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A_∞-MORPHISMS WITH SEVERAL ENTRIES

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ABSTRACT. We show that morphisms from n A_∞-algebras to a single one are maps over an operad module with $n+1$ commuting actions of the operad A_∞, whose algebras are conventional A_∞-algebras. The composition of A_∞-morphisms with several entries is presented as a convolution of a coalgebra-like and an algebra-like structures. Under these notions lie two examples of $\mathcal{C}at$-operads: that of graded modules and of complexes.

It is well-known that operads play a prominent part in the study of A_∞-algebras. In particular, A_∞-algebras in the conventional sense [Sta63] are algebras over the **dg**-operad A_∞, a resolution (a cofibrant replacement) of the **dg**-operad As of associative non-unital **dg**-algebras. Here and elsewhere in this article an operad is a non-symmetric operad unless it is called symmetric. More generally, there is an approach to homotopy algebras over an operad $\mathcal{O}$ as algebras over cofibrant **dg**-resolution $\mathcal{P}$ of this operad [Mar00].

The question arises about morphisms of homotopy algebras, the class of morphisms of $\mathcal{P}$-algebras being too narrow. A possible solution [Lyu11] is to consider a bimodule F over $\mathcal{P}$ and to define homotopy morphisms as "maps" (analogue of "algebras") over F, a cofibrant **dg**-resolution of the bimodule describing $\mathcal{O}$-algebra morphisms. Composition of homotopy morphisms arises as convolution of a coalgebra structure on F and the algebra hom. Practically the same notion called co-ring over an operad was a starting point of research by Hess, Parent and Scott [HPS05].

In the present article we study multicategory of homotopy algebras (when there is one, *e.g.* of A_∞-algebras). Let $\mathcal{V}$ be a bicomplete closed symmetric monoidal category. We are mostly interested in the category of complexes $\mathcal{V} = \mathbf{dg}$. A related choice is the category of graded modules $\mathcal{V} = \mathbf{gr}$. We use also the closed category of essentially small categories. For any $\mathcal{V}$-multicategory C and any object X of C we can produce a $\mathcal{V}$-operad $\mathcal{E}nd\,X$, $(\mathcal{E}nd\,X)(n) = \mathsf{C}(\underbrace{X,\dots,X}_{n};X) = \mathsf{C}(^nX;X)$. Similarly given objects X_1, …, X_n, Y of a symmetric $\mathcal{V}$-multicategory C we can consider the operad $\mathcal{V}$-polymodule (the $n \wedge 1$-operad module) $\mathcal{P} = hom(X_1,\dots,X_n;Y)$. By definition, $\mathcal{P} = (\mathcal{P}(j))_{j\in\mathbb{N}^n}$, where $\mathcal{P}(j) = \mathsf{C}\bigl((^{j^i}X_i)_{i=1}^n;Y\bigr)$ (the argument X_i is repeated j^i times). The collection $\mathcal{P}$ carries commuting left actions of operads $\mathcal{E}nd\,X_i$, $i\in\mathbf{n}$, and right action of $\mathcal{E}nd\,Y$. Formalising properties of these actions we write down the definition of an $n\wedge 1$-operad module over operads $\mathcal{A}_1$, …, $\mathcal{A}_n$, $\mathcal{B}$. When a collection $(F_n)_{n\geqslant 0}$ of $n\wedge 1$-operad modules over an operad $\mathcal{A}$ is given we may consider a multicategory-to-be whose objects are $\mathcal{A}$-alge-

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bras in C (objects A of C with a morphism of operads $\mathcal{A} \to \mathcal{E}nd\, A$) and multimorphisms $A_1, \dots, A_n \to B$ are morphisms of $n \wedge 1$-operad modules $F_n \to \mathit{hom}(A_1, \dots, A_n; B)$ with respect to morphisms of operads $\mathcal{A} \to \mathcal{E}nd\, A_1, \dots, \mathcal{A} \to \mathcal{E}nd\, A_n, \mathcal{A} \to \mathcal{E}nd\, B$. To provide multicategory compositions we equip the collection $(F_n)_{n\geqslant 0}$ with a coassociative comultiplication, making it into a polymodule cooperad. Convolution of this comultiplication and the composition in C gives a composition in the multicategory of $\mathcal{A}$-algebras under construction.

We consider an A_∞-polymodule cooperad $F = (F_n)_{n\geqslant 0}$ responsible for morphisms with several arguments $f : A_1, \dots, A_n \to B$ of A_∞-algebras writing explicit formulas. This cooperad is easy to construct due to absence of signs in expressions and also since degrees of generators are 0.

In a sequel to this paper we shall consider three more examples of polymodule cooperads. The first is A_∞-polymodule cooperad $\mathrm{F} = (\mathrm{F}_n)_{n\geqslant 0}$, which is a cofibrant **dg**-resolution of *As*-polymodule cooperad responsible for composition in multicategory of non-unital associative algebras. This is a signed counterpart of A_∞-polymodule cooperad F, studied in the present article. Notice that A_∞ and A_∞ (resp. F and F) are "isomorphic" via an invertible homomorphism of operads changing degrees. The second is the homotopy unital version F^{hu} of F, which is an A_∞^{hu}-polymodule cooperad for the operad A_∞^{hu} of homotopy unital A_∞-algebras. At last, the third one is the $\mathrm{A}_\infty^{\mathsf{hu}}$-polymodule cooperad F^{hu}, which is a cofibrant **dg**-resolution of *As1*-polymodule cooperad responsible for composition in multicategory of unital associative algebras. The second and the third examples are "isomorphic" via an invertible homomorphism changing degrees. They are responsible for morphisms and their composition in the multicategory of homotopy unital A_∞^{hu}-algebras (resp. $\mathrm{A}_\infty^{\mathsf{hu}}$-algebras).

Contents

We assume that all modules are graded modules over a commutative ring. A lot of signs disappear due to chiral system of notations, see Section 1.1: we use right operators, homogeneous elements of the right homomorphism object in closed symmetric monoidal category of graded modules. We recall basic features of operads in Section 1.2, describe trees which we use in Section 1.6. A_∞-algebras and the related operad A_∞ are recalled in Example 1.9. The approach to A_∞-algebras and A_∞-morphisms via tensor coalgebras with cut comultiplication is reviewed in Sections 1.10–1.11.

The categorical basement to constructions in this article is the notion of $\mathcal{C}at$-multicategories and (co)lax $\mathcal{C}at$-multifunctors, developed in Section 2. A new notion of multinatural transformation of lax $\mathcal{C}at$-multifunctors is proposed in Definition 2.3. The main examples of $\mathcal{C}at$-operads relevant to this article are $\mathcal{C}at$-operad G of graded modules and $\mathcal{C}at$-operad DG of differential graded modules introduced in Section 2.6. Lax $\mathcal{C}at$-multifunctors $\mathsf{1} \to \mathsf{V}$ are identified with $\mathcal{V}$-operads in Section 2.10. One of the main objects of study, operad polymodule, or $n \wedge 1$-operad module, or $\mathbb{N}^n$-indexed collection of objects of $\mathcal{V}$ with n left actions and one right action of operads (all commuting) is defined also as a lax $\mathcal{C}at$-multifunctor $\mathsf{L}_n \to \mathsf{V}$ in Definition 2.12. In particular, for $\mathcal{V} = \mathbf{dg}$ we study the category ${}_n\mathrm{Op}_1$ of $n \wedge 1$-operad modules. Using Crude Tripleability Theorem [BW05, Section 3.5] we prove that the comparison functor for the underlying functor $U : {}_n\mathrm{Op}_1 \to \mathbf{dg}^{n\mathbb{N}\sqcup\mathbb{N}^n\sqcup\mathbb{N}}$ is an isomorphism of categories (Proposition 2.19).

Some pushouts of $n \wedge 1$-operad modules are computed in Section A.8. Starting with a symmetric $\mathcal{V}$-multicategory C we construct in Section 2.21 a lax $\mathcal{C}at$-multifunctor *hom*, which to a sequence $(A_i)_{i\in I}$, B of objects of C and a vector $(n^i)_{i\in I} \in \mathbb{N}^I$ assigns a complex $hom((A_i)_{i\in I}; B)((n^i)_{i\in I}) = \mathsf{C}\big(({}^{n^i}A_i)_{i\in I}; B\big)$ equipped with compositions coming from C. A $\mathcal{C}at$-multicategory M whose objects are operads and categories of morphisms are that of operad polymodules is constructed in Section 2.22. One product $\circledast_{\mathsf{G}}$ it inherits from the $\mathcal{C}at$-operad G and we describe the right and the left actions of operads on this product. Another product $\circledast_{\mathsf{M}}$, quotient of $\circledast_{\mathsf{G}}$, is the tensor product over right and left actions of operads.

The $n \wedge 1$-operad module *hom* is revisited in Section 3.1. In Proposition 3.4 we construct differential graded A_∞-polymodules F_n.

In Section 4 we equip a collection of $n \wedge 1$-operad modules F_n, $n \geqslant 0$, with comultiplication turning it into a polymodule cooperad. First we write comultiplication $\varDelta^{\mathsf{G}}$ for $(A_\infty, F_\bullet)$ which turns it into a graded polymodule cooperad (Proposition 4.10). In Proposition 4.13 we show that comultiplication $\varDelta^{\mathsf{M}}$ targeted at $\circledast_{\mathsf{M}}$ instead of $\circledast_{\mathsf{G}}$ makes $(A_\infty, F_\bullet, \varDelta^{\mathsf{M}})$ into a $\mathbf{dg}$-polymodule cooperad. This comultiplication encodes composition in the multicategory of A_∞-algebras.

In Appendix A we represent certain colimit in the category of T-algebras for a monad $T : \mathcal{C} \to \mathcal{C}$ via a colimit in $\mathcal{C}$. This result is used to find some colimits of operad modules.

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1. Preliminaries

Here we describe notations, recall some notions and results needed in the following parts of the article.

1.1. Notations and conventions. We denote by $\mathbb{N}$ the set of non-negative integers $\mathbb{Z}_{\geqslant 0}$.

Let $\mathcal{V} = (\mathcal{V}, \otimes, \mathbb{1})$ be a complete and cocomplete closed symmetric monoidal category with the right inner hom $\underline{\mathcal{V}}(X, Y)$. For instance, it can be the category $\Bbbk$-mod for a ground commutative ring $\Bbbk$. Tensor product $\otimes_\Bbbk$ will be denoted simply $\otimes$. Other possibilities for $\mathcal{V}$ include the category of (differential) graded $\Bbbk$-modules denoted **gr** (resp. **dg**). In this case we denote the right homomorphism objects by $\underline{\mathbf{gr}}(X,Y)$ (resp. $\underline{\mathbf{dg}}(X,Y)$) for (differential) graded $\Bbbk$-modules X, Y. The differential graded setting will be the main application. When a $\Bbbk$-linear map f is applied to an element x, the result is typically written as $x.f = xf$. The tensor product of two maps of graded $\Bbbk$-modules f, g of certain degree is defined so that for elements x, y of arbitrary degree

$$(x \otimes y).(f \otimes g) = (-1)^{\deg y \cdot \deg f} x.f \otimes y.g.$$

In other words, we follow the Koszul rule imposed by the symmetry. Composition of homogeneous $\Bbbk$-linear maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ is usually denoted $f \cdot g = fg : X \to Z$. For other types of maps composition is often written as $g \circ f = gf$.

We assume that each set is an element of some universe. This universe is not fixed through the whole article. Assume given two universes $\mathscr{U} \in \mathscr{U}'$. Sets in bijection with some element of $\mathscr{U}$ (resp. $\mathscr{U}'$) are called small (resp. large) sets. For instance, the category of categories $\mathcal{C}at$ means the category of large, locally small categories for universes $\mathscr{U} \in \mathscr{U}'$. These universes are used tacitly, without being explicitly mentioned.

We consider the category of totally ordered finite sets and their non-decreasing maps. An arbitrary totally ordered finite set is isomorphic to a unique set $\mathbf{n} = \{1 < 2 < \cdots < n\}$ via a unique isomorphism, $n \geqslant 0$. Functions of totally ordered finite set that we use in this article are assumed to *depend only on the isomorphism class of the set*. Thus, it suffices to define them only for skeletal totally ordered finite sets $\mathbf{n}$. The full subcategory of such sets and their non-decreasing maps is denoted $\mathcal{O}_{\mathsf{sk}}$.

Whenever $I \in \operatorname{Ob} \mathcal{O}_{\mathsf{sk}}$, there is another totally ordered set $[I] = \{0\} \sqcup I$ containing I, where element 0 is the smallest one. Thus, $[n] = [\mathbf{n}] = \{0 < 1 < 2 < \cdots < n\}$.

Let $(I, \leqslant)$, $(X_i, \leqslant)$, $i \in I$, be partially ordered sets. When $\bigsqcup_{i\in I} X_i$ is equipped with the lexicographic order it is denoted $\trianglelefteq_{i\in I} X_i$. Thus $(i,x) < (j,y)$ iff $i < j$ or ($i = j$ and $x < y \in X_i$).

The list $A, \ldots, A$ consisting of n copies of the same object A is denoted ${}^nA$.

For any graded $\Bbbk$-module M denote by $sM = M[1]$ the same module with the grading shifted by 1: $M[1]^k = M^{k+1}$. Denote by $\sigma : M \to M[1]$, $M^k \ni x \mapsto x \in M[1]^{k-1}$ the "identity map" of degree $\deg \sigma = -1$.

1.2. Operads. Category $\mathcal{V}^{\mathbb{N}}$ of collections $(\mathcal{W}(n))_{n\in\mathbb{N}}$ of objects of $\mathcal{V}$ is equipped with the composition tensor product $\odot$:

$$(\mathcal{U} \odot \mathcal{W})(n) = \coprod_{n_1+\cdots+n_k=n}^{k\geqslant 0} \mathcal{U}(n_1) \otimes \cdots \otimes \mathcal{U}(n_k) \otimes \mathcal{W}(k).$$

The unit object $\mathbb{1}$ has $\mathbb{1}(1) = \mathbb{1}$, and $\mathbb{1}(n) = 0$ is the initial object of $\mathcal{V}$ for $n \neq 1$.

A *(non-symmetric) $\mathcal{V}$-operad* $\mathcal{O}$ is a monoid in $(\mathcal{V}^{\mathbb{N}}, \odot)$, say $(\mathcal{O}, \mu : \mathcal{O} \odot \mathcal{O} \to \mathcal{O}, \eta : \mathbb{1} \to \mathcal{O})$. Multiplication consists of substitution compositions

$$\mu : \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \otimes \mathcal{O}(k) \to \mathcal{O}(n_1 + \cdots + n_k) \tag{1.1}$$

(one for each $\Bbbk$-tuple $(n_1, \dots, n_k) \in \mathbb{N}^k$, $k \in \mathbb{N}$), with a two-sided unit $\eta \in \mathcal{V}(\mathbb{1}, \mathcal{O}(1))$ – the identity operation.

1.3. Example. For any object $X \in \mathcal{V}$ there is the $\mathcal{V}$-operad $\mathcal{E}nd\, X$ of its endomorphisms. It has $(\mathcal{E}nd\, X)(n) = \underline{\mathcal{V}}(X^{\otimes n}, X)$. Here $\underline{\mathcal{V}}$ is the category enriched in $\mathcal{V}$ due to closedness of $(\mathcal{V}, \otimes)$.

1.4. Definition. *An algebra X over a $\mathcal{V}$-operad $\mathcal{O}$ is an object X together with a morphism of operads $\mathcal{O} \to \mathcal{E}nd\, X$ (morphism of monoids in $\mathcal{V}^{\mathbb{N}}$).*

1.5. Example. The **dg**-operad As is the $\Bbbk$-linear envelope of the $\mathcal{S}et$-operad as, whose algebras are semigroups without unit ($as(0) = \varnothing$ and $as(n)$ is a singleton for all $n > 0$). They have $As(0) = 0$ and $As(n) = \Bbbk$ for $n > 0$. As-algebras are associative differential graded $\Bbbk$-algebras without unit.

Similarly, the $\mathcal{S}et$-operad $as1$ of semigroups with a unit ($as1(n)$ is a singleton for all $n \geqslant 0$) has the $\Bbbk$-linear envelope – the **dg**-operad $As1$ with $As1(n) = \Bbbk$ for all $n \geqslant 0$. Clearly, $As1$-algebras are associative unital differential graded $\Bbbk$-algebras.

1.6. Trees. A *rooted tree* t can be defined as a *parent map* $P_t : E(t) \to E(t)$, where $E(t)$ is a finite set (of oriented edges), such that $|\operatorname{Im}(P_t^k)| = 1$ for some $k \in \mathbb{N}$. The only element $r \in \operatorname{Im}(P_t^k)$ is called the root edge. An oriented graph without loops G is constructed out of P, whose set of edges (arcs) is $E(t)$. For any edge a which is not a root edge the head of a is glued with the tail of Pa. This defines an equivalence relation on the set of heads and tails of edges from $E(t)$. Equivalence classes are vertices of G. The set of all vertices is denoted $V(t)$. It consist of tails of all edges plus the head of the root edge. The tail rv of the root edge is called the root vertex. Since G is a connected graph, whose number of edges is one less than the number of vertices, it is a tree. There is also the parent map $P : \overline{\mathrm{v}}(t) \to V(t)$, $\operatorname{tail}(e) \mapsto \operatorname{head}(e)$, where $\overline{\mathrm{v}}(t) = V(t) - \{\text{head of root edge}\}$.

Thus the rooted tree is oriented towards the head of the root edge. There is a partial ordering on $V(t) \sqcup E(t)$, namely, $u \preccurlyeq v$ iff v lies on the oriented path connecting u with the head of the root edge. Thus the head of the root edge is the biggest element. For each vertex $p \in V(t)$ denote by $\operatorname{in}(p)$ the set of edges entering p (those whose head is p). The non-negative number $|p| = |\operatorname{in}(p)|$ is called *arity* (=input semi-degree) of the vertex p. Denote by $\operatorname{inV}(p)$ the set of tails of edges from $\operatorname{in}(p)$.

Let $\operatorname{Lv}(t)$ denote the set of leaf vertices, minimal vertices with respect to $\preccurlyeq$. It is in bijection with the set $\operatorname{Le}(t)$ of leaf edges, minimal elements of $(E(t), \preccurlyeq)$. Leaf vertices are precisely tails of leaf edges.

1.7. Definition. *A* rooted tree with inputs *is a rooted tree t with a chosen subset (of input edges)* $\operatorname{Inp}(t)$ *of the set* $\operatorname{Le}(t)$. *The set of input vertices* $\operatorname{Inpv}(t) \cong \operatorname{Inp}(t)$ *is the set of tails of input edges. The* set of internal edges *is defined as* $\mathrm{e}(t) = E(t) - (\operatorname{Inp}(t) \cup \{\textit{root edge}\})$. *It is smaller by one element than the* set of internal vertices $\mathrm{v}(t) = V(t) - \operatorname{Inpv}(t) - \{\textit{head of the root edge}\}$. *A* planar rooted tree *is a rooted tree with a chosen total ordering $\lessdot$ of the set of incoming edges for each vertex. The set of vertices of a planar rooted tree t is equipped with* canonical ordering *denoted $\trianglelefteq$. By definition $x \lhd y$ if either*

$x \prec y$ or $x' \lessdot y'$ in $\mathrm{inV}(z)$*, where $x \to x_1 \to \cdots \to x' \to z \leftarrow y' \leftarrow \cdots \leftarrow y_1 \leftarrow y$ are oriented paths beginning at x and y and merging at some z. An* ordered tree *is a planar rooted tree equipped with a total ordering $\leqslant$ of the set* $\mathrm{v}(t)$ *of internal vertices such that for any two internal vertices x, y the inequality $x \preccurlyeq y$ implies $x \leqslant y$. A* strongly ordered tree *is a planar rooted tree equipped with a total ordering $\leqslant$ of the set $\overline{\mathrm{v}}(t)$ such that* $\mathrm{Inpv}\, t < \mathrm{v}(t)$ *and the restriction of the ordering to* $\mathrm{v}(t)$ *is an ordered tree.*

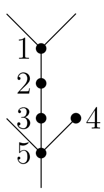
Any ordered tree extends canonically to a strongly ordered tree with $(\overline{\mathrm{v}}(t), \leqslant) = (\mathrm{Inpv}\, t, \trianglelefteq) \boxed{\leqslant} (\mathrm{v}(t), \leqslant)$. From now on unless otherwise stated a tree means an ordered tree canonically extended to a strongly ordered tree.

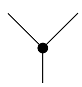
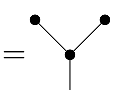
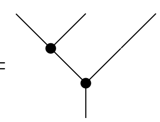
So introduced trees differ from the trees used in [Lyu14] precisely by the added root edge. This allows to use with some care the constructions and notations of [*loc. cit.*]. Thus, the set tr of (isomorphism classes of) planar rooted trees with inputs is partitioned into $\mathrm{tr}(n)$, trees with $n \geqslant 0$ inputs. For each tree $t \in \mathrm{tr}$ there is an operation of substituting trees into internal vertices:

$$I_t : \prod_{p \in \mathrm{v}(t)} \mathrm{tr}\, |p| \to \mathrm{tr}(\mathrm{Inp}\, t)$$

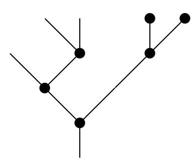
which takes a family $(t_p)_{p \in \mathrm{v}(t)}$ with $|\mathrm{Inp}\, t_p| = |p|$ to the tree $I_t(t_p \mid p \in \mathrm{v}(t)) = I(t; (t_p)_{p \in \mathrm{v}(t)})$ obtained from t by replacing each internal vertex $p \in \mathrm{v}(t)$ with the tree t_p. The collection $(t; (t_p)_{p \in \mathrm{v}(t)})$ is called a 2-cluster tree, see [Lyu14, Definition 3.19]. The details become clear from the following example.

1.8. Example. Consider trees

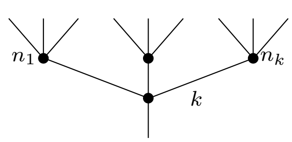
$t =$ 

, $t_1 =$  , $t_2 = t_3 = |$, $t_4 =$  , $t_5 =$  .

Then

$\tau = I_t(t_1, t_2, t_3, t_4, t_5) =$  .

Introduce the notation

$t(n_1, \dots, n_k) = (\mathbf{n_1} + \cdots + \mathbf{n_k} \xrightarrow{g} \mathbf{k} \xrightarrow{\triangleright} \mathbf{1}) =$ 

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where $g^{-1}(j) \cong \mathbf{n_j}$ for all $j \in \mathbf{k}$, $\mathrm{Inp}\, t(n_1, \dots, n_k) = \mathbf{n_1} + \cdots + \mathbf{n_k}$.

1.9. Example. There is a **dg**-operad A_∞, freely generated as a graded operad by n-ary operations b_n of degree 1 for $n \geqslant 2$. The differential is defined as

$$b_n \partial = -\sum_{j+p+q=n}^{1<p<n} (1^{\otimes j} \otimes b_p \otimes 1^{\otimes q}) \cdot b_{j+1+q}.$$

1.10. Tensor coalgebra. The *tensor* $\Bbbk$-*module* of $A[1]$ is $T(A[1]) = \oplus_{n\geqslant 0}T^n(A[1]) = \oplus_{n\geqslant 0}A[1]^{\otimes n}$. Multiplication in an A_∞-algebra A is given by the operations of degree $+1$

$$b_n : T^n(A[1]) = A[1]^{\otimes n} \to A[1], \qquad n \geqslant 1.$$

Recall that $\Bbbk$-linear maps are composed *from left to right*. Operations b_n have to satisfy the A_∞-equations, $n \geqslant 1$:

$$\sum_{r+k+t=n} (1^{\otimes r} \otimes b_k \otimes 1^{\otimes t}) b_{r+1+t} = 0 : T^n(A[1]) \to A[1].$$

Tensor $\Bbbk$-module $T(A[1])$ has a coalgebra structure: the cut comultiplication

$$\Delta(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = \sum_{k=0}^{n} x_1 \otimes \cdots \otimes x_k \bigotimes x_{k+1} \otimes \cdots \otimes x_n.$$

An A_∞-structure on a graded $\Bbbk$-module A is equivalent to $b^2 = 0$, where $b : T(A[1]) \to T(A[1])$ is a coderivation of degree $+1$ given by the formula

$$b = \sum_{r+k+t=n} 1^{\otimes r} \otimes b_k \otimes 1^{\otimes t} : T^n(A[1]) \to T(A[1]), \qquad b_0 = 0.$$

In particular, $b\Delta = \Delta(1 \otimes b + b \otimes 1)$.

1.11. Homomorphisms with n arguments. A_∞-morphisms with several arguments $f : A_1, \dots, A_n \to B$ are defined as augmented $\mathbf{dg}$-coalgebra morphisms

$$\hat{f} : T(A_1[1]) \otimes \cdots \otimes T(A_n[1]) \to T(B[1]).$$

Here both augmented graded coalgebras $(C, \Delta, \varepsilon, \eta)$ are of the form $(\Bbbk \oplus \bar{C}, \Delta(x) = 1 \otimes x + x \otimes 1 + \bar{\Delta}(x)\ \forall\, x \in \bar{C}, \mathrm{pr}_1, \mathrm{in}_1)$, where the non-counital coassociative coalgebra $(\bar{C}, \bar{\Delta})$ is conilpotent (cocomplete [LH03, Section 1.1.2], [Kel06, Section 4.3]). Thus $(\bar{C}, \bar{\Delta})$ is identified with a $T^{\geqslant 1}$-coalgebra [BLM08, Proposition 6.8], see also Proposition 4.6 of the current article. In the category of such augmented graded coalgebras the target $\Bbbk \oplus T^{\geqslant 1}(B[1])$ is cofree, see Corollary 4.7, hence, augmented graded coalgebra morphisms $\hat{f}$ are in bijection with the degree 0 $\Bbbk$-linear maps

$$f : T(A_1[1]) \otimes \cdots \otimes T(A_n[1]) \to B[1],$$

whose restriction to $T^0(A_1[1]) \otimes \dots \otimes T^0(A_n[1]) \simeq \Bbbk$ vanishes. The morphism $\hat{f}$ will be a chain map, $\hat{f}b = b\hat{f}$, if and only if

$$\sum_{q=1}^{n} \sum_{r+c+t=\ell^q}^{c>0} \big[\otimes^{i\in\mathbf{n}} T^{\ell^i} sA_i \xrightarrow{1^{\otimes(q-1)}\otimes(1^{\otimes r}\otimes b_c\otimes 1^{\otimes t})\otimes 1^{\otimes(n-q)}}$$
$$T^{\ell^1} sA_1 \otimes \dots \otimes T^{\ell^{q-1}} sA_{q-1} \otimes T^{r+1+t} sA_q \otimes T^{\ell^{q+1}} sA_{q+1} \otimes \dots \otimes T^{\ell^n} sA_n \xrightarrow{f_{\ell-(c-1)e_q}} sB\big]$$
$$= \sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0\\ j_1+\dots+j_k=\ell}}^{k>0} \big[\otimes^{i\in\mathbf{n}} T^{\ell^i} sA_i \xrightarrow{\sim} \otimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}} T^{j_p^i} sA_i$$
$$\xrightarrow{\sim} \otimes^{p\in\mathbf{k}} \otimes^{i\in\mathbf{n}} T^{j_p^i} sA_i \xrightarrow{\otimes^{p\in\mathbf{k}} f_{j_p}} \otimes^{p\in\mathbf{k}} sB \xrightarrow{b_k} sB\big]. \quad (1.2)$$

2. $\mathcal{C}at$-multicategories

In this section we describe the categorical background to the main subject of morphisms with several entries.

Let S be a set, $S^* = \sqcup_{n\geqslant 0} S^n$. There is a functor $\mathbb{T}_S : \mathcal{C}at^{S^*\times S} \to \mathcal{C}at^{S^*\times S}$,

$$(\mathbb{T}_S P)(X_1,\dots,X_n;Y) = \coprod_{t\in\mathrm{tr}(n)} \coprod_{Z:\mathrm{e}(t)\to S} \prod_{v\in\mathrm{v}(t)} P(Z_{\mathrm{in}(v)}; Z_{\mathrm{ou}(v)}),$$

where $\mathrm{ou} : \mathrm{v}(t) \to E(t)$ assigns to an internal vertex v the only outgoing edge $\mathrm{ou}(v)$, whose tail is v; $\mathrm{in}(v)$ is the $\lessdot$-ordered set of incoming edges for v, whose head is v; $Z_{\mathrm{in}(v)}$ is the corresponding sequence of elements of S. The function Z is extended from $\mathrm{e}(t)$ to $E(t)$ as follows: $Z(\text{root edge}) = Y$, $Z(\mathrm{Inp}(t)) = (X_1,\dots,X_n)$ with preserved total ordering.

When S is small and $\mathcal{C}at$ means the category of (locally) small categories, the functor $\mathbb{T}_S$ has the structure of a strict 2-monad. Multiplication for this 2-monad is given by the functor

$$\big\langle(\mathbb{T}_S^2 P)(X_1,\dots,X_n;Y) = \coprod_{t\in\mathrm{tr}(n)} \coprod_{Z:\mathrm{e}(t)\to S} \prod_{p\in\mathrm{v}(t)} \coprod_{t_p\in\mathrm{tr}\,|p|} \coprod_{U^p:\mathrm{e}(t_p)\to S} \prod_{v\in\mathrm{v}(t_p)} P(U^p_{\mathrm{in}(v)}; U^p_{\mathrm{ou}(v)})$$
$$\cong \coprod_{t\in\mathrm{tr}(n)} \coprod_{(t_p)\in\prod_{p\in\mathrm{v}(t)}\mathrm{tr}\,|p|} \coprod_{(Z,(U^p)):\mathrm{e}(t)\sqcup\coprod_{p\in\mathrm{v}(t)}\mathrm{e}(t_p)\to S} \prod_{(p,v)\in\prod_{r\in\mathrm{v}(t)}\mathrm{v}(t_r)} P(U^p_{\mathrm{in}(v)}; U^p_{\mathrm{ou}(v)})$$
$$\to \coprod_{\tau\in\mathrm{tr}(n)} \coprod_{W:\mathrm{e}(\tau)\to S} \prod_{q\in\mathrm{v}(\tau)} P(W_{\mathrm{in}(q)}; W_{\mathrm{ou}(q)}) = (\mathbb{T}_S P)(X_1,\dots,X_n;Y)\big\rangle = m,$$

which takes a summand indexed by $(t,(t_p),Z,(U^p))$ by the obvious isomorphism to the summand indexed by $(I(t;(t_p)), W = (Z,(U^p)))$. The unit of $\mathbb{T}_S$ is given by the isomorphism of $P(X_1,\dots,X_n;Y)$ to the summand indexed by the n-corolla $t = \tau[n]$.

