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STACKS AND SHEAVES OF CATEGORIES AS FIBRANT OBJECTS, II

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Abstract. We revisit what we call the fibred topology on a fibred category over a site and we prove a few basic results concerning this topology. We give a general result concerning the invariance of a 2-category of stacks under change of base.

1. Introduction

This article is a sequel to [15] and it consists of two parts that can be read independently.

The first part is centered around a certain topology on a fibred category over a site. Let E be a site and F a fibred category over E. Then F, with a suitable notion of refinement of objects, becomes a site (Definition 2.15 and Proposition 2.20). We call this topology the *fibred* topology on F, in order to emphasize that we are in the presence of a fibration and to distinguish it from the *induced* topology on F [1, III, Section 3] (but see below). The fibred topology has a long history. It first appeared in [2, Definition 4.10] for E the category of schemes with the étale topology and F a stack in groupoids. It is also found in [9, page 596], [8, Lemma 1], [11, 3.1], [14, Tag 06NT], and surely in other sources that we are not aware of. However, all the mentioned sources treat only the cases where either the fibred category F is the Grothendieck construction on a presheaf on E of categories or the topology on E is generated by a pretopology. We establish here the existence of the fibred topology in full generality, with no restrictions on E or F. Our approach aims at clarifying the role of cartesian morphisms in the construction of this topology. We prove a few basic results concerning the fibred topology. A first (Corollary 2.31) is that for fibrations in groupoids, the fibred topology coincides with the induced topology. A second (Lemma 2.25) is that for prestacks over a subcanonical site the fibred topology is itself subcanonical, at least for prestacks in groupoids or when the topology on the base category is generated by a pretopology. A third result (Theorem 2.33) is a 'fibred' analogue of the comparison lemma [1, III, Théorème 4.1]. A fourth (Proposition 3.1) is a characterization of stacks over stacks in groupoids inspired by [14, Tag 06NT], which goes as follows. Let G be a stack in groupoids over E. Then a fibration F over G is a stack over G for the fibred topology if and only if F is a stack over E. The proof that we give uses the language of model categories We give the following application to Proposition 3.1. Recall from [15, Theorem 7.4] the localized 'projective' model category

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for presheaves of categories, denoted by $\mathrm{Stack}(\widehat{E})_{proj}$. Let X be a preasheaf on E of groupoids such that ΦX, the Grothendieck construction associated to X, is a stack. We show that the model categories $\mathrm{Stack}(\widehat{\Phi X})_{proj}$ and $(\mathrm{Stack}(\widehat{E})_{proj})_{/X}$ are Quillen equivalent (Theorem 3.4). This result is similar to [12, Theorem 6.2(a)] and [8, Lemma 18], but it is weaker in the sense that *op. cit.* do not require that ΦX is a stack.

The second part of the article deals with the invariance of stacks under change of base. Given a functor $E \to E'$ between sites, it is natural to compare stacks over E' to stacks over E, in other words to give an analogue of the comparison lemma for stacks. The first result in this sense is due to Giraud [5, II, Théorème 3.3.1]: when $E \to E'$ is the functor underlying a morphism of sites $E' \to E$ that induces an equivalence between the associated categories of sheaves, the 2-categories stacks over E' and E are biequivalent. In Theorem 4.7 we give an alternative version to this result, using model categories. Instead of a morphism of sites, we work with a continuous functor that is what we call *locally flat on refinements*, and instead of requiring the equivalence between the categories of sheaves, we ask for a Quillen equivalence between the 'projective' model categories of sheaves of categories [15, Theorem 7.2].

An Appendix is devoted to a general version of the above mentioned comparison lemma.

2. The fibred topology on a fibred category over a site

2.1. Inverse and direct images of sieves. We recall some basic results about inverse and direct images of sieves.

Let E be a category. We shall identify a collection $\mathscr{S}$ of objects of E with the full subcategory of E having as objects the elements of $\mathscr{S}$. We recall that for an object S of E, $E_{/S}$ stands for the category of objects of E over S.

A *sieve* of E is a collection R of objects of E such that for every arrow $x \to y$ of E, $y \in R$ implies $x \in R$. A set theoretical union or intersection of sieves is a sieve.

Let $\mathscr{S}$ be a collection of objects of E. We define $\overline{\mathscr{S}}$ to be the collection of objects x of E for which there is an $S \in \mathscr{S}$ and an arrow $x \to S$. Then $\overline{\mathscr{S}}$ is a sieve of E, called the sieve *generated* by $\mathscr{S}$. If $\mathscr{S}$ is contained in a sieve R, then $\overline{\mathscr{S}} \subseteq R$. The collection $\mathscr{S}$ is a sieve if and only if $\mathscr{S} = \overline{\mathscr{S}}$. The sieve $\overline{\mathscr{S}}$ is the intersection of all sieves of E that contain $\mathscr{S}$.

Let E' be another category and $u\colon E \to E'$ a functor. Then $u(\overline{\mathscr{S}}) \subseteq \overline{u(\mathscr{S})}$ and so $\overline{u(\overline{\mathscr{S}})} = \overline{u(\mathscr{S})}$. For $S \in Ob(E)$, we denote by u_S the natural functor $E_{/S} \to E'_{/u(S)}$. If either u_S is surjective on objects for all $S \in \mathscr{S}$ or u is surjective on objects and full, then $u(\overline{\mathscr{S}}) = \overline{u(\mathscr{S})}$. In particular, if R is a sieve of E then $u(R)$ is a sieve of E' if either u_x is surjective on objects for all $x \in R$ or u is surjective on objects and full.

2.2. Lemma. *Let $S \in Ob(E)$ and $\mathscr{S}$ be a collection of objects of $E_{/S}$. Then $u_S(\overline{\mathscr{S}}) = \overline{u_S(\mathscr{S})}$ if either u_T is surjective on objects for all $T \in Ob(E)$ or u_S is surjective on objects* *and full. In particular, let R be a sieve of $E_{/S}$. Then $u_S(R)$ is a sieve of $E'_{/u(S)}$ if either u_T is surjective on objects for all $T \in Ob(E)$ or u_S is surjective on objects and full.*

Let R' be a sieve of E'. Let R'^u be the collection of objects x of E such that $u(x) \in R'$. Then R'^u is a sieve of E, called the *inverse image* of R' [4, page 78].

2.3. LEMMA. *Let R be a sieve of E and R' a sieve of E'. Then $u(R'^u) \subseteq R'$ and $R \subseteq \overline{u(R)}^u$.*

Let $\varnothing(E)$ be the set of sieves of E, ordered by inclusion. The functor u induces a map of posets $\varnothing(u)\colon \varnothing(E') \to \varnothing(E)$ given by $\varnothing(u)(R') = R'^u$. This data defines a functor $\varnothing\colon CAT^{op} \to POSET$, where $POSET$ is the full subcategory of CAT consisting of posets. From now on we shall write $(-)^u$ instead of $\varnothing(u)$. It follows from Lemma 2.3 that

2.4. COROLLARY. *The functor $(-)^u\colon \varnothing(E') \to \varnothing(E)$ has a left adjoint $\overline{u(-)}$ given by $\overline{u(-)}(R) = \overline{u(R)}$.*

We call $\overline{u(R)}$ the *direct image* of R. From Lemma 2.3 and Corollary 2.4 it follows that the functor $\overline{u(-)}$ is full and faithful if and only if for all $R \in \varnothing(E)$ one has $\overline{u(R)}^u \subseteq R$ and that the functor $(-)^u$ is full and faithful if and only if for all $R' \in \varnothing(E')$ one has $R' \subseteq \overline{u(R'^u)}$.

2.5. LEMMA.

1. *If u is full then $\overline{u(-)}$ is full and faithful.*
2. *Suppose that for all $R' \in \varnothing(E')$ and all $y \in R'$ there are $S \in Ob(E)$ and arrows $u(S) \to y$ and $y \to u(S)$. Then $(-)^u$ is full and faithful.*

In particular, if u is surjective on objects and full then $(u(-), (-)^u)$ is an adjoint equivalence between $\varnothing(E)$ and $\varnothing(E')$.

2.6. COROLLARY. *Let $S \in Ob(E)$. Suppose that u_S is surjective on objects. Then the functor $(-)^{u_S}$ is full and faithful. If, moreover, $(-)^{u_S}$ is full, then the adjoint pair $(u_S(-), (-)^{u_S})$ is an adjoint equivalence between $\varnothing(E_{/S})$ and $\varnothing(E'_{/u(S)})$.*

2.7. EXAMPLE. Here is an example of a functor that is surjective on objects and full. Let Δ be the category whose objects are the ordered sets $[n] = (0, 1, ..., n)$, for every $n \geqslant 0$, and whose maps are the order-preserving functions. Let $\overrightarrow{\Delta}$ be the full subcategory of Δ consisting of the injective maps and $\overleftarrow{\Delta}$ be the full subcategory of Δ consisting of the surjective maps. Every map α of Δ has a unique functorial factorization $\alpha = \overrightarrow{\alpha}\overleftarrow{\alpha}$, where $\overleftarrow{\alpha}$ is in $\overleftarrow{\Delta}$ and $\overrightarrow{\alpha}$ is in $\overrightarrow{\Delta}$ [10, Chapter 15]. For all objects $[n]$ of Δ the inclusion $\overrightarrow{\Delta}_{/[n]} \subset \Delta_{/[n]}$ has a left adjoint

$$\overrightarrow{(-)}\colon \Delta_{/[n]} \to \overrightarrow{\Delta}_{/[n]}$$

that sends an object α to $\overrightarrow{\alpha}$. The functor $\overrightarrow{(-)}$ is surjective on objects and full.

2.8. Remark. To complete somewhat Corollary 2.4, we recall that the inverse image functor $(-)^u$ has also a right adjoint, which we denote by $(-)_u$

$$\varnothing(E) \underset{(-)_u}{\overset{(-)^u}{\leftrightarrows}} \varnothing(E')$$

One has that R_u is the union of all sieves R' such that $R'^u \subseteq R$. However, we shall not use this functor.

2.9. The fibred topology on a fibred category over a site. Let $f: F \to E$ be a fibration. We recall that every map $y \to x$ of F factorizes into a vertical map $y \to x'$ followed by a cartesian map $x' \to x$.

2.10. Definition. Let $f: F \to E$ be a fibration and $x \in Ob(F)$. A sieve R of $F_{/x}$ has *property C* if for all elements $y \to x$ of R, the cartesian morphism in the factorization of $y \to x$ belongs to R.

2.11. Lemma.

1. *For all sieves R' of $E_{/f(x)}$, R'^{f_x} has property C.*
2. *If f is a fibration in groupoids then every sieve of $F_{/x}$ has property C.*
3. *Every sieve of $F_{/x}$ that is generated by a collection of cartesian morphisms of F has property C.*

Proof. Part 2 is clear since all maps of F are cartesian. We prove part 3. Let $\mathscr{S} = (x_i \to x)_{i \in I}$ be a collection of cartesian morphisms of F. We show that the sieve $\overline{\mathscr{S}}$ has property C. Let $\alpha: y \to x$ be a map of $\overline{\mathscr{S}}$. Then there are $i \in I$ and $y \to x_i$ such that the composite $y \to x_i \to x$ is α. The map $y \to x_i$ factorizes into a vertical map $y \to y_i$ followed by a cartesian map $y_i \to x_i$. The composite $y_i \to x_i \to x$ is cartesian and in $\overline{\mathscr{S}}$, so $\overline{\mathscr{S}}$ has property C. ■

2.12. Lemma. *Let $f: F \to E$ be a fibration, $x \in Ob(F)$ and R a sieve of $F_{/x}$. Then R has property C if and only if $f_x(R)^{f_x} \subseteq R$. Consequently, if R has property C, then $f_x(R)^{f_x} = R$.*

Proof. We recall that we always have $R \subseteq f_x(R)^{f_x}$ (Lemma 2.3 and before it). Suppose that R has property C. Let $\alpha: y \to x$ be a map such that $f(\alpha) \in f_x(R)$. Then there is $r: z \to x$ in R such that $f(\alpha) = f(r)$. The map r factorizes into a vertical map $z \to x'$ followed by a cartesian map $c: x' \to x$. Since c is cartesian and $f(\alpha) = f(c)$, there is a unique $\beta: y \to x'$ such that $c\beta = \alpha$ and $f(\beta)$ is the identity. By assumption the map c is in R, therefore α is in R since R is a sieve.

Conversely, let $\alpha: y \to x$ be an element of R. We factorize α into a vertical map $y \to x'$ followed by a cartesian map $c: x' \to x$. We have $f(c) = f(\alpha) \in f_x(R)$, therefore $c \in f_x(R)^{f_x} \subseteq R$, so R has property C. ■

We denote by $\varnothing(F_{/x})_C$ the subset of $\varnothing(F_{/x})$ consisting of those sieves that have property C. Lemma 2.12 says that $(f_x(-), (-)^{f_x})$ is an adjoint equivalence between $\varnothing(F_{/x})_C$ and $\varnothing(E_{/f(x)})$.

