*Theory and Applications of Categories,* Vol. 30, No. 30, 2015, pp. 1017–1031.

# FINITE CATEGORIES WITH PUSHOUTS

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ABSTRACT. Let $C$ be a finite category. For an object $X$ of $C$ one has the hom-functor $\mathrm{Hom}(-, X)$ of $C$ to **Set**. If $G$ is a subgroup of $\mathrm{Aut}(X)$, one has the quotient functor $\mathrm{Hom}(-, X)/G$. We show that any finite product of hom-functors of $C$ is a sum of hom-functors if and only if $C$ has pushouts and coequalizers and that any finite product of hom-functors of $C$ is a sum of functors of the form $\mathrm{Hom}(-, X)/G$ if and only if $C$ has pushouts. These are variations of the fact that a finite category has products if and only if it has coproducts.

## 1. Introduction

It is well-known that in a partially ordered set the infimum of an arbitrary subset exists if and only if the supremum of an arbitrary subset exists. A categorical generalization of this is also known. When a partially ordered set is viewed as a category, infimum and supremum are respectively product and coproduct, which are instances of limits and colimits. A general theorem states that a category has small limits if and only if it has small colimits under certain smallness conditions ([Freyd and Scedrov, 1.837]).

We seek an equivalence of this sort for finite categories. As finite categories having products are just partially ordered sets, we ought to replace the existence of product by some weaker condition.

Let $C$ be a category. For an object $X$ of $C$, $h_X$ denotes the hom-functor $\mathrm{Hom}(-, X)$ of $C$ to the category **Set**. A functor $C^{\mathrm{op}} \to \mathbf{Set}$ is said to be representable if it is isomorphic to $h_X$ for some $X$. A functor is said to be familially representable if it is isomorphic to a sum of hom-functors $h_X$ ([Carboni and Johnstone]). For a subgroup $G$ of $\mathrm{Aut}(X)$, $h_X/G$ denotes the quotient of $h_X$ by the induced action of $G$. We say a functor $C^{\mathrm{op}} \to \mathbf{Set}$ is *nearly representable* if it is isomorphic to $h_X/G$ for some $X$ and $G$.

Existence of product of objects $X$ and $Y$ of $C$ is phrased as representability of $h_X \times h_Y$. Using familial representability and near representability, we obtain the following results for a finite category $C$. Firstly any finite product of hom-functors of $C$ is familially representable if and only if $C$ has pushouts and coequalizers; secondly any finite product of hom-functors is a sum of nearly representable functors if and only if $C$ has pushouts.

In Sections 2 and 3 basic properties of nearly representable functors and their relations to weak pushouts are discussed. In Section 4 we prove a result about connected

Received by the editors 2015-06-26 and, in revised form, 2015-07-18.

Transmitted by Ross Street. Published on 2015-07-27.

2010 Mathematics Subject Classification: 18A30, 18A35.

Key words and phrases: pushout, coequalizer, hom-functor, familially representable functor, nearly representable functor.

components of powers of a set-valued functor. In Section 5 we prove the above-mentioned results. In Section 6 we give a construction of a finite category having pushouts and coequalizers by using powers of a functor. In Section 7 we give an example of a finite category having pushouts based on a partially ordered set with group action.

## 2. Nearly representable functors

Let $C$ be a category. The category of sets is denoted by **Set**. The category of functors $C^{\mathrm{op}} \to \mathbf{Set}$ is denoted by $[C^{\mathrm{op}}, \mathbf{Set}]$. Limits and colimits in this category are given pointwise. For instance, the product $F \times F'$ of $F, F' \in [C^{\mathrm{op}}, \mathbf{Set}]$ is given by $(F \times F')(X) = F(X) \times F'(X)$. A final object of $[C^{\mathrm{op}}, \mathbf{Set}]$ is the functor $\mathbf{1}$ given by $\mathbf{1}(X) = \{1\}$ for all $X \in C$. The sum $F \coprod F'$ of $F$ and $F'$ is given by $(F \coprod F')(X) = F(X) \coprod F'(X)$, and an initial object is the functor $\boldsymbol{\emptyset}$ given by $\boldsymbol{\emptyset}(X) = \emptyset$.

If a group $G$ acts on a set $E$, $E/G$ denotes the quotient set of $E$ under the action. If $G$ acts on a functor $F \in [C^{\mathrm{op}}, \mathbf{Set}]$, $F/G$ denotes the quotient functor of $F$: $(F/G)(X) = F(X)/G$ for $X \in C$. For an object $X$ of $C$, $h_X$ denotes the hom-functor $\mathrm{Hom}(-, X)$ on $C$. If $G$ acts on an object $X$ of $C$, then $G$ acts on $h_X$ so we have the quotient $h_X/G$. We call a functor isomorphic to $h_X/G$ a *nearly representable functor*. For a functor $F\colon C^{\mathrm{op}} \to \mathbf{Set}$ and an object $Z$ of $C$, an element $c \in F(Z)$ corresponds to a morphism $\gamma\colon h_Z \to F$ by Yoneda's Lemma. If a subgroup $G$ of $\mathrm{Aut}(Z)$ leaves $c$ invariant, then $\gamma$ induces a morphism $h_Z/G \to F$. Thus a $G$-invariant element in $F(Z)$ bijectively corresponds to a morphism $h_Z/G \to F$.

