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REFLEXIVITY AND DUALIZABILITY IN CATEGORIFIED LINEAR ALGEBRA

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Abstract. The "linear dual" of a cocomplete linear category $\mathcal{C}$ is the category of all cocontinuous linear functors $\mathcal{C} \to \text{VECT}$. We study the questions of when a cocomplete linear category is reflexive (equivalent to its double dual) or dualizable (the pairing with its dual comes with a corresponding copairing). Our main results are that the category of comodules for a countable-dimensional coassociative coalgebra is always reflexive, but (without any dimension hypothesis) dualizable if and only if it has enough projectives, which rarely happens. Along the way, we prove that the category $\text{QCOH}(X)$ of quasi-coherent sheaves on a stack X is not dualizable if X is the classifying stack of a semisimple algebraic group in positive characteristic or if X is a scheme containing a closed projective subscheme of positive dimension, but is dualizable if X is the quotient of an affine scheme by a virtually linearly reductive group. Finally we prove tensoriality (a type of Tannakian duality) for affine ind-schemes with countable indexing poset.

1. Introduction

Fix a field $\mathbb{K}$. For cocomplete $\mathbb{K}$-linear categories $\mathcal{C}$ and $\mathcal{D}$, let $\text{HOM}(\mathcal{C},\mathcal{D}) = \text{HOM}_{c,\mathbb{K}}(\mathcal{C},\mathcal{D})$ denote the cocomplete $\mathbb{K}$-linear category of cocontinuous $\mathbb{K}$-linear functors and natural transformations from $\mathcal{C}$ to $\mathcal{D}$, and let $\mathcal{C} \boxtimes \mathcal{D} = \mathcal{C} \boxtimes_{c,\mathbb{K}} \mathcal{D}$ denote the universal cocomplete $\mathbb{K}$-linear category receiving a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ which is cocontinuous and $\mathbb{K}$-linear in each variable (while holding the other variable fixed). We will denote the image of $(C, D) \in \mathcal{C}\times\mathcal{D}$ under this functor by $C\boxtimes D \in \mathcal{C}\boxtimes\mathcal{D}$. If $\mathcal{C}$ and $\mathcal{D}$ are both locally presentable (see Definition 2.1), then $\text{HOM}(\mathcal{C},\mathcal{D})$ and $\mathcal{C} \boxtimes \mathcal{D}$ both exist as locally presentable $\mathbb{K}$-linear categories, and HOM and $\boxtimes$ satisfy a hom-tensor adjunction. The unit for $\boxtimes$ is $\text{VECT} = \text{VECT}_{\mathbb{K}}$; the image of $V \boxtimes C$ under the equivalence $\text{VECT} \boxtimes \mathcal{C} \simeq \mathcal{C}$ deserves to be called $V \otimes C$.

Thus the bicategory of locally presentable $\mathbb{K}$-linear categories provides one possible categorification of linear algebra. It includes as a full sub-bicategory the Morita bicategory ALG of associative algebras, bimodules, and intertwiners; the inclusion sends an algebra A to the category $\mathcal{M}_A$ of right A-modules. But locally presentable $\mathbb{K}$-linear cat-

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egories are more general: among them are, for any scheme or any Artin stack X over $\mathbb{K}$, the category $\mathrm{QCOH}(X)$ of quasi-coherent sheaves of $\mathcal{O}_X$-modules, and also for any coassociative coalgebra C over $\mathbb{K}$, the category $\mathcal{M}^C$ of right C-comodules.

In any generalization of linear algebra, it is interesting to ask what the "finite-dimensional" objects are. Finite-dimensionality of vector spaces is closely related to good behavior under taking dual vector spaces. In the context of cocomplete $\mathbb{K}$-linear categories, the "linear dual" of $\mathcal{C}$ is $\mathcal{C}^* = \mathrm{HOM}(\mathcal{C}, \mathrm{VECT})$. There are canonical functors

$$\begin{aligned} \mathcal{C} \boxtimes \mathcal{C}^* \to \mathrm{END}(\mathcal{C}) : &\quad C \boxtimes F \mapsto (D \mapsto F(C) \otimes D), \\ \mathcal{C} \to (\mathcal{C}^*)^* : &\quad C \mapsto (F \mapsto F(C)). \end{aligned}$$

Each of these functors corresponds to a possible generalization of finite-dimensionality:

1.1. DEFINITION. *A locally presentable $\mathbb{K}$-linear category $\mathcal{C}$ is called* dualizable *if the canonical functor $\mathcal{C} \boxtimes \mathcal{C}^* \to \mathrm{END}(\mathcal{C})$ is an equivalence. A locally presentable $\mathbb{K}$-linear category $\mathcal{C}$ is called* reflexive *if the canonical functor $\mathcal{C} \to (\mathcal{C}^*)^*$ is an equivalence.*

Dualizability implies reflexivity, but the converse is generally false. Dualizability is particularly important in light of [BD95, Lur09]. For example, a well-known corollary of the Eilenberg–Watts theorem, which asserts that $\mathrm{HOM}(\mathcal{M}_A, \mathcal{M}_B) \simeq \mathcal{M}_{A^{\mathrm{op}} \otimes B}$ (see Example 2.14), answers the question of dualizability in the affirmative for objects of the Morita bicategory ALG:

1.2. THEOREM. [Folklore] *For any associative algebra A, the category $\mathcal{M}_A$ of right A-modules is dualizable. The dual is $(\mathcal{M}_A)^* \simeq \mathcal{M}_{A^{\mathrm{op}}}$.* ∎

There is no Eilenberg–Watts theorem for coassociative coalgebras. Thus the questions of reflexivity and dualizability are more subtle. Our main results on dualizability are:

1.3. THEOREM. *Let C be a coassociative coalgebra. Then the category $\mathcal{M}^C$ of right C-comodules is dualizable if and only if it has enough projectives.*

Such a coalgebra is called *right semiperfect* in [Lin77] (generalizing the notion from [Bas60]).

1.4. THEOREM. *Let X be a $\mathbb{K}$-scheme. If X has a closed projective subscheme of positive dimension, then $\mathrm{QCOH}(X)$ is not dualizable.*

In order to state the next result, recall ([Don96]) that a linear algebraic group is *linearly reductive* if its category of representations is semisimple. Examples include the classical groups $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, etc., in characteristic 0. It is *virtually linearly reductive* if it has a linearly reductive normal algebraic subgroup such that the quotient is a finite group scheme. Finite groups in characteristic dividing the order of the group are examples of virtually linearly reductive groups that are not linearly reductive. The additive group $\mathbb{G}_a = \mathrm{Spec}(\mathbb{K}[x])$ is not virtually linearly reductive (see Example 3.7).

1.5. Theorem. *Let X be an affine scheme over $\mathbb{K}$ and G a virtually linearly reductive group over $\mathbb{K}$ acting on X. Let $[X/G]$ denote the corresponding quotient stack. Then* $\mathrm{QCOH}([X/G])$ *is dualizable.*

Theorems 1.3, 1.4, and 1.5 are proved in Section 3. (Section 2 reviews some of the theory of locally presentable categories.) In Theorem 1.5, the condition on G is important. Indeed, according to [Don96], corrected in [Don98], $\mathrm{QCOH}([\mathrm{Spec}(\mathbb{K})/G]) \simeq \mathrm{REP}(G) \simeq \mathcal{M}^{\mathcal{O}(G)}$ contains a non-zero projective if and only if G is virtually linearly reductive, and moreover if any non-zero injective fails to be projective, then there are no non-zero projectives. In particular, with Theorem 1.3 this implies that $\mathrm{QCOH}([\mathrm{Spec}(\mathbb{K})/G])$ is not dualizable for G a semisimple group in positive characteristic, nor is it dualizable for $G = \mathbb{G}_a$. Such nondualizability results are in stark contrast with [BZFN10], where it is shown that for many stacks of geometric interest, the corresponding *derived* category of quasi-coherent sheaves *is* dualizable (for the derived version of $\boxtimes$).

We address reflexivity in Section 4. Our main result there is:

1.6. Theorem. *Suppose that $A = (\cdots \twoheadrightarrow A_2 \twoheadrightarrow A_1 \twoheadrightarrow A_0)$ is an $\mathbb{N}$-indexed projective system of associative $\mathbb{K}$-algebras. Let $\mathcal{C}$ denote the category of injective systems $(M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \dots)$ where each M_i is a right A_i-module and each inclusion $M_i \hookrightarrow M_{i+1}$ identifies M_i as the maximal A_{i+1}-submodule on which the A_{i+1}-action factors through A_i. Let B be any associative $\mathbb{K}$-algebra. Then the "double dual" functor*

$$\mathcal{C} \boxtimes \mathcal{M}_B \to \mathrm{HOM}(\mathcal{C}^*, \mathcal{M}_B),$$

which maps $X \boxtimes M$ to $(F \mapsto F(X) \otimes M)$, is an equivalence.

An immediate corollary establishes reflexivity for some categories that are not dualizable (as usual, the dimension of a coalgebra refers to the underlying vector space):

1.7. Corollary. *Let C be a countable-dimensional coassociative coalgebra. Then $\mathcal{M}^C$ is reflexive.*

Proof. By [Swe69, Theorem 2.2.1], there exists an increasing sequence of finite-dimensional subcoalgebras $C_0 \subseteq C_1 \subseteq \dots$ with $C = \bigcup C_i$. Let $A_i = C_i^*$ and $\mathcal{C}$ as in Theorem 1.6. Then $\mathcal{C} \simeq \mathcal{M}^C$ by Example 2.15. Theorem 1.6 with $B = \mathbb{K}$ completes the proof. ∎

We do not know if countable dimensionality in Corollary 1.7 can be dropped. Note that the coalgebra $C = \mathbb{K}[x]$ with $\Delta(x^n) = \sum_{i=0}^n x^i \otimes x^{n-i}$ provides an example of a countable-dimensional coalgebra for which, by Theorem 1.3, $\mathcal{M}^C$ is not dualizable. (See Example 3.7 for an elaboration.) In particular, Theorem 1.3 and Corollary 1.7 together illustrate that dualizability and reflexivity are very different categorifications of finite-dimensionality.

The main step in the proof of Theorem 1.6 is a description given in Proposition 4.8 of $\mathrm{HOM}(\mathcal{C}^*, \mathcal{M}_B)$ in terms of certain modules over certain algebras. We end by applying this description to prove the following result, which can be thought of as a symmetric monoidal version of Theorem 1.6:

1.8. Theorem. *Affine ind-schemes indexed by countable posets are tensorial.*

Tensoriality for schemes was introduced in [Bra11] and studied for quasi-compact quasi-separated schemes and Artin stacks in [BC14, HR14]. We review the definition, along with its natural extension to ind-schemes, in Definition 4.10. Roughly speaking, a geometric object is tensorial if it can be recovered functorially in a "Tannakian" way from its symmetric monoidal category of quasi-coherent sheaves.

2. Recollections on locally presentable categories

We begin by recalling a few basic facts about locally presentable $\mathbb{K}$-linear categories. The primary reference on locally presentable categories is [AR94]. Many of the results in this section appear as early as [GU71].

2.1. Definition. *For an infinite cardinal λ, a partially ordered set $\mathcal{I}$ is* λ-directed *if any subset of $\mathcal{I}$ of cardinality strictly less than λ has an upper bound. A* λ-directed colimit *is a colimit of a diagram indexed by a λ-directed partially ordered set. An object X in a category $\mathcal{C}$ is* λ-presentable *(also called "λ-compact") if* $\hom(X,-)$ *commutes with λ-directed colimits. A colimit is* λ-small *if its indexing diagram has strictly fewer than λ arrows (including identity arrows).*

A set of objects Γ in $\mathcal{C}$ is strongly generating *if the closure of Γ in $\mathcal{C}$ under small colimits is all of $\mathcal{C}$.*

A $\mathbb{K}$-linear locally small category $\mathcal{C}$ is locally presentable *if it is cocomplete (i.e. has all small colimits) and admits a strongly generating set consisting entirely of objects that are λ-presentable for some λ. We may imagine $\mathcal{C}$ as a "categorified vector space over $\mathbb{K}$", since coproducts are categorified sums. Notice, however, that we don't have any additive inverses, and that sums in a vector space have no universal property and therefore are additional structure, in contrast to colimits in a category.*

Locally presentable $\mathbb{K}$-linear categories are the objects of a bicategory $\mathrm{Pres}_{\mathbb{K}}$ *whose 1-morphisms are cocontinuous $\mathbb{K}$-linear functors and whose 2-morphisms are natural transformations. The special adjoint functor theorem implies that every cocontinuous functor $\mathcal{C} \to \mathcal{D}$ with $\mathcal{C}$ locally presentable has a right adjoint (in the bicategory of all functors).*

2.2. Remark. A set of objects Γ in $\mathcal{C}$ is *generating* if $\prod_{X\in\Gamma} \hom(X,-)$ is faithful. Strongly generating sets are generating, and the converse holds if the category in question is abelian, since then all epimorphisms are coequalizers. But the converse generally fails: the category of two-term filtered vector spaces $(V_0 \hookrightarrow V_1)$ is locally presentable, but $(\{0\} \hookrightarrow \mathbb{K})$ is generating but not strongly generating.