2.13. Corollary. *Let $f: F \to E$ be a fibration and $\mathscr{S} = (x_i \to x)_{i \in I}$ a collection of cartesian morphisms of F. Then $f_x(\overline{\mathscr{S}})^{f_x} = \overline{\mathscr{S}}$.*

Proof. This follows from Lemmas 2.11(3) and 2.12. ∎

2.14. Lemma. *Let E be a category and $u: F \to G$ a map of $\mathrm{Fib}(E)$. Let $x \in Ob(F)$ and R be a sieve of $F_{/x}$. If R has property C, then the natural functor $R \to \overline{u_x(R)}$ is final.*

Proof. Let $k: w \to u(x)$ be an element of $\overline{u_x(R)}$. Consider the category $\mathcal{F}(k)$ whose objects are factorizations $k = u(\alpha)l: w \to u(y) \to u(x)$, where $\alpha: y \to x$ is an element of R, and whose morphisms $(l_1, \alpha_1) \to (l_2, \alpha_2)$ are maps $\delta: y_1 \to y_2$ such that $\alpha_2 \delta = \alpha_1$ and $u(\delta)l_1 = l_2$ We have to show that $\mathcal{F}(k)$ is connected. The category $\mathcal{F}(k)$ is not empty since $k \in \overline{u_x(R)}$. Consider the commutative diagram

$$\begin{tikzcd}
& w \arrow[dl, "l_1"'] \arrow[dr, "l_2"] \arrow[dd, "k"] & \\
u(y_1) \arrow[dr, "u(\alpha_1)"'] & & u(y_2) \arrow[dl, "u(\alpha_2)"] \\
& u(x) &
\end{tikzcd}$$

in which $\alpha_i \in R$. Let $f: F \to E$ be the structure map of F and $g: G \to E$ the structure map of G. Let $c_i: z_i \to y_i$ be a cartesian morphism over $g(l_i)$. Then $f(\alpha_1 c_1) = f(\alpha_2 c_2)$. The map $\alpha_i c_i$ factorizes into a vertical map $t_i: z_i \to w_i$ followed by a cartesian morphism $d_i: w_i \to x$. Since R has property C, the map d_i is in R. Since $u(c_i)$ is cartesian, there is a vertical morphism $p_i: w \to u(z_i)$ such that $u(c_i)p_i = l_i$. Thus, in the previous commutative diagram we may assume that α_1 and α_2 are cartesian over the same map. Then, there is a vertical isomorphism $\delta: y_1 \to y_2$ such that $\alpha_2 \delta = \alpha_1$. Since $u(\alpha_2)$ is cartesian it follows that $u(\delta)l_1 = l_2$. This completes the proof. ∎

For convenience we recall from [5, 0, Définition 1.2] the notion of site. Let E be a category. A *topology* on E is an application which associates to each $S \in Ob(E)$ a non-empty collection $J(S)$ of sieves of $E_{/S}$ that satisfies the following axioms:

(T_I) for each morphism $f: T \to S$ and each $R \in J(S)$ we have $R^f \in J(T)$;

(T_{II}) for each object S of E, each $R \in J(S)$ and each sieve R' of $E_{/S}$, we have that $R' \in J(S)$ as soon as for each element $f: T \to S$ of R we have $R'^f \in J(T)$.

The elements of $J(S)$ are called *refinements* of S. A *site* is a category endowed with a topology.

2.15. Definition. Let E be a site. Let $f\colon F \to E$ be a fibration and $x \in Ob(F)$. A sieve R of $F_{/x}$ is said to be a *refinement* of x if there is $R_0 \in J(f(x))$ such that $R_0^{f_x} \subseteq R$.

We shall show in Proposition 2.20 that the application that associates to an object x of F the collection $J(x)$ of refinements of x defines a topology on F.

2.16. Corollary. *Let E be a site, $f\colon F \to E$ a fibration, $x \in Ob(F)$ and R a sieve of $F_{/x}$.*

1. *If R is refinement of x, then $f_x(R)$ is a refinement of $f(x)$. The converse holds provided that R has property C.*
2. *Suppose that the topology on E is generated by a pretopology. Then R is a refinement of x if and only if R contains a collection of cartesian morphisms $(x_i \to x)_{i\in I}$ such that $(f(x_i) \to f(x))_{i\in I}$ is a covering family.*

Proof. (1) Let $R_0 \in J(f(x))$ be such that $R_0^{f_x} \subseteq R$. We have $R_0 \subseteq f_x(R_0^{f_x}) \subseteq f_x(R)$ so $f_x(R)$ is a refinement. The converse follows from Lemma 2.12.

(2) Suppose that R is a refinement of x. Let $R_0 \in J(f(x))$ be such that $R_0^{f_x} \subseteq R$. Let $\mathscr{S}_0 = (S_i \to f(x))_{i\in I}$ be a covering family such that $\overline{\mathscr{S}_0} \subseteq R_0$. For each $i \in I$, let $x_i \to x$ be a cartesian morphism over $S_i \to f(x)$. We put $\mathscr{S} = (x_i \to x)_{i\in I}$. Then we have $\overline{\mathscr{S}} = f_x(\overline{\mathscr{S}})^{f_x} = \overline{f_x(\mathscr{S})}^{f_x} \subseteq R_0^{f_x}$, where the first equality holds by Corollary 2.13.

The converse is proved along the same lines. ∎

2.17. Proposition.

1. (*Refinement preservation*) *Let E be a site and $u\colon F \to G$ a map of* $\mathrm{Fib}(E)$. *Let $x \in Ob(F)$ and R be a refinement of x. Then $\overline{u_x(R)}$ is a refinement of $u(x)$.*
2. (*Cocontinuity*) *Let E be a site and $u\colon F \to G$ a map of* $\mathrm{Fib}(E)$. *Then for all $x \in Ob(F)$ and all refinements R' of $u(x)$, R'^{u_x} is a refinement of x.*
3. (*Transitivity*) *Let E be a site and $g\colon G \to E, f\colon F \to G$ be fibrations. Let $x \in Ob(F)$ and R be a sieve of $F_{/x}$. If R is a refinement of x with respect to gf then $f_x(R)$ is a refinement of $f(x)$. The converse holds provided f is a fibration in groupoids.*

Proof. (1) Let $f\colon F \to E$ be the structure map of F and $g\colon G \to E$ the structure map of G. Let $R_0 \in J(f(x))$ be such that $R_0^{f_x} \subseteq R$. We claim that $R_0^{g_{u(x)}} \subseteq \overline{u_x(R)}$. Let $\beta\colon z \to u(x)$ be an element of $R_0^{g_{u(x)}}$. Let $\alpha\colon y \to x$ be a cartesian morphism over $g(\beta)$. Then $f(\alpha) = g(\beta) \in R_0$, so $\alpha \in R$ and therefore $u(\alpha) \in u_x(R)$. Since $u(\alpha)$ is cartesian, there is a unique $\gamma\colon z \to u(y)$ such that $u(\alpha)\gamma = \beta$ and $g(\gamma)$ is the identity. In particular $\beta \in \overline{u_x(R)}$.

(2) Let $R_0 \in J(f(x))$ be such that $R_0^{g_{u(x)}} \subseteq R'$. We have $R_0^{f_x} = R_0^{g_{u(x)}u_x} \subseteq R'^{u_x}$, so R'^{u_x} is a refinement of x.

(3) Let $R_0 \in J(gf(x))$ be such that $R_0^{(gf)_x} \subseteq R$. We have

$$R_0^{g_{f(x)}} \subseteq f_x((R_0^{g_{f(x)}})^{f_x}) \subseteq f_x(R)$$

so $f_x(R)$ is a refinement of $f(x)$. Conversely, let $R_0 \in J(gf(x))$ be such that $R_0^{g_{f(x)}} \subseteq f_x(R)$. We have $R_0^{(gf)_x} \subseteq f_x(R)^{f_x} \subseteq R$, where the last inclusion holds by Lemmas 2.11(2) and 2.12. Therefore R is a refinement of x. ■

Let now $f\colon F \to E$ be a functor such that f_x is surjective on objects for all $x \in Ob(F)$. Let $\alpha\colon y \to x$ be a map of F. Then α induces natural functors $\alpha\colon F_{/y} \to F_{/x}$ and $f(\alpha)\colon E_{/f(y)} \to E_{/f(x)}$. We have the following diagram of adjunctions (Lemma 2.2 and Corollary 2.4)

$$\begin{tikzcd}
\varnothing(F_{/y}) \arrow[r, shift left, "\alpha(-)"] \arrow[d, shift left, "f_y(-)"] & \varnothing(F_{/x}) \arrow[l, shift left, "(-)^{\alpha}"] \arrow[d, shift left, "f_x(-)"] \\
\varnothing(E_{/f(y)}) \arrow[u, shift left, "(-)^{f_y}"] \arrow[r, shift left, "f(\alpha)(-)"] & \varnothing(E_{/f(x)}) \arrow[u, shift left, "(-)^{f_x}"] \arrow[l, shift left, "(-)^{f(\alpha)}"]
\end{tikzcd}$$

in which $(-)^{f_y}(-)^{f(\alpha)} = (-)^{\alpha}(-)^{f_x}$. Therefore there is an induced natural transformation

$$[\alpha]\colon \alpha(-)(-)^{f_y} \to (-)^{f_x} f(\alpha)(-)\colon \varnothing(E_{/f(y)}) \to \varnothing(F_{/x})$$

In other words, for all sieves R of $E_{/f(y)}$, one has $\alpha(R^{f_y}) \subseteq f(\alpha)(R)^{f_x}$.

2.18. Lemma. *The natural transformation $[\alpha]$ is the identity if and only if for all sieves R of $E_{/f(y)}$, every diagram*

$$\begin{tikzcd}
 & y \arrow[d, "\alpha"] \\
z \arrow[r] & x
\end{tikzcd}$$

such that the diagram

$$\begin{tikzcd}
 & f(y) \arrow[d, "f(\alpha)"] \\
f(z) \arrow[r] & f(x)
\end{tikzcd}$$

has a diagonal filler that belongs to R, has itself a diagonal filler d such that $f(d)$ belongs to R.

Proof. The proof is straightforward. ■

2.19. Corollary. *Suppose that α is a cartesian morphism. Then the natural transformation $[\alpha]$ is the identity.*

Proof. This follows from Lemma 2.18. ■

2.20. Proposition. *Let E be a site and $f\colon F \to E$ a fibration. The application that associates to an object x of F the collection $J(x)$ of refinements of x defines a topology on F. We call this topology the* fibred topology.

Proof. First of all, if $x \in Ob(F)$ then $J(x)$ is non-empty since $J(f(x))$ is so. Let $\alpha\colon y \to x$ be a map of F and R be a refinement of x. Then there is $R_0 \in J(f(x))$ such that $R_0^{f_x} \subseteq R$. We have $(R_0^{f(\alpha)})^{f_y} = (R_0^{f_x})^{\alpha} \subseteq R^{\alpha}$ so R^{α} is a refinement of y. Finally, let $x \in Ob(F)$ and R, R' be sieves of $F_{/x}$ such that R is a refinement of x. Moreover, assume that for all elements $\alpha\colon y \to x$ of R, R'^{α} is a refinement of y. We have to prove that R' is a refinement of x.

By assumption, for all objects $\alpha\colon y \to x$ of R there is a refinement R_α of $f(y)$ such that $R_\alpha^{f_y} \subseteq R'^{\alpha}$. Therefore

$$\alpha(R_\alpha^{f_y}) \subseteq \alpha(R'^{\alpha}) \subseteq R'$$

Let R_0 be a refinement of $f(x)$ such that $R_0^{f_x} \subseteq R$. We put

$$\mathfrak{S} = \bigcup_{\alpha \in R_0^{f_x}, \alpha \ cartesian} f(\alpha)(R_\alpha)$$

Then $\mathfrak{S}$ is a sieve of $E_{/f(x)}$. We claim that $\mathfrak{S}$ is a refinement of $f(x)$ and that $\mathfrak{S}^{f_x} \subseteq R'$. We have

$$\mathfrak{S}^{f_x} = \bigcup_{\alpha \in R_0^{f_x}, \alpha \ cartesian} f(\alpha)(R_\alpha)^{f_x} = \bigcup_{\alpha \in R_0^{f_x}, \alpha \ cartesian} \alpha(R_\alpha^{f_y}) \subseteq R'$$

where the second equality holds by Corollary 2.19. Let now $S \to f(x)$ be an element of R_0. Let $\alpha_0\colon y \to x$ be a cartesian map over $S \to f(x)$. Then we have

$$R_{\alpha_0} \subseteq f(\alpha_0)(R_{\alpha_0})^{f(\alpha_0)} \subseteq \bigcup_{\alpha \in R_0^{f_x}, \alpha \ cartesian} f(\alpha)(R_\alpha)^{f(\alpha_0)} = \mathfrak{S}^{f(\alpha_0)}$$

Therefore $\mathfrak{S}^{f(\alpha_0)}$ is a refinement since it contains a refinement, hence $\mathfrak{S}$ is a refinement. ∎

In Proposition 2.20, if E has the discrete topology then the fibred topology on F is the discrete topology, and if E has the coarse topology then the fibred topology on F is the coarse topology provided E has pullbacks or F is fibred in groupoids.

2.21. Example. Let E be a site and A a category. The projection $A \times E \to E$ is a fibration. The refinements of $(a, S) \in Ob(A \times E)$ are of the form $A_{/a} \times R$, where R is a refinement of S. Every refinement of (a, S) has property C.

Let E be a site whose topology is generated by a pretopology. Let $f\colon F \to E$ be a fibration. We recall [11, 3.1], [14, Tag 06NT] that F becomes a site with the *inherited topology* from E. A family $(x_i \to x)$ of maps of F is covering for the inherited topology from E if and only if each $x_i \to x$ is cartesian and the family $(f(x_i) \to f(x))$ is covering for the topology on E.

2.22. LEMMA. *Let E be a site whose topology is generated by a pretopology and $F \to E$ a fibration. Then the fibred topology on F coincides with the inherited topology from E.*

PROOF. This follows from Corollary 2.16(2). ∎

2.23. LEMMA. *Let E be a category and J, J' be two topologies on E. Suppose that J is finer than J'. Let $F \to E$ be a fibration. Then the fibred topology on F with respect to J is finer than the fibred topology on F with respect to J'.*

PROOF. The proof is straightforward. ∎

In light of Corollary 2.31, the next result extends [1, III, Proposition 5.2(3)].

2.24. LEMMA. *Let E be a site and $g: G \to E, f: F \to G$ be fibrations. Suppose that f is a fibration in groupoids. Then the fibred topology on F obtained using gf coincides with the fibred topology obtained using f and the fibred topology on G.*

PROOF. This follows from Proposition 2.17(3) and Lemma 2.12. ∎

The next result, partly combined with Corollary 2.31, extends [3, IV, Corollaire 4.5.3]; its proof highlights the role of property C (Definition 2.10).