2.1. Proposition. *Let $G$ be a subgroup of* $\mathrm{Aut}(X)$. *Let $N_{\mathrm{Aut}(X)}(G)$ denote the normalizer of $G$ in* $\mathrm{Aut}(X)$. *Then we have an isomorphism of groups*

$$\mathrm{Aut}(h_X/G) \cong N_{\mathrm{Aut}(X)}(G)/G.$$

Proof. The action of $N_{\mathrm{Aut}(X)}(G)$ on $h_X$ passes to the action on $h_X/G$. This yields the homomorphism $N_{\mathrm{Aut}(X)}(G) \to \mathrm{Aut}(h_X/G)$, which we call $\theta$. One sees $\mathrm{Ker}\,\theta = G$. To show the surjectivity of $\theta$, take any automorphism $\alpha\colon h_X/G \to h_X/G$. Then $\alpha$ lifts to a morphism $h_X \to h_X$, which comes from a morphism $\sigma\colon X \to X$. Similarly $\alpha^{-1}$ lifts to a morphism $h_X \to h_X$, which comes from some $\sigma'\colon X \to X$. Then $\sigma'\sigma$ induces the identity on $h_X/G$, so $\sigma'\sigma \in G$. Thus $\sigma$ is an automorphism. Also for any $\tau \in G$, $\sigma\tau\sigma^{-1}$ induces the identity on $h_X/G$, so $\sigma\tau\sigma^{-1} \in G$. Thus $\sigma$ normalizes $G$. As $\theta(\sigma) = \alpha$, we conclude that $\theta$ is surjective. Hence the desired isomorphism follows.

∎

From this we obtain the following.

2.2. Proposition. *Let $H$ be a subgroup of* $\mathrm{Aut}(h_X/G)$. *Let $\tilde{H}$ be the inverse image of $H$ under the natural map $N_{\mathrm{Aut}(X)}(G) \to$* $\mathrm{Aut}(h_X/G)$. *Then we have an isomorphism $(h_X/G)/H \cong h_X/\tilde{H}$.*

An object $F \in [C^{\mathrm{op}}, \mathbf{Set}]$ is said to be connected if $F \neq \emptyset$ and if $F \cong F_1 \amalg F_2$ implies $F_1 = \emptyset$ or $F_2 = \emptyset$. Representable functors are connected and so are images of them, especially nearly representable functors.

Let $F_1, \ldots, F_n$ be connected objects of $[C^{\mathrm{op}}, \mathbf{Set}]$ and put $F = F_1 \coprod \cdots \coprod F_n$. Let $G$ be a subgroup of $\mathrm{Aut}(F)$. Then for $\sigma \in G$ and $i \in \{1, \ldots, n\}$ there exist $\sigma(i) \in \{1, \ldots, n\}$ and $\sigma_i \colon F_i \to F_{\sigma(i)}$ such that $\sigma$ is the sum of $\sigma_i$ for $i = 1, \ldots, n$. The map $(\sigma, i) \mapsto \sigma(i)$ defines an action of $G$ on $\{1, \ldots, n\}$. Let $G_i = \{\sigma \in G \mid \sigma(i) = i\}$. The map $\sigma \mapsto \sigma_i$ gives a homomorphism $G_i \to \mathrm{Aut}(F_i)$. Take a representative $R$ of $G$-orbits in $\{1, \ldots, n\}$. Then

$$F/G \cong \coprod_{i \in R} F_i/G_i.$$

Using this fact and the preceding proposition, we have the following.

2.3. Proposition. *If $F$ is a finite sum of nearly representable functors and $G$ is a subgroup of $\mathrm{Aut}(F)$, then $F/G$ is a finite sum of nearly representable functors.*

We next review comma categories ([MacLane]). Let $F \colon C^{\mathrm{op}} \to \mathbf{Set}$ be a functor. One has the comma category $C/F$. An object of $C/F$ is a pair $(X, a)$ for $X \in C$ and $a \in F(X)$. A morphism $(X, a) \to (Y, b)$ of $C/F$ is a morphism $u \colon X \to Y$ of $C$ such that $F(u)(b) = a$. We note that $C/F$ is usually called the category of elements of $F$.

One has also the comma category $[C^{\mathrm{op}}, \mathbf{Set}]/F$. An object of $[C^{\mathrm{op}}, \mathbf{Set}]/F$ is a morphism $K \to F$ of $[C^{\mathrm{op}}, \mathbf{Set}]$. A morphism $(K \to F) \to (K' \to F)$ of $[C^{\mathrm{op}}, \mathbf{Set}]/F$ is a morphism $K \to K'$ of $[C^{\mathrm{op}}, \mathbf{Set}]$ making the triangle commutative.

As is well-known, $[C^{\mathrm{op}}, \mathbf{Set}]/F$ is equivalent to $[(C/F)^{\mathrm{op}}, \mathbf{Set}]$. An equivalence $\Phi \colon [C^{\mathrm{op}}, \mathbf{Set}]/F \to [(C/F)^{\mathrm{op}}, \mathbf{Set}]$ is given as follows. Let $u \colon K \to F$ be an object of $[C^{\mathrm{op}}, \mathbf{Set}]/F$. Then $\Phi(u) \colon (C/F)^{\mathrm{op}} \to \mathbf{Set}$ is defined by

$$\Phi(u)(X, a) = u(X)^{-1}(a) \quad \text{for } (X, a) \in C/F.$$

We note that $u$ is an isomorphism in $[C^{\mathrm{op}}, \mathbf{Set}]$ if and only if the unique morphism $\Phi(u) \to \mathbf{1}$ of $[(C/F)^{\mathrm{op}}, \mathbf{Set}]$ is an isomorphism.

