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A FUNCTORIAL APPROACH TO DEDEKIND COMPLETIONS AND THE REPRESENTATION OF VECTOR LATTICES AND ℓ-ALGEBRAS BY NORMAL FUNCTIONS

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ABSTRACT. Unlike the uniform completion, the Dedekind completion of a vector lattice is not functorial. In order to repair the lack of functoriality of Dedekind completions, we enrich the signature of vector lattices with a proximity relation, thus arriving at the category $\boldsymbol{pdv}$ of proximity Dedekind vector lattices. We prove that the Dedekind completion induces a functor from the category $\boldsymbol{bav}$ of bounded archimedean vector lattices to $\boldsymbol{pdv}$, which in fact is an equivalence. We utilize the results of Dilworth [14] to show that every proximity Dedekind vector lattice $\mathfrak{D}$ is represented as the normal real-valued functions on the compact Hausdorff space associated with $\mathfrak{D}$. This yields a contravariant adjunction between $\boldsymbol{pdv}$ and the category **KHaus** of compact Hausdorff spaces, which restricts to a dual equivalence between **KHaus** and the proper subcategory of $\boldsymbol{pdv}$ consisting of those proximity Dedekind vector lattices in which the proximity is uniformly closed. We show how to derive the classic Yosida Representation [40], Kakutani-Krein Duality [24, 26], Stone-Gelfand-Naimark Duality [35, 16], and Stone-Nakano Theorem [35, 32] from our approach.

1. Introduction

Among completions of vector lattices and ℓ-algebras, uniform completions and Dedekind completions are the most studied. Let $\boldsymbol{bav}$ be the category of bounded archimedean vector lattices and let $\boldsymbol{ubav}$ be the full subcategory of $\boldsymbol{bav}$ consisting of uniformly complete objects of $\boldsymbol{bav}$ (see Section 2 for the definitions). The uniform completion $\overline{A}$ of $A \in \boldsymbol{bav}$ extends to a functor $\boldsymbol{bav} \to \boldsymbol{ubav}$ which is left adjoint to the inclusion functor $\boldsymbol{ubav} \to \boldsymbol{bav}$.

The uniform completion functor can conveniently be described by utilizing the Yosida Representation of bounded archimedean vector lattices. For $A \in \boldsymbol{bav}$, let $Y(A)$ be the compact Hausdorff space of maximal ℓ-ideals of A. The Yosida Representation Theorem [40] asserts there is an embedding of A into the vector lattice $C(Y(A))$ of continuous real-valued functions on $Y(A)$. By the Kakutani-Krein Theorem [24, 26], this embedding is an isomorphism iff A is uniformly complete. The assignment $A \mapsto Y(A)$ induces a functor Y from $\boldsymbol{bav}$ to the category **KHaus** of compact Hausdorff spaces and continuous

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maps. Composing this with the functor $C : \mathbf{KHaus} \to \boldsymbol{ubav}$, induced by $X \mapsto C(X)$, yields the uniform completion functor.

By contrast, Dedekind completion is not functorial, at least not with respect to vector lattice homomorphisms (see Remarks 2.14 and 4.16). Specifically, a vector lattice homomorphism $\alpha : A \to B$ in $\boldsymbol{bav}$ need not lift to a vector lattice homomorphism $D(A) \to D(B)$, where $D(-)$ indicates Dedekind completion. Some authors [2, 39, 34] have attempted to remedy the lack of functoriality for the Dedekind completion by restricting to the non-full subcategory of $\boldsymbol{bav}$ consisting of the same objects but whose morphisms are normal homomorphisms (i.e., preserve existing joins, and hence existing meets). The normal homomorphisms in $\boldsymbol{bav}$ lift to normal homomorphisms in $\boldsymbol{bav}$ (see Theorem 7.6), so this repairs the lack of functoriality of the Dedekind completion, but at the expense of a more rigid notion of morphism.

Our approach is to work with the category $\boldsymbol{bav}$ and not sacrifice any of its morphisms. To do so we view the image of $D(-)$ as residing in a category enriched with a proximity relation. More formally, let $A, B \in \boldsymbol{bav}$ and let $\alpha : A \to B$ be a vector lattice homomorphism. Then the lift of α to $D(\alpha) : D(A) \to D(B)$, given by

$$D(\alpha)(x) = \bigvee\{\alpha(a) : a \in A \ \& \ a \leq x\},$$

is a function that extends α but need not be a vector lattice homomorphism. Our first goal is to describe axiomatically $D(\alpha)$. We do this by considering proximity-like relations on $D(A)$ and $D(B)$ induced by A and B, respectively. Proximity-like relations have a long history in topology (see, e.g., [31]), and have been extended to the point-free setting [13, 3, 18, 4]. In [5], they were further generalized to the setting of idempotent generated algebras. In this paper, we define the concept of proximity on Dedekind vector lattices, thus obtaining a new object $(D, \prec)$, a proximity Dedekind vector lattice consisting of a Dedekind complete object D in $\boldsymbol{bav}$ and a proximity relation $\prec$ on D. Our axiomatization of the maps $D(\alpha)$ then suggests the notion of a proximity morphism between proximity Dedekind vector lattices. We show that if $\alpha : A \to B$ is a morphism in $\boldsymbol{bav}$, a mapping $\beta : D(A) \to D(B)$ has the property that $\beta = D(\alpha)$ iff β is a proximity morphism that extends α. It follows that $D(\alpha)$ is the unique proximity morphism extending α. With these objects (the proximity Dedekind vector lattices) and morphisms (the proximity morphisms), we obtain a category, which we denote $\boldsymbol{pdv}$, although composition has to be defined carefully. Thus, while Dedekind completion does not induce a functor from $\boldsymbol{bav}$ to $\boldsymbol{bav}$, it induces a functor from $\boldsymbol{bav}$ to $\boldsymbol{pdv}$. In fact, we prove that the functor $D : \boldsymbol{bav} \to \boldsymbol{pdv}$ is an equivalence.

Having thus interpreted Dedekind completion in a categorical context, we turn next to the issue of representation for the objects in $\boldsymbol{bav}$ and $\boldsymbol{pdv}$. The classical Yosida Representation [40] of bounded archimedean vector lattices by real-valued functions on compact Hausdorff spaces can be expressed functorially as having a contravariant adjunction $Y : \boldsymbol{bav} \to \mathbf{KHaus}$ and $C : \mathbf{KHaus} \to \boldsymbol{bav}$ such that each $A \in \boldsymbol{bav}$ embeds in $C(Y(A))$ and each $X \in \mathbf{KHaus}$ is homeomorphic to $Y(C(X))$. On the one hand, the embedding $A \to C(Y(A))$ yields the Yosida Representation of each $A \in \boldsymbol{bav}$ by means of real-valued functions on $Y(A)$. On the other hand, the homeomorphism $X \to Y(C(X))$ yields the Kakutani-Krein Duality [24, 26] between **KHaus** and the image of $C \circ Y$ in $\boldsymbol{bav}$. In this paper we show that a similar situation arises between $\boldsymbol{pdv}$ and **KHaus** by building an appropriate contravariant adjunction $X : \boldsymbol{pdv} \to \mathbf{KHaus}$ and $\mathfrak{N} : \mathbf{KHaus} \to \boldsymbol{pdv}$, which is based on Dilworth's work [14], rather than that of Yosida-Kakutani-Krein. In fact, the Yosida-Kakutani-Krein theory follows directly from our results.

While Kakutani-Krein Duality implies uniformly complete objects in $\boldsymbol{bav}$ are isomorphic to the continuous real-valued functions on compact Hausdorff spaces, the Stone-Nakano theorem [35, 36, 32] yields that Dedekind complete such objects are isomorphic to the continuous real-valued functions on extremally disconnected compact Hausdorff spaces. As was pointed out in [17], the Dedekind completion $D(A)$ of A can also be realized by continuous real-valued functions, albeit on a different space than $Y(A)$. Namely, if $A \in \boldsymbol{bav}$, then $D(A)$ is isomorphic to $C\left(\widetilde{Y(A)}\right)$, where $\widetilde{Y(A)}$ is the Gleason cover of $Y(A)$.

By relaxing the restriction that the representation involves continuous functions, Dilworth [14] gave a representation of the Dedekind completion of the lattice $C^*(X)$ of bounded continuous functions on a completely regular space X as the lattice $N(X)$ of bounded normal functions on X. We develop $N(-)$ into a functor $\mathfrak{N} : \mathbf{KHaus} \to \boldsymbol{pdv}$ that for each $X \in \mathbf{KHaus}$ produces the proximity Dedekind vector lattice $\mathfrak{N}(X) := (N(X), \prec_{C^*(X)})$, whose proximity $\prec_{C^*(X)}$ is given by $f \prec_{C^*(X)} g$ iff there is $h \in C^*(X)$ such that $f \le h \le g$. Note that since X is compact, $C^*(X) = C(X)$. We describe the image of $\mathfrak{N}$ in $\boldsymbol{pdv}$ and show that there is a functor $X : \boldsymbol{pdv} \to \mathbf{KHaus}$ such that the pair $(\mathfrak{N}, X)$ yields a contravariant adjunction between **KHaus** and $\boldsymbol{pdv}$, which restricts to a duality between **KHaus** and the image of $\mathfrak{N}$ in $\boldsymbol{pdv}$.

Putting all this together we obtain a setting for $\boldsymbol{pdv}$ that closely parallels that of $\boldsymbol{bav}$. Each $\mathfrak{D} := (D, \prec) \in \boldsymbol{pdv}$ is represented by normal functions on the compact Hausdorff space $X(\mathfrak{D})$, and each $X \in \mathbf{KHaus}$ is homeomorphic to $X(\mathfrak{N}(X))$. The embedding $\mathfrak{D} \to \mathfrak{N}(X(\mathfrak{D}))$ is an isomorphism in $\boldsymbol{pdv}$ iff the set of reflexive elements of $\mathfrak{D}$ is uniformly complete, and the image of $\mathfrak{N}$ in $\boldsymbol{pdv}$ is dually equivalent to **KHaus**.

Having established representation and duality for $\boldsymbol{pdv}$, we show how the classical Yosida Representation and Kakutani-Krein Duality for $\boldsymbol{bav}$ follow from our results. This gives an alternative view on uniform completion from a perspective in which Dedekind complete objects play the primary role. We also show how to derive the Stone-Nakano Theorem, and prove that a vector lattice homomorphism is normal iff the continuous map between the corresponding Yosida spaces is skeletal. This yields an alternate proof of a result of Rump [34] for compact Hausdorff spaces. In the last section of the paper, we show how multiplication can be incorporated into the picture so that the primary category of interest becomes $\boldsymbol{pd\ell}$, the category of proximity Dedekind ℓ-algebras with proximity ℓ-algebra morphisms. We show $\boldsymbol{pd\ell}$ is a full subcategory of $\boldsymbol{pdv}$, and we prove there is a contravariant adjunction with **KHaus** from which we derive Stone-Gelfand-Naimark Duality [35, 16] between **KHaus** and the category of uniformly complete bounded archimedean ℓ-algebras.

2. Preliminaries

In this section we recall all the needed definitions and facts to make the article self-contained. We use [9] and [27] as our basic references. Throughout all groups are assumed to be abelian.

2.1. Definition.

(1) *A group A with a partial order $\leq$ is an* ℓ-group *if $(A,\leq)$ is a lattice and $a \leq b$ implies $a+c \leq b+c$ for all $a,b,c \in A$.*

(2) *An ℓ-group A is a* vector lattice *if A is an $\mathbb{R}$-vector space and for each $0 \leq a \in A$ and $0 \leq \lambda \in \mathbb{R}$, we have $\lambda a \geq 0$.*

(3) *An ℓ-group A is* archimedean *if for each $a,b \in A$, whenever $n \cdot a \leq b$ for each $n \in \mathbb{N}$, then $a \leq 0$.*

(4) *An ℓ-group A has a* strong order unit *if there is $u \in A$ such that for each $a \in A$ there is $n \in \mathbb{N}$ with $a \leq n \cdot u$.*

(5) *A vector lattice is* bounded *if it has a strong order unit.*

We will often use basic properties of vector lattices without mention. For example, if A is a vector lattice, $\{a_i : i \in I\}$ is a family in A for which $c := \bigvee_i a_i$ exists in A, $b \in A$, and $0 \leq \lambda \in \mathbb{R}$, then

$$\begin{aligned} c \wedge b &= \bigvee_{i\in I}(a_i \wedge b) \\ c + b &= \bigvee_{i\in I}(a_i + b) \\ \lambda c &= \bigvee_{i\in I}(\lambda a_i) \\ -c &= \bigwedge_{i\in I}(-a_i) \end{aligned}$$

(see, e.g., [27, Thms. 12.2, 13.1]). The dual statement for each of these equations also holds.

2.2. Convention. We assume that all vector lattices are bounded and archimedean and all vector lattice homomorphisms are *unital* (preserve the designated strong order unit).

2.3. Notation. We denote by $\boldsymbol{bav}$ the category of bounded archimedean vector lattices and unital vector lattice homomorphisms. For each $A \in \boldsymbol{bav}$, we denote the designated strong order unit of A by 1.

The objects in $\boldsymbol{bav}$ can be viewed as normed spaces in the usual way. Let $A \in \boldsymbol{bav}$. If $a \in A$, then the *positive* and *negative* parts of a are defined as $a^+ = a \vee 0$ and $a^- = (-a) \vee 0$, and we have $a = a^+ - a^-$. Also, the *absolute value* of a is defined as $|a| = a \vee (-a)$, and we have $|a| = a^+ + a^-$ (see, e.g., [27, Def. 11.6, Thm. 11.7]). The *uniform norm* on A is given by

$$\|a\| = \inf\{\lambda \in \mathbb{R} : |a| \leq \lambda\}.$$

Since A is bounded and archimedean, $\|\cdot\|$ is a well-defined norm on A.

2.4. Definition. *A vector lattice A is* uniformly complete *if it is complete with respect to the uniform norm.*

2.5. Example. For a compact Hausdorff space X, let $C(X)$ denote the vector lattice of continuous (necessarily bounded) real-valued functions. Then $C(X) \in \boldsymbol{ubav}$ (see, e.g., [27, Thm. 43.1, p. 282]). The *sup norm* on $C(X)$, which, by the completeness of the reals, coincides with the uniform norm on $C(X)$, is defined by setting, for each $f \in C(X)$,

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

2.6. Definition. *Let A be a vector lattice.*

(1) *A is* Dedekind complete *if every subset of A bounded above has a least upper bound, and hence every subset of A bounded below has a greatest lower bound.*

(2) *A is a* Dedekind vector lattice *if A is Dedekind complete.*

2.7. Remark. A Dedekind complete bounded vector lattice is archimedean [9, Cor. 2, p. 313] and uniformly complete [27, Thm. 42.6, p. 280].

If $A \in \boldsymbol{bav}$, then there is up to isomorphism a unique Dedekind complete vector lattice $D(A) \in \boldsymbol{bav}$ such that A embeds as a vector lattice in $D(A)$ and A is join dense in $D(A)$; see [11, Thm. 1.1].

2.8. Definition. *For $A \in \boldsymbol{bav}$, we call $D(A)$ the* Dedekind completion *of A. Throughout this paper we will identify A with its image in $D(A)$.*

2.9. Remark. The Dedekind completion $D(A)$ of A can be constructed as the set of normal ideals of A; see, e.g., [9, Ch. V.9]. Nakano [33, §30] uses the dual of this description.

2.10. Notation. Let $\boldsymbol{ubav}$ denote the full subcategory of $\boldsymbol{bav}$ consisting of uniformly complete objects of $\boldsymbol{bav}$ and let $\boldsymbol{dbav}$ denote the full subcategory of $\boldsymbol{ubav}$ consisting of Dedekind complete objects of $\boldsymbol{bav}$.

An *ℓ-ideal* of a vector lattice A is a subgroup I of A satisfying $a \in I$ and $|b| \leq |a|$ imply $b \in I$. An ℓ-ideal of A is necessarily a subspace of A [9, Lem. 1, p. 349]. An ℓ-ideal I is *proper* if $I \neq A$, and it is *maximal* if it is maximal among proper ℓ-ideals of A. If M is a maximal ℓ-ideal of A, then $A/M \cong \mathbb{R}$ [9, Thm. XV.2.2]. As a consequence, if $\alpha : A \to B$ is a morphism in $\boldsymbol{bav}$ and M is a maximal ℓ-ideal of B, then $\alpha^{-1}(M)$ is a maximal ℓ-ideal of A.

2.11. Remark. Let M be a maximal ℓ-ideal of $A \in \boldsymbol{bav}$ and let $a \notin M$. Since $A/M \cong \mathbb{R}$, it is totally ordered, so $a+M > 0+M$ or $a+M < 0+M$. We show that $a+M > 0+M$ iff $a^- \in M$ and $a^+ \notin M$. If $a^- \in M$ and $a^+ \notin M$, then $a + M = a^+ + M > 0 + M$. Conversely, suppose $a+M > 0+M$. Because A/M is totally ordered, M is a prime ideal. Therefore, as $a^+ \wedge a^- = 0$ [9, Thm. XIII.4.7], either $a^+ \in M$ or $a^- \in M$. If $a^+ \in M$, then $a + M = -a^- + M \leq 0 + M$, a contradiction. Thus, $a^+ \notin M$ and $a^- \in M$. A similar argument shows that $a + M < 0 + M$ iff $a^+ \in M$ and $a^- \notin M$.

Let $A, B \in \boldsymbol{bav}$. We recall that a monomorphism $\alpha : A \to B$ is *essential* if for each nonzero ℓ-ideal I of B, the ideal $\alpha^{-1}(I)$ of A is nonzero (see, e.g., [10]). If $\alpha : A \to B$ is essential, then we call B an *essential extension* of A, and if α is the inclusion, then we call A an *essential vector sublattice* of B. We say that A is *dense* in B if for each $b \in B$ with $0 < b$ there is $a \in A$ with $0 < a \leq b$ (see, e.g., [11, Sec. 2]).

2.12. Proposition. *Let $A \in \boldsymbol{bav}$, $B \in \boldsymbol{dbav}$, and A be a vector sublattice of B. The following conditions are equivalent.*

(1) *A is dense in B.*

(2) *A is essential in B.*

(3) *A is join dense in B.*

(4) *A is meet dense in B.*

(5) *B is isomorphic to the Dedekind completion of A.*

Proof. (1)⇒(2). Let I be a nonzero ℓ-ideal of B. Then there is $b \in B$ with $0 < b$ and $b \in I$. Since A is dense in B, there is $a \in A$ with $0 < a \leq b$. Because I is convex, $a \in I$. Therefore, $I \cap A \neq (0)$. Thus, A is essential in B.