To each map $f : R \to S$ there corresponds the functor $\mathcal{C}at^{f^* \times f}$ and a natural transformation

$$\begin{tikzcd}
\mathcal{C}at^{S^*\times S} \arrow[r, "\mathbb{T}_S"] \arrow[d, "\mathcal{C}at^{f^*\times f}"'] & \mathcal{C}at^{S^*\times S} \arrow[d, "\mathcal{C}at^{f^*\times f}"] \\
\mathcal{C}at^{R^*\times R} \arrow[r, "\mathbb{T}_R"] \arrow[ur, phantom, "\overset{\tau^f}{\Nearrow}"] & \mathcal{C}at^{R^*\times R}
\end{tikzcd}$$

which takes identically the summand indexed by $X : \mathrm{e}(t) \to R$ to the summand indexed by $X \cdot f : \mathrm{e}(t) \to S$. Clearly, to the composition of maps $Q \xrightarrow{g} R \xrightarrow{f} S$ corresponds the vertical pasting of τ^f and τ^g.

2.1. Definition. *A small (strong) $\mathcal{C}at$-multicategory C is a pair (S, C) consisting of a small set of objects S and a strong $\mathbb{T}_S$-algebra $\mathsf{C} = (\mathsf{C}, \mu : \mathbb{T}_S\mathsf{C} \to \mathsf{C}, \alpha, \iota)$*

$$\begin{tikzcd}
\mathbb{T}_S^2\mathsf{C} \arrow[r, "\mathbb{T}_S\mu"] \arrow[d, "m"'] & \mathbb{T}_S\mathsf{C} \arrow[d, "\mu"] \arrow[dl, phantom, "\overset{\alpha}{\underset{\cong}{\Swarrow}}"] \\
\mathbb{T}_S\mathsf{C} \arrow[r, "\mu"] & \mathsf{C}
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathsf{C} \arrow[rr, equal] \arrow[dr, "i"'] & \arrow[d, phantom, "\iota \Downarrow \cong"] & \mathsf{C} \\
& \mathbb{T}_S\mathsf{C} \arrow[ur, "\mu"'] &
\end{tikzcd},$$

where $\Longrightarrow$ means identity 1-morphism. The equations that have to be satisfied by α, ι are well-known. They can be found, for instance, in [Lyu14, Section 2]. We write $S = \mathrm{Ob}\,\mathsf{C}$. The same equations are supposed to hold when S is large, then we do not call the $\mathcal{C}at$-multicategory C small.

Assume given a map $F : R \to S$ of small sets. A $\mathcal{C}at$-multicategory C with $\mathrm{Ob}\,\mathsf{C} = S$ gives rise to a $\mathcal{C}at$-multicategory $\mathsf{C}_F = \mathcal{C}at^{F^*\times F}(\mathsf{C})$ with $\mathrm{Ob}\,\mathsf{C}_F = R$, namely, $\mathsf{C}_F(X_\bullet; Y) = \mathsf{C}(FX_\bullet; FY)$ is equipped with the action

$$\mu^{\mathsf{C}_F} = \bigl\langle \mathbb{T}_R(\mathsf{C}_F) \xrightarrow{\tau^F} (\mathbb{T}_S\mathsf{C})_F \xrightarrow{\mu^{\mathsf{C}}_F} \mathsf{C}_F \bigr\rangle$$

and natural transformations

$$\alpha^{\mathsf{C}_F} = \begin{tikzcd}
\mathbb{T}_R^2(\mathsf{C}_F) \arrow[r, "\mathbb{T}_R\tau^F"] \arrow[ddd, "m"] & \mathbb{T}_R((\mathbb{T}_S\mathsf{C})_F) \arrow[r, "\mathbb{T}_R(\mu_F)"] \arrow[d, "\tau^F"] \arrow[dr, phantom, "="] & \mathbb{T}_R(\mathsf{C}_F) \arrow[d, "\tau^F"] \\
\arrow[r, phantom, "="] & (\mathbb{T}_S^2\mathsf{C})_F \arrow[r, "(\mathbb{T}_S\mu)_F"] \arrow[dd, "m_F"] & (\mathbb{T}_S\mathsf{C})_F \arrow[dd, "\mu_F"] \arrow[ddl, phantom, "\overset{\alpha^{\mathsf{C}}_F}{\underset{\cong}{\Swarrow}}"] \\
& & \\
\mathbb{T}_R(\mathsf{C}_F) \arrow[r, "\tau^F"] & (\mathbb{T}_S\mathsf{C})_F \arrow[r, "\mu_F"] & \mathsf{C}_F
\end{tikzcd},$$

$$\iota^{\mathsf{C}_F} = \begin{tikzcd}
\mathsf{C}_F \arrow[rr, equal] \arrow[d, "i"] \arrow[dr, "i_F"] & \arrow[d, phantom, "\iota_F \Downarrow \cong"] & \mathsf{C}_F \\
\mathbb{T}_R(\mathsf{C}_F) \arrow[r, "\tau^F"] & (\mathbb{T}_S\mathsf{C})_F \arrow[ur, "\mu_F"'] &
\end{tikzcd}.$$

The construction of C_F extends easily to the case when R and S are large.

2.2. Definition. *A (co)lax $\mathcal{C}at$-multifunctor $F : \mathsf{B} \to \mathsf{C}$ is a pair $(\operatorname{Ob} F, (F, \phi))$ consisting of a map $F = \operatorname{Ob} F : R = \operatorname{Ob} \mathsf{B} \to \operatorname{Ob} \mathsf{C} = S$ and a (co)lax $\mathbb{T}_R$-morphism $(F : \mathsf{B} \to \mathsf{C}_F, \phi)$, where*

$$
\begin{tikzcd}
\mathbb{T}_R\mathsf{B} \arrow[r, "\mathbb{T}_R F"] \arrow[d, "\mu^{\mathsf{B}}"'] & \mathbb{T}_R(\mathsf{C}_F) \arrow[d, "\mu^{\mathsf{C}_F}"] \arrow[dl, Rightarrow, "\phi"'] \\
\mathsf{B} \arrow[r, "F"'] & \mathsf{C}_F
\end{tikzcd}
\qquad
\left(
\text{resp.}
\quad
\begin{tikzcd}
\mathbb{T}_R\mathsf{B} \arrow[r, "\mathbb{T}_R F"] \arrow[d, "\mu^{\mathsf{B}}"'] & \mathbb{T}_R(\mathsf{C}_F) \arrow[d, "\mu^{\mathsf{C}_F}"] \\
\mathsf{B} \arrow[r, "F"'] \arrow[ur, Rightarrow, "\phi"] & \mathsf{C}_F
\end{tikzcd}
\right).
$$

Equations that have to be satisfied by a $\mathbb{T}_R$-morphism are well-known, see [Lac10, Section 4.1] or [Lyu14, Definitions 2.1, 2.2] for concrete presentation.

$\mathcal{C}at$-multicategories and (co)lax $\mathcal{C}at$-multifunctors form a category. Composition of lax $\mathcal{C}at$-multifunctors $(H, \eta) = \bigl(\mathsf{B} \xrightarrow{(F,\phi)} \mathsf{C} \xrightarrow{(G,\psi)} \mathsf{D}\bigr)$ is determined by

$$
\operatorname{Ob} H = \bigl(\operatorname{Ob} \mathsf{B} = R \xrightarrow{\operatorname{Ob} F} \operatorname{Ob} \mathsf{C} = S \xrightarrow{\operatorname{Ob} G} \operatorname{Ob} \mathsf{D} = Q\bigr),
$$

$$
H_{X_\bullet;Y} = \bigl\langle \mathsf{B}(X_\bullet; Y) \xrightarrow{F_{X_\bullet;Y}} \mathsf{C}(FX_\bullet; FY) \xrightarrow{G_{FX_\bullet;FY}} \mathsf{D}(GFX_\bullet; GFY) \bigr\rangle,
$$

$$
\eta =
\begin{tikzcd}
\mathbb{T}_R\mathsf{B} \arrow[r, "\mathbb{T}_R F"] \arrow[dd, "\mu^{\mathsf{B}}"'] & \mathbb{T}_R(\mathsf{C}_F) \arrow[rr, "\mathbb{T}_R(G_F)"] \arrow[d, "\tau^F"] \arrow[ddl, Rightarrow, "\phi"'] \arrow[dr, phantom, "="] & & \mathbb{T}_R(\mathsf{D}_{G\circ F}) \arrow[d, "\tau^{G\circ F}"] \arrow[dl, "\tau^F"'] \arrow[d, phantom, shift right=6, "="] \\
 & (\mathbb{T}_S\mathsf{C})_F \arrow[r, "(\mathbb{T}_S G)_F"] \arrow[d, "\mu^{\mathsf{C}}_F"] & (\mathbb{T}_S(\mathsf{D}_G))_F \arrow[r, "\tau^G_F"] & (\mathbb{T}_R\mathsf{D})_{G\circ F} \arrow[d, "\mu^{\mathsf{D}}_{G\circ F}"] \arrow[dll, Rightarrow, "\psi_F"'] \\
\mathsf{B} \arrow[r, "F"'] & \mathsf{C}_F \arrow[rr, "G_F"'] & & \mathsf{D}_{G\circ F}
\end{tikzcd}
$$

Note that $\mathsf{D}_{G\circ F} = (\mathsf{D}_G)_F$. Composition of colax $\mathcal{C}at$-multifunctors is given by the same formulae with reversed 2-arrows.

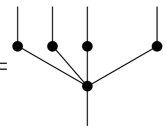
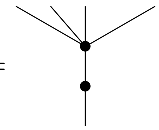
Let t be a tree. A *cut* of this tree is a subset $\mathfrak{c} \subset E(t)$ such that

— any path e, $P_t e$, $P_t^2 e$, ... contains no more than one element of $\mathfrak{c}$;
— when $e \in \operatorname{Inp}(t)$, the path e, $P_t e$, $P_t^2 e$, ... contains exactly one element of $\mathfrak{c}$.

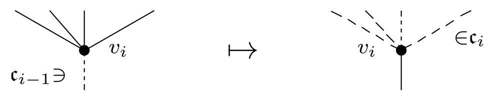
2.3. Definition. *A multinatural transformation of lax $\mathcal{C}at$-multifunctors $r : (F, \phi) \to (G, \psi) : \mathsf{C} \to \mathsf{D}$ is a collection of objects $r_X \in \operatorname{Ob} \mathsf{D}(FX; GX)$, $X \in \operatorname{Ob} \mathsf{C}$, and a collection of natural transformations*

$$
\begin{tikzcd}
\mathsf{C}((X_i)_{i\in I}; Y) \arrow[rr, "\cong"] \arrow[d, "\cong"'] & & \mathsf{C}((X_i)_{i\in I}; Y) \times 1 \arrow[r, "F\times \dot r_Y"] & \mathsf{D}(FX_\bullet; FY) \times \mathsf{D}(FY; GY) \arrow[d, "\mu_{t(|I|)}"] \arrow[dlll, Rightarrow, "\rho"'] \\
1^I \times \mathsf{C}(X_\bullet; Y) \arrow[r, "(\dot r_{X_i})_\bullet \times G"'] & (\mathsf{D}(FX_i; GX_i))_\bullet \times \mathsf{D}(GX_\bullet; GY) \arrow[rr, "\mu_{t(I_1)}"'] & & \mathsf{D}((FX_i)_{i\in I}; GY)
\end{tikzcd}
$$

where $X_\bullet$ means $(X_i)_{i\in I}$, I is a finite totally ordered set, and other instances of $\bullet$ are interpreted similarly. The trees occurring here are

$t({}^I 1) =$  , $t(|I|) =$  . (2.1)

Let $\trianglelefteq$ be the canonical ordering of the set $\mathrm{v}(t)$ described in Section 1.6. Write down the list of internal vertices of t as $v_1 \triangleright v_2 \triangleright \cdots \triangleright v_N$, $N = |\,\mathrm{v}(t)|$. Associate with it the sequence of cuts of t $\mathfrak{c}_0 = \{$root edge$\}$, $\mathfrak{c}_1, \dots, \mathfrak{c}_N = \mathrm{Inp}(t)$ such that $\mathfrak{c}_{i-1}$ and $\mathfrak{c}_i$ differ for $1 \leqslant i \leqslant N$ only at edges adjacent to v_i, namely,



(including vertices v_i of arity 0). Here edges belonging to a cut are dashed. Geometric picture that should be kept in mind is the following. Draw the tree t on the plane so that all vertices have distinct heights (ordinates), directions of all oriented edges of t deviated from the negative ordinate axis no more than by $\pi/2$. Edges intersecting the horizontal strip between vertices v_i and v_{i+1} are drawn dashed. This describes $\mathfrak{c}_i$.

Associate with $\mathfrak{c}_i$ the functor

$$\Phi_i = \Big\langle \prod_{j=N}^{1} \mathsf{C}(Z_{\mathrm{in}(v_j)}; Z_{\mathrm{ou}(v_j)}) \xrightarrow{\prod_{j=N}^{i+1} F \times \prod_{e\in\mathfrak{c}_i} \dot{r}_{Z_e} \times \prod_{j=i}^{1} G}$$
$$\prod_{j=N}^{i+1} \mathsf{D}(FZ_{\mathrm{in}(v_j)}; FZ_{\mathrm{ou}(v_j)}) \times \prod_{e\in\mathfrak{c}_i} \mathsf{D}(FZ_e; GZ_e) \times \prod_{j=i}^{1} \mathsf{D}(GZ_{\mathrm{in}(v_j)}; GZ_{\mathrm{ou}(v_j)}) \xrightarrow{\mu_{t_i}} \mathsf{D}((FX_k)_1^n; GY) \Big\rangle,$$

where t_i is the tree t with added unary vertices in the middle of edges $e \in \mathfrak{c}_i$. Let us construct a natural morphism $\Phi_{i-1} \to \Phi_i$:

$$\Phi_{i-1} \xrightarrow{\alpha^{-1}} \Big\langle \prod_{j=N}^{1} \mathsf{C}(Z_{\mathrm{in}(v_j)}; Z_{\mathrm{ou}(v_j)}) \xrightarrow{\prod_{j=N}^{i} F \times \prod_{e\in\mathfrak{c}_{i-1}} \dot{r}_{Z_e} \times \prod_{j=i-1}^{1} G}$$
$$\prod_{j=N}^{i+1} \mathsf{D}(FZ_{\mathrm{in}(v_j)}; FZ_{\mathrm{ou}(v_j)}) \times \mathsf{D}(FZ_{\mathrm{in}(v_i)}; FZ_{\mathrm{ou}(v_i)}) \times \mathsf{D}(FZ_{\mathrm{ou}(v_i)}; GZ_{\mathrm{ou}(v_i)})$$
$$\times \prod_{e\in\mathfrak{c}'_{i-1}} \mathsf{D}(FZ_e; GZ_e) \times \prod_{j=i-1}^{1} \mathsf{D}(GZ_{\mathrm{in}(v_j)}; GZ_{\mathrm{ou}(v_j)}) \xrightarrow{(1\times\mu_{t(|v_i|)}\times 1)\cdot\mu_{t'_{i-1}}} \mathsf{D}((FX_k)_1^n; GY) \Big\rangle$$
$$\xrightarrow{(\prod_{j=N}^{i+1} F \times \prod_{e\in\mathfrak{c}'_{i-1}} \dot{r}_{Z_e} \times \prod_{j=i-1}^{1} G)\cdot(1\times\rho\times 1)\cdot\mu_{t'_i}} \Big\langle \prod_{j=N}^{1} \mathsf{C}(Z_{\mathrm{in}(v_j)}; Z_{\mathrm{ou}(v_j)}) \xrightarrow{\prod_{j=N}^{i+1} F \times \prod_{e\in\mathfrak{c}_i} \dot{r}_{Z_e} \times \prod_{j=i}^{1} G}$$

$$\prod_{j=N}^{i+1} \mathsf{D}(FZ_{\mathrm{in}(v_j)};FZ_{\mathrm{ou}(v_j)}) \times \prod_{e\in\mathfrak{c}''_i} \mathsf{D}(FZ_e;GZ_e) \times \prod_{e\in\mathrm{in}(v_i)} \mathsf{D}(FZ_e;GZ_e) \times \mathsf{D}(GZ_{\mathrm{in}(v_i)};GZ_{\mathrm{ou}(v_i)})$$
$$\times \prod_{j=i-1}^{1} \mathsf{D}(GZ_{\mathrm{in}(v_j)};GZ_{\mathrm{ou}(v_j)}) \xrightarrow{(1\times\mu_{t(|v_i|_1)}\times 1)\cdot\mu_{t''_i}} \mathsf{D}((FX_k)_1^n;GY)\Big\rangle \xrightarrow{\alpha} \Phi_i. \quad (2.2)$$

Here $\mathfrak{c}'_{i-1} = \mathfrak{c}_{i-1} - \{\mathrm{ou}(v_i)\}$, t'_{i-1} is the tree t with added unary vertices in the middle of edges $e \in \mathfrak{c}'_{i-1}$, $\mathfrak{c}''_i = \mathfrak{c}_i - \{\mathrm{in}(v_i)\}$, t''_i is the tree t with added unary vertices in the middle of edges $e \in \mathfrak{c}''_i$. Actually, $\mathfrak{c}'_{i-1} = \mathfrak{c}''_i$ and $t'_{i-1} = t''_i$.

At last we formulate equations that have to be satisfied by (r,ρ) in order to be a multinatural transformation:

$$\begin{tikzcd}
& \prod\limits_{v\in\mathrm{v}(t)} \mathsf{C}(Z_{\mathrm{in}(v)};Z_{\mathrm{ou}(v)}) \arrow[rr, "\prod F"] \arrow[dl, "\mu_t"] & & \prod\limits_{v\in\mathrm{v}(t)} \mathsf{D}(FZ_{\mathrm{in}(v)};FZ_{\mathrm{ou}(v)}) \arrow[dlll, Rightarrow, "\phi"'] \arrow[dl, "\mu_t"'] \arrow[dd, "(1\times\dot r)\cdot\mu_{t_0}"] \\
\mathsf{C}(X_\bullet;Y) \arrow[rr, "F"'] \arrow[dr, "G"'] & & \mathsf{D}(FX_\bullet;FY) \arrow[dl, Rightarrow, "\rho"'] \arrow[dr, "-\bullet r"] \arrow[r, phantom, "\overset{\alpha^{-1}}{\Longleftarrow}"] & {} \\
& \mathsf{D}(GX_\bullet;GY) \arrow[rr, "(r_{X_i})_\bullet\bullet-"'] & & \mathsf{D}(FX_\bullet;GY)
\end{tikzcd}$$

$$= \begin{tikzcd}
& \prod\limits_{v\in\mathrm{v}(t)} \mathsf{C}(Z_{\mathrm{in}(v)};Z_{\mathrm{ou}(v)}) \arrow[rr, "\prod F"] \arrow[dl, "\mu_t"] \arrow[dr, "\prod G"'] & & \prod\limits_{v\in\mathrm{v}(t)} \mathsf{D}(FZ_{\mathrm{in}(v)};FZ_{\mathrm{ou}(v)}) \arrow[dd, "(1\times\dot r)\cdot\mu_{t_0}"] \\
\mathsf{C}(X_\bullet;Y) \arrow[dr, "G"'] & & \prod\limits_{v\in\mathrm{v}(t)} \mathsf{D}(GZ_{\mathrm{in}(v)};GZ_{\mathrm{ou}(v)}) \arrow[ll, Rightarrow, "\psi"] \arrow[dl, "\mu_t"'] \arrow[dr, "((r_{X_i})_\bullet\bullet-)\mu_{t_N}"] \arrow[u, phantom, "\textit{composition}"] \arrow[d, phantom, "\cong"] & \\
& \mathsf{D}(GX_\bullet;GY) \arrow[rr, "(r_{X_i})_\bullet\bullet-"'] & & \mathsf{D}(FX_\bullet;GY)
\end{tikzcd},$$

where composition is that of $\Phi_0 \to \Phi_1 \to \cdots \to \Phi_N$ (each morphism is one of (2.2)). Due to 2-category property of $\mathcal{C}at$ the composition $\Phi_0 \to \Phi_N$ does not depend on the choice of ordering $\preccurlyeq$ of $\mathrm{v}(t)$.

Definition of multinatural transformation of colax $\mathcal{C}at$-multifunctors is similar.

$\mathcal{C}at$-multicategories, (co)lax $\mathcal{C}at$-multifunctors and multinatural transformations form a 2-category. In fact, all data for it come from the 2-category $\mathcal{C}at$.

(Strong) $\mathcal{C}at$-operads D are small $\mathcal{C}at$-multicategories with $\mathrm{Ob}\,\mathsf{D} = \mathbf{1}$. Hence, they are precisely $\mathbb{T}_{\mathbf{1}}$-algebras, (co)lax $\mathcal{C}at$-multifunctors between them are precisely (co)lax $\mathbb{T}_{\mathbf{1}}$-morphisms. Instead of $\mathsf{D}({}^I*;*)$, $I \in \mathrm{Ob}\,\mathcal{O}_{\mathsf{sk}}$, the notation $\mathsf{D}(I)$ is used. Among multinatural transformations $(r,\rho) : (F,\phi) \to (G,\psi) : \mathsf{C} \to \mathsf{D}$ of (co)lax $\mathcal{C}at$-multifunctors from a $\mathcal{C}at$-multicategory C to a $\mathcal{C}at$-operad D we single out those with $r \in \operatorname{Ob}\mathsf{D}(1)$ which is the unit 1 with respect to operadic compositions in D. This makes sense since $\operatorname{Ob}F = \operatorname{Ob}G = \triangleright : \operatorname{Ob}\mathsf{C} = S \to \operatorname{Ob}\mathsf{D} = \mathbf{1}$ is the only map. For $(1,\rho)$ the second component reduces to a 2-morphism $\rho : F \to G : \mathsf{C} \to \mathsf{D}$. We call such special multinatural transformations *operadic transformations*. They are precisely $\mathbb{T}_S$-transformations, that is, satisfy the equation from [Lac10, Section 4.1] or [Lyu14, Definition 2.3]: for lax $\mathcal{C}at$-multifunctors

$$
\begin{tikzcd}
\mathbb{T}_S\mathsf{C} \arrow[r, shift left=2, "\mathbb{T}_SF"] \arrow[r, shift right=2, "\mathbb{T}_SG"'] \arrow[r, phantom, "\Downarrow\mathbb{T}_S\rho"] \arrow[d, "\mu^{\mathsf{C}}"'] & \mathbb{T}_S(\mathsf{D}_\triangleright) \arrow[d, "\mu^{\mathsf{D}_\triangleright}"] \arrow[dl, Rightarrow, "\psi"'] \\
\mathsf{C} \arrow[r, "G"'] & \mathsf{D}_\triangleright
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{T}_S\mathsf{C} \arrow[r, "\mathbb{T}_SF"] \arrow[d, "\mu^{\mathsf{C}}"'] & \mathbb{T}_S(\mathsf{D}_\triangleright) \arrow[d, "\mu^{\mathsf{D}_\triangleright}"] \arrow[dl, Rightarrow, "\phi"] \\
\mathsf{C} \arrow[r, shift left=2, "F"] \arrow[r, shift right=2, "G"'] \arrow[r, phantom, "\Downarrow\rho"] & \mathsf{D}_\triangleright
\end{tikzcd}
, \qquad (2.3)
$$

and for colax $\mathcal{C}at$-multifunctors

$$
\begin{tikzcd}
\mathbb{T}_S\mathsf{C} \arrow[r, shift left=2, "\mathbb{T}_SG"] \arrow[r, shift right=2, "\mathbb{T}_SF"'] \arrow[r, phantom, "\Uparrow\mathbb{T}_S\rho"] \arrow[d, "\mu^{\mathsf{C}}"'] & \mathbb{T}_S(\mathsf{D}_\triangleright) \arrow[d, "\mu^{\mathsf{D}_\triangleright}"] \\
\mathsf{C} \arrow[r, "F"'] \arrow[ur, Rightarrow, "\phi"] & \mathsf{D}_\triangleright
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{T}_S\mathsf{C} \arrow[r, "\mathbb{T}_SG"] \arrow[d, "\mu^{\mathsf{C}}"'] & \mathbb{T}_S(\mathsf{D}_\triangleright) \arrow[d, "\mu^{\mathsf{D}_\triangleright}"] \\
\mathsf{C} \arrow[r, shift left=2, "G"] \arrow[r, shift right=2, "F"'] \arrow[r, phantom, "\Uparrow\rho"] \arrow[ur, Rightarrow, "\psi"'] & \mathsf{D}_\triangleright
\end{tikzcd}
.
$$

2.4. Definition. *A modification of multinatural transformations $f : (q,\kappa) \to (r,\rho) : (F,\phi) \to (G,\psi) : \mathsf{C} \to \mathsf{D}$ is a collection of morphisms $f_X \in \mathsf{D}(FX;GX)(q_X,r_X)$, $X \in \operatorname{Ob}\mathsf{C}$, such that*

$$
\begin{tikzcd}
\mathsf{C}((X_i)_{i\in I};Y) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathsf{C}((X_i)_{i\in I};Y)\times 1 \arrow[r, shift left=2, "F\times\dot{q}_Y"] \arrow[r, shift right=2, "F\times\dot{r}_Y"'] \arrow[r, phantom, "\Downarrow F\times\dot{f}_Y"] & \mathsf{D}(FX_\bullet;FY)\times\mathsf{D}(FY;GY) \arrow[d, "\mu_{t(|I|)}"] \arrow[dl, Rightarrow, "\rho"'] \\
1^I\times\mathsf{C}(X_\bullet;Y) \arrow[r, "(\dot{r}_{X_i})_\bullet\times G"'] & (\mathsf{D}(FX_i;GX_i))_\bullet\times\mathsf{D}(GX_\bullet;GY) \arrow[r, "\mu_{t(I_1)}"'] & \mathsf{D}((FX_i)_{i\in I};GY)
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
\mathsf{C}((X_i)_{i\in I};Y) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathsf{C}((X_i)_{i\in I};Y)\times 1 \arrow[r, "F\times\dot{q}_Y"] & \mathsf{D}(FX_\bullet;FY)\times\mathsf{D}(FY;GY) \arrow[d, "\mu_{t(|I|)}"] \arrow[dl, Rightarrow, "\kappa"'] \\
1^I\times\mathsf{C}(X_\bullet;Y) \arrow[r, shift left=2, "(\dot{q}_{X_i})_\bullet\times G"] \arrow[r, shift right=2, "(\dot{r}_{X_i})_\bullet\times G"'] \arrow[r, phantom, "\Downarrow(\dot{f}_{X_i})_\bullet\times G"] & (\mathsf{D}(FX_i;GX_i))_\bullet\times\mathsf{D}(GX_\bullet;GY) \arrow[r, "\mu_{t(I_1)}"'] & \mathsf{D}((FX_i)_{i\in I};GY)
\end{tikzcd}
.
$$

2.5. Remark. $\mathcal{C}at$-multicategories, lax $\mathcal{C}at$-multifunctors, multinatural transformations and modifications form a strict 3-category.

2.6. Weak $\mathcal{C}at$-operad V. Let us construct an example of a weak $\mathcal{C}at$-operad V coming from the closed symmetric monoidal category $\mathcal{V}$. When $\mathcal{V} = \mathbf{gr}$ (resp. $\mathcal{V} = \mathbf{dg}$) the $\mathcal{C}at$-operad V will be denoted G (resp. DG). We define $\mathsf{V}(I) = \mathcal{V}^{\mathbb{N}^I}$. For a tree t we are going to construct a functor

$$\circledast(t) = \mu_t : \prod_{v\in \mathrm{v}(t)} \mathsf{V}(|v|) \to \mathsf{V}(\operatorname{Inp} t), \qquad (\mathcal{P}_v)_{v\in\mathrm{v}(t)} \mapsto \circledast(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)}.$$

A t-tree is a functor $\tau : t \to \mathcal{O}_{\mathsf{sk}}$ such that $\tau(\mathrm{rv}) = \mathbf{1}$, where the poset t is the free category built on the quiver $t - \{\text{root edge}\}$ oriented towards the root vertex. It has the set of objects $\overline{\mathrm{v}}(t) = V(t) - \{\text{head of the root edge}\}$, morphisms are oriented paths, and the root vertex is the terminal object of t.

Thus, for an ordered tree t, collection $\mathcal{P}_v \in \operatorname{Ob}\mathsf{V}(|v|) = \operatorname{Ob}\mathcal{V}^{\mathbb{N}^{|v|}}$ and $z \in \mathbb{N}^{\operatorname{Inp} t}$

$$\circledast(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)}(z) = \coprod_{\forall a\in\operatorname{Inpv} t\, |\tau(a)|=z^a}^{t-\text{tree }\tau} \bigotimes^{v\in\mathrm{v}(t)} \bigotimes^{p\in\tau(v)} \mathcal{P}_v\Bigl(\bigl(|\tau(e)^{-1}(p)|\bigr)_{e\in\mathrm{in}(v)}\Bigr).$$

Let us construct the natural isomorphism α. The expression $\mathbb{T}^2_{\mathbf{1}}\mathsf{V}$ is the disjoint union over 2-cluster trees, collections of trees $(t;(t_v)_{v\in\mathrm{v}(t)})$ such that $t_v \in \mathrm{tr}(|v|)$. Multiplication m applied to this summand ends up in the summand indexed by $\theta = I_t(t_v \mid v \in \mathrm{v}(t))$. We have to construct the natural isomorphism

$$\begin{tikzcd}
\prod\limits_{v\in\mathrm{v}(t)} \prod\limits_{q\in\mathrm{v}(t_v)} \mathsf{V}(|q|) \arrow[r, "\prod_{v\in\mathrm{v}(t)}\circledast(t_v)"] \arrow[d, "m"', "\cong"] & \prod\limits_{v\in\mathrm{v}(t)} \mathsf{V}(|v|) \arrow[d, "\circledast(t)"] \arrow[dl, Rightarrow, "\alpha"', "\cong"] \\
\prod\limits_{r\in\mathrm{v}(\theta)} \mathsf{V}(|r|) \arrow[r, "\circledast(\theta)"] & \mathsf{V}(\operatorname{Inp} t)
\end{tikzcd}$$

And in fact, we have

$$\begin{aligned}
&\circledast(t)\Bigl(\prod_{v\in\mathrm{v}(t)} \circledast(t_v)(\mathcal{P}^v_q)^{v\in\mathrm{v}(t)}_{q\in\mathrm{v}(t_v)}\Bigr)(z) \\
&\qquad = \coprod_{\forall a\in\operatorname{Inpv} t\, |\tau(a)|=z^a}^{t-\text{tree }\tau} \bigotimes^{v\in\mathrm{v}(t)} \bigotimes^{p\in\tau(v)} \circledast(t_v)(\mathcal{P}^v_q)_{q\in\mathrm{v}(t_v)}\Bigl(\bigl(|\tau(u)^{-1}(p)|\bigr)_{u\in\mathrm{in}(v)}\Bigr) \\
&= \coprod_{\forall a\in\operatorname{Inpv} t\, |\tau(a)|=z^a}^{t-\text{tree }\tau} \bigotimes^{v\in\mathrm{v}(t)} \bigotimes^{p\in\tau(v)} \coprod_{\forall u\in\operatorname{Inpv}(t_v)=\mathrm{inV}(v)\, |\tau^p_v(u)|=|\tau(u\to v)^{-1}(p)|}^{t_v-\text{tree }\tau^p_v} \bigotimes^{q\in\mathrm{v}(t_v)} \bigotimes^{r\in\tau^p_v(q)} \mathcal{P}^v_q\Bigl(\bigl(|\tau^p_v(y)^{-1}(r)|\bigr)_{y\in\mathrm{in}(q)}\Bigr) \\
&\quad \xrightarrow{\cong} \coprod_{\forall a\in\operatorname{Inpv}\theta\, |T(a)|=z^a}^{\theta-\text{tree }T} \bigotimes^{w\in\mathrm{v}(\theta)} \bigotimes^{x\in T(w)} \mathcal{P}_w\Bigl(\bigl(|T(j)^{-1}(x)|\bigr)_{j\in\mathrm{in}(w)}\Bigr) = \circledast(\theta)(\mathcal{P}_w)_{w\in\mathrm{v}(\theta)}(z).
\end{aligned} \tag{2.4}$$

Note that a vertex w of θ is an equivalence class of $(v,q) \in \mathrm{v}(t) \times \overline{\mathrm{v}}(t_v)$. The θ-tree T is obtained from t, (t_v), τ, (τ_v^p) as follows. The associated to w by T set is $T(w) = T(v,q) = \bigsqcup^{\triangleleft}_{p\in\tau(v)} \tau_v^p(q)$, lexicographically ordered.