For general categories, our definition of "strongly generating" is stronger than the usual one, but the two definitions are the same for locally presentable categories [Kel05, Propostion 3.40]. Our definition of local presentability is equivalent to the usual one by [AR94, Theorem 1.20].

Note that the forgetful functor $\mathrm{Vect} \to \mathrm{Set}$ preserves λ-directed colimits, so there is no difference between Set- and Vect-valued homs for the purposes of defining notions like λ-presentability. A λ-presentable object $X \in \mathcal{C}$ has "size less than λ"; even "smaller" than this are the objects $X \in \mathcal{C}$ such that $\hom(X,-) : \mathcal{C} \to \mathrm{V{\small ECT}}$ preserves *all* colimits, and not just the λ-directed ones. If $\mathcal{C}$ is abelian, such objects are precisely the compact projective ones: in a general category, an object is *compact* if it is $\aleph_0$-presentable; in an abelian category, an object X is *projective* if $\hom(X,-)$ preserves coequalizers; all colimits are compositions of finite direct sums, $\aleph_0$-filtered colimits and coequalizers. As such, for $\mathcal{C}$ an arbitrary locally presentable $\mathbb{K}$-linear category, we will call an object $X \in \mathcal{C}$ *compact projective* if $\hom(X,-) : \mathcal{C} \to \mathrm{V{\small ECT}}$ is cocontinuous. Compact projectivity is called "small projectivity" in [Kel05].

The fundamental theorem of locally presentable categories is:

2.3. Proposition. *Let $\mathcal{C}$ be a locally presentable $\mathbb{K}$-linear category with a strongly generating set consisting of λ-presentable objects, for λ a regular infinite cardinal. Then the full subcategory $\mathcal{C}_{<\lambda} \subseteq \mathcal{C}$ of λ-presentable objects is essentially small and closed in $\mathcal{C}$ under λ-small colimits. Moreover,*

$$\mathcal{C} \xrightarrow[Y \,\mapsto\, \text{restriction of } \hom(-,Y) \text{ to } \mathcal{C}_{<\lambda}]{\text{“Yoneda” functor}} \boxed{\begin{array}{c}\mathbb{K}\text{-linear functors}\\ \mathcal{C}_{<\lambda}^{\mathrm{op}} \to \mathrm{V{\small ECT}}\\ \text{preserving } \lambda\text{-}\\ \text{small limits}\end{array}}$$

is an equivalence, as is

$$\boxed{\begin{array}{c}\text{cocontinuous}\\ \mathbb{K}\text{-linear func-}\\ \text{tors } \mathcal{C} \to \mathcal{D}\end{array}} \xrightarrow{\text{restriction to } \mathcal{C}_{<\lambda}} \boxed{\begin{array}{c}\mathbb{K}\text{-linear functors}\\ \mathcal{C}_{<\lambda} \to \mathcal{D} \text{ preserving}\\ \lambda\text{-small colimits}\end{array}}$$

for any cocomplete $\mathbb{K}$-linear category $\mathcal{D}$.

Recall that a cardinal λ is *regular* if it is not the union of strictly fewer than λ sets, each of which has cardinality strictly less than λ. See [AR94, Exercise 1.b] for why one may assume λ to be regular in Definition 2.1.

Proof. The statement consists of Theorems 1.20 and 1.46 and Proposition 1.45 from [AR94] (an error in the proof was corrected in [AHR99]). To incorporate linearity is straightforward: one can systematically develop a $\mathbb{K}$-linear theory exactly parallel to the usual theory with no surprises, making only a few changes as necessary; see [Kel82, Kel05]. (Indeed, one can systematically develop a version for categories enriched in any closed symmetric monoidal locally presentable category, although at that level of generality one must use weighted colimits and not just colimits.) ∎

An important corollary of Proposition 2.3 is that $\mathrm{H{\small OM}}_{\mathrm{P{\scriptsize RES}}_{\mathbb{K}}}(\mathcal{C},\mathcal{D})$ is locally small, being equivalent to a full subcategory of the category of functors from a small category to a locally small category.

We will use the following variant of Proposition 2.3:

2.4. Proposition. [Kel05, Theorem 5.26] *Let $\mathcal{C}$ be a locally presentable $\mathbb{K}$-linear category with a strongly generating set Γ consisting of compact projective objects. Then*

$$\mathcal{C} \xrightarrow[Y \mapsto \text{restriction of } \hom(-,Y) \text{ to } \Gamma]{\text{“Yoneda” functor}} \boxed{\mathbb{K}\text{-linear functors } \Gamma^{\mathrm{op}} \to \mathrm{VECT}}$$

is an equivalence, as is

$$\boxed{\begin{array}{c}\text{cocontinuous}\\ \mathbb{K}\text{-linear func-}\\ \text{tors } \mathcal{C} \to \mathcal{D}\end{array}} \xrightarrow{\text{restriction to } \Gamma} \boxed{\mathbb{K}\text{-linear functors } \Gamma \to \mathcal{D}}$$

for any cocomplete $\mathbb{K}$-linear category $\mathcal{D}$. ■

Conversely, note that for any small $\mathbb{K}$-linear category Γ, in the category $\mathrm{FUN}_{\mathbb{K}}(\Gamma^{\mathrm{op}}, \mathrm{VECT})$ of all $\mathbb{K}$-linear functors, every representable functor $\hom(-, X)$ for $X \in \Gamma$ is compact projective, and the Yoneda Lemma implies that the representable functors are a strongly generating set. Thus Proposition 2.4 identifies categories of the form $\mathrm{FUN}_{\mathbb{K}}(\Gamma^{\mathrm{op}}, \mathrm{VECT})$ as precisely those $\mathbb{K}$-linear categories strongly generated by a set of compact projectives.

2.5. Lemma. [Bir84],[AR94, Exercise 2.n],[CJF13, Proposition 2.1.11] *The bicategory $\mathrm{PRES}_{\mathbb{K}}$ of locally presentable $\mathbb{K}$-linear categories has all small 2-limits and all small 2-colimits:*

- *To compute a 2-limit in $\mathrm{PRES}_{\mathbb{K}}$, simply compute the same 2-limit in the bicategory CAT of categories, ignoring that the arrows happen to be left adjoints.*
- *To compute a 2-colimit in $\mathrm{PRES}_{\mathbb{K}}$, replace every 1-morphism by its right adjoint — this is a contravariant bifunctor $\mathrm{PRES}_{\mathbb{K}} \to \mathrm{CAT}$ — and compute the corresponding 2-limit in CAT, ignoring that the arrows happen to be right adjoints.* ■

What we call “2-limits” and “2-colimits” are also called “bilimits” and “bicolimits” in the literature. We will often drop the prefixes “2-,” calling them just “limits” and “colimits.”

2.6. Remark. We will be particularly interested in (2-)limits and (2-)colimits indexed by partially ordered sets $\mathcal{I}$. Let $\mathcal{C} : \mathcal{I} \to \mathrm{PRES}_{\mathbb{K}}$ be an $\mathcal{I}$-indexed diagram. Thus it consists of categories $\mathcal{C}_i \in \mathrm{PRES}_{\mathbb{K}}$ for each $i \in \mathcal{I}$, $\mathbb{K}$-linear cocontinuous functors $F_{i<j} : \mathcal{C}_i \to \mathcal{C}_j$ for each $i < j$, and natural isomorphisms $F_{i<k} \cong F_{j<k} \circ F_{i<j}$ which are compatible for 4-tuples in $\mathcal{I}$.

Consider first the limit $\varprojlim_{i\in\mathcal{I}} \mathcal{C}_i$ in $\mathrm{PRES}_{\mathbb{K}}$. By Lemma 2.5, it is nothing but the corresponding limit in CAT. Thus an object of $\varprojlim_{i\in\mathcal{I}} \mathcal{C}_i$ consists of an object $X_i \in \mathcal{C}_i$ for each $i \in \mathcal{I}$ and an isomorphism $X_j \cong F_{i<j}(X_i)$ for each $i < j$ such that these isomorphisms commute for triples $i < j < k$. (Such commutativity uses the isomorphisms $F_{i<k} \cong F_{j<k} \circ F_{i<j}$.) A morphism $\{X_i\}_{i\in\mathcal{I}} \to \{Y_i\}_{i\in\mathcal{I}}$ in $\varprojlim_{i\in\mathcal{I}} \mathcal{C}_i$ consists of a morphism

$X_i \to Y_i$ in $\mathcal{C}_i$ for each i, with the ith morphism mapping under $F_{i<j}$ to the jth morphism for $i < j$.

Let $F_{i<j}^R : \mathcal{C}_j \to \mathcal{C}_i$ denote the right adjoint to $F_{i<j}$. Under the adjunction, the isomorphism $F_{i<j}(X_i) \xrightarrow{\sim} X_j$ in $\mathcal{C}_j$ corresponds to a map $X_i \to F_{i<j}^R(X_j)$ which is isomorphic to the unit-of-the-adjunction map $X_i \to F_{i<j}^R(F_{i<j}(X_i))$. It therefore realizes $F_{i<j}^R(X_j)$ as the object in the image of $F_{i<j}^R$ which is universal (among morphisms in the image of $F_{i<j}^R$) for receiving a morphism from X_i. In many cases of interest, the map $X_i \to F_{i<j}^R(X_j)$ is an epimorphism. In general, the object $\{X_i\}_{i\in\mathcal{I}} \in \varprojlim_i \mathcal{C}_i$ is a "formal limit" $\varprojlim X_i$ along units of adjunctions.

Consider second the colimit $\varinjlim_{i\in\mathcal{I}} \mathcal{C}_i$ in $\mathrm{P{\scriptsize RES}}_{\mathbb{K}}$. By Lemma 2.5 this is the limit in $\mathrm{C{\scriptsize AT}}$ of the $\mathcal{I}^{\mathrm{op}}$-indexed diagram $i \mapsto \mathcal{C}_i$ along the functors $F_{i<j}^R : \mathcal{C}_j \to \mathcal{C}_i$. An object of $\varinjlim_{i\in\mathcal{I}} \mathcal{C}_i$ is therefore an object $X_i \in \mathcal{C}_i$ for each i along with compatible isomorphisms $X_i \cong F_{i<j}^R(X_j)$. Under the adjunction, these isomorphisms correspond to maps $F_{i<j}(X_i) \to X_j$ realizing $F_{i<j}(X_i)$ as the universal object in the image of $F_{i<j}$ that maps to X_j; in many cases of interest, these maps are monomorphisms. In general, the object $\{X_i\}_{i\in\mathcal{I}} \in \varinjlim_i \mathcal{C}_i$ can be thought of as a "formal colimit" $\varinjlim X_i$ along counits of adjunctions.

One corollary of Lemma 2.5 is that if $\mathcal{I}$ is just a set then products and coproducts in $\mathrm{P{\scriptsize RES}}_{\mathbb{K}}$ indexed by $\mathcal{I}$ agree (and agree with the product, but not the coproduct, of underlying categories). It thus makes sense to call this (co)product the *direct sum* $\bigoplus_{i\in\mathcal{I}} \mathcal{C}_i$.

Just like vector spaces, in addition to a direct sum, locally presentable $\mathbb{K}$-linear categories also admit a tensor product:

2.7. Lemma. [Kel05, Section 6.5],[AR94, Exercise 1.l]

1. *For $\mathcal{C}, \mathcal{D} \in \mathrm{P{\scriptsize RES}}_{\mathbb{K}}$, the category $\mathrm{H{\scriptsize OM}}(\mathcal{C}, \mathcal{D})$ is also locally presentable and $\mathbb{K}$-linear.*
2. *For each $\mathcal{C}$, the bifunctor $\mathrm{H{\scriptsize OM}}(\mathcal{C}, -) : \mathrm{P{\scriptsize RES}}_{\mathbb{K}} \to \mathrm{P{\scriptsize RES}}_{\mathbb{K}}$ has a left adjoint $(-) \boxtimes \mathcal{C}$, making $\mathrm{P{\scriptsize RES}}_{\mathbb{K}}$ into a closed symmetric monoidal bicategory.*
3. *The* tensor product *$\mathcal{C} \boxtimes \mathcal{D}$ satisfies the following universal property: For any cocomplete $\mathbb{K}$-linear category $\mathcal{E}$, cocontinuous functors $\mathcal{C} \boxtimes \mathcal{D} \to \mathcal{E}$ correspond to functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that are cocontinuous and $\mathbb{K}$-linear in each variable, holding the other variable fixed.* ■

2.8. Remark. One can present $\mathcal{C} \boxtimes \mathcal{D}$ from part (3) of Lemma 2.7 in a number of ways. For example, it is (up to canonical equivalence) the category of continuous functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Indeed, choose any regular cardinal λ for which $\mathcal{C}$ admits a strongly generating set consisting of λ-presentable objects. The first part of Proposition 2.3 identifies $\mathcal{C}^{\mathrm{op}}$ with the category of functors $\mathcal{C}_{<\lambda} \to \mathrm{V{\scriptsize ECT}}^{\mathrm{op}}$ preserving λ-small limits, and the second part identifies this with $\mathrm{H{\scriptsize OM}}(\mathcal{C}, \mathrm{V{\scriptsize ECT}}^{\mathrm{op}})$, so that together two parts of Proposition 2.3 provide a canonical equivalence $\mathcal{C} \simeq (\mathrm{H{\scriptsize OM}}(\mathcal{C}, \mathrm{V{\scriptsize ECT}}^{\mathrm{op}}))^{\mathrm{op}}$ identifying $C \in \mathcal{C}$ with $\hom(-, C)$. (As in the introduction, we write $\mathrm{H{\scriptsize OM}}(\mathcal{C}, \mathrm{V{\scriptsize ECT}}^{\mathrm{op}})$ for the category of cocontinuous $\mathbb{K}$-linear functors $\mathcal{C} \to \mathrm{V{\scriptstyle ECT}}^{\mathrm{op}}$, even though $\mathrm{V{\scriptstyle ECT}}^{\mathrm{op}}$ is not locally presentable.) Then we have equivalences

$$\mathcal{C} \boxtimes \mathcal{D} \simeq \big(\mathrm{H{\scriptstyle OM}}(\mathcal{C} \boxtimes \mathcal{D}, \mathrm{V{\scriptstyle ECT}}^{\mathrm{op}})\big)^{\mathrm{op}} \simeq \big(\mathrm{H{\scriptstyle OM}}(\mathcal{C}, \mathrm{H{\scriptstyle OM}}(\mathcal{D}, \mathrm{V{\scriptstyle ECT}}^{\mathrm{op}}))\big)^{\mathrm{op}} \simeq \big(\mathrm{H{\scriptstyle OM}}(\mathcal{C}, \mathcal{D}^{\mathrm{op}})\big)^{\mathrm{op}},$$

from which the claim follows.