(2)⇒(4). This is proved in [8, Thm. 3.2].

(3)⇔(4). Suppose A is join dense in B. If $b \in B$, then $-b$ is the join of $\{a \in A : a \leq -b\}$, and so b is the meet of $\{-a \in A : a \leq -b\} = \{c \in A : b \leq c\}$. Thus, A is meet dense in B. A similar argument gives the converse.

(3)⇔(5). This is obvious since (3)⇔(4).

(3)⇒(1). Let $0 < b \in B$. By (3) we may write b as the join of $\{a \in A : a \leq b\}$. Note that if $a \leq b$, then $a \leq a \vee 0 \leq b$. Therefore, we may write $b = \bigvee\{a \in A : 0 \leq a \leq b\}$. Since $b > 0$, there is $a \in A$ with $0 < a \leq b$. Thus, A is dense in B. ■

2.13. Remark. For $A, B \in \boldsymbol{bav}$, if A is a uniformly dense vector sublattice of B, then A is dense in B. To see this, let $0 < b \in B$. Since A is uniformly dense in B, there is a sequence $\{a_n\}$ in A such that $a_n \to b$ and $0 \leq a_n \leq b$. Since $b > 0$, we must have $a_n > 0$ for some n. Therefore, $0 < a_n \leq b$, showing that A is dense in B.

As we pointed out in the introduction, taking the uniform completion extends to a functor $\boldsymbol{bav} \to \boldsymbol{ubav}$ which is left adjoint to the inclusion functor $\boldsymbol{ubav} \to \boldsymbol{bav}$. On the other hand, taking the Dedekind completion is not functorial. For example, as we will see in Remark 4.16, the join lift $D(\alpha) : D(A) \to D(B)$ of a vector lattice homomorphism $\alpha : A \to B$ need not be a vector lattice homomorphism. More generally, there does not exist a functor from $\boldsymbol{bav}$ to $\boldsymbol{dbav}$ that behaves similarly to D in the sense made precise in the next remark.

2.14. Remark. Motivated by a similar observation in [8, Sec. 3], we claim there does not exist a functor $F : \boldsymbol{bav} \to \boldsymbol{bav}$ such that the following conditions hold:

(i) $F(A) = D(A)$ for all $A \in \boldsymbol{bav}$;

(ii) There is a natural transformation $\eta : 1_{\boldsymbol{bav}} \to F$ whose component maps $\eta_A : A \to F(A)$ are the inclusion mappings $A \to D(A)$;

(iii) The induced natural transformation $F \to F \circ F$ is componentwise epic.

To see this, since F satisfies (ii) and (iii), [1, Lem. 3.3] yields that each η_A is an epimorphism. Choose $A \in \boldsymbol{bav}$ such that A is not uniformly dense in $D(A)$. (For such an example, see Remark 8.17.) This implies that the inclusion $A \to D(A)$ is not an epimorphism (see Remark 8.17). Therefore, η_A is not an epimorphism. The obtained contradiction shows that no such functor F exists.

To repair this lack of functoriality of D, in the next section we introduce proximity relations into the category $\boldsymbol{dbav}$ and obtain a different categorical setting from $\boldsymbol{dbav}$ in which Dedekind completion becomes functorial.

3. Proximity vector lattices

Let $A \in \boldsymbol{bav}$ and let $D(A) \in \boldsymbol{dbav}$ be the Dedekind completion of A. We define $\prec_A$ on $D(A)$ by setting

$$x \prec_A y \text{ iff } \exists a \in A \text{ with } x \leq a \leq y.$$

As we will see in Lemma 3.3, the relation $\prec_A$ satisfies the following conditions.

3.1. Definition.

(1) *Let $D \in \boldsymbol{dbav}$. We call a binary relation $\prec$ on D a* proximity *if the following axioms are satisfied:*

- (P1) $0 \prec 0$ *and* $1 \prec 1$.
- (P2) $a \prec b$ *implies* $a \leq b$.
- (P3) $a \leq b \prec c \leq d$ *implies* $a \prec d$.
- (P4) $a \prec b, c$ *implies* $a \prec b \wedge c$.
- (P5) $a \prec b$ *implies there is* $c \in D$ *with* $a \prec c \prec b$.
- (P6) $a > 0$ *implies there is* $0 < b \in D$ *with* $b \prec a$.

(P7) *$a \prec b$ implies $-b \prec -a$.*

(P8) *$a \prec b$ and $c \prec d$ imply $a + c \prec b + d$.*

(P9) *$a \prec b$ implies $\lambda a \prec \lambda b$ for $0 < \lambda \in \mathbb{R}$.*

(2) *Suppose $\prec$ is a proximity on $D \in$ **dbav**. We call $a \in D$* reflexive *if $a \prec a$, and we call $\prec$* reflexive *if* (P5) *is strengthened to the following axiom:*

(SP5) *$a \prec b$ implies there is a reflexive $c \in D$ such that $a \prec c \prec b$.*

(3) *We call a pair $\mathfrak{D} := (D, \prec)$ a* proximity Dedekind vector lattice *if $D \in$ **dbav** and $\prec$ is a reflexive proximity on D.*

(4) *For a proximity Dedekind vector lattice $\mathfrak{D}$, let $R(\mathfrak{D})$ denote the reflexive elements of $\mathfrak{D}$.*

3.2. Remark.

(1) Motivated by the work of de Vries [13], the notion of proximity on idempotent generated algebras was introduced in [5]. An important distinction with Definition 3.1 is that Axiom (SP5) does not occur in [5, Def. 4.2]. Nevertheless, there is a close connection between the two approaches, which will be addressed in [6].

(2) If L is a lattice and $\prec$ is a binary relation satisfying (P2), (P3), (P4), (P5) and the additional condition that $a, b \prec c$ implies $a \vee b \prec c$, then $\prec$ is a called a *Katětov relation* in [19, Sec. 6]. Conditions (P4) and (P7) imply this additional condition involving join, so a proximity is a special case of a Katětov relation. While our notion of proximity has its roots in de Vries duality, Katětov relations give a lattice-theoretic framework for the study of the Katětov-Tong and Stone insertion theorems for normal and extremally disconnected spaces, respectively [19, Sec. 6 and 7]. We use the Katětov-Tong Theorem to interpret an important proximity for us in Remark 4.14 and Theorem 6.6.

3.3. Lemma. *Suppose $A \in$ **bav**. Then $\mathfrak{D}(A) := (D(A), \prec_A)$ is a proximity Dedekind vector lattice and $A = R(\mathfrak{D}(A))$.*

Proof. Clearly $D(A) \in$ **dbav** and it is straightforward to verify that all the proximity axioms hold. For example, we see that (P6) holds because A is (isomorphic to) an essential vector sublattice of $D(A)$ (see Proposition 2.12). That $A = R(\mathfrak{D}(A))$ is clear from the definition of $\prec_A$. ∎

3.4. Remark. As an immediate consequence of Lemma 3.3, we obtain that $D \in$ **dbav** implies $(D, \leq)$ is a proximity Dedekind vector lattice.

3.5. Lemma. *If $\mathfrak{D} = (D, \prec)$ is a proximity Dedekind vector lattice, then $R(\mathfrak{D})$ is an essential vector sublattice of D, and hence D is isomorphic to the Dedekind completion of $R(\mathfrak{D})$.*

PROOF. Let $A = R(\mathfrak{D})$. That A is a vector sublattice of D easily follows from the proximity axioms. For example, if $a, b \in A$, then $a \prec a$ and $b \prec b$, so $a + b \prec a + b$ by (P8), and hence $a + b \in A$. All other statements follow for similarly simple reasons. To see that A is an essential vector sublattice of D, let $a \in D$ with $a > 0$. By (P6), there is $0 < b \in D$ with $b \prec a$. Since $\prec$ is reflexive, (SP5) implies there is $c \in A$ with $b \prec c \prec a$. Therefore, by (P2), $0 < c \le a$, which proves A is a dense vector sublattice of D. Thus, Proposition 2.12 yields that A is essential in D and D is isomorphic to the Dedekind completion of A. ∎

Let $A, B \in \boldsymbol{bav}$ and let $\alpha : A \to B$ be a vector lattice homomorphism. Define $D(\alpha) : D(A) \to D(B)$ by setting

$$D(\alpha)(x) = \bigvee\{\alpha(a) : a \in A \ \&\ a \le x\}.$$

While in general $D(\alpha)$ is not a vector lattice homomorphism, it satisfies the following conditions, as we will see in Lemma 3.9

3.6. DEFINITION. *Let $\mathfrak{D} = (D, \prec)$ and $\mathfrak{E} = (E, \prec)$ be proximity Dedekind vector lattices. We call a map $\alpha : D \to E$ a* proximity morphism *provided, for all $a, b \in D$, $c \in R(\mathfrak{D})$, and $0 < \lambda \in \mathbb{R}$, we have:*

(M1) $\alpha(0) = 0$ *and* $\alpha(1) = 1$.

(M2) $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$.

(M3) *If $a \prec b$, then $-\alpha(-a) \prec \alpha(b)$.*

(M4) $\alpha(b) = \bigvee\{\alpha(a) : a \prec b\}$.

(M5) $\alpha(a \vee c) = \alpha(a) \vee \alpha(c)$.

(M6) $\alpha(a + c) = \alpha(a) + \alpha(c)$.

(M7) $\alpha(\lambda a) = \lambda\alpha(a)$.

3.7. REMARK.

(1) Proximity morphisms for idempotent generated algebras were introduced in [5], and were motivated by [13]. Axioms (M5)–(M7) of Definition 3.6 appear to be stronger than the corresponding axioms in [5, Def. 6.4]. However, as we will show in [6], the two definitions are equivalent in the setting of idempotent generated algebras.

(2) In general, a proximity morphism need not be a vector lattice homomorphism; see Example 4.12. It follows from a recent result of Toumi [38, Thm. 4] that if $A, B \in \boldsymbol{bav}$ and $\alpha : A \to B$ is a lattice homomorphism preserving the $\mathbb{R}$-action of positive scalars, then α is a vector lattice homomorphism. In light of Toumi's theorem and (M1), (M2), and (M7), we conclude that a proximity morphism is a vector lattice homomorphism iff $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$ for all $a, b \in D$. Thus, the lack of a join axiom is what gives the notion of a proximity morphism its subtlety.

3.8. Lemma. *Suppose $\mathfrak{D} = (D, \prec), \mathfrak{E} = (E, \prec)$ are proximity Dedekind vector lattices and $\alpha : D \to E$ is a proximity morphism. Then α restricts to a (unital) vector lattice homomorphism $R(\mathfrak{D}) \to R(\mathfrak{E})$.*

Proof. Let $a \in R(\mathfrak{D})$. Then $a \prec a$. Applying (M3) gives $-\alpha(-a) \prec \alpha(a)$. By (M1) and (M6), $0 = \alpha(0) = \alpha(a + (-a)) = \alpha(a) + \alpha(-a)$. Therefore, $-\alpha(-a) = \alpha(a)$. This implies that $\alpha(a) \prec \alpha(a)$, and so $\alpha(a) \in R(\mathfrak{E})$. Thus, the restriction $\alpha|_{R(\mathfrak{D})}$ is a well-defined map $R(\mathfrak{D}) \to R(\mathfrak{E})$. By (M6) and (M7), $\alpha|_{R(\mathfrak{D})}$ is a linear transformation, by (M2) and (M5), it is a lattice homomorphism, and by (M1), it is unital. Consequently, $\alpha|_{R(\mathfrak{D})}$ is a unital vector lattice homomorphism. ∎

3.9. Lemma. *Suppose $\mathfrak{D} = (D, \prec), \mathfrak{E} = (E, \prec)$ are proximity Dedekind vector lattices and $\alpha : R(\mathfrak{D}) \to R(\mathfrak{E})$ is a vector lattice homomorphism. Define $\beta : D \to E$ by*

$$\beta(x) = \bigvee\{\alpha(a) : a \in R(\mathfrak{D}) \ \&\ a \leq x\}.$$

Then β is a proximity morphism such that $\beta|_{R(\mathfrak{D})} = \alpha$.

Proof. It is clear that β is well defined and restricts to α on $R(\mathfrak{D})$. We verify the axioms of a proximity morphism.

(M1): We have $\beta(0) = 0$ and $\beta(1) = 1$ since $0, 1 \in R(\mathfrak{D})$ and β extends α.

(M2): If $x, y \in D$, then

$$\begin{aligned}\beta(x) \wedge \beta(y) &= \bigvee\{\alpha(a) : a \leq x\} \wedge \bigvee\{\alpha(b) : b \leq y\}\\ &= \bigvee\{\alpha(a) \wedge \alpha(b) : a \leq x, b \leq y\}\\ &= \bigvee\{\alpha(a \wedge b) : a \leq x, b \leq y\}\\ &= \bigvee\{\alpha(c) : c \leq x \wedge y\}\\ &= \beta(x \wedge y).\end{aligned}$$

(M3): Let $x \prec y$. By (SP5) and (P2), there is $a \in R(\mathfrak{D})$ with $x \leq a \leq y$. By definition, $\alpha(a) \leq \beta(y)$. Moreover, $-a \leq -x$, so $\alpha(-a) \leq \beta(-x)$. Therefore, $-\alpha(a) \leq \beta(-x)$, so $-\beta(-x) \leq \alpha(a)$. As $\alpha(a) \in R(\mathfrak{E})$, this yields $-\beta(-x) \prec \beta(y)$.

(M4): By definition, $\beta(x) = \bigvee\{\alpha(a) : a \leq x\}$. From $a \in R(\mathfrak{D})$ it follows that $a \leq x$ is equivalent to $a \prec x$. Therefore, since $\beta(a) = \alpha(a)$, we see that $\beta(x) = \bigvee\{\beta(y) : y \prec x\}$.

(M6): We have

$$\begin{aligned}\beta(a + x) &= \bigvee\{\alpha(b) : b \leq a + x\} = \bigvee\{\alpha(b) : b - a \leq x\}\\ &= \bigvee\{\alpha(c + a) : c \leq x\} = \bigvee\{\alpha(c) + \alpha(a) : c \leq x\}\\ &= \alpha(a) + \bigvee\{\alpha(c) : c \leq x\} = \alpha(a) + \beta(x) = \beta(a) + \beta(x).\end{aligned}$$

(M5): We first observe that for any $x, y \in D$, we have

$$\begin{aligned}\beta(x) + \beta(y) &= \bigvee\{\alpha(a) : a \leq x\} + \bigvee\{\alpha(b) : b \leq y\}\\ &= \bigvee\{\alpha(a) + \alpha(b) : a \leq x, b \leq y\} = \bigvee\{\alpha(a + b) : a \leq x, b \leq y\}\\ &\leq \bigvee\{\alpha(c) : c \leq x + y\} = \beta(x + y).\end{aligned}$$

Therefore, $\beta(x)+\beta(y) \le \beta(x+y)$. Since $a+x = a\vee x + a\wedge x$ (see, e.g., [9, p. 293]), applying (M2), (M6), and the previous inequality yields

$$\begin{aligned}\beta(a\vee x)+\beta(a)\wedge\beta(x) &= \beta(a\vee x)+\beta(a\wedge x)\\ &\le \beta(a\vee x + a\wedge x) = \beta(a+x)\\ &= \beta(a)+\beta(x) = \beta(a)\vee\beta(x)+\beta(a)\wedge\beta(x).\end{aligned}$$

Thus, $\beta(a\vee x) \le \beta(a)\vee\beta(x)$. The reverse inequality is obvious since β is order preserving by (M2), so $\beta(a\vee x) = \beta(a)\vee\beta(x)$.

(M7): Let $0 < \lambda \in \mathbb{R}$. For $x \in D$, we have

$$\begin{aligned}\beta(\lambda x) &= \bigvee\{\alpha(a) : a \le \lambda x\} = \bigvee\{\alpha(a) : \lambda^{-1}a \le x\}\\ &= \bigvee\{\lambda\alpha(b) : b \le x\} = \lambda\beta(x).\end{aligned}$$

Thus, β is a proximity morphism. ∎

From Lemmas 3.8 and 3.9, we obtain the following characterization of proximity morphisms.

3.10. THEOREM. *Suppose $\mathfrak{D} = (D,\prec), \mathfrak{E} = (E,\prec)$ are proximity Dedekind vector lattices and $\beta : D \to E$ is a map such that $\beta(R(\mathfrak{D})) \subseteq R(\mathfrak{E})$. Set $\alpha = \beta|_{R(\mathfrak{D})}$. Then β is a proximity morphism iff α is a vector lattice homomorphism and $\beta(x) = \bigvee\{\alpha(a) : a \in R(\mathfrak{D}), a \le x\}$ for all $x \in D$.*

PROOF. First suppose that β is a proximity morphism. By Lemma 3.8, α is a vector lattice homomorphism. Let $x \in D$. By (M4), $\beta(x) = \bigvee\{\beta(y) : y \prec x\}$. By (SP5) and (P2), there is $a \in R(\mathfrak{D})$ with $y \le a \le x$. Since β is order preserving and $\alpha = \beta|_{R(\mathfrak{D})}$, we see that $\beta(y) \le \alpha(a) \le \beta(x)$. Therefore, $\beta(x) = \bigvee\{\alpha(a) : a \in R(\mathfrak{D}), a \le x\}$. Conversely, suppose α is a vector lattice homomorphism and $\beta(x) = \bigvee\{\alpha(a) : a \in R(\mathfrak{D}), a \le x\}$. Then by Lemma 3.9, β is a proximity morphism. ∎

3.11. COROLLARY. *Suppose $A, B \in$ **bav** and $\alpha : A \to B$ is a vector lattice homomorphism. Then a map $\beta : D(A) \to D(B)$ is a proximity morphism extending α iff $\beta(x) = \bigvee\{\alpha(a) : a \le x\}$ for all $x \in D(A)$.*

We next show that proximity Dedekind vector lattices and proximity morphisms form a category in which composition of two morphisms need not be function composition.