2.25. LEMMA. *Let E be a site whose topology is less fine than the canonical topology and $p: F \to E$ a fibration. If the fibred topology on F is less fine than the canonical topology then F is a prestack. Conversely, suppose that F is a prestack. Then the fibred topology on F is less fine than the canonical topology in either of the following cases:*

(*a*) *the topology on E is generated by a pretopology;*

(*b*) *F is fibred in groupoids.*

PROOF. We prove the Lemma in the case $F = \Phi X$, that is, F is the Grothendieck construction associated to a functor $X: E^{op} \to CAT$. The general case follows using the surjective equivalence $\Phi \mathcal{S} F \to F$ [15, 2.4(4)].

Suppose that the fibred topology on ΦX is less fine than the canonical topology. Let $S \in Ob(E)$, $x, y \in Ob(X(S))$ and R_0 be a refinement of S. Then $R_0^{p(S,x)}$ is a refinement of $(S, x) \in Ob(\Phi X)$. We have to prove that the natural map

$$X(S)(x, y) \to \lim_{R_0^{op}} [(T, f: T \to S) \mapsto X(T)(f^*(x), f^*(y))]$$

is bijective. Let $\alpha, \beta: x \to y$ be such that $f^*(\alpha) = f^*(\beta)$ for each $f \in R_0$. By assumption, the natural map

$$\Phi X(-, (S, y))(S, x) \to \lim_{(R_0^{p(S,x)})^{op}} [((f, u): (T, z) \to (S, x)) \mapsto \Phi X((T, z), (S, y))]$$

is a bijection. Since the elements $(Id_S, \alpha), (Id_S, \beta)$ of $\Phi X((S, x), (S, y))$ are sent to the same element $(f, f^*(\alpha)u)$, it follows that $\alpha = \beta$. Let now $v_f: f^*(x) \to f^*(y)$, where

$f: T \to S$ is an element of R_0, be a compatible family, in the sense that whenever we have a commutative diagram

$$\begin{tikzcd} U \arrow[rr, "h"] \arrow[dr, "g"'] & & T \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

with $f, g \in R_0$, then $h^*(v_f) = v_g$. Consider the commutative diagram

$$\begin{tikzcd} (U,t) \arrow[rr, "(h,w)"] \arrow[dr, "(g,\omega)"'] & & (T,z) \arrow[dl, "(f,u)"] \\ & (S,x) & \end{tikzcd}$$

where (f,u) and (g,ω) are elements of $R_0^{p(S,x)}$, so that $fh = g$ and $h^*(u)w = \omega$. Then $(g, v_g\omega) = (h,w)^*((f, v_f u))$. In other words the family $(f, v_f u): (T,z) \to (S,y)$ indexed over $(f,u): (T,z) \to (S,x)$, with $(f,u) \in R_0^{p(S,x)}$, is compatible. Therefore there is a unique element $(a,\alpha): (S,x) \to (S,y)$ such that $(af, f^*(\alpha)u) = (f, v_f u)$ for each element (f,u) of $R_0^{p(S,x)}$. In particular, $(af, f^*(\alpha)) = (f, v_f)$ for each $f \in R_0$. By considering the bijection

$$E(-,S)(S) \to \lim_{R_0^{op}}[(T, f: T \to S) \mapsto E(T,S)]$$

it follows that $a = 1_S$, therefore $\alpha: x \to y$ and $f^*(\alpha) = v_f$ for each $f \in R_0$.

We now prove the converse. Let us assume for the moment that the topology on E is an arbitrary one and that ΦX is an arbitrary prestack. Let $(T,y), (S,x)$ be two objects of ΦX and R a refinement of (S,x). We prove that the natural map

$$\Phi X(-,(T,y))(S,x) \to \lim_{R^{op}}(\Phi X(-,(T,y))|R)$$

is bijective. We first show that the map is injective. Let $(f,u), (g,u'): (S,x) \to (T,y)$ be such that $(f,u)(h,v) = (g,u')(h,v)$ for each element $(h,v): (U,z) \to (S,x)$ of R. Since there is a refinement R_0 of S such that $R_0^{p(S,x)} \subseteq R$, we have in particular that $fh = gh$ for each element h of R_0. Since $E(-,T)$ is a sheaf, it follows that $f = g$. So we have that the composite $(fh)^*(y) \overset{h^*(u)}{\to} h^*(x) \overset{v}{\to} z$ is equal to the composite $(fh)^*(y) \overset{h^*(u')}{\to} h^*(x) \overset{v}{\to} z$. Choosing (h,v) to be $(h, 1_{h^*(x)}): (U, h^*(x)) \to (S,x)$, where $h: U \to S$ is an element of R_0, we have that $h^*(u) = h^*(u')$ for each element h of R_0. Since ΦX is a prestack, the natural map

$$X(S)(f^*(y),x) \to \lim_{R_0^{op}}[(h: U \to S) \mapsto X(U)(h^*(f^*(y)), h^*(x))]$$

is bijective. Therefore $u = u'$. We now wish to show the surjectivity of our original natural map. Let $(h_{(f,u)}, v_{(f,u)}): (U,z) \to (T,y)$ be a compatible family indexed over the elements $(f,u): (U,z) \to (S,x)$ of R, so that $h_{(f,u)}: U \to T$, $v_{(f,u)}: h^*_{(f,u)}(y) \to z$, and compatibility means that whenever we have a commutative diagram

$$\begin{tikzcd}
(U',z') \arrow[rr, "{(g,w)}"] \arrow[dr, "{(f',u')}"'] & & (U,z) \arrow[dl, "{(f,u)}"] \\
& (S,x) &
\end{tikzcd}$$

with $(f,u),(f',u')$ elements of R, then $(h_{(f',u')}, v_{(f',u')}) = (h_{(f,u)}g, wg^*(v_{(f,u)}))$. We have to produce a map $h\colon S \to T$ and map $v\colon h^*(y) \to x$ of $X(S)$ such that $(hf, uf^*(v)) = (h_{(f,u)}, v_{(f,u)})$ for each element (f,u) of R. Since $E(-,T)$ is a sheaf and $p_{(S,x)}(R)$ is a refinement of S (Proposition 2.17(1)), the natural map

$$E(-,T)(S) \to \lim_{p_{(S,x)}(R)^{op}} (E(-,T)|p_{(S,x)}(R))$$

is bijective. The family $(h_{(f,u)})$ is compatible by the above, therefore there is a unique $h\colon S \to T$ such that $hf = h_{(f,u)}$ for each element (f,u) of R. Consider now the bijection

$$X(S)(h^*(y),x) \to \lim(p_{(S,x)}(R)^{op} \to SET)$$

where the functor $p_{(S,x)}(R)^{op} \to SET$ sends $(f\colon U \to S)$, for $(f,u) \in R$, to the set $X(U)(f^*(h^*(y)), f^*(x)) = X(U)(h^*_{(f,u)}(y), f^*(x))$. The family $(v_{(f,1_{f^*(x)})}\colon h^*_{(f,u)}(y) \to f^*(x))$ is compatible provided $(f,u) \in R$ implies $(f, 1_{f^*(x)}) \in R$, in other words provided R has property C, for then the diagram

$$\begin{tikzcd}
(U',f'^*(x)) \arrow[rr, "{(g,1_{g^*(f^*(x))})}"] \arrow[dr, "{(f',1_{f'^*(x)})}"'] & & (U,f^*(x)) \arrow[dl, "{(f,1_{f^*(x)})}"] \\
& (S,x) &
\end{tikzcd}$$

is commutative whenever $(g,w)\colon (f',u') \to (f,u)$ is a map in R as in the previous triangle diagram above, and so in this case $v_{(f',1_{f'^*(x)})} = g^*(v_{(f,1_{f^*(x)})})$. But then we have a map

$$\begin{tikzcd}
(U,z) \arrow[rr, "{(1_U,u)}"] \arrow[dr, "{(f,u)}"'] & & (U,f^*(x)) \arrow[dl, "{(f,1_{f^*(x)})}"] \\
& (S,x) &
\end{tikzcd}$$

in R, so in particular $v_{(f,u)} = uv_{(f,1_{f^*(x)})}$. Therefore, if R has property C, there is a unique $v\colon h^*(y) \to x$ such that $f^*(v) = v_{(f,1_{f^*(x)})}$ for each element (f,u) of R, and so $v_{(f,u)} = uf^*(v)$ in this case. This finishes the proof that our original natural map is surjective. Finally, by Lemma 2.11 we know that R has property C in the cases (a) and (b). ∎

2.26. Functoriality of the fibred topology. Let E, E' be sites and $u\colon E \to E'$ a functor. We recall some definitions from [1, III]. One says that u is *continuous* if the composition with u functor $u^*\colon \widehat{E'} \to \widehat{E}$ sends sheaves to sheaves. One says that u is *cocontinuous* if for all $S \in Ob(E)$ and all refinements R' of $u(S)$, R'^{u_S} is a refinement of S. This is equivalent to saying that for all $S \in Ob(E)$ and all refinements R' of $u(S)$, there is a refinement R of S such that $u_S(R) \subseteq R'$. One says that u is *refinement preserving* if for all $S \in Ob(E)$ and all refinements R of S, $\overline{u_S(R)}$ is a refinement of $u(S)$.

If u is cocontinuous and refinement preserving, then for all $S \in Ob(E)$ the adjoint pair $(\overline{u_S(-)}, (-)^{u_S})$ (Corollary 2.4) restricts to an adjoint pair between the refinements of S and the refinements of $u(S)$.

In the next result, all fibred categories over a site will be considered as having the fibred topology. We recall that $\mathrm{Fibg}(E)$ is the full subcategory of $\mathrm{Fib}(E)$ whose objects are the categories fibred in groupoids.

2.27. Corollary. *Let E be a site.*

1. *Every arrow of* $\mathrm{Fib}(E)$ *is cocontinuous and refinement preserving. If the topology on E is generated by a pretopology, then every arrow of* $\mathrm{Fib}(E)$ *is continuous.*
2. *Every arrow of* $\mathrm{Fibg}(E)$ *is cocontinuous and continuous.*

Proof. The first part of part 1 is a consequence of Proposition 2.17((2) and (1)). We prove the last part. Let $u\colon F \to G$ be a map in $\mathrm{Fib}(E)$. Let $x \in Ob(F)$, R be a sieve of $F_{/x}$ and Z a sheaf on G. Consider the following commutative diagram

$$\begin{tikzcd} R^{op} \arrow[r] \arrow[d] & (F_{/x})^{op} \arrow[r] \arrow[d] & F^{op} \arrow[r, "u^*Z"] \arrow[d] & SET \\ \overline{u_x(R)}^{op} \arrow[r] & (G_{/u(x)})^{op} \arrow[r] & G^{op} \arrow[ur, "Z"'] & \end{tikzcd}$$

We denote by $(u^*Z|R)$ the composite of the top horizontal arrows and by $(Z|\overline{u_x(R)})$ the composite of the bottom horizontal arrows followed by Z. We have a natural map

$$\varprojlim_{\overline{u_x(R)}^{op}} (Z|\overline{u_x(R)}) \to \varprojlim_{R^{op}} (u^*Z|R)$$

To show that u^*Z is a sheaf we can assume, using Lemma 2.22, that R is generated by a collection of cartesian morphisms. But then the natural map displayed above is bijective by Lemmas 2.14 and 2.11(3). We conclude by Proposition 2.17(1). The continuity in part 2 is proved similarly, using the same diagram. The map $R \to \overline{u_x(R)}$ is now an equivalence. ∎

2.28. Corollary. *Let E be a site and $f: F \to E$ a fibration. Then the fibred topology on F is the least fine topology for which f is cocontinuous.*

For the next results we need to fix some notation. Let E be a category. We denote by $\widehat{E}$ the category of presheaves on E and by $\eta: E \to \widehat{E}$ the Yoneda embedding. For $S \in Ob(E)$ and a sieve R of $E_{/S}$, we denote by R^p the associated subfunctor of $\eta(S)$. Let $u: E \to E'$ be a functor. We denote by $u_!$ the left adjoint to the composition with u functor $u^*: \widehat{E'} \to \widehat{E}$. There is a natural map $u_!(R^p) \to \eta(u(S))$.

2.29. Lemma. *Let $S \in Ob(E)$ and R be a sieve of $E_{/S}$. There is a natural factorization*

$$\begin{tikzcd} u_!(R^p) \arrow[rr] \arrow[dr, "e"'] & & \eta(u(S)) \\ & \overline{u_S(R)}^p \arrow[ur] & \end{tikzcd}$$

with e an epimorphism. If the natural functor $R \to \overline{u_S(R)}$ is final, then e is an isomorphism.

Proof. Let $y \in Ob(E')$. Then $\overline{u_S(R)}^p(y)$ is the set of maps $y \to u(S)$ that factorize as $y \to u(T) \stackrel{u(f)}{\to} u(S)$ for some element f of R, and $u_!(R^p)(y)$ is the colimit of the functor $P_y: (y \downarrow u)^{op} \to SET$ that sends an object $(T, \alpha: y \to u(T))$ to the set of arrows $f: T \to S$ that belong to R. The map that sends f to $u(f)\alpha$ induces a map

$$e_y: \operatorname*{colim}_{(y \downarrow u)^{op}} P_y \to \overline{u_S(R)}^p(y)$$

This map is clearly natural in y, so it induces the map e. Notice that by construction the map e_y is surjective, so e is an epimorphism. We prove that e_y is injective. The colimit of P_y is the quotient of the set

$$\coprod_{(T, \alpha: y \to u(T))} \{f: T \to S, f \in R\}$$

by the following equivalence relation. Two elements $(T, \alpha: y \to u(T), f: T \to S, f \in R)$ and $(T', \alpha': y \to u(T'), f': T' \to S, f' \in R)$ are equivalent when there is a finite sequence of elements

$$(T_0, \alpha_0: y \to u(T_0), f_0: T_0 \to S, f_0 \in R), ..., (T_n, \alpha_n: y \to u(T_n), f_n: T_n \to S, f_n \in R)$$

such that $T_0 = T, \alpha_0 = \alpha, f_0 = f, T_n = T', \alpha_n = \alpha', f_n = f'$ and for all $0 \leqslant i \leqslant n-1$ there is $g_i: T_{i+1} \to T_i$ (possibly an identity) such that $u(g_i)\alpha_{i+1} = \alpha_i$ and $f_i g_i = f_{i+1}$ or there is $g_i: T_i \to T_{i+1}$ (possibly an identity) such that $u(g_i)\alpha_i = \alpha_{i+1}$ and $f_{i+1} g_i = f_i$.

