Theory and Applications of Categories, Vol. 31, No. 16, 2016, pp. 444–461.

RELATIVE INTERNAL ACTIONS

JAMES RICHARD ANDREW GRAY AND TAMAR JANELIDZE-GRAY

ABSTRACT. For a relative exact homological category $(\mathbf{C}, \mathbf{E})$, we define *relative points* over an arbitrary object in $\mathbf{C}$, and show that they form an exact homological category. In particular, it follows that the full subcategory of nilpotent objects in an exact homological category is an exact homological category. These nilpotent objects are defined with respect to a Birkhoff subcategory in $\mathbf{C}$ as defined by T. Everaert and T. Van der Linden [9]. In addition, we introduce *relative internal actions* and show that, just as in the classical case, there is an equivalence of categories between the category of relative points over an object and the category of relative internal actions for the same object.

1. Introduction

It is well known that every surjection of groups $p : E \to B$ has its domain bijective (as a set) to the product $\mathbf{Ker}(p) \times B$. This fact eventually leads to the fact that every split extension p together with a chosen splitting s determines and is determined up to isomorphism (via a semi-direct product) by an action of B on $\mathbf{Ker}(p)$. D. Bourn and G. Janelidze showed in [5] how this fact can be understood categorically for groups and more generally for a semi-abelian category [14]. In order to understand this, let us first recall some terminology. For each object B in $\mathbf{C}$, following D. Bourn [4], we denote by $\mathbf{Pt}_B(\mathbf{C})$ the category defined as follows: the objects are triples (A, α, β), where A is an object in $\mathbf{C}$, and $\alpha : A \to B$ and $\beta : B \to A$ are morphisms in $\mathbf{C}$ such that $\alpha\beta = 1_B$; a morphism $f : (A, \alpha, \beta) \to (A', \alpha', \beta')$ in $\mathbf{Pt}_B(\mathbf{C})$ is a morphism $f : A \to A'$ in $\mathbf{C}$ such that $f\beta = \beta'$ and $\alpha' f = \alpha$. When $\mathbf{C}$ has pullbacks of split epimorphisms along arbitrary morphisms, each morphism $p : E \to B$ determines a functor $\mathbf{Pt}_B(\mathbf{C}) \to \mathbf{Pt}_E(\mathbf{C})$ which is usually denoted p^* and is defined on objects by $p^*(A, \alpha, \beta) = (E \times_B A, \pi_1, \langle 1, \beta p\rangle)$ where $(E \times_A A, \pi_1, \pi_2)$ is a pullback of α and p. In [5] D. Bourn and G. Janelidze called a category with pullbacks of split epimorphisms along arbitrary morphisms a category with semi-direct products when for each p in $\mathbf{C}$ the pullback functor $p^* : \mathbf{Pt}_B(\mathbf{C}) \to \mathbf{Pt}_E(\mathbf{C})$ is monadic. Theorem 3.4 (b) of [5] implies that every semi-abelian category has semi-direct products. In particular this means that for each B in the category of groups the pullback

The second author's research was partially supported by the University of South Africa postdoctoral fellowship and by the Shota Rustaveli National Science Foundation Grant DI/18/5-113/13.

Received by the editors 2014-12-11 and, in final form, 2016-05-24.

Transmitted by Stephen Lack. Published on 2016-05-30.

2010 Mathematics Subject Classification: 18G50, 18G25, 18A20, 18A05, 18A25, 18A32, 18C20.

Key words and phrases: relative semi-abelian category, relative homological category, semi-abelian category, homological category, category of relative points, relative internal actions.

functor along $0 \to B$ (usually called a kernel functor) is monadic and the algebras over this monad turn out to be equivalent to group actions. The monads obtained from kernel functors have been studied further (see e.g. [2] [3]).

Relative homological / semi-abelian categories were introduced by the second author in [16, 17] as a generalization of homological [1] / semi-abelian categories. A relative homological / semi-abelian category consists of a pair $(\mathbf{C}, \mathbf{E})$ where $\mathbf{C}$ is a category and $\mathbf{E}$ is a chosen class of regular epimorphisms in $\mathbf{C}$ with conditions imposed on both $\mathbf{C}$ and $\mathbf{E}$. These conditions are such that:

(a) if $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$, then $\mathbf{C}$ is a homological / semi-abelian category;

(b) if $\mathbf{E}$ is the class of all isomorphisms in $\mathbf{C}$, then these conditions reduce to requiring that $\mathbf{C}$ be finitely complete and has certain colimits.

As explained in [16] the idea of replacing a category with a pair consisting of a category and a chosen class of regular epimorphisms goes back to N. Yoneda [24], whose notion of a quasi-abelian category generalizes the notion of an abelian category. A quasi-abelian category can be defined as pair $(\mathbf{C}, \mathbf{E})$, where $\mathbf{C}$ is an additive category and $\mathbf{E}$ is a class of regular epimorphisms satisfying certain properties. These conditions are such that if $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$, then $\mathbf{C}$ is an abelian category.

In Section 2 for a pair $(\mathbf{C}, \mathbf{E})$ where $\mathbf{C}$ is a finitely complete category and $\mathbf{E}$ is a class of regular epimorphisms in $\mathbf{C}$ containing all isomorphisms, we define the category of relative points over an arbitrary object from $\mathbf{C}$. We show that if $(\mathbf{C}, \mathbf{E})$ is relative exact homological then each category of relative points is exact homological (see Theorem 2.13). The second author has shown [18] that if $\mathbf{C}$ is a pointed exact Mal'tsev category with cokernels and $\mathbf{E}$ is the class of all composites of central extensions in the sense of [12] defined with respect to an adjunction between a semi-abelian Birkhoff subcategory of $\mathbf{C}$ and $\mathbf{C}$, then the pair $(\mathbf{C}, \mathbf{E})$ is relative semi-abelian. This means that each category of relative points defined with respect to such a pair $(\mathbf{C}, \mathbf{E})$ is homological, and in particular, as explained in Example 2.15 that the category of nilpotent objects, as defined by T. Everaert and T. Van der Linden in [9] (see also [6]), is homological.

In Section 3 we define an additional condition (Condition 3.1) that a pair $(\mathbf{C}, \mathbf{E})$ may satisfy. For a relative semi-abelian category $(\mathbf{C}, \mathbf{E})$ satisfying Condition 3.1 such that $\mathbf{C}$ has pushouts we show that each category of relative points is semi-abelian (see Theorem 3.11) and that each pullback functor between relative points is monadic (see Theorem 3.17).

Finally in Section 4 we give many examples of relative semi-abelian categories satisfying Condition 3.1.