3.12. THEOREM. *Proximity Dedekind vector lattices with proximity vector lattice morphisms form a category **pdv**, where the composition $\beta_2 \star \beta_1$ of proximity morphisms $\beta_1 : (D_1,\prec_1) \to (D_2,\prec_2)$ and $\beta_2 : (D_2,\prec_2) \to (D_3,\prec_3)$ is defined by*

$$(\beta_2 \star \beta_1)(y) = \bigvee\{\beta_2(\beta_1(x)) : x \prec_1 y\}.$$

Proof. Set $A_i = R(D_i, \prec_i)$. Since $\prec_1$ is reflexive, $(\beta_2 \star \beta_1)|_{A_1} = \beta_2|_{A_2} \circ \beta_1|_{A_1}$. Therefore, by Lemma 3.8, $(\beta_2 \star \beta_1)|_{A_1}$ is a vector lattice homomorphism $A_1 \to A_3$. Moreover, we may describe $\beta_2 \star \beta_1$ as $(\beta_2 \star \beta_1)(y) = \bigvee\{\beta_2(\beta_1(a)) : a \in A_1, a \leq y\}$. Consequently, by Lemma 3.9, $\beta_2 \star \beta_1$ is a proximity morphism.

The only nontrivial step remaining to prove is associativity. Suppose $\beta_1 : (D_1, \prec_1) \to (D_2, \prec_2)$, $\beta_2 : (D_2, \prec_2) \to (D_3, \prec_3)$, and $\beta_3 : (D_3, \prec_3) \to (D_4, \prec_4)$ are proximity morphisms. Then $\beta_3 \star (\beta_2 \star \beta_1), (\beta_3 \star \beta_2) \star \beta_1 : (D_1, \prec_1) \to (D_4, \prec_4)$ are proximity morphisms. They both restrict to the same vector lattice homomorphism α on A_1. Therefore, they agree, since for all $y \in D_1$, by the definition of $\star$, we have

$$(\beta_3 \star (\beta_2 \star \beta_1))(y) = \bigvee\{\alpha(a) : a \in A_1, a \leq y\} = ((\beta_3 \star \beta_2) \star \beta_1)(y).$$

■

3.13. Remark. As follows from the proof of Theorem 3.12, if β_1, β_2 are proximity morphisms, then $\beta_2 \star \beta_1$ restricted to the reflexive elements is simply function composition.

We will see in Example 4.18 that set-theoretic composition of proximity morphisms need not be a proximity morphism as it may fail to satisfy (M4). Although the composition in ***pdv*** is not function composition, isomorphisms in ***pdv*** are structure preserving bijections, as we will show next.

3.14. Proposition. *Let $\mathfrak{D} = (D, \prec), \mathfrak{E} = (E, \prec)$ be objects of **pdv** and $\beta : D \to E$ be a morphism of **pdv**. Then β is an isomorphism in **pdv** iff β is a vector lattice isomorphism such that $x \prec y$ in $\mathfrak{D}$ iff $\beta(x) \prec \beta(y)$ in $\mathfrak{E}$.*

Proof. The "$\Leftarrow$" direction is straightforward. For the converse, we first note that as in the verification of (M5) in the proof of Lemma 3.9, we have $\beta(a) + \beta(-a) \leq \beta(a + (-a)) = \beta(0) = 0$. Therefore, $\beta(a) \leq -\beta(-a)$. Thus, by (M3), if $a \prec b$, then $\beta(a) \prec \beta(b)$. Now, since $\beta : D \to E$ is a proximity isomorphism, there is a proximity morphism $\beta' : E \to D$ satisfying $\beta' \star \beta = \mathrm{Id}_{\mathfrak{D}}$ and $\beta \star \beta' = \mathrm{Id}_{\mathfrak{E}}$. Set $A = R(\mathfrak{D})$ and $B = R(\mathfrak{E})$. Then $\alpha := \beta|_A$ and $\alpha' := \beta'|_B$ are vector lattice homomorphisms by Lemma 3.8. It follows from Remark 3.13 that $(\beta' \star \beta)|_{\mathfrak{D}} = \alpha' \circ \alpha$ and $(\beta \star \beta')|_{\mathfrak{E}} = \alpha \circ \alpha'$. So α and α' are inverses of each other, and hence A, B are isomorphic vector lattices. Since D, E are isomorphic to Dedekind completions of A, B, there is a unique vector lattice isomorphism from D to E extending α (see, e.g., [27, pp. 185-186]), which is β since it preserves arbitrary joins. Thus, β is a vector lattice isomorphism. Its inverse is the unique vector lattice isomorphism extending α'. Hence, it is β'. Therefore, β and β' are inverse vector lattice isomorphisms. Since they both preserve proximity, we conclude that $x \prec y$ in $\mathfrak{D}$ iff $\beta(x) \prec \beta(y)$ in $\mathfrak{E}$. ■

We next show that taking Dedekind completion and reflexive elements yield functors that provide a category equivalence of ***bav*** and ***pdv***.

3.15. Theorem. *Define $R : \textbf{pdv} \to \textbf{bav}$ that sends $\mathfrak{D} = (D, \prec) \in \textbf{pdv}$ to $R(\mathfrak{D})$, and a proximity morphism $\alpha : D \to E$ to the restriction $R(\alpha) := \alpha|_{R(\mathfrak{D})}$. Then R is a well-defined covariant functor.*

Proof. Apply Lemmas 3.5, 3.8 and Remark 3.13. ■

3.16. THEOREM. *Define* $\mathfrak{D} : \boldsymbol{bav} \to \boldsymbol{pdv}$ *that sends* $A \in \boldsymbol{bav}$ *to* $\mathfrak{D}(A) = (D(A), \prec_A)$, *and a vector lattice homomorphism* $\alpha : A \to B$ *to* $D(\alpha) : D(A) \to D(B)$ *given by* $D(\alpha)(x) = \bigvee\{\alpha(a) : a \le x\}$. *Then* $\mathfrak{D}$ *is a well-defined covariant functor.*

PROOF. By Lemma 3.3 and Corollary 3.11, $\mathfrak{D}$ is well defined. It is clear that $\mathfrak{D}$ sends identity maps to identity maps. Let $\alpha_1 : A_1 \to A_2$ and $\alpha_2 : A_2 \to A_3$ be morphisms in $\boldsymbol{bav}$. Then $D(\alpha_2 \circ \alpha_1)$ and $D(\alpha_2) \star D(\alpha_1)$ are proximity morphisms which agree on A_1 by Remark 3.13. Therefore, $D(\alpha_2 \circ \alpha_1) = D(\alpha_2) \star D(\alpha_1)$ by (M4). Thus, $\mathfrak{D}$ is a functor. ■

3.17. THEOREM. *The functors* $\mathfrak{D}, R$ *yield an equivalence of* $\boldsymbol{bav}$ *and* $\boldsymbol{pdv}$.

PROOF. Let $(D, \prec) \in \boldsymbol{pdv}$. By Lemma 3.5, $A \coloneqq R(D, \prec)$ is an essential vector sublattice of D, and hence $(D, \prec) \cong (D(A), \prec_A)$. We next show that $\mathfrak{D}$ is full and faithful. Let $\beta : (D(A), \prec_A) \to (D(B), \prec_B)$ be a proximity morphism and $\alpha = \beta|_A$. By Corollary 3.11, $\beta = D(\alpha)$. Therefore, $\mathfrak{D}$ is full. Next, suppose that $\beta_1, \beta_2 : (D(A), \prec_A) \to (D(B), \prec_B)$ are proximity morphisms with $\beta_1|_A = \beta_2|_A \coloneqq \alpha$. Then, for each $x \in D(A)$, we have

$$\beta_1(x) = \bigvee\{\alpha(a) : a \in A, a \le x\} = \beta_2(x),$$

so $\beta_1 = \beta_2$. Thus, $\mathfrak{D}$ is faithful. Consequently, $\mathfrak{D}$ is an equivalence of categories by [28, Thm. IV.4.1]. ■

3.18. REMARK. Our focus throughout this section has been on vector lattices. Proximity Dedekind ℓ-groups with strong order unit can be defined as in Definition 3.1 by omitting axiom (P9). Similarly, proximity morphisms between proximity Dedekind ℓ-groups can be defined by omitting (M7) from Definition 3.6. With these modifications, it is straightforward to see that the obvious ℓ-group analogues of the results in this section hold with little or no modification to their proofs. However, the notion of essential ℓ-subgroup (e.g., in Lemma 3.5) should be replaced with that of dense ℓ-subgroup.

4. Normal functions and the Dilworth functor

We next develop a representation of proximity Dedekind vector lattices, which relies on Dilworth's representation of the Dedekind completion of $C(X)$ for a compact Hausdorff space X [14].[1] We will see in Section 7 that our representation is closely related to Yosida's representation of bounded archimedean vector lattices [40].

For a set X, let $B(X)$ denote the set of bounded functions $X \to \mathbb{R}$. It is straightforward to see that $B(X) \in \boldsymbol{dbav}$, where the operations on $B(X)$ are defined pointwise.

If X is compact Hausdorff, then there are two operators on $B(X)$, the lower and upper limit function operators, that are fundamental in Dilworth's treatment of normal functions.[2] Dăneţ points out in [12] that these operators are typically called the Baire operators on $B(X)$ in honor of Baire, who was the first to introduce them.

[1] In fact, Dilworth's representation works in the more general setting of completely regular spaces, but in this paper we are only interested in compact Hausdorff spaces.

[2] Again, these operators can be defined in the more general setting of completely regular spaces, but we restrict our attention to compact Hausdorff spaces.

4.1. Notation. Let X be a compact Hausdorff space. For each $x \in X$, let $\mathscr{N}_x$ be the collection of open neighborhoods of x. For each $f \in B(X)$ and $x \in X$, define

$$f_*(x) = \sup_{U \in \mathscr{N}_x} \inf_{y \in U} f(y) \quad \text{and} \quad f^*(x) = \inf_{U \in \mathscr{N}_x} \sup_{y \in U} f(y).$$

The Baire operators can also be interpreted as joins and meets, a fact that we collect in the next lemma, along with several other properties needed in later sections.

4.2. Lemma. (Dilworth [14, Lems. 3.1 and 4.1]) *Let X be a compact Hausdorff space, and let $f, g \in B(X)$.*

(1) $f_* = \bigvee\{g \in C(X) : g \le f\}$, *where the join is taken in $B(X)$.*

(2) $f^* = \bigwedge\{g \in C(X) : g \ge f\}$, *where the meet is taken in $B(X)$.*

(3) *If $f \le g$, then $f_* \le g_*$ and $f^* \le g^*$.*

(4) $f_* \le f \le f^*$, $(f_*)_* = f_*$, *and* $(f^*)^* = f^*$.

4.3. Remark. Recall (see, e.g., [14, Sec. 3]) that a real-valued function f is lower semicontinuous if $f^{-1}(\lambda, +\infty)$ is open for each $\lambda \in \mathbb{R}$. If f is bounded, this is equivalent to $f = f_*$. One can define upper semicontinuous functions similarly. Let $\mathrm{LSC}(X)$ and $\mathrm{USC}(X)$ be the posets of lower semicontinuous and upper semincontinuous functions on X, respectively. Then the order preserving functions

$$(-)^* : \mathrm{LSC}(X) \to \mathrm{USC}(X) \text{ and } (-)_* : \mathrm{USC}(X) \to \mathrm{LSC}(X)$$

form a Galois correspondence; that is, for $f \in \mathrm{USC}(X)$ and $g \in \mathrm{LSC}(X)$, we have

$$f_* \le g \text{ iff } f \le g^*.$$

Dilworth's representation of the Dedekind completion of $C(X)$ is in terms of normal functions, the definition of which we recall next.

4.4. Definition. *Let X be compact Hausdorff and $f \in B(X)$. The function $f^\# := (f^*)_*$ is the* normalization *of f, and f is* normal *if $f = f^\#$.*

Using Remark 4.3, it is easy to see that $f^\#$ is a normal function for each $f \in B(X)$.

4.5. Notation. For a compact Hausdorff space X, we denote by $N(X)$ the set of all normal functions in $B(X)$.

4.6. Remark. Dilworth worked with normal upper semicontinuous functions, which correspond to normal filters in the lattice $C(X)$. Our preference is to work with lower semicontinuous functions instead because they correspond to normal ideals of $C(X)$.

4.7. Proposition. *If X is compact Hausdorff, then $N(X) \in$ **dbav**, where the operations on $N(X)$ are given by the normalization of the pointwise operations. In fact, $N(X)$ is the Dedekind completion of $C(X)$.*

Proof. That $N(X)$ is the Dedekind completion of $C(X)$ with respect to normalized pointwise joins and meets was proved by Dilworth [14, Thm. 4.1]. That $N(X)$ is a vector lattice under the specified operations is proved by Dăneţ in [12, Thm. 5.1, Cor. 6.2]. ∎

4.8. Remark. Let $f \in B(X)$, $c \in C(X)$, and $0 < \lambda \in \mathbb{R}$. By the formulas for the Baire operators, it is easy to see that $(c+f)^* = c + f^*$, $(c+f)_* = c + f_*$, $(\lambda f)^* = \lambda f^*$, and $(\lambda f)_* = \lambda f_*$. From this it follows that in $N(X)$ addition by $c \in C(X)$ and multiplication by $0 < \lambda \in \mathbb{R}$ are pointwise.

4.9. Remark. Let $A \in \boldsymbol{bav}$. If A is a uniformly dense vector sublattice of $C(X)$, then by Remark 2.13, A is dense in $C(X)$. Therefore, since $C(X)$ is dense in $N(X)$, we see that A is dense in $N(X)$. Thus, by Proposition 2.12, $N(X)$ is isomorphic to the Dedekind completion of A.

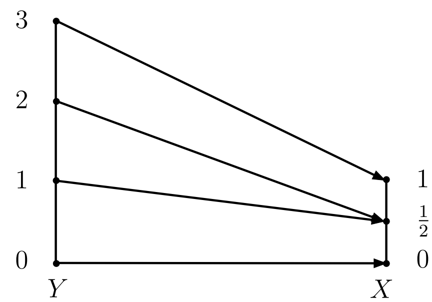
4.10. Remark. Several other interpretations of $D(C(X))$ and $N(X)$ have been given. Hardy [20, Sec. 2] has shown that $N(X)$ is a direct limit of the bounded continuous functions over dense G_δ subsets of X; the operations on $N(X)$ induced by the pointwise operations in the components of the direct limit coincide with those of Proposition 4.7 [20, p. 162]. In [39, Sec. 3], van Haandel and van Rooij show that $D(C(X))$ is isomorphic as a vector lattice to the vector lattice of bounded Baire functions on X modulo the ℓ-ideal of Baire functions whose cozero sets are small. Viewing $N(X)$ as an ℓ-algebra (see Section 8), $N(X)$ can be identified with the ℓ-algebra of bounded elements of the complete ring of quotients of $C(X)$; see Hardy [20, Prop. 2.3] and Fine, Gillman, and Lambek [15]. In addition, Dilworth [14, Thm. 6.1] showed $D(C(X))$ is isomorphic as a lattice to $C(Y)$, where Y is the Gleason cover of X; that this is also an isomorphism of vector lattices was proved by Gierz [17, p. 448]. To see that it is also an isomorphism of ℓ-algebras, consult for example [8, Cor. 3.2]. Finally, a pointfree approach to $N(X)$ was recently developed by Carollo, García, and Picado [29, 30].

4.11. Example. In general, $N(X)$ is not closed under pointwise operations. To see this, for $U \subseteq X$, let χ_U be the characteristic function of U. It is straightforward to see that $(\chi_U)^* = \chi_{\overline{U}}$ and $(\chi_U)_* = \chi_{\mathrm{Int}(U)}$. Therefore, $(\chi_U)^\# = \chi_{\mathrm{Int}(\overline{U})}$. Thus, $\chi_U \in N(X)$ iff U is regular open. Now, let $X = [0,1]$, $U = [0, \frac{1}{2})$, and $V = (\frac{1}{2}, 1]$. Then $\chi_U, \chi_V \in N(X)$. The pointwise sum of χ_U and χ_V is the characteristic function of $U \cup V$, which is not a normal function. In fact, the sum of these functions in $N(X)$ is 1. The same example shows that the join $\chi_U \vee \chi_V$ is not the pointwise join of these two functions.

Proposition 4.7 suggests the question of whether N can be extended to a contravariant functor $\mathbf{KHaus} \to \boldsymbol{dbav}$. If $\sigma : Y \to X$ is a continuous map between compact Hausdorff spaces, recall that $C(\sigma) : C(X) \to C(Y)$, defined by $C(\sigma)(f) = f \circ \sigma$, is a morphism in $\boldsymbol{bav}$. It is natural to define $N(\sigma) : N(X) \to N(Y)$ by $N(\sigma)(f) = (f \circ \sigma)^\#$. If $f \in C(Y)$, then $N(\sigma)(f) = C(\sigma)(f)$. The next example shows that $N(\sigma)$ is not a vector lattice homomorphism, and so this assignment does not define a functor.

4.12. Example. Let $X = [0,1]$, $Y = [0,3]$, and $\sigma : Y \to X$ be given by

$$\sigma(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \le x \le 1, \\ \frac{1}{2} & \text{if } 1 \le x \le 2, \\ \frac{x-1}{2} & \text{if } 2 \le x \le 3. \end{cases}$$



It is obvious that σ is continuous. Let $U = [0, \frac{1}{2})$ and $V = (\frac{1}{2}, 1]$. Then $\chi_U, \chi_V \in N(X)$, $N(\sigma)(\chi_U) = \chi_{[0,1)}$, and $N(\sigma)(\chi_V) = \chi_{(2,3]}$. Therefore, while $\chi_U + \chi_V = 1$ in $N(X)$, so $N(\sigma)(\chi_U + \chi_V) = 1$ in $N(Y)$, we see that $N(\sigma)(\chi_U) + N(\sigma)(\chi_V) = \chi_{[0,1)\cup(2,3]} \neq 1$ in $N(Y)$. Thus, $N(\sigma)$ does not preserve addition, and so is not a vector lattice homomorphism. The same example shows that $N(\sigma)$ does not preserve binary joins.

This lack of functoriality for N can be repaired if we add proximity to the structure of $N(X)$. A natural proximity to work with in this setting is $\prec_{C(X)}$.