The natural transformation ι is the inverse to the isomorphism $\circledast(\tau[n])(\mathcal{P})(z) \cong \mathcal{P}\bigl((z^u)_{u\in\mathrm{inV}(v)}\bigr) = \mathcal{P}(z)$. Equations for α and ι hold true due to combinatorial reasons.

2.7. Proposition. *Let* C *be a* $\mathcal{C}at$*-operad. This induces a monoidal structure on the category* $\mathsf{C}(1)$.

Proof. Define a *linear tree* corresponding to a set $I \in \operatorname{Ob}\mathcal{O}_{\mathsf{sk}}$ as the functor $lt_I : [I] \to \mathcal{O}_{\mathsf{sk}}$, $[I] \ni i \mapsto \mathbf{1}$. We may view $lt_I = (\mathbf{1} \to \mathbf{1} \to \cdots \to \mathbf{1})$ as a synonym for $[I]$. Restricting multiplication functor μ_t to the linear tree $t = lt_I$ we get tensor multiplication

$$\odot^I \overset{\text{def}}{=} \circledast(lt_I)(1) : \mathsf{C}(1)^I \to \mathsf{C}(1).$$

Notice that if trees t and t_v, $v \in \mathrm{v}(t)$ are linear, then so is $\theta = I_t(t_v \mid v \in \mathrm{v}(t))$. The 2-submonad M of $\mathbb{T}_{\mathbf{1}}$ containing summands indexed by linear trees is precisely the free monoid 2-monad $\mathsf{C}(1) \mapsto \bigsqcup_{I=0}^{\infty} \mathsf{C}(1)^I$ on $\mathcal{C}at$. Therefore, for any $\mathbb{T}_{\mathbf{1}}$-algebra C the category $\mathsf{C}(1)$ is an M-algebra, that is, an unbiased monoidal category [Lei03, Definition 3.1.1]. ■

2.8. Example. The Monoidal product in the category $\mathsf{V}(1) = \mathcal{V}^{\mathbb{N}}$ of collections $\mathcal{A}_h$ is isomorphic to $\circledast(lt_I)(\mathcal{A}_h)_{h\in I}(z)$, $z \in \mathbb{N}$:

$$(\odot^{h\in I}\mathcal{A}_h)(z) = \coprod_{|\tau(0)|=z}^{\text{staged tree } \tau:[I]\to\mathcal{O}_{\mathsf{sk}}} \bigotimes^{h\in I} \bigotimes^{p\in\tau(h)} \mathcal{A}_h(|\tau_h^{-1}p|).$$

This turns $\mathcal{V}^{\mathbb{N}}$ into the familiar monoidal category $(\mathcal{V}^{\mathbb{N}}, \odot)$. Algebras in this monoidal category are precisely $\mathcal{V}$-operads.

2.9. Proposition. *1. Any (co)lax* $\mathcal{C}at$*-multifunctor* $(F,\phi) : \mathsf{L} \to \mathsf{M}$ *between* $\mathcal{C}at$*-operads induces a (co)lax monoidal functor* $(F(1),\phi(1)) : \mathsf{L}(1) \to \mathsf{M}(1)$.

2. Let L, M *be* $\mathcal{C}at$*-operads. Any operadic transformation* $\xi : (F,\phi) \to (G,\psi) : \mathsf{L} \to \mathsf{M}$ *between (co)lax* $\mathcal{C}at$*-multifunctors induces a Monoidal transformation* $\xi(1) : (F(1),\phi(1)) \to (G(1),\psi(1)) : \mathsf{L}(1) \to \mathsf{M}(1)$.

Proof. 1. Follows from the proof of Proposition 2.7.

2. Restricting the transformations in the lax case to linear trees we get

$$\begin{tikzcd} \mathsf{L}(1)^I \arrow[r, bend left, "F(1)^I"] \arrow[r, bend right, "G(1)^I"'] \arrow[r, phantom, "\Downarrow\xi(1)^I"] \arrow[d, "\odot^I"'] & \mathsf{M}(1)^I \arrow[d, "\odot^I"] \arrow[dl, Rightarrow, "\psi^I(1)"'] \\ \mathsf{L}(1) \arrow[r, "G(1)"'] & \mathsf{M}(1) \end{tikzcd} = \begin{tikzcd} \mathsf{L}(1)^I \arrow[r, "F(1)^I"] \arrow[d, "\odot^I"'] & \mathsf{M}(1)^I \arrow[d, "\odot^I"] \arrow[dl, Rightarrow, "\phi^I(1)"] \\ \mathsf{L}(1) \arrow[r, bend left, "F(1)"] \arrow[r, bend right, "G(1)"'] \arrow[r, phantom, "\Downarrow\xi(1)"] & \mathsf{M}(1) \end{tikzcd} . \qquad (2.5)$$

These equations say that the transformation $\xi(1)$ is Monoidal, see [BLM08, Definition 2.20]. ■

2.10. $\mathcal{V}$-OPERADS ARE LAX $\mathcal{C}at$-MULTIFUNCTORS $\mathsf{1} \to \mathsf{V}$. Let C be an arbitrary $\mathcal{C}at$-operad. Denote by $\mathsf{1}$ the $\mathcal{C}at$-operad with

$$\mathsf{1}(n) = \begin{cases} \mathit{1} = \text{terminal (1-morphism) category}, & \text{if } n = 1, \\ \varnothing = \text{initial (empty) category}, & \text{if } n \neq 1. \end{cases}$$

A multiquiver morphism $\mathsf{1} \to \mathsf{C}$ is a functor $\mathit{1} \to \mathsf{C}(1)$, so it is just an object of $\mathsf{C}(1)$. In particular, a $\mathcal{C}at$-multiquiver map $\mathsf{1} \to \mathsf{V}$ is the same as a functor $\mathit{1} \to \mathsf{V}(1) = \mathcal{V}^{\mathbb{N}}$.

Proposition 2.9 implies that a lax $\mathcal{C}at$-multifunctor $\mathsf{1} \to \mathsf{V}$ is the same as a lax Monoidal functor $\mathit{1} \to \mathsf{V}(1) = (\mathcal{V}^{\mathbb{N}}, \odot^I)$. By Definition 2.25 and Proposition 2.28 of [BLM08] this is the same as an algebra in $(\mathcal{V}^{\mathbb{N}}, \odot^I)$, that is, an operad.

By Proposition 2.9 an operadic transformation $\xi : (F, \phi) \to (G, \psi) : \mathsf{1} \to \mathsf{V}$ is the same as a Monoidal transformation $\xi(1) : \mathcal{O} = (F(1), \phi(1)) \to (G(1), \psi(1)) = \mathcal{P} : \mathit{1} \to (\mathcal{V}^{\mathbb{N}}, \odot^I)$. Here operads $\mathcal{O}$ and $\mathcal{P}$ are identified with the image of $1 \in \operatorname{Ob} \mathit{1}$ under corresponding functors. Equations (2.5) for $\xi(1)$ translate to

$$\big(\odot^I \mathcal{O} \xrightarrow{\odot^I \xi(1)} \odot^I \mathcal{P} \xrightarrow{\psi^I(1)} \mathcal{P} \big) = \big(\odot^I \mathcal{O} \xrightarrow{\phi^I(1)} \mathcal{O} \xrightarrow{\xi(1)} \mathcal{P} \big),$$

that is, $\xi(1) : \mathcal{O} \to \mathcal{P}$ is a morphism of operads.

2.11. $n \wedge 1$-OPERAD MODULES ARE LAX $\mathcal{C}at$-MULTIFUNCTORS. Consider a $\mathcal{C}at$-multicategory L_n with $\operatorname{Ob} \mathsf{L}_n = \{0, 1, 2, \dots, n\}$ such that

$$\begin{aligned} \mathsf{L}_n(i; i) &= \mathit{1} && \text{for } 0 \leqslant i \leqslant n, \\ \mathsf{L}_n(1, 2, \dots, n; 0) &= \mathit{1} && \text{and} \\ \mathsf{L}_n(k_1, k_2, \dots, k_m; k_0) &= \varnothing && \text{for other lists of arguments.} \end{aligned}$$

Components of the category $\mathbb{T}_{\mathbf{1}} \mathsf{L}_n$ either are empty or indexed by trees of two kinds:

- labelled linear trees $(lt_{\mathbf{k}}, {}^k i) = (i \to i \to \cdots \to i)$, whose all vertices are labelled with the same $i \in [n]$;
- labelled trees with strings of various length

$$\succ = \begin{tikzcd}[row sep=small] & n \arrow[r] & \cdots \arrow[r] & n \arrow[r] & n \arrow[ddr] \\ & & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[dr] \\ 2 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & 2 \arrow[r] & 2 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \\ & 1 \arrow[r] & \cdots \arrow[r] & 1 \arrow[r] & 1 \arrow[ur] \end{tikzcd}$$

The non-vanishing components are terminal categories $\mathit{1}$. The multiplication functor $\mu : \mathbb{T}_{\mathbf{1}} \mathsf{L}_n \to \mathsf{L}_n$ is Id_{1} on any non-vanishing component of the source. It takes the component indexed by a tree of the first kind to the category $\mathsf{L}_n(i; i)$. The component indexed by a tree of the second kind goes to the category $\mathsf{L}_n(1, 2, \dots, n; 0)$.

2.12. DEFINITION. *An* $n \wedge 1$-operad module *is a lax* $\mathcal{C}at$*-multifunctor* $\mathcal{P} : \mathsf{L}_n \to \mathsf{V}$*. A* morphism of $n \wedge 1$-operad modules $r : \mathcal{P} \to \mathcal{Q}$ *is an operadic transformation* $r : \mathcal{P} \to \mathcal{Q} : \mathsf{L}_n \to \mathsf{V}$*. The disjoint union* M *over* $n \geqslant 0$ *of so defined categories* ${}_n\mathrm{Op}_1$ *of* $n \wedge 1$*-operad modules is the category of operad polymodules.*

Let us describe the structure of an $n \wedge 1$-operad module. A multifunctor $\mathsf{L}_n \to \mathsf{V}$ presumes a sequence $(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$, where $\mathcal{B}, \mathcal{A}_i \in \operatorname{Ob} \mathcal{V}^{\mathbb{N}}$ for $i \in \mathbf{n}$, and $\mathcal{P} \in \operatorname{Ob} \mathcal{V}^{\mathbb{N}^n}$.

2.13. EXAMPLE. Particular cases of $\circledast$ for V will obtain a special notation. In addition to the above assume that $\mathcal{A}_i^h \in \operatorname{Ob} \mathcal{V}^{\mathbb{N}}$. We denote

$$\begin{gathered}
\mathcal{P} \odot_0 \mathcal{B} = \circledast(\mathbf{n} \to \mathbf{1} \to \mathbf{1})(\mathcal{P}; \mathcal{B}), \\
\odot_{>0}((\mathcal{A}_i)_{i \in \mathbf{n}}; \mathcal{P}) = \circledast(\mathbf{n} \xrightarrow{1} \mathbf{n} \to \mathbf{1})((\mathcal{A}_i)_{i \in \mathbf{n}}; \mathcal{P}), \\
\odot_{\geqslant 0}((\mathcal{A}_i)_{i \in \mathbf{n}}; \mathcal{P}; \mathcal{B}) = \circledast(\mathbf{n} \xrightarrow{1} \mathbf{n} \to \mathbf{1} \to \mathbf{1})((\mathcal{A}_i)_{i \in \mathbf{n}}; \mathcal{P}; \mathcal{B}).
\end{gathered}$$

All these expressions describe actions of several copies of the category $\mathcal{V}^{\mathbb{N}}$ on the category $\mathcal{V}^{\mathbb{N}^n}$. In isomorphic form these actions are given by the graded components, $\ell \in \mathbb{N}^n$,

$$(\mathcal{P} \odot_0 \mathcal{B})(\ell) \simeq \coprod_{t_1 + \dots + t_m = \ell}^{m \geqslant 0} \Bigl(\bigotimes_{r=1}^{m} \mathcal{P}(t_r) \Bigr) \otimes \mathcal{B}(m),$$

$$\odot_{>0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P})(\ell) \simeq \coprod_{k \in \mathbb{N}^n} \coprod_{j_1^i + \dots + j_{k^i}^i = \ell^i}^{\forall\, i \in \mathbf{n}} \Bigl[\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k^i} \mathcal{A}_i(j_p^i) \Bigr] \otimes \mathcal{P}(k),$$

$$\begin{aligned}
&\odot_{\geqslant 0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})(\ell) \\
&\quad \simeq \coprod_{m=0}^{\infty} \coprod_{k_1, \dots, k_m \in \mathbb{N}^n} \coprod_{\sum_{p=1}^{k_1^i + \dots + k_m^i} j_p^i = \ell^i}^{\forall\, i \in \mathbf{n}} \Bigl(\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k_1^i + \dots + k_m^i} \mathcal{A}_i(j_p^i) \Bigr) \otimes \Bigl(\bigotimes_{r=1}^{m} \mathcal{P}(k_r) \Bigr) \otimes \mathcal{B}(m) \\
&\quad \simeq \coprod_{m=0}^{\infty} \coprod_{t_1 + \dots + t_m = \ell} \coprod_{k_1, \dots, k_m \in \mathbb{N}^n} \coprod_{y_{r,1}^i + \dots + y_{r,k_r^i}^i = t_r^i}^{\forall\, i \in \mathbf{n},\, r \in \mathbf{m}} \bigotimes_{r=1}^{m} \Bigl[\Bigl(\bigotimes_{i=1}^{n} \bigotimes_{v=1}^{k_r^i} \mathcal{A}_i(y_{r,v}^i) \Bigr) \otimes \mathcal{P}(k_r) \Bigr] \otimes \mathcal{B}(m).
\end{aligned}$$

Actually, the action $\odot_{>0}$ can be presented as a combination of partial actions $\odot_i$ for $1 \leqslant i \leqslant n$ defined as

$$\mathcal{A} \odot_i \mathcal{P} = \odot_{>0}(\mathbb{1}, \dots, \mathbb{1}, \mathcal{A}, \mathbb{1}, \dots, \mathbb{1}; \mathcal{P}), \qquad \mathcal{A} \text{ on } i\text{-th place},$$

where the operad $\mathbb{1}$ has $\mathbb{1}(1) = \Bbbk$ and $\mathbb{1}(m) = 0$ for $m \neq 1$. Explicit presentation of this action is

$$(\mathcal{A} \odot_i \mathcal{P})(\ell) = \coprod_{j_1 + \dots + j_q = \ell^i}^{q \geqslant 0} \Bigl(\bigotimes_{p=1}^{q} \mathcal{A}(j_p) \Bigr) \otimes \mathcal{P}(\ell, \ell^i \mapsto q),$$

where $(\ell, \ell^i \mapsto q) = (\ell^1, \dots, \ell^{i-1}, q, \ell^{i+1}, \dots, \ell^n)$.

Iterating these actions I times we get the following expressions

$$\odot_0^{[I]}(\mathcal{P};(\mathcal{B}_i)_{i\in I}) = \circledast(\mathbf{n}\to\underbrace{\mathbf{1}\to\mathbf{1}\cdots\to\mathbf{1}}_{\mathbf{1}\sqcup I})(\mathcal{P};(\mathcal{B}_i)_{i\in I}),$$

$$\odot_{>0}^{I\sqcup\mathbf{1}}(((\mathcal{A}_i^h)_{h\in\mathbf{n}})_{i\in I};\mathcal{P}) = \circledast(\underbrace{\mathbf{n}\xrightarrow{1}\mathbf{n}\xrightarrow{1}\mathbf{n}\dots\xrightarrow{1}\mathbf{n}\xrightarrow{1}\mathbf{n}}_{\mathbf{1}\sqcup I}\to\mathbf{1})(((\mathcal{A}_i^h)_{h\in\mathbf{n}})_{i\in I};\mathcal{P}),$$

$$\odot_{\geqslant0}^{[I]}(((\mathcal{A}_i^h)_{h\in\mathbf{n}})_{i\in I};\mathcal{P};(\mathcal{B}_i)_{i\in I}) = \circledast(\underbrace{\mathbf{n}\dots\xrightarrow{1}\mathbf{n}}_{[I]}\to\underbrace{\mathbf{1}\to\dots\mathbf{1}}_{[I]})(((\mathcal{A}_i^h)_{h\in\mathbf{n}})_{i\in I};\mathcal{P};(\mathcal{B}_i)_{i\in I}).$$

Here the last three trees are functors $\mathbf{1}\sqcup[I]\to\mathcal{O}_{\mathsf{sk}}$, $\mathbf{1}\sqcup[I]^{\mathrm{op}}\to\mathcal{O}_{\mathsf{sk}}$ and $\mathbf{1}\sqcup[I]^{\mathrm{op}}\cup_{0\sim0}[I]\to\mathcal{O}_{\mathsf{sk}}$ respectively, where $[I]^{\mathrm{op}}\cup_{0\sim0}[I]$ is obtained by identifying elements $0\in[I]^{\mathrm{op}}$ and $0\in[I]$.

One can show that these actions are Monoidal and that actions $\odot_i$ for different $0\leqslant i\leqslant n$ commute up to isomorphisms that satisfy coherence conditions. Furthermore, the action $\odot_{\geqslant0}$ can be presented as a combination of partial actions $\odot_i$ for $0\leqslant i\leqslant n$. To be rigorous this approach requires more definitions, and we have chosen to avoid it.

All strings $i\to i\to\dots\to i$ of $\succ$ have different length. Since the actions are Monoidal, we can extend strings to the same length by adding action of several units $1\in\mathcal{A}_i(1)$. Equivalently, we may iterate the monad $\mathcal{P}\mapsto\odot_{\geqslant0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$.

2.14. Definition. *An* $n\wedge1$-operad module *can be defined also as a family* $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$, *consisting of* $n+1$ *operads* $\mathcal{A}_i$, $\mathcal{B}$ *and an object* $\mathcal{P}\in\mathbf{gr}^{\mathbb{N}^n}$ *(resp.* $\mathcal{P}\in\mathbf{dg}^{\mathbb{N}^n}$*), equipped with an algebra structure*

$$\alpha:\odot_{\geqslant0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})\to\mathcal{P}$$

for the monad $\mathcal{Q}\mapsto\odot_{\geqslant0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{Q};\mathcal{B})$.

An action is specified by a collection of maps given for each $m\in\mathbb{N}$, each family $k_1,\dots,k_m\in\mathbb{N}^n$ and each family of non-negative integers $\bigl((j_p^i)_{p=1}^{k_1^i+\dots+k_m^i}\bigr)_{i=1}^n$

$$\begin{aligned}\alpha:{}&\circledast(\mathbf{n}\xrightarrow{1}\mathbf{n}\to\mathbf{1}\to\mathbf{1})((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{P};\mathcal{B})(\tau)\\ &=\Bigl(\bigotimes_{i=1}^n\bigotimes_{p=1}^{k_1^i+\dots+k_m^i}\mathcal{A}_i(j_p^i)\Bigr)\otimes\Bigl(\bigotimes_{r=1}^m\mathcal{P}(k_r)\Bigr)\otimes\mathcal{B}(m)\to\mathcal{P}\Bigl(\Bigl(\sum_{p=1}^{k_1^i+\dots+k_m^i}j_p^i\Bigr)_{i=1}^n\Bigr),\end{aligned}\tag{2.6}$$

$$\tau_\alpha=\begin{array}{ccc} \sum_{p=1}^{k_1^n+\dots+k_m^n}\mathbf{j}_p^n\longrightarrow\mathbf{k}_1^n+\dots+\mathbf{k}_m^n & & \\ \dots\longrightarrow\dots & \searrow & \\ & & \mathbf{m}\longrightarrow\mathbf{1} \\ \sum_{p=1}^{k_1^2+\dots+k_m^2}\mathbf{j}_p^2\longrightarrow\mathbf{k}_1^2+\dots+\mathbf{k}_m^2 & \nearrow & \\ \sum_{p=1}^{k_1^1+\dots+k_m^1}\mathbf{j}_p^1\longrightarrow\mathbf{k}_1^1+\dots+\mathbf{k}_m^1 & & \end{array}.\tag{2.7}$$

Associativity of α can be formulated via contraction of trees.

Restricting the action α to submonads $\odot_{>0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{Q})\hookrightarrow\odot_{\geqslant0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{Q};\mathcal{B})$, $\mathcal{Q}\odot_0\mathcal{B}\hookrightarrow\odot_{\geqslant0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{Q};\mathcal{B})$, $\mathcal{A}_i\odot_i\mathcal{Q}\hookrightarrow\odot_{\geqslant0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{Q};\mathcal{B})$, $1\leqslant i\leqslant n$, obtained via insertion of operad units η, we get the partial actions

$$\rho = \rho_{(k_r)} : \circledast(\mathbf{n} \to \mathbf{1} \to \mathbf{1})(\mathcal{P};\mathcal{B})(\tau_{(k_r)}) = \Big(\bigotimes_{r=1}^{m} \mathcal{P}(k_r)\Big) \otimes \mathcal{B}(m) \to \mathcal{P}\Big(\sum_{r=1}^{m} k_r\Big),$$

$$\tau_\rho = \tau_{(k_r)} = \begin{array}{l} \mathbf{k}_1^n + \dots + \mathbf{k}_m^n \\ \cdots \\ \mathbf{k}_1^2 + \dots + \mathbf{k}_m^2 \\ \mathbf{k}_1^1 + \dots + \mathbf{k}_m^1 \end{array} \longrightarrow \mathbf{m} \longrightarrow \mathbf{1} \qquad , \tag{2.8}$$

$$\lambda = \lambda_{k,(j_p^i)} : \circledast(\mathbf{n} \xrightarrow{1} \mathbf{n} \to \mathbf{1})((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{P})(\tau_{k,(j_p^i)})$$
$$= \Big[\bigotimes_{i=1}^{n}\bigotimes_{p=1}^{k^i} \mathcal{A}_i(j_p^i)\Big] \otimes \mathcal{P}\big((k^i)_{i=1}^n\big) \to \mathcal{P}\Big(\Big(\sum_{p=1}^{k^i} j_p^i\Big)_{i=1}^n\Big),$$

$$\tau_\lambda = \tau_{k,(j_p^i)} = \begin{array}{l} \sum_{p=1}^{k^n} \mathbf{j}_p^n \longrightarrow \mathbf{k}^n \\ \cdots \longrightarrow \cdots \\ \sum_{p=1}^{k^2} \mathbf{j}_p^2 \longrightarrow \mathbf{k}^2 \\ \sum_{p=1}^{k^1} \mathbf{j}_p^1 \longrightarrow \mathbf{k}^1 \end{array} \longrightarrow \mathbf{1} \qquad , \tag{2.9}$$

$$\lambda^i = \lambda^i_{k,(j_p)} : \Big[\bigotimes_{p=1}^{k^i} \mathcal{A}_i(j_p)\Big] \otimes \mathcal{P}(k) \to \mathcal{P}\Big(k^1,\dots,k^{i-1},\sum_{p=1}^{k^i} j_p, k^{i+1},\dots,k^n\Big). \tag{2.10}$$

On the other hand, given n left actions λ^i of $\mathcal{A}_i$ and a right action ρ of $\mathcal{B}$, all pairwise commuting, we can restore the total action α.

Assume that $f_i : \mathcal{C}_i \to \mathcal{A}_i$, $g : \mathcal{D} \to \mathcal{B}$ are morphisms of operads. They imply a morphism of monads $\odot_{\geqslant 0}(\mathcal{C}_1,\dots,\mathcal{C}_n;\mathcal{Q};\mathcal{D}) \to \odot_{\geqslant 0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{Q};\mathcal{B})$. An algebra $\mathcal{P}$ over the latter monad becomes an algebra over the former monad denoted ${}_{f_1,\dots,f_n}\mathcal{P}_g$.

The category of $n \wedge 1$-operad modules ${}_n\mathrm{Op}_1$ has morphisms

$$(f_1,\dots,f_n;h;f_0) : (\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{A}_0) \to (\mathcal{C}_1,\dots,\mathcal{C}_n;\mathcal{Q};\mathcal{C}_0),$$

where $f_i : \mathcal{A}_i \to \mathcal{C}_i$, $0 \leqslant i \leqslant n$, are morphisms of $\mathcal{V}$-operads and $h : \mathcal{P} \to {}_{f_1,\dots,f_n}\mathcal{Q}_{f_0} \in \mathcal{V}^{\mathbb{N}^n}$ is a module morphism with respect to actions of all $\mathcal{A}_i$. In fact, a morphism of $n \wedge 1$-operad modules is by definition an operadic transformation $\xi : \mathcal{P} = (\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{A}_0) \to (\mathcal{C}_1,\dots,\mathcal{C}_n;\mathcal{Q};\mathcal{C}_0) = \mathcal{Q} : \mathsf{L}_n \to \mathsf{V}$. It consists of morphisms $f_i : \mathcal{A}_i \to \mathcal{C}_i \in \mathcal{V}^{\mathbb{N}}$, $0 \leqslant i \leqslant n$, and $h : \mathcal{P} \to \mathcal{Q} \in \mathcal{V}^{\mathbb{N}^n}$. Equation (2.3) reads on $(lt_{\mathbf{k}}, {}^k i)$ as

$$\big(\odot^{\mathbf{k}}\mathcal{A}_i \xrightarrow{\odot^{\mathbf{k}} f_i} \odot^{\mathbf{k}}\mathcal{B}_i \xrightarrow{\mu^k} \mathcal{B}_i\big) = \big(\odot^{\mathbf{k}}\mathcal{A}_i \xrightarrow{\mu^k} \mathcal{A}_i \xrightarrow{f_i} \mathcal{B}_i\big),$$

that is, f_i is a morphism of operads. On $\succ$ the equation gives

$$\big[\circledast^{\mathbf{k}}_{\mathsf{V}}(\succ)(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{A}_0) \xrightarrow{\circledast^{\mathbf{k}}_{\mathsf{V}}(\succ)(f_1,\dots,f_n;h;f_0)} \circledast^{\mathbf{k}}_{\mathsf{V}}(\succ)(\mathcal{C}_1,\dots,\mathcal{C}_n;\mathcal{Q};\mathcal{C}_0) \xrightarrow{\alpha} \mathcal{Q}\big]$$
$$= \big[\circledast^{\mathbf{k}}_{\mathsf{V}}(\succ)(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{A}_0) \xrightarrow{\alpha} \mathcal{P} \xrightarrow{h} \mathcal{Q}\big],$$

that is, $h : \mathcal{P} \to {}_{f_1,\dots,f_n}\mathcal{Q}_{f_0}$ is a module morphism with respect to actions of all $\mathcal{A}_i$, $0 \leqslant i \leqslant n$.

Objects of the category $\mathcal{V}^{n\mathbb{N}\sqcup\mathbb{N}^n\sqcup\mathbb{N}}$ are also written as tuples $(\mathcal{U}_1,\dots,\mathcal{U}_n;\mathcal{X};\mathcal{W})$. The free algebra functor for the monad $\odot_{\geqslant 0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\text{-};\mathcal{B})$ is the functor $\mathcal{V}^{\mathbb{N}^n} \to \mathcal{A}_1\text{-}\cdots\text{-}\mathcal{A}_n\text{-mod-}\mathcal{B}$, $\mathcal{X} \mapsto \odot_{\geqslant 0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{X};\mathcal{B})$, left adjoint to the underlying functor $\mathcal{A}_1\text{-}\cdots\text{-}\mathcal{A}_n\text{-mod-}\mathcal{B} \to \mathcal{V}^{\mathbb{N}^n}$. Hence, there is also a pair of adjoint functors $F : \mathcal{V}^{n\mathbb{N}\sqcup\mathbb{N}^n\sqcup\mathbb{N}} \rightleftarrows {}_n\mathrm{Op}_1 : U$,

$$F(\mathcal{U}_1,\dots,\mathcal{U}_n;\mathcal{X};\mathcal{W}) = (T\mathcal{U}_1,\dots,T\mathcal{U}_n;\odot_{\geqslant 0}(T\mathcal{U}_1,\dots,T\mathcal{U}_n;\mathcal{X};T\mathcal{W});T\mathcal{W}).$$

The module part is indexed by trees with the top floor describing $\mathcal{X}(\text{-}_1)\otimes\cdots\otimes\mathcal{X}(\text{-}_k)$, lower floors indexed by $\mathcal{W}(\text{-})$ and n forests indexed by $\mathcal{U}_i(\text{-})$ attached to each of k leaves.