Let $(T, \alpha: y \to u(T), f: T \to S, f \in R)$ and $(T', \alpha': y \to u(T'), f': T' \to S, f' \in R)$ be such that $u(f)\alpha = u(f')\alpha'$. The fact that $R \to \overline{u_S(R)}$ is final implies then that there is a diagram

$$
\begin{tikzcd}
 & & & y \arrow[dlll, "\alpha"'] \arrow[dll] \arrow[dl] \arrow[d] \arrow[drr] \arrow[drrr, "\alpha'"] & & & \\
u(T) \arrow[r, dash, "u(g_1)"] \arrow[drrr, "u(f)"'] & u(T_1) \arrow[r, dash, "u(g_2)"] \arrow[drr, "u(f_1)"] & u(T_2) \arrow[r, dash, "u(g_3)"] \arrow[dr, "u(f_2)"] & u(T_3) \arrow[r, dash] \arrow[d, "u(f_3)"] & \dots \arrow[r, dash, "u(g_{n-1})"] & u(T_{n-1}) \arrow[r, dash, "u(g_n)"] \arrow[dll, "u(f_{n-1})"'] & u(T') \arrow[dlll, "u(f')"] \\
 & & & u(S) & & &
\end{tikzcd}
$$

in which each map g_i points to the left or to the right, each map f_i is an element of R, all triangles having y as a vertex and one horizontal side commute, and all triangles having $u(S)$ as a vertex and one horizontal side commute before applying the functor u to them. In particular, $(T, \alpha\colon y \to u(T), f\colon T \to S, f \in R)$ and $(T', \alpha'\colon y \to u(T'), f'\colon T' \to S, f' \in R)$ are equivalent, so e_y is injective. ∎

The next result is meant to extend [1, III, Proposition 1.6].

2.30. Proposition. *Let E, E' be sites and $u\colon E \to E'$ a functor. If u is continuous then u is refinement preserving. The converse holds provided for all $S \in Ob(E)$ and all refinements R of S, the natural functor $R \to \overline{u_S(R)}$ is final.*

Proof. Suppose that u is continuous. Consider the natural factorization from Lemma 2.29. By [1, III, Proposition 1.2] the map $u_!(R^p) \to \eta(u(S))$ is a local isomorphism, so the map $\overline{u_S(R)}^p \to \eta(u(S))$ is a local epimorphism, therefore u is refinement preserving by [5, 0, Proposition 3.5.2(iii)]. The converse proof is similar to the proof of Corollary 2.27(1). Let $S \in Ob(E)$, R be a refinement of S and Z a sheaf on E'. Consider the following commutative diagram

$$
\begin{tikzcd}
R^{op} \arrow[r] \arrow[d] & (E_{/S})^{op} \arrow[r] \arrow[d] & E^{op} \arrow[r, "u^* Z"] \arrow[d] & SET \\
\overline{u_S(R)}^{op} \arrow[r] & (E'_{/u(S)})^{op} \arrow[r] & E'^{op} \arrow[ur, "Z"'] &
\end{tikzcd}
$$

We denote by $(u^*Z|R)$ the composite of the top horizontal arrows and by $(Z|\overline{u_S(R)})$ the composite of the bottom horizontal arrows followed by Z. The natural map

$$
\varprojlim_{\overline{u_S(R)}^{op}} (Z|\overline{u_S(R)}) \to \varprojlim_{R^{op}} (u^*Z|R)
$$

is bijective by assumption. It follows that u^*Z is a sheaf on E. ∎

In Proposition 2.30, the condition 'for all $S \in Ob(E)$ and all refinements R of S, the natural functor $R \to \overline{u_S(R)}$ is final' is fulfilled if u has a left adjoint. See Lemma 4.18 for a result similar to Proposition 2.30.

Let now E' be a site and $u: E \to E'$ a functor. We recall [1, III, 3.1] that the *induced topology on E by the functor u* is the finest topology for which u is continuous.

2.31. Corollary. *Let E be a site and $f: F \to E$ a fibration in groupoids. Then the fibred topology on F coincides with the induced topology on F by the functor f.*

Proof. The functor f is continuous by Corollary 2.27(2). Therefore the fibred topology is less fine than the induced topology. Conversely, let $x \in Ob(F)$ and R be a refinement of x for the induced topology. By Proposition 2.30, using Lemmas 2.11(2) and 2.14 and Proposition 2.17(1), it follows that $f_x(R)$ is a refinement of $f(x)$. Since $f_x(R)^{f_x} \subseteq R$ (Lemmas 2.12 and 2.11(2)), it follows that R is a refinement of x for the fibred topology. ∎

2.32. Example. Continuing Example 2.21, the projection $A \times E \to E$ is refinement preserving, continuous and cocontinuous. The fibred topology on $A \times E$ coincides with the induced topology by the projection. It is well-known that the category of sheaves on $A \times E$ is equivalent to the functor category $[A^{op}, \widetilde{E}]$, where $\widetilde{E}$ is the category of sheaves on E, and the former is in turn the category of sheaves on E with values in $\widehat{A}$.

2.33. Theorem. *[une lemme de comparaison fibrée] Let E be a site and $u: F \to G$ a map in* Fib(E)*. Suppose that:*

1. *u is continuous;*
2. *for each object x of F, the map $F(-, x) \to G(u(-), u(x))$ is a local monomorphism;*
3. *for each object x of F, the map $F(-, x) \to G(u(-), u(x))$ is a local epimorphism;*
4. *each object y of G has a refinement whose objects are of the form $y' \to u(x) \to y$.*

Then the composition with u functor $u^*: \widehat{G} \to \widehat{F}$ induces and equivalence between the categories of sheaves $\widetilde{G}$ and $\widetilde{F}$.

Proof. This follows from Proposition 4.16, using Corollary 2.27. ∎

2.34. Corollary. *Let E be a site whose topology is generated by a pretopology and less fine than the canonical topology. Let F, G be prestacks over E and $u: F \to G$ a map in* Fib(E)*. The composition with u functor $u^*: \widehat{G} \to \widehat{F}$ induces and equivalence between the categories of sheaves $\widetilde{G}$ and $\widetilde{F}$ if and only if u is full and faithful and each object y of G has a refinement whose objects are of the form $y' \to u(x) \to y$.*

Proof. This follows from Proposition 4.17 and Theorem 2.33, using Corollary 2.27 and Lemma 2.25. ∎

3. Stacks over stacks in groupoids

Let E be a site and $g\colon G \to E$ a fibration. Since G is a site (Proposition 2.20), it is natural to ask: what are the stacks over G? The purpose of this section is to answer this question in the case G is a stack in groupoids (Proposition 3.1). We give one application to our result (Theorem 3.4).

For a site E, we denote by $\mathit{Stack}(E)$ the full subcategory of $\mathrm{Fib}(E)$ whose objects are the stacks over E.

3.1. Proposition. *Suppose that G is a stack in groupoids. Let $(\mathit{Stack}(E)_{/G})_{fib}$ be the full subcategory of $\mathit{Stack}(E)_{/G}$ whose objects are the pairs $(F, F \to G)$ with $F \to G$ a fibration. Then we have an isomorphism of categories*

$$\mathit{Stack}(G) \cong (\mathit{Stack}(E)_{/G})_{fib}$$

that is the identity on objects.

Proposition 3.1 can be compared to [14, Tag 06NT] and can be restated as: the natural functor $g^{\bullet}\colon \mathrm{Fib}(G) \to \mathrm{Fib}(E)$ preserves and reflects stacks provided G is a stack in groupoids.

Proof. Let $f\colon F \to E$ be a stack over E and $u\colon F \to G$ a fibration. We shall show that F is a stack over G. Let $z \in \mathit{Ob}(G)$ and R be a refinement of z. We shall show that the map

$$\mathbf{Cart}_G(G_{/z}, F) \to \mathbf{Cart}_G(R, F)$$

is an equivalence. Let $g^{\bullet}\colon \mathrm{Fib}(G) \to \mathrm{Fib}(E)$ be the natural functor. The natural map $g_z\colon G_{/z} \to E_{/g(z)}$ can be viewed as a map $g_z\colon g^{\bullet}(G_{/z}) \to E_{/g(z)}$ in $\mathrm{Fib}(E)$. Then we have a commutative diagram

$$\begin{tikzcd}
g^{\bullet}(R) \arrow[r] \arrow[d, "g_{z|R}"'] & g^{\bullet}(G_{/z}) \arrow[d, "g_z"] \\
g_z(R) \arrow[r] & E_{/g(z)}
\end{tikzcd}$$

in which the horizontal arrows are inclusions and the vertical ones are trivial fibrations since g is a fibration in groupoids. Consider the solid arrow diagram

$$\begin{tikzcd}
 & & g^{\bullet}(R) \arrow[dr] \arrow[dd, "g_{z|R}"] & \\
 & & & g^{\bullet}(G_{/z}) \arrow[dd, "g_z"] \\
g_z(R) \arrow[rr, equal] \arrow[dr] \arrow[uurr, dotted, "s'"] & & g_z(R) \arrow[dr] & \\
 & E_{/g(z)} \arrow[rr, equal] \arrow[uurr, dotted] & & E_{/g(z)}
\end{tikzcd}$$

There is a diagonal filler s' since $g_{z|_R}$ is a trivial fibration. There is then a diagonal filler $s\colon E_{/g(z)} \to g^\bullet(G_{/z})$ since $g_z(R) \to E_{g(z)}$ is a cofibration and g_z is a trivial fibration. We can now define a natural transformation

$$\nu\colon \mathbf{Cart}_G(G_{/z}, -) \to \mathbf{Cart}_E(E_{/g(z)}, g^\bullet(-))\colon \mathrm{Fib}(G) \to CAT$$

as $\nu_F(v) = g^\bullet(v)s$, and a natural transformation

$$\nu'\colon \mathbf{Cart}_G(R, -) \to \mathbf{Cart}_E(g_z(R), g^\bullet(-))\colon \mathrm{Fib}(G) \to CAT$$

as $\nu'_F(v) = g^\bullet(v)s'$. Since (s', s) is a diagonal filler, the diagram

$$\begin{tikzcd}
\mathbf{Cart}_G(G_{/z}, F) \arrow[r, "\nu_F"] \arrow[d] & \mathbf{Cart}_E(E_{/g(z)}, g^\bullet(F)) \arrow[d] \\
\mathbf{Cart}_G(R, F) \arrow[r, "\nu'_F"] & \mathbf{Cart}_E(g_z(R), g^\bullet(F))
\end{tikzcd}$$

is commutative. Consider the commutative cubic diagram

$$\begin{tikzcd}
\mathbf{Cart}_G(G_{/z}, F) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbf{Cart}_G(G_{/z}, G) \arrow[dr] \arrow[dd] & \\
& \mathbf{Cart}_E(E_{/g(z)}, g^\bullet(F)) \arrow[rr] \arrow[dd] & & \mathbf{Cart}_E(E_{/g(z)}, G) \arrow[dd] \\
\mathbf{Cart}_G(R, F) \arrow[rr] \arrow[dr] & & \mathbf{Cart}_G(R, G) \arrow[dr] & \\
& \mathbf{Cart}_E(g_z(R), g^\bullet(F)) \arrow[rr] & & \mathbf{Cart}_E(g_z(R), G)
\end{tikzcd}$$

The two vertical arrows of the front face of the cube are weak equivalences by Corollary 2.16(1) and the fact that F and G are stacks. The map $\mathbf{Cart}_G(G_{/z}, G) \to \mathbf{Cart}_G(R, G)$ is the identity map of the terminal category. To finish the proof it suffices, by [10, Proposition 13.3.14], to show that the top and bottom faces of the cube are homotopy fiber squares. The top face is the outer diagram

$$\begin{tikzcd}
\mathbf{Cart}_G(G_{/z}, F) \arrow[r] \arrow[d] & \mathbf{Cart}_G(G_{/z}, G) = * \arrow[d] \\
\mathbf{Cart}_E(g^\bullet(G_{/z}), g^\bullet(F)) \arrow[r] \arrow[d] & \mathbf{Cart}_E(g^\bullet(G_{/z}), G) \arrow[d] \\
\mathbf{Cart}_E(E_{/g(z)}, g^\bullet(F)) \arrow[r] & \mathbf{Cart}_E(E_{/g(z)}, G)
\end{tikzcd}$$

Using the maps $u\colon g^\bullet(F) \to G$ and $s\colon E_{/g(z)} \to g^\bullet(G_{/z})$ it follows from [15, Proposition 4.10] that in the previous diagram the bottom part is a homotopy fiber square. One can easily check that the top part is a pullback; since it is a pullback along a fibration, it is a homotopy fiber square. Therefore the previous diagram is a homotopy fiber square by [10, Proposition 13.3.15].

The bottom face of the cubic diagram is dealt with similarly, using the maps $u\colon g^\bullet(F) \to G$ and $s'\colon g_z(R) \to g^\bullet(R)$. Therefore F is a stack over G.