4.13. Lemma. *If X is compact Hausdorff, then $\mathfrak{N}(X) := (N(X), \prec_{C(X)}) \in$ **pdv**.*

Proof. By Proposition 4.7, $N(X) \in$ ***dbav*** and $N(X)$ is the Dedekind completion of $C(X)$. Therefore, by Lemma 3.3, $\mathfrak{N}(X) \in$ ***pdv***. ∎

4.14. Remark. By the celebrated Katětov-Tong Theorem [25, 37], for $f, g \in N(X)$, we have

$$f \prec_{C(X)} g \text{ iff } f^* \leq g.$$

4.15. Lemma. *Let $X, Y \in$ **KHaus** and $\sigma : Y \to X$ be a continuous map. Then*

$$N(\sigma)(f) = \bigvee\{C(\sigma)(c) : c \in C(X), c \leq f\}.$$

Proof. Let $f \in N(X)$. By [14, Lem. 4.1], f is the pointwise join of those $c \in C(X)$ with $c \leq f$. Therefore, using sup for pointwise joins, we have

$$\begin{aligned}
N(\sigma)(f) &= (f \circ \sigma)^\# \\
&= (\sup\{c \in C(X) : c \leq f\} \circ \sigma)^\# \\
&= (\sup\{c \circ \sigma : c \in C(X), c \leq f\})^\# \\
&= (\sup\{C(\sigma)(c) : c \in C(X), c \leq f\})^\# \\
&= \bigvee\{C(\sigma)(c) : c \in C(X), c \leq f\}.
\end{aligned}$$

∎

4.16. Remark. Let $\sigma : Y \to X$ be a continuous map between compact Hausdorff spaces. By Proposition 4.7, $N(X) = D(C(X))$, and by Lemma 4.15, $D(C(\sigma)) = N(\sigma)$.

$$\begin{tikzcd}
C(X) \arrow[r, "C(\sigma)"'] \arrow[d, hook] & C(Y) \arrow[d, hook] \\
N(X) \arrow[r, "N(\sigma)"] \arrow[d, equal] & N(Y) \arrow[d, equal] \\
D(C(X)) \arrow[r, "D(C(\sigma))"] & D(C(Y))
\end{tikzcd}$$

It then follows from Example 4.12 that $D : \boldsymbol{bav} \to \boldsymbol{dbav}$ is not a functor.

4.17. Theorem. *Define $\mathfrak{N} : \mathbf{KHaus} \to \boldsymbol{pdv}$ by sending $X \in \mathbf{KHaus}$ to $\mathfrak{N}(X)$ and $\sigma : X \to Y$ to $N(\sigma) : N(Y) \to N(X)$. Then $\mathfrak{N}$ is a well-defined contravariant functor.*

Proof. By Lemma 4.13, $\mathfrak{N}(X) \in \boldsymbol{pdv}$. By Lemmas 3.9 and 4.15, if σ is continuous, then $N(\sigma)$ is a proximity morphism. It is clear that if σ is an identity map, then so is $N(\sigma)$. It remains to prove that N preserves composition. Let $\sigma : X \to Y$ and $\rho : Y \to Z$ be continuous maps. Suppose $f \in N(Z)$. By Lemma 4.15,

$$N(\rho \circ \sigma)(f) = \bigvee\{c \circ (\rho \circ \sigma) : c \in C(Z), c \leq f\}.$$

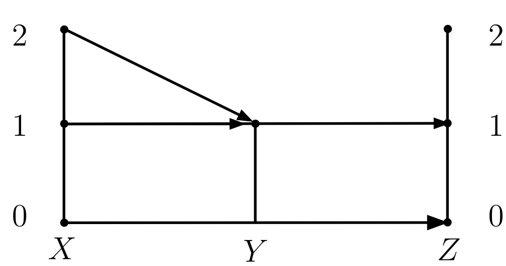
On the other hand,

$$\begin{aligned}
(N(\sigma) \star N(\rho))(f) &= \bigvee\{N(\sigma)(N(\rho)(g)) : g \in N(Z), g \prec f\} \\
&= \bigvee\{N(\sigma)(N(\rho)(c)) : c \in C(Z), c \leq f\} \\
&= \bigvee\{(c \circ \rho) \circ \sigma : c \in C(Z), c \leq f\}.
\end{aligned}$$

Thus, $N(\rho \circ \sigma) = N(\sigma) \star N(\rho)$. ∎

We conclude this section by showing that function composition of proximity morphisms may not be a proximity morphism, thus fulfilling the promise from Section 3.

4.18. Example. Let $X = [0, 2] = Z$ and $Y = [0, 1]$. Define $\sigma : X \to Y$ by $\sigma(x) = x$ if $x \in Y$ and $\sigma(x) = 1$ otherwise. Also, define $\rho : Y \to Z$ by $\rho(x) = x$.



Clearly both σ and ρ are continuous. By Theorem 4.17, $\mathfrak{N}(Z), \mathfrak{N}(Y), \mathfrak{N}(X) \in \boldsymbol{pdv}$ and $N(\rho), N(\sigma)$ are proximity morphisms.

We first show that $N(\sigma) \circ N(\rho) \neq N(\rho \circ \sigma)$. Let $f = \chi_{[0,1)}$. Then $f \in N(Z)$ and $N(\rho)(f) = \chi_{[0,1]}$. Therefore, $(N(\sigma) \circ N(\rho))(f) = \chi_{[0,2]}$. On the other hand, $N(\rho \circ \sigma)(f) = \chi_{[0,1]}$. Thus, $N(\sigma) \circ N(\rho) \neq N(\rho \circ \sigma)$.

Next we show that $N(\sigma) \circ N(\rho)$ does not satisfy (M4). For $f = \chi_{[0,1)}$, we have $f \in N(Z)$ and $N(\sigma)(N(\rho)(f)) = \chi_{[0,2]}$. On the other hand, if $c \in C(Z)$ with $0 \leq c \leq f$, then $c(1) = 0$. Therefore, $N(\sigma)(N(\rho)(c)) = 0$ on $[1,2]$, and it follows that $N(\sigma)(N(\rho)(c)) \leq \chi_{[0,1)} \in N(X)$. Thus, the join in $N(X)$ of all such functions is bounded by $\chi_{[0,1)}$. Consequently,

$$\begin{aligned}\bigvee\{N(\sigma)(N(\rho)(g)) : g \in N(Z), g \prec_{C(Z)} f\} &= \bigvee\{N(\sigma)(N(\rho)(c)) : c \in C(Z), c \leq f\} \\ &\leq \chi_{[0,1)} < \chi_{[0,2]} = N(\sigma)(N(\rho)(f)).\end{aligned}$$

This yields that $N(\sigma) \circ N(\rho)$ does not satisfy (M4), and hence is not a proximity morphism.

5. The end functor

In the previous section we constructed the contravariant functor $\mathfrak{N} : \mathbf{KHaus} \to \boldsymbol{pdv}$. In this section we construct a contravariant functor in the other direction. Let $\mathfrak{D} = (D, \prec) \in \boldsymbol{pdv}$. We will describe the dual space of $\mathfrak{D}$ by means of maximal round ideals of $\mathfrak{D}$.

5.1. Definition. *Let $\mathfrak{D} = (D, \prec) \in \boldsymbol{pdv}$ and let I be an ℓ-ideal of D.*

(1) *I is a* round ideal *of $\mathfrak{D}$ if for each $x \in I$, there exists $y \in I$ with $|x| \prec y$.*

(2) *I is an* end ideal *of $\mathfrak{D}$ if I is a maximal proper round ideal. Let $X(\mathfrak{D})$ be the set of end ideals of $\mathfrak{D}$.*

Suppose $\mathfrak{D} = (D, \prec) \in \boldsymbol{pdv}$. A Zorn's lemma argument shows that each proper round ideal of D is contained in an end of D. For $S \subseteq D$, let

$$\mathop{\downarrow\!\!\!\downarrow} S = \{a \in D : \exists b \in S \text{ with } |a| \prec |b|\}.$$

Also, for $A \in \boldsymbol{bav}$, let $Y(A)$ denote the set of maximal ℓ-ideals of A.

5.2. Lemma. *Let $\mathfrak{D} \in \boldsymbol{pdv}$ and let $A = R(\mathfrak{D})$. Then the following hold for all $P \in X(\mathfrak{D})$ and $M, N \in Y(D)$.*

(1) *$\mathop{\downarrow\!\!\!\downarrow} M \cap A = M \cap A$.*

(2) *$\mathop{\downarrow\!\!\!\downarrow} M = \mathop{\downarrow\!\!\!\downarrow} N$ iff $M \cap A = N \cap A$.*

(3) *P is generated as an ℓ-ideal by $P \cap A$.*

(4) *$P \cap A$ is a maximal ℓ-ideal of A.*

PROOF. (1). Since M is an ℓ-ideal, $\Downarrow M \subseteq M$, so $\Downarrow M \cap A \subseteq M \cap A$. For the reverse inclusion, let $a \in M \cap A$. This intersection is an ℓ-ideal of A, so $|a| \in M \cap A$. Because $|a| \prec |a|$, we see that $a \in \Downarrow M$. Thus, $a \in \Downarrow M \cap A$.

(2). Suppose that $M \cap A = N \cap A$. If $x \in \Downarrow M$, then $|x| \prec y$ for some $y \in M$. Then there is $a \in A$ with $|x| \leq a \leq y$. This implies $a \in M \cap A = N \cap A$. Therefore, $|x| \leq a$ yields $x \in \Downarrow N$. Reversing M and N gives the other inclusion, so $\Downarrow M = \Downarrow N$. Conversely, suppose that $\Downarrow M = \Downarrow N$. Then $\Downarrow M \cap A = \Downarrow N \cap A$, so $M \cap A = N \cap A$ by (1).

(3). Suppose $x \in P$ is nonzero. Then there is $y \in P$ with $|x| \prec y$. Therefore, there is $a \in A$ with $|x| \leq a \leq y$. Since P is convex, $a \in P$, so $a \in P \cap A$. The inequality $|x| \leq a$ shows that x lies in the ℓ-ideal of D generated by $P \cap A$.

(4). Suppose I is a proper ℓ-ideal of A with $P \cap A \subseteq I$. Let J be the ℓ-ideal of D generated by I; that is, $J = \{b \in D : \exists a \in I \text{ with } |b| \leq a\}$ (see, e.g., [27, p. 96]). Since $A = R(\mathfrak{D})$, it is clear that J is a round ideal of $\mathfrak{D}$. By (3), P is generated as an ℓ-ideal by $P \cap A$, so $P \subseteq J$. Since P is an end, $P = J$, which yields $I \subseteq P \cap A$. This proves that $P \cap A$ is a maximal ℓ-ideal. ∎

5.3. LEMMA. *If $\mathfrak{D} \in$ **pdv**, then $X(\mathfrak{D}) = \{\Downarrow M : M \in Y(D)\}$.*

PROOF. Let $A = R(\mathfrak{D})$, let $P \in X(\mathfrak{D})$, and let $M \in Y(D)$ with $P \subseteq M$. By Lemma 5.2(4), $P \cap A = M \cap A$. Therefore, by Lemma 5.2(3), $P = \{d \in D : \exists a \in M \cap A \text{ with } |d| \leq a\} = \Downarrow M$. Thus, $X(\mathfrak{D}) \subseteq \{\Downarrow M : M \in Y(D)\}$.

To prove the reverse inclusion, let $M \in Y(D)$. Then

$$\Downarrow M = \{d \in D : \exists a \in M \cap A \text{ with } |d| \leq a\},$$

so that $\Downarrow M$ is the ℓ-ideal of D generated by $M \cap A$. Now $\Downarrow M$ is round, so there is an end P such that $\Downarrow M \subseteq P$. By Lemma 5.2(1), $M \cap A \subseteq P \cap A$. Therefore, $M \cap A = P \cap A$ since $M \in Y(D)$. By Lemma 5.2(3), P is generated as an ℓ-ideal by $M \cap A$, which forces $\Downarrow M = P$. This proves that $\{\Downarrow M : M \in Y(D)\} \subseteq X(\mathfrak{D})$. ∎

5.4. REMARK. Let $\mathfrak{D} \in$ **pdv** and set $A = R(\mathfrak{D})$.

(1) For each $M \in Y(A)$, there is $P \in X(\mathfrak{D})$ with $P \cap A = M$. To see this, since D has a strong order unit, if M is a maximal ℓ-ideal of A, then the ℓ-ideal of D generated by M is proper, and so is contained in a maximal ℓ-ideal N of D. Let $P = \Downarrow N$. Then $P \in X(\mathfrak{D})$ by Lemma 5.3, and $P \cap A = N \cap A = M$ by Lemma 5.2(1).

(2) We have $\bigcap X(\mathfrak{D}) = (0)$. Indeed, $\bigcap Y(D) = (0)$ by [40, Lem. 4]. Since $\Downarrow M \subseteq M$ for each $M \in Y(D)$, we get $\bigcap X(\mathfrak{D}) = (0)$ by Lemma 5.3.

For each $d \in D$, let $\varphi(d) = \{P \in X(\mathfrak{D}) : \exists e \geq 0 \text{ with } e \prec |d| \text{ and } e \notin P\}$. Clearly $\varphi(d) = \varphi(|d|)$.

5.5. Lemma. *Let $\mathfrak{D} \in$ **pdv** and set $A = R(\mathfrak{D})$.*

(1) $\varphi(1) = X(\mathfrak{D})$ *and* $\varphi(0) = \varnothing$.

(2) *If* $d, e \in D$*, then* $\varphi(d) \cap \varphi(e) = \varphi(|d| \wedge |e|)$.

(3) *If* $a \in R(\mathfrak{D})$*, then* $\varphi(a) = \{P \in X(\mathfrak{D}) : a \notin P\}$.

(4) *If* $d \in D$*, then* $\varphi(d) = \bigcup\{\varphi(a) : |a| \le |d|\}$.

(5) *The sets* $\{\varphi(d) : d \in D\}$ *and* $\{\varphi(a) : a \in A\}$ *both form a basis of the same topology on* $X(\mathfrak{D})$.

Proof. (1). This is clear since each end is a proper ideal, so contains 0 but not 1.

(2). Since ends are ℓ-ideals, it is clear that if $|d| \le |d'|$, then $\varphi(d) \subseteq \varphi(d')$. Therefore, $\varphi(|d| \wedge |e|) \subseteq \varphi(d) \cap \varphi(e)$. For the reverse inclusion, let $P \in \varphi(d) \cap \varphi(e)$. Then there are $d', e' \ge 0$ with $d' \prec |d|$, $e' \prec |e|$, and $d', e' \notin P$. Therefore, there are $a, b \in A$ with $d' \le a \le |d|$ and $e' \le b \le |e|$. It follows that $a, b \notin P$. Since $P \cap A$ is a maximal ℓ-ideal of A, and hence a prime ideal, $a \wedge b \notin P \cap A$. Thus, $d' \wedge e' \notin P$. Since $d' \wedge e' \prec |d| \wedge |e|$, we conclude that $P \in \varphi(|d| \wedge |e|)$.

(3). Let $a \in A$. Since $a \prec |a|$, it follows that $a \notin P$ iff $P \in \varphi(a)$, so (3) holds.

(4). One inclusion is clear. For the other inclusion, let $P \in \varphi(d)$. Then there is $e \ge 0$ with $e \prec |d|$ and $e \notin P$. Therefore, there is $a \in A$ with $e \le a \le |d|$. Thus, $a \notin P$, so $P \in \varphi(a)$.

(5). By (1) and (2), the set $\{\varphi(d) : d \in D\}$ forms a basis for a topology on $X(\mathfrak{D})$. By (4), $\{\varphi(a) : a \in A\}$ is also a basis for the same topology. ∎

5.6. Theorem. *If $\mathfrak{D} \in$ **pdv**, then $X(\mathfrak{D})$ is a compact Hausdorff space.*

Proof. We first show that $X(\mathfrak{D})$ is compact. Suppose that we have an open cover of $X(\mathfrak{D})$. We may assume that the cover consists of basic open sets. So, say $X(\mathfrak{D}) = \bigcup_i \varphi(a_i)$, where each $a_i \in A = R(\mathfrak{D})$. If the a_i generate a proper ℓ-ideal of A, then they lie in a maximal ℓ-ideal M of A. By Remark 5.4(1), there is an end P of $\mathfrak{D}$ such that $M = P \cap A$. Since $P \in \bigcup_i \varphi(a_i)$, we have $a_i \notin P$ for some i, which means $a_i \notin M$. The obtained contradiction proves that the ℓ-ideal in A generated by the a_i is A. This means there is a finite number of the a_i and $b_i \in A$ with $1 \le |b_1 a_1 + \cdots + b_n a_n|$. Then $1 \le |b_1||a_1| + \cdots + |b_n||a_n|$. Let $Q \in X(\mathfrak{D})$. If all $a_i \in Q$, then $1 \in Q$, a contradiction. Thus, $X(\mathfrak{D}) = \varphi(a_1) \cup \cdots \cup \varphi(a_n)$, proving compactness.

We next show that $X(\mathfrak{D})$ is Hausdorff. Let P, Q be distinct elements of $X(\mathfrak{D})$. By Lemma 5.2, there is $a \in A$ with $0 \le a$, $a \in P$, and $a \notin Q$. Let $M = P \cap A$ and $N = Q \cap A$. Then M, N are maximal ℓ-ideals of A with $a \in M$ and $a \notin N$. Since A/M and A/N are isomorphic to $\mathbb{R}$, there is $0 < \lambda \in \mathbb{R}$ with $a + N = \lambda + N$. Therefore, if $b = a - \lambda/2$, then $b + N > 0 + N$ and $b + M < 0 + M$. Thus, $b^- \notin M$ and $b^+ \notin N$ (see Remark 2.11). Since $b^+ \wedge b^- = 0$, we have $P \in \varphi(b^-)$, $Q \in \varphi(b^+)$, and $\varphi(b^+) \cap \varphi(b^-) = \varphi(b^+ \wedge b^-) = \varphi(0) = \varnothing$ by Lemma 5.5. Consequently, we have separated P, Q with disjoint open sets, so $X(\mathfrak{D})$ is Hausdorff. ∎

5.7. Lemma. *Let $\mathfrak{D}, \mathfrak{E} \in$ **pdv** and let $\alpha : \mathfrak{D} \to \mathfrak{E}$ be a proximity morphism. Define $X(\alpha) : X(\mathfrak{E}) \to X(\mathfrak{D})$ by $X(\alpha)(P) = \downarrow\alpha^{-1}(P)$. Then $X(\alpha)$ is a well-defined continuous map.*

Proof. For $P \in X(\mathfrak{E})$ let $I := \downarrow\alpha^{-1}(P)$. To see that I is an ℓ-ideal, let $x, y \in I$. Then $|x| \prec x'$ and $|y| \prec y'$ for some x', y' with $\alpha(x'), \alpha(y') \in P$. Therefore, there are $a, b \in A = R(\mathfrak{D})$ with $|x| \le a \le x'$ and $|y| \le b \le y'$. Thus, $|x \pm y| \le |x| + |y| \le a + b$. By Lemma 3.8, $\alpha|_A$ is a vector lattice homomorphism, so

$$\alpha(a + b) = \alpha(a) + \alpha(b) \le \alpha(x') + \alpha(y').$$

Since $\alpha(x') + \alpha(y') \in P$, we see that $\alpha(a + b) \in P$, and hence $x \pm y \in I$. If $x, y \in D$ with $|x| \le |y|$ and $y \in I$, then it is clear from the definition that $x \in I$. Therefore, I is an ℓ-ideal. It is clearly round. Set $B = R(\mathfrak{E})$. By Lemma 5.2(4), $N := P \cap B \in Y(B)$. Thus, as $\alpha|_A$ is a vector lattice homomorphism, $\alpha^{-1}(N) \in Y(A)$, so by Lemma 5.3, $\downarrow\alpha^{-1}(N) \in X(\mathfrak{D})$. But $\downarrow\alpha^{-1}(N) \subseteq \downarrow\alpha^{-1}(P) = I$. Therefore, to see that I is an end, we only need to show that I is proper. If not, then $1 \in I$, so $1 \prec d$ for some d with $\alpha(d) \in P$. Since $1 \prec d$, we have $1 = \alpha(1) \prec \alpha(d)$, which implies $1 \in P$, a contradiction. Thus, I is proper, and hence is an end. This shows $X(\alpha)$ is well defined.