2.9. Definition. *A* symmetric monoidal locally presentable $\mathbb{K}$-linear category *is by definition a symmetric monoidal category $\mathcal{S} = (\mathcal{C}, \otimes)$ whose underlying category $\mathcal{C}$ is locally presentable $\mathbb{K}$-linear such that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is cocontinuous and $\mathbb{K}$-linear in each variable, so that it extends to a 1-morphism $\mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$. In fact, $\mathcal{S}$ is a symmetric pseudomonoid in the symmetric monoidal bicategory $(\mathrm{P{\scriptstyle RES}}_{\mathbb{K}}, \boxtimes)$. Therefore, we may imagine $\mathcal{S}$ as a "categorified commutative algebra over $\mathbb{K}$."*

We denote by $\mathrm{P{\scriptstyle RES}}_{\otimes,\mathbb{K}}$ the bicategory of symmetric monoidal locally presentable $\mathbb{K}$-linear categories, where for two such objects $\mathcal{C}, \mathcal{D}$ the category $\mathrm{H{\scriptstyle OM}}_{\otimes,c,\mathbb{K}}(\mathcal{C}, \mathcal{D})$ consists of cocontinuous strong symmetric monoidal $\mathbb{K}$-linear functors and symmetric monoidal natural transformations. We will often simply write $\mathrm{H{\scriptstyle OM}}_{\otimes}$ in place of $\mathrm{H{\scriptstyle OM}}_{\otimes,c,\mathbb{K}}$, as $\mathbb{K}$ will often be implicit and we will never use non-cocontinuous symmetric monoidal functors.

*Let $\mathcal{S} = (\mathcal{C}, \otimes) \in \mathrm{P{\scriptstyle RES}}_{\otimes,\mathbb{K}}$. An $\mathcal{S}$-*module in $\mathrm{P{\scriptstyle RES}}_{\mathbb{K}}$ *is a locally presentable $\mathbb{K}$-linear category $\mathcal{D}$ together with an* action $\mathcal{C} \boxtimes \mathcal{D} \to \mathcal{D}$*, denoted by $X \boxtimes M \mapsto X \triangleright M$, as well as unit and associativity data making the appropriate triangles and pentagons commute.*

2.10. Remark. Module categories are in particular enriched: For any object $M \in \mathcal{D}$, the functor $(-) \triangleright M : \mathcal{C} \to \mathcal{D}$ is cocontinuous, hence has a right adjoint $\hom(M, -)_{\mathcal{S}} : \mathcal{D} \to \mathcal{C}$. It is straightforward to check that the associativity and unit data make $\hom(-,-)_{\mathcal{S}}$ into an $\mathcal{S}$-enrichment of $\mathcal{D}$. See for example [GP97, Theorem 3.7].

2.11. Remark. For a field extension $\mathbb{K} \hookrightarrow \mathbb{L}$, a locally presentable $\mathbb{L}$-linear category is precisely a $\mathrm{V{\scriptstyle ECT}}_{\mathbb{L}}$-module in $\mathrm{P{\scriptstyle RES}}_{\mathbb{K}}$. It follows that we have a 2-functor

$$\bullet \boxtimes \mathrm{V{\scriptstyle ECT}}_{\mathbb{L}} : \mathrm{P{\scriptstyle RES}}_{\mathbb{K}} \to \mathrm{P{\scriptstyle RES}}_{\mathbb{L}}$$

(the tensor product is in $\mathrm{P{\scriptstyle RES}}_{\mathbb{K}}$) which preserves $\boxtimes$ in the sense that it lifts to a symmetric monoidal 2-functor.

The following enriched version of compact projectivity will be useful in the course of the proof of Theorem 1.5.

2.12. Definition. *Let $\mathcal{S} = (\mathcal{C}, \otimes) \in \mathrm{P{\scriptstyle RES}}_{\otimes,\mathbb{K}}$ and $(\mathcal{D}, \triangleright)$ be an $\mathcal{S}$-module in the sense of Definition 2.9. An object $M \in \mathcal{D}$ is* compact projective over $\mathcal{S}$ *if the enriched hom functor $\hom(M, -)_{\mathcal{S}} : \mathcal{D} \to \mathcal{C}$ is cocontinuous. A set Γ of objects in $\mathcal{D}$ is* strongly generating over $\mathcal{S}$ *if $\mathcal{D}$ is the closure of Γ under $\mathcal{S}$-weighted colimits.*

2.13. Proposition. *With the notations above, assume that $\mathcal{C}$ admits a strongly generating set of compact projective (over $\mathrm{V{\scriptstyle ECT}}$) objects, and that $\mathcal{D}$ admits a strongly generating set over $\mathcal{S}$ of objects which are all compact projective over $\mathcal{S}$. Then $\mathcal{D}$ is strongly generated by compact projective (over $\mathrm{V{\scriptstyle ECT}}$) objects.*

PROOF. Let $\{c_i\}$ be a strongly generating set of compact projective objects in $\mathcal{C}$ and $\{d_j\}$ a strongly generating over $\mathcal{S}$ set of compact projective over $\mathcal{S}$ objects in $\mathcal{D}$. We have

$$\hom_{\mathcal{D}}(c_i \triangleright d_j, -) \cong \hom_{\mathcal{C}}(c_i, \hom_{\mathcal{D}}(d_j, -)_{\mathcal{S}}) : \mathcal{D} \to \mathrm{VECT}.$$

The right hand side is cocontinuous by assumption, so the left hand side is as well. Hence, $c_i \triangleright d_j$ are compact projective. The fact that they strongly generate follows from the fact that c_i and d_j strongly generate and an unpacking of the notion of weighted colimit. ■

We may now introduce our main examples:

2.14. EXAMPLE. Let A be an associative algebra over $\mathbb{K}$. Then the category $\mathcal{M}_A$ of all right A-modules is a locally presentable $\mathbb{K}$-linear category. Since every module has a presentation, the strongly generating set Γ may be taken to consist of the rank-one free module A, which is compact projective. Proposition 2.4 implies that for any cocomplete $\mathbb{K}$-linear category $\mathcal{D}$, cocontinuous $\mathbb{K}$-linear functors $\mathcal{M}_A \to \mathcal{D}$ are equivalent to $\mathbb{K}$-linear functors $\Gamma \to \mathcal{D}$. It then follows from Lemma 2.7 that there is a canonical equivalence of categories $\mathrm{HOM}(\mathcal{M}_A, \mathcal{D}) \simeq {}_A\mathcal{M} \boxtimes \mathcal{D}$ for any $\mathcal{D} \in \mathrm{PRES}_{\mathbb{K}}$, where ${}_A\mathcal{M} \cong \mathcal{M}_{A^{\mathrm{op}}}$ is the category of left A-modules. The Eilenberg–Watts theorem and its corollary Theorem 1.2 follow.

2.15. EXAMPLE. Let C be a coassociative coalgebra over $\mathbb{K}$. We will describe in some detail the category $\mathcal{M}^C$ of right C-comodules, showing in particular that it is locally presentable.

By the so-called fundamental theorem of coalgebras ([Swe69, Theorem 2.2.1]), $C \simeq \varinjlim_{i \in \mathcal{I}} C_i$, where $\{C_i\}_{i \in \mathcal{I}}$ is the partially ordered set of finite-dimensional sub-coalgebras of C_i. We claim first that $\mathcal{M}^{C_i}$, the category of right C_i-comodules, is locally presentable for each i, and second that

$$\mathcal{M}^C \simeq \varinjlim_{i \in \mathcal{I}} \mathcal{M}^{C_i},$$

where the colimit is computed in $\mathrm{PRES}_{\mathbb{K}}$ along the scalar corestriction functors $\mathcal{M}^{C_i} \to \mathcal{M}^{C_j}$ for the inclusions $C_i \leq C_j$.

To see the first claim, give the linear dual C_i^* the algebra structure $(\alpha \cdot \beta)(c) = \sum \alpha(c_{(2)}) \otimes \beta(c_{(1)})$, where the comultiplication on C_i in Sweedler's notation is $c \mapsto c_{(1)} \otimes c_{(2)}$. Then the category $\mathcal{M}^{C_i}$ of right C_i-comodules is canonically equivalent to the category $\mathcal{M}_{C_i^*}$ of right C_i^*-modules, hence locally presentable by Example 2.14.

To see the second claim, recall that the fundamental theorem of coalgebras moreover asserts that every right C-comodule X is canonically a colimit $X = \varinjlim_{i \in \mathcal{I}} X_i$ where X_i is the largest subcomodule for which the coaction $X_i \to X_i \otimes C$ factors through $X_i \otimes C_i$, and conversely the union of any filtered system of C_i-comodules like this is a C-comodule. Moreover, every C-comodule morphism $f : X \to Y$ restricts to a system of compatible C_i-comodule morphisms $f_i : X_i \to Y_i$, and any such compatible system $\{f_i : X_i \to Y_i\}_{i \in \mathcal{I}}$ defines a C-comodule morphism f.

Thus $\mathcal{M}^C$ is equivalent to the category of $\mathcal{I}$-indexed systems of vector spaces $\{X_i\}_{i \in \mathcal{I}}$ such that for each i, X_i is a C_i-comodule, and for $i \leq j$, there maps $X_i \hookrightarrow X_j$ (compatible for $i \le j \le k$) realizing X_i as the largest C_i-sub-comodule of X_j. Said another way, X_i, scalar corestricted to become a C_j-comodule, is the universal object in the image of scalar corestriction $\mathcal{M}^{C_i} \to \mathcal{M}^{C_j}$ that maps to X_j. But Remark 2.6 identifies such systems $\{X_i\}_{i\in\mathcal{I}}$ as the objects of $\varinjlim \mathcal{M}^{C_i}$, and systems $\{f_i : X_i \to Y_i\}_{i\in\mathcal{I}}$ as the morphisms of $\varinjlim \mathcal{M}^{C_i}$.

2.16. Example. Let X be a scheme. We can present X as a colimit of open affine subschemes: $X = \varinjlim_{i\in\mathcal{I}} \mathrm{Spec}(A_i)$. Let $\mathrm{QCOH}(X)$ denote the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules. We have

$$\mathrm{QCOH}(X) \simeq \varprojlim_{i\in\mathcal{I}^{\mathrm{op}}} \mathrm{QCOH}(\mathrm{Spec}(A_i)) \simeq \varprojlim_{i\in\mathcal{I}^{\mathrm{op}}} {}_{A_i}\mathcal{M},$$

since a quasicoherent sheaf is a module on each open in an affine open cover along with compatibility data on overlaps.

Even though a priori this limit is computed in CAT, the pullback functors ${}_{A_i}\mathcal{M} \to {}_{A_j}\mathcal{M}$ involved are cocontinuous, and so by Lemma 2.5 it is also the limit in $\mathrm{PRES}_{\mathbb{K}}$. In particular, $\mathrm{QCOH}(X)$ is locally presentable. More generally, any Artin stack X can be presented as a 2-colimit of affine schemes, and a similar argument applies to show that $\mathrm{QCOH}(X)$ is locally presentable, using faithfully flat descent of quasi-coherent sheaves.

3. (Non)dualizability

In this section we will prove Theorems 1.3, 1.4, and 1.5. First, let us briefly unpack the notion of dualizability from Definition 1.1.