2. Relative points

Throughout this paper we assume that $(\mathbf{C}, \mathbf{E})$ is a pair where $\mathbf{C}$ is a pointed finitely complete category and $\mathbf{E}$ is a class of regular epimorphisms in $\mathbf{C}$ containing all isomorphisms.

Consider the following conditions on $(\mathbf{C}, \mathbf{E})$:

2.1. Condition.

(a) The class $\mathbf{E}$ *is closed under composition;*

(b) If $f \in \mathbf{E}$ *and* $gf \in \mathbf{E}$ *then* $g \in \mathbf{E}$*;*

(c) The class $\mathbf{E}$ *is pullback stable;*

(d) If a morphism f *in* $\mathbf{C}$ *factors as* $f = em$ *in which* e *is in* $\mathbf{E}$ *and* m *is a monomorphism, then it also factors (essentially uniquely) as* $f = m'e'$ *in which* m' *is a monomorphism and* e' *is in* $\mathbf{E}$*;*

(e) The $\mathbf{E}$*-Short Five Lemma holds in* $\mathbf{C}$*. That is, for each commutative diagram*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B, \end{tikzcd} \tag{2.1}$$

if $k = \ker(f)$*,* $k' = \ker(f')$*, and* f *and* f' *are in* $\mathbf{E}$*, then* w *is an isomorphism;*

(f) Every equivalence $\mathbf{E}$*-relation in* $\mathbf{C}$ *is* $\mathbf{E}$*-effective. That is, every equivalence relation* $(R, r_1, r_2) : A \to A$ *with* $r_1 : R \to A$ *and* $r_2 : R \to A$ *in* $\mathbf{E}$*, is the kernel pair of some morphism* $f : A \to B$ *in* $\mathbf{E}$*.*

The following two lemmas should be compared to Lemma 2.3 in [18]:

2.2. Lemma. *If* $(\mathbf{C}, \mathbf{E})$ *satisfies Conditions 2.1(c) and 2.1(d), then for any composable pair of morphisms* f *and* g *in* $\mathbf{C}$*, such that the composite* gf *is in* $\mathbf{E}$*, the morphism* f *factors as* me *where* e *is in* $\mathbf{E}$ *and* m *a monomorphism.*

Proof. Let $f : A \to B$ and $g : B \to C$ be morphisms in $\mathbf{C}$ such that gf is in $\mathbf{E}$. Consider the diagram

$$\begin{tikzcd} A \arrow[r, dashed, "e"] \arrow[d, "\langle 1_A, f\rangle"'] \arrow[dr, "f"] & S \arrow[d, dashed, "m"] \\ A \times_C B \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "gf"'] & C \end{tikzcd}$$

in which the lower square is a pullback. Since gf is in $\mathbf{E}$, π_2 is also in $\mathbf{E}$ by Condition 2.1(c), and therefore by Condition 2.1(d) we obtain the factorization $\pi_2\langle 1_A, f\rangle = me$ where e in $\mathbf{E}$ and m is a monomorphism. ∎

The following lemma immediately follows from Lemma 2.2.

2.3. Lemma. *If* $(\mathbf{C}, \mathbf{E})$ *satisfies Conditions 2.1(a), 2.1(c) and 2.1(d), then for any composable pair of morphisms f and g in $\mathbf{C}$, such that f is an extremal epimorphism and the composite gf is in $\mathbf{E}$, the morphism f is in $\mathbf{E}$.*

2.4. Remark. Note that Condition 2.1(a) in the above lemma could be replaced by the weaker condition that if f and g are a composable pair with f in $\mathbf{E}$ and g an isomorphism then gf is in $\mathbf{E}$.

We recall:

2.5. Definition. *An arbitrary pair $(\mathbf{C}, \mathbf{E})$ where $\mathbf{C}$ is a category and $\mathbf{E}$ is a class of regular epimorphism in $\mathbf{C}$ containing all isomorphisms is:*

(A) a relative regular category *if $\mathbf{C}$ is finitely complete and Conditions 2.1(a)-2.1(d) hold for $(\mathbf{C}, \mathbf{E})$;*

(B) a relative exact category *if it is relative regular and Condition 2.1(f) holds for $(\mathbf{C}, \mathbf{E})$;*

(C) a relative homological category *if it is relative regular, $\mathbf{C}$ is pointed with cokernels and Condition 2.1(e) holds for $(\mathbf{C}, \mathbf{E})$;*

(D) a relative semi-abelian category *if it is relative homological, relative exact and $\mathbf{C}$ has finite coproducts.*

For each object B in $\mathbf{C}$, we denote by $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ the category of relative points defined as follows: the objects are triples (A, α, β) where A is an object in $\mathbf{C}$, $\alpha : A \to B$ is a morphism in $\mathbf{E}$ and $\beta : B \to A$ is a morphism in $\mathbf{C}$ such that $\alpha\beta = 1_B$; a morphism $f : (A, \alpha, \beta) \to (A', \alpha', \beta')$ in $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ is a morphism $f : A \to A'$ in $\mathbf{C}$ such that $f\beta = \beta'$ and $\alpha' f = \alpha$. Note that $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ is a full subcategory of $\mathbf{Pt}_B(\mathbf{C})$ introduced by D. Bourn in [4].

2.6. Lemma. *For each object B in $\mathbf{C}$, we have:*

(i) $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ has a zero object;

(ii) if Conditions 2.1(a) and 2.1(c) hold for $(\mathbf{C}, \mathbf{E})$, then $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ has binary products;

(iii) if Conditions 2.1(a) and 2.1(d) hold for $(\mathbf{C}, \mathbf{E})$, then $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ has equalizers.

Proof. (i): It is easy to check that since $\mathbf{E}$ contains all isomorphisms, $(B, 1_B, 1_B)$ is the zero object in $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$.
(ii): Let (A, α, β) and (A', α', β') be objects in $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ and consider the pullback

$$\begin{tikzcd} A \times_B A' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A' \arrow[d, "\alpha'"] \\ A \arrow[r, "\alpha"'] & B. \end{tikzcd}$$