If $Q \in X(\mathfrak{E})$ and $a \in A$, then $Q \in X(\alpha)^{-1}(\varphi(a))$ iff $\downarrow\alpha^{-1}(Q) \in \varphi(a)$ iff $a \notin \downarrow\alpha^{-1}(Q)$ iff $\alpha(a) \notin Q$. Therefore, $X(\alpha)^{-1}(\varphi(a)) = \varphi(\alpha(a))$, and hence $X(\alpha)$ is continuous. ∎

5.8. Theorem. *Define $X :$ **pdv** $\to$ **KHaus** by sending $\mathfrak{D} \in$ **pdv** to $X(\mathfrak{D})$ and $\beta : \mathfrak{D} \to \mathfrak{E}$ to $X(\beta) : X(\mathfrak{E}) \to X(\mathfrak{D})$. Then X is a well-defined contravariant functor.*

Proof. By Theorem 5.6, $X(\mathfrak{D}) \in$ **KHaus** for each $\mathfrak{D} \in$ ***pdv***; also, by Lemma 5.7, X sends proximity morphisms to continuous maps. It is clear that X sends identity maps to identity maps. It remains to prove that if $\beta_1 : \mathfrak{D}_1 \to \mathfrak{D}_2$ and $\beta_2 : \mathfrak{D}_2 \to \mathfrak{D}_3$ are proximity morphisms, then $X(\beta_2 \star \beta_1) = X(\beta_1) \circ X(\beta_2)$. Let $Q \in X(\mathfrak{D}_3)$. We need to prove that $\downarrow(\beta_2 \star \beta_1)^{-1}(Q) = \downarrow\beta_1^{-1}(\downarrow\beta_2^{-1}(Q))$. Since the proximities are reflexive and the restriction of $\beta_2 \star \beta_1$ to the reflexive elements is function composition (see Remark 3.13), it is straightforward to see that $\downarrow(\beta_2 \star \beta_1)^{-1}(Q) = \downarrow\beta_1^{-1}(\beta_2^{-1}(Q)) = \downarrow\beta_1^{-1}(\downarrow\beta_2^{-1}(Q))$, as desired. ∎

Consequently, we have two contravariant functors $\mathfrak{N} :$ **KHaus** $\to$ ***pdv*** and $X :$ ***pdv*** $\to$ **KHaus**. In the next section we will see that these two functors yield a contravariant adjunction between ***pdv*** and **KHaus** that restricts to a dual equivalence between **KHaus** and a full subcategory of ***pdv***, which we will describe explicitly.

6. Contravariant adjunction and duality

In this section we show that the functors $\mathfrak{N} :$ **KHaus** $\to$ ***pdv*** and $X :$ ***pdv*** $\to$ **KHaus** yield a contravariant adjunction, which restricts to a dual equivalence between **KHaus** and the image of $\mathfrak{N} \circ X$. We characterize this image as those $(D, \prec) \in$ ***pdv***, where $\prec$ is uniformly closed in the product $D \times D$.

6.1. Theorem.

(1) *For* $X \in \mathbf{KHaus}$*, define* $\varepsilon : X \to X(\mathfrak{N}(X))$ *by* $\varepsilon(x) = \{f \in N(X) : |f|^*(x) = 0\}$*. Then* ε *is a well-defined homeomorphism.*

(2) $1_{\mathbf{KHaus}} \cong X \circ \mathfrak{N}$.

Proof. (1). To see that ε is well-defined, let $x \in X$ and let $f, g \in \varepsilon(x)$. Because $(-)^*$ is order preserving, $|f \pm g|^* \leq (|f| + |g|)^*$. Since the sum of upper semicontinuous functions is upper semicontinuous, $|f|^* + |g|^*$ is upper semicontinuous. Therefore, as $|f| + |g| \leq |f|^* + |g|^*$, we have $(|f| + |g|)^* \leq |f|^* + |g|^*$. Thus, $|f \pm g|^* \leq |f|^* + |g|^*$. Because $f, g \in \varepsilon(x)$, we have $|f|^*(x) = 0 = |g|^*(x)$. Consequently, $|f \pm g|^*(x) = 0$, and so $f \pm g \in \varepsilon(x)$. Next, suppose that $|f| \leq |g|$ and $g \in \varepsilon(x)$. Then $0 \leq |f|^*(x) \leq |g|^*(x) = 0$. Therefore, $|f|^*(x) = 0$, so $f \in \varepsilon(x)$. This shows that $\varepsilon(x)$ is an ℓ-ideal of $N(X)$. To see that $\varepsilon(x)$ is round, let $f \in \varepsilon(x)$. Then $|f|^*(x) = 0$. By the definition of $|f|^*$, if $\epsilon > 0$, then there is an open neighborhood U of x with $U \subseteq |f|^{-1}(-\epsilon, \epsilon)$, so $|f|$ is continuous at x. Hence, by [37, Thm. 2], there is $c \in C(X)$ with $|f|^* \leq c$ and $c(x) = 0$. Therefore, $c \in \varepsilon(x)$ and $|f| \prec c$. Thus, $\varepsilon(x)$ is round.

To see that $\varepsilon(x)$ is an end, suppose that I is a round ideal properly containing $\varepsilon(x)$. Take $f \in I \setminus \varepsilon(x)$. Then $|f|^*(x) > 0$. Since I is round, there is $g \in I$ with $f \prec g$. Consequently, there is $c \in C(X)$ with $f \leq c \leq g$. Since I is convex, $c \in I$. This means that $I \cap C(X)$ properly contains $\varepsilon(x) \cap C(X) = M_x := \{f \in C(X) : f(x) = 0\}$. But M_x is a maximal ideal of $C(X)$. Therefore, $I \cap C(X) = C(X)$, and so $1 \in I$. Thus, $I = N(X)$. This proves that $\varepsilon(x)$ is an end of $N(X)$.

To see that ε is onto, suppose P is an end of $\mathfrak{N}(X)$. By Lemma 5.2(4), $P \cap C(X)$ is a maximal ℓ-ideal of $C(X)$. Therefore, there is $x \in X$ with $P \cap C(X) = M_x$. Let $f \in \varepsilon(x)$. Then, by the argument above, there is $c \in M_x$ with $|f|^* \leq c$. Since $c \in P$, we see that $f \in P$. This yields $\varepsilon(x) \subseteq P$. As $\varepsilon(x)$ is an end, we conclude that $P = \varepsilon(x)$. To see that ε is 1-1, suppose $x \in X$. Then $\varepsilon(x) \cap C(X) = M_x$, and so if $x \neq y$, then $\varepsilon(x) \neq \varepsilon(y)$ because $M_x \neq M_y$.

Finally, since both X and $X(\mathfrak{N}(X))$ are compact Hausdorff, to prove that ε is a homeomorphism it is sufficient to show that ε is continuous. For this we observe that if $f \in C(X)$, then

$$\varepsilon^{-1}(\varphi(f)) = \{x \in X : f \notin \varepsilon(x)\} = \{x \in X : |f|^*(x) > 0\} = X \setminus |f|^{-1}(0)$$

is open.

(2). We show that ε yields a natural equivalence between $1_{\mathbf{KHaus}}$ and $X \circ \mathfrak{N}$. For $X \in \mathbf{KHaus}$ we will write ε_X for the homeomorphism $X \to X(\mathfrak{N}(X))$. Let $\sigma : X \to Y$ be a continuous map with $X, Y \in \mathbf{KHaus}$. We need to show that the following diagram is commutative.

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\varepsilon_X"'] & Y \arrow[d, "\varepsilon_Y"] \\ X(\mathfrak{N}(X)) \arrow[r, "X(N(\sigma))"'] & X(\mathfrak{N}(Y)) \end{tikzcd}$$

Let $x \in X$. For $c \in C(Y)$, we have $c \in \varepsilon_Y(\sigma(x))$ iff $|c|(\sigma(x)) = 0$ iff $|c(\sigma(x))| = 0$. This happens iff $|c \circ \sigma|(x) = 0$, which happens iff $|N(\sigma)(c)|(x) = 0$, which is equivalent to $N(\sigma)(c) \in \varepsilon_X(x)$. Therefore, $c \in \varepsilon_Y(\sigma(x))$ iff $c \in (N(\sigma))^{-1}(\varepsilon_X(x))$. On other hand, if $\rho = X(N(\sigma))$, then since $\rho(\varepsilon_X(x)) = {\downarrow} N(\sigma)^{-1}(\varepsilon_X(x))$, we have by Lemma 5.2(1) that $\rho(\varepsilon_X(x)) \cap C(Y) = N(\sigma)^{-1}(\varepsilon_X(x)) \cap C(Y)$. Thus, $\varepsilon_Y(\sigma(x)) \cap C(Y) = \rho(\varepsilon_X(x)) \cap C(Y)$, so $\varepsilon_Y(\sigma(x)) = \rho(\varepsilon_X(x))$ by Lemma 5.2(3). ∎

6.2. Theorem.

(1) *For $\mathfrak{D} \in$ **pdv**, there is a vector lattice isomorphism $\beta : D \to N(X(\mathfrak{D}))$ in **bav** such that $d \prec e$ implies $\beta(d) \prec_{C(X(\mathfrak{D}))} \beta(e)$.*

(2) *If $R(\mathfrak{D}) \in$ **ubav**, then β is a proximity isomorphism.*

(3) *There is a natural transformation $1_{\boldsymbol{pdv}} \to \mathfrak{N} \circ X$.*

Proof. (1). Let $a \in A = R(\mathfrak{D})$. We define a real-valued function f_a on $X(\mathfrak{D})$ by $f_a(P) = \lambda$, where $\lambda \in \mathbb{R}$ satisfies $\lambda + P = a + P$. To see that f_a is well defined, by Lemma 5.2(4), if $P \in X(\mathfrak{D})$, then $P \cap A$ is a maximal ℓ-ideal of A, and $A/(P \cap A)$ embeds as a vector lattice in D/P. But, $A/(P \cap A)$ is isomorphic to $\mathbb{R}$, and so $a + P$ is in the image of $\mathbb{R} \to D/P$. If (λ, μ) is an open interval in $\mathbb{R}$, then Remark 2.11 implies that $f_a^{-1}(\lambda, \mu) = \varphi((a - \lambda)^+) \cap \varphi((\mu - a)^+)$, which is open in $X(\mathfrak{D})$. Therefore, f_a is continuous. It is also straightforward to see that the map $\alpha : A \to C(X(\mathfrak{D}))$ sending a to f_a is a morphism in **bav**, and it is injective by Remark 5.4(2). The image of A separates points since if $P \neq Q$, then $P \cap A$ and $Q \cap A$ are distinct maximal ℓ-ideals of A by Lemma 5.2, so there is $a \in A$ with $a \in (P \cap A) \smallsetminus (Q \cap A)$. Therefore, $f_a(P) \neq f_a(Q)$. Thus, by the Stone-Weierstrass Theorem, $\alpha(A)$ is a uniformly dense vector sublattice of $C(X(\mathfrak{D}))$. Consequently, by Lemma 3.5 and Remark 4.9, both D and $N(X(\mathfrak{D}))$ are Dedekind completions of A. So, α extends to an isomorphism $\beta : D \to N(X(\mathfrak{D}))$ in **bav** (see, e.g., [27, pp. 185-186]). From this (1) follows since $\prec$ on D is a reflexive proximity and $\alpha(A) \subseteq C(X(\mathfrak{D}))$.

(2). Suppose $A = D(\mathfrak{D})$ is uniformly complete. By (1), $\alpha : A \to C(X(\mathfrak{D}))$ is 1-1 and $\alpha(A)$ is uniformly dense in $C(X(\mathfrak{D}))$. Therefore, α is an isomorphism in **bav**. From this it is clear that $d \prec e$ iff $\beta(d) \prec_{C(X(\mathfrak{D}))} \beta(e)$, and so β is a proximity isomorphism by Proposition 3.14.

(3). Let $\beta : \mathfrak{D} \to \mathfrak{E}$ be a proximity morphism. We denote by $\beta_{\mathfrak{D}}$ the proximity morphism $\mathfrak{D} \to \mathfrak{N}(X(\mathfrak{D}))$ defined in (1). We need to show that the diagram

$$\begin{tikzcd}
\mathfrak{D} \arrow[r, "\beta"] \arrow[d, "\beta_{\mathfrak{D}}"'] & \mathfrak{E} \arrow[d, "\beta_{\mathfrak{E}}"] \\
\mathfrak{N}(X(\mathfrak{D})) \arrow[r, "N(X(\beta))"'] & \mathfrak{N}(X(\mathfrak{E}))
\end{tikzcd}$$

is commutative. Let $A = R(\mathfrak{D})$ and $B = R(\mathfrak{E})$. By Theorem 3.10, proximity morphisms are determined by their action on reflexive elements, so by Remark 3.13 it is enough to show that $\beta_{\mathfrak{E}}(\beta(a)) = N(X(\beta))(\beta_{\mathfrak{D}}(a))$ for each $a \in A$. We have $\beta_{\mathfrak{E}}(\beta(a)) = f_{\beta(a)}$. On the other hand, $\beta_{\mathfrak{D}}(a) = f_a$, so $N(X(\beta))$ sends f_a to $f_a \circ X(\beta)$. Let $Q \in X(\mathfrak{E})$ and set $P = X(\beta)(Q) = \downarrow\!\beta^{-1}(Q)$. If $\lambda \in \mathbb{R}$ with $a - \lambda \in P$, then $f_a(P) = \lambda$. Since $a - \lambda \in A$, we see that $a - \lambda \in \beta^{-1}(Q)$, so $\beta(a) - \lambda \in Q$. Therefore, $f_{\beta(a)}(Q) = \lambda$. Thus, $(f_a \circ X(\beta))(Q) = f_a(P) = \lambda = f_{\beta(a)}(Q)$. This yields $f_{\beta(a)} = f_a \circ X(\beta)$, as desired. ∎

6.3. Remark. Our use of the Stone-Weierstrass Theorem in the proof of Theorem 6.2(1) is crucial. For more on the role that this theorem plays in our approach to bounded archimedean vector lattices and ℓ-algebras, see [7]. The use of the Stone-Weierstrass Theorem here also highlights a difference between the ℓ-group and vector lattice cases: If G is an ℓ-group and $Y(G)$ is the Yosida space of G, then although the image of G in $C(Y(G))$ separates points, G need not be uniformly dense in $C(Y(G))$.

6.4. Remark. The proof of Theorem 6.2 gives a different type of functorial representation of objects in $\boldsymbol{pdv}$. Let $\mathfrak{D} = (D, \prec), \mathfrak{E} = (E, \prec) \in \boldsymbol{pdv}$ with $A = R(\mathfrak{D})$ and $B = R(\mathfrak{E})$. Suppose $\beta : \mathfrak{D} \to \mathfrak{E}$ is a proximity morphism. Then there are isomorphisms $\gamma : D \to N(X(\mathfrak{D}))$ and $\eta : E \to N(X_{\mathfrak{E}})$ in $\boldsymbol{bav}$ such that $\gamma(A) \subseteq C(X(\mathfrak{D}))$ and $\eta(B) \subseteq C(X(\mathfrak{E}))$. It is obvious that $(N(X(\mathfrak{D})), \prec_{\gamma(A)}), (N(X(\mathfrak{E})), \prec_{\eta(B)}) \in \boldsymbol{pdv}$, that $\gamma : \mathfrak{D} \to (N(X(\mathfrak{D})), \prec_{\gamma(A)})$, $\eta : \mathfrak{E} \to (N(X(\mathfrak{E})), \prec_{\eta(B)})$ are proximity isomorphisms, and that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{D} \arrow[r, "\beta"] \arrow[d, "\gamma"'] & \mathfrak{E} \arrow[d, "\eta"] \\
(N(X(\mathfrak{D})), \prec_{\gamma(A)}) \arrow[r, "N(X(\beta))"'] & (N(X(\mathfrak{E})), \prec_{\eta(B)})
\end{tikzcd}$$

This makes it possible when working with objects and morphisms in $\boldsymbol{pdv}$ to reduce to the case that the object is of the form $(N(X), \prec)$, where the reflexive elements of $\prec$ are in $C(X)$. This differs from the representation afforded by $\mathfrak{N} \circ X$, which always produces proximity Dedekind vector lattices with closed proximities, which we define next.

6.5. Definition. *Let $\mathfrak{D} = (D, \prec) \in \boldsymbol{pdv}$. We call $\prec$ a* closed proximity *provided the graph of $\prec$ is uniformly closed in the product topology on $D \times D$.*

6.6. Theorem. *Let $\mathfrak{D} = (D, \prec) \in \boldsymbol{pdv}$ and let $A = R(\mathfrak{D})$. Then $\prec$ is a closed proximity iff $A \in \boldsymbol{ubav}$.*

Proof. Suppose that $\prec$ is a closed proximity. If $\{a_n\}$ is a sequence of elements in A converging to $a \in D$, then since $a_n \prec a_n$ for all n, the fact that $\prec$ is closed implies that $a \prec a$, and hence $a \in A$. Therefore, A is uniformly closed in D. Since D is a complete metric space with respect to the uniform norm and A is uniformly closed in D, it follows that A is uniformly complete. Thus, $A \in \boldsymbol{ubav}$.