We will make use of the notion of adjunction between bicategories, referring the reader to [Fio06, Chapter 9] for background. The term used in that reference is "biadjunction," but we will simply say "adjunction." The following Lemma is implicit in [DS97, Proposition 3] and reproduces at the 2-categorical level facts that are essentially standard within symmetric monoidal 1-categories:

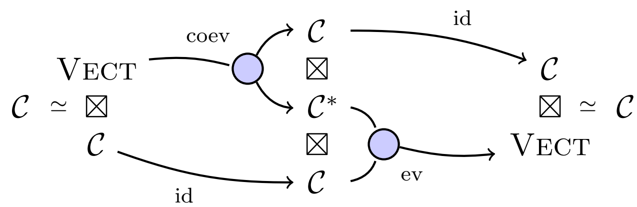
3.1. Lemma. *For a locally presentable $\mathbb{K}$-linear category $\mathcal{C}$ the following conditions are equivalent:*

1. *$\mathcal{C}$ is dualizable in the sense of Definition 1.1.*
2. *For any $\mathcal{D} \in \mathrm{PRES}_{\mathbb{K}}$ the canonical functor $\mathrm{can}_{\mathcal{D}} : \mathcal{D} \boxtimes \mathcal{C}^* \to \mathrm{HOM}(\mathcal{C}, \mathcal{D})$ is an equivalence.*
3. *If $\mathrm{ev} : \mathcal{C}^* \boxtimes \mathcal{C} \to \mathrm{VECT}$ is the standard evaluation pairing, then*

$$(\bullet \boxtimes \mathcal{C}^*) \boxtimes \mathcal{C} \simeq \bullet \boxtimes (\mathcal{C}^* \boxtimes \mathcal{C}) \xrightarrow{-\boxtimes \mathrm{ev}} \mathrm{id} \qquad (*)$$

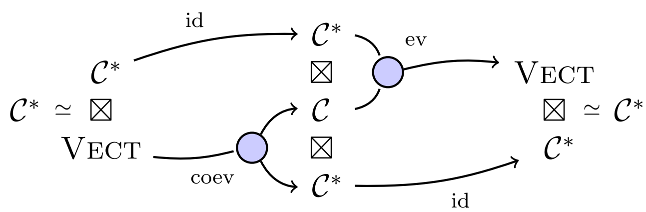
is the counit of an adjunction between the 2-endofunctors $\bullet \boxtimes \mathcal{C}$ and $\bullet \boxtimes \mathcal{C}^*$ of $\mathrm{PRES}_{\mathbb{K}}$.

4. *There is a cocontinuous linear functor* coev : $\mathrm{V{\small ECT}} \to \mathcal{C} \boxtimes \mathcal{C}^*$ *(the* coevaluation*) such that the two compositions*



(†)

and



(‡)

are naturally isomorphic to the identity.

5. *The identity functor* $\mathrm{id}_{\mathcal{C}}$ *is in the essential image of the canonical functor* $\mathcal{C} \boxtimes \mathcal{C}^* \to \mathrm{E{\small ND}}(\mathcal{C})$*.*

6. *The 2-endofunctor* $\boxtimes\mathcal{C}$ *of* $\mathrm{P{\small RES}}_{\mathbb{K}}$ *has a right adjoint of the form* $\boxtimes\mathcal{C}'$ *for some* $\mathcal{C}' \in \mathrm{P{\small RES}}_{\mathbb{K}}$*.*

P{\small ROOF}. **(1) ⇒ (5)** This is immediate.

(5) ⇒ (4) An object $x \in \mathcal{C}\boxtimes\mathcal{C}^*$ that maps onto $\mathrm{id}_{\mathcal{C}}$ through $\mathcal{C}\boxtimes\mathcal{C}^* \to \mathrm{E{\small ND}}(\mathcal{C})$ induces a left adjoint coev : $\mathrm{V{\small ECT}} \to \mathcal{C} \boxtimes \mathcal{C}^*$, $\mathbb{K}^{\oplus\alpha} \mapsto x^{\oplus\alpha}$. We claim that as the name suggests, coev is a coevaluation in the sense of (4). In order to verify this, we have to show that the functors (†) and (‡) are (naturally isomorphic to) identities.

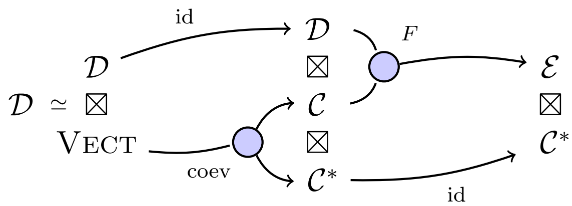
For (†) this is simply an unpacking of the fact that the image of x in $\mathrm{E{\small ND}}(\mathcal{C})$ is the identity. Indeed, the right-hand half of (†) is simply the $\mathcal{C}$-valued pairing of $\mathcal{C} \boxtimes \mathcal{C}^*$ with $\mathcal{C}$ obtained by first mapping the former into $\mathrm{E{\small ND}}(\mathcal{C})$ and then evaluating $\mathcal{C}$-endofunctors at a given object in $\mathcal{C}$.

The verification is almost as simple for (‡). The desired isomorphism can be tested against $\mathcal{C}$ by pairing via ev; in other words, it is enough to show that the composition $\mathrm{ev} \circ ((‡) \boxtimes \mathrm{id}_{\mathcal{C}})$ is naturally isomorphic to $\mathrm{ev} : \mathcal{C}^* \boxtimes \mathcal{C} \to \mathrm{V{\small ECT}}$. A diagram chase shows that this composition is isomorphic to $\mathrm{ev} \circ (\mathrm{id}_{\mathcal{C}^*} \boxtimes (†))$, which in turn is isomorphic to ev because $(†) \cong \mathrm{id}_{\mathcal{C}}$.

(4) ⇒ (3) For every $\mathcal{D}, \mathcal{E} \in \mathrm{P{\small RES}}_{\mathbb{K}}$, the functors ev and coev induce functors

$$\begin{tikzcd} \mathrm{H{\small OM}}(\mathcal{D} \boxtimes \mathcal{C}, \mathcal{E}) \arrow[r, bend left, "R_{\mathcal{D},\mathcal{E}}"] & \mathrm{H{\small OM}}(\mathcal{D}, \mathcal{E} \boxtimes \mathcal{C}^*), \arrow[l, bend left, "L_{\mathcal{D},\mathcal{E}}"] \end{tikzcd}$$

natural in $\mathcal{D}$ and $\mathcal{E}$ in the obvious sense. The functor $R_{\mathcal{D},\mathcal{E}}$, for instance, sends $F : \mathcal{D}\boxtimes\mathcal{C} \to \mathcal{E}$ to



,

while $L_{\mathcal{D},\mathcal{E}}$ is defined similarly using ev.

Now, the conditions in (4) imply that $R_{\mathcal{D},\mathcal{E}}$ and $L_{\mathcal{D},\mathcal{E}}$ are mutually inverse, so in particular each $R_{\mathcal{D},\mathcal{E}}$ is an equivalence. Collectively, the $R_{\mathcal{D},\mathcal{E}}$ implement an adjunction between $\bullet \boxtimes \mathcal{C}$ and $\bullet \boxtimes \mathcal{C}^*$ (where the former is the left adjoint) as in [Fio06, Definition 9.8]. The identification of the counit with ev as in ($*$) is now easy.

(3) ⇒ (2) By definition $\mathrm{Hom}(\mathcal{C}, \bullet)$ is a right adjoint to $\bullet \boxtimes \mathcal{C}$. By the uniqueness of adjoints between bicategories (e.g. [Fio06, Theorem 9.20]), we can find an equivalence $\eta_{\mathcal{D}} : \mathcal{D} \boxtimes \mathcal{C}^* \simeq \mathrm{Hom}(\mathcal{C}, \mathcal{D})$, natural in $\mathcal{D}$, that intertwines the counits of the two adjunctions: the diagram

$$\begin{tikzcd} & \mathrm{Hom}(\mathcal{C},\mathcal{D}) \boxtimes \mathcal{C} \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{D} \boxtimes \mathcal{C}^* \boxtimes \mathcal{C} \arrow[ur, "\eta_{\mathcal{D}}\boxtimes \mathrm{id}_{\mathcal{C}}"] \arrow[rr, "\mathrm{id}_{\mathcal{D}}\boxtimes \mathrm{ev}"'] & {} & \mathcal{D} \end{tikzcd}$$

commutes up to natural isomorphism for every $\mathcal{D} \in \mathrm{Pres}_{\mathbb{K}}$. Moreover, we get a similar commutative diagram if we substitute $\mathrm{can}_{\mathcal{D}}$ for $\eta_{\mathcal{D}}$. But then, by the universality of the counit of an adjunction (the so-called biuniversality of [Fio06, Definition 9.4], or rather its dual), the two functors $\eta_{\mathcal{D}}$ and $\mathrm{can}_{\mathcal{D}}$ are naturally isomorphic; see e.g. [Fio06, Lemma 9.7].

(2) ⇒ (1) Indeed, Definition 1.1 is the particular instance of (2) obtained by taking $\mathcal{D} = \mathcal{C}$.

(3) ⇒ (6) Simply set $\mathcal{C}' = \mathcal{C}^*$.

(6) ⇒ (3) The equivalence

$$\mathcal{C}' \simeq \mathrm{Hom}(\mathrm{Vect}, \mathcal{C}') \simeq \mathrm{Hom}(\mathcal{C}, \mathrm{Vect})$$

resulting from the adjunction identifies $\mathcal{C}'$ with $\mathcal{C}^*$ in such a way that the counit $\mathcal{C}' \boxtimes \mathcal{C} \to \mathrm{Vect}$ gets identified with the evaluation $\mathrm{ev} : \mathcal{C}^* \boxtimes \mathcal{C} \to \mathrm{Vect}$. ∎

3.2. Remark. Condition (6) in Lemma 3.1 makes it clear that the base change 2-functor $\bullet \boxtimes \mathrm{Vect}_{\mathbb{L}} : \mathrm{Pres}_{\mathbb{K}} \to \mathrm{Pres}_{\mathbb{L}}$ from Remark 2.11 preserves dualizability. Indeed, $\bullet \boxtimes (\mathcal{C}' \boxtimes \mathrm{Vect}_{\mathbb{L}})$ is right adjoint to $\bullet \boxtimes (\mathcal{C} \boxtimes \mathrm{Vect}_{\mathbb{L}})$ on $\mathrm{Pres}_{\mathbb{L}}$ whenever $\bullet \boxtimes \mathcal{C}'$ is right adjoint to $\bullet \boxtimes \mathcal{C}$ on $\mathrm{Pres}_{\mathbb{K}}$.

3.3. Remark. Let $(\mathcal{C}_i)_{i\in\mathcal{I}}$ be a family of dualizable locally presentable $\mathbb{K}$-linear categories. Then its direct sum $\bigoplus_{i\in I}\mathcal{C}_i$ is dualizable. This follows from (1) $\Leftrightarrow$ (2) in Lemma 3.1 and Remark 2.6.

For future use, we note the following consequence of Lemma 3.1.

3.4. Corollary. *Suppose that $\iota : \mathcal{D} \to \mathcal{C}$ is a cocontinuous $\mathbb{K}$-linear functor between locally presentable $\mathbb{K}$-linear categories, and $\pi : \mathcal{C} \to \mathcal{D}$ a cocontinuous $\mathbb{K}$-linear functor with $\pi \circ \iota \cong \mathrm{id}_{\mathcal{D}}$. If $\mathcal{C}$ is dualizable, then so is $\mathcal{D}$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{D} \boxtimes \mathcal{C}^* \arrow[r, "\simeq"] \arrow[d, "\mathrm{id}_{\mathcal{D}}\boxtimes\iota^*"'] & \textsc{Hom}(\mathcal{C},\mathcal{D}) \arrow[d, "\iota^*"] \\
\mathcal{D} \boxtimes \mathcal{D}^* \arrow[r] & \textsc{Hom}(\mathcal{D},\mathcal{D}),
\end{tikzcd}$$

where the vertical arrows are both given by restriction along $\iota : \mathcal{D} \to \mathcal{C}$.