Conversely, let $u\colon F \to G$ be a stack over G. We shall show that $g^\bullet(F)$ is a stack over E. Let $S \in Ob(E)$ and R be a refinement of S. Consider the solid arrow diagram

$$\begin{tikzcd}
\mathbf{Cart}_E(E_{/S}, g^\bullet(F)) \arrow[r] \arrow[d] & \mathbf{Cart}_E(E_{/S}, G) \arrow[r] \arrow[d, shift right] & G_S \\
\mathbf{Cart}_E(R, g^\bullet(F)) \arrow[r] & \mathbf{Cart}_E(R, G) \arrow[u, dotted, shift right] &
\end{tikzcd}$$

Since G is a stack, the right vertical arrow is a surjective equivalence. The dotted arrow is a section of it. If G_S is the empty category then the left vertical arrow is the identity arrow of the empty category. Hence we may assume that G_S is nonempty. Let z be such that $g(z) = S$. Then $R = g_z(R^{g_z})$ (Lemmas 2.11(2) and 2.12) and R^{g_z} is a refinement of z. Recall from the first part of the proof the diagonal filler (s', s):

$$\begin{tikzcd}
 & & g^\bullet(R^{g_z}) \arrow[dd, "g_{z|R^{g_z}}"] \arrow[dr] & & \\
 & & & g^\bullet(G_{/z}) \arrow[dd, "g_z"] & \\
R \arrow[uurr, dotted, "s'"] \arrow[rr, equal] \arrow[dr] & & R \arrow[dr] & & \\
 & E_{/g(z)} \arrow[rr, equal] \arrow[uurr, dotted] & & E_{/g(z)} &
\end{tikzcd}$$

For every fibration H over E we have then a commutative diagram

$$\begin{tikzcd}
\mathbf{Cart}_E(E_{/g(z)}, H) \arrow[r] \arrow[d] & \mathbf{Cart}_E(g^\bullet(G_{/z}), H) \arrow[r] \arrow[d] & \mathbf{Cart}_E(E_{/g(z)}, H) \arrow[d] \\
\mathbf{Cart}_E(R, H) \arrow[r] & \mathbf{Cart}_E(g^\bullet(R^{g_z}), H) \arrow[r] & \mathbf{Cart}_E(R, H)
\end{tikzcd}$$

in which the top horizontal composite and the bottom horizontal composite are the identity. For $H = g^\bullet(F)$ the middle vertical map in the previous diagram is

$$\mathbf{Cart}_E(g^\bullet(G_{/z}), g^\bullet(F)) \to \mathbf{Cart}_E(g^\bullet(R^{g_z}), g^\bullet(F))$$

which is isomorphic to the map

$$\mathbf{Cart}_G(G_{/z}, g_\bullet(g^\bullet(F))) \to \mathbf{Cart}_G(R^{g_z}, g_\bullet(g^\bullet(F)))$$

since the 2-functor $g^\bullet$ has a right 2-adjoint $g_\bullet$ [15, 2.4]. The object $g_\bullet(g^\bullet(F))$ is calculated from the pullback diagrams

$$\begin{tikzcd} g_\bullet(g^\bullet(F)) \arrow[r] \arrow[d] & F \arrow[d, "u"] \\ g_\bullet(G) \arrow[r] \arrow[d] & G \arrow[d, "g"] \\ G \arrow[r, "g"'] & E \end{tikzcd}$$

in other words $g_\bullet(g^\bullet(F)) = g_\bullet(G) \times F$. Since $g_\bullet$ preserves stacks [15, Proposition 5.17], it follows that $g_\bullet(g^\bullet(F))$ is a stack over G. Consequently, the map

$$\mathbf{Cart}_E(E_{/S}, g^\bullet(F)) \to \mathbf{Cart}_E(R, g^\bullet(F))$$

is a retract of an equivalence, hence it is an equivalence. ■

Let now P be a presheaf on E. We denote by $E_{/P}$ the Grothendieck construction associated to P, often called the category of elements of P. Every map $F \to E_{/P}$ of $\mathrm{Fib}(E)$ is an isofibration by [15, Lemma 2.3(1)]. From Lemma 3.3 we deduce that

$$\mathrm{Fib}(E_{/P}) \cong \mathrm{Fib}(E)_{/E_{/P}}$$

The next result can be compared with [14, Tag 04WT].

3.2. Corollary. *Suppose that P is a sheaf on E. Then we have an isomorphism of categories*

$$\mathit{Stack}(E_{/P}) \cong \mathit{Stack}(E)_{/E_{/P}}$$

Proof. This is a consequence of Proposition 3.1 and of the above considerations. ■

3.3. Lemma. *Let $g\colon G \to E$ be a category fibred in groupoids such that for all $S \in Ob(E)$ the fibre category G_S is rigid, meaning that the group of automorphisms of every object of G_S is trivial. Then* $\mathrm{Fib}(G)$ *is isomorphic to the full subcategory of* $\mathrm{Fib}(E)_{/G}$ *consisting of the 'fibrant' objects, meaning that the map to G is an isofibration.*

Proof. A fibration over G is an isofibration. Conversely, let $u\colon F \to G$ be a map in $\mathrm{Fib}(E)$ that is an isofibration. We show that u is a fibration. Let $\beta\colon a \to u(x)$ be a map in G. Let $\alpha\colon y \to x$ be a cartesian (with respect to gu) map over $g(\beta)$. Since g is a fibration in groupoids there is an isomorphism $\theta\colon a \to u(y)$ in $G_{g(a)}$ such that $u(\alpha)\theta = \beta$. Since $u_{g(a)}$ is an isofibration there is an isomorphism $\gamma\colon y' \to y$ in $F_{g(a)}$ such that $u(\gamma) = \theta$. Therefore $u(\alpha\gamma) = \beta$. We show that $\alpha\gamma$ is cartesian with respect to u. Let $h\colon z \to x$ and $t\colon u(z) \to u(y')$ be such that $\beta t = u(h)$. Then $gu(\alpha\gamma)g(t) = gu(h)$ and since $\alpha\gamma$ is cartesian there is a unique map $\delta\colon z \to y'$ such that $\alpha\gamma\delta = h$ and $gu(\delta) = g(t)$. Since g is a fibration in groupoids there is an isomorphism $\xi\colon u(z) \to u(z)$ in $G_{gu(z)}$ such that $u(\delta)\xi = t$. By assumption ξ must be the identity. ■

For the next result, recall from [15, Theorem 7.4] the model category $\mathrm{Stack}(\widehat{E})_{proj}$.

3.4. THEOREM. *Let* $X\colon E^{op} \to GRPD$ *be such that its Grothendieck construction* ΦX *is a stack. Then the model categories* $\mathrm{Stack}(\widehat{\Phi X})_{proj}$ *and* $(\mathrm{Stack}(\widehat{E})_{proj})_{/X}$ *are Quillen equivalent.*

Before giving the proof we recall the functors that give the Quillen equivalence. Let E be a category and $X\colon E^{op} \to CAT$. There is an adjoint pair

$$[(\Phi X)^{op}, CAT] \underset{\mathfrak{G}_X}{\overset{\mathcal{B}_X}{\rightleftarrows}} [E^{op}, CAT]_{/X}$$

The functor $\mathfrak{G}_X$ is defined in [12, Definition 4.2]; one has that $\mathfrak{G}_X(P, f\colon P \to X)$ is the functor $(\Phi X)^{op} \to CAT$ that sends an object (S, x) to the fibre $P(S)_x$ of $f_S\colon P(S) \to X(S)$ at x. The left adjoint $\mathcal{B}_X$ to $\mathfrak{G}_X$ is defined as follows. Let $Z\colon (\Phi X)^{op} \to CAT$ and $S \in Ob(E)$. We denote by $(Z|S)$ the composite functor $X(S)^{op} \to (\Phi X)^{op} \xrightarrow{Z} CAT$. Then $\mathcal{B}_X Z(S)$ is the Grothendieck construction associated to $(Z|S)$. Alternatively, $\mathcal{B}_X Z(S)$ can be described as the coend of the functor $X(S)^{op} \times X(S) \to CAT$ that sends (x, y) to $(Z|S)(x) \times X(S)_{/y}$ [7, Lemma 3.2]. The description of $\mathcal{B}_X Z(S)$ as a coend is also present in [12, Definition 4.1]. For all objects Z of $[(\Phi X)^{op}, CAT]$, the unit map $Z \to \mathfrak{G}_X \mathcal{B}_X Z$ is an isomorphism.

3.5. THEOREM. *Suppose that* $X\colon E^{op} \to GRPD$*. Then the adjoint pair* $(\mathcal{B}_X, \mathfrak{G}_X)$ *is a Quillen equivalence between* $[(\Phi X)^{op}, CAT]_{proj}$ *and* $([E^{op}, CAT]_{proj})_{/X}$*.*

Theorem 3.5 is proved in [12, Theorem 4.4(b)] in the case when $[E^{op}, CAT]$ is regarded as having the injective model category. For completeness we shall give a simpler proof. We first recall a few facts about

Fibrations over groupoids. Let E be a groupoid. Then a functor $f\colon F \to E$ is a fibration if and only if f is an isofibration. In this case F^{cart} is the maximal groupoid associated to F and $\mathrm{Fib}(E)$ is a full subcategory of $CAT_{/E}$, namely the subcategory of fibrant objects.

PROOF OF THEOREM 3.5. It is straightforward from the definition that $\mathfrak{G}_X$ preserves fibrations and trivial fibrations. We claim that $\mathcal{B}_X$ preserves and reflects weak equivalences. Let $u\colon Z \to Z'$ be a map in $[(\Phi X)^{op}, CAT]$. For all $S \in Ob(E)$ the map $(\mathcal{B}_X u)_S$ is a map in $\mathrm{Fib}(X(S))$. By [15, Proposition 4.3] $(\mathcal{B}_X u)_S$ is an $X(S)$-equivalence if and only if for all $x \in Ob(X(S))$ the map $u_{(S,x)}$ is an equivalence. The claim follows. To complete the proof it suffices to show that for all fibrant objects (P, f) of $([E^{op}, CAT]_{proj})_{/X}$ the map $\mathcal{B}_X \mathfrak{G}_X(P, f) \to (P, f)$ is a weak equivalence.

Let $(P, f\colon P \to X)$ be a fibrant object of $([E^{op}, CAT]_{proj})_{/X}$. Since X takes values in groupoids, this means that for all $S \in Ob(E)$ the map f_S is a fibration. The map $\mathcal{B}_X \mathfrak{G}_X(P, f) \to (P, f)$ is a weak equivalence if and only if for all $S \in Ob(E)$ the map $\Phi(\mathfrak{G}_X(P, f)|S) \to P(S)$ is an equivalence, where $\Phi(\mathfrak{G}_X(P, f)|S)$ is the Grothendieck construction associated to the functor $(\mathfrak{G}_X(P, f)|S)\colon X(S)^{op} \to CAT$ that sends x to the fibre $P(S)_x$ of f_S at x. By [15, Proposition 4.3] the map $\Phi(\mathfrak{G}_X(P,f)|S) \to P(S)$ is an equivalence if and only if for all $x \in Ob(X(S))$ the map $\Phi(\mathfrak{G}_X(P,f)|S)_x \to P(S)_x$ is an equivalence. But this map is the identity. ■

Proof of Theorem 3.4. Let $m\colon \Phi X \to E$ be the natural map. We have the following diagram of adjunctions

$$\begin{tikzcd}
{[(\Phi X)^{op}, CAT]} \arrow[r, shift left, "\mathcal{B}_X"] \arrow[d, shift left, "\Phi_X"] & {[E^{op}, CAT]_{/X}} \arrow[l, shift left, "\mathfrak{G}_X"] \arrow[d, shift left, "U"] \\
\mathrm{Fib}(\Phi X) \arrow[u, shift left, "\mathcal{S}_X"] \arrow[r, shift left, "m^\bullet"] & \mathrm{Fib}(E) \arrow[l, shift left, "m_\bullet"] \arrow[u, shift left, "\mathcal{S}(-)\times X"]
\end{tikzcd}$$

The functor U sends an object $(P, P \to X)$ to the composite $\Phi P \to \Phi X \to E$. There is a natural isomorphism $U\mathcal{B}_X \cong m^\bullet \Phi_X$.

We now make visible in the previous diagram the model categories we are interested in:

$$\begin{tikzcd}
\mathrm{Stack}(\widehat{\Phi X})_{proj} \arrow[r, shift left, "\mathcal{B}_X"] \arrow[d, shift left, "\Phi_X"] & (\mathrm{Stack}(\widehat{E})_{proj})_{/X} \arrow[l, shift left, "\mathfrak{G}_X"] \arrow[d, shift left, "U"] \\
\mathrm{Champ}(\Phi X) \arrow[u, shift left, "\mathcal{S}_X"] \arrow[r, shift left, "m^\bullet"] & \mathrm{Champ}(E) \arrow[l, shift left, "m_\bullet"] \arrow[u, shift left, "\mathcal{S}(-)\times X"]
\end{tikzcd}$$

We prove that the functor $\mathcal{B}_X$ preserves weak equivalences. The functor Φ_X preserves weak equivalences. Since $(m^\bullet, m_\bullet)$ is a Quillen pair, the functor $m^\bullet$ preserves weak equivalences. It is clear that U reflects weak equivalences. Therefore $\mathcal{B}_X$ preserves weak equivalences. From Theorem 3.5 it follows then that $(\mathcal{B}_X, \mathfrak{G}_X)$ is a Quillen pair.

To prove that $(\mathcal{B}_X, \mathfrak{G}_X)$ is a Quillen equivalence it suffices to show that $\mathcal{B}_X$ preserves fibrant objects. We first notice that an object $(P, P \to X)$ of $[E^{op}, CAT]_{/X}$ is fibrant in $(\mathrm{Stack}(\widehat{E})_{proj})_{/X}$ if and only if $\Phi P \to \Phi X$ is a fibration in Champ(E). Since ΦX is a stack, this is equivalent to: ΦP is a stack and $\Phi P \to \Phi X$ is an isofibration. Let Z be a fibrant object in $\mathrm{Stack}(\widehat{\Phi X})_{proj}$, meaning that $\Phi_X Z$ is a stack over ΦX. By construction, $\mathcal{B}_X Z$ is fibrant in $([E^{op}, CAT]_{proj})_{/X}$. Notice that $\Phi_X Z$ is (naturally isomorphic to) $\Phi\mathcal{B}_X Z$ and that the natural map $\Phi_X Z \to \Phi X$ is the Grothendieck construction Φ applied to the natural map $\mathcal{B}_X Z \to X$. In particular, $\Phi\mathcal{B}_X Z \to \Phi X$ is an isofibration. An application of Proposition 3.1 to the sequence $\Phi_X Z = \Phi\mathcal{B}_X Z \to \Phi X \to E$ finishes the proof. ■

4. On the invariance of stacks under change of base

In [5, II, Théorème 3.3.1 (voir aussi Lemme 3.3.2)], Giraud proves that if $E' \to E$ is a morphism of sites that induces an equivalence between the associated categories of sheaves, then the 2-categories stacks over E' and E are biequivalent. In this section we give an alternative version to this result (Theorem 4.7). We first set up the framework.