Since $\mathbf{E}$ is pullback stable and closed under composition (Conditions 2.1(c) and 2.1(a)), the composite $\alpha\pi_1 = \alpha'\pi_2$ is in $\mathbf{E}$. Therefore, since $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ is a full subcategory of $\mathbf{Pt}_B(\mathbf{C})$, the triple $(A \times_B A, \alpha\pi_1, \langle\beta,\beta'\rangle)$ is the product of (A,α,β) and (A',α',β') in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$.
(iii): Let $f, g : (A,\alpha,\beta) \to (A',\alpha',\beta')$ be a parallel pair of morphisms and let $i : I \to A$ be the equalizer of f and g in $\mathbf{C}$. Since $f\beta = \beta' = g\beta$, it follows that there exists a unique morphism $\sigma : B \to I$ such that $i\sigma = \beta$. Consider the diagram

$$\begin{tikzcd} B \arrow[r, "\sigma"] \arrow[dr, "\beta"'] & I \arrow[r, dashed, "e"] \arrow[d, "i"] & S \arrow[d, dashed, "m"] \\ & A \arrow[r, "\alpha"'] & B \end{tikzcd}$$

where e is in $\mathbf{E}$, m is a monomorphism, and me is the factorization of αi which exists by Condition 2.1(d). It follows that m is an isomorphism and therefore $\alpha i = me$ is in $\mathbf{E}$ by Condition 2.1(a). Since $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ is a full subcategory of $\mathbf{Pt}_B(\mathbf{C})$ it follows that $i : (I, \alpha i, \sigma) \to (A,\alpha,\beta)$ is the equalizer of f and g in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$. ■

As an immediate consequence we obtain:

2.7. Proposition. *If* $(\mathbf{C},\mathbf{E})$ *satisfies Conditions 2.1(a), 2.1(c) and 2.1(d), then for each object* B *in* $\mathbf{C}$ *the category* $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ *is pointed and has finite limits.*

2.8. Lemma. *If* $(\mathbf{C},\mathbf{E})$ *satisfies Conditions 2.1(a), 2.1(c), and 2.1(d), then for each object* B *in* $\mathbf{C}$ *the category* $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ *has a pullback stable (regular epimorphism, monomorphism)-factorization system, where the regular epimorphisms are in* $\mathbf{E}$*.*

Proof. Let $f : (A,\alpha,\beta) \to (A',\alpha',\beta')$ be a morphism in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$. Since $\alpha' f = \alpha$ is in $\mathbf{E}$, it follows by Lemma 2.2 that f factors as me where $e : A \to S$ is in $\mathbf{E}$ and m is a monomorphism. By Condition 2.1(d), the composite $\alpha' m$ factors as a $m'e'$ where e' is in $\mathbf{E}$ and m' is a monomorphism. Since $m'e'e\beta = \alpha' me\beta = \alpha' f\beta = \alpha\beta = 1_B$, it follows that m is an isomorphism and so $\alpha' m = m'e'$ is in $\mathbf{E}$, therefore $(S, \alpha' m, e\beta)$ is in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ and e and m are morphisms in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$, as required.

Since pullbacks in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$, are calculated as in $\mathbf{C}$, pullback stability of factorizations follows directly from Condition 2.1(c). ■

2.9. Proposition. *If* $(\mathbf{C},\mathbf{E})$ *is a relative regular category, or more generally, if* $(\mathbf{C},\mathbf{E})$ *satisfies Conditions 2.1(a), 2.1(c) and 2.1(d), then for each object* B *in* $\mathbf{C}$ *the category* $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ *is a pointed regular category.*

Proof. The proof follows directly from Proposition 2.7 and Lemma 2.8. ■

2.10. Proposition. *If* $(\mathbf{C},\mathbf{E})$ *is a relative exact category, then for each object* B *in* $\mathbf{C}$ *the category* $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ *is a pointed exact category.*

Proof. As follows from Proposition 2.9, in order to prove that for each object B in $\mathbf{C}$ the category $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ is exact, it is sufficient to show that equivalence relations in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ are effective. Let B be an object in $\mathbf{C}$ and let $r_1, r_2 : (R,\rho,\sigma) \to (A,\alpha,\beta)$ be the projections of an equivalence relation in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$. Since each equivalence relation in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ is an equivalence relation in $\mathbf{C}$, it follows from Lemma 2.8 that r_1 and r_2 are the projections of an $\mathbf{E}$-equivalence relation in $\mathbf{C}$. By Condition 2.1(f), r_1 and r_2 are the kernel pair of their coequalizer q in $\mathbf{E}$. Consider the diagram

$$\begin{tikzcd}
R \arrow[rr, shift left, "r_1"] \arrow[rr, shift right, "r_2"'] \arrow[drr, shift left, "\rho"] & & A \arrow[rr, "q"] \arrow[d, shift left, "\alpha"] & & Q \arrow[dll, dashed, shift left, "\gamma"] \\
& & B \arrow[ull, shift left, "\sigma"] \arrow[u, shift left, "\beta"] \arrow[urr, shift left, "\delta"] & &
\end{tikzcd}$$

where $\delta = q\beta$. Since $\alpha r_1 = \rho = \alpha r_2$, by the universal property of q there exists a unique morphism γ such that $\gamma q = \alpha$. Since $\gamma q = \alpha$ is in $\mathbf{E}$, it follows from Condition 2.1(b) that γ is in $\mathbf{E}$, and therefore r_1 and r_2 are the kernel pair of q in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$. ∎

2.11. Proposition. *If* $(\mathbf{C},\mathbf{E})$ *satisfies Conditions 2.1(a), 2.1(c), 2.1(d), and 2.1(e), then* $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ *is a homological category.*

Proof. As follows from Proposition 2.9, in order to prove that for each object B in $\mathbf{C}$ the category $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ is homological it is sufficient to show that the Split Short Five Lemma holds in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$. Consider the diagram

$$\begin{tikzcd}
& & 0 \arrow[drr] \arrow[ddd, equal] & & & \\
& & & & B \arrow[dr] \arrow[ddd, equal] & \\
K \arrow[uurr] \arrow[r, "k"'] \arrow[ddd, equal] & X \arrow[urrr, "\theta"] \arrow[rr, "x"'] \arrow[ddd, equal] & & A \arrow[rr, shift left, "p"] \arrow[ddd, "w"] & & Y \arrow[ll, shift left, "s"] \arrow[ddd, equal] \\
& & 0 \arrow[drr] & & & \\
& & & & B \arrow[dr] & \\
K \arrow[uurr] \arrow[r, "k"'] & X \arrow[urrr, "\theta"] \arrow[rr, "x'"'] & & A' \arrow[rr, shift left, "p'"] & & Y \arrow[ll, shift left, "s'"]
\end{tikzcd}$$

which consists of the Split Short Five Lemma diagram in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ (with some of the structural morphisms omitted), and where k is the kernel of θ in $\mathbf{C}$. It follows that xk and $x'k$ are the kernels of p and p' respectively, and by Lemma 2.3 that p and p' are in $\mathbf{E}$. Therefore, by the $\mathbf{E}$-Split Short Five Lemma (Condition 2.1(e)), it follows that w is an isomorphism. ∎

As an immediate corollary we have:

2.12. Corollary. *If* $(\mathbf{C},\mathbf{E})$ *is a relative homological category, then for each object* B *in* $\mathbf{C}$ *the category* $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ *is homological.*

Combining Proposition 2.10 and Corollary 2.12 we obtain:

2.13. Theorem. *If* $(\mathbf{C},\mathbf{E})$ *is a relative exact homological category, then for each object* B *in* $\mathbf{C}$ *the category* $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ *is exact homological.*

2.14. Remark. Note that for each object B in $\mathbf{C}$ the category $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$ is not necessarily semi-abelian even when $(\mathbf{C},\mathbf{E})$ is relative semi-abelian. In particular, it is well known that the category of nilpotent groups does not have coproducts and therefore is not semi-abelian (see the example below).