If $\mathcal{C}$ is dualizable, then by Lemma 3.1 the upper horizontal functor is an equivalence. The hypotheses imply that $\mathrm{id}_{\mathcal{D}}$ is in the essential image of the right hand vertical arrow (e.g. it is the image of $\pi \in \textsc{Hom}(\mathcal{C},\mathcal{D})$), and hence also in the image of the lower horizontal functor. But then the equivalence between (1) and (5) of Lemma 3.1 applies to prove dualizability. ∎

We may now begin establishing certain categories as either dualizable or not. First, Example 2.14 generalizes immediately to all categories strongly generated by compact projectives:

3.5. Lemma. *Suppose that a locally presentable $\mathbb{K}$-linear category $\mathcal{C}$ has a strongly generating set Γ consisting entirely of compact projective objects. Then $\mathcal{C}$ is dualizable.*

Proof. By Proposition 2.4, the Yoneda functor establishes an equivalence $\mathcal{C} \simeq \textsc{Fun}_{\mathbb{K}}(\Gamma^{\mathrm{op}}, \textsc{Vect})$. Its dual is then $\mathcal{C}^* = \textsc{Hom}(\mathcal{C}, \textsc{Vect}) \simeq \textsc{Fun}_{\mathbb{K}}(\Gamma, \textsc{Vect})$, where the pairing $\mathcal{C} \boxtimes \mathcal{C}^* \to \textsc{Vect}$ is computed as a coend over Γ, and

$$\textsc{Hom}(\textsc{Fun}_{\mathbb{K}}(\Gamma^{\mathrm{op}}, \textsc{Vect}), \mathcal{D}) \simeq \textsc{Fun}_{\mathbb{K}}(\Gamma, \mathcal{D}) \simeq \textsc{Fun}_{\mathbb{K}}(\Gamma, \textsc{Vect}) \boxtimes \mathcal{D}.$$

Both equivalences use Proposition 2.4; the latter also uses Remark 2.8. The composition is nothing but the functor $\textsc{Hom}(\textsc{Fun}_{\mathbb{K}}(\Gamma^{\mathrm{op}}, \textsc{Vect}), \mathcal{D}) \leftarrow \textsc{Fun}_{\mathbb{K}}(\Gamma^{\mathrm{op}}, \textsc{Vect})^* \boxtimes \mathcal{D}$ induced by the pairing. By taking $\mathcal{D} = \mathcal{C}$ we see that $\mathcal{C} \simeq \textsc{Fun}_{\mathbb{K}}(\Gamma^{\mathrm{op}}, \textsc{Vect})$ is dualizable. ∎

3.6. Remark. We know of no dualizable locally presentable category not of this type, and conjecture that dualizability of a locally presentable $\mathbb{K}$-linear category implies that the category is strongly generated by compact projectives. Theorem 1.3 implies that there are no counterexamples to this conjecture among categories of the form $\mathcal{M}^C$ for C a coassociative coalgebra.

Theorem 1.5 is now a simple consequence of Proposition 2.13.

Proof of Theorem 1.5. Let $\mathcal{O}(X)$ be the $\mathbb{K}$-algebra of regular functions on X. It carries an action of G that is compatible with the multiplication; i.e. $\mathcal{O}(X)$ makes sense as an algebra object in $\mathrm{Rep}(G) = \mathcal{M}^{\mathcal{O}(G)}$, the symmetric monoidal category of comodules over the Hopf algebra $\mathcal{O}(G)$ of regular functions on G.

A quasi-coherent sheaf on $[X/G]$ is nothing but a quasi-coherenet sheaf on X with a compatible G-action. Since X is affine, quasi-coherent sheaves on X are just $\mathcal{O}(X)$-modules, and so $\mathrm{Qcoh}([X/G])$ is the category of $\mathcal{O}(X)$-modules with compatible G-action. But an $\mathcal{O}(X)$-module with compatible G-action is nothing but an $\mathcal{O}(X)$-module-object in the symmetric monoidal category $\mathrm{Rep}(G)$. This puts us well within the enriched world of Definition 2.9 and Remark 2.10: in particular, $\mathrm{Qcoh}([X/G])$ is a $\mathrm{Rep}(G)$-module.

We now apply Proposition 2.13 to the present situation, where we let $\mathcal{C} = \mathrm{Qcoh}([X/G])$ and $\mathcal{S} = \mathrm{Rep}(G)$, to conclude that $\mathcal{C}$ is strongly generated by compact projectives over Vect:

First, $\mathcal{O}(X)$ is a $\mathrm{Rep}(G)$-compact-projective $\mathrm{Rep}(G)$-strongly-generating object in $\mathrm{Qcoh}([X/G])$ (this is an enriched version of Example 2.14, and follows from [Kel05, Theorem 5.26]).

Secondly, $\mathrm{Rep}(G)$ is strongly generated by compact projectives over Vect essentially by [Don96]. One result that is part of the main theorem of that paper is that being virtually linearly reductive is equivalent to the monoidal unit $\mathbb{K}$ in $\mathrm{Rep}(G)$ having finite-dimensional injective envelope E. It is easy to see then that E^* is projective in $\mathrm{Rep}(G)$ (and hence compact projective because it is also finite-dimensional).

For every finite-dimensional object $V \in \mathrm{Rep}(G)$ the endofunctor $- \otimes V$ of $\mathrm{Rep}(G)$ is left adjoint to the exact functor $- \otimes V^*$, and hence preserves projectivity. Since $\mathbb{K}$ is surjected upon by E^*, V is surjected upon by the compact projective object $E^* \otimes V$. Since objects in the comodule category $\mathrm{Rep}(G)$ are unions of finite-dimensional subobjects, the conclusion that $\mathrm{Rep}(G)$ is strongly generated by compact projectives follows.

Finally, dualizability of $\mathrm{Qcoh}([X/G])$ is a consequence of Lemma 3.5. ∎

We turn now to the category $\mathcal{M}^C$ of right comodules for C a coassociative coalgebra. Since $\mathcal{M}^C$ is abelian and in abelian categories generating sets are strongly generating, Lemma 3.5 implies that $\mathcal{M}^C$ is dualizable if it is generated by its compact projectives. This, of course, fails in general, as the following well-known example illustrates:

3.7. Example. Consider the coalgebra $\mathbb{K}[x]$ with comultiplication $x^n \mapsto \sum_{i=0}^n x^i \otimes x^{n-i}$. We will show that in the category $\mathcal{M}^{\mathbb{K}[x]}$, there are no non-zero projectives. Indeed, suppose that $V \in \mathcal{M}^{\mathbb{K}[x]}$ is not the zero object. To see that V is not projective, it suffices to witness a surjection $W \twoheadrightarrow V$ that does not split.

The finite-dimensional subcoalgebras of the $\mathbb{K}[x]$ are dual to the algebras $\mathbb{K}[t]/(t^n)$; thus $\mathcal{M}^{\mathbb{K}[x]}$ is the category whose objects are vector spaces V equipped with a locally nilpotent endomorphism t, i.e. an endomorphism t such that for each $v \in V$, there is some $n \in \mathbb{N}$ with $t^n(v) = 0$. In particular, $\ker t$ is not zero if V is not zero. Split V as a vector space as $V = \bar{V} \oplus \ker t$. The action of t then has form:

$$t = \begin{pmatrix} \bar{t} & 0 \\ \tau & 0 \end{pmatrix}$$

for some $\bar{t} : \bar{V} \to \bar{V}$ and $\tau : \bar{V} \to \ker t$. We then define $W = V \oplus \ker t = \bar{V} \oplus \ker t \oplus \ker t$, and give it the locally nilpotent endomorphism

$$t = \begin{pmatrix} \bar{t} & 0 & 0 \\ \tau & 0 & 0 \\ 0 & \mathrm{id}_{\ker t} & 0 \end{pmatrix}.$$

The map $W \twoheadrightarrow V$ kills the second copy of $\ker t$. Any splitting will take $v \in \ker t$ to some element $w \in W$ for which $t(w) = (0, 0, v) \in \bar{V} \oplus \ker t \oplus \ker t$. Thus no splitting is t-linear, and V is not projective.

Note that $\mathbb{K}[x]$ is the Hopf algebra of functions on the (non-reductive) additive group $\mathbb{G}_a$. Thus $\mathcal{M}^{\mathbb{K}[x]} = \mathrm{QCOH}\big((\mathrm{Spec}\mathbb{K})/\mathbb{G}_a\big)$, and so Theorem 1.5 fails without the virtual linear reducibility requirement.

3.8. Lemma. *Let C be a coassociative coalgebra and $\{C_i\}_{i\in\mathcal{I}}$ the partially ordered set of finite-dimensional sub-coalgebras of C. Let $\mathcal{D}$ be a locally presentable linear category. The canonical functor $\mathcal{M}^C \boxtimes \mathcal{D}^* \to \mathrm{HOM}(\mathcal{D}, \mathcal{M}^C)$ is fully faithful. Its essential image consists of those functors F occurring as colimits $F = \varinjlim_i F_i$ where $F_i : \mathcal{D} \to \mathcal{M}^C$ factors through the inclusion $\mathcal{M}^{C_i} \hookrightarrow \mathcal{M}^C$.*

Proof. The hom-tensor adjunction of Lemma 2.7 implies $\boxtimes$ distributes over colimits. Thus there are canonical equivalences:

$$\mathcal{M}^C \boxtimes \mathcal{D}^* \simeq (\varinjlim_i \mathcal{M}^{C_i}) \boxtimes \mathcal{D}^* \simeq \varinjlim_i \big(\mathcal{M}^{C_i} \boxtimes \mathcal{D}^*\big) \simeq \varinjlim_i \mathrm{HOM}(\mathcal{D}, \mathcal{M}^{C_i})$$

In the last step we used dualizability of $\mathcal{M}^{C_i} \cong \mathcal{M}_{C_i^*}$ to imply that the canonical functor from $\mathcal{M}^{C_i} \boxtimes \mathcal{D}^*$ to $\mathrm{HOM}(\mathcal{D}, \mathcal{M}^{C_i})$ induced by the pairing is an equivalence: by Lemma 3.1 part (3) and the hom-tensor adjunction,

$$\begin{aligned} \mathcal{M}^{C_i} \boxtimes \mathcal{D}^* = \mathcal{M}^{C_i} \boxtimes \mathrm{HOM}(\mathcal{D}, \mathrm{VECT}) &\simeq \mathrm{HOM}\big((\mathcal{M}^{C_i})^*, \mathrm{HOM}(\mathcal{D}, \mathrm{VECT})\big) \\ \simeq \mathrm{HOM}\big((\mathcal{M}^{C_i})^* \boxtimes \mathcal{D}, \mathrm{VECT}\big) &\simeq \mathrm{HOM}\big(\mathcal{D}, \mathrm{HOM}\big((\mathcal{M}^{C_i})^*, \mathrm{VECT}\big)\big) \simeq \mathrm{HOM}\big(\mathcal{D}, \mathcal{M}^{C_i}\big). \end{aligned}$$

Since the pairing-induced functor $\mathcal{C} \boxtimes \mathcal{D}^* \to \mathrm{HOM}(\mathcal{D}, \mathcal{C})$ is natural in $\mathcal{C}$, we conclude that the inclusion

$$\varinjlim_i \mathrm{HOM}(\mathcal{D}, \mathcal{M}^{C_i}) \simeq \mathcal{M}^C \boxtimes \mathcal{D}^* \to \mathrm{HOM}(\mathcal{D}, \mathcal{M}^C)$$

is the one induced by the inclusions $\mathcal{M}^{C_i} \hookrightarrow \mathcal{M}^C$. Lemma 2.5 and Remark 2.6 complete the proof. ∎

We are now equipped to prove Theorem 1.3, which asserts that $\mathcal{M}^C$ is dualizable if and only if it has enough projectives.

Proof of Theorem 1.3. According to part (d) of the main theorem of [Lin77], the category $\mathcal{M}^C$ of right C-comodules has enough projectives if and only if every finite-dimensional right C-comodule has a projective cover; inspection of the proof reveals the attested projective cover to be finite-dimensional. The finite-dimensional right C-comodules are precisely the compact ones, and so we see that $\mathcal{M}^C$ has enough projectives if and only if it is generated by its compact projective objects. Lemma 3.5 then implies one direction of the claim.

Suppose now that $\mathcal{M}^C$ does not have enough projectives. Then, again by [Lin77] (this time part (c) of the main theorem), there exists a simple left C-comodule S such that the injective hull of S is infinite-dimensional. It follows that we can find essential extensions $S \hookrightarrow T$ with $\dim T$ finite but arbitrarily large (as otherwise there would be a maximal such T, which would therefore be injective). Recall that an extension $S \hookrightarrow T$ is *essential* if any non-zero subobject of T intersects S nontrivially; the dual notion (for abelian categories) is an *essential projection* $Q \twoheadrightarrow P$, which is a surjection for which every proper subobject of Q fails to surject onto P. Thus, by dualization, we have found a simple right C-comodule $S^* \in \mathcal{M}^C$ with essential projections $T^* \twoheadrightarrow S^*$ of arbitrarily large dimension. (The map $S \mapsto S^*$ is a contravariant equivalence between the abelian category of *finite-dimensional* left C-comodules and the abelian category of finite dimensional right C-comodules.)

Let $F \in \mathrm{Hom}(\mathcal{M}^C, \mathcal{M}^C)$ be in the essential image of $(\mathcal{M}^C)^* \boxtimes \mathcal{M}^C$. By Lemma 3.8, $F = \varinjlim F_i$, where F_i is the largest subfunctor of F factoring through $\mathcal{M}^{C_i}$. We will prove that $F \ncong \mathrm{id}_{\mathcal{M}^C}$. To do so, consider an arbitrary natural transformation $\theta : F \to \mathrm{id}_{\mathcal{M}^C}$, or, what is equivalent, a system of natural transformations $\theta_i : F_i \to \mathrm{id}_{\mathcal{M}^C}$. Since $\varinjlim(F_i(X)) = (\varinjlim F_i)(X)$ for all $X \in \mathcal{M}^C$, it suffices to prove that $\theta_i(S^*) : F_i(S^*) \to S^*$ vanishes for all sufficiently large i and for S^* the simple with arbitrarily large essential surjections from the previous paragraph. Since $\dim(S^*) < \infty$, for all sufficiently large i we have $S^* \in \mathcal{M}^{C_i}$, and it suffices to consider just these.