Let an element $c \in F(Z)$ correspond to a morphism $\gamma \colon h_Z \to F$. Then we have

$$\Phi(\gamma) = h_{(Z,c)}$$

in $[(C/F)^{\mathrm{op}}, \mathbf{Set}]$. Furthermore let a subgroup $G$ of $\mathrm{Aut}(Z)$ leave $c$ invariant. Then $\gamma \colon h_Z \to F$ induces $\tilde{\gamma} \colon h_Z/G \to F$, and also $G \subset \mathrm{Aut}(Z, c)$. Then we have

$$\Phi(\tilde{\gamma}) = h_{(Z,c)}/G.$$

2.4. Proposition. *Let $F \colon C^{\mathrm{op}} \to \mathbf{Set}$ be a functor. Then $F$ is a sum of nearly representable functors if and only if $\mathbf{1} \colon (C/F)^{\mathrm{op}} \to \mathbf{Set}$ is a sum of nearly representable functors.*

Proof. Let $Z_i$ be objects of $C$ and $c_i \in F(Z_i)$. Let $G_i$ be a subgroup of $\mathrm{Aut}(Z_i)$ leaving $c_i$ invariant. Then $c_i$ induces $\tilde{\gamma}_i\colon h_{Z_i}/G_i \to F$. These morphisms sum to a morphism $u\colon \coprod_i h_{Z_i}/G_i \to F$. Then

$$\Phi(u) = \coprod_i h_{(Z_i,c_i)}/G_i.$$

Therefore $u$ is an isomorphism if and only if the unique morphism $\coprod_i h_{(Z_i,c_i)}/G_i \to \mathbf{1}$ is an isomorphism. This proves the proposition. ∎

For a full subcategory $D$ of $C$, we consider the following conditions.
(F0) If $X \in C$, there exists a morphism $X \to Y$ with $Y \in D$.
(F1) If $f_1\colon X \to Y_1$ and $f_2\colon X \to Y_2$ are morphisms with $Y_1, Y_2 \in D$, then there exists a morphism $g\colon Y_1 \to Y_2$ such that $f_2 = gf_1$.
(F2) If $f_1\colon X \to Y$ and $f_2\colon X \to Y$ are morphisms with $Y \in D$, then $f_1 = f_2$.

The conjunction of (F0) and (F1) is a stronger condition than the *finality* of $D$ in $C$ ([MacLane]).

2.5. Proposition. *(F1) implies that $D$ is a groupoid.*

Proof. Let $f\colon Y_1 \to Y_2$ be a morphism with $Y_1, Y_2 \in D$. By (F1) applied to $f\colon Y_1 \to Y_2$ and $1_{Y_1}\colon Y_1 \to Y_1$, we have $g\colon Y_2 \to Y_1$ such that $1_{Y_1} = gf$. Thus every morphism of $D$ has a left inverse. It then follows that every morphism of $D$ is an isomorphism. ∎

We note that under the assumption of (F1), $D$ satisfies (F2) if and only if $\mathrm{Aut}(Y) = 1$ for all $Y \in D$.

2.6. Proposition. *Let $C$ be a category and $D$ a subcategory satisfying (F0) and (F1). Let $\{Z_i\}$ be a representative system of isomorphism classes of objects of $D$. Then*

$$\mathbf{1} \cong \coprod_i h_{Z_i}/\mathrm{Aut}(Z_i)$$

*in $[C^{\mathrm{op}}, \mathbf{Set}]$.*

Proof. We could use a well-known theorem on a final subcategory ([MacLane, p.217]), but we argue directly. It is enough to show that for every $X \in C$ there exists a unique $i$ such that $\#\mathrm{Hom}(X, Z_i)/\mathrm{Aut}(Z_i) = 1$ and $\mathrm{Hom}(X, Z_j) = \emptyset$ for all $j \neq i$. Let $X \in C$. Using (F0), we take $i$ such that $\mathrm{Hom}(X, Z_i) \neq \emptyset$. By (F1) $\mathrm{Aut}(Z_i)$ acts on $\mathrm{Hom}(X, Z_i)$ transitively. So $\mathrm{Hom}(X, Z_i)/\mathrm{Aut}(Z_i)$ is a one-element set. Let $\mathrm{Hom}(X, Z_j) \neq \emptyset$. Then by (F1) $Z_i \cong Z_j$, so $j = i$. The uniqueness of $i$ is clear. ∎

2.7. Proposition. *Let $\{Z_i\}$ be a family of objects of $C$. Let $G_i$ be a subgroup of* $\mathrm{Aut}(Z_i)$. *Assume we have an isomorphism*

$$\mathbf{1} \cong \coprod_i h_{Z_i}/G_i$$

*in $[C^{\mathrm{op}}, \mathbf{Set}]$. Then $\{Z_i\}$ satisfies (F0) and (F1).*

PROOF. For any $X \in C$ there exists a unique $i$ such that $\#\mathrm{Hom}(X, Z_i)/\mathrm{Aut}(Z_i) = 1$ and $\mathrm{Hom}(X, Z_j) = \emptyset$ for all $j \neq i$. (F0) surely holds. To see (F1), let $f_1: X \to Z_{i_1}$, $f_2: X \to Z_{i_2}$ be morphisms. Then $i_1 = i_2$. Moreover there exists $g \in G_{i_1}$ with $f_2 = gf_1$. Hence (F1) holds. ∎

2.8. PROPOSITION. *Let $F: C^{\mathrm{op}} \to \mathbf{Set}$ be a functor. The category $C/F$ has a subcategory satisfying (F0) and (F1) if and only if $F$ is a sum of nearly representable functors.*

PROOF. This follows from Propositions 2.6, 2.7, and 2.4. ∎

Propositions 2.6, 2.7 and 2.8 specialize to the following well-known facts.