Recall that a Birkhoff subcategory $\mathbf{X}$ of $\mathbf{C}$, is a full and replete subcategory of $\mathbf{C}$ which is closed under subobjects and regular quotients in $\mathbf{C}$.

2.15. Example. As shown in [18], if $\mathbf{C}$ is a pointed exact Mal'tsev category with cokernels and $\mathbf{E}$ is the class of composites of central extensions defined with respect to an adjunction (satisfying certain properties) between a protomodular Birkhoff subcategory of $\mathbf{C}$ and $\mathbf{C}$, then $(\mathbf{C},\mathbf{E})$ is a relative semi-abelian category. It follows from Theorem 2.13 that $\mathbf{Pt}_0(\mathbf{C},\mathbf{E})$, which is essentially the category of nilpotent objects in the sense of T. Everaert and T. Van der Linden [9], is exact homological. Indeed, $\mathbf{Pt}_0(\mathbf{C},\mathbf{E})$ is isomorphic to the full subcategory of $\mathbf{C}$ consisting of those objects from which the unique morphism to 0 is a composite of central extensions.

3. Relative internal actions

If $(\mathbf{C},\mathbf{E})$ satisfies Condition 2.1(c), then each morphism $p : E \to B$ in $\mathbf{C}$ determines a pullback functor $p^* : \mathbf{Pt}_B(\mathbf{C},\mathbf{E}) \to \mathbf{Pt}_E(\mathbf{C},\mathbf{E})$ defined as follows. For each object (A,α,β) in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$, form the pullback

$$\begin{tikzcd} E\times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B, \end{tikzcd}$$

and define $p^*(A,\alpha,\beta) = (E\times_B A, \pi_1, \langle 1_E, \beta p\rangle)$, where $\langle 1_E,\beta p\rangle : E \to E\times_B A$ is the unique morphism with $\pi_1\langle 1_E,\beta p\rangle = 1_E$ and $\pi_2\langle 1_E,\beta p\rangle = \beta p$. For a morphism $f : (A,\alpha,\beta) \to (A',\alpha',\beta')$ in $\mathbf{Pt}_B(\mathbf{C},\mathbf{E})$, define $p^*(f)$ to be the unique morphism such that $\pi_1 p^*(f) = 1_E\pi_1$ and $\pi_2 p^*(f) = f\pi_2$.

Consider the following condition on $(\mathbf{C},\mathbf{E})$:

3.1. Condition. *For each regular epimorphism $f : A \to B$ in $\mathbf{C}$ there exists an initial factorization*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ & E \arrow[ur, "e"'] & \end{tikzcd}$$

where g is a regular epimorphism and e is in $\mathbf{E}$. That is, for each factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g'"'] & & B \\ & E' \arrow[ur, "e'"'] & \end{tikzcd}$$

where g' is a regular epimorphism and e' is $\mathbf{E}$, there exists a unique morphism $i : E \to E'$ such that $ig = g'$ (and therefore $e'i = e$).

3.2. Remark. Let $\mathbf{C}^2$ be the category of morphisms in $\mathbf{C}$. Note that Condition 3.1 is equivalent to requiring that the full sub-category of $\mathbf{C}^2$ with objects those morphisms which are in $\mathbf{E}$, is reflective in the full sub-category of $\mathbf{C}^2$ with objects all regular epimorphisms in $\mathbf{C}$. When the underlying category considered in [8] is in addition exact, Theorem 3.7 (1) of [8] becomes a special case of Condition 3.1 where $\mathbf{E}$ is the class of central extensions - see [18].

3.3. Lemma. *Suppose $\mathbf{C}$ is a regular category and that $(\mathbf{C}, \mathbf{E})$ satisfies Conditions 2.1(c) and 3.1. For each pullback*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

if $ge = f$ and $g'e' = f'$ are the initial factorizations as in Condition 3.1 of f and f' respectively, then there exists a unique morphism ϵ such that the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] \arrow[dd, "\alpha"'] & & B \arrow[dd, "\beta"] \\ & E \arrow[ur, "e"'] \arrow[dd, "\epsilon"'] & \\ A' \arrow[rr, "f'"'] \arrow[dr, "g'"'] & & B' \\ & E' \arrow[ur, "e'"'] & \end{tikzcd}$$

commutes.

Proof. Let $(E' \times_{B'} B, \pi_1, \pi_2)$ be the pullback of e' and β. Since $e'g'\alpha = \beta f$, there exists a unique morphism $\tilde{g} : A \to E' \times_{B'} B$ such that $\pi_1 \tilde{g} = g'\alpha$ and $\pi_2 \tilde{g} = f$. We obtain the commutative diagram

$$\begin{tikzcd}
A \arrow[rrrr, "f"] \arrow[drr, "g"] \arrow[ddrr, "\tilde{g}"'] \arrow[ddd, "\alpha"'] & & & & B \arrow[ddd, "\beta"] \\
& & E \arrow[urr, "e"] \arrow[d, dashed, "\theta"] & & \\
& & E' \times_{B'} B \arrow[uurr, "\pi_2"'] \arrow[dd, "\pi_1"] & & \\
A' \arrow[rrrr, "f'"] \arrow[drr, "g'"'] & & & & B' \\
& & E' \arrow[urr, "e'"'] & &
\end{tikzcd}$$

where $g'\alpha = \pi_1 \tilde{g}$ is a pullback (implying that $\tilde{g}$ is a regular epimorphism), and θ is obtained from Condition 3.1. It follows that $\epsilon = \pi_1 \theta$ satisfies the desired properties. ■

The data in the following proposition is similar to a Galois structure in the sense of G. Janelidze see e.g. [11].