Thus fix $i \in \mathcal{I}$ with $S^* \in \mathcal{M}^{C_i} \cong \mathcal{M}_{C_i^*}$. Since $\dim(C_i^*) < \infty$, there are bounds on the dimensions of essential surjections onto S^* in $\mathcal{M}^{C_i}$: a projective cover is as large as you can get. We can therefore choose T^* mapping essentially onto S^* so large that $T^* \notin \mathcal{M}^{C_i}$. Consider the following commutative diagram:

$$\begin{tikzcd}
F_i(T^*) \arrow[r, "\theta_i(T^*)"] \arrow[d, two heads] & T^* \arrow[d, two heads] \\
F_i(S^*) \arrow[r, "\theta_i(S^*)"] & S^*
\end{tikzcd}$$

The left arrow is a surjection because F_i is cocontinuous. Since $F_i(T^*) \in \mathcal{M}^{C_i}$ but $T^* \notin \mathcal{M}^{C_i}$, the image of $\theta_i(T^*)$ must be a proper sub-co-module of T^*. (Indeed, it is within the largest C_i-subcomodule of T^*.) Since the surjection $T^* \to S^*$ is essential, the composition $F_i(T^*) \to S^*$ cannot be a surjection; since S^* is simple, the composition must vanish. But $F_i(T^*) \to F_i(S^*)$ is a surjection; hence $\theta_i(S^*) = 0$. ∎

A similar argument works for $\mathrm{QCOH}(X)$ when X is a projective scheme over $\mathbb{K}$, and provides the basis of the proof of Theorem 1.4. Recall that X is a *projective scheme* if it is embeddable as a closed subscheme into some projective space $\mathbb{P}^N_{\mathbb{K}}$.

First, we specialize Corollary 3.4 as follows.

3.9. COROLLARY. *If a $\mathbb{K}$-scheme X is such that $\mathrm{QCOH}(X)$ is dualizable, then the same is true of all closed subschemes $i : Y \subseteq X$.*

PROOF. Setting $\mathcal{C} = \mathrm{QCOH}(X)$, $\mathcal{D} = \mathrm{QCOH}(Y)$, $\pi = i^*$ and $\iota = i_*$ places us within the scope of Corollary 3.4.

Indeed, note first that $i_* : \mathrm{QCOH}(Y) \to \mathrm{QCOH}(X)$ is a left adjoint whenever $i : Y \subseteq X$ is a closed subscheme, with right adjoint $\mathcal{H}om(\mathcal{O}_X/\mathcal{I}_Y, -)$ where $\mathcal{I}_Y$ is the sheaf of ideals defining the inclusion $Y \subseteq X$ ([Har77, Proposition II.5.9]) and $\mathcal{H}om$ denotes the sheaf of homomorphisms ([Har77, Definitions preceding II.5.1]).

Second, note that the counit $i^*i_* \to \mathrm{id}$ is an isomorphism whenever i is a closed embedding: i_* implements an equivalence of $\mathrm{QCOH}(Y)$ with the category of quasi-coherent sheaves on X annihilated by the sheaf of ideals $\mathcal{I}_Y$. In particular i_* is fully faithful, hence the claim. ∎

PROOF OF THEOREM 1.4. We make the problem progressively simpler as follows.

Step 1: Reduction to algebraically closed ground fields. We claim that for any commutative $\mathbb{K}$-algebra A we have

$$\mathrm{QCOH}(X \times \mathrm{Spec}(A)) \simeq \mathrm{QCOH}(X) \boxtimes {}_A\mathcal{M}.$$

To see this, note first that the canonical functor

$$\mathrm{QCOH}(X) \boxtimes {}_A\mathcal{M} \to \mathrm{QCOH}(X \times \mathrm{Spec}(A))$$

is an equivalence when $X = \mathrm{Spec}(B)$ is affine by the Eilenberg–Watts theorem. Then, covering X by affine open subschemes $U_i = \mathrm{Spec}(A_i)$, we have canonical functors

$$\mathrm{QCOH}(X) \boxtimes {}_A\mathcal{M} \xrightarrow{\sim} \big(\varprojlim {}_{A_i}\mathcal{M}\big) \boxtimes {}_A\mathcal{M} \xrightarrow{\sim} \varprojlim\big({}_{A_i}\mathcal{M} \boxtimes {}_A\mathcal{M}\big),$$

where the first equivalence follows from Example 2.16 and the second one follows from the fact that ${}_A\mathcal{M}$ is dualizable and hence $- \boxtimes {}_A\mathcal{M}$ preserves limits in $\mathrm{PRES}_{\mathbb{K}}$.

We then further have

$$\varprojlim\big({}_{A_i}\mathcal{M} \boxtimes {}_A\mathcal{M}\big) \xrightarrow{\sim} \varprojlim \mathrm{QCOH}(U_i \times \mathrm{Spec}(A)) \xrightarrow{\sim} \mathrm{QCOH}(X \times \mathrm{Spec}(A))$$

by another application of Example 2.16 to the open affine cover $U_i \times \mathrm{Spec}(A) \cong \mathrm{Spec}(A_i \otimes A)$ of $X \times \mathrm{Spec}(A)$.

Now apply this observation to an algebraic closure $A = \mathbb{L}$ of $\mathbb{K}$ and note that if $\mathrm{QCOH}(X)$ is dualizable over $\mathrm{VECT}_{\mathbb{K}}$, then by Remark 3.2 $\mathrm{QCOH}(X \times \mathrm{Spec}(\mathbb{L})) \simeq$

$\mathrm{QCOH}(X) \boxtimes \mathrm{VECT}_{\mathbb{L}}$ is dualizable over $\mathrm{VECT}_{\mathbb{L}}$. Consequently, throughout the rest of the proof we will assume that $\mathbb{K}$ was algebraically closed to begin with.

Step 2: Reduction to integral projective schemes. Corollary 3.9 allows us to do this by transporting dualizability first from X to a closed projective subscheme of positive dimension and then to the reduced induced subscheme structure on a positive-dimensional irreducible component of that (c.f. [Har77, Example II.3.2.6]).

Step 3: The case of integral projective schemes over algebraically closed fields. Any embedding $X \hookrightarrow \mathbb{P}^N$ into some projective space provides a strongly generating set of line bundles $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n} \in \mathrm{QCOH}(X)$, with $\mathcal{O}(1)$ being the pullback of the twisting sheaf $\mathcal{O}_{\mathbb{P}^N}(1)$ (see [Har77, Definition preceding Proposition II.5.12]; the fact that they generate is a consequence of [Har77, Theorem II.5.17]).

Note that there are no non-zero maps $\mathcal{O}(m) \to \mathcal{O}(n)$ for $m > n$, or in other words that $\mathcal{O}(-n)$ has no non-zero global sections if $n > 0$. Indeed, suppose for contradiction that $\mathcal{O}(-n)$ has a non-zero global section $s : \mathcal{O} \to \mathcal{O}(-n)$. Since X is integral, each line bundle $\mathcal{O}(n)$ is a subsheaf of the constant sheaf $\mathcal{K}$ associated to the field of rational functions on X (see e.g. [Har77, Proposition II.6.15] and surrounding discussion), and so s can be regarded as a rational function, and the section $s^{\otimes k}$ of $\mathcal{O}(-kn) \cong \mathcal{O}(-n)^{\otimes k}$ is simply its kth power and hence again non-zero. Moreover, $\otimes s^{\otimes k} : \mathcal{O}(kn) \to \mathcal{O}(kn) \otimes \mathcal{O}(-kn) \cong \mathcal{O}$ is an embedding on spaces of global sections.

But the function

$$m \mapsto \text{ dimension of the space of sections of } \mathcal{O}(m)$$

is a polynomial of degree $\dim(X)$ for large m [Har77, Theorem I.7.5 and Exercise II.7.6], whereas the only global regular functions on an integral projective scheme over an algebraically closed field are the constants [Har77, Theorem I.3.4]. Thus for k sufficiently large, no map $\mathcal{O}(kn) \to \mathcal{O}$ can be an embedding on global sections. Thus the non-zero section $s : \mathcal{O} \to \mathcal{O}(-n)$ cannot exist.

If $\mathrm{QCOH}(X)_{\geq n}$ is the full, cocomplete subcategory of $\mathrm{QCOH}(X)$ strongly generated by $\{\mathcal{O}(m)\}_{m \geq n}$, then $\mathrm{QCOH}(X) \simeq \varinjlim_{n \to -\infty} \mathrm{QCOH}(X)_{\geq n}$. Without dualizability of $\mathrm{QCOH}(X)_{\geq n}$, the essential image of $\mathrm{QCOH}(X)^* \boxtimes \mathrm{QCOH}(X)$ inside $\mathrm{HOM}(\mathrm{QCOH}(X), \mathrm{QCOH}(X))$ may fail to include all colimits of the form $F = \varinjlim F_n$ for F_n factoring through $\mathrm{QCOH}(X)_{\geq n}$, but the arguments of Lemma 3.8 do imply that any F in the essential image is of this type. Thus, as in the proof of Theorem 1.3, it suffices to find a non-zero object $M \in \mathrm{QCOH}(X)$ such that for all sufficiently negative n, any natural transformation $\theta_n : F_n \to \mathrm{id}_{\mathrm{QCOH}(X)}$ satisfies $\theta_n(M) = 0 : F_n(M) \to M$.

Fix k arbitrarily and set $M = \mathcal{O}(k)$. For any $n \leq k$, we can find some direct sum of $\mathcal{O}(m)$s with $m < n$ that surjects onto $\mathcal{O}(k)$. We thus build a commutative square similar to the one from the proof of Theorem 1.3:

$$\begin{tikzcd}
F_n\big(\bigoplus \mathcal{O}(m)\big) \arrow[r, "\theta_n\left(\bigoplus \mathcal{O}(m)\right)"] \arrow[d] & \bigoplus \mathcal{O}(m) \arrow[d] \\
F_n\big(\mathcal{O}(k)\big) \arrow[r, "\theta_n\left(\mathcal{O}(k)\right)"] & \mathcal{O}(k)
\end{tikzcd}$$

As before, the left arrow is a surjection since F_n is cocontinuous. But $F_n\big(\bigoplus \mathcal{O}(m)\big) \in \mathrm{QCOH}(X)_{\geq n}$, and so $\theta_n\big(\bigoplus \mathcal{O}(m)\big) = 0$. It follows that $\theta_n\big(\mathcal{O}(k)\big) = 0$. ■

4. Reflexivity

The goal of this section is to prove Proposition 4.8, which we will use to prove Theorems 1.6 and 1.8. The arguments in this section apply when $\mathbb{K}$ is not a field but just a commutative ring, in which case "VECT" means the symmetric monoidal category $\mathcal{M}_{\mathbb{K}}$ of all $\mathbb{K}$-modules.

In this section we denote by $(\mathcal{I}, \leq)$ a generic poset which is $\aleph_0$-directed, which means that any two $i, j \in \mathcal{I}$ are dominated by some $k \in \mathcal{I}$. As always, we regard $\mathcal{I}$ as a category with an arrow $i \to j$ for $i \leq j$.

4.1. DEFINITION. *An* $\mathcal{I}$-indexed pro-object *(or just pro-object, when $\mathcal{I}$ is understood) in a category $\mathcal{T}$ is a functor $\mathcal{I}^{\mathrm{op}} \to \mathcal{T}$. An* $\mathcal{I}$-indexed pro-algebra *(or just pro-algebra) is an $\mathcal{I}$-indexed pro-object in the category* ALG *of $\mathbb{K}$-algebras and homomorphisms. The morphisms between an $\mathcal{I}$-indexed pro-algebra $A = \{A_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ and a $\mathcal{J}$-indexed pro-algebra $B = \{B_j\}_{j \in \mathcal{J}^{\mathrm{op}}}$ are*

$$\hom(A, B) = \varprojlim_{j \in \mathcal{J}} \varinjlim_{i \in \mathcal{I}} \hom(A_i, B_j).$$

An acceptable *$\mathcal{I}$-indexed pro-algebra is a pro-algebra whose indexing poset $\mathcal{I}$ has countable cofinality (i.e. it has a countable cofinal subset), and for which the transition maps $A_j \to A_i$ for $i \leq j$ are onto.*

4.2. REMARK. Given a homomorphism $A \to B$ of associative algebras, both the scalar restriction functor $\mathcal{M}_B \to \mathcal{M}_A$ and the scalar extension functor $(-) \otimes_A B : \mathcal{M}_A \to \mathcal{M}_B$ are cocontinuous, and so we have 2-functors Restrict : $\mathrm{ALG}^{\mathrm{op}} \to \mathrm{PRES}_{\mathbb{K}}$ and Extend : $\mathrm{ALG} \to \mathrm{PRES}_{\mathbb{K}}$. Thus any $\mathcal{I}^{\mathrm{op}}$-indexed diagram $A = \{A_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ defines, by using scalar restriction, an $\mathcal{I}$-indexed diagram $\{\mathcal{M}_{A_i}\}_{i \in \mathcal{I}}$ in $\mathrm{PRES}_{\mathbb{K}}$ as well as, by using scalar extension, an $\mathcal{I}^{\mathrm{op}}$-indexed diagram $\{\mathcal{M}_{A_i}\}_{i \in \mathcal{I}^{\mathrm{op}}}$. Whenever we write $\varinjlim_{i \in \mathcal{I}} \mathcal{M}_{A_i}$, we will mean the colimit of the former; when we write $\varprojlim_{i \in \mathcal{I}^{\mathrm{op}}} \mathcal{M}_{A_i}$, we will mean the limit of the latter.