2.9. PROPOSITION. *Let $C$ be a category and $D$ a subcategory satisfying (F0), (F1) and (F2). Let $\{Z_i\}$ be a representative system of isomorphism classes of objects of $D$. Then*

$$\mathbf{1} \cong \coprod_i h_{Z_i}$$

*in $[C^{\mathrm{op}}, \mathbf{Set}]$.*

2.10. PROPOSITION. *Let $\{Z_i\}$ be a family of objects of $C$. Assume we have an isomorphism*

$$\mathbf{1} \cong \coprod_i h_{Z_i}$$

*in $[C^{\mathrm{op}}, \mathbf{Set}]$. Then $\{Z_i\}$ satisfies (F0), (F1) and (F2).*

2.11. PROPOSITION. *Let $F: C^{\mathrm{op}} \to \mathbf{Set}$ be a functor. The category $C/F$ has a subcategory satisfying (F0), (F1) and (F2) if and only if $F$ is familially representable.*

## 3. Weak colimits

Recall that a category $C$ is said to be *pseudo-filtered* if the following conditions hold.
(P1) For every pair of morphisms $f: X \to Y$ and $g: X \to Z$, there exist morphisms $h: Y \to W$ and $k: Z \to W$ such that $hf = kg$.
(P2) For every pair of morphisms $f: X \to Y$ and $g: X \to Y$, there exists a morphism $h: Y \to Z$ such that $hf = hg$.

We shall consider these conditions separately.

3.1. PROPOSITION. *If $C$ has a subcategory satisfying (F0) and (F1), then $C$ satisfies (P1).*

PROOF. Let $D$ be a subcategory satisfying (F0) and (F1). Let $f: X \to Y$, $g: X \to Z$ be morphisms of $C$. Using (F0), we take morphisms $l: Y \to U$, $m: Z \to V$ with $U, V \in D$. Using (F1) for $lf: X \to U$, $mg: X \to V$, we take $n: U \to V$ such that $mg = nlf$. Thus we obtain the morphisms $nl: Y \to V$ and $m: Z \to V$, which satisfy $(nl)f = mg$. This proves (P1).

∎

3.2. Proposition. *If $C$ has a subcategory satisfying (F0), (F1) and (F2), then $C$ satisfies (P1) and (P2).*

Proof. Let $D$ be a subcategory satisfying (F0), (F1) and (F2). We shall verify (P2). Let $f\colon X \to Y$ and $g\colon X \to Y$ be morphisms of $C$. Take a morphism $l\colon Y \to U$ with $U \in D$. Then $lf$ and $lg$ are morphisms $X \to U$, which must coincide by (F2). This proves (P2). ■

3.3. Proposition. *Let $C$ be a finite category. Assume that all morphisms of $C$ are epimorphisms and $C$ satisfies (P1). Then $C$ has a subcategory satisfying (F0) and (F1).*

Proof. Let $D$ be the set of objects $X$ of $C$ such that every morphism $X \to Y$ is an isomorphism. We shall show that $D$ satisfies (F0) and (F1). Since $C$ is finite and all morphisms are epimorphism, $C$ cannot have an infinite sequence of non-isomorphisms $X_1 \to X_2 \to \cdots$. Therefore $D$ satisfies (F0).

Let $f_1\colon X \to Y_1$, $f_2\colon X \to Y_2$ with $Y_1, Y_2 \in D$. As $C$ satisfies (P1), there exist $g_1\colon Y_1 \to Z$, $g_2\colon Y_2 \to Z$ such that $g_1 f_1 = g_2 f_2$. As $Y_2 \in D$, $g_2$ is invertible so we have $f_2 = g_2^{-1} g_1 f_1$. Thus (F1) holds. ■

3.4. Proposition. *Let $C$ be a finite category. Assume that all morphisms of $C$ are epimorphisms and $C$ satisfies (P1) and (P2). Then $C$ has a subcategory satisfying (F0), (F1) and (F2).*

Proof. Let $D$ be as in the preceding proof. Let $f_1\colon X \to Y$ and $f_2\colon X \to Y$ be morphisms with $Y \in D$. By (P2) we take a morphism $h\colon Y \to Z$ such that $hf_1 = hf_2$. Take a morphism $k\colon Z \to V$ with $V \in D$. As $kh\colon Y \to V$ is an isomorphism by Proposition 2.5, we know $f_1 = f_2$. Thus $D$ satisfies (F2). ■

By this proposition and Proposition 2.11 we know that under the assumption that all morphisms of $C$ are epimorphisms, a functor $F\colon C^{\mathrm{op}} \to \mathbf{Set}$ is familially representable if and only if $C/F$ satisfies (P1) and (P2). In fact this holds under a weaker assumption that all idempotent morphisms of $C$ split, as stated in [Leinster].

We next review weak colimits ([Freyd and Scedrov]).

An object $A$ of a category $C$ is called a *weak initial object* if $\mathrm{Hom}(A, X)$ is not empty for all $X \in C$.

A commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "k"'] & W \end{tikzcd}$$

in a category $C$ is called a *weak pushout* if for every pair of morphisms $h'\colon Y \to W'$ and $k'\colon Z \to W'$ satisfying $h'f = k'g$, there exists a morphism $u\colon W \to W'$ such that $h' = uh$, $k' = uk$. A category $C$ is said to have weak pushouts if every diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ Z & \end{tikzcd}$$

in $C$ can be completed into a weak pushout diagram. If $C$ has weak pushouts, then $C$ satisfies (P1).

A commutative diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z$$

is called a *weak coequalizer* if for every morphism $h': Y \to Z'$ satisfying $h'f = h'g$, there exists a morphism $u: Z \to Z'$ such that $h' = uh$. A category $C$ is said to have weak coequalizers if every diagram

$$X \rightrightarrows Y$$

in $C$ can be completed into a weak coequalizer. If $C$ has weak coequalizers, then $C$ satisfies (P2).

3.5. PROPOSITION. *Let $C$ be a finite category. Assume*

*(1) $C \neq \emptyset$.*

*(2) For every pair of objects $A$ and $A'$ there exists an object $B$ such that* $\mathrm{Hom}(B, A) \neq \emptyset$ *and* $\mathrm{Hom}(B, A') \neq \emptyset$.

*Then $C$ has a weak initial object.*

PROOF. Let $A_1, \dots, A_n$ be all objects of $C$. By (1) $n \geq 1$. Using (2) repeatedly, we find an object $B$ such that $\mathrm{Hom}(B, A_i) \neq \emptyset$ for all $i$. Then $B$ is a weak initial object. ∎

3.6. PROPOSITION. *Let $C$ be a finite category. Assume*

*(1) $C$ satisfies (P1).*

*(2) For any $X, Y \in C$, $C/h_X \times h_Y$ satisfies (P1).*

*Then $C$ has weak pushouts.*

PROOF. Let $f: X \to Y$, $g: X \to Z$ be morphisms. We write $h'_X = \mathrm{Hom}(X, -)$, the covariant hom-functor. Let $K$ be the comma category of the functor $h'_Y \times_{h'_X} h'_Z$ on $C$. An object of $K$ is a triple $(V, l, m)$, where $l: Y \to V$ and $m: Z \to V$ are morphisms of $C$ satisfying $lf = mg$. A morphism $(V, l, m) \to (V', l', m')$ of $K$ is a morphism $s: V \to V'$ of $C$ satisfying $l' = sl$, $m' = sm$. Then, for any $h: Y \to W$ and $k: Z \to W$, the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "k"'] & W \end{tikzcd}$$

is a weak pushout in $C$ if and only if $(W, h, k)$ is a weak initial object of $K$.

It is enough to show that $K$ has a weak initial object. We verify that $K$ satisfies the assumption of Proposition 3.5.

First of all $K$ is finite. As $C$ satisfies (P1), $K$ is not empty.

Let $(V, l, m), (V', l', m') \in K$. Consider $F = h_V \times h_{V'}$ in $[C^{\mathrm{op}}, \mathbf{Set}]$. Then

$$(l, l') \in F(Y),\ (m, m') \in F(Z).$$

Put $b = (l, l'), c = (m, m')$. Then

$$f^*(b) = (lf, l'f),\ g^*(c) = (mg, m'g).$$

The right-hand sides are the same element, which we denote by $a$. Then $a \in F(X)$ and we have morphisms in $C/F$:

$$f\colon (X, a) \to (Y, b),\ g\colon (X, a) \to (Z, c).$$

As $C/F$ satisfies (P1), we take a commutative diagram

$$\begin{tikzcd}
(X,a) \arrow[r] \arrow[d] & (Y,b) \arrow[d, "l_1"] \\
(Z,c) \arrow[r, "m_1"'] & (V_1, d_1).
\end{tikzcd}$$

Then

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "l_1"] \\
Z \arrow[r, "m_1"'] & V_1
\end{tikzcd}$$

is commutative, so $(V_1, l_1, m_1) \in K$.

As $d_1 \in F(V_1)$, we write

$$d_1 = (s, s') \text{ with } s\colon V_1 \to V,\ s'\colon V_1 \to V'.$$

As $l_1^*(d_1) = b$, we have

$$sl_1 = l,\ s'l_1 = l',$$

and as $m_1^*(d_1) = c$, we have

$$sm_1 = m,\ s'm_1 = m'.$$

Thus

$$s \in \mathrm{Hom}((V_1, l_1, m_1), (V, l, m)),\ s' \in \mathrm{Hom}((V_1, l_1, m_1), (V', l', m')).$$

By Proposition 3.5 $K$ has a weak initial object.

∎

3.7. Proposition. *Let $C$ be a finite category. Assume*

*(1) $C$ satisfies (P2).*

*(2) For any $X, Y \in C$, $C/h_X \times h_Y$ satisfies (P2).*

*Then $C$ has weak coequalizers.*

This is proved similarly to the preceding proposition.

The following are well-known.

3.8. Proposition. *Let $C$ be a category and $F\colon C^{\mathrm{op}} \to \mathbf{Set}$ a functor. Assume that $C$ has pushouts. Then the following are equivalent.*

*(i) $C/F$ has pushouts.*

*(ii) $F$ turns pushouts into pullbacks.*

3.9. Proposition. *Let $C$ be a category and $F\colon C^{\mathrm{op}} \to \mathbf{Set}$ a functor. Assume that $C$ has coequalizers. Then the following are equivalent.*

*(i) $C/F$ has coequalizers.*

*(ii) $F$ turns coequalizers into equalizers.*

The following is known as well.