In particular,

$$F(\mathcal{U}_1,\dots,\mathcal{U}_n;0;\mathcal{W}) = (T\mathcal{U}_1,\dots,T\mathcal{U}_n;(T\mathcal{W})(0);T\mathcal{W}).$$

2.15. The monad of free $n\wedge 1$-operad modules. Recall [BW05, Section 3.3.6] that a parallel pair of morphisms $f,g : A \to B \in \mathcal{C}$ is called *reflexive* if there is a morphism $r : B \to A \in \mathcal{C}$ such that $f\circ r = \mathrm{id}_B = g\circ r$. Recall that a *contractible coequalizer* [BW05, Section 3.3.3] (= a *split fork* [Mac88, Section VI.6]) is a diagram in a category $\mathcal{D}$

$$A' \underset{d^1}{\overset{d^0}{\rightrightarrows}} B' \xrightarrow{d} C', \qquad t : B' \to A', \quad s : C' \to B'$$

such that $d^0\circ t = \mathrm{id}$, $d^1\circ t = s\circ d$, $d\circ s = \mathrm{id}$, and $d\circ d^0 = d\circ d^1$. Suppose there is a functor $U : \mathcal{C} \to \mathcal{D}$. Then a pair $f,g : A \to B \in \mathcal{C}$ is called *U-contractible coequalizer pair* if $d^0 = Uf, d^1 = Ug : UA \to UB$ extend to a contractible coequalizer in $\mathcal{D}$. One says that U *creates* U-contractible coequalizers [Mac88, Section VI.7] if for any pair $f,g : A \to B \in \mathcal{C}$ and any contractible coequalizer in $\mathcal{D}$

$$UA \underset{Ug}{\overset{Uf}{\rightrightarrows}} UB \xrightarrow{d} C', \qquad t : UB \to UA, \quad s : C' \to UB$$

† there is a unique morphism $h : B \to C \in \mathcal{C}$ such that $C' = UC$, $d = Uh$, and

‡ h is a coequalizer of (f,g) in $\mathcal{C}$.

The following statement is Exercise 3.3.(PPTT) of [BW05].

2.16. Theorem. *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor which has a left adjoint F. Then the comparison functor $\Phi : \mathcal{C} \to \mathcal{D}^\top$, $A \mapsto (UA, U\varepsilon : UFUA \to UA)$, for the monad $\top = U\circ F$ in $\mathcal{D}$ is an isomorphism of categories if and only if U creates coequalizers of reflexive U-contractible coequalizer pairs in $\mathcal{C}$.*

Here $\mathcal{D}^\top$ is the category of $\top$-algebras. The condition of the theorem applied to $f = \mathrm{id} = g$ implies that U reflects isomorphisms. The proof of this theorem is contained in the proof of (PTT), Beck's Precise Tripleability Theorem [BW05, Theorem 3.3.14]. We shall use the following corollary to Theorem 2.16.

One says that (*cf.* [BW05, Section 3.5])

(CTT′) $U : \mathcal{C} \to \mathcal{D}$ creates coequalizers for reflexive pairs (f, g) for which (Uf, Ug) has a coequalizer,

if for any reflexive pair $f, g : A \to B \in \mathcal{C}$ and any coequalizer $d : UB \to C'$ of (Uf, Ug) in $\mathcal{D}$ conclusions † and ‡ before Theorem 2.16 hold.

2.17. Corollary. [Crude Tripleability Theorem] *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor which satisfies* (CTT′) *and has a left adjoint F. Then the comparison functor $\Phi : \mathcal{C} \to \mathcal{D}^\top$, $\top = U \circ F$ is an isomorphism of categories.*

2.18. Definition. *An* ideal *of an object $\tilde{\mathcal{A}} = (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{A}_0) \in {}_n\mathrm{Op}_1$ is a subobject $\tilde{\mathcal{I}} = (\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{K}; \mathcal{I}_0)$ of $U\tilde{\mathcal{A}}$ in $\mathcal{V}^S$, $S = n\mathbb{N} \sqcup \mathbb{N}^n \sqcup \mathbb{N} = \mathbb{N} \sqcup \dots \sqcup \mathbb{N} \sqcup \mathbb{N}^n \sqcup \mathbb{N}$, stable under all multiplications in operads $\mathcal{A}_j$, $0 \leqslant j \leqslant n$, and under the action on $\mathcal{P}$ from* (2.6)*. Namely if at least one $\otimes$-argument of multiplication or action is in $\tilde{\mathcal{I}}$, then the result is in $\tilde{\mathcal{I}}$ as well. Equivalently, for all values of indices*

$$\lambda^i_{k,(j_p)}\Bigl(\Bigl[\bigotimes_{p=1}^{k^i} \mathcal{A}_i(j_p)\Bigr] \otimes \mathcal{K}(k)\Bigr) \subset \mathcal{K}\Bigl(k^1, \dots, k^{i-1}, \sum_{p=1}^{k^i} j_p, k^{i+1}, \dots, k^n\Bigr),$$

$$\rho\Bigl(\Bigl[\Bigl(\bigotimes_{r=1}^{t-1} \mathcal{P}(k_r)\Bigr) \otimes \mathcal{K}(k_t) \otimes \Bigl(\bigotimes_{r=t+1}^{m} \mathcal{P}(k_r)\Bigr)\Bigr] \otimes \mathcal{A}_0(m)\Bigr) \subset \mathcal{K}\Bigl(\sum_{r=1}^{m} k_r\Bigr),$$

and $\mathcal{I}_j$ are ideals of operads $\mathcal{A}_j$ in a similar sense.

Assume in addition that $\mathcal{V}$ is abelian. Ideals are precisely kernels in $\mathcal{V}^S$ of Uh for morphisms $h : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}} \in {}_n\mathrm{Op}_1$. If $\tilde{\mathcal{I}}$ is an ideal of $\tilde{\mathcal{A}}$, then the quotient $U\tilde{\mathcal{A}}/U\tilde{\mathcal{I}}$ in the abelian category $\mathcal{V}^S$ admits a unique structure of an $n \wedge 1$-operad module such that the quotient map $q : \mathcal{A} \to \tilde{\mathcal{A}}/\tilde{\mathcal{I}}$ is in ${}_n\mathrm{Op}_1$.

For any subcomplex $\mathcal{N} \subset \tilde{\mathcal{A}} \in \mathcal{V}^S$ there is the smallest ideal $\tilde{\mathcal{I}}$ of $\tilde{\mathcal{A}}$ containing $\mathcal{N}$. It is spanned as a graded $\Bbbk$-submodule of $\tilde{\mathcal{A}}$ by results of multiplications or actions containing an element of $\mathcal{N}$ among its $\otimes$-arguments. So obtained $\tilde{\mathcal{I}}$ is indeed an ideal due to associativity of the action. In particular, for a pair of parallel arrows $f, g : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}} \in {}_n\mathrm{Op}_1$ there is the image $\mathcal{N} = \mathrm{Im}(f - g)$ in the abelian category $\mathcal{V}^S$. If $\tilde{\mathcal{I}}$ is the smallest ideal of $\tilde{\mathcal{A}}$ containing $\mathcal{N}$, then the quotient $\tilde{\mathcal{A}}/\tilde{\mathcal{I}}$ is the coequalizer of f and g in ${}_n\mathrm{Op}_1$.

2.19. Proposition. *The comparison functor for the underlying functor $U : {}_n\mathrm{Op}_1 \to \mathcal{V}^{n\mathbb{N} \sqcup \mathbb{N}^n \sqcup \mathbb{N}}$ is an isomorphism of categories.*

Thus ${}_n\mathrm{Op}_1$ is isomorphic to the category of $\top$-algebras for the monad $\top = U \circ F$ in $\mathcal{V}^{n\mathbb{N} \sqcup \mathbb{N}^n \sqcup \mathbb{N}}$.

Proof. Let us prove that U satisfies condition (CTT′). First (as a warm-up) we show it for $U : \mathrm{Op} \to \mathcal{V}^{\mathbb{N}}$. Let a reflexive pair $f, g : \mathcal{A} \leftrightarrows \mathcal{B} : r$ in Op be given together with a coequalizer $d : UB \to \mathcal{C}'$ in $\mathcal{V}^{\mathbb{N}}$ of (Uf, Ug). The subobject $\mathcal{K} = \mathrm{Im}(f - g) = \mathrm{Ker}\, d \in \mathcal{V}^{\mathbb{N}}$

of UB is an ideal of the operad $\mathcal{B}$. In fact, for any multiplication $\mu : \mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_k)\otimes\mathcal{O}(k) \to \mathcal{O}(n_1+\cdots+n_k)$ for the operads $\mathcal{B}$ and $\mathcal{A}$ we have

$$\begin{aligned}
&\mu^{\mathcal{B}}(b_1\otimes\cdots\otimes b_{i-1}\otimes(f-g)a\otimes b_{i+1}\otimes\cdots\otimes b_k\otimes b)\\
&= \mu^{\mathcal{B}}(frb_1\otimes\cdots\otimes frb_{i-1}\otimes fa\otimes frb_{i+1}\otimes\cdots\otimes frb_k\otimes frb)\\
&- \mu^{\mathcal{B}}(grb_1\otimes\cdots\otimes grb_{i-1}\otimes ga\otimes grb_{i+1}\otimes\cdots\otimes grb_k\otimes grb)\\
&= (f-g)\mu^{\mathcal{A}}(rb_1\otimes\cdots\otimes rb_{i-1}\otimes a\otimes rb_{i+1}\otimes\cdots\otimes rb_k\otimes rb)\in\mathcal{K}
\end{aligned}$$

for all $1\leqslant i\leqslant k$. Similarly,

$$\mu^{\mathcal{B}}(b_1\otimes\cdots\otimes b_k\otimes(f-g)a) = (f-g)\mu^{\mathcal{A}}(rb_1\otimes\cdots\otimes rb_k\otimes a)\in\mathcal{K}.$$

Thus the quotient operad $\mathcal{B}/\mathcal{K}$ exists together with the quotient map $q : \mathcal{B}\to\mathcal{B}/\mathcal{K}\in\mathrm{Op}$. The map d factorises as $d = \bigl(U\mathcal{B}\xrightarrow{Uq}U(\mathcal{B}/\mathcal{K})\xrightarrow[\sim]{\phi}\mathcal{C}'\bigr)$ for a unique isomorphism $\phi\in\mathcal{V}^{\mathbb{N}}$. Transferring the operad structure from $\mathcal{B}/\mathcal{K}$ to $\mathcal{C}'$ along ϕ we make the latter into an operad $\mathcal{C}$, make d into a morphism of operads. Clearly, properties † and ‡ on page 1520 hold true.

Consider now a reflexive pair in ${}_n\mathrm{Op}_1$

$$\tilde f = ((f_i)_0^n; f), \tilde g = ((g_i)_0^n; g) : \tilde{\mathcal{A}} = ((\mathcal{A}_i)_1^n;\mathcal{P};\mathcal{A}_0) \leftrightarrows \tilde{\mathcal{B}} = ((\mathcal{B}_i)_1^n;\mathcal{Q};\mathcal{B}_0) : \tilde r = ((r_i)_0^n; r).$$

Then $\tilde{\mathcal{I}} = ((\mathcal{I}_i)_1^n;\mathcal{K};\mathcal{I}_0) = \mathrm{Im}(U\tilde f - U\tilde g)$ is an ideal in $\tilde{\mathcal{B}}$. In fact it suffices to take in the source of action map (2.6) one of the $\otimes$-arguments equal to $(\tilde f-\tilde g)x$ for $x\in\mathcal{A}_i$ or $\mathcal{P}$ or $\mathcal{A}_0$. Then

$$\begin{aligned}
&\alpha(\cdots\otimes b_i^p\otimes\cdots\otimes(\tilde f-\tilde g)x\otimes\cdots\otimes q_j\otimes\cdots\otimes b_0)\\
&= \alpha(\cdots\otimes f_ir_ib_i^p\otimes\cdots\otimes\tilde fx\otimes\cdots\otimes frq_j\otimes\cdots\otimes f_0r_0b_0)\\
&- \alpha(\cdots\otimes g_ir_ib_i^p\otimes\cdots\otimes\tilde gx\otimes\cdots\otimes grq_j\otimes\cdots\otimes g_0r_0b_0)\\
&= (f-g)\alpha(\cdots\otimes r_ib_i^p\otimes\cdots\otimes x\otimes\cdots\otimes rq_j\otimes\cdots\otimes r_0b_0)\in\mathcal{K}.
\end{aligned}$$

Thus $\tilde{\mathcal{B}}/\tilde{\mathcal{I}}$ is an $n\wedge 1$-operad module and the rest of the proof goes similarly to the case of operads. ∎

2.20. Corollary. *The category ${}_n\mathrm{Op}_1$ is complete and cocomplete.*

Proof. In fact, $\mathcal{D} = \mathcal{V}^S$ is complete and cocomplete. Completeness of ${}_n\mathrm{Op}_1 \simeq \mathcal{D}^\top$ follows by [BW05, Corollary 3.4.3]. We have seen that the category ${}_n\mathrm{Op}_1$ has coequalizers. Therefore ${}_n\mathrm{Op}_1$ is cocomplete by [BW05, Corollary 9.3.3]. ∎

2.20.1. QUOTIENTS. The category $(\mathcal{A}_1,\dots,\mathcal{A}_n)$-mod-$\mathcal{B}$ of operad $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B})$-modules is a subcategory of ${}_n\mathrm{Op}_1$, whose objects are $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ (shortly $\mathcal{P}$) and morphisms are $(1_{\mathcal{A}_1},\dots,1_{\mathcal{A}_n};h;1_{\mathcal{B}})$. It has an initial object $\mathcal{I}=\mathcal{B}(0)$ with

$$\mathcal{I}(k)=\begin{cases}\mathcal{B}(0), & \text{if } k=0\\ 0, & \text{if } k\neq 0\end{cases}, \quad k\in\mathbb{N}^n.$$

Actions are given by

$$\lambda_0^i=\mathrm{id}:\mathcal{B}(0)\to\mathcal{B}(0),$$
$$\rho=\mu^{\mathcal{B}}_{0,\dots,0}:(\otimes^{\mathbf{m}}\mathcal{B}(0))\otimes\mathcal{B}(m)\to\mathcal{B}(0).$$

A *system of relations* in a $\mathbf{gr}$-operad $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B})$-module $\mathcal{P}$ means a set (of relations) R, arity function $a:R\to\mathbb{N}^n$, grade function $g:R\to\mathbb{Z}$ and for each $r\in R$ elements $x_r,y_r\in\mathcal{P}(a(r))^{g(r)}$ supposed to be identified by the relation $x_r=y_r$. A system of relations gives rise to a free graded $\Bbbk$-module $\Bbbk R\in\mathbf{gr}^{\mathbb{N}^n}$ with

$$\Bbbk R(a)^g=\Bbbk\{r\in R\mid a(r)=a,\ g(r)=g\},\quad \text{for } a\in\mathbb{N}^n,\ g\in\mathbb{Z},$$

and to a map $\Bbbk R\to\mathcal{P}$, $r\mapsto x_r-y_r$. Denote by $\mathcal{N}$ the image of this map in abelian category $\mathbf{gr}^{\mathbb{N}^n}$. Let $\mathcal{K}=(0,\dots,0;\mathcal{K};0)$ be the graded ideal of $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ generated by $\mathcal{N}$. If $\mathcal{V}=\mathbf{dg}$ and $\mathcal{N}\partial\subset\mathcal{K}$, then $\mathcal{K}$ is a differential graded ideal and the quotient $\mathcal{P}/\mathcal{K}$ in $\mathcal{V}^{\mathbb{N}^n}$ is an operad $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B})$-module. This is the quotient of an operad module by a system of relations.

2.21. THE LAX $\mathcal{C}at$-MULTIFUNCTOR *hom*. Starting with an arbitrary symmetric $\mathcal{V}$-multicategory C we construct a lax $\mathcal{C}at$-multifunctor $hom:\mathsf{B}\to\mathsf{V}$, where the $\mathcal{C}at$-multicategory B has $\operatorname{Ob}\mathsf{B}=\operatorname{Ob}\mathsf{C}$. For any sequence $(A_i)_{i\in I}$, B of objects of B the category $\mathsf{B}((A_i)_{i\in I};B)$ is the terminal category $\mathbb{1}$. An arbitrary lax $\mathcal{C}at$-multifunctor $\mathsf{B}\to\mathsf{V}$ assigns an object of $\mathcal{V}^{\mathbb{N}^I}$ to a sequence $(A_i)_{i\in I}$, B. In the case of *hom* this is the object $hom((A_i)_{i\in I};B)\in\operatorname{Ob}\mathcal{V}^{\mathbb{N}^I}$ given by

$$hom((A_i)_{i\in I};B)((n^i)_{i\in I})=\mathsf{C}\big(({}^{n^i}A_i)_{i\in I};B\big).$$

For each tree t and each t-tree $\tau:t\to\mathcal{O}_{\mathsf{sk}}$ we choose a 2-morphism:

$$\begin{aligned}\mathrm{comp}_\tau:\ &\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)} hom\big(A_{\mathrm{in}(v)};A_{\mathrm{ou}(v)}\big)\big((|\tau(e)^{-1}(p)|)_{e\in\mathrm{in}(v)}\big)\\ &=\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)}\mathsf{C}\big(({}^{|\tau(e)^{-1}(p)|}A_e)_{e\in\mathrm{in}(v)};A_{\mathrm{ou}(v)}\big)\\ &\xrightarrow{\cong}\bigotimes^{r\in\mathrm{v}(\tilde\tau)}\mathsf{C}\big((A_{\tau^*(\varepsilon)})_{\varepsilon\in\mathrm{in}(r)};A_{\tau^*(\mathrm{ou}(r))}\big)\xrightarrow{\mu^{\tilde\tau}_{\mathsf{C}}}\mathsf{C}\big((A_{\tau^*(b)})_{b\in\mathrm{Inp}(\tilde\tau)};A_{\tau^*(\text{root edge}(\tilde\tau))}\big)\\ &=\mathsf{C}\big(({}^{|\tau(\mathrm{tail}\,a)|}A_a)_{a\in\mathrm{Inp}\,t};A_{\text{root edge}(t)}\big)=hom\big(A_{\mathrm{Inp}\,t};A_{\text{root edge}(t)}\big)\big((|\tau(a)|)_{a\in\mathrm{Inpv}\,t}\big).\end{aligned}\tag{2.11}$$

The essential map here is the multiplication $\mu_{\mathsf{C}}^{\tilde{\tau}}$ in symmetric $\mathcal{V}$-multicategory C associated with the strongly ordered tree $\tilde{\tau}$ which is obtained from the t-tree τ. All its vertices except the head of the root edge constitute the totally ordered set $\overline{\mathrm{v}}(\tilde{\tau}) = \bigsqcup^{\leqslant}_{v\in\overline{\mathrm{v}}(t)} \tau(v)$. The root vertex of $\tilde{\tau}$ is the only element of $\tau(\mathrm{rv}(t))$. The subset of internal vertices is $\mathrm{v}(\tilde{\tau}) = \bigsqcup^{\leqslant}_{v\in\mathrm{v}(t)} \tau(v)$. Correspondingly, $\mathrm{Inpv}(\tilde{\tau}) = \bigsqcup^{\leqslant}_{a\in\mathrm{Inpv}(t)} \tau(a)$. Sets $\mathrm{inV}_{\tilde{\tau}}(v,p)$, $p\in\tau(v)$, get a total ordering being subsets of lexicographically ordered sets $\bigsqcup^{\leqslant}_{u\in(\mathrm{inV}(v),\lessdot)} \tau(u)$. The tree map $\tau^* : \tilde{\tau} \to t$ takes $\tau(v)$ to $v\in\overline{\mathrm{v}}(t)$ and the head of the root edge to the head of the root edge. Edges of $\tilde{\tau}$ are of the form $(u,q)\to(v,p)$, where $(e: u\to v)\in E(t)$ and $p=\tau(e).q$. Under τ^* this edge goes to $e: u\to v$. The equation for the lax $\mathcal{C}at$-multifunctor hom follows from [BLM08, equation (2.25.1)] written for the algebra C in the lax Monoidal category ${}^{\mathrm{Ob}\,\mathsf{C}}\mathcal{PMQ}_{\mathcal{V}}$.

There is an equation for $\theta = I_t(t_v \mid v\in\mathrm{v}(t))$

$$\begin{aligned}
\Big[&\circledast_{\mathsf{DG}}(\theta)\big(hom((A_e)_{e\in\mathrm{in}(q)};A_{\mathrm{ou}(q)})\big)_{q\in\mathrm{v}(\theta)} \\
&\xrightarrow{\cong} \circledast_{\mathsf{DG}}(t)\big(\circledast_{\mathsf{DG}}(t_v)(hom((A_e)_{e\in\mathrm{in}(p)};A_{\mathrm{ou}(p)}))_{p\in\mathrm{v}(t_v)}\big)_{v\in\mathrm{v}(t)} \\
&\xrightarrow{\circledast_{\mathsf{DG}}(t)\,\mathrm{comp}(t_v)} \circledast_{\mathsf{DG}}(t)\big(hom((A_e)_{e\in\mathrm{in}(v)};A_{\mathrm{ou}(v)})\big)_{v\in\mathrm{v}(t)} \\
&\xrightarrow{\mathrm{comp}(t)} hom((A_a)_{a\in\mathrm{Inp}\,t};A_{\mathrm{root\ edge}(t)})\Big] = \mathrm{comp}(\theta). \qquad (2.12)
\end{aligned}$$

2.22. Multicategory of operad modules. Let $\mathcal{V}$ be abelian. Note that $n\wedge 1$-operad modules form a $\mathcal{C}at$-multiquiver M whose objects are operads. It is actually a $\mathcal{C}at$-multicategory. To operads $\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B}$ there is associated the category $\mathsf{M}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B}) = (\mathcal{A}_1,\dots,\mathcal{A}_n)$-mod-$\mathcal{B}$ of $n\wedge 1$-operad polymodules. There are two multicategory compositions of interest: $\circledast_{\mathsf{G}}$ and $\circledast_{\mathsf{M}}$, the tensor product of operad modules. The first is described as follows.

Consider a family of operad polymodules $\mathcal{P}_v\in\mathcal{A}_{\mathrm{in}(v)}$-mod-$\mathcal{A}_{\mathrm{ou}(v)}$, $v\in\mathrm{v}(t)$. When $t\neq |$, the right action of $\mathcal{B}=\mathcal{A}_{\mathrm{root\ edge}}$ on $\mathcal{P}=\circledast_{\mathsf{G}}(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)}$ for $j_1+\dots+j_m=j\in\mathbb{N}^{\mathrm{Inp}\,t}$ is

$$\begin{aligned}
\rho = \sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau_r(a)|=j_r^a}^{t-\mathrm{trees}\ (\tau_r)_{r=1}^m} \Big\langle &\Big(\bigotimes_{r=1}^m\big(\circledast_{\mathsf{G}}(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)}(j_r)\big)\Big)\otimes\mathcal{B}(m) \xrightarrow{(\otimes_{r=1}^m \mathrm{pr}_{\tau_r})\otimes 1} \\
&\Big(\bigotimes_{r=1}^m \bigotimes^{v\in\mathrm{v}(t)} \bigotimes^{p\in\tau_r(v)} \mathcal{P}_v\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes\mathcal{B}(m) \xrightarrow{\varkappa\otimes 1} \\
&\Big(\bigotimes^{v\in\mathrm{v}(t)} \bigotimes_{r=1}^m \bigotimes^{p\in\tau_r(v)} \mathcal{P}_v\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes\mathcal{B}(m) \xrightarrow{\cong} \\
&\Big(\bigotimes^{v\in\mathrm{v}(t)-\{\mathrm{rv}\}} \bigotimes_{r=1}^m \bigotimes^{p\in\tau_r(v)} \mathcal{P}_v\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes \\
&\qquad\otimes\Big(\bigotimes_{r=1}^m \mathcal{P}_{\mathrm{rv}}\big(|\tau_r(x)|_{x\in\mathrm{inV}(\mathrm{rv})}\big)\Big)\otimes\mathcal{B}(m) \xrightarrow{1\otimes\rho}
\end{aligned}$$

$$
\begin{aligned}
&\Big(\bigotimes^{v\in\mathrm{v}(t)-\{\mathrm{rv}\}}\ \bigotimes^{p\in\tau(v)} \mathcal{P}_v\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes\mathcal{P}_{\mathrm{rv}}\big(|\tau(x)|_{x\in\mathrm{inV}(\mathrm{rv})}\big)\xrightarrow{\cong}\\
&\bigotimes^{v\in\mathrm{v}(t)}\ \bigotimes^{p\in\tau(v)} \mathcal{P}_v\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\xrightarrow{\mathrm{in}_\tau}\ \circledast_{\mathsf{G}}\,(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)}(j)=\mathcal{P}(j)\Big\rangle,
\end{aligned}
\tag{2.13}
$$

where $\tau=\tau_1+\dots+\tau_m$ is the t-tree such that $\tau(v)=\trianglelefteq\!\rfloor_{r=1}^{m}\tau_r(v)$ for all $v\in\overline{\mathrm{v}}(t)-\{\mathrm{rv}\}$. Maps $\tau(u\to v)$ are $\trianglelefteq\!\rfloor_{r=1}^{m}\tau_r(u\to v)$ if v is not the root vertex.

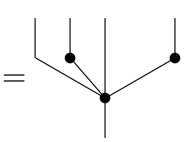
In particular, the map $\rho^{\mathcal{P}}_{\varnothing}:\mathcal{B}(0)\to\mathcal{P}(0)$ sends $\mathcal{B}(0)$ to the summand $\mathcal{P}(\tau_0)\simeq\mathcal{P}_{\mathrm{rv}}(0)$ via $\rho^{\mathcal{P}_{\mathrm{rv}}}_{\varnothing}:\mathcal{B}(0)\to\mathcal{P}_{\mathrm{rv}}(0)$, where $\tau_0(v)=\varnothing$ for $v\in\overline{\mathrm{v}}(t)\setminus\{\mathrm{rv}\}$, while $\tau_0(\mathrm{rv})=\mathbf{1}$.

Consider the same family of operad polymodules $\mathcal{P}_v\in\mathcal{A}_{\mathrm{in}(v)}$-mod-$\mathcal{A}_{\mathrm{ou}(v)}$, $v\in\mathrm{v}(t)$. Identify $\mathrm{Inp}\,t$ with $\mathbf{n}$, then $i\in\mathbf{n}$ means the i-th input edge of t. The left i-th action of $\mathcal{A}_i$ on $\mathcal{P}=\circledast_{\mathsf{G}}(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)}$ for $j_1+\dots+j_m=j\in\mathbb{N}$ is

$$
\begin{aligned}
\lambda^i=&\sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=k^a}^{t-\text{tree }\tau}\Big\langle\Big(\bigotimes_{q=1}^{k^i}\big(\mathcal{A}_i(j_q)\big)\Big)\otimes\circledast_{\mathsf{G}}(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)}(k)\xrightarrow{1\otimes\mathrm{pr}_\tau}\\
&\Big(\bigotimes^{p\in\tau(\mathrm{head}(i))}\ \bigotimes^{q\in\tau(i)^{-1}(p)}\mathcal{A}_i(j_q)\Big)\otimes\bigotimes^{v\in\mathrm{v}(t)}\ \bigotimes^{p\in\tau(v)}\mathcal{P}_v\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\underset{\cong}{\xrightarrow{\zeta}}\\
&\bigotimes^{v\in\mathrm{v}(t)}\ \bigotimes^{p\in\tau(v)}\Big[\Big(\text{if } v=\mathrm{head}(i),\ \text{then}\ \bigotimes^{q\in\tau(i)^{-1}(p)}\mathcal{A}_i(j_q)\otimes\Big)\mathcal{P}_v\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big]\xrightarrow{\otimes^v\otimes^p[1\text{ or }\lambda^i]}\\
&\bigotimes^{v\in\mathrm{v}(t)}\ \bigotimes^{p\in\tau(v)}\mathcal{P}_v\Big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)};\text{if } v=\mathrm{head}(i),e=i\text{ then }|\tau(i)^{-1}(p)|\mapsto\sum_{q\in\tau(i)^{-1}(p)}j_q\Big)\\
&\xrightarrow{\mathrm{in}_{\tau'}}\ \circledast_{\mathsf{G}}\,(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)}(k;k^i\mapsto j)\Big\rangle.
\end{aligned}
\tag{2.14}
$$

In order to describe the second product $\circledast_{\mathsf{M}}$, for an arbitrary tree $t\in\mathrm{tr}(n)$ define another tree $t^*\in\mathrm{tr}(n)$, which is t with a unary vertex added in the middle of each internal edge of t. Choose a function $\mathcal{A}:E(t)\to\mathrm{Ob}\,\mathrm{Op}$, $e\mapsto\mathcal{A}_e$. Consider a family of operad polymodules $\mathcal{P}_v\in\mathcal{A}_{\mathrm{in}(v)}$-mod-$\mathcal{A}_{\mathrm{ou}(v)}$, $v\in\mathrm{v}(t)$. Extend family $\mathcal{P}$ to a family $\mathcal{P}^*$ indexed by $v\in\mathrm{v}(t^*)$ assigning operads $\mathcal{A}_e$ to new unary vertices $e\in\mathrm{e}(t)=\mathrm{v}(t^*)-\mathrm{v}(t)$.