The notions of "pro-object" and "ind-object" are not restricted to categories — they also make sense in any bicategory. Since (2-)functors take pro- and ind-objects to pro- and ind-objects, for each pro-algebra $\{A_i\}_{i \in \mathcal{I}}$, we get, via either restricting or extending, an ind-object $\{\mathcal{M}_{A_i}\}_{i \in \mathcal{I}}$ in $\mathrm{PRES}_{\mathbb{K}}$ and a pro-object $\{\mathcal{M}_{A_i}\}_{i \in \mathcal{I}^{\mathrm{op}}}$ in $\mathrm{PRES}_{\mathbb{K}}$, both varying functorially for morphisms of pro-algebras. Moreover, just as in the case of 1-categories, the bicategorical $\varinjlim$ and $\varprojlim$ are functors from ind- and pro-objects, respectively, to objects. It follows in particular that the assignments

$$\{A_i\}_{i\in\mathcal{I}^{\mathrm{op}}} \mapsto \varinjlim_i \mathcal{M}_{A_i}$$

and

$$\{A_i\}_{i\in\mathcal{I}^{\mathrm{op}}} \mapsto \varprojlim_i \mathcal{M}_{A_i}$$

take isomorphic pro-algebras to equivalent categories.

4.3. Remark. The cofinal countability of $\mathcal{I}$ for acceptable pro-algebras will come up in a number of ways:

First, it allows us to assume that $(\mathcal{I}, \leq)$ is $\mathbb{N} = \{0, 1, \ldots\}$ with the usual order, as pro-algebras indexed by varying posets form a category, and every pro-object whose indexing poset has countable cofinality is isomorphic in this category to an $\mathbb{N}$-indexed one. Since Remark 4.2 assures that the discussion below is invariant under isomorphism in the category of pro-algebras, we will make such a substitution in the sequel.

Second, for an acceptable $\mathbb{N}$-indexed pro-algebra $\{A_i\}$ the map from the limit $\widehat{A} := \varprojlim_i A_i$ to each A_j is onto. In general, when $\mathcal{I}$ has uncountable cofinality, it is possible for each A_i to be nontrivial (i.e. not the ground field) and all transition maps $A_j \to A_i$ to be onto but nevertheless for $\varprojlim A_i$ to be trivial; see e.g. [Ber, Corollary 8].

4.4. Definition. *A module V over a filtered limit $\widehat{A} = \varprojlim_i A_i$ of algebras is called* discrete *if for every $v \in V$ there is some $i \in \mathbb{N}$ for which v is killed by* $\ker(\widehat{A} \twoheadrightarrow A_i)$.

4.5. Lemma. *If $\{A_i\}$ is an acceptable pro-algebra, the canonical functor*

$$(\varinjlim_i \mathcal{M}_{A_i}) \boxtimes \mathcal{M}_B \simeq \varinjlim_i (\mathcal{M}_{A_i} \boxtimes \mathcal{M}_B) \to \mathcal{M}_{\widehat{A}} \boxtimes \mathcal{M}_B \simeq \mathcal{M}_{\widehat{A}\otimes B},$$

obtained by restricting scalars along $\widehat{A} \to A_i$, is full and faithful. Its essential image consists of those $(\widehat{A} \otimes B)$-modules V whose underlying $\widehat{A}$-modules are discrete.

Proof. According to Lemma 2.5 and Remark 2.6, an object of $\varinjlim_i(\mathcal{M}_{A_i} \boxtimes \mathcal{M}_B)$ consists of a sequence $\{M_i\}_{i\in\mathbb{N}}$, where, for each i, M_i is an A_i-module in $\mathcal{M}_B$, together with compatible isomorphisms

$$M_i \cong \hom_{A_j}(A_i, M_j) \leq M_j$$

for each $i \leq j$, realizing M_i as the maximal submodule of M_j on which the A_j action factors through $A_j \to A_i$.

The functor $\varinjlim_i(\mathcal{M}_{A_i} \boxtimes \mathcal{M}_B) \to \mathcal{M}_{\widehat{A}} \boxtimes \mathcal{M}_B$ takes such a sequence to the vector space $M = \varinjlim_i M_i$ along with an action by $\widehat{A} \otimes B$ that factors through $A_i \otimes B$ when acting on the subvector space M_i. Note that, for each i, M_i can be recovered from M as the maximal subspace on which the $\widehat{A}$-action factors through A_i:

$$M_i \cong \hom_{\widehat{A}}(A_i, M)$$

Conversely, any $\widehat{A}\otimes B$-module V determines a filtered vector space whose ith filtered piece V_i is the maximal subspace of V on which the $\widehat{A}$-action factors through A_i. Discreteness of V is equivalent to the canonical map $\varinjlim V_i \to V$ being an isomorphism. Thus the discrete modules are precisely the ones in the essential image of $\varinjlim_i(\mathcal{M}_{A_i} \boxtimes \mathcal{M}_B) \to \mathcal{M}_{\widehat{A}} \boxtimes \mathcal{M}_B$.

Suppose that $M = \varinjlim M_i$ and $N = \varinjlim N_i$ are discrete, and that $f : M \to N$ is an $\widehat{A} \otimes B$-linear map. Then f restricts to an $A_i \otimes B$-linear map $f_i : M_i \to N_i$ for each i, and the sequence $\{f_i\}_{i\in\mathbb{N}}$ defines a morphism $\{M_i\}_{i\in\mathbb{N}} \to \{N_i\}_{i\in\mathbb{N}}$, whose image under the functor $\varinjlim_i \mathcal{M}_{A_i} \to \mathcal{M}_{\widehat{A}}$ is f. Full faithfulness follows. ∎

Note that we have used surjectivity of $\widehat{A} \to A_i$ from Remark 4.3; Lemma 4.5 can fail for $\mathcal{I}$ uncountable.

Since $(-)^* = \textsc{Hom}(-, \textsc{Vect})$ turns colimits into limits, we have

$$(\varinjlim \mathcal{M}_{A_i})^* \simeq \varprojlim(\mathcal{M}_{A_i})^* \simeq \varprojlim({}_{A_i}\mathcal{M})$$

(where the second equivalence uses the Eilenberg–Watts theorem). An object of this limit is a sequence $\{M_i\}_{i\in\mathbb{N}}$ whose ith entry is a left A_i-module, along with isomorphisms $M_i \cong A_i \otimes_{A_j} M_j$ for $j > i$, compatible for triples $k > j > i$. Notice that this implies that the corresponding maps $M_j \to M_i$ of A_j-modules are onto. An example is the *regular* module $A = \{A_i\}_{i\in\mathbb{N}}$ with the canonical isomorphisms, which when thought of as an object of $(\varinjlim \mathcal{M}_{A_i})^*$ is nothing but the forgetful functor $\varinjlim \mathcal{M}_{A_i} \to \textsc{Vect}$. It enjoys $\hom(A, A) \cong \widehat{A}$.

More generally, any $M \in \varprojlim({}_{A_i}\mathcal{M})$ has an underlying pro-vector space $M = \{M_i\}_{i\in\mathbb{N}^{\mathrm{op}}}$, and $\hom(A, M) \cong \widehat{M} = \varprojlim M_i$ is its limit in $\textsc{Vect}$. It consists of *compatible systems* of elements $\{m_i \in M_i\}_{i\in\mathbb{N}}$, i.e. m_j is the image of m_i under the projection $M_i \twoheadrightarrow A_j \otimes_{A_i} M_i \cong M_j$.

For the remainder of this section we let $\mathcal{M} = \varprojlim_i {}_{A_i}\mathcal{M}$.

4.6. Lemma. *Let $M = \{M_i\}_{i\in\mathbb{N}}$ and $N = \{N_i\}_{i\in\mathbb{N}}$ be objects in $\mathcal{M}$, and $f : M \to N$ be an epimorphism (i.e. all components $f_i : M_i \to N_i$ are onto). Let also $m_j \in \ker(f_j)$ be an element for some fixed $j \in \mathbb{N}$. Then m_j can be expanded to a compatible system of elements $m_i \in \ker(f_i)$, $i \in \mathbb{N}$.*

Proof. We may as well assume $j = 0$. Choose any preimage $m_1' \in M_1$, and consider its image $n_1' \in N_1$. It lies in $\ker(N_1 \to N_0) = \ker(A_1 \to A_0) \cdot N_1$, which thus lifts to some element in $\ker(A_1 \to A_0) \cdot M_1$. Subtracting this element from m_1', we get our desired lift $m_1 \in M_1$. Now repeat to construct m_2, etc. ∎

4.7. Lemma. *The objects of $\mathcal{M}$ admit functorial free resolutions of the form*

$$N \mapsto (A^{\oplus S_1(N)} \to A^{\oplus S_0(N)} \to N \to 0),$$

where S_i are functors $\mathcal{M} \to \textsc{Set}$.

PROOF. Define $S_0(N) = \widehat{N}$, and the natural epimorphism $A^{\oplus S_0(N)} \to N$ in the obvious way: For $n \in \widehat{N} \cong \hom_{\mathcal{M}}(A, N)$, the map from the summand of $A^{\oplus \widehat{N}}$ indexed by $n \in \widehat{N}$ to N is simply n.

The resulting map $A^{S_0(N)} \to N$ is surjective at each level $j \in \mathbb{N}$, because any $n_j \in N_j$ can be lifted to a compatible system of elements $n_i \in N_i$, $i \in \mathbb{N}$.

Next, let $S_1(N)$ be the set of compatible systems of elements $a_i \in \ker\big(A_i^{\oplus S_0(N)} \to N_i\big)$ and define $A^{\oplus S_1(N)} \to A^{\oplus S_0(N)}$ similarly as before. Taking f to be our morphism $A^{\oplus S_0(N)} \to N$, Lemma 4.6 ensures that

$$A^{\oplus S_1(N)} \to A^{\oplus S_0(N)} \to N \to 0$$

is exact at each level, i.e. that its components in the categories $\mathcal{M}_{A_i}$ are exact. This implies that the diagram is a cokernel in $\mathcal{M}$, as desired. ∎

We are now equipped to prove the following proposition, of which Theorems 1.6 and 1.8 are consequences.

4.8. PROPOSITION. *Let $A = \{A_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ be an acceptable pro-algebra and B be any $\mathbb{K}$-algebra. The functor*

$$\eta : \mathrm{HOM}\Big(\varprojlim({}_{A_i}\mathcal{M}), \mathcal{M}_B\Big) \to \mathcal{M}_{\widehat{A} \otimes B}, \quad F \mapsto F(A)$$

is fully faithful. Its essential image consists of those $\widehat{A} \otimes B$-modules which are discrete over $\widehat{A}$ in the sense of Definition 4.4.

PROOF. As before, we will abbreviate $\mathcal{M} := \varprojlim({}_{A_i}\mathcal{M})$, and use Remark 4.3 to set $\mathcal{I} = \mathbb{N}$.

Step 1: The essential image of η contains all $\widehat{A}$-discrete $\widehat{A} \otimes B$-modules.

Given some $\widehat{A}$-discrete $\widehat{A} \otimes B$-module V, let $V_i := \{v \in V : \ker(\widehat{A} \to A_i) \cdot v = 0\}$ be the largest A_i-submodule of V. Then $i \leq j$ implies $V_i \subseteq V_j$ and we have $V = \varinjlim_i V_i$. If $M \in \mathcal{M}$, we obtain a B-linear map

$$V_i \otimes_{A_i} M_i \cong V_i \otimes_{A_i} (A_i \otimes_{A_j} M_j) \cong V_i \otimes_{A_j} M_j \to V_j \otimes_{A_j} M_j.$$

Define the functor $F : \mathcal{M} \to \mathcal{M}_B$ by $F(M) := \varinjlim_i V_i \otimes_{A_i} M_i$. Clearly, F is cocontinuous and satisfies $F(A) \cong V$.

Step 2: The essential image of η contains only $\widehat{A}$-discrete $\widehat{A} \otimes B$-modules.

Let $F \in \mathrm{HOM}\big(\mathcal{M}, \mathcal{M}_B\big)$. We want to show that for every $v \in F(A)$, there is an $i \in \mathbb{N}$ such that v is killed by $\ker\big(\widehat{A} \twoheadrightarrow A_i\big)$.