3.4. Proposition. *Let* $(\mathbf{C}', \mathbf{E}')$ *be a pair consisting of a category together with a class of morphisms contained in all regular epimorphisms and containing all isomorphisms. Suppose that* $H : \mathbf{C}' \to \mathbf{C}$ *is a full and faithful functor with left adjoint* I *such that* $IH = 1_{\mathbf{C}'}$*, suppose in addition that* H *preserves regular epimorphisms, and* $H(\mathbf{E}') \subset \mathbf{E}$ *and* $I(\mathbf{E}) \subset \mathbf{E}'$*. If* f *is a regular epimorphism in* $\mathbf{C}'$ *whose image under* H *admits a factorization as in Condition 3.1, then* f *admits a factorization as in Condition 3.1.*

Proof. Let $f : A \to A'$ be a regular epimorphism in $\mathbf{C}'$ such that there exist morphisms $g : H(A) \to E$ in $\mathbf{E}$ and $e : E \to H(B)$ a regular epimorphism making the diagram

$$\begin{tikzcd}
H(A) \arrow[rr, "H(f)"] \arrow[dr, "g"'] & & H(B) \\
& E \arrow[ur, "e"'] &
\end{tikzcd}$$

an initial factorization of $H(f)$. We will show that the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "I(g)"'] & & B \\
& I(E) \arrow[ur, "I(e)"'] &
\end{tikzcd}$$

is an initial factorization of f. Let $u : A \to W$ be a regular epimorphism in $\mathbf{C}'$ and let $v : W \to B$ be a morphism in $\mathbf{E}'$ such that $f = vu$. Since $H(f) = H(vu) = H(v)H(u)$, it follows by the universal property of eg that there exists a unique morphism $w : E \to H(W)$ such that $wg = H(u)$. It follows by adjunction that there exists a unique morphism $\tilde{w} : I(C) \to W$ such that $\tilde{w}I(g) = u$, as required. ∎

3.5. PROPOSITION. *If $\mathbf{C}$ is a regular category with coequalizers of equivalence relations and $\mathbf{E}$ is pullback stable (i.e $(\mathbf{C}, \mathbf{E})$ satisfies Condition 2.1(c)) and pushout stable along regular epimorphisms, then Condition 3.1 is equivalent to the same condition where the first instance of "regular epimorphism" is replaced by "split epimorphism".*

PROOF. Trivially Condition 3.1 implies the same condition where the first instance of "regular epimorphism" is replaced by "split epimorphism". Conversely, let $f : A \to B$ be a regular epimorphism and let $r_1, r_2 : R \to A$ be the kernel pair of f. Let $\mathbf{EqRel}(\mathbf{C})$ be the category of equivalence relations in $\mathbf{C}$. Choose $\mathbf{E}'$ to be the class of all morphisms in $(\mathbf{EqRel}(\mathbf{C}) \downarrow (R, A, r_1, r_2))$ whose underlying morphisms are in $\mathbf{E}$. It is easy to check that the functor $H : (\mathbf{C} \downarrow B) \to (\mathbf{EqRel}(\mathbf{C}) \downarrow (R, A, r_1, r_2))$, sending an object to its *descent data* (see e.g. [15]), has a left adjoint and that this functor satisfies the requirements of the previous proposition. Since $H(f) : H(A, f) \to H(B, 1_B)$ has underlying split epimorphisms, it follows from Lemma 3.3 that it admits an initial factorization as in Condition 3.1, and hence by Proposition 3.4 f admits an initial factorization, as required. ∎

The following well-known result follows from the fact that pulling back along a regular epimorphism in a regular category reflects isomorphisms [19]

3.6. LEMMA. *Let $\mathbf{C}$ be a pointed regular category with finite limits and let*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd}$$

be a commutative diagram in $\mathbf{C}$. If w is a regular epimorphism, k is the kernel of f, and $k'u = wk$ is a pullback, then k' is the kernel of f'.

Recall that a subcategory of $\mathbf{X}$ of a category $\mathbf{C}$ is called extension closed if for each short exact sequence

$$0 \longrightarrow X \longrightarrow A \longrightarrow B \longrightarrow 0$$

in $\mathbf{C}$, if X and B are in $\mathbf{X}$, then A is also in $\mathbf{X}$. The conditions in the proposition below seem to be related to [13].

3.7. PROPOSITION. *Let $\mathbf{C}$ be a semi-abelian category and let*

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "I"] & \mathbf{X} \arrow[l, shift left, "H"] \end{tikzcd}$$

be an adjunction where $\mathbf{X}$ is an extension closed Birkhoff subcategory of $\mathbf{C}$. The pair $(\mathbf{C}, \mathbf{E})$, where $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$ whose kernels are in $\mathbf{X}$, is a relative semi-abelian category satisfying Condition 3.1.

Proof. As shown in [17], the pair $(\mathbf{C}, \mathbf{E})$ is a relative semi-abelian category. Therefore, we only need to show that Condition 3.1 holds for $(\mathbf{C}, \mathbf{E})$. Let p be a regular epimorphism. Consider the diagram

$$\begin{tikzcd}
\mathbf{Ker}(\eta_K) \arrow[d, "\mathbf{ker}(\eta_K)"'] & & \\
K \arrow[r, "k"] \arrow[d, "\eta_K"'] & E \arrow[r, "p"] \arrow[d, "q"] & B \\
HI(K) & \overline{E} \arrow[ur, "\overline{p}"'] &
\end{tikzcd}$$

in which k is the kernel of p, η is the unit of the above adjunction, q is the cokernel of $k\mathbf{ker}(\eta_K)$, and $\overline{p}$ is the induced unique morphism which makes the diagram commute. Note that $\overline{p}$ is a regular epimorphism since p is. We will show that the triangle on the right of the above diagram gives the desired factorization. Let us first prove that $\overline{p}$ is in $\mathbf{E}$. Since $\overline{p}$ is a regular epimorphism it suffices to prove that the kernel of $\overline{p}$ is in the image of H. Consider the diagram

$$\begin{tikzcd}
 & \mathbf{Ker}(\eta_K) \arrow[r, "j"] \arrow[d, "\mathbf{ker}(\eta_K)"'] & \mathbf{Ker}(q) \arrow[dl, "\kappa"'] \arrow[d, "\mathbf{ker}(q)"] & \\
 & K \arrow[r, "k"] \arrow[d, "\eta_K"'] \arrow[ddl, bend right, "\mathbf{coker}(\kappa)"'] & E \arrow[r, "p"] \arrow[d, "q"] & B \\
 & HI(K) \arrow[dl, "\lambda"'] & \overline{E} \arrow[ur, "\overline{p}"'] & \\
\mathbf{Coker}(\kappa) \arrow[urr, bend right, "m"'] & & &
\end{tikzcd}$$

consisting of the previous diagram with the additional morphisms defined as follows:

- since $qk\mathbf{ker}(\eta_K) = 0$, there exists a unique morphism j such that $k\mathbf{ker}(\eta_K) = \mathbf{ker}(q)j$;
- since $p\mathbf{ker}(q) = \overline{p}q\mathbf{ker}(q) = 0$, there exists a unique morphism κ such that $k\kappa = \mathbf{ker}(q)$, and so $\kappa j = \mathbf{ker}(\eta_K)$ since k is a monomorphism;
- since $\mathbf{coker}(\kappa)\mathbf{ker}(\eta_K) = \mathbf{coker}(\kappa)\kappa j = 0$, there exists a unique morphism λ such that $\lambda\eta_K = \mathbf{coker}(\kappa)$;

- since $qk\kappa = q\mathbf{ker}(q) = 0$, there exists a unique morphism m such that $m\mathbf{coker}(\kappa) = qk$.