3.10. Proposition. *Let $C$ be a finite category. Assume that $C$ has coproducts. Then $C$ is a preordered set.*

Proof. Let $X, Y \in C$. Let $X_n$ be a coproduct of $n$ copies of $X$. The set of objects $X_n$ for all $n \geq 1$ is finite. Hence the set of integers $\#\mathrm{Hom}(X_n, Y)$ for all $n \geq 1$ is finite. But $\#\mathrm{Hom}(X_n, Y) = (\#\mathrm{Hom}(X, Y))^n$. This forces that $\#\mathrm{Hom}(X, Y) = 1$ or $0$. Thus $C$ is a preordered set. ∎

3.11. Proposition. *Let $C$ be a finite category. Assume that $C$ has pushouts. Then every morphism of $C$ is an epimorphism.*

Proof. Let $A \in C$. The comma category $A\backslash C$, that is, the category of morphisms $A \to X$, has coproducts. By the preceding proposition $A\backslash C$ is a preordered set. This means that every morphism $A \to X$ is an epimorphism. ∎

## 4. Powers of functors

Let $K$ be a finite category. An object $F \in [K, \mathbf{Set}]$ is said to be connected if $F \neq \emptyset$ and if $F \cong F_1 \amalg F_2$ implies $F_1 = \emptyset$ or $F_2 = \emptyset$. If $F \in [K, \mathbf{Set}]$ has values in finite sets, $F$ is a finite sum of connected subobjects, which we call connected components of $F$.

Let $X \in [K, \mathbf{Set}]$ and assume $X(k)$ are finite for all $k \in K$.

4.1. Proposition. *If connected components of $X^n$ for all $n$ have only finitely many isomorphism classes, then $X(\alpha)$ is injective for every morphism $\alpha$ of $K$.*

Proof. Suppose that $X(\alpha)$ is not injective for a morphism $\alpha\colon i \to j$. Take $b \in X(j)$ such that $\#X(\alpha)^{-1}(b) > 1$. Put $p = \#X(\alpha)^{-1}(b)$. Put $b_n = (b, \ldots, b) \in X(j)^n$. Then $\#X^n(\alpha)^{-1}(b_n) = p^n$. Take a connected component $Y_n$ of $X^n$ such that $b_n \in Y_n(j)$. Then $\#Y_n(\alpha)^{-1}(b_n) = p^n$. Thus the integers $\#Y_n(i)$ for all $n \geq 1$ are unbound. It follows that the set of isomorphism classes of $Y_n$ is infinite. This proves the proposition.

∎

4.2. Proposition. *If $X(\alpha)$ is injective for every morphism $\alpha$ of $K$, then the connected components of $X^n$ for all $n$ have only finitely many isomorphism classes.*

Proof. For sets $A$ and $B$ write

$$\mathrm{Sur}(A,B)=\{A\to B \mid \text{surjection}\},\quad \mathrm{Inj}(A,B)=\{A\to B\mid \text{injection}\}.$$

Write $[m]=\{1,2,\dots,m\}$. For a set $S$ and $n\geq 0$ we have a natural decomposition

$$S^n \cong \coprod_{\alpha:[n]\to[m]} \mathrm{Inj}([m],S),$$

where $\alpha$ ranges over representatives of $\mathrm{Aut}([m])$-orbits in $\mathrm{Sur}([n],[m])$. An element $(\alpha,\beta)$ of the right-hand side with $\alpha\in\mathrm{Sur}([n],[m])$ and $\beta\in\mathrm{Inj}([m],S)$ corresponds to the map $\beta\alpha\colon[n]\to S$ regarded as an element of the left-hand side.

Now assume $X(\alpha)$ is injective for every morphism $\alpha$ of $K$. Define the set $\mathrm{Inj}([m],X)(k)$ for $k\in K$ by

$$\mathrm{Inj}([m],X)(k)=\mathrm{Inj}([m],X(k)).$$

For a morphism $\alpha\colon i\to j$ of $K$ the injection $X(\alpha)$ induces the map

$$\mathrm{Inj}([m],X(i))\to\mathrm{Inj}([m],X(j)).$$

Defining $\mathrm{Inj}([m],X)(\alpha)$ to be this map, we have an object $\mathrm{Inj}([m],X)$ of $[K,\mathbf{Set}]$.

The natural decomposition of $n$-th power of sets yields the decomposition

$$X^n\cong\coprod_{\alpha}\mathrm{Inj}([m],X)$$

in $[K,\mathbf{Set}]$.

If $m>\max\{\#X(k)\mid k\in K\}$, then $\mathrm{Inj}([m],X)=\boldsymbol{\emptyset}$. Hence connected components of $X^n$ for all $n$ are isomorphic to subobjects of $\mathrm{Inj}([m],X)$ for $m\leq\max\{\#X(k)\mid k\in K\}$. Consequently they have only finitely many isomorphism classes.

■

## 5. Proof of the theorems

5.1. Theorem. *Let $C$ be a finite category. The following are equivalent.*

*(1) The functors $\mathbf{1}$ and $h_X\times h_Y$ for all $X,Y\in C$ are sums of nearly representable functors .*

*(2) $C$ has pushouts.*

PROOF. Assume (1). Let $X, Y \in C$ and $G \subset \mathrm{Aut}(X)$, $H \subset \mathrm{Aut}(Y)$. Then

$$h_X/G \times h_Y/H \cong (h_X \times h_Y)/(G \times H).$$

As $h_X \times h_Y$ is a sum of nearly representable functors, so is $(h_X \times h_Y)/(G \times H)$ by Proposition 2.3. Thus we know that the product of two nearly representable functors is a sum of nearly representable functors.