Conversely, suppose $A \in \boldsymbol{ubav}$. By the proof of Theorem 6.2(2), A is isomorphic to $C(X(\mathfrak{D}))$ and $\mathfrak{D}$ is isomorphic to $(N(X(\mathfrak{D})), \prec_{C(X(\mathfrak{D}))})$. We may thus identify A with $C(X(\mathfrak{D}))$ and D with $N(X(\mathfrak{D}))$. Let $\{(f_n, g_n)\}$ be a sequence in $D \times D$ such that $f_n \prec g_n$ for all n and $\{(f_n, g_n)\}$ converges to $(f, g) \in D \times D$. Then $f_n \to f$ and $g_n \to g$. Therefore, for $\varepsilon > 0$, there is a positive integer M such that $\|f_n - f\| < \varepsilon$ for each $n \geq M$. This implies $f_n - \varepsilon \leq f \leq f_n + \varepsilon$, and hence $f_n^* - \varepsilon \leq f^* \leq f_n^* + \varepsilon$. Thus, $\|f_n^* - f^*\| \leq \varepsilon$, which proves $f_n^* \to f^*$. Since $f_n \prec g_n$, there is $c_n \in A$ such that $f_n \leq c_n \leq g_n$. Because c_n is a continuous function, $f_n^* \leq c_n$, so $f_n^* \leq g_n$ for each n. Since the partial order on $N(X(\mathfrak{D}))$ is pointwise, if $x \in X$, then $f_n^*(x) \leq g_n(x)$ for each n. Because uniform convergence implies pointwise convergence, we get $f^*(x) \leq g(x)$ for each x, and so $f^* \leq g$. Applying the Katětov-Tong Theorem (see Remark 4.14) then yields $f \prec g$. Thus, $\prec$ is a closed proximity. ∎

6.7. Definition. *Let* ***cpdv*** *be the full subcategory of* ***pdv*** *whose objects are proximity Dedekind vector lattices with a closed proximity.*

6.8. Corollary. *The equivalence* ***bav*** $\to$ ***pdv*** *restricts to an equivalence* ***ubav*** $\to$ ***cpdv****.*

Proof. Apply Theorems 3.17 and 6.6. ∎

6.9. Corollary. *The image of the functor* $\mathfrak{N} : \mathbf{KHaus} \to$ ***pdv*** *is* ***cpdv****.*

Proof. For $X \in \mathbf{KHaus}$, since $R(\mathfrak{N}(X)) = C(X)$, which is uniformly complete, $\mathfrak{N}(X) \in$ ***cpdv*** by Theorem 6.6. ∎

6.10. Theorem. *The functors* X : ***pdv*** $\to$ **KHaus** *and* $\mathfrak{N}$: **KHaus** $\to$ ***pdv*** *yield a contravariant adjunction, which restricts to a dual equivalence between* ***cpdv*** *and* **KHaus**.

$$
\begin{tikzcd}
\boldsymbol{cpdv} \arrow[rr, hook] \arrow[dr, leftrightarrow] & & \boldsymbol{pdv} \arrow[dl, shift left, "X"] \\
& \mathbf{KHaus} \arrow[ur, shift left, "\mathfrak{N}"] &
\end{tikzcd}
$$

Proof. By Theorem 6.1(2) we have a natural isomorphism $1_{\mathbf{KHaus}} \to X \circ \mathfrak{N}$. By Theorem 6.2(3), there is a natural transformation $1_{\boldsymbol{pdv}} \to \mathfrak{N} \circ X$, which restricts to a natural isomorphism $1_{\boldsymbol{cpdv}} \to \mathfrak{N} \circ X$ by the proof of Theorem 6.6. Thus, $\mathfrak{N}$ and X yield a contravariant adjunction between ***pdv*** and **KHaus** which restricts to a dual equivalence between ***cpdv*** and **KHaus**. ∎

6.11. Definition.

(1) *Let* $\mathfrak{D} = (D, \prec) \in$ ***pdv****. We call the proximity* $\prec$ trivial *if* $\prec$ *is equal to* $\leq$*, and we call* $\mathfrak{D}$ trivial *if* $\prec$ *is trivial.*

(2) *Let* ***tpdv*** *be the full subcategory of* ***pdv*** *consisting of trivial objects of* ***pdv****.*

(3) *Let* **ED** *be the full subcategory of* **KHaus** *consisting of extremally disconnected objects of* **KHaus**.

If $\mathfrak{D} = (D, \leq) \in$ ***tpdv***, then $D = R(\mathfrak{D})$, so $R(\mathfrak{D})$ is uniformly complete by Remark 2.7, and hence $\mathfrak{D} \in$ ***cpdv*** by Theorem 6.6. Thus, ***tpdv*** is a full subcategory of ***cpdv***.

6.12. Corollary.

(1) *The equivalence* $\boldsymbol{bav} \to \boldsymbol{pdv}$ *restricts to an equivalence* $\boldsymbol{dbav} \to \boldsymbol{tpdv}$.

(2) *The functors* $\mathfrak{N}$ *and* X *yield a dual equivalence between* $\boldsymbol{tpdv}$ *and* $\mathbf{ED}$.

Proof. (1). If $A \in \boldsymbol{dbav}$, then $\mathfrak{D}(A) = A$ and $\prec_A$ is equal to $\le_A$, so $\mathfrak{D}(A) \in \boldsymbol{tpdv}$. If $\mathfrak{D} = (D, \le) \in \boldsymbol{tpdv}$, then $R(\mathfrak{D}) = D \in \boldsymbol{dbav}$. Now apply Theorem 3.17.

(2). Let $\mathfrak{D} = (D, \le) \in \boldsymbol{tpdv}$. Since $\boldsymbol{tpdv} \subseteq \boldsymbol{cpdv}$, Theorem 6.2(2) implies that $\mathfrak{D} \cong (\mathfrak{N}(X(\mathfrak{D})), \prec_{C(X(\mathfrak{D}))})$, so $\prec_{C(X(\mathfrak{D}))}$ is equal to $\le$. This yields that $C(X(\mathfrak{D})) = N(X(\mathfrak{D}))$, so $X(\mathfrak{D}) \in \mathbf{ED}$ by [14, Cor. to Thm. 3.2]. Conversely, if $X \in \mathbf{ED}$, then $C(X) = N(X)$ by [14, Cor. to Thm. 3.2], so $\prec_{C(X)}$ is the partial order $\le$ on $N(X)$. Therefore, $\mathfrak{N}(X) \in \boldsymbol{tpdv}$. Now apply Theorem 6.10. ∎

7. Yosida Representation and Kakutani-Krein Duality

In this section we apply our analysis of the functors $\mathfrak{N}, X, R$, and $\mathfrak{D}$ to show how the classical Yosida Representation and Kakutani-Krein Duality can be derived in our setting.

7.1. Definition. [27, §45] *Let* $A \in \boldsymbol{bav}$. *The* Yosida space *of* A *is the set* $Y(A)$ *of maximal* ℓ*-ideals of* A *equipped with the topology whose closed sets are the sets of the form*

$$\{M \in Y(A) : I \subseteq M \text{ is an } \ell\text{-ideal of } A\}.$$

Lemma 5.2(3) and Remark 5.4(1) show there is a bijection $X(\mathfrak{D}(A)) \to Y(A)$ given by $P \mapsto P \cap A$. That this is a homeomorphism is clear by Lemma 5.5(5). Thus, $X(\mathfrak{D}(A))$ and $Y(A)$ are homeomorphic. From this and Theorem 5.6 we deduce the classical fact that the Yosida space $Y(A)$ of each $A \in \boldsymbol{bav}$ is compact Hausdorff [40]. This observation suggests a functor from $\boldsymbol{bav}$ to $\mathbf{KHaus}$, which we define next. We clarify the relationship between this functor and the functor X in Lemma 7.3.

7.2. Notation.

(1) We denote by $Y : \boldsymbol{bav} \to \mathbf{KHaus}$ the contravariant functor that sends $A \in \boldsymbol{bav}$ to $Y(A)$ and a morphism $A \to B$ in $\boldsymbol{bav}$ to the induced continuous map $Y(B) \to Y(A)$.

(2) We denote by $C : \mathbf{KHaus} \to \boldsymbol{bav}$ the contravariant functor that sends $X \in \mathbf{KHaus}$ to $C(X)$ and a continuous map $X \to Y$ to the induced morphism $C(Y) \to C(X)$ in $\boldsymbol{bav}$.

The following diagram illustrates the functors we consider in this section.

$$\begin{tikzcd}
\boldsymbol{bav} \arrow[rr, shift left, "\mathfrak{D}"] \arrow[dr, shift right, "Y"'] & & \boldsymbol{pdv} \arrow[ll, shift left, "R"] \arrow[dl, shift left, "X"] \\
& \mathbf{KHaus} \arrow[ul, shift right, "C"'] \arrow[ur, shift left, "\mathfrak{N}"] &
\end{tikzcd}$$

Lemma 7.3 relates the functors Y and C to $\mathfrak{N}, X, \mathfrak{D}$, and R. Using this lemma and the fact from Theorem 3.17 that $\mathfrak{D}\circ R \cong 1_{\boldsymbol{pdv}}$ and $R\circ\mathfrak{D}\cong 1_{\boldsymbol{bav}}$, the reader can deduce additional relationships. However, note that in general the diagram does not commute, since $\mathfrak{D}$ is not isomorphic to $\mathfrak{N}\circ Y$ (the former has image ***pdv*** while the latter has image ***cpdv***).

7.3. Lemma. *$\mathfrak{N}\cong\mathfrak{D}\circ C$ and $X\cong Y\circ R$.*

Proof. First we claim that $\mathfrak{N}\cong\mathfrak{D}\circ C$. Suppose $X\in$ **KHaus**. By Proposition 4.7, $N(X)$ is the Dedekind completion of $C(X)$. Thus, $\mathfrak{N}(X)\cong(\mathfrak{D}\circ C)(X)$. If $\sigma: Y\to X$ is continuous, Lemma 4.15 implies that the following diagram commutes, so it follows that $\mathfrak{N}\cong\mathfrak{D}\circ C$.

$$\begin{tikzcd}
\mathfrak{N}(X) \arrow[r, "\mathfrak{N}(\sigma)"] \arrow[d] & \mathfrak{N}(Y) \arrow[d] \\
(\mathfrak{D}\circ C)(X) \arrow[r, "(\mathfrak{D}\circ C)(\sigma)"'] & (\mathfrak{D}\circ C)(Y)
\end{tikzcd}$$

Next we show that $X\cong Y\circ R$. For each $\mathfrak{D}\in\boldsymbol{pdv}$, let $\eta_{\mathfrak{D}}: X(\mathfrak{D})\to Y(R(\mathfrak{D}))$ be defined by $P\mapsto P\cap R(\mathfrak{D})$. As noted before Notation 7.2, $\eta_{\mathfrak{D}}$ is a homeomorphism. If $\beta:\mathfrak{D}\to\mathfrak{E}$ is a proximity morphism, let $A=R(\mathfrak{D})$, $B=R(\mathfrak{E})$, and $\alpha=R(\beta)$. We consider the diagram

$$\begin{tikzcd}
X(\mathfrak{E}) \arrow[r, "X(\beta)"] \arrow[d, "\eta_{\mathfrak{E}}"'] & X(\mathfrak{D}) \arrow[d, "\eta_{\mathfrak{D}}"] \\
Y(B) \arrow[r, "Y(\alpha)"'] & Y(A)
\end{tikzcd}$$

Let $P\in X(\mathfrak{E})$. Then $(Y(\alpha)\circ\eta_{\mathfrak{E}})(P)=Y(\alpha)(P\cap B)=\alpha^{-1}(P\cap B)$. On the other hand,

$$(\eta_{\mathfrak{D}}\circ X(\beta))(P)=\eta_{\mathfrak{D}}(\twoheaddownarrow\beta^{-1}(P))=(\twoheaddownarrow\beta^{-1}(P))\cap A=\beta^{-1}(P)\cap A=\alpha^{-1}(P\cap B).$$

Thus, the diagram commutes, and since $\eta_{\mathfrak{E}}$ and $\eta_{\mathfrak{D}}$ are homeomorphisms, we have $X\cong Y\circ R$. ■

7.4. Theorem.

(1) (Yosida Representation [40]) *If $A\in$ **bav**, then there is an injective homomorphism $\gamma_A: A\to C(Y(A))$ in **bav** such that the image of γ_A is uniformly dense in $C(Y(A))$. In fact, the functors C and Y yield a contravariant adjunction.*

(2) (Kakutani-Krein Duality [24, 26]) *The functors C and Y restrict to a duality between **ubav** and* **KHaus**.

(3) (Stone-Nakano Theorem [35, 36, 32]) *The functors C and Y restrict further to a duality between **dbav** and* **ED**.

PROOF. (1). By Theorem 6.2(3), there is a natural transformation $1_{\boldsymbol{pdv}} \to \mathfrak{N} \circ X$. By Theorem 3.17, R and $\mathfrak{D}$ form a category equivalence, so there is a natural transformation $1_{\boldsymbol{bav}} \cong R \circ \mathfrak{D} \cong R \circ 1_{\boldsymbol{pdv}} \circ \mathfrak{D} \to R \circ \mathfrak{N} \circ X \circ \mathfrak{D}$. By Lemma 7.3, $X \cong Y \circ R$. Therefore, there is a natural transformation $1_{\boldsymbol{bav}} \to R \circ \mathfrak{N} \circ (Y \circ R) \circ \mathfrak{D} \cong R \circ \mathfrak{N} \circ Y$. Since $R \circ \mathfrak{N} = C$, we obtain a natural transformation $\gamma : 1_{\boldsymbol{bav}} \to C \circ Y$. By the proof of Theorem 6.2(1), for each $A \in \boldsymbol{bav}$, the component map $\gamma_A : A \to C(Y(A))$ is injective and the image is uniformly dense in $C(Y(A))$. Moreover, by Theorem 6.1(2) and Lemma 7.3, $1_{\mathbf{KHaus}} \cong X \circ \mathfrak{N} \cong Y \circ R \circ \mathfrak{N} \cong Y \circ C$. It follows that Y and C form a contravariant adjunction.

(2). By Corollary 6.8, the functors R and $\mathfrak{D}$ restrict to an equivalence between ***ubav*** and ***cpdv***. By Theorem 6.10, the functors $\mathfrak{N}$ and X restrict to a dual equivalence between ***cpdv*** and **KHaus**. Therefore, $X \circ \mathfrak{D}$ and $R \circ \mathfrak{N}$ yield a dual equivalence between ***ubav*** and **KHaus**. By Lemma 7.3, $X \circ \mathfrak{D} \cong Y \circ R \circ \mathfrak{D} \cong Y$ and $R \circ \mathfrak{N} \cong R \circ \mathfrak{D} \circ C \cong C$. Thus, Y and C yield a dual equivalence between ***ubav*** and **KHaus**.

(3). By Corollary 6.12(1), the functors R and $\mathfrak{D}$ further restrict to an equivalence between ***dbav*** and ***tpdv***. By Corollary 6.12(2), the functors $\mathfrak{N}$ and X further restrict to a dual equivalence between ***tpdv*** and **ED**. Therefore, $X \circ \mathfrak{D}$ and $R \circ \mathfrak{N}$ yield a dual equivalence between ***dbav*** and **ED**. By the proof of (2), $X \circ \mathfrak{D} \cong Y$ and $R \circ \mathfrak{N} \cong C$. Thus, the restrictions of Y and C yield a dual equivalence between ***dbav*** and ***tpdv***. ∎

7.5. REMARK. In light of Theorem 7.4, we make explicit now an idea that underlies our approach in this article. The classical Yosida representation $A \to C(Y(A))$ of Theorem 7.4 can be viewed as a functorial representation of uniform completion via continuous functions. Instead of focusing on uniform completion and making the Yosida representation the primary tool, we have instead emphasized the Dedekind completion and what can be termed the *Dilworth representation*: $A \to N(Y(A))$. However, neither Dedekind completion nor the Dilworth representation extends to a functor from ***bav*** to ***dbav***. So we have adjusted the Dilworth representation to $A \to \mathfrak{N}(X(\mathfrak{D}(A)))$ and worked in ***pdv*** rather than ***dbav***. In this way, both the Dilworth representation and Dedekind completion become functorial. Corollary 6.8 and Theorem 7.4 are examples of how uniform completion and the Yosida representation can be encompassed by this approach.

We conclude this section by two more applications of our approach that connect skeletal maps with normal homomorphisms. For topological spaces X, Y we recall that a continuous map $\sigma : X \to Y$ is *skeletal* provided F nowhere dense in Y implies that $\sigma^{-1}(F)$ is nowhere dense in X. We also recall that σ is *quasi-open* if U nonempty open in X implies that $\operatorname{Int} f(U) \neq \varnothing$. It is well known that each skeletal map is quasi-open and that the two concepts coincide in **KHaus**.

For $A, B \in \boldsymbol{bav}$ we recall that a map $\alpha : A \to B$ is a *normal homomorphism* if it is a morphism in ***bav*** that preserves all existing joins (and hence all existing meets). The next theorem relates the two concepts.

7.6. THEOREM. *Let $A, B \in \boldsymbol{bav}$ and let $\alpha : A \to B$ be a morphism in* ***bav***. *The following are equivalent.*

(1) *α is a normal homomorphism.*

(2) $D(\alpha)$ *is a normal homomorphism.*

(3) $Y(\alpha) : Y(B) \to Y(A)$ *is skeletal.*

Proof. (1) $\Rightarrow$ (2). This follows from [2, Thm. 1].