Since $\mathbf{coker}(\kappa)$ and q are normal epimorphisms and $\mathbf{C}$ is a pointed protomodular category, it easily follows that the square $m\mathbf{coker}(\kappa) = qk$ is a pullback. Applying Lemma 3.6 to the diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "\mathbf{coker}(\kappa)"'] & E \arrow[r, "p"] \arrow[d, "q"'] & B \\ \mathbf{Coker}(\kappa) \arrow[r, "m"'] & \overline{E} \arrow[ur, "\overline{p}"'] & \end{tikzcd}$$

we conclude that m is the kernel of $\overline{p}$. Since $\mathbf{coker}(\kappa)$ is a regular epimorphisms, it follows that λ is a regular epimorphism, and therefore $\mathbf{Coker}(\kappa)$ is in the image of H. To prove that the above factorization is universal, let

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "g"'] & & B \\ & D \arrow[ur, "h"'] & \end{tikzcd}$$

be a factorization of p, where g is a regular epimorphism and h is in $\mathbf{E}$. Consider the diagram

$$\begin{tikzcd} \mathbf{Ker}(\eta_K) \arrow[d, "\mathbf{ker}(\eta_K)"'] & & \\ K \arrow[r, "k"] \arrow[d, "\eta_K"] \arrow[dd, bend right=60, "\overline{g}"'] & E \arrow[r, "p"] \arrow[d, "q"] \arrow[dd, bend right=60, "g"'] & B \\ HI(K) \arrow[d, "H(\tilde{i})"] & \overline{E} \arrow[ur, "\overline{p}"'] \arrow[d, dashed, "i"] & \\ \mathbf{Ker}(h) \arrow[r, "\mathbf{ker}(h)"'] & D \arrow[uur, bend right=60, "h"'] & \end{tikzcd}$$

where the morphisms $\overline{g}$ and $\tilde{i}$ are defined as follows:

- since $hgk = pk = 0$, there exists a unique morphism $\overline{g}$ such that $\mathbf{ker}(h)\overline{g} = gk$;

- since by the assumption $\mathbf{Ker}(h)$ is in the image of H, there exists a unique morphism $\tilde{i}$ (in $\mathbf{X}$) such that $H(\tilde{i})\eta_K = \overline{g}$.

Finally, since $gk\mathbf{ker}(\eta_K) = \mathbf{ker}(h)\overline{g}\mathbf{ker}(\eta_K) = \mathbf{ker}(h)H(\tilde{i})\eta_K\mathbf{ker}(\eta_K) = 0$, there exists a unique morphism i such that $iq = g$, as desired. ∎

3.8. Remark. Note that in Proposition 3.7 we only need $\mathbf{C}$ to be a homological category with cokernels to prove that $(\mathbf{C}, \mathbf{E})$ satisfies Condition 3.1. Note also that Condition (iv) of Example 3.7 in [17] holds when $\mathbf{C}$ is an exact homological category. It follows that if we require $\mathbf{C}$ to have cokernels and replace "semi-abelian" with "exact homological" everywhere in the statement of Proposition 3.7, then it still remains true.

3.9. Proposition. *If* $(\mathbf{C}, \mathbf{E})$ *satisfies Conditions 2.1(c) and 3.1, then for each object* B *in* $\mathbf{C}$ *the inclusion* J_B *of* $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ *into* $\mathbf{Pt}_B(\mathbf{C})$ *has a left adjoint.*

Proof. Let B be an object in $\mathbf{C}$ and let (A, α, β) be an object in $\mathbf{Pt}_B(\mathbf{C})$. It is easy to check that $((\overline{A}, \overline{\alpha}, \overline{\beta}), q)$, where

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "q"'] & & B \\ & \overline{A} \arrow[ur, "\overline{\alpha}"'] & \end{tikzcd}$$

is the initial factorization of α obtained from Condition 3.1 and $\overline{\beta} = q\beta$, is the initial object in the comma category $((A, \alpha, \beta) \downarrow J_B)$. ■

Since for each object B in $\mathbf{C}$ the category $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ is a full subcategory of $\mathbf{Pt}_B(\mathbf{C})$, as an immediate corollary we obtain:

3.10. Proposition. *If* $\mathbf{C}$ *has pushouts and* $(\mathbf{C}, \mathbf{E})$ *satisfies Conditions 2.1(c) and 3.1, then for each object* B *in* $\mathbf{C}$ *the category* $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ *has finite coproducts.*

Combining Theorem 2.13 and Proposition 3.10 we obtain:

3.11. Theorem. *If* $\mathbf{C}$ *has pushouts,* $(\mathbf{C}, \mathbf{E})$ *is a relative semi-abelian category satisfying Condition 3.1, then for each* B *the category* $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ *is semi-abelian.*

Recall the following well-known result:

3.12. Lemma. *Let* $\mathbf{A}$, $\mathbf{X}$ *and* $\mathbf{Y}$ *be categories and let* $U : \mathbf{A} \to \mathbf{X}$, $H : \mathbf{Y} \to \mathbf{X}$ *and* $V : \mathbf{A} \to \mathbf{Y}$ *be functors such that* $HV = U$*. If* H *is fully faithful and* U *has a left adjoint* F*, then* V *has a left adjoint* G *such that* $G = FH$*.*

Proof. For each Y in $\mathbf{Y}$ and A in $\mathbf{A}$, since H is fully faithful the map sending $f : Y \to V(A)$ in $\hom(Y, V(A))$ to $H(f)$ in $\hom(H(Y), HV(A))$ is a bijection. Therefore, the composite

$$\begin{aligned} \hom(Y, V(A)) &\cong \hom(H(Y), HV(A)) \\ &= \hom(H(Y), U(A)) \\ &\cong \hom(FH(Y), A) \end{aligned}$$

is a bijection natural in Y and A. ■

3.13. Proposition. *Suppose* $\mathbf{C}$ *has pushouts and* $(\mathbf{C}, \mathbf{E})$ *satisfies Condition 2.1(c). For each* $p : E \to B$ *in* $\mathbf{C}$ *if the functor* J_B*, as defined in Proposition 3.9, has a left adjoint, then the pullback functor* $p^* : \mathbf{Pt}_B(\mathbf{C}, \mathbf{E}) \to \mathbf{Pt}_E(\mathbf{C}, \mathbf{E})$ *has a left adjoint.*