Let E, E' be sites and $u\colon E \to E'$ a functor. Consider the following solid arrow diagram of adjunctions

$$\begin{tikzcd}
\mathrm{Fib}(E) \arrow[r, dotted, shift left, "u^\bullet = \Phi' u_! \mathsf{L}"] \arrow[d, shift right=2, "\mathcal{S}"'] \arrow[d, shift left=4, "\mathsf{L}"] & \mathrm{Fib}(E') \arrow[l, shift left, "u_\bullet"] \arrow[d, shift right, "\mathcal{S}'"'] \\
{[E^{op}, CAT]} \arrow[u, "\Phi"] \arrow[r, shift left, "u_!"] & {[E'^{op}, CAT]} \arrow[l, shift left, "u^*"] \arrow[u, shift right, "\Phi'"']
\end{tikzcd}$$

in which $u_!$ is the left adjoint to the composition with u functor u^*, Φ and Φ' are the Grothendieck construction functors, L is the left adjoint to Φ and $\mathcal{S}$ and $\mathcal{S}'$ are the right adjoints to Φ and Φ'. We have $\Phi u^* = u_\bullet \Phi'$. Let $u^\bullet = \Phi' u_! \mathsf{L}$; then $u^\bullet$ is a 2-functor. By [5, I, Théorème 2.5.2] there is, for all fibrations F over E and all fibrations F' over E', a natural equivalence of categories

$$\mathbf{Cart}_{E'}(u^\bullet(F), F') \xrightarrow{\simeq} \mathbf{Cart}_E(F, u_\bullet(F')) \tag{1}$$

We recall how this is obtained. All the functors in the preceding diagram of adjunctions are 2-functors, and all the adjunctions are 2-adjunctions. Therefore there is a natural isomorphism

$$\mathbf{Cart}_{E'}(u^\bullet(F), F') \cong \mathbf{Cart}_E(F, \Phi u^* \mathcal{S}'(F'))$$

By [4, Remarque 5.12] there is a natural transformation

$$\xi\colon u^*\mathcal{S}' \to \mathcal{S}u_\bullet\colon \mathrm{Fib}(E') \to [E^{op}, CAT] \tag{2}$$

such that for all fibrations F' over E', $\xi_{F'}$ is objectwise an equivalence of categories. The desired natural equivalence is then obtained as the composite

$$\mathbf{Cart}_E(F, \Phi u^* \mathcal{S}'(F')) \to \mathbf{Cart}_E(F, \Phi \mathcal{S} u_\bullet(F')) \to \mathbf{Cart}_E(F, u_\bullet(F'))$$

where the first arrow is induced by $\Phi(\xi_{F'})$ and the second by $\mathsf{v}u_\bullet(F')$.

4.1. Lemma. *Let $u\colon E \to E'$ be a functor.*

1. *For all fibrations F over E there is a natural map $F \to u_\bullet u^\bullet(F)$.*
2. *For all fibrations F' over E' there is a map $u^\bullet u_\bullet(F') \to F'$.*

Proof. (1) The natural map is the composite

$$F \to \Phi \mathsf{L} F \to \Phi u^* u_! \mathsf{L} F = u_\bullet u^\bullet F$$

(2) The map $\mathsf{v}u_\bullet(F')\colon \Phi\mathcal{S}u_\bullet(F') \to u_\bullet(F')$ is a surjective equivalence, hence it has a section

$$\mathsf{v}u_\bullet(F')^{-1}\colon u_\bullet(F') \to \Phi\mathcal{S}u_\bullet(F')$$

By [15, Proposition 4.13] the adjunct $\mathsf{L}u_\bullet(F') \to \mathcal{S}u_\bullet(F')$ to this map is a weak equivalence in $[E^{op}, CAT]_{proj}$. Consider the following solid arrow diagram

$$\begin{tikzcd} & u^*\mathcal{S}'(F') \arrow[d, "\xi_{F'}"] \\ \mathsf{L}u_\bullet(F') \arrow[r] \arrow[ur, dotted, "d"] & \mathcal{S}u_\bullet(F') \end{tikzcd}$$

in $[E^{op}, CAT]$. The vertical map is a weak equivalence in $[E^{op}, CAT]_{proj}$ between fibrant objects and $\mathsf{L}u_\bullet(F')$ is cofibrant, hence there is a (dotted) map d such that $\xi_{F'}d$ is homotopic to the horizontal map. It follows that d is a weak equivalence. The (composite) adjunct to d is a map $\Phi' u_! \mathsf{L}u_\bullet(F') \to F'$, in other words a map $u^\bullet u_\bullet(F') \to F'$. ■

Let E be a category. We recall that $\eta: E \to \widehat{E}$ denotes the Yoneda embedding and that for $S \in Ob(E)$ and a sieve R of $E_{/S}$, R^p denotes the associated subfunctor of $\eta(S)$. We denote by $D: \widehat{E} \to [E^{op}, CAT]$ the functor induced by the discrete category functor $D: SET \to CAT$. One has $R = \Phi D R^p$.

We recall that a functor $u: E \to E'$ is said to be *flat* if for all $S' \in Ob(E')$ the category $E^{\backslash S'}$ (whose objects are pairs $(S, S' \to u(S))$) is cofiltered. We say that u is *flat on sieves* if for all sieves R of E the natural functor $R \to \overline{u(R)}$ is flat. We say that u is *locally flat on sieves* if for all $S \in Ob(E)$ the functor $u_S: E_{/S} \to E'_{/u(S)}$ is flat on sieves.

4.2. Example. Every flat functor is locally flat on sieves.

The next result is an alternative to [5, II, Proposition 3.1.1] and can be compared to [14, Tag 04WA].

4.3. Proposition. *Let E, E' be sites and $u: E \to E'$ a continuous functor that is locally flat on refinements. Then the functor $u^\bullet: \mathrm{Fib}(E') \to \mathrm{Fib}(E)$ preserves stacks.*

Proof. Let F' be a stack over E', S an object of E and R a refinement of S. Since $(\Phi, \mathcal{S})$ is a 2-adjunction we have a natural isomorphism

$$\mathbf{Cart}_E(R, u_\bullet(F')) \cong \underline{Hom}(DR^p, \mathcal{S}u_\bullet(F'))$$

Using the natural equivalence (1) and the counit map $\mathsf{L}\Phi D\eta(S) \to D\eta(S)$ we have maps

$$\mathbf{Cart}_{E'}(\Phi' u_! D\eta(S), F') \to \mathbf{Cart}_{E'}(\Phi' u_! \mathsf{L}\Phi D\eta(S), F') \xrightarrow{\simeq} \mathbf{Cart}_E(E_{/S}, u_\bullet F')$$

We claim that the first map is an equivalence. Indeed, the map $\mathsf{L}\Phi D\eta(S) \to D\eta(S)$ is a weak equivalence in $[E^{op}, CAT]_{proj}$ between cofibrant objects, hence the map

$$\Phi' u_! \mathsf{L}\Phi D\eta(S) \to \Phi' u_! D\eta(S)$$

is an E'-equivalence. The claim is proved.

Let $D': \widehat{E} \to [E'^{op}, CAT]$ be the functor induced by the discrete category functor and $u_!$ be the left adjoint to the composition with u functor $\widehat{E'} \to \widehat{E}$. Then $u_! D = D' u_!$. Using this and the fact that the composite adjunction $(\Phi' u_!, u^*\mathcal{S}')$ is a 2-adjunction, we have maps

$$\mathbf{Cart}_E(\Phi' u_! D\eta(S), F') \to \mathbf{Cart}_E(\Phi' D' u_! R^p, F') \cong \underline{Hom}(DR^p, u^*\mathcal{S}'(F'))$$

with the first map an equivalence by [1, III, Proposition 1.2] since F' is a stack.

Putting all of these maps together we see that it remains to prove that the map

$$\underline{Hom}(DR^p, \xi_{F'})\colon \underline{Hom}(DR^p, u^*\mathcal{S}'(F')) \to \underline{Hom}(DR^p, \mathcal{S}u_\bullet(F'))$$

is an equivalence. But this follows from Lemma 4.4. ■

4.4. Lemma. *Let $u\colon E \to E'$ be a functor that is locally flat on sieves. Then for all fibrations F' over E', all $S \in Ob(E)$ and all sieves R of $E_{/S}$, the map*

$$\underline{Hom}(DR^p, \xi_{F'})\colon \underline{Hom}(DR^p, u^*\mathcal{S}'(F')) \to \underline{Hom}(DR^p, \mathcal{S}u_\bullet(F'))$$

is an equivalence.

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
R^{op} \arrow[r] \arrow[d, "u_S^{op}"'] & (E_{/S})^{op} \arrow[r] \arrow[d] & E^{op} \arrow[r, "u^*\mathcal{S}'(F')"] \arrow[d] & CAT \\
\overline{u_S(R)}^{op} \arrow[r] & (E'_{/u(S)})^{op} \arrow[r] & E'^{op} \arrow[ur, "\mathcal{S}'(F')"'] &
\end{tikzcd}$$

We denote by $(u^*\mathcal{S}'(F')|R)$ the composite of the top horizontal arrows and by $(\mathcal{S}'(F')|\overline{u_S(R)})$ the composite of the bottom horizontal arrows followed by $\mathcal{S}'(F')$. By [15, Proposition 4.13] the object $\mathcal{S}'(F')$ is fibrant in $[E'^{op}, CAT]_{inj}$. By Proposition 4.5 the object

$$u_S^*(\mathcal{S}'(F')|\overline{u_S(R)}) = (u^*\mathcal{S}'(F')|R)$$

is fibrant in $[R^{op}, CAT]_{inj}$, where u_S^* is the composition with u_S^{op} functor. By [15, Proposition 4.13], the object $\mathcal{S}u_\bullet(F')$ is fibrant in $[E^{op}, CAT]_{inj}$. We denote by $(\mathcal{S}u_\bullet(F')|R)$ the composite of $\mathcal{S}u_\bullet(F')$ and $R^{op} \to E^{op}$; then by Proposition 4.5 the object $(\mathcal{S}u_\bullet(F')|R)$ is fibrant in $[R^{op}, CAT]_{inj}$. Thus, we have a weak equivalence

$$(u^*\mathcal{S}'(F')|R) \to (\mathcal{S}u_\bullet(F')|R)$$

in $[R^{op}, CAT]_{inj}$ between fibrant objects. Since the limit functor is a right Quillen functor $[R^{op}, CAT]_{inj} \to CAT$, the result follows. ■

4.5. Proposition. *Let $u\colon E \to E'$ be a functor that is either flat or a discrete fibration. Then the adjoint pair*

$$[E^{op}, CAT]_{inj} \underset{u^*}{\overset{u_!}{\rightleftarrows}} [E'^{op}, CAT]_{inj}$$

is a Quillen pair.

Proof. We show that $u_!$ preserves cofibrations and weak equivalences. Let X be a presheaf on E with values in CAT and $S' \in Ob(E')$. Then $u_!X(S')$ is calculated as the colimit of a presheaf on $E^{\setminus S'}$ with values in CAT. Suppose that u is flat. Then the result follows from the fact that a filtered colimit of monomorphisms of sets is a monomorphism and that a filtered colimit of equivalences is an equivalence. Suppose that u is a discrete fibration. Let $E_{S'}$ be the fibre category of u at S'. The natural functor $E_{S'} \to E^{\setminus S'}$ has a right adjoint hence it is cofinal. The result follows then from the fact that a coproduct of monomorphisms of sets is a monomorphism and that a coproduct of equivalences is an equivalence. ∎

Let $u\colon E \to E'$ be a continuous functor between sites. Consider the solid arrow diagram

$$\begin{tikzcd}
\mathrm{Fib}(E) \arrow[d, shift left, "\mathsf{L}"] & \mathrm{Fib}(E') \arrow[l, "u_\bullet"'] \arrow[d, shift left, "\mathsf{L}'"] \\
{[E^{op}, CAT]} \arrow[u, shift left, "\Phi"] \arrow[r, shift left, "u_!"] \arrow[d, shift left, "a"] & {[E'^{op}, CAT]} \arrow[u, shift left, "\Phi'"] \arrow[l, shift left, "u^*"] \arrow[d, shift left, "a'"] \\
\mathrm{Cat}(\widetilde{E}) \arrow[u, shift left, "i"] & \mathrm{Cat}(\widetilde{E'}) \arrow[u, shift left, "i'"] \arrow[l, dotted, "u^*"]
\end{tikzcd}$$

where $u_!$ is the left adjoint to the functor u^* obtained by composing with u and a and a' are induced by the associated sheaf functors. The dotted arrow u^* is induced by the composition with u functor since u is continuous. One has $\Phi i u^* = u_\bullet \Phi' i'$. This u^* has the composite functor $a' u_! i$ as left adjoint. Recall from [15, Theorem 7.2] the model category $\mathrm{Stack}(\widetilde{E})_{proj}$.

4.6. Corollary. *Let E, E' be sites and $u\colon E \to E'$ a continuous functor that is locally flat on refinements. Then the functor $u^*\colon \mathrm{Cat}(\widetilde{E'}) \to \mathrm{Cat}(\widetilde{E})$ is the right adjoint of a Quillen pair between $\mathrm{Stack}(\widetilde{E'})_{proj}$ and $\mathrm{Stack}(\widetilde{E})_{proj}$.*

Proof. It suffices to show that u^* preserves trivial fibrations and fibrations between fibrant objects. From the definition of fibration and weak equivalence of $\mathrm{Stack}(\widetilde{E})_{proj}$ [15, Theorem 7.2], we see that it suffices to prove that $u_\bullet$ preserves trivial fibrations, isofibrations and stacks. The first two requirements are clear, the last one is Proposition 4.3. ∎

For the next result we denote by $\mathit{Stack}(E)$ the full sub-2-category of $\mathit{Fib}(E)$ whose objects are the stacks over E.