(2) $\Rightarrow$ (3). Set $\sigma = Y(\alpha)$. By Theorem 7.4(1), we may assume that $A \subseteq C(Y(A))$, $B \subseteq C(Y(B))$, and $\alpha = C(\sigma)|_A$. We also identify $D(A) = N(Y(A))$, $D(B) = N(Y(B))$, and $D(\alpha) = N(\sigma)$. To show σ is skeletal, it is sufficient to show that if U is open dense in $Y(A)$, then $\sigma^{-1}(U)$ is dense in $Y(B)$. Since $Y(A)$ is compact Hausdorff, we may write $U = \bigcup_i V_i$ with each V_i regular open. Therefore, $\chi_{V_i} \in N(Y(A))$ (see Example 4.11). The pointwise join of the χ_{V_i} is χ_U. So $\bigvee_i \chi_{V_i} = (\chi_U)^\# = \chi_{\mathrm{Int}(\overline{U})} = 1$. Since $N(\sigma)$ is a normal homomorphism, $\bigvee_i N(\sigma)(\chi_{V_i}) = 1$. For each i we have $N(\sigma)(\chi_{V_i}) = (\chi_{V_i} \circ \sigma)^\# = \chi_{W_i}$, where $W_i = \mathrm{Int}\left(\overline{\sigma^{-1}(V_i)}\right)$. Thus, $\bigvee_i \chi_{W_i} = 1$. Let $W = \bigcup_i W_i$. Then $1 = \bigvee_i \chi_{W_i} = \chi_{\mathrm{Int}(\overline{W})}$. Therefore, W is dense in $Y(B)$ (see Example 4.11). We claim that this forces $\sigma^{-1}(U)$ to be dense in $Y(B)$. To see this, let V be open in $Y(B)$. Then there is i with $V \cap W_i \neq \varnothing$. Therefore, $V \cap \overline{\sigma^{-1}(V_i)} \neq \varnothing$. Since V is open, $V \cap \sigma^{-1}(V_i) \neq \varnothing$. Thus, $V \cap \sigma^{-1}(U) \neq \varnothing$, and so $\sigma^{-1}(U)$ is dense. Consequently, σ is a skeletal map.

(3) $\Rightarrow$ (1). Set $\sigma = Y(\alpha)$. By Theorem 7.4(1), we may assume that $A \subseteq C(Y(A))$, $B \subseteq C(Y(B))$, and $\alpha = C(\sigma)|_A$. So $\alpha(a) = a \circ \sigma$ for all $a \in A$. Suppose $a = \bigvee_i a_i$ in A. We claim that $\alpha(a) = \bigvee_i \alpha(a_i)$. Since $\bigwedge_i (a - a_i) = 0$, by setting $b_i = a - a_i$, we have $\bigwedge_i b_i = 0$. It is then sufficient to prove that $\bigwedge_i \alpha(b_i) = 0$. As α is order preserving and $\alpha(0) = 0$, we have that 0 is a lower bound for the $\alpha(b_i)$ in B. Suppose 0 is not the greatest lower bound for the $\alpha(b_i)$ in B. Then there is $c \in B$ such that $0 < c \leq \alpha(b_i)$ for all i. Since $Y(B)$ is compact Hausdorff and the open set $\mathrm{Coz}(c) = \{y \in Y(B) : c(y) \neq 0\}$ is nonempty, there is a nonempty regular open subset V of $Y(B)$ whose closure $\overline{V}$ is contained in $\mathrm{Coz}(c)$. Let $\lambda = \inf\{c(y) : y \in \overline{V}\}$. Since $\overline{V}$ is compact, $\lambda \in \{c(y) : y \in \overline{V}\}$, and so, since $\overline{V} \subseteq \mathrm{Coz}(c)$, we have $0 < \lambda$.

Let $U = \mathrm{Int}\left(\overline{\sigma(V)}\right)$. Since σ is skeletal, and hence quasi-open, U is nonempty. As U is regular open in $Y(A)$, we have $\lambda\chi_U \in N(Y(A))$. We claim $\lambda\chi_U \leq a_i$ for each i. Since $0 \leq a_i$, it suffices to show that $\lambda \leq a_i(u)$ for all $u \in U$. Let $u \in U$. Since $Y(A)$ and $Y(B)$ are compact Hausdorff, σ is a closed map, and so $U \subseteq \overline{\sigma(V)} \subseteq \sigma\left(\overline{V}\right)$. Therefore, there is $y \in \overline{V}$ such that $\sigma(y) = u$. The fact that $b \leq \alpha(a_i)$ implies

$$\lambda \leq b(y) \leq (\alpha(a_i))(y) = a_i(\sigma(y)) = a_i(u),$$

which shows that $\lambda\chi_U \leq a_i$. Thus, $0 < \lambda\chi_U \leq \bigwedge_i a_i = 0$, a contradiction which shows that 0 is the greatest lower bound of the $\alpha(a_i)$. Consequently, α is a normal homomorphism. ∎

The next corollary gives a different proof for compact Hausdorff spaces of a result due to Rump for the category of topological spaces with skeletal maps [34, Cor. 2, p. 167].

7.7. Corollary. *Let* $\mathbf{KHaus}_{sk}$ *denote the category of compact Hausdorff spaces and skeletal maps, and let* $\mathbf{ED}_{sk}$ *denote the full subcategory consisting of extremally disconnected spaces. Then* $\mathbf{ED}_{sk}$ *is a coreflective subcategory of* $\mathbf{KHaus}_{sk}$.

PROOF. By Theorems 7.4(1) and 7.6, there is a contravariant adjunction between $\mathbf{KHaus}_{sk}$ and the category $\boldsymbol{bav}_{nor}$ of bounded archimedean vector lattices and normal homomorphisms. By Theorem 7.4(3), the image under the functor $\mathbf{KHaus}_{sk} \to \boldsymbol{bav}_{nor}$ of $\mathbf{ED}_{sk}$ is the full subcategory $\boldsymbol{dbav}_{nor}$ of $\boldsymbol{bav}_{nor}$. Since $\boldsymbol{dbav}_{nor}$ is a reflective subcategory of $\boldsymbol{bav}_{nor}$ [2], we conclude that $\mathbf{ED}_{sk}$ is a coreflective subcategory of $\mathbf{KHaus}_{sk}$. ∎

8. Proximity Dedekind ℓ-algebras

The Yosida representation, along with the fact that for X a compact Hausdorff space $C(X)$ is not only a vector lattice but an ℓ-algebra, means that multiplication is present in a natural way on a number of the objects we consider. In this section we show how to incorporate multiplication into the categorical picture developed in the previous sections.

8.1. CONVENTION. We assume that all rings are commutative and unital.

8.2. DEFINITION.

(1) *A ring A with a partial order $\le$ is an* ℓ-ring *if A is an ℓ-group and $0 \le a, b$ implies $0 \le ab$.*

(2) *An ℓ-ring A is an* ℓ-algebra *if A is a vector lattice and an $\mathbb{R}$-algebra.*

(3) *An ℓ-ring A is* bounded *if 1 is a strong order unit.*

(4) *An ℓ-algebra A is a* Dedekind ℓ-algebra *if A is Dedekind complete.*

8.3. NOTATION. Denote by $\boldsymbol{ba\ell}$ the category of bounded archimedean ℓ-algebras and unital ℓ-algebra homomorphisms.

8.4. EXAMPLE. Let $X \in \mathbf{KHaus}$.

(1) $C(X) \in \boldsymbol{ba\ell}$, where multiplication is defined pointwise. In fact, $C(X)$ is a uniformly complete object of $\boldsymbol{ba\ell}$.

(2) $N(X) \in \boldsymbol{ba\ell}$. Since $N(X)$ is the Dedekind completion of the ℓ-algebra $C(X)$, we have that $N(X)$ is an ℓ-algebra (apply [32, Sec. 31] and [23, Lem. 1]; see [8, Thm. 3.1]) with respect to the binary operation $\cdot$ which is first defined for all $0 \le f, g \in N(X)$ by

$$f \cdot g = \bigvee \{ab : 0 \le a, b \in C(X), a \le f, b \le g\},$$

and then is extended to all $f, g \in N(X)$ by

$$f \cdot g = f^+ \cdot g^+ + f^- \cdot g^- - (f^+ \cdot g^- + f^- \cdot g^+).$$

(3) The multiplication $\cdot$ on $N(X)$ is in fact the normalization of the pointwise multiplication. To see this, let $f, g \in N(X)$ and let $\odot$ denote pointwise multiplication in $B(X)$. First suppose $f, g \geq 0$. We use sup for pointwise joins in $B(X)$. By Lemma 4.2(1), $f = \sup\{a \in C(X) : 0 \leq a \leq f\}$ and $g = \sup\{b \in C(X) : 0 \leq b \leq g\}$. A short calculation shows that $f \odot g = \sup\{ab : a, b \in C(X), 0 \leq a \leq f, 0 \leq b \leq g\}$. Therefore,

$$\begin{aligned} f \cdot g &= \bigvee\{ab : 0 \leq a, b \in C(X), a \leq f, b \leq g\} \\ &= (\sup\{ab : 0 \leq a, b \in C(X), a \leq f, b \leq g\})^{\#} \\ &= (f \odot g)^{\#}, \end{aligned}$$

which implies that for $0 \leq f, g \in N(X)$, we have $f \cdot g = (f \odot g)^{\#}$.

Now let $f, g \in N(X)$ be arbitrary. We write $\oplus$ for the pointwise addition in $B(X)$, and $+$ for the addition in $N(X)$ obtained by normalization of pointwise addition. There are $0 \leq \lambda, \mu \in \mathbb{R}$ with $0 \leq f + \lambda, g + \mu$. Since addition of a scalar is the same in both $B(X)$ and $N(X)$ (see Remark 4.8), we have

$$\begin{aligned} (f + \lambda) \cdot (g + \mu) &= ((f + \lambda) \odot (g + \mu))^{\#} = ((f \odot g) \oplus \lambda g \oplus \mu f \oplus \lambda\mu)^{\#} \\ &= (f \odot g)^{\#} + \lambda g + \mu f + \lambda\mu. \end{aligned}$$

On the other hand, $(f + \lambda) \cdot (g + \mu) = fg + \lambda g + \mu f + \lambda\mu$. Comparing this with the equation above, we get $fg = (f \odot g)^{\#}$.

By forgetting multiplication and designating 1 as the strong order unit, we identify $\boldsymbol{ba\ell}$ as a subcategory of $\boldsymbol{bav}$. It can be deduced from [22, Thm. 5.4] that $\boldsymbol{ba\ell}$ is a full subcategory of $\boldsymbol{bav}$. In our context, this also follows easily from the Yosida representation.

8.5. Proposition. *We have* $\boldsymbol{ubav} \subseteq \boldsymbol{ba\ell} \subseteq \boldsymbol{bav}$, *where* $\subseteq$ *represents that the first category is a full subcategory of the second.*

Proof. Let $A \in \boldsymbol{ubav}$. By Theorem 7.4(2), the embedding $\gamma_A : A \to C(Y(A))$ is an isomorphism. Since $C(Y(A)) \in \boldsymbol{ba\ell}$, this isomorphism allows us to define a multiplication on A so that $A \in \boldsymbol{ba\ell}$. To complete the proof it suffices to show that every vector lattice homomorphism between objects in $\boldsymbol{ba\ell}$ is an ℓ-algebra homomorphism. Let $\alpha : A \to B$ be a vector lattice homomorphism with $A, B \in \boldsymbol{ba\ell}$. As noted in the proof of Theorem 7.4(1), there is a natural transformation $\gamma : 1_{\boldsymbol{bav}} \to C \circ Y$. Thus, we have the following commutative diagram in $\boldsymbol{bav}$:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma_A"'] & B \arrow[d, "\gamma_B"] \\ C(Y(A)) \arrow[r, "C(\alpha)"'] & C(Y(B)) \end{tikzcd}$$

Since A and B are ℓ-algebras, the vector lattice homomorphisms γ_A and γ_B are easily seen to be ℓ-algebra homomorphisms. Because $C(\alpha)$ is a morphism in $\boldsymbol{ba\ell}$, commutativity of this diagram implies that α preserves multiplication, and so α is a morphism in $\boldsymbol{ba\ell}$. ∎

8.6. Corollary. *Let $A \in \boldsymbol{bav}$. If $A \in \boldsymbol{ba\ell}$, then there is a unique multiplication on A such that $A \in \boldsymbol{ba\ell}$.*

Proof. Suppose $\cdot$ and $\odot$ are two multiplications on A, each of which make $A \in \boldsymbol{ba\ell}$. Let $1_A : A \to A$ denote the identity map. Then 1_A is a vector lattice isomorphism. Since $(A,+,\cdot),(A,+,\odot) \in \boldsymbol{ba\ell}$, Proposition 8.5 implies that 1_A is an isomorphism of ℓ-algebras. Thus, $\cdot$ and $\odot$ define the same binary operation on A. ∎

In [7] the full subcategory of uniformly complete objects in $\boldsymbol{ba\ell}$ is denoted $\boldsymbol{uba\ell}$. By Proposition 8.5, $\boldsymbol{uba\ell} = \boldsymbol{ubav}$. Thus, Theorem 7.4 and Proposition 8.5 yield Stone-Gelfand-Naimark Duality [35, 16] (for a history of this theorem, see for example [7]):

8.7. Corollary. (Stone-Gelfand-Naimark Duality) *The functors C and Y induce a contravariant adjunction between $\boldsymbol{ba\ell}$ and* **KHaus**. *This adjunction restricts to a dual equivalence between $\boldsymbol{uba\ell}$ and* **KHaus**.

By Theorem 3.17, $\mathfrak{D} : \boldsymbol{bav} \to \boldsymbol{pdv}$ is an equivalence. We describe next the image under $\mathfrak{D}$ of $\boldsymbol{ba\ell}$ in $\boldsymbol{pdv}$. To distinguish the proximities we defined earlier and those below, we will refer to a proximity on a Dedekind vector lattice as a vector lattice proximity. We note that if $D \in \boldsymbol{dbav}$, then $D \in \boldsymbol{ubav} = \boldsymbol{uba\ell}$, and so D is a Dedekind ℓ-algebra. We let $\boldsymbol{dba\ell}$ be the full subcategory of $\boldsymbol{uba\ell}$ consisting of Dedekind ℓ-algebras.

8.8. Definition. *Let $D \in \boldsymbol{dbav}$ and $\prec$ be a vector lattice proximity on D.*

(1) *We call $\prec$ an* ℓ-algebra proximity *if, in addition to* (P1)–(P9), *$\prec$ also satisfies*

(P10) *$a,b,c,d \geq 0$ with $a \prec b$ and $c \prec d$ imply $ac \prec bd$.*

(2) *We call a pair $\mathfrak{D} = (D,\prec)$ a* proximity Dedekind ℓ-algebra *if D is a Dedekind ℓ-algebra and $\prec$ is a reflexive ℓ-algebra proximity on D.*

(3) *Suppose $\mathfrak{D}$ and $\mathfrak{E}$ are proximity Dedekind ℓ-algebras and $\alpha : \mathfrak{D} \to \mathfrak{E}$ is a proximity vector lattice morphism. We call α a* proximity ℓ-algebra morphism *if in addition to* (M1)–(M7) *we also have*

(M8) *$\alpha(ca) = \alpha(c)\alpha(a)$ for all $a \in D$ and $0 \leq c \in R(\mathfrak{D})$.*

8.9. Remark. To show that $\alpha : \mathfrak{D} \to \mathfrak{E}$ is a proximity ℓ-algebra morphism, it is sufficient to verify (M8) only for $a \geq 0$ in $\mathfrak{D}$. For, suppose that $\alpha(ca) = \alpha(c)\alpha(a)$ for all $0 \leq c \in R(\mathfrak{D})$ and $0 \leq a \in D$. Let $b \in D$. Since D is bounded, there is $\lambda \in \mathbb{R}$ with $b + \lambda \geq 0$. Therefore, $\alpha(c)\alpha(b+\lambda) = \alpha(c(b+\lambda)) = \alpha(cb + \lambda c)$. By (M1), (M6), and (M7) this simplifies to $\alpha(c)(\alpha(b)+\lambda) = \alpha(cb) + \lambda\alpha(c)$. Thus, $\alpha(c)\alpha(b) = \alpha(cb)$, and so (M8) holds.

8.10. Lemma.

(1) *Let $\mathfrak{D} \in \boldsymbol{pdv}$. Then $\mathfrak{D}$ is a proximity Dedekind ℓ-algebra iff $R(\mathfrak{D}) \in \boldsymbol{ba\ell}$.*

(2) *Let $A \in \boldsymbol{bav}$. Then $\mathfrak{D}(A) = (D(A), \prec_A) \in \boldsymbol{pd\ell}$ iff $A \in \boldsymbol{ba\ell}$.*

PROOF. (1). Suppose $\mathfrak{D} = (D, \prec)$ is a proximity Dedekind ℓ-algebra. By Lemma 3.5, $R(\mathfrak{D}) \in \boldsymbol{bav}$, so it suffices to show that it is closed under multiplication. If $a, b \in R(\mathfrak{D})$ with $0 \le a, b$, then $a \prec a$ and $b \prec b$ imply that $ab \prec ab$ by (P10), so $ab \in R(\mathfrak{D})$. Now, for arbitrary $a, b \in R(\mathfrak{D})$, there are scalars λ, μ with $a + \lambda, b + \mu \ge 0$. By the previous case, $(a+\lambda)(b+\mu) \in R(\mathfrak{D})$. Therefore, $ab + (\mu a + \lambda b + \lambda\mu) \in R(\mathfrak{D})$. Since $\mu a + \lambda b + \lambda\mu \in R(\mathfrak{D})$, this implies $ab \in R(\mathfrak{D})$. Thus, $R(\mathfrak{D}) \in \boldsymbol{ba\ell}$.

Conversely, if $R(\mathfrak{D}) \in \boldsymbol{ba\ell}$ and $a, b, c, d \in D$ with $a, b, c, d \ge 0$, $a \prec b$, and $c \prec d$, then there exist $r, s \in R(\mathfrak{D})$ with $a \le r \le b$ and $c \le s \le d$. Therefore, since all elements involved are in the positive cone of D, we have $ac \le rs \le bd$. As $R(\mathfrak{D})$ is a ring, $rs \in R(\mathfrak{D})$, and so $ac \prec bd$. This verifies (P10), and hence $\mathfrak{D}$ is a proximity Dedekind ℓ-algebra.

(2). This follows from (1) since $A = R(\mathfrak{D}(A))$. ∎

8.11. EXAMPLE. If $X \in \mathbf{KHaus}$, then $\mathfrak{D} = (N(X), \prec_{C(X)})$ is a proximity Dedekind ℓ-algebra. To see this, by Example 8.4(2), $N(X)$ is a Dedekind ℓ-algebra. By Lemma 4.13, $\mathfrak{D} \in \boldsymbol{pdv}$. Since $R(\mathfrak{D}) = C(X)$, Lemma 8.10(1) implies that $\mathfrak{D}$ is a proximity Dedekind ℓ-algebra.

8.12. THEOREM. *A proximity vector lattice morphism between proximity Dedekind ℓ-algebras is a proximity ℓ-algebra morphism. Thus, the proximity Dedekind ℓ-algebras with proximity ℓ-algebra morphisms form a full subcategory of* $\boldsymbol{pdv}$.