There are two natural ways to get t^* from a 2-cluster tree based on t. The first list of subtrees of t^* consists of trees

$t'_v=$  ,

which is the corolla $\tau[|v|]\subset t$ with unary vertices added on internal edges from $\mathrm{in}(v)\cap\mathrm{e}(t)$. The second list of subtrees of t^* consists of trees $t''_v=t(|v|)$ (see (2.1)) if $v\in\mathrm{v}(t)-\{\mathrm{rv}\}$ and of $t''_{\mathrm{rv}}=\tau[|\,\mathrm{rv}\,|]$. Clearly, $t^*=I_t(t'_v\mid v\in\mathrm{v}(t))=I_t(t''_v\mid v\in\mathrm{v}(t))$. This gives two morphisms in $\mathcal{A}_{\mathrm{Inp}(t)}$-mod-$\mathcal{A}_{\text{root edge}(t)}$ along pairs of edges of the square

$$\begin{tikzcd} \circledast_{\mathsf{V}}(t^*)(\mathcal{P}^*_v)_{v\in\mathrm{v}(t^*)} \arrow[r, "\alpha", "\cong"'] \arrow[d, "\alpha"', "\cong"] & \circledast_{\mathsf{V}}(t)(\circledast_{\mathsf{V}}(t''_v)(\mathcal{P}^*_u)_{u\in\mathrm{v}(t''_v)})_{v\in\mathrm{v}(t)} \arrow[d, "\circledast_{\mathsf{V}}(t)(\rho_v)"] \\ \circledast_{\mathsf{V}}(t)(\circledast_{\mathsf{V}}(t'_v)(\mathcal{P}^*_u)_{u\in\mathrm{v}(t'_v)})_{v\in\mathrm{v}(t)} \arrow[r, "\circledast_{\mathsf{V}}(t)(\lambda_v)"] & \circledast_{\mathsf{V}}(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)} \end{tikzcd} \tag{2.15}$$

where the morphism of $\mathcal{A}_{\mathrm{in}(v)}$-mod-$\mathcal{A}_{\mathrm{ou}(v)}$

$$\lambda_v = \left\langle \circledast_{\mathsf{V}}(t'_v)((\mathcal{A}_e)_{e\in\mathrm{in}(v)\cap\mathrm{e}(t)};\mathcal{P}_v) \to \circledast_{\mathsf{V}}(t(^{|v|}1))((\mathcal{A}_e)_{e\in\mathrm{in}(v)};\mathcal{P}_v) \xrightarrow{\lambda} \mathcal{P}_v \right\rangle$$

is the left action including insertion of operad units on places indexed by $e \in \mathrm{in}(v)\backslash\mathrm{e}(t)$ (see (2.1) and (2.10)), $\rho_v : \circledast_{\mathsf{V}}(t''_v)(\mathcal{P}_v;\mathcal{A}_{\mathrm{ou}(v)}) \to \mathcal{P}_v \in \mathcal{A}_{\mathrm{in}(v)}$-mod-$\mathcal{A}_{\mathrm{ou}(v)}$ is right action (2.10) if v is not a root vertex, and $\rho_{\mathrm{rv}} = \mathrm{id}_{\mathcal{P}_{\mathrm{rv}}}$. By definition, for $t \neq |$ the tensor product $\circledast_{\mathsf{M}}(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)}$ is the coequaliser of pair of morphisms (2.15) in $\mathcal{A}_{\mathrm{Inp}(t)}$-mod-$\mathcal{A}_{\text{root edge}(t)}$. In the case of $t = |$ we define $\circledast_{\mathsf{M}}(|)()(z) = \mathcal{A}_{\text{root edge}(|)}(z)$, $z \in \mathbb{N}$, the regular $\mathcal{A}_{\text{root edge}(|)}$-bimodule.

When t is a corolla, then $t^* = t$ and all maps in (2.15) are identity maps. Thus, there is an isomorphism $\iota = \left\langle \mathcal{P} \xrightarrow{\cong} \circledast_{\mathsf{V}}(\tau[n])(\mathcal{P}) \xrightarrow{\cong} \circledast_{\mathsf{M}}(\tau[n])(\mathcal{P})\right\rangle$. Isomorphism α between iterated tensor product and a single product follows from properties of colimits. For the same reasons isomorphisms α and ι satisfy necessary equations, turning M into a $\mathcal{C}at$-multicategory.

Starting with an arbitrary symmetric $\mathcal{V}$-multicategory C we have a lax $\mathcal{C}at$-multifunctor $hom : \mathsf{B} \to \mathsf{V}$, see Section 2.21. There is also a lax $\mathcal{C}at$-multifunctor $\mathsf{M} \to \mathsf{V}$, $((\mathcal{A}_i)_{i\in I};\mathcal{P};\mathcal{B}) \mapsto \mathcal{P}$. The component $\pi : \circledast_{\mathsf{V}}(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)} \to \circledast_{\mathsf{M}}(t)(\mathcal{P}_v)_{v\in\mathrm{v}(t)}$ is the canonical morphism. We lift hom to a lax $\mathcal{C}at$-multifunctor $\mathcal{H}om : \mathsf{B} \to \mathsf{M}$ so that $\left(\mathsf{B} \xrightarrow{\mathcal{H}om} \mathsf{M} \longrightarrow \mathsf{V}\right) = hom$. Namely, $\mathcal{H}om : \mathsf{B} \to \mathsf{M}$, $B \mapsto \mathcal{E}nd\, B$, and

$$((A_i)_{i\in I};B) \mapsto ((\mathcal{E}nd\, A_i)_{i\in I}; hom((A_i)_{i\in I};B);\mathcal{E}nd\, B) = \mathcal{H}om((A_i)_{i\in I};B).$$

3. Morphisms with several entries

Here we give support to the observation that morphisms with n entries of algebras over operads form an $n\wedge 1$-operad module. In particular, we find this module for A_∞-algebras. From now on we assume tacitly that $\mathcal{V} = \mathbf{dg}$. When the differential is not concerned we may use $\mathcal{V} = \mathbf{gr}$.

3.1. Main source of $n \wedge 1$-operad modules. Starting with an arbitrary symmetric $\mathbf{dg}$-multicategory C we get a $\mathbf{dg}$-operad $\mathcal{E}(X) = \mathcal{E}nd\, X$ for any object X and an $n\wedge 1$-module $\mathcal{H}om = (\mathcal{E}nd\, A_1, \dots, \mathcal{E}nd\, A_n; \mathcal{H}; \mathcal{E}nd\, B)$ for any family $A_1, \dots, A_n, B$ in $\mathrm{Ob}\,\mathsf{C}$

$$(\mathcal{E}nd\, X)(v) = \mathsf{C}(^vX;X),$$
$$\mathcal{H}(j^1,\dots,j^n) = hom(A_1,\dots,A_n;B)(j^1,\dots,j^n) = \mathsf{C}\big((^{j^i}A_i)_{i=1}^n;B\big).$$

The right action

$$\rho_{(j^i_p)} : \Big[\bigotimes_{p\in\mathbf{k}} \mathcal{H}\big((j^i_p)_{i\in\mathbf{n}}\big)\Big] \otimes (\mathcal{E}nd\, B)(k) = \Big[\bigotimes_{p\in\mathbf{k}} \mathsf{C}\big(({}^{j^i_p}A_i)^n_{i=1}; B\big)\Big] \otimes \mathsf{C}({}^kB; B)$$
$$\to \mathsf{C}\big(({}^{\ell^i}A_i)^n_{i=1}; B\big) = \mathcal{H}\big((\ell^i)^n_{i=1}\big),$$

where $\ell^i = \sum_{p=1}^k j^i_p$, equals to the multicategory composition μ_ϕ, which corresponds to the map $\phi : \mathbf{l}^1 \sqcup \dots \sqcup \mathbf{l}^n \to \mathbf{k}$, whose restriction to $\mathbf{l}^i$ is isotonic and sends exactly j^i_p elements to $p \in \mathbf{k}$.

The left action

$$\lambda_{(j^i_p)} : \Big[\bigotimes_{i\in\mathbf{n}} \bigotimes_{p=1}^{k^i} (\mathcal{E}nd\, A_i)(j^i_p)\Big] \otimes \mathcal{H}\big((k^i)^n_{i=1}\big) \to \mathcal{H}\Big(\Big(\sum_{p=1}^{k^i} j^i_p\Big)^n_{i=1}\Big),$$

that is,

$$\lambda_{(j^i_p)} : \Big[\bigotimes_{i\in\mathbf{n}} \bigotimes_{p=1}^{k^i} \mathsf{C}({}^{j^i_p}A_i; A_i)\Big] \otimes \mathsf{C}\big(({}^{k^i}A_i)^n_{i=1}; B\big) \to \mathsf{C}\big(({}^{\ell^i}A_i)^n_{i=1}; B\big)$$

with $\ell^i = \sum_{p=1}^{k^i} j^i_p$, equals to the multicategory composition μ_ψ, corresponding to the isotonic map $\psi = \sqcup \sqcup \triangleright : \sqcup_{i=1}^n \sqcup_{p=1}^{k^i} \mathbf{j}^i_p \to \sqcup_{i=1}^n \sqcup_{p=1}^{k^i} \mathbf{1}$, which sends exactly j^i_p elements to the element of the target indexed by (i,p). Notice that $\rho_\varnothing : (\mathcal{E}nd\, Y)(0) = \mathsf{C}(;Y) = \mathcal{H}(0,\dots,0)$ is the identity map.

3.2. Example. In particular, reasoning of Section 3.1 applies to the symmetric **dg**-multicategory $\mathsf{C} = \underline{\mathsf{C}}_\Bbbk$ and for any $(n+1)$-tuple $(X_1,\dots,X_n;Y)$ of complexes gives an $n \wedge 1$-operad module

$$(\mathcal{E}nd\, X_1, \dots, \mathcal{E}nd\, X_n; hom(X_1,\dots,X_n;Y); \mathcal{E}nd\, Y),$$
$$\mathcal{H}(j^1,\dots,j^n) = hom(X_1,\dots,X_n;Y)(j^1,\dots,j^n) = \underline{\mathsf{C}}_\Bbbk\big(({}^{j^i}X_i)^n_{i=1}; Y\big).$$

The case of $n = 0$ gives $\mathcal{H} = Y$. The left action

$$\lambda_{(j^i_p)} : \Big[\bigotimes_{i\in\mathbf{n}} \bigotimes_{p=1}^{k^i} \underline{\mathsf{C}}_\Bbbk({}^{j^i_p}A_i; A_i)\Big] \otimes \underline{\mathsf{C}}_\Bbbk\big(({}^{k^i}A_i)^n_{i=1}; B\big) \to \underline{\mathsf{C}}_\Bbbk\big(({}^{\ell^i}A_i)^n_{i=1}; B\big), \quad (\otimes_i \otimes_p g^i_p) \otimes f \mapsto h$$

with $\ell^i = \sum_{p=1}^{k^i} j^i_p$ is found as

$$h = \big[\otimes^{i\in\mathbf{n}} T^{\ell^i}\mathcal{A}_i \xrightarrow{\otimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \otimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}^i} T^{j^i_p}\mathcal{A}_i \xrightarrow{\otimes^{i\in\mathbf{n}}\otimes^{p\in\mathbf{k}^i} g^i_p} \otimes^{i\in\mathbf{n}} T^{k^i}\mathcal{A}_i \xrightarrow{f} \mathcal{B}\big]$$

The right action

$$\rho_{(j^i_p)} : \Big[\bigotimes_{p\in\mathbf{k}} \underline{\mathsf{C}}_\Bbbk\big(({}^{j^i_p}A_i)^n_{i=1}; B\big)\Big] \otimes \underline{\mathsf{C}}_\Bbbk({}^kB; B) \to \underline{\mathsf{C}}_\Bbbk\big(({}^{\ell^i}A_i)^n_{i=1}; B\big), \quad (\otimes_p f^p) \otimes g \mapsto h,$$

where $\ell^i = \sum_{p=1}^k j_p^i$, is found as (see [BLM08, Eq. (6.1.1)] for isomorphism $\overline{\varkappa}$)

$$h = \bigl[\otimes^{i\in\mathbf{n}} T^{\ell^i} A_i \xrightarrow{\otimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \otimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}} T^{j_p^i} A_i \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{k}} \otimes^{i\in\mathbf{n}} T^{j_p^i} A_i \xrightarrow{\otimes^{p\in\mathbf{k}} f^p} \otimes^{p\in\mathbf{k}} \mathcal{B} \xrightarrow{g} \mathcal{B}\bigr].$$

Given an operad $\mathcal{O}$ and an $n \wedge 1$-operad $\mathcal{O}$-module $\mathcal{F}_n$ for each $n \geqslant 0$ we define a morphism of $\mathcal{O}$-algebras with n arguments $X_1, \dots, X_n \to Y$ as a morphism of ${}_n\mathrm{Op}_1$

$$(\mathcal{O}, \dots, \mathcal{O}; \mathcal{F}_n; \mathcal{O}) \to (\mathcal{E}nd\, X_1, \dots, \mathcal{E}nd\, X_n; \mathit{hom}(X_1, \dots, X_n; Y); \mathcal{E}nd\, Y).$$

3.3. A_∞-MORPHISMS WITH SEVERAL ENTRIES.

3.4. PROPOSITION. *There is the $n \wedge 1$-operad A_∞-module $F_n = \odot_{\geqslant 0}({}^n A_\infty; \Bbbk\{f_j \mid j \in \mathbb{N}^n - 0\}; A_\infty)$ freely generated as a graded module by elements $f_{j^1,\dots,j^n} \in F_n(j^1, \dots, j^n)$, $(j^1, \dots, j^n) \in \mathbb{N}^n - 0$, of degree 0. The differential for it is given by*

$$f_\ell \partial = \sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>1} \lambda^q_{({}^r 1, x, {}^t 1)}({}^r 1, b_x, {}^t 1; f_{\ell-(x-1)e_q}) - \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>1} \rho_{(j_p^i)}((f_{j_p})_{p=1}^k; b_k). \qquad (3.1)$$

The first arguments of λ are all $1 \in A_\infty(1)$ except b_x on the only place $p = r+1$. F_n-maps are A_∞-algebra morphisms $A_1, \dots, A_n \to B$ (for algebras written with operations b_n).

PROOF. Notice that $F_0 = A_\infty(0) = 0$ by Lemma A.9.

The following lemma is verified straightforwardly.

3.5. LEMMA. *For* **dg***-operads $\mathcal{A}_1, \dots, \mathcal{A}_n$ there is a* **dg***-category $\mathcal{A}_1$-$\cdots$-$\mathcal{A}_n$-*mod*, whose objects are left n-operad $\mathcal{A}_1$-$\cdots$-$\mathcal{A}_n$-modules and degree t morphisms $f : \mathcal{P} \to \mathcal{Q}$ are collections of $\Bbbk$-linear maps $f(k^1, \dots, k^n) : \mathcal{P}(k^1, \dots, k^n) \to \mathcal{Q}(k^1, \dots, k^n)$ of degree t such that*

$$\begin{array}{ccc}
\Bigl[\bigotimes_{i\in\mathbf{n}} \bigotimes_{p=1}^{k^i} \mathcal{A}_i(j_p^i)\Bigr] \otimes \mathcal{P}\bigl((k^i)_{i=1}^n\bigr) & \xrightarrow{\lambda_{(j_p^i)}} & \mathcal{P}\Bigl(\Bigl(\sum_{p=1}^{k^i} j_p^i\Bigr)_{i=1}^n\Bigr) \\
{\scriptstyle [\otimes\otimes 1]\otimes f}\big\downarrow & = & \big\downarrow{\scriptstyle f} \\
\Bigl[\bigotimes_{i\in\mathbf{n}} \bigotimes_{p=1}^{k^i} \mathcal{A}_i(j_p^i)\Bigr] \otimes \mathcal{Q}\bigl((k^i)_{i=1}^n\bigr) & \xrightarrow{\lambda_{(j_p^i)}} & \mathcal{Q}\Bigl(\Bigl(\sum_{p=1}^{k^i} j_p^i\Bigr)_{i=1}^n\Bigr)
\end{array}$$

The differential is $f \mapsto [f, \partial] = f\partial - (-1)^f \partial f$.

A *connection* on a graded $n \wedge 1$-operad module $\mathcal{P}$ over **dg**-operads $\mathcal{A}_1, \dots, \mathcal{A}_n$, $\mathcal{B}$ is a collection of $\Bbbk$-linear maps $\partial : \mathcal{P}(j) \to \mathcal{P}(j)$ of degree 1, $j \in \mathbb{N}^n$, which can be viewed as functors $\mathbb{Z} \to \Bbbk$-mod, $p \mapsto \mathcal{P}(j)^p$, where the category $\mathbb{Z}$ comes from the ordered set $\mathbb{Z}$. All action maps λ^i, ρ from (2.10) are required to be natural (chain) transformations with respect to the sum of maps $1^{\otimes a} \otimes \partial \otimes 1^{\otimes b}$ in the source, where ∂ denotes the connection on the module or the differential in an operad. Equivalently, action maps λ, ρ are chain transformations, or, equivalently, action maps α from (2.6) are chain transformations. A connection on a freely generated module $\mathcal{P}$ is unambiguously fixed by its value on generators.

The square ∂^2 of a connection ∂ is also a connection (of degree 2). It makes all actions into chain transformations with respect to the sum of maps $1^{\otimes a} \otimes \partial^2 \otimes 1^{\otimes b}$ in the source (where ∂^2 vanishes if applied to an operad). In particular, $\partial^2 : \mathcal{P} \to \mathcal{P}$ is a morphism of graded left n-operad $\mathcal{A}_1$-$\cdots$-$\mathcal{A}_n$-modules of degree 2 as defined in Lemma 3.5. If ∂^2 vanishes, $(\mathcal{P}, \partial)$ becomes an $n \wedge 1$-operad **dg**-module.

3.6. Lemma. *F_n is an $n \wedge 1$-operad* **dg**-*module.*

Proof. Recall that the differential in the operad A_∞ is given by

$$b_n.\partial = - \sum_{a+p+c=n}^{p>1,\, a+c>0} \mu({}^a1, b_p, {}^c1; b_{a+1+c}).$$

Let us prove that $\partial^2 = 0$ for connection ∂ given by (3.1). Let us verify this on generators:

$$\begin{aligned}
f_\ell \partial^2 &= \sum_{q=1}^{n} \sum_{k+y+m=\ell^q}^{y>1} \lambda^q({}^k1, b_y, {}^m1; f_{\ell-(y-1)e_q}.\partial) + \sum_{q=1}^{n} \sum_{u+c+v=\ell^q}^{c>1} \lambda^q({}^u1, b_c.\partial, {}^v1; f_{\ell-(c-1)e_q}) \\
&- \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>1} \rho((f_{j_p})_{p=1}^k; b_k.\partial) + \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>1} \sum_{p=1}^{k} \rho(f_{j_1}, \dots, f_{j_{p-1}}, f_{j_p}.\partial, f_{j_{p+1}}, \dots, f_{j_k}; b_k) \\
&= \sum_{q=1}^{n} \sum_{k+y+m=\ell^q}^{y>1} \sum_{p=1}^{n} \sum_{u+h+v=\ell^q-y+1}^{y>1} \lambda^q({}^k1, b_y, {}^m1; \lambda^p({}^u1, b_h, {}^v1; f_{\ell-(y-1)e_q-(h-1)e_p})) \qquad (3.2) \\
&- \sum_{q=1}^{n} \sum_{c>1} \sum_{r+c+t=\ell^q} \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell-(c-1)e_q}}^{k>1} \lambda^q({}^r1, b_c, {}^t1; \rho((f_{j_p})_p; b_k)) \qquad (3.3) \\
&- \sum_{q=1}^{n} \sum_{u+c+v=\ell^q}^{c>1} \sum_{x+y+z=c}^{y>1} \lambda^q({}^{u+x}1, b_y, {}^{z+v}1; \lambda^q({}^u1, b_{x+1+z}, {}^v1; f_{\ell-(c-1)e_q})) \qquad (3.4) \\
&- \sum_{\substack{j_1,\dots,j_s \in \mathbb{N}^n - 0 \\ j_1+\dots+j_s=\ell}}^{s>1} \sum_{x+m+z=s}^{m>1} \rho((f_{j_p})_{p=1}^s; \mu({}^x1, b_m, {}^z1; b_{x+1+z})) \qquad (3.5) \\
&+ \sum_{q=1}^{n} \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>1} \sum_{p=1}^{k} \sum_{x+c+z=j_p^q}^{c>1} \rho(f_{j_1}, \dots, f_{j_{p-1}}, \lambda^q({}^x1, b_c, {}^z1; f_{j_p-(c-1)e_q}), f_{j_{p+1}}, \dots, f_{j_k}; b_k) \\
&\qquad (3.6)
\end{aligned}$$

$$- \sum_{\substack{y_1,\dots,y_k\in\mathbb{N}^n-0\\ y_1+\dots+y_k=\ell}}^{k>1} \sum_{p=1}^{k} \sum_{\substack{t_1,\dots,t_m\in\mathbb{N}^n-0\\ t_1+\dots+t_m=y_p}}^{m>1} \rho(f_{y_1},\dots,f_{y_{p-1}},\rho(f_{t_1},\dots,f_{t_m};b_m),f_{y_{p+1}},\dots,f_{y_k};b_k). \quad (3.7)$$

Summands of (3.2) pairwise cancel each other if $p \neq q$. Also summands of (3.2) with $p = q$ pairwise cancel each other if output of b_y does not become an input of b_h. The remainder of (3.2) cancels with sum (3.4). Sums (3.3) and (3.6) cancel each other. Identifying in sums (3.5) and (3.7) the index s with $k+m-1$ and the sequence $(j_1,\dots,j_s)$ with the sequence $(y_1,\dots,y_{p-1},t_1,\dots,t_m,y_{p+1},\dots,y_k)$ we see that they cancel. Therefore, ∂^2 vanishes. ∎

The image of $f_\ell\partial$ in $hom((sA_i)_i;sB)$ is

$$\sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>1} \big[\otimes^{i\in\mathbf{n}} T^{\ell^i} sA_i \xrightarrow{1^{\otimes(q-1)}\otimes(1^{\otimes r}\otimes b_x\otimes 1^{\otimes t})\otimes 1^{\otimes(n-q)}}$$

$$T^{\ell^1}sA_1\otimes\dots\otimes T^{\ell^{q-1}}sA_{q-1}\otimes T^{r+1+t}sA_q\otimes T^{\ell^{q+1}}sA_{q+1}\otimes\dots\otimes T^{\ell^n}sA_n \xrightarrow{f_{\ell-(x-1)e_q}} sB\big]$$

$$- \sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0\\ j_1+\dots+j_k=\ell}}^{k>1} \big[\otimes^{i\in\mathbf{n}} T^{\ell^i} sA_i \xrightarrow{\otimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \otimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}} T^{j^i_p} sA_i \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{k}} \otimes^{i\in\mathbf{n}} T^{j^i_p} sA_i$$

$$\xrightarrow{\otimes^{p\in\mathbf{k}} f_{j_p}} \otimes^{p\in\mathbf{k}} sB \xrightarrow{b_k} sB\big].$$

Isomorphisms λ^{γ_i} and $\overline{\varkappa}$ are the obvious ones, see [BLM08] for details.

An F_n-algebra map is specified by A_∞-algebras A_1, ..., A_n, B, and a collection of $\Bbbk$-linear degree 0 maps $f_j : \otimes^{i\in\mathbf{n}} T^{j^i} sA_i \to sB$ assigned to generators $(f_j)_{j\in\mathbb{N}^n-0}$. It suffices to satisfy on generators the only requirement that $F_n \to hom((A_i[1])_{i=1}^n; B[1])$ be a chain map. The latter means that the equation holds for all $\ell\in\mathbb{N}^n-0$:

$$f_\ell b_1 - \Big[\sum_{q=1}^{n}\sum_{r+1+t=\ell^q} 1^{\otimes(q-1)}\otimes(1^{\otimes r}\otimes b_1\otimes 1^{\otimes t})\otimes 1^{\otimes(n-q)}\Big] f_\ell = f_\ell\partial.$$

Explicitly this equation says

$$\sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>0} \big[\otimes^{i\in\mathbf{n}} T^{\ell^i} sA_i \xrightarrow{1^{\otimes(q-1)}\otimes(1^{\otimes r}\otimes b_x\otimes 1^{\otimes t})\otimes 1^{\otimes(n-q)}}$$

$$T^{\ell^1}sA_1\otimes\dots\otimes T^{\ell^{q-1}}sA_{q-1}\otimes T^{r+1+t}sA_q\otimes T^{\ell^{q+1}}sA_{q+1}\otimes\dots\otimes T^{\ell^n}sA_n \xrightarrow{f_{\ell-(x-1)e_q}} sB\big]$$

$$= \sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0\\ j_1+\dots+j_k=\ell}}^{k>0} \big[\otimes^{i\in\mathbf{n}} T^{\ell^i} sA_i \xrightarrow{\otimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \otimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}} T^{j^i_p} sA_i$$

$$\xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{k}} \otimes^{i\in\mathbf{n}} T^{j^i_p} sA_i \xrightarrow{\otimes^{p\in\mathbf{k}} f_{j_p}} \otimes^{p\in\mathbf{k}} sB \xrightarrow{b_k} sB\big]. \quad (3.8)$$

Collections $(f_j)_{j\in\mathbb{N}^n-0}$ are in bijection with augmented coalgebra morphisms $f:\otimes^{i\in\mathbf{n}}TsA_i\to TsB$. Coherence with augmentation means that

$$\bigl(\Bbbk = \otimes^{i\in\mathbf{n}}T^0sA_i \xrightarrow{f|} TsB\bigr) = \bigl(\Bbbk = T^0sB \hookrightarrow TsB\bigr).$$

Tensor quivers TsB of A_∞-algebras B are **dg**-coalgebras, whose differential $b:TsB\to TsB$ has the components $b_k:T^ksB\to sB$. Equation (3.8) can be rewritten as

$$\bigl(\otimes^{i\in\mathbf{n}}TsA_i \xrightarrow{f} TsB \xrightarrow{b} sB\bigr) = \bigl(\otimes^{i\in\mathbf{n}}TsA_i \xrightarrow{\sum_{i=1}^n 1^{\otimes(i-1)}\otimes b\otimes 1^{\otimes(n-i)}} \otimes^{i\in\mathbf{n}}TsA_i \xrightarrow{f} sB\bigr).$$

In other terms, f is an augmented **dg**-coalgebra morphism. These are A_∞-morphisms $A_1,\dots,A_n\to B$ by definition, see [BLM08]. ∎

4. Composition of morphisms with several arguments

The mechanism which provides an associative composition of morphisms with several arguments is that of convolution product in the module of linear maps from a coalgebra to an algebra. The part of a coalgebra is played by a colax $\mathcal{C}at$-multifunctor. A lax $\mathcal{C}at$-multifunctor $\mathcal{H}om$ comes in place of an algebra. The convolution product of these multifunctors gives a multicategory structure to the collection of A_∞-algebras and A_∞-morphisms with several arguments.

We shall show that A_∞-modules F_n form a *polymodule cooperad*, that is, a colax $\mathcal{C}at$-multifunctor $F:\mathsf{F}\to\mathsf{M}$, where the category M of operad polymodules is described in Definition 2.12. Here (strict) $\mathcal{C}at$-operad F has (1-element set of objects), $\mathsf{F}(I)=\mathit{1}$ is the terminal category for any $I\in\mathcal{O}_{\mathsf{sk}}$, 1-morphism $\circledast:\mathbb{T}_{\mathbf{1}}\mathsf{F}\to\mathsf{F}$ is the unique one.

A general polymodule cooperad amounts to the following data: an operad $\mathcal{A}=F(*)$, for each $I\in\mathcal{O}_{\mathsf{sk}}$ an $I\wedge 1$-$\mathcal{A}$-module F_I, for each tree t a morphism $\Delta(t):F_{\operatorname{Inp}t}\to\circledast_{\mathsf{M}}(t)(F_{|v|})_{v\in\mathrm{v}(t)}$ of $\operatorname{Inp}(t)\wedge 1$-$\mathcal{A}$-modules such that for any corolla $t=\tau[n]$ *normalization* holds:

$$\bigl(F_n \xrightarrow{\Delta(\tau[n])} \circledast_{\mathsf{M}}(\tau[n])(F_n) \xrightarrow{\cong} F_n\bigr) = 1, \tag{4.1}$$

for all 2-cluster trees $(t;(t_v)_{v\in\mathrm{v}(t)})$ assembled to $\theta=I_t(t_v\mid v\in\mathrm{v}(t))$ *multiplicativity* holds:

$$\begin{tikzcd} F_{\operatorname{Inp}t} \arrow[r, "\Delta(t)"] \arrow[d, "\Delta(\theta)"'] \arrow[dr, phantom, "="] & \circledast_{\mathsf{M}}(t)(F_{|v|})_{v\in\mathrm{v}(t)} \arrow[d, "\circledast_{\mathsf{M}}(t)(\Delta(t_v))"] \\ \circledast_{\mathsf{M}}(\theta)(F_{|w|})_{w\in\mathrm{v}(\theta)} \arrow[r, "\cong"] & \circledast_{\mathsf{M}}(t)\bigl(\circledast_{\mathsf{M}}(t_v)(F_{|u|})_{u\in\mathrm{v}(t_v)}\bigr)_{v\in\mathrm{v}(t)} \end{tikzcd} \tag{4.2}$$

We are interested in the cases of $\mathcal{A}=A_\infty$, A_∞, A_∞^{hu} and $\mathrm{A}_\infty^{\mathsf{hu}}$.

4.1. Exercise. Write down explicitly equation (4.2) in two cases:

(a) $t=t({}^n1)=(\mathbf{n}\xrightarrow{\mathrm{id}}\mathbf{n}\to\mathbf{1})$, $t_v=|$ for $v\in\mathbf{n}$, $t_{\mathrm{rv}(t)}=\tau[n]=(\mathbf{n}\to\mathbf{1})$;

(b) $t=t(n)=(\mathbf{n}\to\mathbf{1}\to\mathbf{1})$, $t_1=\tau[n]$, $t_{\mathrm{rv}(t)}=|$.

Conclude that for the tree $|$ the operad $\mathcal{A}$-bimodule map $\Delta(|):F_1\to\mathcal{A}$ plays the part of a counit for Δ.