4.7. THEOREM. *Let E, E' be sites and $u\colon E \to E'$ a continuous functor that is locally flat on refinements. Suppose that the functor $u^*\colon \mathrm{Cat}(\widetilde{E'}) \to \mathrm{Cat}(\widetilde{E})$ is the right adjoint of a Quillen equivalence between $\mathrm{Stack}(\widetilde{E'})_{proj}$ and $\mathrm{Stack}(\widetilde{E})_{proj}$. Then the map $u_\bullet\colon \mathit{Stack}(E') \to \mathit{Stack}(E)$ is a biequivalence of 2-categories.*

PROOF. Let F' and G' be stacks over E'. We show that the map

$$\mathbf{Cart}_{E'}(F', G') \to \mathbf{Cart}_E(u_\bullet(F'), (u_\bullet(G')) \tag{3}$$

is an equivalence. Let $\varepsilon_{F'}\colon u^\bullet u_\bullet(F') \to F'$ be the map constructed in Lemma 4.1(2). Consider the composite

$$\mathbf{Cart}_{E'}(F', G') \to \mathbf{Cart}_{E'}(u^\bullet u_\bullet(F'), G') \xrightarrow{\simeq} \mathbf{Cart}_E(u_\bullet(F'), (u_\bullet(G')) \tag{4}$$

where the first map is induced by $\varepsilon_{F'}$ and the second one by (1). This composite is not the map (3), but it is homotopic, in other words naturally isomorphic, to it. Hence to prove that (3) is an equivalence it suffices to prove that the map induced by $\varepsilon_{F'}$ is an equivalence. To prove this it suffices to prove that $\varepsilon_{F'}$ is a bicovering map, since G' is a stack. Going back to the construction of $\varepsilon_{F'}$ in the proof of Lemma 4.1(2), recall that there is a weak equivalence

$$d\colon \mathsf{L}u_\bullet(F') \to u^*\mathcal{S}'(F')$$

in $[E^{op}, CAT]_{proj}$. By [15, Proposition 4.13] the object $\mathsf{L}u_\bullet(F')$ is cofibrant in $[E^{op}, CAT]_{proj}$. Since F' is a stack, the object $\mathcal{S}'(F')$ is fibrant in $\mathrm{Stack}(\widehat{E'})_{proj}$ [15, Theorem 7.4]. By hypothesis and [15, Theorems 7.2 and 7.4], the model categories $\mathrm{Stack}(\widehat{E})_{proj}$ and $\mathrm{Stack}(\widehat{E'})_{proj}$ are Quillen equivalent, therefore the adjunct of d, which is a map $u_!\mathsf{L}u_\bullet(F') \to \mathcal{S}'(F')$, is a weak equivalence in $\mathrm{Stack}(\widehat{E'})_{proj}$. This implies that $\varepsilon_{F'}$ is a bicovering map.

Let now F be a stack over E. We need to show that there is a stack F' over E' and an equivalence $u_\bullet(F') \simeq F$ in $\mathit{Stack}(E)$. Equivalently, we need to show that there is a stack F' over E' and an E-equivalence $u_\bullet(F') \to F$. The unit $F \to \Phi\mathsf{L}F$ of the adjunction (L, Φ) is an E-equivalence, therefore $\mathsf{L}F$ is fibrant in $\mathrm{Stack}(\widehat{E})_{proj}$ [15, Theorem 7.4]. The object $\mathsf{L}F$ is also cofibrant. By hypothesis and [15, Theorems 7.2 and 7.4], the model categories $\mathrm{Stack}(\widehat{E})_{proj}$ and $\mathrm{Stack}(\widehat{E'})_{proj}$ are Quillen equivalent. Therefore, if $u_!\mathsf{L}F \to \widehat{u_!\mathsf{L}F}$ is a fibrant approximation to $u_!\mathsf{L}F$ in $\mathrm{Stack}(\widehat{E'})_{proj}$, then the composite map $\mathsf{L}F \to u^*u_!\mathsf{L}F \to u^*\widehat{u_!\mathsf{L}F}$ is a weak equivalence in $\mathrm{Stack}(\widehat{E})_{proj}$. The composite map is a weak equivalence between fibrant objects, therefore it is objectwise an equivalence of categories. It follows that the composite

$$F \to \Phi\mathsf{L}F \to u_\bullet(\Phi'\widehat{u_!\mathsf{L}F})$$

is an E-equivalence, with $\Phi'\widehat{u_!\mathsf{L}F}$ a stack over E'. ∎

Appendix

On the comparison lemma

Let E and E' be two sites. In this section we shall give some sufficient conditions under which a *continuous* functor $E \to E'$ (2.26) induces an equivalence between the categories of sheaves $\widetilde{E'}$ and $\widetilde{E}$ (Proposition 4.16). These are needed in the proof of Theorem 2.33. The original result of this kind is the so-called comparison lemma [1, III, Théorème 4.1]. To the best of our knowledge, generalizations of the comparison lemma have been given in [13, page 152], [9, C2.2, Theorem 2.2.3] and [14, Tag 039Z]. Our conditions coincide with those of [14], but to arrive at them we use a different approach, one that we hope is 'plus naturelle'. We also work in full generality, meaning that the topologies are not necessarily generated by pretopologies. An effort was made to make the presentation less dependent on the results of [1, III].

We begin by considering the following solid arrow diagram of categories and functors

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "F"] \arrow[d, shift left, "a"] & \mathcal{M}' \arrow[l, shift left, "G"] \arrow[d, shift left, "a'"] \\ \mathcal{N} \arrow[u, shift left, "i"] \arrow[r, shift left, dotted, "F'"] & \mathcal{N}' \arrow[l, shift left, "G'"] \arrow[u, shift left, "i'"] \end{tikzcd}$$

We assume that F is left adjoint to G, a is left adjoint to i, a' is left adjoint to i', $iG' = Gi'$, and i, i' are full and faithful. Then one can easily check that $F' = a'Fi$ is left adjoint to G'. Our goal is to provide some extra assumptions on the categories and functors in the above diagram that will make the adjoint pair (F', G') an adjoint equivalence (Corollary 4.11).

We say that a map f of $\mathcal{M}$ is a *local isomorphism/epimorphism/monomorphism* if $a(f)$ is an isomorphism/epimorphism/monomorphism in $\mathcal{N}$. We say the same about a map of $\mathcal{M}'$.

There is a natural transformation

$$\nu\colon aG \to G'a'\colon \mathcal{M}' \to \mathcal{N}$$

defined as $\nu_Y = aG(\eta'_Y)$, where $\eta'_Y\colon Y \to i'a'(Y)$ is the unit of the adjunction.

4.8. Lemma. *The natural transformation ν is a natural isomorphism if and only if G preserves local isomorphisms.*

Proof. Suppose that ν is a natural isomorphism. Let $Y \to Y'$ be a local isomorphism. From the commutative diagram

$$\begin{tikzcd} aG(Y) \arrow[r, "\nu_Y"] \arrow[d] & G'a'(Y) \arrow[d] \\ aG(Y') \arrow[r, "\nu_{Y'}"] & G'a'(Y') \end{tikzcd}$$

it follows that $G(Y) \to G(Y')$ is a local isomorphism. Conversely, for each object Y of $\mathcal{M}'$, the unit map $\eta'_Y : Y \to i'a'(Y)$ is a local isomorphism. ∎

4.9. Lemma. *Consider the following two properties:*

(u) *for each object X of $\mathcal{N}$, the unit map $i(X) \to GFi(X)$ is a local isomorphism, and*

(c) *for each object Y of $\mathcal{N}'$, the counit map $FGi'(Y) \to i'(Y)$ is a local isomorphism.*

1. *If* (u) *holds and G reflects local isomorphisms then* (c) *holds.*
2. *Assume that G preserves local isomorphisms. Then (F', G') is an adjoint equivalence if and only if both* (u) *and* (c) *hold.*

Proof. Suppose that (u) holds and that G reflects local isomorphisms. It suffices to show that the map $GFGi'(Y) \to Gi'(Y)$ is a local isomorphism. The composite $Gi'(Y) \to GFGi'(Y) \to Gi'(Y)$ is the identity and the first map in the previous sequence is obtained by putting $X = G'(Y)$ in (u), hence the conclusion.

If (F', G') is an adjoint equivalence then clearly (u) and (c) hold. Conversely, let $Y \in Ob(\mathcal{N}')$. The counit map $F'G'(Y) \to Y$ is the composite

$$a'FGi'(Y) \to a'i'(Y) \to Y$$

so it is an isomorphism by (c). Let $X \in Ob(\mathcal{N})$. The unit map $X \to G'F'(X)$ is the composite

$$X \to ai(X) \to aGFi(X) \to G'a'Fi(X)$$

so it is an isomorphism by (u) and Lemma 4.8. ∎

We recall that a category is said to have kernel (cokernel) pairs when the pullback (pushout) of any arrow along itself exists.

4.10. Lemma. *Assume that:*

1. *the categories $\mathcal{M}, \mathcal{M}', \mathcal{N}$ and $\mathcal{N}'$ have kernel pairs and the functors a, a' preserve kernel pairs;*
2. *in $\mathcal{N}$ and $\mathcal{N}'$, a morphism is an isomorphism if and only if it is both an epimorphism and a monomorphism.*

If G preserves (reflects) local epimorphisms, then G preserves (reflects) local isomorphisms.

PROOF. Let $f\colon X \to Y$ be an arbitrary map in $\mathcal{N}$. We claim that

(i) $aG(f)$ is a monomorphism if and only if $G(X \to X \times_Y X)$ is a local epimorphism, and

(ii) $a'(f)$ is a monomorphism if and only if $X \to X \times_Y X$ is a local epimorphism.

The claims are true since a and a' preserve kernel pairs.

Suppose that G preserves local epimorphisms. Let $f\colon X \to Y$ be a local isomorphism of $\mathcal{N}$. Since $a'(f)$ is an epimorphism, so is $aG(f)$. Since $a'(f)$ is a monomorphism, the map $X \to X \times_Y X$ is a local epimorphism by the above, therefore $G(X \to X \times_Y X)$ is a local epimorphism, so again by the above the map $aG(f)$ is a monomorphism. Therefore $G(f)$ is a local isomorphism. Suppose now that G reflects local epimorphisms. Let $f\colon X \to Y$ be a map in $\mathcal{N}$ such that $G(f)$ is a local isomorphism. Since $aG(f)$ is an epimorphism, the map f is a local epimorphism. Since $aG(f)$ is a monomorphism, the map $G(X \to X \times_Y X)$ is a local epimorphism by the above, therefore $X \to X \times_Y X$ is a local epimorphism, so again by the above the map $a'(f)$ is a monomorphism. ∎

Summing up the last three Lemmas we obtain

4.11. COROLLARY. *Assume that:*

1. *the categories $\mathcal{M}, \mathcal{M}', \mathcal{N}$ and $\mathcal{N}'$ have kernel pairs and the functors a, a' preserve kernel pairs;*
2. *in $\mathcal{N}$ and $\mathcal{N}'$, a morphism is an isomorphism if and only if it is both an epimorphism and a monomorphism;*
3. *the functor G preserves and reflects local epimorphisms;*
4. *for each object X of $\mathcal{N}$, the unit map $i(X) \to GFi(X)$ is a local isomorphism.*

Then the adjoint pair (F', G') is an adjoint equivalence.

4.12. LEMMA. *Assume that the categories $\mathcal{M}, \mathcal{M}', \mathcal{N}$ and $\mathcal{N}'$ have cokernel pairs. Then G reflects local epimorphisms if and only if for each object Y of $\mathcal{M}'$, the counit map $FG(Y) \to Y$ is a local epimorphism.*

PROOF. Suppose that G reflects local epimorphisms. Let $Y \in Ob(\mathcal{M}')$. It suffices to show that the map $GFG(Y) \to G(Y)$ is a local epimorphism. The composite $G(Y) \to GFG(Y) \to G(Y)$ is the identity, hence the map $GFG(Y) \to G(Y)$ is an epimorphism, hence a local epimorphism. Conversely, let $Y \to Y'$ be a map of $\mathcal{N}$ such that $G(Y) \to G(Y')$ is a local epimorphism. From the commutative diagram

$$\begin{tikzcd} FG(Y) \arrow[r] \arrow[d] & FG(Y') \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

it suffices to show that F preserves local epimorphisms. But this is true since $a'F \cong F'a$. ∎

We now return to our original problem. Let $u\colon E \to E'$ be a continuous functor. We denote by $\widehat{E}$ and $\widehat{E'}$ the categories of presheaves on E and on E'. The composition with u functor $u^*\colon \widehat{E'} \to \widehat{E}$ sends sheaves to sheaves. The functor u^* has a left adjoint $u_!$ and a right adjoint u_*. We denote by y both Yoneda embeddings $E \to \widehat{E}$ and $E' \to \widehat{E'}$. For $S \in Ob(E)$ and a sieve R of $E_{/S}$, we denote by R^p the associated subfunctor of $\mathsf{y}(S)$. We wish to apply Corollary 4.11 to the diagram

$$\begin{tikzcd}
\widehat{E} \arrow[r, shift left, "u_!"] \arrow[d, shift left, "a"] & \widehat{E'} \arrow[l, shift left, "u^*"] \arrow[d, shift left, "a'"] \\
\widetilde{E} \arrow[u, shift left, "i"] \arrow[r, dotted, shift left, "\tilde{u}_!"] & \widetilde{E'} \arrow[l, shift left, "u^*"] \arrow[u, shift left, "i'"]
\end{tikzcd}$$

in which a, a' are the associated sheaf functors and $\tilde{u}_! = a'u_!i$. Conditions (1) and (2) are satisfied. In the next three Lemmas we deal with conditions (3) and (4). Recall from 2.26 the notion of cocontinuous functor.