It follows that for every $X \in C$ and positive integer $n$, $h_X^n$ is a sum of nearly representable functors; in other words, the connected components of $h_X^n$ are nearly representable. But $C$ being finite, nearly representable functors on $C$ have only finitely many isomorphism classes. Proposition 4.1 then tells us that $h_X(f)$ is injective for every morphism $f$ of $C$. Hence all morphisms of $C$ are epimorphisms.

As $\mathbf{1}$ is a sum of nearly representables, $C$ has a subcategory satisfying (F0) and (F1) by Proposition 2.7. Hence $C$ satisfies (P1) by Proposition 3.1.

Let $X, Y \in C$. As $h_X \times h_Y$ is a sum of nearly representables, $C/h_X \times h_Y$ has a subcategory satisfying (F0) and (F1) by Proposition 2.8. Hence $C/h_X \times h_Y$ satisfies (P1). By Proposition 3.6 it follows that $C$ has weak pushouts. All morphisms of $C$ being epimorphisms, weak pushout is true pushout. So $C$ has pushouts.

Conversely assume (2). By Proposition 3.11 all morphisms of $C$ are epimorphisms, and by Proposition 3.3 $C$ has a subcategory satisfying (F0) and (F1). Then $\mathbf{1}$ is a sum of nearly representable functors by Proposition 2.6.

Let $X, Y \in C$. As $h_X$ and $h_Y$ turn pushouts into pullbacks, so does $h_X \times h_Y$. By Proposition 3.8, $C/h_X \times h_Y$ has pushouts. Hence by Proposition 3.3 $C/h_X \times h_Y$ has a subcategory satisfying (F0) and (F1). Then $h_X \times h_Y$ is a sum of nearly representable functors by Proposition 2.8.

■

By the same argument as above, (2) implies that finite limits of hom-functors are sums of nearly representable functors. Also, by a result of [Paré], (2) implies that $C$ has finite simply connected colimits.

5.2. THEOREM. *Let $C$ be a finite category. The following are equivalent.*

*(1) The functors $\mathbf{1}$ and $h_X \times h_Y$ for all $X, Y \in C$ are familially representable.*

*(2) $C$ has pushouts and coequalizers.*

This is proved similarly to Theorem 5.1 by using Propositions 2.11 and 3.7 and 3.4 and 3.9.

We discuss the case where the condition for $\mathbf{1}$ of the theorem is deleted.

Any category $C$ can be enlarged to a category with final object. Define a category $D$ as follows. Objects of $D$ are objects of $C$ and a new object $\omega$. Morphisms of $D$ between objects of $C$ are the same as morphisms of $C$. There is a unique morphism from every object of $C$ to $\omega$, and there is no morphism from $\omega$ to objects of $C$. Thus $D$ is a category containing $C$ and $\omega$ is a final object of $D$. So $\mathbf{1} \in [D^{\mathrm{op}}, \mathbf{Set}]$ is representable.

Here we write a hom-functor on $C$ as $h_X^C$ and a hom-functor on $D$ as $h_X^D$. It is easily verified that for $X, Y \in C$, $h_X^C \times h_Y^C$ is familially representable if and only if $h_X^D \times h_Y^D$ is familially representable. Also it is verified that a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ Z & \end{tikzcd}$$

in $C$ can be completed to a pushout diagram in $D$ if and only if it can be completed to a pushout diagram in $C$ or never completed to a commutative square in $C$. A similar equivalence for coequalizer holds.

Using these facts and applying Theorem 5.2 to $D$, we obtain the following.

5.3. Theorem. *Let $C$ be a finite category. The following (1) and (2) are equivalent.*

*(1) $h_X \times h_Y$ for all $X, Y \in C$ are familially representable.*

*(2) (a) Every diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ Z & \end{tikzcd}$$

*in $C$ can be completed to a pushout diagram if it can be completed to a commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & V. \end{tikzcd}$$

*(b) Every diagram*

$$X \rightrightarrows Y$$

*in $C$ can be completed to a coequalizer diagram if it can be completed to a commutative diagram*

$$X \rightrightarrows Y \longrightarrow Z.$$

An example of categories satisfying (1) of this theorem is the orbit category of a fusion system for a finite group ([Diaz and Libman, Proposition 2.9]), which is treated also in [Oda] in connection to the generalized Burnside ring.

## 6. Construction of a finite category with pushouts and coequalizers

Let $K$ be a finite category. Let $X \in [K, \mathbf{Set}]$ be a functor valued in finite sets. Suppose that $X$ takes all morphisms of $K$ to injective maps. By Proposition 4.2 the connected components of $X^n$ for all $n$ have only finitely many isomorphism classes. Let $C$ be a representative system of isomorphism classes of those components.

For any objects $U, V$ of $C$, we have an isomorphism in $[K, \mathbf{Set}]$

$$U \times V \cong \coprod W_i$$

for some $W_i \in C$. Also

$$\mathbf{1} \cong \coprod Z_i$$

for some $Z_i \in C$. Regard $C$ as a full subcategory of $[K, \mathbf{Set}]$. Then

$$h_U \times h_V \cong \coprod h_{W_i}, \mathbf{1} \cong \coprod h_{Z_i}$$

in $[C^{\mathrm{op}}, \mathbf{Set}]$. Thus $C$ satisfies conditions (1) of Theorem 5.2 and so $C$ has pushouts and coequalizers.