4.2. Definition. *A system of polymodules and their maps* $(\mathcal{A}, F_\bullet, \Delta)$ *is called* coassociative *if for all 2-cluster trees* $(t; (t_v)_{v\in \mathrm{v}(t)})$ *such that all (but one) trees* t_v *are corollas* $\tau[|v|]$, $v \in \mathrm{v}(t) - \{x\}$, *equation* (4.2) *holds.*

4.3. Proposition. *Assume that coassociative* $(\mathcal{A}, F_\bullet, \Delta)$ *satisfies normalization* (4.1). *Then* $(\mathcal{A}, F_\bullet, \Delta)$ *satisfies multiplicativity* (4.2) *for all 2-cluster trees, thus, it is a polymodule cooperad.*

Proof. Let us enumerate internal vertices of t as $\{v_1, \dots, v_k\} = \mathrm{v}(t)$. Associate with a 2-cluster tree $(t; (t_v))$ a sequence of trees and 2-cluster trees: $t^0 = t$, $(t^0; (t^1_v)_{v\in\mathrm{v}(t^0)}) = (t; t_{v_1}, \text{corollas})$ (2-cluster tree in which t_{v_1} is associated to the vertex v_1 and corollas $\tau[|v|]$ are associated to other vertices v), $t^1 = I(t; t_{v_1}, \text{corollas})$ with $\{v_i \mid i > 1\} \hookrightarrow \mathrm{v}(t^1)$, $(t^1; (t^2_v)_{v\in\mathrm{v}(t^1)}) = (t^1; t_{v_2}, \text{corollas})$, $t^2 = I(t^1; t_{v_2}, \text{corollas})$ with $\{v_i \mid i > 2\} \hookrightarrow \mathrm{v}(t^2)$, and so on, $(t^{k-1}; (t^k_v)_{v\in\mathrm{v}(t^{k-1})}) = (t^{k-1}; t_{v_k}, \text{corollas})$, $t^k = I(t^{k-1}; t_{v_k}, \text{corollas}) = I_t(t_{v_i} \mid 1 \leqslant i \leqslant k)$. Let us write several coassociativity relations omitting the structure isomorphisms for the product $\circledast_{\mathsf{M}}$:

$$
\begin{tikzcd}
F_{\operatorname{Inp} t} \arrow[r, "\Delta(t)"] \arrow[d, equal] \arrow[dr, phantom, "="] & \circledast_{\mathsf{M}}(t)(F_{|v|})_{v\in\mathrm{v}(t)} \arrow[d, "\circledast_{\mathsf{M}}(t)(\Delta(t_{v_1}),1,\dots,1)"] \\
F_{\operatorname{Inp} t} \arrow[r, "\Delta(t^1)"] \arrow[d, equal] \arrow[dr, phantom, "="] & \circledast_{\mathsf{M}}(t)\big(\circledast_{\mathsf{M}}(t^1_v)(F_{|u|})_{u\in\mathrm{v}(t^1_v)}\big)_{v\in\mathrm{v}(t)} \arrow[d, "\circledast_{\mathsf{M}}(t^1)(1,\Delta(t_{v_2}),\dots,1)"] \\
F_{\operatorname{Inp} t} \arrow[r, "\Delta(t^2)"] \arrow[d, equal] \arrow[dr, phantom, "\dots"] & \circledast_{\mathsf{M}}(t^1)\big(\circledast_{\mathsf{M}}(t^2_v)(F_{|u|})_{u\in\mathrm{v}(t^2_v)}\big)_{v\in\mathrm{v}(t^1)} \arrow[d, "\dots"] \\
F_{\operatorname{Inp} t} \arrow[r, "\Delta(t^{k-1})"] \arrow[d, equal] \arrow[dr, phantom, "="] & \circledast_{\mathsf{M}}(t^{k-2})\big(\circledast_{\mathsf{M}}(t^{k-1}_v)(F_{|u|})_{u\in\mathrm{v}(t^{k-1}_v)}\big)_{v\in\mathrm{v}(t^{k-2})} \arrow[d, "\circledast_{\mathsf{M}}(t^{k-1})(1,\dots,1,\Delta(t_{v_k}))"] \\
F_{\operatorname{Inp} t} \arrow[r, "\Delta(t^k)"] & \circledast_{\mathsf{M}}(t^{k-1})\big(\circledast_{\mathsf{M}}(t^k_v)(F_{|u|})_{u\in\mathrm{v}(t^k_v)}\big)_{v\in\mathrm{v}(t^{k-1})}
\end{tikzcd}
$$

Composition in the right column equals $\circledast_{\mathsf{M}}(t)(\Delta(t_v))$, therefore, (4.2) is deduced. ∎

Viewing (system of $\mathcal{A}$-modules) $F : \mathsf{F} \to \mathsf{M}$ as a coalgebra and $\mathcal{H}om : \mathsf{B} \to \mathsf{M}$ (coming from a symmetric multicategory C) as an algebra we consider homomorphisms between them (in the sense of M) and they have to form an algebra as well. So we define a multiquiver H whose objects are $\mathcal{A}$-algebras $(B, \alpha_B : \mathcal{A} \to \mathcal{E}nd\, B)$ with

$$
\begin{aligned}
&\mathsf{H}((A_i, \alpha_{A_i})_{i\in I}; (B, \alpha_B)) \\
&\qquad = \{((\alpha_{A_i})_{i\in I}; \phi; \alpha_B) \in \mathsf{M}\big(({}^I\mathcal{A}; F_I; \mathcal{A}), ((\mathcal{E}nd\, A_i)_{i\in I}; hom((A_i)_{i\in I}; B); \mathcal{E}nd\, B)\big)\}.
\end{aligned}
$$

Let us define a multicategory composition for it. For any tree t and any collection of $\mathcal{A}$-algebras $\alpha_e : \mathcal{A} \to \mathcal{E}nd\, A_e$, $e \in E(t)$, assume given $|v| \wedge 1$-operad module morphisms for $v \in \mathrm{v}(t)$:

$$\big((\alpha_e)_{e\in\mathrm{in}(v)};g_v;\alpha_v\big) : \big({}^{\mathrm{in}(v)}\mathcal{A};F_{\mathrm{in}(v)};\mathcal{A}\big) \\ \to \big((\mathcal{E}nd\,A_e)_{e\in\mathrm{in}(v)};hom((A_e)_{e\in\mathrm{in}(v)};A_{\mathrm{ou}(v)});\mathcal{E}nd\,A_{\mathrm{ou}(v)}\big).$$

Then their composition is defined as $\big((\alpha_e)_{e\in\mathrm{Inp}\,t};\mathrm{comp}(t)(g_v);\alpha_{\text{root edge}(t)}\big)$, where

$$\mathrm{comp}(t)(g_v) = \big[F_{\mathrm{Inp}\,t} \xrightarrow{\Delta(t)} \circledast_{\mathsf{M}}(t)(F_{|v|})_{v\in\mathrm{v}(t)} \xrightarrow{\circledast_{\mathsf{M}}(t)(g_v)} \\ \circledast_{\mathsf{M}}(t)\big(\mathcal{H}om((A_e)_{e\in\mathrm{in}(v)};A_{\mathrm{ou}(v)})\big)_{v\in\mathrm{v}(t)} \xrightarrow{\mathrm{comp}(t)} \mathcal{H}om((A_a)_{a\in\mathrm{Inp}\,t};A_{\text{root edge}(t)})\big]. \quad (4.3)$$

4.4. Exercise. The composition in H is strictly associative.

4.5. Comultiplication under A_∞. Taking tensor coalgebra of a graded $\Bbbk$-module gives morphism of multiquivers to multicategory of differential graded augmented counital coassociative coalgebras $Ts : \mathsf{H} \hookrightarrow \mathbf{dgac}$. We wish to define a colax $\mathcal{C}at$-multifunctor $F : \mathsf{F} \to \mathsf{M}$ such that Ts becomes a multifunctor. The following statements follow from results of [BLM08].

4.6. Proposition. [See Proposition 6.8 of [BLM08]] *A $T^{\geqslant 1}$-coalgebra C in* $\mathbf{dg}$ *is a coassociative coalgebra* $(C,\overline{\Delta} : C \to C\otimes C)$ *in* $\mathbf{dg}$ *such that*

$$C = \operatorname*{colim}_{k\to\infty} \mathrm{Ker}\big(\overline{\Delta}^{(k)} : C \to C^{\otimes k}\big).$$

4.7. Corollary. [See Corollary 6.11 of [BLM08]] *Let C be a $T^{\geqslant 1}$-coalgebra in* $\mathbf{dg}$*, and let $B \in \mathrm{Ob}\,\mathbf{dg}$ be a complex. Then there is a natural bijection*

$$\mathbf{dg}_{T^{\geqslant 1}}(C,T^{\geqslant 1}B) \to \mathbf{dg}(C,B), \qquad (f : C \to T^{\geqslant 1}B) \mapsto \big(C \xrightarrow{f} T^{\geqslant 1}B \xrightarrow{\mathrm{pr}_1} B\big),$$

where $\mathbf{dg}_{T^{\geqslant 1}}$ is the category of $T^{\geqslant 1}$-coalgebras in $\mathbf{dg}$.

4.8. Proposition. [See Corollary 6.18 and Proposition 6.19 of [BLM08]] *The full and faithful functor*

$$T^{\leqslant 1} : \mathbf{dg}_{T^{\geqslant 1}} \to \mathbf{dgac}, \qquad (C,\overline{\Delta}) \mapsto \\ \big(\Bbbk\oplus C, \Delta_0 = \mathrm{pr}_1\cdot\overline{\Delta}\cdot(\mathrm{in}_1\otimes\mathrm{in}_1) + \mathrm{id}\otimes\mathrm{in}_0 + \mathrm{in}_0\otimes\mathrm{id} - \mathrm{pr}_0\cdot(\mathrm{in}_0\otimes\mathrm{in}_0), \varepsilon = \mathrm{pr}_0, \eta = \mathrm{in}_0\big)$$

makes $\mathbf{dg}_{T^{\geqslant 1}}$ into a symmetric Monoidal subcategory of $\mathbf{dgac}$.

4.9. Corollary. *An arbitrary augmented* $\mathbf{dg}$*-coalgebra $A = \otimes^{i\in I}TA_i$ comes from a $T^{\geqslant 1}$-coalgebra $A \ominus \Bbbk$ and there is a natural bijection*

$$\mathbf{dgac}(\otimes^{i\in I}TA_i, TB) \to \mathbf{dg}((\otimes^{i\in I}TA_i)\ominus\Bbbk, B), \\ (f : \otimes^{i\in I}TA_i \to TB) \mapsto \big((\otimes^{i\in I}TA_i)\ominus\Bbbk \xrightarrow{f|} T^{\geqslant 1}B \xrightarrow{\mathrm{pr}_1} B\big).$$

Recall full and faithful embedding $Ts : A_\infty \hookrightarrow \mathbf{dgac}$ of multicategory of A_∞-algebras into the multicategory of differential graded augmented counital coassociative coalgebras over $\Bbbk$. In this way F obtains a unique colax $\mathcal{C}at$-multifunctor structure $\Delta(t)$. Let us describe the details.

An A_∞-algebra B is taken by Ts to the tensor coalgebra $(TsB, \Delta_0, \varepsilon, \eta)$, where $TsB = \oplus_{n=0}^\infty T^n sB = \oplus_{n=0}^\infty (B[1])^{\otimes n}$, Δ_0 is the cut comultiplication

$$\Delta_0(x_1 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n),$$

$\varepsilon = \mathrm{pr}_0 : TsB \longrightarrow T^0 sB = \Bbbk$ is the counit and $\eta = \mathrm{in}_0 : \Bbbk = T^0 sB \hookrightarrow TsB$ is the augmentation.

The differential $b : TsB \to TsB$, $\deg b = 1$, has matrix entries $b^{n,k} : T^n sB \to T^k sB$,

$$b^{n,k} = \sum_{a+p+c=n}^{a+1+c=k} 1^{\otimes a} \otimes b_p \otimes 1^{\otimes c}, \quad \text{where} \quad b_p = (-1)^n (\sigma^{\otimes p})^{-1} \cdot m_p \cdot \sigma : T^p sB \to sB$$

for $p \geqslant 1$, $b_0 = 0$, and $\sigma : B \to sB = B[1]$, $x \mapsto x$, is the shift map (the suspension), $\deg \sigma = -1$, $\deg b_p = 1$. Here $m_p : T^p B \to B$, $\deg m_p = 2-p$, are linear maps representing generators $m_p \in \mathrm{A}_\infty(p)$ for $p \geqslant 2$ and $m_1 : B \to B$ is the differential in the complex B.

A_∞-algebra morphisms $f : (A_i)_{i \in \mathbf{n}} \to B$, are taken by Ts to the augmented coalgebra chain homomorphisms $f : \otimes^{i \in \mathbf{n}} TsA_i \to TsB$, whose compositions with the projections $\mathrm{pr}_l : TsB \to T^l sB$ are given by

$$\begin{aligned} f \cdot \mathrm{pr}_l = \big[TsA_1 \otimes \cdots \otimes TsA_n \xrightarrow{\Delta_0^{(l)} \otimes \cdots \otimes \Delta_0^{(l)}} (TsA_1)^{\otimes l} \otimes \cdots \otimes (TsA_n)^{\otimes l} \\ \xrightarrow{\overline{\varkappa}_{n,l}} (TsA_1 \otimes \cdots \otimes TsA_n)^{\otimes l} \xrightarrow{\check{f}^{\otimes l}} (sB)^{\otimes l} \big], \end{aligned} \tag{4.4}$$

where the restriction of $\check{f}$ to $T^{j^1} sA_1 \otimes \cdots \otimes T^{j^n} sA_n$ is given by the component

$$f_j = f \cdot \mathrm{pr}_1 : T^{j^1} sA_1 \otimes \cdots \otimes T^{j^n} sA_n \to sB.$$

It is the linear map that represents the generator $f_j \in F_n(j^1, \dots, j^n)$. The symmetry $\overline{\varkappa}_{n,l} = c_{s_{n,l}}$ corresponds to the permutation $s_{n,l}$ of the set $\{1, 2, \dots, nl\}$,

$$s_{n,l}(1 + t + kl) = 1 + k + tn \quad \text{for} \quad 0 \leqslant t < l, \; 0 \leqslant k < n.$$

Detailed description of map (4.4) on direct summands is

$$\begin{aligned} \Big[\bigotimes_{a \in \mathbf{n}} T^{j^a} sA_a \xrightarrow{\otimes^{\mathbf{n}} \Delta_0^{(l)}} \bigotimes_{a \in \mathbf{n}} \bigoplus_{\sum_{q=1}^l r_q^a = j^a} \bigotimes_{p \in \mathbf{l}} T^{r_p^a} sA_a \xrightarrow{\sim} \bigoplus_{\sum_{q=1}^l r_q^a = j^a} \bigotimes_{a \in \mathbf{n}} \bigotimes_{p \in \mathbf{l}} T^{r_p^a} sA_a \\ \xrightarrow{\oplus \overline{\varkappa}_{n,l}} \bigoplus_{\sum_{q=1}^l r_q^a = j^a} \bigotimes_{p \in \mathbf{l}} \bigotimes_{a \in \mathbf{n}} T^{r_p^a} sA_a \xrightarrow{\sum \otimes^{p \in \mathbf{l}} f_{(r_p^a)_{a \in \mathbf{n}}}} \bigotimes_{p \in \mathbf{l}} sB = T^l sB \Big]. \end{aligned} \tag{4.5}$$

In order to describe the composition of A_∞-morphisms with several entries let us forget about differentials for a moment. Thus we consider graded augmented coalgebras TsA, $A \in \Bbbk$-mod, equipped with the cut comultiplication. The multicategory gaC of graded augmented coalgebras comes from the symmetric monoidal category of those, $\mathrm{gaC}((C_i)_{i=1}^n; D) = \mathrm{gaC}(\otimes_{i=1}^n C_i; D)$. Multicategory composition along a tree t

$$\mu^{\mathrm{gaC}}(t) : \prod_{v\in \mathrm{v}(t)} \mathrm{gaC}((C_e)_{e\in \mathrm{in}(v)}; C_{\mathrm{ou}(v)}) \to \mathrm{gaC}((C_a)_{a\in \mathrm{Inp}\, t}; C_{\text{root edge}})$$

is the composition of graded augmented coalgebra homomorphisms, tensored with identity morphisms,

$$\mu^{\mathrm{gaC}}(t) : \prod_{v\in \mathrm{v}(t)} \mathrm{gaC}\big(\bigotimes_{e\in \mathrm{in}(v)} C_e, C_{\mathrm{ou}(v)}\big) \to \mathrm{gaC}\big(\bigotimes_{a\in \mathrm{Inp}\, t} C_a, C_{\text{root edge}}\big),\ (g^v) \mapsto \mu^{\mathrm{gaC}}(t)(g^v)_{v\in \mathrm{v}(t)}. \tag{4.6}$$

In particular, this holds for $C_e = TsA_e$. Then (4.4)–(4.5) describe a bijection

$$\begin{aligned}\mathrm{gaC}\big(\bigotimes_{e\in \mathrm{in}(v)} TsA_e, TsA_{\mathrm{ou}(v)}\big) &\cong \prod_{k\in \mathbb{N}^{\mathrm{in}(v)}-0} \mathbf{gr}\big(\bigotimes_{e\in \mathrm{in}(v)} T^{k^e} sA_e, sA_{\mathrm{ou}(v)}\big) \\ &= \prod_{k\in \mathbb{N}^{\mathrm{in}(v)}-0} \mathit{hom}((sA_e)_{e\in \mathrm{in}(v)}; sA_{\mathrm{ou}(v)})(k)^0,\end{aligned}$$

with the help of the $\mathbf{gr}$-multicategory $\mathsf{C} = \underline{\mathbf{gr}}$.

We are going to provide comultiplication Δ^{G} in F such that $\mu^{\mathrm{gaC}}(t)(g^v)_{v\in \mathrm{v}(t)} \cdot \mathrm{pr}_j$, $j \in \mathbb{N}^{\mathrm{Inp}\, t}$, is the image of f_j under the map

$$\begin{aligned}F_{\mathrm{Inp}\, t}(j) \xrightarrow{\Delta^{\mathsf{G}}(t)} \circledast_{\mathsf{G}}(t)(F_{|v|})_{v\in \mathrm{v}(t)}(j) &\xrightarrow{\circledast_{\mathsf{G}}(t)(g_v)} \circledast_{\mathsf{G}}(t)(\mathit{hom}((sA_e)_{e\in \mathrm{in}(v)}; sA_{\mathrm{ou}(v)}))_{v\in \mathrm{v}(t)}(j) \\ &\xrightarrow{\mathrm{comp}(t)} \mathit{hom}((sA_a)_{a\in \mathrm{Inp}\, t}; sA_{\text{root edge}(t)})(j).\end{aligned} \tag{4.7}$$

Let $j \in \mathbb{N}^{\mathrm{Inp}\, t}$ and let τ denote a t-tree $t \to \mathcal{O}_{\mathsf{sk}}$ such that $|\tau(a)| = j^a$ for all $a \in \mathrm{Inpv}\, t$. By definition

$$\circledast_{\mathsf{G}}(t)(F_{|v|})_{v\in \mathrm{v}(t)}(j) = \coprod_{\forall a\in \mathrm{Inpv}\, t\, |\tau(a)|=j^a}^{t\text{-tree } \tau} \bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} F_{|v|}\Big(\big(|\tau(e)^{-1}(p)|\big)_{e\in \mathrm{in}(v)}\Big), \tag{4.8}$$

$$\begin{aligned}&\circledast_{\mathsf{G}}(t)(\mathit{hom}((A_e)_{e\in \mathrm{in}(v)};A_{\mathrm{ou}(v)}))_{v\in \mathrm{v}(t)}(j) \\ &\qquad = \coprod_{\forall a\in \mathrm{Inpv}\, t\, |\tau(a)|=j^a}^{t\text{-tree } \tau} \bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} \underline{\mathbf{dg}}(\otimes^{e\in \mathrm{in}(v)} T^{|\tau(e)^{-1}(p)|} A_e, A_{\mathrm{ou}(v)}).\end{aligned}$$

A tree r is *surjective* if $|v| > 0$ for all $v \in \mathrm{v}(r)$. Since $F_0 = 0$ and $F_n(0) = 0$, $n > 0$, the summation in expression (4.8) extends precisely over t-trees τ such that

— $\forall\, v \in \mathrm{v}(t)\ |v| = 0 \implies \tau(v) = \varnothing$;

— $\forall\, v \in \mathrm{v}(t)\ |v| > 0 \implies \bigl(\forall\, p \in \tau(v)\ \exists\, e \in \mathrm{in}(v) \quad \tau(e)^{-1}(p) \neq \varnothing\bigr)$.

The two conditions can be combined into a single one:
— the obvious map $\sqcup_{u\in\mathrm{inV}(v)}\tau(u) \to \tau(v)$ is surjective for all internal vertices v of t. Equivalently, the tree $\tilde{\tau}$ is surjective. In this case we say that τ is *surjective*.

The number of surjective trees $\tilde{\tau}$ is finite. Hence, the number of tree mappings $\tilde{\tau} \to t$ is finite, and the number of surjective t-trees τ is finite as well. Thus direct sum (4.8) is finite.

4.10. Proposition. *Define for a tree t the degree 0 graded $\mathrm{Inp}(t) \wedge 1$-A_∞-module homomorphism $\Delta^{\mathsf{G}}(t)(j) : F_{\mathrm{Inp}\,t}(j) \to \circledast_{\mathsf{G}}(t)(F_{|v|})_{v\in\mathrm{v}(t)}(j)$ on generators as*

$$\Delta^{\mathsf{G}}(t)(f_j) = \sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=j^a}^{\text{surjective } t\text{-tree } \tau} \otimes^{v\in\mathrm{v}(t)} \otimes^{p\in\tau(v)} f_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}}. \tag{4.9}$$

In particular, for the tree $t = |$ the A_∞-bimodule homomorphism $\Delta^{\mathsf{G}}(|)(j) : F_1(j) \to \Bbbk(j) = \delta_{j1}\Bbbk$ is determined by $\Delta^{\mathsf{G}}(|)(f_j) = \delta_{j1}$, $j \geqslant 1$. Then this comultiplication is normalized and multiplicative in the sense of (4.1), (4.2) with $\circledast_{\mathsf{M}}$ replaced with $\circledast_{\mathsf{G}}$. Thus, $(A_\infty, F_\bullet, \Delta^{\mathsf{G}})$ is a graded polymodule cooperad.

Proof. First we show that (4.7) takes f_j to $\mu^{\mathrm{gaC}}(t)(g^v)_{v\in\mathrm{v}(t)} \cdot \mathrm{pr}_j$. That is, we get it applying $\mathrm{comp}(t)$ to

$$\sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=j^a}^{\text{surjective } t\text{-tree } \tau} \otimes^{v\in\mathrm{v}(t)} \otimes^{p\in\tau(v)} \bigl(g^v_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}} : \otimes^{e\in\mathrm{in}(v)} T^{|\tau(e)^{-1}(p)|} sA_e \to sA_{\mathrm{ou}(v)}\bigr), \tag{4.10}$$

where $\mathrm{comp}(t)$ restricted to the summand indexed by τ is given by (2.11). Here $\deg g^v_* = 0$. This is a form of a recipe how to compose graded augmented coalgebra morphisms of the type $\widehat{g} : \otimes_{e=1}^k TA_e \to TB$ of tensor coalgebras with cut comultiplication Δ_0. In order to elucidate this recipe, let us represent $\widehat{g^v}$ via its components g^v in two ways. The first formula is a direct consequence of the augmented coalgebra morphism properties, see [BLM08, diagram (8.25.1)]. Thus, the morphism $\widehat{g}$ of augmented coalgebras is recovered from its components $g_{(n^e)_e} : \otimes_{e=1}^k T^{n^e} A_e \to B$, $(n^e)_e \in \mathbb{N}^k$, $g_0 = 0$, as

$$\widehat{g} \cdot \mathrm{pr}_m = \Bigl(\bigotimes_{e=1}^k T^{n^e} A_e \xrightarrow{\otimes_{e=1}^k \Delta_0^{(m)}} \bigotimes_{e=1}^k \sum_{\sum_p r_p^e = n^e} \bigotimes_{p=1}^m T^{r_p^e} A_e$$
$$\xrightarrow{\cong} \sum_{\sum_p r_p^e = n^e} \bigotimes_{p=1}^m \bigotimes_{e=1}^k T^{r_p^e} A_e \xrightarrow{\sum \otimes_{p=1}^m g_{(r_p^e)_e}} \bigotimes_{p=1}^m B\Bigr).$$

This can be written in second form close to (4.10) for corolla $t = \tau[k]$:

$$\widehat{g} \cdot \mathrm{pr}_m = \sum_{\substack{\forall 1\leqslant e\leqslant k\,|\tau(e)|=n^e\\ |\tau(\mathrm{rv})|=m}}^{\text{surjective } t\text{-tree } \tau} \otimes^{v\in\mathrm{v}(\tau[k])} \otimes^{p\in\tau(v)} \bigl(g^v_{|\tau(e)^{-1}(p)|_{e=1}^k} : \otimes_{e=1}^k T^{|\tau(e)^{-1}(p)|} A_e \to B\bigr)$$

with significant distinction: $\tau(\mathrm{rv})$ is not a singleton here. Of course, $\mathrm{v}(\tau[k]) = \{\mathrm{rv}\}$ and $\tau(\mathrm{rv}) = \mathbf{m}$.

This observation allows to compute some subsums of (4.10) and this expression takes the form

$$\sum_{k:\mathrm{v}(t)\to\mathbb{N}}^{k(\mathrm{rv})=1} \otimes^{v\in\mathrm{v}(t)} \bigl(\widehat{g^v} : \otimes^{u\in\mathrm{inV}(v)} T^{K(u)} sA_{\mathrm{ou}(u)} \to T^{k(v)} sA_{\mathrm{ou}(v)}\bigr), \tag{4.11}$$

where the function $K : \overline{\mathrm{v}}(t) \to \mathbb{N}$ is the only extension of k such that $K|_{\mathrm{v}(t)} = k$, $K(a) = j^a$ for $a \in \mathrm{Inpv}(t)$. Here augmented coalgebra homomorphisms $\widehat{g^v} : \otimes^{u\in\mathrm{inV}(v)} TsA_{\mathrm{ou}(u)} \to TsA_{\mathrm{ou}(v)}$ come into play. Application of $\mathrm{comp}(t)$ to (4.11) will give $\mu^{\mathrm{gaC}}(t)(\widehat{g^v}) \cdot \mathrm{pr}_1$ due to imposed condition $k(\mathrm{rv}) = 1$. Besides, without this restriction (4.11) would give the composition $\mu_{\mathrm{gaC}}(t)(\widehat{g^v})$.

Multiplicativity (4.2) of comultiplication Δ^{G} it suffices to check on generators f_j. We have

$$\circledast_{\mathsf{G}}(t)(\Delta^{\mathsf{G}}(t_v))(\Delta^{\mathsf{G}}(t)(f_j)) = \circledast_{\mathsf{G}}(t)(\Delta^{\mathsf{G}}(t_v)) \sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=j^a}^{\text{surjective } t\text{-tree } \tau} \otimes^{v\in\mathrm{v}(t)} \otimes^{p\in\tau(v)} f_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}}$$

$$= \sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=j^a}^{\text{surjective } t\text{-tree } \tau} \otimes^{v\in\mathrm{v}(t)} \otimes^{p\in\tau(v)} \sum_{\substack{\forall c\in\mathrm{Inpv}\,t_v=\mathrm{inV}(v)\\ \tau_v^p(c)=\tau(\mathrm{ou}(c))^{-1}(p)}}^{\text{surjective } t_v\text{-tree } \tau_v^p} \otimes^{u\in\mathrm{v}(t_v)} \otimes^{q\in\tau_v^p(u)} f_{|\tau_v^p(e)^{-1}(q)|_{e\in\mathrm{in}(u)}}. \tag{4.12}$$

The natural isomorphism, interchanging the second and the third tensor products, identifies this with

$$\Delta^{\mathsf{G}}(\theta)(f_j) = \sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=j^a}^{\substack{\theta\text{-tree } T\\ \tilde\tau,\tilde\tau_v^p-\text{surjective}}} \otimes^{w\in\mathrm{v}(\theta)} \otimes^{r\in T(w)} f_{|T(e)^{-1}(r)|_{e\in\mathrm{in}(w)}} \tag{4.13}$$

once we explain how to construct T departing from t, τ, (t_v), $(\tau_v^p)_v^p$.

The first and the third tensor products from (4.12) combine to $\otimes^{\mathrm{v}(\theta)}$ due to known relation $\mathrm{v}(\theta) = \biguplus_{v\in\mathrm{v}(t)}^{<} \mathrm{v}(t_v)$ for $\theta = I_t(t_v \mid v \in \mathrm{v}(t))$. The second and the fourth tensor products from (4.12) combine to $\otimes^{T(w)}$ provided $T(w) = T(v,u) = \biguplus_{p\in\tau(v)}^{<} \tau_v^p(u)$, where $w = (v,u)$ is an internal vertex of θ. The same formula is used when $w = (v,u)$ is an input vertex of θ, namely, $v \in \mathrm{v}(t)$ and $u \in \mathrm{Inpv}\,t_v \cap \mathrm{Inpv}\,\theta \cong \mathrm{inV}(v) \cap \mathrm{Inpv}\,t$. For such (v,u) we have

$$T(v,u) = \biguplus_{p\in\tau(v)}^{<} \tau_v^p(u) \cong \biguplus_{p\in\tau(v)}^{<} \tau(u \to v)^{-1}(p) = \tau(u). \tag{4.14}$$

For $\bar{E}(t) = E(t) - \{\text{root edge}\}$ we have $\bar{E}(\theta) = \sqcup_{v\in\mathrm{v}(t)} \bar{E}(t_v)$. Thus each edge in $\bar{E}(\theta)$ is represented by a vertex $v \in \mathrm{v}(t)$ and an edge $(e : u \to w) \in \bar{E}(t_v)$. The collection $T(\_,\_)$ is made into a θ-tree using maps

$$\biguplus_{p\in\tau(v)}^{<} \tau_v^p(e) : T(v,u) = \biguplus_{p\in\tau(v)}^{<} \tau_v^p(u) \to \biguplus_{p\in\tau(v)}^{<} \tau_v^p(w) = T(v,w).$$

It is clear how to get τ, $(\tau_v^p)_v^p$ from T. This identifies expressions (4.12) and (4.13). Surjectivity of $\tilde{\tau}$ is necessary for surjectivity of $\tilde{T}$. Since surjectivity condition means non-vanishing of indices of $f_\bullet$, $\tilde{T}$ is surjective iff $\tilde{\tau}$ and $(\tilde{\tau}_v^p)_v^p$ are. ∎

4.11. Remark. Let us show that T satisfies also the relation $\tilde{T} = I_{\tilde{\tau}}(\tilde{\tau}_v^p \mid v \in \mathrm{v}(t),\, p \in \tau(v))$. An input vertex $(u,q) \in \operatorname{Inpv} \tilde{\tau}_v^p$ consists of a vertex $u \in \operatorname{Inpv} t_v \cong \mathrm{inV}_t(v)$ and an element $q \in \tau_v^p(u) \cong \tau(u \to v)^{-1}(p)$, which is the same as an edge $(u,q) \to (v,p)$ in $\tilde{\tau}$, that is, $(u,q) \in \mathrm{inV}_{\tilde{\tau}}(v,p)$. Thus, $\operatorname{Inpv} \tilde{\tau}_v^p \cong \mathrm{inV}_{\tilde{\tau}}(v,p)$ and $(\tilde{\tau}; (\tilde{\tau}_v^p) \mid v \in \mathrm{v}(t),\, p \in \tau(v))$ is a 2-cluster tree. The set of internal vertices of $I(\tilde{\tau}; (\tilde{\tau}_v^p))$ is

$$\begin{aligned}\mathrm{v}(I(\tilde{\tau}; (\tilde{\tau}_v^p))) = \bigsqcup_{(v,p)\in \mathrm{v}(\tilde{\tau})} \mathrm{v}(\tilde{\tau}_v^p) = \bigsqcup_{v\in\mathrm{v}(t)} \bigsqcup_{p\in\tau(v)} \bigsqcup_{u\in\mathrm{v}(t_v)} \tau_v^p(u) \\ \cong \bigsqcup_{v\in\mathrm{v}(t)} \bigsqcup_{u\in\mathrm{v}(t_v)} \bigsqcup_{p\in\tau(v)} \tau_v^p(u) = \bigsqcup_{w\in\mathrm{v}(\theta)} T(w) = \mathrm{v}(\tilde{T}).\end{aligned}$$

Due to (4.14) input edges satisfy

$$\operatorname{Inp} I(\tilde{\tau}; (\tilde{\tau}_v^p)) \cong \operatorname{Inp} \tilde{\tau} \cong \operatorname{Inp} \tilde{T}.$$

Equip $\mathrm{v}(\tilde{\tau}) = \sqcup_{v\in\mathrm{v}(t)}\tau(v)$ with partial order as disjoint union of totally ordered sets. This order is the extra datum which allows to restore τ from $\tilde{\tau}$. Similarly we do for $\overline{\mathrm{v}}(\tilde{\tau}_v^p)$, $v \in \mathrm{v}(t)$, $p \in \tau(v)$ and for $\overline{\mathrm{v}}(\tilde{T})$. In order to describe the total order on fibres $T(v,u)$ of the map $\tilde{T} \to I(t,(t_v))$ represent its set of vertices as

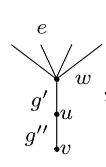
$$\overline{\mathrm{v}}(\tilde{T}) - \{\mathrm{rv}(\tilde{T})\} = \bigsqcup_{(v,p)\in\mathrm{v}(\tilde{\tau})} (\overline{\mathrm{v}}(\tilde{\tau}_v^p) - \{\mathrm{rv}(\tilde{\tau}_v^p)\}).$$

Although the partial order on $\overline{\mathrm{v}}(\tilde{T}) - \{\mathrm{rv}(\tilde{T})\}$ differs from the order on $\trianglelefteq_{(v,p)\in\mathrm{v}(\tilde{\tau})}(\overline{\mathrm{v}}(\tilde{\tau}_v^p) - \{\mathrm{rv}(\tilde{\tau}_v^p)\})$, these two agree on fibres $T(v,u) = \trianglelefteq_{p\in\tau(v)} \tau_v^p(u)$. In fact, for $q \in \tau_v^p(u)$, $q' \in \tau_v^{p'}(u)$ the inequality in the second sense $(v,p,u,q) < (v,p',u,q')$ holds iff $p < p'$ or ($p = p'$ and $q < q'$) iff $(p,q) < (p',q')$ in $\trianglelefteq_{p\in\tau(v)} \tau_v^p(u)$.

