrightarrow{d_{f,g}} \tilde{C}_{f,\tilde{g}} \xrightarrow{\tilde{t}_{f,\tilde{g}}} \tilde{X}_{i+2}$$

by (22). Thus we have deduced the following composite

$$\begin{array}{ccccc}
 X & \overset{\text{-----}}{\longrightarrow} & X_{i+1} & & \\
 \parallel & & \downarrow c_{f,g} & & \\
 X & \overset{\text{---}}{\longrightarrow} & X_{i+1} \xrightarrow{c_{f,g}} & C_{fg} & \\
 \downarrow u & & \downarrow d_{f,g} & \downarrow \tilde{t}_{\tilde{f},\tilde{g}} & \\
 \tilde{X} & \overset{\text{-----}}{\longrightarrow} & \tilde{X}_{i+2} & & \\
 & & \downarrow \tilde{t}_{\tilde{f},\tilde{g}} & & \\
 & & \tilde{C}_{\tilde{f},\tilde{g}} & & 
 \end{array}$$

The upper square is (41). Moreover,  $c_{f,g}$  is an epimorphism (being a coinsertion). Thus UPPER CANCELLATION yields the deduction of the lower square. This is the desired composite square (45): indeed, see (23). ■

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