Proof. Let $p : E \to B$ be a morphism in $\mathbf{C}$ and suppose J_B has a left adjoint. Since J_E is fully faithful and the composite of $p^* : \mathbf{Pt}_B(\mathbf{C}) \to \mathbf{Pt}_E(\mathbf{C})$ and $J_B : \mathbf{Pt}_B(\mathbf{C}, \mathbf{E}) \to \mathbf{Pt}_B(\mathbf{C})$ has a left adjoint, it follows from Lemma 3.12 that $p^* : \mathbf{Pt}_B(\mathbf{C}, \mathbf{E}) \to \mathbf{Pt}_E(\mathbf{C}, \mathbf{E})$ has a left adjoint. ■

Recall:

3.14. LEMMA. *Let* $\mathbf{C}$ *be a regular Mal'tsev category. Each reflexive pair* $X \underset{x_2}{\overset{x_1}{\rightrightarrows}} A$ *factors as*

$$\begin{tikzcd} X \arrow[rr, shift left, "x_1"] \arrow[rr, shift right, "x_2"'] \arrow[dr, "e"'] & & A \\ & R \arrow[ur, shift left, "r_1"] \arrow[ur, shift right, "r_2"'] & \end{tikzcd}$$

where e *is a regular epimorphism and* r_1 *and* r_2 *are the projections of an equivalence relation.*

PROOF. Let $X \underset{x_2}{\overset{x_1}{\rightrightarrows}} A$ be a reflexive pair and let

$$\begin{tikzcd} X \arrow[rr, "\langle x_1,x_2\rangle"] \arrow[dr, "e"'] & & A \times A \\ & R \arrow[ur, "\langle r_1,r_2\rangle"'] & \end{tikzcd}$$

be the factorization of $\langle x_1, x_2\rangle$ as a regular epimorphism followed by a monomorphism. It easily follows that $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A$ is a reflexive relation and therefore an equivalence relation since $\mathbf{C}$ is a Mal'tsev category. ■

3.15. LEMMA. *Let* $\mathbf{C}$ *be an exact Mal'tsev category and let* $U : \mathbf{C} \to \mathbf{D}$ *be a functor that preserves regular epimorphisms and finite limits. The functor* U *preserves reflexive coequalizers.*

PROOF. Let $X \underset{x_2}{\overset{x_1}{\rightrightarrows}} A$ be a reflexive pair and let $c : A \to C$ be the coequalizer of x_1 and x_2. By Lemma 3.14 there exists a factorization

$$\begin{tikzcd} X \arrow[rr, shift left, "x_1"] \arrow[rr, shift right, "x_2"'] \arrow[dr, "e"'] & & A \\ & R \arrow[ur, shift left, "r_1"] \arrow[ur, shift right, "r_2"'] & \end{tikzcd}$$

where e is a regular epimorphism and r_1 and r_2 are the projections of an equivalence relation. Since e is an epimorphism it follows that c is the coequalizer of r_1 and r_2, and therefore since $\mathbf{C}$ is an exact category, r_1 and r_2 are the kernel pair of c. Consider the diagram

$$\begin{tikzcd} U(X) \arrow[rr, shift left, "U(x_1)"] \arrow[rr, shift right, "U(x_2)"'] \arrow[dr, "U(e)"'] & & A \arrow[r, "U(c)"] & U(C) \\ & U(R). \arrow[ur, shift left, "U(r_1)"] \arrow[ur, shift right, "U(r_2)"'] & & \end{tikzcd}$$

Since U preserves regular epimorphisms it follows that $U(e)$ and $U(c)$ are regular epimorphisms, and since U preserves limits it follows that $U(r_1)$ and $U(r_2)$ are the kernel pair of $U(c)$ and so $U(c)$ is the coequalizer of $U(r_1)$ and $U(r_2)$. Therefore, since $U(e)$ is an epimorphism, it follows that $U(c)$ is the coequalizer of $U(x_1)$ and $U(x_2)$, as required. ∎

3.16. PROPOSITION. *Let* $(\mathbf{C}, \mathbf{E})$ *be a relative exact homological category and let* $p : E \to B$ *be a morphism in* $\mathbf{C}$. $p^* : \mathbf{Pt}_B(\mathbf{C}, \mathbf{E}) \to \mathbf{Pt}_E(\mathbf{C}, \mathbf{E})$ *is monadic if and only if it has a left adjoint.*

PROOF. It easily follows from Condition 2.1(e) that p^* reflects isomorphisms, and, by Lemma 3.15 it follows that p^* preserves reflexive coequalizers. Therefore, by Beck's monadicity theorem, p^* is monadic whenever it has a left adjoint. ∎

Combining Propositions 3.9, 3.11, 3.13 and 3.16, we obtain:

3.17. THEOREM. *If* $\mathbf{C}$ *has pushouts,* $(\mathbf{C}, \mathbf{E})$ *is a relative semi-abelian category satisfying Condition 3.1, then each category of relative points is semi-abelian and each pullback functor between categories of relative points is monadic. Furthermore under these assumptions each category* $\mathbf{Pt}_B(\mathbf{C}, \mathbf{E})$ *is equivalent to algebras over a monad in* $\mathbf{Pt}_0(\mathbf{C}, \mathbf{E})$.

4. Examples

According to Proposition 3.7 and Theorem 3.17, each Birkhoff subcategory $\mathbf{X}$ of a semi-abelian category $\mathbf{C}$, which is extension closed in $\mathbf{C}$, determines a relative semi-abelian category $(\mathbf{C}, \mathbf{E})$ for which every pullback functor between relative points is monadic. In this section we recall examples of extension closed Birkhoff subcategories of semi-abelian categories.

4.1. EXAMPLE. Let $\mathbf{C}$ be the category of (not necessarily unital) rings. Let P be a non-empty finite set of prime numbers, for each $p \in P$ let $N(p)$ be a finite non-empty set of positive integers, and let $\mathbf{X}_{P,N}$ be the subvariety of rings satisfying the following conditions:

1. $(\prod_{p \in P} p)x = 0$;
2. for each $p \in P$,
$$\hat{p}x \prod_{n \in N(p)} \left(x^{p^n - 1} - 1\right) = 0$$
where $\hat{p} = \prod_{q \in P, q \neq p} q$.