4.12. Example. Consider the $\mathcal{V}$-operad $\mathit{As1}$ with $\mathit{As1}(n) = \mathbb{1} \in \operatorname{Ob}\mathcal{V}$ for $n \geqslant 0$. Consider the $n \wedge 1$-operad $\mathit{As1}$-module $\mathit{FAs1}_n$ with $\mathit{FAs1}_n(j) = \mathbb{1}$ for all $j \in \mathbb{N}^n$. Actions of $\mathit{As1}$ on $\mathit{FAs1}_n$ are given by multiplication for $\mathbb{1}$. We claim that there is a canonical isomorphism

$$\circledast_{\mathsf{M}}(t)(\mathit{FAs1}_{|v|})_{v\in\mathrm{v}(t)} \cong \mathit{FAs1}_{\operatorname{Inp} t}\,.$$

In fact this holds true for $t = |$ or corolla t, see Section 2.22. If g is an internal edge of t with $\mathrm{tail}(g) = w$, $\mathrm{head}(g) = v$ there is a unary vertex $u \in \mathrm{v}(t^*)$ as shown:



$$, \qquad g = g' \cdot g''.$$

Correspondingly, coequaliser (2.15) contains the following pieces (partial coequalisers). For any t^*-tree τ^* denote $\tau = \tau^*\big|_t$. One morphism is

$$\bigotimes^{p\in\tau(w)} FAs1_{|w|}\big(|\tau^*(e)^{-1}(p)|_{e\in\mathrm{in}_{t^*}(w)}\big) \otimes \bigotimes^{q\in\tau^*(u)} As1\big(|\tau^*(g')^{-1}(q)|\big)\otimes$$
$$\bigotimes^{r\in\tau(v)} FAs1_{|v|}\big(|\tau^*(f)^{-1}(r)|_{f\in\mathrm{in}_{t^*}(v)}\big) \xrightarrow{\cong}$$
$$\bigotimes^{p\in\tau(w)} FAs1_{|w|}\big(|\tau^*(e)^{-1}(p)|_{e\in\mathrm{in}_{t^*}(w)}\big)\otimes$$
$$\bigotimes^{r\in\tau(v)}\Big[\bigotimes^{q\in\tau^*(g'')^{-1}(r)} As1\big(|\tau^*(g')^{-1}(q)|\big) \otimes FAs1_{|v|}\big(|\tau^*(f)^{-1}(r)|_{f\in\mathrm{in}_{t^*}(v)}\big)\Big] \xrightarrow{1\otimes\otimes^r\lambda_v}$$
$$\bigotimes^{p\in\tau(w)} FAs1_{|w|}\big(|\tau^*(e)^{-1}(p)|_{e\in\mathrm{in}_{t^*}(w)}\big)\otimes \bigotimes^{r\in\tau(v)} FAs1_{|v|}\big(|\tau(g)^{-1}(r)|, |\tau^*(f)^{-1}(r)|_{f\in\mathrm{in}_{t^*}(v)-\{g''\}}\big)$$

due to

$$\sum_{q\in\tau^*(g'')^{-1}(r)} |\tau^*(g')^{-1}(q)| = |\tau(g)^{-1}(r)|.$$

Another morphism is

$$\bigotimes^{p\in\tau(w)} FAs1_{|w|}\big(|\tau^*(e)^{-1}(p)|_{e\in\mathrm{in}_{t^*}(w)}\big) \otimes \bigotimes^{q\in\tau^*(u)} As1\big(|\tau^*(g')^{-1}(q)|\big)\otimes$$
$$\bigotimes^{r\in\tau(v)} FAs1_{|v|}\big(|\tau^*(f)^{-1}(r)|_{f\in\mathrm{in}_{t^*}(v)}\big) \xrightarrow{\cong}$$
$$\bigotimes^{q\in\tau^*(u)}\Big[\bigotimes^{p\in\tau^*(g')^{-1}(q)} FAs1_{|w|}\big(|\tau^*(e)^{-1}(p)|_{e\in\mathrm{in}_{t^*}(w)}\big) \otimes As1\big(|\tau^*(g')^{-1}(q)|\big)\Big]\otimes$$
$$\bigotimes^{r\in\tau(v)} FAs1_{|v|}\big(|\tau^*(f)^{-1}(r)|_{f\in\mathrm{in}_{t^*}(v)}\big) \xrightarrow{(\otimes^q\rho_w)\otimes 1}$$
$$\bigotimes^{q\in\tau^*(u)} FAs1_{|w|}\big(|\tau^*(e\cdot g')^{-1}(q)|_{e\in\mathrm{in}_{t^*}(w)}\big) \otimes \bigotimes^{r\in\tau(v)} FAs1_{|v|}\big(|\tau^*(f)^{-1}(r)|_{f\in\mathrm{in}_{t^*}(v)}\big)$$

due to

$$\sum_{p\in\tau^*(g')^{-1}(q)} |\tau^*(e)^{-1}(p)|_{e\in\mathrm{in}_{t^*}(w)} = |\tau^*(e\cdot g')^{-1}(q)|_{e\in\mathrm{in}_{t^*}(w)}.$$

These morphisms tensored with identity morphisms are to be coequalised. Notice that all tensor factors are actually $\mathbb{1}$. Thus coequaliser (2.15) has the form

$$\coprod_{\forall a\in\mathrm{Inpv}\,t\,|\tau^*(a)|=z^a}^{t^*-\text{tree }\tau^*} \mathbb{1} \overset{f}{\underset{g}{\rightrightarrows}} \coprod_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=z^a}^{t-\text{tree }\tau} \mathbb{1} \longrightarrow \circledast_{\mathsf{M}}(t)(FAs1_{|v|})_{v\in\mathrm{v}(t)}(z).$$

The summand $\mathbb{1}$ indexed by τ^* is mapped by f (resp. g) identically to the summand $\mathbb{1}$ indexed by $\tau = \tau^*\big|_t$ (resp. by t-tree τ' described as follows). Consider a full subcategory (also a tree) $t' \subset t^*$, whose vertices belong to

$$\overline{\mathrm{v}}(t') = (\mathrm{Inpv}\, t \cup \{\mathrm{rv}\}) \sqcup \{\mathrm{middle}(e) \mid e \text{ is an internal edge of } t\}.$$

Clearly there is an isomorphism $t \cong t'$, taking a vertex w to itself if $w \in \mathrm{Inpv}\, t \cup \{\mathrm{rv}\}$ or to the middle of $\mathrm{ou}(w) \in E(t)$ if this edge is internal. Thus, the t'-tree $\tau' = \tau^*\big|_{t'}$ can be viewed as a t-tree. Therefore, the part of the equivalence relations indexed by τ^* identifies the summands $\mathbb{1}$ of $\circledast_{\mathsf{V}}(t)(FAs1_{|v|})$ indexed by t-trees $\tau = \tau^*\big|_t$ and $\tau' = \tau^*\big|_{t'}$.

With given t-tree τ we associate canonically the t^*-tree τ^* such that $\tau^*\big|_{\mathrm{Inp}\, t} = \tau\big|_{\mathrm{Inp}\, t}$ and each internal edge g is composed of $g' = g$ and $g'' = \mathrm{id}$. Thus, $\tau^*\big|_t$ coincides with τ. For this τ^* each internal vertex v joined by an edge with the root vertex is mapped by τ' to $\tau'(v) = \mathbf{1}$. Repeating this again we get t-tree with value $\mathbf{1}$ for all internal vertices joined by a path of two edges with the root vertex, and so on. Finally we conclude that summand $\mathbb{1}$ indexed by any t-tree τ is equivalent to the summand $\mathbb{1}$ indexed by τ_{std} with $\tau_{\mathrm{std}}(v) = \mathbf{1}$ for all $v \in \mathrm{v}(t)$. Thus, $\circledast_{\mathsf{M}}(t)(FAs1_{|v|})_{v\in\mathrm{v}(t)}(z) = \mathbb{1} = FAs1_{\mathrm{Inp}\, t}(z)$ for any $z \in \mathbb{N}^{\mathrm{Inp}\, t}$.

As $\Delta(t) : FAs1_{\mathrm{Inp}\, t} \to \circledast_{\mathsf{M}}(t)(FAs1_{|v|})_{v\in\mathrm{v}(t)}$ we take the identity map. Thus, equation (4.2) holds, and $(As1, FAs1_\bullet, \Delta)$ is a polymodule cooperad. Composition (4.3) is nothing else but the composition in the multicategory of monoids in $\mathcal{V}$. As an exercise the reader might check directly that any summand of $\circledast_{\mathsf{V}}$ will stand for $\circledast_{\mathsf{M}}$ in (4.3).

The case of non-unital algebras is similar. Again $\circledast_{\mathsf{M}}(t)(FAs_{|v|})_{v\in\mathrm{v}(t)} \cong FAs_{\mathrm{Inp}\, t}$. The left hand side is computed using surjective t-trees τ and it can be shown that any such τ is equivalent in $\circledast_{\mathsf{M}}(t)$ to $\tau^{\mathrm{sur}}_{\mathrm{std}}$ with

$$\tau^{\mathrm{sur}}_{\mathrm{std}}(v) = \begin{cases} \mathbf{1}, & \text{if } v \in \mathrm{v}(t) \setminus \mathrm{Lv}(t), \\ \varnothing, & \text{if } v \in \mathrm{v}(t) \cap \mathrm{Lv}(t). \end{cases}$$

Again $\Delta(t) = \mathrm{id} : FAs_{\mathrm{Inp}\, t} \to \circledast_{\mathsf{M}}(t)(FAs_{|v|})_{v\in\mathrm{v}(t)}$ makes $(As, FAs_\bullet)$ into a polymodule cooperad.

We may sum up Proposition 4.10 as follows. Composition (4.6) for $C_e = TsA_e$ written in components

$$\mu^{\mathrm{gaC}}(t) : \prod_{v\in\mathrm{v}(t)} \prod_{k\in\mathbb{N}^{\mathrm{in}(v)}-0} \mathbf{gr}\Big(\bigotimes_{e\in\mathrm{in}(v)} T^{k^e} sA_e, sA_{\mathrm{ou}(v)}\Big) \to \prod_{j\in\mathbb{N}^{\mathrm{Inp}\, t}-0} \mathbf{gr}\Big(\bigotimes_{a\in\mathrm{Inp}\, t} T^{j^a} sA_a, sA_{\text{root edge}}\Big) \tag{4.15}$$

composed with the projection pr_j is the map

$$\big((g^v_k)_{k\in\mathbb{N}^{\mathrm{in}(v)}-0}\big)_{v\in\mathrm{v}(t)} \mapsto \mathrm{comp} \sum_{\forall a\in\mathrm{Inpv}\, t\, |\tau(a)|=j^a}^{\text{surjective } t\text{-tree } \tau} \otimes^{v\in\mathrm{v}(t)} \otimes^{p\in\tau(v)} g^v_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}}.$$

Now let us recall the differentials.

4.13. Proposition. *Define comultiplication $\Delta^{\mathsf{M}}(t)$ for the A_∞-polymodule $F_\bullet$ and a tree t via composition with canonical map π*

$$\Delta^{\mathsf{M}}(t)(j) = \big[F_{\operatorname{Inp} t}(j) \xrightarrow{\Delta^{\mathsf{G}}(t)} \circledast_{\mathsf{G}}(t)(F_{|v|})_{v\in \mathrm{v}(t)}(j) \xrightarrow{\pi} \circledast_{\mathsf{M}}(t)(F_{|v|})_{v\in \mathrm{v}(t)}(j)\big]$$

(on generators it is still given by (4.9)). For the tree $t = |$ define $\Delta^{\mathsf{M}}(|)(j) : F_1(j) \to A_\infty(j)$, $f_j \mapsto \delta_{j1}$, $j \geqslant 1$. Then all $\Delta^{\mathsf{M}}(t)(j)$ are chain maps, thus, $(A_\infty, F_\bullet, \Delta^{\mathsf{M}})$ is a **dg***-polymodule cooperad.*

Proof. First of all, comultiplication Δ^{M} is multiplicative — it satisfies (4.2), since Δ^{G} satisfies it. For corollas t the morphism $\Delta^{\mathsf{M}}(t) = \Delta^{\mathsf{G}}(t)$ satisfies (4.1).

For any 2-cluster tree $(t; (t_v))$ if $\Delta^{\mathsf{M}}(t)$ and $\Delta^{\mathsf{M}}(t_v)$, $v \in \mathrm{v}(t)$, are chain maps, then so is $\Delta^{\mathsf{M}}(\theta)$ for $\theta = I_t(t_v \mid v \in \mathrm{v}(t))$. In fact, multiplicativity equation (4.2) represents $\Delta^{\mathsf{M}}(\theta)$ as composition of $\Delta^{\mathsf{M}}(t)$ and tensor product of $\Delta^{\mathsf{M}}(t_v)$. When t is a corolla, $\Delta^{\mathsf{M}}(t)$ is an isomorphism, and even an identity morphism (after certain identification). Therefore, it suffices to prove that $\Delta^{\mathsf{M}}(t)$ is a chain map for $t = |$ and trees t with two internal vertices.

It suffices also to prove the commutation of $\Delta^{\mathsf{M}}(t)$ and ∂ on generators:

$$f_j.\Delta^{\mathsf{M}}(t)\partial = f_j.\partial\Delta^{\mathsf{M}}(t) \text{ for all trees } t \text{ and all indices } j \in \mathbb{N}^{\operatorname{Inp} t} - 0. \tag{4.16}$$

In fact, any element of F_n, $n = |\operatorname{Inp} t|$, is a sum of elements of the form $\alpha\big((\otimes_{i=1}^n \otimes_{p=1}^{k_1^i+\dots+k_m^i} \omega_p^i) \otimes (\otimes_{r=1}^m f_{k_r}) \otimes \omega\big)$, where $\omega_p^i \in A_\infty(j_p^i)$, $\omega \in A_\infty(m)$, see the first row of the following diagram

$$\begin{tikzcd}
\Big(\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k_1^i+\dots+k_m^i} A_\infty(j_p^i)\Big) \otimes \Big(\bigotimes_{r=1}^{m} F_{\operatorname{Inp} t}(k_r)\Big) \otimes A_\infty(m) \arrow[r, "\alpha"] \arrow[d, "1\otimes(\otimes_{r=1}^m \Delta^{\mathsf{M}}(t))\otimes 1"'] & F_{\operatorname{Inp} t}\Big(\Big(\sum_{p=1}^{k_1^i+\dots+k_m^i} j_p^i\Big)_{i=1}^{n}\Big) \arrow[d, "\Delta^{\mathsf{M}}(t)"] \\
\Big(\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k_1^i+\dots+k_m^i} A_\infty(j_p^i)\Big) \otimes \Big(\bigotimes_{r=1}^{m} \circledast_{\mathsf{M}}(t)(F_{|v|})_{v\in \mathrm{v}(t)}(k_r)\Big) \otimes A_\infty(m) \arrow[r, "\alpha"] & \circledast_{\mathsf{M}}(t)(F_{|v|})_{v\in \mathrm{v}(t)}\Big(\Big(\sum_{p=1}^{k_1^i+\dots+k_m^i} j_p^i\Big)_{i=1}^{n}\Big)
\end{tikzcd}$$

This square commutes since $\Delta^{\mathsf{M}}(t)$ is a $\operatorname{Inp}(t) \wedge 1$-$A_\infty$-module homomorphism of degree 0. Use this square as the top and the bottom faces of a cubical diagram whose vertical edges are given by the differential ∂. We know that α is a chain map. Apply all 3-arrow-paths in this cube to the element $x = (\otimes_{i=1}^n \otimes_{p=1}^{k_1^i+\dots+k_m^i} \omega_p^i) \otimes (\otimes_{r=1}^m f_{k_r}) \otimes \omega$ from the top vertex. The equation $x.\big(1 \otimes (\otimes_{r=1}^m \Delta^{\mathsf{M}}(t)) \otimes 1\big)\partial = x.\partial(1 \otimes (\otimes_{r=1}^m \Delta^{\mathsf{M}}(t)) \otimes 1\big)$ holds by assumption, hence, the equation $x.\alpha\Delta^{\mathsf{M}}(t)\partial = x.\alpha\partial\Delta^{\mathsf{M}}(t)$ holds as well.

For the tree $t = |$ and a positive integer j we have $f_j.\Delta^{\mathsf{M}}(|)\partial = \delta_{j1}.\partial = 0$. On the other hand,

$$f_j.\partial\Delta^{\mathsf{M}}(|) = \sum_{r+n+t=j}^{n>1} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) f_{r+1+t}.\Delta^{\mathsf{M}}(|) - \sum_{i_1+\dots+i_l=j}^{l>1} (f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_l}) b_l.\Delta^{\mathsf{M}}(|)$$
$$= b_j \chi(j > 1) - b_j \chi(j > 1) = 0.$$

Thus, $\Delta^{\mathsf{M}}(|)$ is a chain map.

Let us prove equation (4.16) for $t = \{\mathbf{q} \xrightarrow{\beta} \mathbf{n} \to \mathbf{1}\}$, $\beta(\mathbf{q}) \subset \{c\} \subset \mathbf{n}$, $\mathrm{v}(t) = \{c\} \sqcup \mathbf{1}$. Thus, $\mathrm{Inp}(t) = \mathbf{q} \sqcup \{h \in \mathbf{n} \mid h \neq c\}$. Elements of $\mathbb{N}^{\mathrm{Inp}(t)} = \mathbb{N}^q \times \mathbb{N}^{n-1}$ are written as $j = (i^g, u^h \mid g \in \mathbf{q},\ h \in \mathbf{n} - \{c\})$. Summands of (4.9) are indexed by t-trees

$$\tau = \begin{tikzcd}
 & \mathbf{u}^n \arrow[ddr] & & \\
\sqcup_{p=1}^{u^c} \mathbf{r}_p^q \arrow[dr] & \cdots \arrow[dr] & & \\
\cdots \arrow[r] & \mathbf{u}^c \arrow[r] & \mathbf{1} \arrow[r] & \mathbf{1} \\
\sqcup_{p=1}^{u^c} \mathbf{r}_p^1 \arrow[ur] & \cdots \arrow[ur] & & \\
 & \mathbf{u}^1 \arrow[uur] & &
\end{tikzcd} \tag{4.17}$$

The tree τ occurs in expansion of $\Delta^{\mathsf{M}}(t)(f_j)$ if $\sum_{p=1}^{u^c} r_p^g = i^g$. Denote $r_p = (r_p^g)_{g \in \mathbf{q}} \in \mathbb{N}^q$. Thus,

$$f_j.\Delta^{\mathsf{M}}(t) = \sum_{u^c=0}^{\infty} \sum_{\sum_{p=1}^{u^c} r_p = i}^{r_1,\dots,r_{u^c} \in \mathbb{N}^q - 0} \Bigl(\bigotimes_{p=1}^{u^c} f_{r_p} \Bigr) \otimes f_u.$$

We find

$$\begin{aligned} f_j.\Delta^{\mathsf{M}}(t)\partial = \sum_{u^c=0}^{\infty} \sum_{\sum_{p=1}^{u^c} r_p = i}^{r_1,\dots,r_{u^c} \in \mathbb{N}^q - 0} \sum_{h=1}^{u_c} \sum_{g=1}^{q} \sum_{a+x+m=r_h^g}^{x>1} \\ \Bigl(\bigotimes_{p=1}^{h-1} f_{r_p} \Bigr) \otimes \lambda^g({}^a1, b_x, {}^m1; f_{r_h-(x-1)e_g}) \otimes \Bigl(\bigotimes_{p=h+1}^{u^c} f_{r_p} \Bigr) \otimes f_u \end{aligned} \tag{4.18}$$

$$- \sum_{k^c=0}^{\infty} \sum_{\sum_{p=1}^{k^c} s_p = i}^{s_1,\dots,s_{k^c} \in \mathbb{N}^q - 0} \sum_{z=1}^{k_c} \sum_{w=2}^{\infty} \sum_{l_1+\dots+l_w=s_z}^{l_1,\dots,l_w \in \mathbb{N}^q - 0} \Bigl(\bigotimes_{p=1}^{z-1} f_{s_p} \Bigr) \otimes \rho((f_{l_\nu})_{\nu=1}^{w}; b_w) \otimes \Bigl(\bigotimes_{p=z+1}^{k^c} f_{s_p} \Bigr) \otimes f_k \tag{4.19}$$

$$+ \sum_{u^c=0}^{\infty} \sum_{\sum_{p=1}^{u^c} r_p = i}^{r_1,\dots,r_{u^c} \in \mathbb{N}^q - 0} \sum_{h=1}^{n} \sum_{a+w+m=u^h}^{w>1} \Bigl(\bigotimes_{p=1}^{u^c} f_{r_p} \Bigr) \otimes \lambda^h({}^a1, b_w, {}^m1; f_{u-(w-1)e_h}) \tag{4.20}$$

$$- \sum_{u^c=0}^{\infty} \sum_{\sum_{p=1}^{u^c} r_p = i}^{r_1,\dots,r_{u^c} \in \mathbb{N}^q - 0} \sum_{w=2}^{\infty} \sum_{u_1+\dots+u_w=u}^{u_1,\dots,u_w \in \mathbb{N}^n - 0} \Bigl(\bigotimes_{p=1}^{u^c} f_{r_p} \Bigr) \otimes \rho((f_{u_v})_{v=1}^{w}; b_w). \tag{4.21}$$

Also by (3.1)

$$f_j.\partial\Delta^{\mathsf{M}}(t) = \sum_{g=1}^{q} \sum_{k+x+l=i^g}^{x>1} \lambda^g({}^k1, b_x, {}^l1; f_{j-(x-1)(e_g,0)}.\Delta^{\mathsf{M}}(t))$$

$$+ \sum_{h\in\mathbf{n}}^{h\neq c} \sum_{a+w+m=u^h}^{w>1} \lambda^h({}^a1, b_w, {}^m1; f_{j-(w-1)(0,e_h)}.\Delta^{\mathsf{M}}(t)) - \sum_{w=2}^{\infty} \sum_{j_1+\dots+j_w=j}^{j_1,\dots,j_w\in\mathbb{N}^{\operatorname{Inp} t}-0} \rho((f_{j_v}.\Delta^{\mathsf{M}}(t))_{v=1}^w; b_w)$$

$$= \sum_{g=1}^{q} \sum_{k+x+l=i^g}^{x>1} \sum_{u^c=0}^{\infty} \sum_{\sum_{p=1}^{u^c} s_p=i-(x-1)e_g}^{s_1,\dots,s_{u^c}\in\mathbb{N}^q-0} \lambda^g\Bigl({}^k1, b_x, {}^l1; \Bigl(\bigotimes_{p=1}^{u^c} f_{s_p}\Bigr)\otimes f_u\Bigr) \tag{4.22}$$

$$+ \sum_{h\in\mathbf{n}}^{h\neq c} \sum_{a+w+m=u^h}^{w>1} \sum_{u^c=0}^{\infty} \sum_{\sum_{p=1}^{u^c} r_p=i}^{r_1,\dots,r_{u^c}\in\mathbb{N}^q-0} \Bigl(\bigotimes_{p=1}^{u^c} f_{r_p}\Bigr)\otimes\lambda^h\bigl({}^a1, b_w, {}^m1; f_{u-(w-1)e_h}\bigr) \tag{4.23}$$

$$- \sum_{w=2}^{\infty} \sum_{j_1+\dots+j_w=j}^{j_1,\dots,j_w\in\mathbb{N}^{\operatorname{Inp} t}-0} \rho\Biggl(\Biggl(\sum_{u_v^c\in\mathbb{N}} \sum_{\sum_{p=u_1^c+\dots+u_{v-1}^c+1}^{u_1^c+\dots+u_{v-1}^c+u_v^c} r_p=i_v}^{\forall p\, r_p\in\mathbb{N}^q-0} \Bigl(\bigotimes_{p=1+\sum_{\alpha=1}^{v-1}u_\alpha^c}^{\sum_{\alpha=1}^{v}u_\alpha^c} f_{r_p}\Bigr)\otimes f_{u_v}\Biggr)_{v=1}^{w}; b_w\Biggr). \tag{4.24}$$

In the last sum we denote $\sum_{v=1}^w u_v^c$ by u^c, thus, $\sum_{v=1}^w u_v = u$.

We have that sum (4.18) is equal to sum (4.22) because due to (2.14)

$$\lambda^g\bigl({}^k1, b_x, {}^l1; (\otimes_{p=1}^{u^c} f_{s_p})\otimes f_u\bigr) = \bigl(\otimes_{p=1}^{h-1} f_{s_p}\bigr)\otimes\lambda^g({}^a1, b_x, {}^m1; f_{s_h})\otimes\bigl(\otimes_{p=h+1}^{u^c} f_{s_p}\bigr)\otimes f_u,$$

where $h\in\mathbb{N}$ is found from the inequalities

$$1\leqslant h\leqslant u^c, \quad a \overset{\text{def}}{=} k-\sum_{p=1}^{h-1} s_p^g \geqslant 0, \quad m \overset{\text{def}}{=} \sum_{p=1}^{h} s_p^g - k - 1 \geqslant 0.$$

It suffices to identify $s_p = r_p$ for $p\neq h$ and $s_h = r_h-(x-1)e_g$.

Sum (4.19) equals the subsum of (4.20) with $h=c$. In fact, identify the following indices: $u^v = k^v$ for $v\in\mathbf{n}-\{c\}$, $u^c = k^c+w-1$, $s_p = r_p$ for $1\leqslant p<z$, $l_\nu = r_{\nu+z-1}$ for $1\leqslant\nu\leqslant w$, $s_p = r_{p+w-1}$ for $z<p\leqslant k^c$. Then (4.19) and $(4.20)_{h=c}$ become equal in the tensor product over the operad A_∞ (containing elements b_w).

The subsum of (4.20) corresponding to $h\neq c$ coincides with (4.23).

Finally, sum (4.21) equals sum (4.24). Thus, $f_j.\Delta^{\mathsf{M}}(t)\partial = f_j.\partial\Delta^{\mathsf{M}}(t)$. ∎

A. Colimits of algebras over monads

Let $\top:\mathcal{C}\to\mathcal{C}$ be a monad, and let $F:\mathcal{C}\leftrightarrows\mathcal{C}^\top:U$ be the associated adjunction. Assume that $\mathcal{C}$ is cocomplete and $\mathcal{C}^\top$ has coequalizers. The latter condition is satisfied in each of the two following cases:

- $\mathcal{C}$ is a complete, regular, regularly co-well-powered category with coequalizers, and $\top$ is a monad which preserves regular epimorphisms [BW05, Proposition 9.3.8].

- $\mathcal{C}$ has finite colimits and equalizers of arbitrary sets of maps (with the same source and target), and $\top$ is a monad in $\mathcal{C}$ which preserves colimits along countable chains [BW05, Theorem 9.3.9].

When $\mathcal{C}^\top$ has coequalizers, the category $\mathcal{C}^\top$ is cocomplete by a result of Barr and Wells [BW05, Corollary 9.3.3]. Our goal in this section is to reprove this result expressing the colimit in $\mathcal{C}^\top$ through the colimit in $\mathcal{C}$ via sufficiently explicit recipe.

A.1. Proposition. *Assume that $\mathcal{C}$ is cocomplete and $\mathcal{C}^\top$ has coequalizers. Then the category $\mathcal{C}^\top$ is cocomplete.*

Proof. Let I be a small category and let $I \ni i \mapsto P_i \in \mathcal{C}^\top$ be a diagram in $\mathcal{C}^\top$. Denote by C the colimit (coequalizer) of the following diagram in $\mathcal{C}^\top$

$$\begin{tikzcd}
F \operatorname{colim}_i UFUP_i \arrow[d, "F \operatorname{colim} \alpha_i"'] \arrow[dr, "F \operatorname{can}"] & & \\
F \operatorname{colim}_i UP_i \arrow[d, "F \operatorname{colim} \eta"'] & FUF \operatorname{colim}_i UP_i \arrow[r, "\varepsilon"] & F \operatorname{colim}_i UP_i \\
F \operatorname{colim}_i UFUP_i \arrow[ur, "F \operatorname{can}"'] & &
\end{tikzcd} \tag{A.1}$$

where 'can' means any canonical map. Equip $C = (C, \operatorname{can} : F \operatorname{colim}_i UP_i \to C)$ with maps in $\mathcal{C}$ going through the rightmost vertex

$$\begin{aligned}
\operatorname{In}_i &= \bigl(UP_i \xrightarrow{\operatorname{in}_i} \operatorname{colim}_i UP_i \xrightarrow{\eta} UF \operatorname{colim}_i UP_i \xrightarrow{U \operatorname{can}} UC\bigr) \\
&= \bigl(UP_i \xrightarrow{\eta} UFUP_i \xrightarrow{UF \operatorname{in}_i} UF \operatorname{colim}_i UP_i \xrightarrow{U \operatorname{can}} UC\bigr).
\end{aligned} \tag{A.2}$$

We claim that $\operatorname{In}_i \in \mathcal{C}^\top$ and $(C, \operatorname{In}_i : P_i \to C \mid i \in I)$ is the colimiting cocone of the diagram $i \mapsto P_i$ in $\mathcal{C}^\top$.