PROOF. Let $\mathfrak{D}, \mathfrak{E}$ be proximity Dedekind ℓ-algebras, and let $\beta : \mathfrak{D} \to \mathfrak{E}$ be a proximity vector lattice morphism. It is sufficient to show that β satisfies (M8). Let $A = R(\mathfrak{D})$, $B = R(\mathfrak{E})$, $X = X(\mathfrak{D})$, and $Y = X(\mathfrak{E})$. By Lemma 8.10(1), $A, B \in \boldsymbol{ba\ell}$. Therefore, by Proposition 8.5, $R(\beta) : A \to B$ is a morphism in $\boldsymbol{ba\ell}$. By Remark 6.4, we may assume without loss of generality that $\mathfrak{D} = (N(X), \prec_A)$ and $\mathfrak{E} = (N(Y), \prec_B)$, together with $A \subseteq C(X)$ and $B \subseteq C(Y)$. Let $0 \le a \in A$ and $f \in N(X)$. By Remark 8.9, we may assume $f \ge 0$. Since β is a proximity vector lattice morphism, we have $\beta(f) = \bigvee\{\beta(b) : b \in A, 0 \le b \le f\}$. As $A \subseteq C(X)$ and β is order preserving, $\beta(f) = \bigvee\{\beta(b) : 0 \le b \le f, b \in C(X)\}$. Applying the functor $R \circ \mathfrak{N} \circ X$ to β, we see that β restricts to a vector lattice homomorphism from $C(X)$ to $C(Y)$.

First suppose that $a \ge 1$. Then $a^{-1} \in C(X)$. Therefore, if $c \in C(X)$, then $c \le af$ iff $a^{-1}c \le f$. Thus,

$$\begin{aligned}
\beta(af) &= \bigvee\{\beta(c) : 0 \le c \in C(X), c \le af\} \\
&= \bigvee\{\beta(c) : 0 \le c \in C(X), a^{-1}c \le f\} \\
&= \bigvee\{\beta(a)\beta(d) : d \in C(X), 0 \le d \le f\} \\
&= \beta(a)\left(\bigvee\{\beta(d) : 0 \le d \le f\}\right) \\
&= \beta(a)\beta(f),
\end{aligned}$$

where the third equality follows from the fact noted above that the restriction of β to $C(X)$ is a morphism in $\boldsymbol{ba\ell}$, and the fourth equality holds by [23, Lem. 1] since $\beta(a)$ is positive.

Next, let $a \geq 0$. By the previous paragraph,

$$\beta((1+a)f) = \beta(1+a)\beta(f) = (1+\beta(a))\beta(f) = \beta(f) + \beta(a)\beta(f).$$

On the other hand, $\beta((1+a)f) = \beta(f+af)$. Therefore,

$$\beta(f+af) = \beta(f) + \beta(a)\beta(f).$$

As noted in the proof of (M5) in Lemma 3.9, $\beta(f) + \beta(g) \leq \beta(f+g)$ for all $g \in N(X)$. In particular, $\beta(f) + \beta(af) \leq \beta(f+af) = \beta(f) + \beta(a)\beta(f)$, so $\beta(af) \leq \beta(a)\beta(f)$. For the reverse inequality, we have

$$\begin{aligned}\beta(a)\beta(f) &= \beta(a)\left(\bigvee\{\beta(b) : b \in A, 0 \leq b \leq f\}\right) = \bigvee\{\beta(ab) : 0 \leq b \in A, b \leq f\}\\ &\leq \bigvee\{\beta(c) : c \in A, 0 \leq c \leq af\} = \beta(af).\end{aligned}$$

Thus, $\beta(af) = \beta(a)\beta(f)$. This shows that (M8) holds for β, completing the proof. ∎

From Theorem 8.12 we obtain as an analogue of Theorem 3.10 a characterization of proximity ℓ-algebra morphisms in terms of how they lift reflexive elements.

8.13. Corollary. *Suppose $\mathfrak{D} = (D, \prec), \mathfrak{E} = (E, \prec)$ are proximity Dedekind ℓ-algebras and $\beta : D \to E$ is a map such that $\beta(R(\mathfrak{D})) \subseteq R(\mathfrak{E})$. Set $\alpha = \beta|_{R(\mathfrak{D})}$. Then the following are equivalent.*

(1) *β is a proximity ℓ-algebra morphism.*

(2) *α is a vector lattice homomorphism and $\beta(x) = \bigvee\{\alpha(a) : a \in R(\mathfrak{D}), a \leq x\}$.*

(3) *α is an ℓ-algebra homomorphism and $\beta(x) = \bigvee\{\alpha(a) : a \in R(\mathfrak{D}), a \leq x\}$.*

Proof. Apply Theorems 3.10 and 8.12. ∎

8.14. Corollary. *Suppose $A, B \in$ **bal** and $\alpha : A \to B$ is an ℓ-algebra homomorphism. Then a map $\beta : D(A) \to D(B)$ is a proximity ℓ-algebra morphism extending α iff $\beta(x) = \bigvee\{\alpha(a) : a \leq x\}$ for all $x \in D(A)$.*

8.15. Definition. *We denote by **pdl** the category of proximity Dedekind ℓ-algebras with proximity ℓ-algebra morphisms. Let **cpdl** be the full subcategory of **pdl** whose objects are proximity Dedekind ℓ-algebras with a closed proximity. Let **tpdl** be the full subcategory of **cpdl** whose objects are proximity Dedekind ℓ-algebras with a trivial proximity.*

8.16. Corollary. *The restrictions of the functors $\mathfrak{D}, R$ yield an equivalence **bal** $\to$ **pdl**, which restricts to an equivalence **ubal** $\to$ **cpdl**, which further restricts to an equivalence **dbal** $\to$ **tpdl**.*

Proof. By Lemma 8.10 and Corollary 8.13, R restricts to a functor from **pdl** to **bal** and $\mathfrak{D}$ restricts to a functor from **bal** to **pdl**. Thus, the first assertion of the corollary follows from Theorem 3.17, the second from Corollary 6.8, and the third from Corollary 6.12(1). ∎

8.17. Remark. Returning to the discussion at the end of Section 2, we clarify further why for $A \in \boldsymbol{bav}$ the inclusion mapping $\iota_A : A \to D(A)$ need not be an epimorphism in $\boldsymbol{bav}$. By Proposition 8.5, $\boldsymbol{ba\ell}$ is a full subcategory of $\boldsymbol{bav}$, so it suffices to consider when the inclusion mapping $\iota_A : A \to D(A)$ is an epimorphism in $\boldsymbol{ba\ell}$. By [7, Prop. 3.3], ι_A is an epimorphism iff A is uniformly dense in $D(A)$. Thus, choosing X to be the interval $[0,1]$ and $A = C(X)$, we have that A is uniformly complete, and so A is uniformly closed in $D(A) = N(X)$. Since $C(X) \subsetneq N(X)$ (as the characteristic function of every proper nonempty regular open set is in $N(X)$ but not in $C(X)$), it follows that A is not uniformly dense in $D(A)$, and hence ι_A is not an epimorphism.

The table below lists all of the categories we have considered in this paper.

Category	Description
$\boldsymbol{bav}$	bounded archimedean vector lattices
$\boldsymbol{ubav}$	uniformly complete objects in $\boldsymbol{bav}$
$\boldsymbol{dbav}$	Dedekind complete objects in $\boldsymbol{bav}$
$\boldsymbol{pdv}$	proximity Dedekind vector lattices
$\boldsymbol{cpdv}$	objects in $\boldsymbol{pdv}$ with a closed proximity
$\boldsymbol{tpdv}$	objects in $\boldsymbol{pdv}$ with a trivial proximity
$\boldsymbol{ba\ell}$	bounded archimedean ℓ-algebras
$\boldsymbol{uba\ell}$	uniformly completely objects in $\boldsymbol{ba\ell}$
$\boldsymbol{dba\ell}$	Dedekind complete objects in $\boldsymbol{ba\ell}$
$\boldsymbol{pd\ell}$	proximity Dedekind ℓ-algebras
$\boldsymbol{cpd\ell}$	objects in $\boldsymbol{pd\ell}$ with a closed proximity
$\boldsymbol{tpd\ell}$	objects in $\boldsymbol{pd\ell}$ with a trivial proximity
$\boldsymbol{bav}_{nor}$	objects in $\boldsymbol{bav}$ with normal homomorphisms
$\boldsymbol{dbav}_{nor}$	objects in $\boldsymbol{dbav}$ with normal homomorphisms
KHaus	compact Hausdorff spaces
ED	extremally disconnected objects in **KHaus**
$\mathbf{KHaus}_{sk}$	objects in **KHaus** with skeletal maps
$\mathbf{ED}_{sk}$	objects in **ED** with skeletal maps

The following picture summarizes the connections between some of the categories we have considered in the paper.

$$
\begin{tikzcd}
& & & & \boldsymbol{bav} \arrow[rrr, leftrightarrow] & & & \boldsymbol{pdv} \\
& \mathbf{KHaus} \arrow[rr, leftrightarrow] & & \boldsymbol{ubav} \arrow[ur, hook] \arrow[rrr, leftrightarrow] \arrow[ddd, equal] & & & \boldsymbol{cpdv} \arrow[ur, hook] \arrow[ddd, equal] & \\
\mathbf{ED} \arrow[ur, hook] \arrow[rr, leftrightarrow] & & \boldsymbol{dbav} \arrow[ur, hook] \arrow[rrr, leftrightarrow] \arrow[ddd, equal] & & & \boldsymbol{tpdv} \arrow[ur, hook] \arrow[ddd, equal] & & \\
& & & & \boldsymbol{ba\ell} \arrow[uuu, hook] \arrow[rrr, leftrightarrow] & & & \boldsymbol{pd\ell} \arrow[uuu, hook] \\
& & & \boldsymbol{uba\ell} \arrow[ur, hook] \arrow[rrr, leftrightarrow] & & & \boldsymbol{cpd\ell} \arrow[ur, hook] & \\
& & \boldsymbol{dba\ell} \arrow[ur, hook] \arrow[rrr, leftrightarrow] & & & \boldsymbol{tpd\ell} \arrow[ur, hook] & &
\end{tikzcd}
$$

8.18. Remark. In [7], we studied several reflectors and coreflectors in the category $\boldsymbol{ba\ell}$. In doing so, we sought to individuate the full subcategory $\boldsymbol{uba\ell}$ in $\boldsymbol{ba\ell}$ by its categorical properties. For example, it turns out that $\boldsymbol{uba\ell}$ is the smallest reflective full subcategory of $\boldsymbol{ba\ell}$. These considerations were motivated by the goal of unraveling the algebraic, topological, and analytic properties that all play a role in the uniform completion functor in $\boldsymbol{ba\ell}$. In the sequel [8], we considered categorical properties of $\boldsymbol{dba\ell}$, and showed, for example, that the objects in $\boldsymbol{dba\ell}$ are the injective objects in the category $\boldsymbol{ba\ell}$, thus finding a categorical characterization of $\boldsymbol{dba\ell}$ inside $\boldsymbol{ba\ell}$. The present paper continues this sequence of ideas by finding a functorial description of Dedekind completion to accompany the well-known description of the uniform completion functor. As with the classical case of uniform completion, we have given a corresponding representation and duality theory for Dedekind completions.

In [7] we also considered an important mono-coreflective full subcategory of $\boldsymbol{ba\ell}$ consisting of the $\mathbb{R}$-algebras generated by their idempotents. These algebras, which we termed Specker algebras, have a purely algebraic description and play a role, for those algebras in $\boldsymbol{ba\ell}$ whose Yosida space is a Stone space, that is roughly analogous to that played by piecewise polynomials in $C([0,1])$. In [6], we use Specker algebras to give an algebraic foundation for issues involving both uniform and Dedekind completions inside $\boldsymbol{ba\ell}$. In doing so we bring the Boolean algebras of idempotents of Dedekind ℓ-algebras to the forefront. When equipped with an appropriate proximity, the Boolean algebra of idempotents becomes a de Vries algebra. Extending this proximity to the Specker algebra generated by these idempotents involves working with finitely-valued normal functions on the dual compact Hausdorff space. Taking limits then yields the algebra of all normal functions equipped with a proximity as in the sense of the present paper. The resulting proximity Dedekind ℓ-algebra thus gives a unified setting for de Vries duality and Katětov-like insertion theorems. Finitely-valued normal functions were studied in depth in [5], and in [6] we draw on that paper and the present article to help complete our picture of a categorical and algebraic features of uniform and Dedekind completions in $\boldsymbol{ba\ell}$.

References

[1] J. Adámek, H. Herrlich, J. Rosický, and W. Tholen, *Injective hulls are not natural*, Algebra Universalis **48** (2002), no. 4, 379–388.

[2] C. D. Aliprantis and E. Langford, *Order completions of Archimedean Riesz spaces and l-groups*, Algebra Universalis **19** (1984), no. 2, 151–159.

[3] B. Banaschewski, *Compactification of frames*, Math. Nachr. **149** (1990), 105–115.

[4] G. Bezhanishvili and J. Harding, *Proximity frames and regularization*, Appl. Categ. Structures **22** (2014), no. 1, 43–78.

[5] G. Bezhanishvili, V. Marra, P. J. Morandi, and B. Olberding, *De Vries powers: a generalization of Boolean powers for compact Hausdorff spaces*, J. Pure Appl. Algebra **219** (2015), no. 9, 3958–3991.

[6] G. Bezhanishvili, P. J. Morandi, and B. Olberding, *Specker algebras: an algebraic approach to uniform completions*, in preparation.

[7] ———, *Bounded Archimedean ℓ-algebras and Gelfand-Neumark-Stone duality*, Theory Appl. Categ. **28** (2013), No. 16, 435–475.

[8] ———, *Dedekind completions of bounded Archimedean ℓ-algebras*, J. Algebra Appl. **12** (2013), no. 1, 1250139, 16 pages.

[9] G. Birkhoff, *Lattice theory*, third ed., American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, Providence, R.I., 1979.

[10] P. Conrad, *The essential closure of an Archimedean lattice-ordered group*, Duke Math. J. **38** (1971), 151–160.

[11] P. Conrad and D. MacAlister, *The completion of a lattice-ordered group*, J. Aus. Math. Soc. **9** (1969), 182–208.

[12] N. Dăneţ, *Riesz spaces of normal semicontinuous functions*, Mediterr. J. Math. **12** (2015), no. 4, 1345–1355.

[13] H. de Vries, *Compact spaces and compactifications. An algebraic approach*, Ph.D. thesis, University of Amsterdam, 1962.

[14] R. P. Dilworth, *The normal completion of the lattice of continuous functions*, Trans. Amer. Math. Soc. **68** (1950), 427–438.

[15] N. J. Fine, L. Gillman, and J. Lambek, *Rings of quotients of rings of functions*, McGill Univ. Press, Montreal, Que., 1966.

[16] I. Gelfand and M. Naimark, *On the imbedding of normed rings into the ring of operators in Hilbert space*, Rec. Math. [Mat. Sbornik] N.S. **12(54)** (1943), 197–213.

[17] G. Gierz, *The normal completion of the lattice of continuous functions*, The Dilworth theorems, Contemp. Mathematicians, Birkhäuser Boston, Boston, MA, 1990, pp. 445–449.

[18] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous lattices and domains*, Cambridge University Press, Cambridge, 2003.

[19] J. Gutiérrez García and J. Picado, *On the parallel between normality and extremal disconnectedness*, J. Pure Appl. Algebra **218** (2014), no. 5, 784–803.

[20] K. Hardy, *On normal functions*, General Topology and Appl. **2** (1972), 157–163.

[21] M. Henriksen and D. G. Johnson, *On the structure of a class of Archimedean lattice-ordered algebras*, Fund. Math. **50** (1961/1962), 73–94.

[22] C. B. Huijsmans and B. de Pagter, *Subalgebras and Riesz subspaces of an f-algebra*, Proc. London Math. Soc. (3) **48** (1984), no. 1, 161–174.

[23] D. G. Johnson, *The completion of an archimedean f-ring*, J. London Math. Soc. **40** (1965), 493–496.

[24] S. Kakutani, *Weak topology, bicompact set and the principle of duality*, Proc. Imp. Acad. Tokyo **16** (1940), 63–67.

[25] M. Katětov, *On real-valued functions in topological spaces*, Fund. Math. **38** (1951), 85–91.

[26] M. Krein and S. Krein, *On an inner characteristic of the set of all continuous functions defined on a bicompact Hausdorff space*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **27** (1940), 427–430.

[27] W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces. Vol. I*, North-Holland Publishing Co., Amsterdam, 1971, North-Holland Mathematical Library.

[28] S. MacLane, *Categories for the working mathematician*, Springer-Verlag, New York, 1971, Graduate Texts in Mathematics, Vol. 5.

[29] I. Mozo Carollo, J. Gutiérrez García, and J. Picado, *On the Dedekind completion of function rings*, Forum Math. **27** (2015), no. 5, 2551–2585.

[30] ______, *Normal semicontinuity and the Dedekind completion of pointfree function rings*, Algebra Universalis **75** (2016), no. 3, 301–330.

[31] S. A. Naimpally and B. D. Warrack, *Proximity spaces*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 59, Cambridge University Press, London-New York, 1970.

[32] H. Nakano, *Über das System aller stetigen Funktionen auf einem topologischen Raum*, Proc. Imp. Acad. Tokyo **17** (1941), 308–310.

[33] ______, *Modern Spectral Theory*, Maruzen Co. Ltd., Tokyo, 1950.

[34] W. Rump, *The absolute of a topological space and its application to abelian l-groups*, Appl. Categ. Structures **17** (2009), no. 2, 153–174.

[35] M. H. Stone, *A general theory of spectra. I*, Proc. Nat. Acad. Sci. U. S. A. **26** (1940), 280–283.

[36] ______, *Boundedness properties in function-lattices*, Canadian J. Math. **1** (1949), 176–186.

[37] H. Tong, *Some characterizations of normal and perfectly normal spaces*, Duke Math. J. **19** (1952), 289–292.

[38] M. A. Toumi, *When lattice homomorphisms of Archimedean vector lattices are Riesz homomorphisms*, J. Aust. Math. Soc. **87** (2009), no. 2, 263–273.

[39] M. B. J. G. van Haandel and A. C. M. van Rooij, *The Dedekind σ-completion of a Riesz space*, Indag. Math. (N.S.) **17** (2006), no. 2, 231–242.

[40] K. Yosida, *On vector lattice with a unit*, Proc. Imp. Acad. Tokyo **17** (1941), 121–124.

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