Every category satisfying (1) of Theorem 5.2 arises this way. Indeed, suppose that $C$ satisfies the condition. The Yoneda functor $X \mapsto h_X$ embeds $C$ into $[C^{\mathrm{op}}, \mathbf{Set}]$. Since every morphism of $C$ is an epimorphism, $h_X$ takes every morphism of $C$ to an injective map. Let $M$ be the sum of $h_X$ for all $X \in C$. Then for every $n$ the power $M^n$ is a sum of $h_Y$ for $Y \in C$. Thus the full subcategory consisting of representatives of isomorphism classes of connected components of $M^n$ for all $n$ is equivalent to $C$.

## 7. Example of a category with pushouts

For finite categories with pushouts, we do not have a unified construction. We only give a special construction from a partially ordered set with pushouts and group action. A partially ordered set could be replaced by a small category, but we confine ourselves to the simpler case.

Let $P$ be a partially ordered set and $G$ a group. Suppose that $G$ acts on $P$, namely a map $P \times G \to P$ taking $(x, \sigma)$ to $x^\sigma$ is given so that

$$(x^\sigma)^\tau = x^{\sigma\tau}$$

and

$$x \le y \Rightarrow x^\sigma \le y^\sigma.$$

Suppose further that for each $x \in P$ a subgroup $K_x$ of $G$ is given so that the following conditions hold.

(i) $\sigma \in K_x \Rightarrow x^\sigma = x$.
(ii) $x \le y \Rightarrow K_x \subset K_y$.
(iii) $\sigma^{-1} K_x \sigma = K_{x^\sigma}$.

We define a category $C$ as follows. Objects of $C$ are elements of $P$. For $x, y \in P$ the hom-set $\mathrm{Hom}(x, y)$ of $C$ is given by

$$\mathrm{Hom}(x, y) = \{\sigma \mid \sigma \in G, x \le y^\sigma\}/K_y.$$

Owing to (i) the set $\{\sigma \mid \sigma \in G, x \leq y^\sigma\}$ is stable under the left multiplication of $K_y$, and the right-hand side is the quotient of this set under the action of $K_y$. We denote the class of $\sigma$ in $\mathrm{Hom}(x, y)$ by $[\sigma]$. The composition of $D$ is defined by

$$[\tau][\sigma] = [\tau\sigma].$$

7.1. Proposition. *$C$ is a category.*

Proof. We verify that the composition is well-defined. Let $x \leq y^\sigma$, $y \leq z^\tau$. Then $x \leq z^{\tau\sigma}$. Suppose $[\sigma] = [\sigma']$ in $\mathrm{Hom}(x, y)$ and $[\tau] = [\tau']$ in $\mathrm{Hom}(y, z)$. Then $\sigma' = \alpha\sigma$, $\tau' = \beta\tau$ for some $\alpha \in K_y$, $\beta \in K_z$. Then

$$\tau'\sigma' = \beta\tau\alpha\sigma = \beta\tau\alpha\tau^{-1}\tau\sigma.$$

By $K_y \subset K_{z^\tau} = \tau^{-1}K_z\tau$, we know $\alpha \in \tau^{-1}K_z\tau$, so $\tau\alpha\tau^{-1} \in K_z$. Hence $\beta\tau\alpha\tau^{-1} \in K_z$. Thus $[\tau'\sigma'] = [\tau\sigma]$ in $\mathrm{Hom}(x, z)$.

∎

7.2. Proposition. *If $P$ has pushouts, then so does $C$.*

Proof. Let a diagram

$$\begin{tikzcd} x \arrow[r, "{[\sigma]}"] \arrow[d, "{[\tau]}"'] & y \\ z & \end{tikzcd}$$

in $C$ be given. We must complete this into a pushout diagram. Since $[\sigma]\colon x \to y$ is the composite of $[1]\colon x \to y^\sigma$ and an isomorphism $[\sigma]\colon y^\sigma \to y$, we may assume $\sigma = 1$. Similarly we may assume $\tau = 1$. Then $x \leq y$, $x \leq z$ in $P$. Take a pushout

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ z \arrow[r] & w \end{tikzcd} \tag{1}$$

in $P$. We shall show the diagram

$$\begin{tikzcd} x \arrow[r, "{[1]}"] \arrow[d, "{[1]}"'] & y \arrow[d, "{[1]}"] \\ z \arrow[r, "{[1]}"'] & w \end{tikzcd} \tag{2}$$

is a pushout in $C$.

This is a commutative diagram. Suppose that

$$\begin{tikzcd} x \arrow[r, "{[1]}"] \arrow[d, "{[1]}"'] & y \arrow[d, "{[\lambda]}"] \\ z \arrow[r, "{[\mu]}"'] & v \end{tikzcd}$$

is a commutative diagram in $C$. Then $y \leq v^\lambda, z \leq v^\mu$, and as $[\lambda] = [\mu]$ in $\mathrm{Hom}(x, v)$ we have $v^\lambda = v^\mu$ and $\lambda = \alpha\mu$ for some $\alpha \in K_v$. Since (1) is a pushout in $P$, we have $w \leq v^\lambda$. Then $[\lambda]\colon y \to v$ factors as $y \xrightarrow{[1]} w \xrightarrow{[\lambda]} v$ and $[\mu]\colon z \to v$ factors as $z \xrightarrow{[1]} w \xrightarrow{[\mu]} v$. The uniqueness of such a morphism $w \to v$ is clear. Thus (2) is a pushout. This completes the proof.

∎

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