Note that in 2 above 1 is used only to simplify its presentation (i.e we do not assume the existence of 1). According to [10] the following are equivalent for a subvariety $\mathbf{X}$ of $\mathbf{C}$:

(i) the objects of $\mathbf{X}$ form a semi-simple radical class;

(ii) the objects of $\mathbf{X}$ form a homomorphically closed semi-simple radical class;

(iii) $\mathbf{X}$ is extension closed in $\mathbf{C}$;

(iv) $\mathbf{X}$ has attainable identities in $\mathbf{C}$;

(v) $\mathbf{X} = \mathbf{X}_{P,N}$ for some P and N.

It follows that in particular Boolean rings are extension closed inside of rings and more generally that the subvarieties of rings consisting of those objects which satisfy the equation $x^n = x$ for a fixed natural number n are all extension closed in rings.

4.2. Example. For pointed full subcategories $\mathbf{X}$ and $\mathbf{Y}$ of a pointed category $\mathbf{C}$ a new full subcategory (containing both) can be constructed by taking those objects in $\mathbf{C}$ which appear in an extension with kernel in $\mathbf{X}$ and codomain in $\mathbf{Y}$. According to the paper [20] Neumann, Neumann and Neumann and independently Smel'kin have shown that the proper subvarieties of groups under this operation form a free monoid (see [21], [22], [23]). The variety of Brouwerian semilattices has the same property [20]. It easily follows that there are no non-trivial extension closed subvarieties in both cases.

Recall that a pair $(\mathbf{T}, \mathbf{F})$ of full replete subcategories of a pointed category $\mathbf{C}$ is called a torsion theory (see e.g. [8]) if

(a) every morphism with domain in $\mathbf{T}$ and codomain in $\mathbf{F}$ is a zero morphism;

(b) for each object C in $\mathbf{C}$ there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow C \longrightarrow F \longrightarrow 0$$

where T is $\mathbf{T}$ and F is in $\mathbf{F}$.

As mentioned in [8] the category $\mathbf{F}$ is closed under extensions and hence is an example of such a subcategory whenever it is Birkhoff.

4.3. Example. For a semi-abelian category $\mathbf{C}$, let us denote by $\mathbf{Gpd}(\mathbf{C})$ the category of internal groupoids in $\mathbf{C}$, and by $D : \mathbf{C} \to \mathbf{Gpd}(\mathbf{C})$ the discrete functor associating to each object C the discrete equivalence relation on C (considered as a groupoid). According to [7], this functor determines a torsion theory $(\mathbf{T}, \mathbf{F})$ in $\mathbf{Gpd}(\mathbf{C})$ where $\mathbf{F}$ is the image of $\mathbf{C}$ under D. Moreover, the category $\mathbf{F}$ is a Birkhoff subcategory of $\mathbf{Gpd}(\mathbf{C})$.

References

[1] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Kluwer Academic Publishers, 2004.

[2] F. Borceux, G. Janelidze, and G. M. Kelly, *Internal object actions*, Commentationes Mathematicae Universitatis Carolinae **46**(2), 235–255, 2005.

[3] F. Borceux, G. Janelidze, and G. M. Kelly, *On the representability of actions in a semi-abelian category*, Theory and Applications of Categories **14**(11), 244–286, 2005.

[4] D. Bourn, *Mal'cev categories and fibration of pointed objects*, Applied Categorical Structures **4**(2-3), 307–327, 1996.

[5] D. Bourn and G. Janelidze, *Protomodularity, descent and semidirect products*, Theory and Applications of Categories **4**(2), 37–46, 1998.

[6] T. Everaert, *An approach to non-abelian homology based on categorical galois theory*, Ph.D. thesis, Vrijie Universiteit Brussel, 2007.

[7] T. Everaert and M. Gran, *Homology of n-fold groupoids*, Theory and Applications of Categories **23**(2), 22–41, 2010.

[8] T. Everaert and M. Gran, *Protoadditive functors, derived torsion theories and homology*, Journal of Pure Applied Algebra **219**, 3629–3676, 2015.

[9] T. Everaert and T. Van der Linden, *Baer invariants in semi-abelian categories I*, Theory and Application of Categories **12**, 1–33, 2004.

[10] B. Gardner and P. N. Stewart, *On semi-simple radical classes*, Bull. Austral. Math. Soc. **13**, 349–353, 1975.

[11] G. Janelidze, *Galois connections and applications*, ch. Categorical Galois Theory: Revision and Some Recent Developments, pp. 139–171, Springer Netherlands, Dordrecht, 2004.

[12] G. Janelidze and G. M. Kelly, *Galois theory and a general notion of central extension*, Journal of Pure and Applied Algebra **97**(2), 135–161, 1994.

[13] G. Janelidze, L. Márki, and W. Tholen, *Locally semisimple coverings*, Journal of Pure and Applied Algebra **128**(3), 281–289, 1998.

[14] G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, Journal of Pure and Applied Algebra **168**, 367–386, 2002.

[15] G. Janelidze and W. Tholen, *Facets of descent, I*, Applied Categorical Structures **2**(3), 245–281, 1994.

[16] T. Janelidze, *Relative homological categories*, Journal of Homotopy and Related Structures **1**(1), 185–194, 2006.

[17] T. Janelidze, *Relative semi-abelian categories*, Applied Categorical Structures **17**(4), 373–386, 2009.

[18] T. Janelidze-Gray, *Composites of central extensions form a relative semi-abelian category*, Applied Categorical Structures **22**(5), 857–872, 2014.

[19] G. M. Kelly, *Basic consepts of enriched category theory*, Cambridge University Press, 1982.

[20] P. Köhler, *The semigroup of varieties of brouwerian semilattices*, Trasactions of the american mathematical society **241**, 331–342, 1978.

[21] B. H. Neumann, H. Neumann, and P. M. Neumann, *Wreath products and varieties of groups*, Math. Z. **80**, 44–62, 1962.

[22] H. Neumann, *Varieties of groups*, Springer-Verlag, 1967.

[23] A. L. Shmel'kin, *The semigroup of group manifolds*, Soviet Math, Dockl **4**, 449–451, 1963.

[24] N. Yoneda, *On ext and exact sequences*, Journal of the Faculty of Science, Section I **8**, 507–576, 1961.

James Richard Andrew Gray
Department of Mathematical Sciences
University of Stellenbosch
Stellenbosch
South Africa.
email: jamesgray@sun.ac.za

Tamar Janelidze-Gray
Department of Mathematics and Applied Mathematics
University of Cape Town
Cape Town
South Africa.
email: tamar.janelidze-gray@uct.ac.za

This article may be accessed at `http://www.tac.mta.ca/tac/`