4.13. Lemma. *Let E, E' be sites and $u\colon E \to E'$ a functor. The following are equivalent:*

1. *u is cocontinuous;*
2. *the functor $u^*\colon \widehat{E'} \to \widehat{E}$ preserves local epimorphisms;*
3. *the functor u_* preserves sheaves.*

Proof. We prove that (1) $\Leftrightarrow$ (2). Suppose that u is cocontinuous. Let $f\colon X \to Y$ be a local epimorphism in $\widehat{E'}$. Let $S \in Ob(E)$ and $y \in Y(u(S))$. There is a refinement R of $u(S)$ such that for each element $\alpha\colon S' \to u(S)$ of R, there is $x \in X(S')$ such that $\alpha^*(y) = f(x)$. By assumption, R^{u_S} is a refinement of S. It follows that $u^*(f)$ is a local epimorphism. Conversely, let $S \in Ob(E)$ and R' be a refinement of $u(S)$. Then $R'^p \to \mathsf{y}(u(S))$ is a local epimorphism, so $u^*(R'^p) \to u^*(\mathsf{y}(u(S)))$ is a local epimorphism. Choosing the identity map of S in $u^*(\mathsf{y}(u(S)))(S)$, it follows that there is a refinement R of S such that for each element $\alpha\colon T \to S$ of R, there is $\beta\colon u(T) \to u(S)$ in R' such that $u(\alpha) = \beta$. This means that $u_S(R) \subseteq R'$, so u is cocontinuous.

The equivalence of (1) with (3) is [1, III, Proposition 2.2]. Here is another proof. Suppose that u is cocontinuous. Let $S' \in Ob(E)$ and R be a refinement of S'. Let j_R be the map $R^p \to \mathsf{y}(S')$ and X a sheaf on E. We have a natural bijection $\widehat{E'}(j_R, u_*(X)) \cong \widehat{E}(u^*(j_R), X)$. Since $u^*(j_R)$ is a local isomorphism, it follows that $u_*(X)$ is a sheaf. Conversely, let $f\colon X \to Y$ be a local epimorphism. By factoring f into an epimorphism followed by a monomorphism, we may assume that f is a monomorphism as well. Then $u^*(f)$ is a local isomorphism if and only if for each sheaf Z on E the map $\widehat{E}(u^*(f), Z)$ is bijective if and only if for each sheaf Z on E the map $\widehat{E'}(f, u_*(Z))$ is bijective. But this is true by assumption. ∎

It follows from Lemma 4.13 that for a continuous and cocontinuous functor $u: E \to E'$ between sites, the adjoint pair (u^*, u_*) restricts to an adjoint pair between $\widetilde{E'}$ and $\widetilde{E}$.

4.14. Lemma. *Let E, E' be sites and $u: E \to E'$ a continuous functor. The following are equivalent:*

1. *the functor $u^*: \widehat{E'} \to \widehat{E}$ reflects local epimorphisms;*
2. *for each presheaf Z on E', the counit map $u_! u^*(Z) \to Z$ is a local epimorphism;*
3. *each object S' of E' has a refinement whose objects are of the form $T' \to u(S) \to S'$.*

Proof. By Lemma 4.12 it suffices to prove that (2) $\Leftrightarrow$ (3). We recall that for an object S' of E', the counit map $u_! u^*(Z)(S') \to Z(S')$ sends an equivalence class $[S \in Ob(E), \alpha: S' \to u(S), z \in Z(u(S))]$ to $\alpha^*(z)$. Assume (2). Let S' be an object of E'. Taking Z to be the representable $\mathsf{y}(S')$, we have that for each object T' of E' and each map $\gamma: T' \to S'$, there is a refinement R' of T' such that for each element $g: T'' \to T'$ of R' there is an element $[S \in Ob(E), \alpha: T'' \to u(S), \beta: u(S) \to S']$ such that $\beta\alpha = \gamma g$. In particular, taking $T' = S'$ and γ to be the identity map of S', we obtain the desired refinement of S'. Assume (3). Since every presheaf on E' is a colimit of representables, the functor u^* preserves colimits and a colimit of local epimorphisms is a local epimorphism, it suffices to show that for each object S' of E', the counit map $u_! u^*(\mathsf{y}(S')) \to \mathsf{y}(S')$ is a local epimorphism. Let T' be an object of E' and $\gamma: T' \to S'$. There is a refinement R of T' whose objects are of the form $T'' \xrightarrow{\alpha} u(S) \xrightarrow{\beta} T'$. For each such object we have the element $[S, \alpha: T'' \to u(S), \gamma\beta: u(S) \to S']$ of $u_! u^*(\mathsf{y}(S'))(T'')$. This element is sent to $\gamma\beta\alpha$, so $u_! u^*(\mathsf{y}(S')) \to \mathsf{y}(S')$ is a local epimorphism. ∎

Suppose that in Lemma 4.14 the category E' has the discrete topology. Then condition (3) means 'every object of E' is a retract of an object in the image of u'.

4.15. Lemma. *Let E, E' be sites and $u: E \to E'$ a continuous and cocontinuous functor. Assume that:*

1. *for each object S of E, the unit map $\mathsf{y}(S) \to u^* u_!(\mathsf{y}(S))$ is a local monomorphism;*
2. *for each object S of E, the unit map $\mathsf{y}(S) \to u^* u_!(\mathsf{y}(S))$ is a local epimorphism.*

Then the functor $u_*: \widetilde{E} \to \widetilde{E'}$ is full and faithful.

Proof. We recall that (1) means that for each pair of maps $f, g: T \to S$ of E such that $u(f) = u(g)$, there is a refinement R of T such that for each element $h: U \to T$ of R we have $fh = gh$; (2) means that for each map $\alpha: u(T) \to u(S)$ of E', there is a refinement R of T such that for each element $f: U \to T$ of R there is a map $h: U \to S$ such that $\alpha u(f) = u(h)$.

To prove the Lemma it suffices to prove that for each sheaf X on E, the counit map $u^* u_*(X) \to X$ is an isomorphism in $\widehat{E}$. Recall that $u_*(X)(S')$ is the limit of the functor $(u \downarrow S')^{op} \to SET$ that sends an object $(T, \alpha: u(T) \to S')$ to $X(T)$. We prove that $u^*u_*(X) \to X$ is an epimorphism. Let $S \in Ob(E)$ and $x \in X(S)$. Let us fix an object $(T, \alpha: u(T) \to u(S))$ of $(u \downarrow u(S))$. By (2) there is a refinement $R_{(T,\alpha)}$ of T such that for each element $f: U \to T$ of $R_{(T,\alpha)}$ there is a map $h_f: U \to S$ such that $\alpha u(f) = u(h_f)$. We claim that the family $(h_f^*(x))_f$ is an element in the limit of the functor $(X|R_{(T,\alpha)}): R_{(T,\alpha)}^{op} \to SET$. Let $g: (U', f') \to (U, f)$ be a map in $R_{(T,\alpha)}$. We have to show that $g^*(h_f^*(x)) = h_{f'}^*(x)$ in $X(U')$. Since $u(h_f g) = u(h_{f'})$, there is by (1) a refinement R' of U' such that for each element $e: V \to U'$ of R' we have $h_f g e = h_{f'} e$. Since X is a sheaf, the natural map

$$X(U') \to \lim_{R'^{op}} (X|R')$$

is bijective. It follows that $g^*(h_f^*(x)) = h_{f'}^*(x)$. Again since X is a sheaf there is a unique $x_{(T,\alpha)}$ in $X(T)$ such that $f^*(x_{(T,\alpha)}) = h_f^*(x)$. Next, we claim that the family $(x_{(T,\alpha)})_{(T,\alpha)}$ belongs to $u_*(X)(u(S))$. Let $g: (T', \alpha') \to (T, \alpha)$ be a map in $(u \downarrow u(S))$. Then $R_{(T',\alpha')} \cap R^g_{(T,\alpha)}$ is a refinement of T'. For each element $e: U' \to T'$ of $R_{(T',\alpha')} \cap R^g_{(T,\alpha)}$ we have $e^*(g^*(x_{(T,\alpha)})) = h_{ge}^*(x)$ and $e^*(x_{(T',\alpha')}) = (h'_e)^*(x)$ in $X(U')$. We have $u(h_{ge}) = u(h'_e)$ by construction, so by (1) there is a refinement R' of U' such that for each element $k: V \to U'$ of R' we have $h_{ge}k = h'_e k$. Since X is a sheaf it follows that $h_{ge}^*(x) = (h'_e)^*(x)$. Again since X is a sheaf it follows that $g^*(x_{(T,\alpha)}) = x_{(T',\alpha')}$, and the claim is proved. The uniqueness of $x_{(T,\alpha)}$ implies that $x_{(S,1_{u(S)})} = x$.

We now prove that $u^*u_*(X) \to X$ is a monomorphism. Let S be an object of E and $(x_{(T,\alpha)})_{(T,\alpha)}, (x'_{(T,\alpha)})_{(T,\alpha)}$ be two elements of $u_*(X)(u(S))$ such that $x_{(S,1_{u(S)})} = x'_{(S,1_{u(S)})}$. By (2) there is a refinement $R_{(T,\alpha)}$ of T such that for each element $f: U \to T$ of $R_{(T,\alpha)}$ there is a map $h_f: U \to S$ such that $\alpha u(f) = u(h_f)$. Then, for each element $f: U \to T$ of $R_{(T,\alpha)}$ we have, by naturality, $f^*(x_{(T,\alpha)}) = x_{(U,u(h_f))} = h_f^*(x_{(S,1_{u(S)})}) = h_f^*(x'_{(S,1_{u(S)})}) = x'_{(U,u(h_f))} = f^*(x'_{(T,\alpha)})$. Since X is a sheaf it follows that $x_{(T,\alpha)} = x'_{(T,\alpha)}$. ∎

4.16. Proposition. *Let E, E' be sites and $u: E \to E'$ a continuous and cocontinuous functor. Assume that:*

1. *for each object S of E, the unit map $\mathsf{y}(S) \to u^*u_!(\mathsf{y}(S))$ is a local monomorphism;*
2. *for each object S of E, the unit map $\mathsf{y}(S) \to u^*u_!(\mathsf{y}(S))$ is a local epimorphism;*
3. *each object S' of E' has a refinement whose objects are of the form $T' \to u(S) \to S'$.*

Then the functor $u^*: \widehat{E'} \to \widehat{E}$ induces an equivalence between the categories of sheaves $\widetilde{E'}$ and $\widetilde{E}$.

Proof. We shall use Corollary 4.11. Conditions (1) and (2) are clear. Condition (3) follows from Lemmas 4.13 and 4.14. Condition (4) follows from Lemma 4.15, using the fact that if an arbitrary functor has both a left and right adjoint, then the left adjoint is full and faithful if and only if the right adjoint is so. ∎

Concerning the assumptions of Proposition 4.16, we recall that there are familiar situations when u satisfies the conditions (1) to (3) without being continuous and cocontinuous. For example, E is the category of affine schemes equipped with a topology finer than the Zariski topology, E' is the category of schemes with the Zariski topology, and u is the inclusion. In these situations one can try to construct a finer topology on E' in order to make u continuous and cocontinuous; this works for schemes [3, IV, Proposition 6.2.1].

The next result is a converse to Proposition 4.16.

4.17. Proposition. *Let E, E' be sites having topologies less fine than the canonical topology. Let $u\colon E \to E'$ be a continuous and cocontinuous functor. If the functor $u^*\colon \widehat{E'} \to \widehat{E}$ induces an equivalence between the categories of sheaves $\widetilde{E'}$ and $\widetilde{E}$, then u is full and faithful and each object S' of E' has a refinement whose objects are of the form $T' \to u(S) \to S'$.*

Proof. We shall apply the second part of Lemma 4.9. Let $S \in Ob(E)$. The unit map $\mathsf{y}(S) \to u^*u_!(\mathsf{y}(S))$ is a local isomorphism between sheaves, hence an isomorphism, therefore u is full and faithful. Let $S' \in Ob(E')$. The counit map $u_!u^*(\mathsf{y}(S')) \to \mathsf{y}(S')$ is a local isomorphism, hence a local epimorphism, and then we can reason as in the proof of Lemma 4.14. ■

4.18. Lemma. *Let E, E' be sites and $u\colon E \to E'$ a functor. Suppose that*

1. *u is refinement preserving;*
2. *u is full and faithful;*
3. *each object S' of E' has a refinement whose objects are of the form $T' \to u(S) \to S'$.*

Then u is continuous.

Proof. Let $S \in Ob(E)$ and R be a refinement of S. Consider the natural factorization from Lemma 2.29

$$\begin{tikzcd} u_!(R^p) \arrow[rr] \arrow[dr, "e"'] & & \mathsf{y}(u(S)) \\ & \overline{u_S(R)}^p \arrow[ur] & \end{tikzcd}$$

We will show that the map e is a local monomorphism, and conclude by [1, III, Proposition 1.2] and [5, 0, Proposition 3.5.2(iii)]. Let $S' \in Ob(E')$ and $[T \in Ob(E), \alpha\colon S' \to u(T), f\colon T \to S, f \in R]$, $[U \in Ob(E), \alpha'\colon S' \to u(U), g\colon U \to S, g \in R]$ be two elements of $u_!(R^p)(S')$ such that $u(f)\alpha = u(g)\alpha'$. There is a refinement R' of S' whose objects are of the form $S'' \xrightarrow{i_g} u(V) \xrightarrow{i_g} S'$. For each such object we consider the diagram

$$\begin{tikzcd}
 & S'' \arrow[d, "\gamma"] & \\
 & u(V) \arrow[d, "\delta"] & \\
 & S' \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\
u(T) \arrow[dr, "u(f)"'] & & u(U) \arrow[dl, "u(g)"] \\
 & u(S) &
\end{tikzcd}$$

The assumption that u is full and faithful implies that we have a commutative diagram

$$\begin{tikzcd}
 & S'' \arrow[dl] \arrow[d, "\gamma"] \arrow[dr] & \\
u(T) \arrow[dr, "u(f)"'] & u(V) \arrow[l, "u(h)"'] \arrow[r, "u(h')"] \arrow[d] & u(U) \arrow[dl, "u(g)"] \\
 & u(S) &
\end{tikzcd}$$

This implies that $[T \in Ob(E), \alpha\gamma\delta\colon S'' \to u(T), f\colon T \to S, f \in R] = [U \in Ob(E), \alpha'\gamma\delta\colon S'' \to u(U), g\colon T \to S, f \in R]$ in $u_!(R^p)(S'')$, so e is a local monomorphism. ■

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