Let us verify that In_i are morphisms of $\top$-algebras. The exterior of the following diagram commutes

$$\begin{tikzcd}
UFUP_i \arrow[r, "UF\eta"] \arrow[d, "\alpha_i"'] \arrow[drr, "UF \operatorname{in}_i"] & UFUFUP_i \arrow[r, "UFUF \operatorname{in}_i"] \arrow[d, phantom, "="] & UFUF \operatorname{colim}_i UP_i \arrow[r, "UFU \operatorname{can}"] \arrow[d, "U\varepsilon"] \arrow[dr, phantom, "="] & UFUC \arrow[d, "\alpha_C"] \\
UP_i \arrow[r, "\eta"] & UFUP_i \arrow[r, "UF \operatorname{in}_i"] & UF \operatorname{colim}_i UP_i \arrow[r, "U \operatorname{can}"] & UC
\end{tikzcd} \tag{A.3}$$

if and only if

$$\begin{tikzcd}
UFUP_i \arrow[r, "UF\,\mathrm{in}_i"] \arrow[d, "\alpha_i"'] & UF\operatorname{colim}_i UP_i \arrow[rr, "U\,\mathrm{can}"] & & UC \\
UP_i \arrow[r, "\eta"] & UFUP_i \arrow[r, "UF\,\mathrm{in}_i"] & UF\operatorname{colim}_i UP_i \arrow[ur, phantom, ""] \arrow[r, phantom, ""] & \\
\end{tikzcd} \tag{A.4}$$

where the right vertical arrow $UF\operatorname{colim}_i UP_i \xrightarrow{U\,\mathrm{can}} UC$ closes the square ($=$).

Schematically this is the equation $f = \bigl(A \xrightarrow{g} A \xrightarrow{f} C\bigr)$, where $f = UF\,\mathrm{in}_i \cdot U\,\mathrm{can} : A \to C \in \mathcal{C}^\top$ but $g = \alpha_i \cdot \eta \in \mathcal{C}$. By the freeness of $\top$-algebra $\top A$ (see the proof of [BW05, Theorem 3.2.1]) this is equivalent to equation

$$\bigl(\top A \xrightarrow{\alpha_A} A \xrightarrow{f} C\bigr) = \bigl(\top A \xrightarrow{\top g} \top A \xrightarrow{\alpha_A} A \xrightarrow{f} C\bigr).$$

In detail it is the equation

$$\begin{tikzcd}
UFUFUP_i \arrow[r, "U\varepsilon"] \arrow[d, "UF\alpha_i"'] & UFUP_i \arrow[r, "UF\,\mathrm{in}_i"] & UF\operatorname{colim}_i UP_i \arrow[r, "U\,\mathrm{can}"] & UC \\
UFUP_i \arrow[r, "UF\eta"] & UFUFUP_i \arrow[r, "U\varepsilon"] & UFUP_i \arrow[r, "UF\,\mathrm{in}_i"] & UF\operatorname{colim}_i UP_i \arrow[u, "U\,\mathrm{can}"']
\end{tikzcd} \tag{A.5}$$

Removing the unnecessary U we write it as an equation in $\mathcal{C}^\top$:

$$\begin{tikzcd}
FUFUP_i \arrow[r, "FUF\,\mathrm{in}_i"] \arrow[d, "F\alpha_i"'] & FUF\operatorname{colim}_i UP_i \arrow[r, "\varepsilon"] & F\operatorname{colim}_i UP_i \arrow[r, "\mathrm{can}"] & C \\
FUP_i \arrow[r, "F\eta"] & FUFUP_i \arrow[r, "FUF\,\mathrm{in}_i"] & FUF\operatorname{colim}_i UP_i \arrow[r, "\varepsilon"] & F\operatorname{colim}_i UP_i \arrow[u, "\mathrm{can}"']
\end{tikzcd} \tag{A.6}$$

which holds due to (C, can) being coequalizer of (A.1).

Clearly, $\mathrm{In}_i : P_i \to C$ is a cocone from the diagram $i \mapsto P_i$. Let us prove that it is an initial one. Let $\phi_i : P_i \to Q \in \mathcal{C}^\top$ be an arbitrary cocone from the diagram $i \mapsto P_i$. There is a unique map $\beta : \operatorname{colim}_i UP_i \to UQ \in \mathcal{C}$ such that $U\phi_i = \bigl(UP_i \xrightarrow{\mathrm{in}_i} \operatorname{colim}_i UP_i \xrightarrow{\beta} UQ\bigr)$. It has an adjunct $\gamma = {}^t\beta = \bigl(F\operatorname{colim}_i UP_i \xrightarrow{F\beta} FUQ \xrightarrow{\varepsilon} Q\bigr) \in \mathcal{C}^\top$, so that ${}^t(U\phi_i) = \bigl(FUP_i \xrightarrow{F\,\mathrm{in}_i} F\operatorname{colim}_i UP_i \xrightarrow{\gamma} Q\bigr)$. Consequently,

$$U\phi_i = \bigl(UP_i \xrightarrow{\eta} UFUP_i \xrightarrow{UF\,\mathrm{in}_i} UF\operatorname{colim}_i UP_i \xrightarrow{U\gamma} UQ\bigr).$$

Since $\phi_i \in \mathcal{C}^\top$ the exterior of diagram (A.3) commutes, where can and C are replaced with γ and Q. Therefore, equation (A.4) with the same replacement holds. As explained above this implies equations (A.5) and (A.6) with the same modification. Therefore, both paths in diagram (A.1) postcomposed with $\gamma : F\operatorname{colim}_i UP_i \to Q$ from the top vertex $F\operatorname{colim}_i UFUP_i$ to Q are equal to each other. Hence, γ factorizes as $F\operatorname{colim}_i UP_i \xrightarrow{\mathrm{can}} C \xrightarrow{\psi} Q$ for a unique $\psi \in \mathcal{C}^\top$. ■

A.2. Remark. It is shown in the proof of the above proposition that $\operatorname{colim}_i P_i$ is the biggest quotient of $F \operatorname{colim}_i UP_i$ via a regular epimorphism $\mathrm{can} : F \operatorname{colim}_i UP_i \to C = \operatorname{colim}_i P_i$ such that morphisms $\mathrm{In}_i : UP_i \to UC \in \mathcal{C}$ from (A.2) are morphisms of $\top$-algebras.

A.3. Proposition. *Assume that $\mathcal{C}$ is cocomplete and $\mathcal{C}^\top$ has coequalizers. Let $X \in \operatorname{Ob}\mathcal{C}$ and $A = (UA, \alpha : UFUA \to UA) \in \operatorname{Ob}\mathcal{C}^\top$. Then the colimit $C = (C, \mathrm{can} : F(X \sqcup UA) \to C)$ of the diagram in $\mathcal{C}^\top$*

$$\begin{tikzcd}
FUFUA \arrow[d, "F\alpha"'] \arrow[drr, "FUF\,\mathrm{in}_2"] & & \\
FUA \arrow[d, "F\eta"'] & & FUF(X \sqcup UA) \arrow[r, "\varepsilon"] & F(X \sqcup UA) \\
FUFUA \arrow[urr, "FUF\,\mathrm{in}_2"'] & &
\end{tikzcd} \tag{A.7}$$

equipped with the morphisms of $\top$-algebras

$$\begin{aligned}
\mathrm{In}_1 &= \big(FX \xrightarrow{F\,\mathrm{in}_1} F(X \sqcup UA) \xrightarrow{\mathrm{can}} C\big), \\
\mathrm{In}_2 &= \big(UA \xrightarrow{\mathrm{in}_2} X \sqcup UA \xrightarrow{\eta} UF(X \sqcup UA) \xrightarrow{U\,\mathrm{can}} UC\big) \\
&= \big(UA \xrightarrow{\eta} UFUA \xrightarrow{UF\,\mathrm{in}_2} UF(X \sqcup UA) \xrightarrow{U\,\mathrm{can}} UC\big)
\end{aligned} \tag{A.8}$$

is the coproduct $FX \sqcup A$ in $\mathcal{C}^\top$.

Proof. Let us verify that In_2 is a morphism of $\top$-algebras. This is equivalent to commutativity of the exterior of the following diagram

$$\begin{tikzcd}
UFUA \arrow[r, "UF\eta"] \arrow[d, "\alpha"'] \arrow[drr, "UF\,\mathrm{in}_2"] & UFUFUA \arrow[r, "UFUF\,\mathrm{in}_2"] \arrow[d, phantom, "="] & UFUF(X \sqcup UA) \arrow[r, "UFU\,\mathrm{can}"] \arrow[d, "U\varepsilon"] \arrow[dr, phantom, "="] & UFUC \arrow[d, "\alpha_C"] \\
UA \arrow[r, "\eta"] & UFUA \arrow[r, "UF\,\mathrm{in}_2"] & UF(X \sqcup UA) \arrow[r, "U\,\mathrm{can}"] & UC
\end{tikzcd} \tag{A.9}$$

which holds if and only if

$$\begin{tikzcd}
UFUA \arrow[r, "UF\,\mathrm{in}_2"] \arrow[d, "\alpha"'] & UF(X \sqcup UA) \arrow[r, "U\,\mathrm{can}"] \arrow[d, phantom, "="] & UC \\
UA \arrow[r, "\eta"] & UFUA \arrow[r, "UF\,\mathrm{in}_2"] & UF(X \sqcup UA) \arrow[u, "U\,\mathrm{can}"']
\end{tikzcd} \tag{A.10}$$

Schematically this is the equation $f = \big(B \xrightarrow{g} B \xrightarrow{f} C\big)$, where $f = UF\,\mathrm{in}_2 \cdot U\,\mathrm{can} : B \to C \in \mathcal{C}^\top$ but $g = \alpha \cdot \eta \in \mathcal{C}$. By the freeness of $\top$-algebra $\top B$ (see the proof of [BW05, Theorem 3.2.1]) this is equivalent to equation

$$\big(\top B \xrightarrow{\alpha_B} B \xrightarrow{f} C\big) = \big(\top B \xrightarrow{\top g} \top B \xrightarrow{\alpha_B} B \xrightarrow{f} C\big).$$

In detail it is the equation

$$\begin{tikzcd} UFUFUA \arrow[r, "U\varepsilon"] \arrow[d, "UF\alpha"'] & UFUA \arrow[r, "UF\,\mathrm{in}_2"] & UF(X \sqcup UA) \arrow[r, "U\,\mathrm{can}"] & UC \\ UFUA \arrow[r, "UF\eta"] & UFUFUA \arrow[r, "U\varepsilon"] & UFUA \arrow[r, "UF\,\mathrm{in}_2"] & UF(X \sqcup UA) \arrow[u, "U\,\mathrm{can}"'] \end{tikzcd} \tag{A.11}$$

Removing the unnecessary U we write it as an equation in $\mathcal{C}^\top$:

$$\begin{tikzcd} FUFUA \arrow[r, "FUF\,\mathrm{in}_2"] \arrow[d, "F\alpha"'] & FUF(X \sqcup UA) \arrow[r, "\varepsilon"] & F(X \sqcup UA) \arrow[r, "\mathrm{can}"] & C \\ FUA \arrow[r, "F\eta"] & FUFUA \arrow[r, "FUF\,\mathrm{in}_2"] & FUF(X \sqcup UA) \arrow[r, "\varepsilon"] & F(X \sqcup UA) \arrow[u, "\mathrm{can}"'] \end{tikzcd} \tag{A.12}$$

which holds due to (C, can) being coequalizer of (A.7).

Let us prove that $(C, \mathrm{In}_1 : FX \to C, \mathrm{In}_2 : A \to C)$ is the coproduct $FX \sqcup A$ in $\mathcal{C}^\top$. Let $\phi_1 : FX \to Q \in \mathcal{C}^\top$ and $\phi_2 : A \to Q \in \mathcal{C}^\top$. The maps $\delta = \phi_1^t = \bigl(X \xrightarrow{\eta} UFX \xrightarrow{U\phi_1} UQ\bigr)$ and $U\phi_2 : UA \to UQ$ determine a unique map $\beta : X \sqcup UA \to UQ$ in $\mathcal{C}$. It has an adjunct $\gamma = {}^t\beta = \bigl(F(X \sqcup UA) \xrightarrow{F\beta} FUQ \xrightarrow{\varepsilon} Q\bigr) \in \mathcal{C}^\top$, so that $\phi_1 = {}^t\delta = \bigl(FX \xrightarrow{F\,\mathrm{in}_1} F(X \sqcup UA) \xrightarrow{\gamma} Q\bigr)$, ${}^t(U\phi_2) = \bigl(FUA \xrightarrow{F\,\mathrm{in}_2} F(X \sqcup UA) \xrightarrow{\gamma} Q\bigr)$. Consequently,

$$U\phi_2 = \bigl(UA \xrightarrow{\eta} UFUA \xrightarrow{UF\,\mathrm{in}_2} UF(X \sqcup UA) \xrightarrow{U\gamma} UQ\bigr).$$

Since $\phi_2 \in \mathcal{C}^\top$ the exterior of diagram (A.9) commutes, where can and C are replaced with γ and Q. Therefore, equation (A.10) with the same replacement holds. As explained above this implies equations (A.11) and (A.12) with the same modification. Therefore, both paths in diagram (A.7) postcomposed with $\gamma : F(X \sqcup UA) \to Q$ from the top vertex $FUFUA$ to Q are equal to each other. Hence, γ factorizes as $F(X \sqcup UA) \xrightarrow{\mathrm{can}} C \xrightarrow{\psi} Q$ for a unique $\psi \in \mathcal{C}^\top$. We get

$$\phi_1 = \bigl(FX \xrightarrow{F\,\mathrm{in}_1} F(X \sqcup UA) \xrightarrow{\mathrm{can}} C \xrightarrow{\psi} Q\bigr) = \bigl(FX \xrightarrow{\mathrm{In}_1} C \xrightarrow{\psi} Q\bigr),$$
$$U\phi_2 = \bigl(UA \xrightarrow{\eta} UFUA \xrightarrow{UF\,\mathrm{in}_2} UF(X \sqcup UA) \xrightarrow{U\,\mathrm{can}} UC \xrightarrow{U\psi} UQ\bigr),$$

hence, $\phi_2 = \bigl(A \xrightarrow{\mathrm{In}_2} C \xrightarrow{\psi} Q\bigr)$. This shows that $(C, \mathrm{In}_1, \mathrm{In}_2)$ is the coproduct $FX \sqcup A$ in $\mathcal{C}^\top$. ■

A.4. Corollary. *$FX \sqcup A$ is the biggest quotient of $F(X \sqcup UA)$ via a regular epimorphism* $\mathrm{can} : F(X \sqcup UA) \to C = FX \sqcup A$ *such that the morphism* $\mathrm{In}_2 : UA \to UC \in \mathcal{C}$ *from* (A.8) *is a morphism of* $\top$*-algebras.*

A.5. Induced modules. Assume that $f_i : \mathcal{A}_i \to \mathcal{B}_i$ are morphisms of operads for $i \in [n]$. For any $n \wedge 1$-operad module $((\mathcal{A}_i)_{i\in[n]}; \mathcal{P})$ there is a $(\mathcal{B}_i)_{i\in[n]}$-module $\mathcal{Q} = \bigcirc_{i=0}^{n} \mathcal{B}_i \odot^i_{\mathcal{A}_i} \mathcal{P}$, which is the colimit of the diagram (the coequalizer)

$$\begin{tikzcd}
\odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]}; \odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]}; \mathcal{P})) \arrow[r, "\odot_{\geqslant 0}({}^{[n]}1;\alpha)"] \arrow[d, "\odot_{\geqslant 0}({}^{[n]}1;\odot_{\geqslant 0}((f_i)_{i\in[n]};1))"'] & \odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]}; \mathcal{P}) \\
\odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]}; \odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]}; \mathcal{P})) \arrow[ur, "\mu"] \arrow[r, "\sim"', "="] & \odot_{\geqslant 0}((\mathcal{B}_i \odot \mathcal{B}_i)_{i\in[n]}; \mathcal{P}) \arrow[u, "\odot_{\geqslant 0}({}^{[n]}\mu;1)"']
\end{tikzcd}$$

This can be described also using monads $\mathbb{A} : \mathcal{X} \mapsto \odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]}; \mathcal{X})$, $\mathbb{B} : \mathcal{X} \mapsto \odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]}; \mathcal{X})$ in $\mathbf{dg}^{\mathbb{N}^n}$ and a morphism $f = \odot_{\geqslant 0}((f_i)_{i\in[n]}; 1) : \mathbb{A} \to \mathbb{B}$ between them. Denote $\alpha^{\mathcal{P}} : \mathbb{A}\mathcal{P} \to \mathcal{P}$ the action on the $\mathbb{A}$-module $\mathcal{P}$. The induced module $\mathcal{Q}$ is the colimit $\mathbb{B}_{\mathbb{A}}\mathcal{P}$ of the diagram in the category of $\mathbb{B}$-modules (the coequalizer of)

$$\begin{tikzcd}
\mathbb{B}\mathbb{A}\mathcal{P} \arrow[rr, "\mathbb{B}\alpha^{\mathcal{P}}"] \arrow[dr, "\mathbb{B}f"'] & & \mathbb{B}\mathcal{P} \\
& \mathbb{B}\mathbb{B}\mathcal{P} \arrow[ur, "\mu^{\mathbb{B}}"'] &
\end{tikzcd}$$

If $\mathcal{P} = \odot_{\geqslant 0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{X}; \mathcal{A}_0)$ is a free $n\wedge 1$-module, then the $n\wedge 1$-module $\bigcirc_{i=1}^{n} \mathcal{B}_i \odot^i_{\mathcal{A}_i} \mathcal{P} \odot^0_{\mathcal{A}_0} \mathcal{B}_0 = \odot_{\geqslant 0}(\mathcal{B}_1, \dots, \mathcal{B}_n; \mathcal{X}; \mathcal{B}_0)$ is free as well. It is also generated by $\mathcal{X}$. In fact, the coequalizer in question has the structure of a contractible coequalizer [BW05, Definition 3.3.3] with $d^1 = \mathbb{B}f \cdot \mu^{\mathbb{B}} : \mathbb{B}\mathbb{A}\mathbb{A}\mathcal{X} \to \mathbb{B}\mathbb{A}\mathcal{X}$ and $d = \mathbb{B}f \cdot \mu^{\mathbb{B}} : \mathbb{B}\mathbb{A}\mathcal{X} \to \mathbb{B}\mathcal{X}$

$$\begin{tikzcd}
\mathbb{B}\mathbb{A}\mathbb{A}\mathcal{X} \arrow[rr, shift left, "d^0=\mathbb{B}\mu^{\mathbb{A}}"] \arrow[dr, "\mathbb{B}f"'] & & \mathbb{B}\mathbb{A}\mathcal{X} \arrow[ll, shift left, "t=\mathbb{B}\mathbb{A}\eta"] \arrow[r, "\mathbb{B}f"] & \mathbb{B}\mathbb{B}\mathcal{X} \arrow[d, "\mu^{\mathbb{B}}"] \\
& \mathbb{B}\mathbb{B}\mathbb{A}\mathcal{X} \arrow[ur, "\mu^{\mathbb{B}}"'] & & \mathbb{B}\mathcal{X} \arrow[ul, "s=\mathbb{B}\eta"]
\end{tikzcd}$$

The following is a part of the theory of modules in general categories, not necessarily linear.

A.6. Proposition. *So defined functor* $(\mathcal{A}_i)_{i\in[n]}$-mod $\to (\mathcal{B}_i)_{i\in[n]}$-mod, $\mathcal{P} \mapsto \bigcirc_{i=0}^{n} \mathcal{B}_i \odot^i_{\mathcal{A}_i} \mathcal{P}$, *is left adjoint to the base change functor* $(\mathcal{B}_i)_{i\in[n]}$-mod $\to (\mathcal{A}_i)_{i\in[n]}$-mod, $\mathcal{R} \mapsto {}_{f_1,\dots,f_n}\mathcal{R}_{f_0}$. *Thus, the mapping*

$$(\mathcal{B}_i)_{i\in[n]}\text{-mod}(\bigcirc_{i=0}^{n} \mathcal{B}_i \odot^i_{\mathcal{A}_i} \mathcal{P}, \mathcal{R}) \xrightarrow{\sim} (\mathcal{A}_i)_{i\in[n]}\text{-mod}(\mathcal{P}, {}_{f_1,\dots,f_n}\mathcal{R}_{f_0}), \qquad h \mapsto \eta \cdot h,$$

is a bijection, where $\eta : \mathcal{P} \to \bigcirc_{i=0}^{n} \mathcal{B}_i \odot^i_{\mathcal{A}_i} \mathcal{P}$ *comes from the unit of the monad* $\odot_{\geqslant 0}((\mathcal{B}_i); -)$.

PROOF. Since $\mathcal{Q} = \bigodot_{i=0}^{n} \mathcal{B}_i \odot^{i}_{\mathcal{A}_i} \mathcal{P}$ is a colimit (a coequalizer), the following set of morphisms is a limit (an equalizer of a pair of maps):

$$
\begin{aligned}
&(\mathcal{B}_i)_{i\in[n]}\text{-mod}(\mathcal{Q},\mathcal{R}) \\
&= \lim \left[
\begin{tikzcd}
(\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((\mathcal{B}_i);\mathcal{P}),\mathcal{R}) \arrow[r, "{(\odot_{\geqslant 0}((1);\alpha),1)}"] \arrow[d, "{(\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((\mu_i);1),1)}"'] & (\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((\mathcal{B}_i);\odot_{\geqslant 0}((\mathcal{A}_i);\mathcal{P})),\mathcal{R}) \\
(\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((\mathcal{B}_i\odot\mathcal{B}_i);\mathcal{P}),\mathcal{R}) \arrow[r, "\sim"] & (\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((\mathcal{B}_i);\odot_{\geqslant 0}((\mathcal{B}_i);\mathcal{P})),\mathcal{R}) \arrow[u, "{(\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((1);\odot_{\geqslant 0}((f_i);1)),1)}"']
\end{tikzcd}
\right] \\
&= \lim \left[
\begin{tikzcd}
\mathbf{dg}^{\mathbb{N}^n}(\mathcal{P},\mathcal{R}) \arrow[r, "{\mathbf{dg}^{\mathbb{N}^n}(\alpha,1)}"] \arrow[d, "{\odot_{\geqslant 0}((\mathcal{B}_i);-)}"'] & \mathbf{dg}^{\mathbb{N}^n}(\odot_{\geqslant 0}((\mathcal{A}_i);\mathcal{P}),\mathcal{R}) \\
\mathbf{dg}^{\mathbb{N}^n}(\odot_{\geqslant 0}((\mathcal{B}_i);\mathcal{P}),\odot_{\geqslant 0}((\mathcal{B}_i);\mathcal{R})) \arrow[r, "{\mathbf{dg}^{\mathbb{N}^n}(1,\alpha)}"] & \mathbf{dg}^{\mathbb{N}^n}(\odot_{\geqslant 0}((\mathcal{B}_i);\mathcal{P}),\mathcal{R}) \arrow[u, "{\mathbf{dg}^{\mathbb{N}^n}(\odot_{\geqslant 0}((f_i);1),1)}"']
\end{tikzcd}
\right].
\end{aligned}
$$

Thus, a morphism $u : \mathcal{P} \to \mathcal{R} \in \mathbf{dg}^{\mathbb{N}^n}$ belongs to the equalizer iff

$$
\begin{aligned}
&\big[\odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]};\mathcal{P}) \xrightarrow{\odot_{\geqslant 0}((f_i);1)} \odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]};\mathcal{P}) \xrightarrow{\odot_{\geqslant 0}((1);u)} \odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]};\mathcal{R}) \xrightarrow{\alpha} \mathcal{R}\big] \\
&\qquad = \big[\odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]};\mathcal{P}) \xrightarrow{\alpha} \mathcal{P} \xrightarrow{u} \mathcal{R}\big].
\end{aligned}
$$

Equivalently, the mapping $u : \mathcal{P} \to {}_{f_1,\dots,f_n}\mathcal{R}_{f_0}$ is a homomorphism of $(\mathcal{A}_i)_{i\in[n]}$-modules. ■

Together with Lemma A.9 this proposition implies the following

A.7. COROLLARY. *In assumptions of Section A.5 denote* $\mathcal{Q} = \bigodot_{i=0}^{n} \mathcal{B}_i \odot^{i}_{\mathcal{A}_i} \mathcal{P}$. *Then the diagram in the category* ${}_n\mathrm{Op}_1$

$$
\begin{tikzcd}
(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{A}_0(0);\mathcal{A}_0) \arrow[r, "{(1,\dots,1;\rho_\varnothing;1)}"] \arrow[d, "{(f_1,\dots,f_n;f_0(0);f_0)}"'] & (\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{A}_0) \arrow[d, "{(f_1,\dots,f_n;\eta;f_0)}"] \\
(\mathcal{B}_1,\dots,\mathcal{B}_n;\mathcal{B}_0(0);\mathcal{B}_0) \arrow[r, "{(1,\dots,1;\rho_\varnothing;1)}"] \arrow[ur, phantom, "\ulcorner", very near end] & (\mathcal{B}_1,\dots,\mathcal{B}_n;\mathcal{Q};\mathcal{B}_0)
\end{tikzcd}
$$

is a pushout square.

A.8. SOME COLIMITS OF OPERAD MODULES.

A.9. LEMMA. *Let* $f_i : \mathcal{A}_i \to \mathcal{B}_i$ *be morphisms of operads for* $i \in [n]$. *Let* $\mathcal{P}$ *be a* $(\mathcal{B}_i)_{i\in[n]}$-*module. Then there is a unique morphism* ϕ *such that*

$$((f_i)_{i\in[n]};\phi) : ((\mathcal{A}_i)_{i\in[n]};\mathcal{A}_0(0)) \to ((\mathcal{B}_i)_{i\in[n]};\mathcal{P}) \in {}_n\mathrm{Op}_1\,.$$

Proof. The morphism ϕ is recovered from the equation

$$\big(\mathcal{A}_0(0) \xrightarrow[1]{\rho_\varnothing^{\mathcal{A}_0(0)}} \mathcal{A}_0(0) \xrightarrow{\phi} \mathcal{P}(0)\big) = \big(\mathcal{A}_0(0) \xrightarrow{f_0(0)} \mathcal{B}_0(0) \xrightarrow{\rho_\varnothing^{\mathcal{P}}} \mathcal{P}(0)\big)$$

in the unique possible form $\phi = f_0(0) \cdot \rho_\varnothing^{\mathcal{P}}$. It is compatible with the action ρ because the diagram

$$\begin{tikzcd}
\mathcal{A}_0(0)^{\otimes m} \otimes \mathcal{A}_0(m) \arrow[r, "f_0(0)^{\otimes m} \otimes f_0(m)"] \arrow[d, "\mu^{\mathcal{A}_0}"'] \arrow[dr, phantom, "="] & \mathcal{B}_0(0)^{\otimes m} \otimes \mathcal{B}_0(m) \arrow[r, "\rho_\varnothing^{\otimes m} \otimes 1"] \arrow[d, "\mu^{\mathcal{B}_0}"'] \arrow[dr, phantom, "="] & \mathcal{P}_0(0)^{\otimes m} \otimes \mathcal{B}_0(m) \arrow[d, "\rho_{m0}^{m}"] \\
\mathcal{A}_0(0) \arrow[r, "f_0(0)"] & \mathcal{B}_0(0) \arrow[r, "\rho_\varnothing"] & \mathcal{P}_0(0)
\end{tikzcd}$$

commutes. For the second square this follows by associativity of the action. ■

A.10. Corollary. *For arbitrary operads $\mathcal{C}_i$, $\mathcal{A}_i$, $i \in [n]$, there is an isomorphism in ${}_n\mathrm{Op}_1$*

$$((\mathcal{C}_i)_{i\in[n]}; \mathcal{C}_0(0)) \sqcup ((\mathcal{A}_i)_{i\in[n]}; \mathcal{A}_0(0)) \simeq ((\mathcal{C}_i \sqcup \mathcal{A}_i)_{i\in[n]}; (\mathcal{C}_0 \sqcup \mathcal{A}_0)(0)).$$

References

[BLM08] Yu. Bespalov, V. V. Lyubashenko, and O. Manzyuk, *Pretriangulated A_∞-categories*, Proc. Inst. Math. NAS of Ukraine. Math. and its Appl., vol. 76, Inst. of Math. of NAS of Ukraine, Kyiv, 2008, http://www.math.ksu.edu/~lub/papers.html.

[BW05] M. Barr and C. Wells, *Toposes, triples and theories*, Theory Appl. Categ. **278** (1985, 2005), no. 12, x+288 pp. (electronic), http://www.tac.mta.ca/tac/reprints/articles/12/tr12abs.html Corrected reprint of the 1985 original, Grundlehren der mathematischen Wissenschaften.

[HPS05] K. P. Hess, P.-E. Parent, and J. A. Scott, *Co-rings over operads characterize morphisms*, 2005, arXiv:math/0505559.

[Kel06] B. Keller, *A-infinity algebras, modules and functor categories*, Trends in representation theory of algebras and related topics, Contemp. Math., vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 67–93.

[Lac10] S. Lack, *A 2-categories companion*, Towards higher categories, IMA Vol. Math. Appl., vol. 152, Springer, New York, 2010, pp. 105–191.

[Lei03] T. Leinster, *Higher operads, higher categories*, London Math. Soc. Lect. Notes Series, Cambridge Univ. Press, Boston, Basel, Berlin, 2003.

[LH03] K. Lefèvre-Hasegawa, *Sur les A_∞-catégories*, Ph.D. thesis, Université Paris 7, U.F.R. de Mathématiques, 2003, arXiv:math.CT/0310337.

[Lyu11] V. V. Lyubashenko, *Homotopy unital A_∞-algebras*, J. Algebra **329** (2011), no. 1, 190–212, Special Issue Celebrating the 60th Birthday of Corrado De Concini.

[Lyu14] V. V. Lyubashenko, *Homotopy cooperads*, 2014.

[Mac88] S. Mac Lane, *Categories for the working mathematician*, GTM, vol. 5, Springer-Verlag, New York, 1971, 1988.

[Mar00] M. Markl, *Homotopy algebras via resolutions of operads*, Rend. Circ. Mat. Palermo (2) Suppl. (2000), no. 63, 157–164, The Proc. of the 19th Winter School "Geometry and Physics" (Srní, 1999).

[Sta63] J. D. Stasheff, *Homotopy associativity of H-spaces I & II*, Trans. Amer. Math. Soc. **108** (1963), 275–292, 293–312.

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