Consider the direct sum $A^{\oplus \mathbb{N}}$ of $\aleph_0$-many copies of A in $\mathcal{M}$. The object A, although a generator of $\mathcal{M}$, is not compact when the pro-algebra A does not "stabilize" (i.e. when for infinitely many i the maps $A_{i+1} \to A_i$ are not isomorphisms). A manifestation of this is that $\hom(A, A^{\oplus \mathbb{N}})$ is not $\widehat{A}^{\oplus \mathbb{N}}$ but rather the projective limit $\varprojlim_i (A_i^{\oplus \mathbb{N}})$ as computed in VECT. In other words, a homomorphism $A \to A^{\oplus \mathbb{N}}$ is the same as a system of homomorphisms $\{f_j : A \to A\}_{j \in \mathbb{N}}$ with the property that for all $i \in \mathbb{N}$, all but finitely many of the f_js belong to $\ker\big(\widehat{A} \twoheadrightarrow A_i\big)$.

Suppose now that $v \in F(A)$ is such that for every $i \in \mathbb{N}$, there is some $f_i \in \ker\big(\widehat{A} \to A_i\big)$ which does not annihilate v. These f_is define a homomorphism $f : A \to A^{\oplus \mathbb{N}}$. Since F is cocontinuous, we get a linear map

$$F(f) : F(A) \to F\big(A^{\oplus \mathbb{N}}\big) \cong F(A)^{\oplus \mathbb{N}}$$

for which every component of $F(f)(v)$ is non-zero. This is absurd, since $F(f)(v)$ is supposed to sit inside the direct sum $F(A)^{\oplus \mathbb{N}}$.

Step 3: The functor η is faithful.

This follows immediately from Lemma 4.7 and the fact that colimit-preserving functors preserve epimorphisms.

Step 4: The functor η is full.

Let $F, G \in \textsc{Hom}(\mathcal{M} \to \mathcal{M}_B)$. We have to show that any $\widehat{A} \otimes B$-morphism $\phi : F(A) \to G(A)$ is the A-component θ_A of a natural transformation $\theta : F \to G$.

Recall from Lemma 4.6 that we have functorial free resolutions for objects in $\mathcal{M}$. We claim that to prove fullness of η, it suffices to check, for all sets S and T and all morphisms $f : A^{\oplus S} \to A^{\oplus T}$, the commutativity of

$$\begin{tikzcd} F(A)^{\oplus S} \arrow[r, "F(f)"] \arrow[d, "\phi^{\oplus S}"'] & F(A)^{\oplus T} \arrow[d, "\phi^{\oplus T}"] \\ G(A)^{\oplus S} \arrow[r, "G(f)"] & G(A)^{\oplus T} \end{tikzcd} \qquad (\S)$$

Indeed, we can then set $\theta_{A^{\oplus S}} = \phi^{\oplus S}$ for every set S and extend this to a natural transformation θ via the functorial resolutions.

The right hand vertical arrow in (§) embeds into

$$F(A)^{\times T} \xrightarrow{\phi^{\times T}} G(A)^{\times T}$$

(direct product rather than direct sum). For each $s \in S$ and each $t \in T$, let $\iota_s : A \to A^{\oplus S}$ denote the inclusions onto the sth summand, and $\pi_t : A^{\oplus T} \to A$ the projection onto the tth summand (the latter exists in any additive category). Then the (t, s)th matrix entry in the composition $F(A)^{\oplus S} \xrightarrow{F(f)} F(A)^{\oplus T} \to F(A)^{\times T}$ is just $F(f')$ for $f' = \pi_t \circ f \circ \iota_s$; the (t, s)th matrix entry in $G(A)^{\oplus S} \to G(A)^{\times T}$ is $G(f')$ for the same $f' \in \hom_{\mathcal{M}}(A, A)$. Thus, by replacing f with f' in (§), we can assume S and T are singletons.

Finally, the commutativity of

$$\begin{tikzcd} F(A) \arrow[r, "F(f')"] \arrow[d, "\phi"'] & F(A) \arrow[d, "\phi"] \\ G(A) \arrow[r, "G(f')"] & G(A) \end{tikzcd}$$

follows from the fact that $F(f')$ and $G(f')$ are nothing but the actions of $f' \in \hom_{\mathcal{M}}(A, A) \cong \widehat{A}$ on $F(A)$ and $G(A)$, while ϕ is by definition a morphism of $\widehat{A}$-modules. ∎

4.9. Remark. Actually Proposition 4.8 can be proven in a more general setting and provides a partial universal property of $\varprojlim({}_{A_i}\mathcal{M})$: If $\mathcal{C}$ is a locally finitely presentable $\mathbb{K}$-linear category (i.e. locally presentable and strongly generated by its compact objects), then $\mathrm{Hom}\big(\varprojlim({}_{A_i}\mathcal{M}), \mathcal{C}\big)$ is equivalent to the category of discrete $\widehat{A}$-module objects in $\mathcal{C}$. These are objects $T \in \mathcal{C}$ equipped with a homomorphism of $\mathbb{K}$-algebras $\widehat{A} \to \mathrm{End}_{\mathcal{C}}(T)$ such that $T = \varinjlim_i T_i$, where $T_i := \ker\big(T \to \prod_{a \in \ker(\widehat{A} \to A_i)} T\big)$ is the largest A_i-submodule object of T. The same proof as before works. Only in Step 2 we have to replace v by a morphism $P \to F(A)$, where P is any compact object.

Proof of Theorem 1.6. Lemma 2.5 identifies the category $\mathcal{C}$ from the statement of Theorem 1.6 with the colimit $\varinjlim_i \mathcal{M}_{A_i}$ in $\mathrm{Pres}_{\mathbb{K}}$. Consider the triangle

$$\begin{tikzcd}
\big(\varinjlim_i \mathcal{M}_{A_i}\big) \boxtimes \mathcal{M}_B \arrow[rr, "\mathrm{dd}"] \arrow[dr, hook, "\text{inclusion}"'] & & \mathrm{Hom}\Big(\big(\varinjlim_i \mathcal{M}_{A_i}\big)^*, \mathcal{M}_B\Big) \arrow[dl, "\eta"] \\
& \mathcal{M}_{\widehat{A} \otimes B} &
\end{tikzcd}$$

of functors. First, we claim that it commutes (up to canonical natural isomorphism).

To see this, note first that dd factors through $\big(\varinjlim_i \mathcal{M}_{A_i}\big)^{**} \boxtimes \mathcal{M}_B$ and the triangle is obtained simply by applying $- \boxtimes \mathcal{M}_B$ to the case $B = \mathbb{K}$. There is hence no loss of generality if we just assume $B = \mathbb{K}$ for the purpose of verifying the claim.

By definition, $\eta \circ \mathrm{dd}$ sends an object $\{M_i\}$ as in Lemma 4.5 to its image through $A = \{A_i\}$ regarded as an object of $\big(\varinjlim_i \mathcal{M}_{A_i}\big)^* \simeq \varprojlim_i {}_{A_i}\mathcal{M}$. Since $\{M_i\}$ is the colimit in $\varinjlim_i \mathcal{M}_{A_i}$ of M_i, this image is $\varinjlim(M_i \otimes_{A_i} A_i) \cong \varinjlim M_i$.

In other words, the composition $\eta \circ \mathrm{dd}$ simply assembles all M_is together via the identifications $M_i \leq M_j$ into a filtered vector space with compatible right actions by A_i on the pieces of the filtration. But this is exactly the identification of $\varinjlim_i \mathcal{M}_{A_i}$ with a subcategory of $\mathcal{M}_{\widehat{A}}$ from Lemma 4.5. This concludes the justification of the claim.

Now, according to Lemma 4.5 and Proposition 4.8, both vertical functors are fully faithful with essential image the category of $\widehat{A}$-discrete $\widehat{A} \otimes B$-modules. It follows that dd is an equivalence of categories. ∎

We conclude this section with a proof of Theorem 1.8. Recall that for a stack X over $\mathbb{K}$, the category $\mathrm{Qcoh}(X)$ is not just $\mathbb{K}$-linear and locally presentable, but also symmetric monoidal in the sense of Definition 2.9.

Following [Bra11, HR14], call a stack X over $\mathbb{K}$ *tensorial* if for any affine scheme $\mathrm{Spec}(B)$ over $\mathbb{K}$, the functor

$$\hom\big(\mathrm{Spec}(B), X\big) \to \mathrm{Hom}_{\otimes}\big(\mathrm{Qcoh}(X), \mathcal{M}_B\big), \quad f \mapsto f^*$$

is an equivalence. Such stacks are called "2-affine" in [CJF13, BC14], where symmetric monoidal locally presentable (resp. cocomplete) categories are called "commutative 2-rings."

In particular, we can talk about tensorial schemes, a notion which we now extend to ind-schemes in the obvious manner. Recall that an ind-scheme $X = \varinjlim_i X_i$ is a formal directed colimit of schemes X_i where the transition morphisms $X_i \to X_j$ $(i \leq j)$ are closed immersions. One defines $\mathrm{Q{\scriptsize COH}}(X) := \varprojlim_i \mathrm{Q{\scriptsize COH}}(X_i)$. It inherits a symmetric monoidal structure from the symmetric monoidal structures on the $\mathrm{Q{\scriptsize COH}}(X_i)$s.

4.10. Definition. *An ind-scheme $X = \varinjlim_{i \in \mathcal{I}} X_i$ is* tensorial *if for every commutative algebra B the canonical functor*

$$\varinjlim_i \hom\bigl(\mathrm{Spec}(B), X_i\bigr) = \hom\bigl(\mathrm{Spec}(B), X\bigr) \to \mathrm{H{\scriptsize OM}}_\otimes\bigl(\mathrm{Q{\scriptsize COH}}(X), \mathcal{M}_B\bigr), \quad f \mapsto f^*$$

is an equivalence of categories. Unpacking this, given an acceptable commutative pro-algebra $\{A_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$, the corresponding affine ind-scheme $\varinjlim \mathrm{Spec}(A_i)$ is tensorial if for every commutative algebra B the canonical functor

$$\varinjlim_i \hom\bigl(A_i, B\bigr) \to \mathrm{H{\scriptsize OM}}_\otimes\Bigl(\varprojlim_i \mathcal{M}_{A_i}, \mathcal{M}_B\Bigr)$$

sending an algebra homomorphism $\phi : A_i \to B$ to the composition

$$\mathcal{M} \longrightarrow \mathcal{M}_{A_i} \xrightarrow{-\otimes_{A_i} B} \mathcal{M}_B,$$

is an equivalence of categories.

Proof of Theorem 1.8. Let $\{A_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ and B be as above and $\mathcal{M} := \varprojlim_i \mathcal{M}_{A_i}$. We tackle the full faithfulness and essential surjectivity of $\varinjlim \hom(A_i, B) \to \mathrm{H{\scriptsize OM}}_\otimes(\mathcal{M}, \mathcal{M}_B)$ separately.

Step 1: Full faithfulness. Any functor whose domain is equivalent to a set is automatically faithful, and any functor whose codomain is equivalent to a set is automatically full. Since $\varinjlim \hom(A_i, B)$ is already a set, it thus suffices to check that $\mathrm{H{\scriptsize OM}}_\otimes(\mathcal{M}, \mathcal{M}_B)$ is equivalent to a set. But a morphism $F \to G$ of cocontinuous symmetric monoidal functors $\mathcal{M} \to \mathcal{M}_B$ is unique, if it exists, and in that case an isomorphism, because, by Proposition 4.8, it is completely determined by $F(A) \to G(A)$, and this has to be the composition of the distinguished isomorphisms $F(A) \cong B$ and $B \cong G(A)$.

Step 2: Essential surjectivity. We want to show that every $F \in \mathrm{H{\scriptsize OM}}_\otimes(\mathcal{M}, \mathcal{M}_B)$ factors as $\mathcal{M} \to \mathcal{M}_{A_i} \overset{-\otimes_{A_i} B}{\longrightarrow} \mathcal{M}_B$ as a symmetric monoidal functor. Proposition 4.8 identifies the category $\mathrm{H{\scriptsize OM}}_\otimes(\mathcal{M}, \mathcal{M}_B)$ with the category of $\widehat{A}$-discrete $\widehat{A} \otimes B$-modules V together with a symmetric monoidal structure on the corresponding cocontinuous functor $F : \mathcal{M} \to \mathcal{M}_B$ with $F(A) = V$.

Symmetric monoidality provides a distinguished isomorphism of B-modules $F(A) \cong B$, so that we may assume that the underlying B-module of V is just B. The $\widehat{A}$-module structure on B comes from the algebra homomorphism $\widehat{A} = \mathrm{End}(A) \to \mathrm{End}(F(A)) \cong \mathrm{End}(B) \cong B$. Discreteness means that every element of B is killed by some $\ker(\widehat{A} \to A_i)$, but it clearly suffices to demand this for $1 \in B$. In other words, $\widehat{A} \to B$ factors through some A_i. ∎

4.11. Corollary. *The functor from acceptable commutative pro-algebras to symmetric monoidal locally presentable categories, mapping $\{A_i\}$ to $\varprojlim_i \mathcal{M}_{A_i}$, is fully faithful.*

Following [CJF13, BC14], this may be interpreted as the statement that affine ind-schemes (indexed by countable posets) are 2-affine.

Proof. If $A = \{A_i\}_{i\in\mathcal{I}^{\mathrm{op}}}$ and $B = \{B_j\}_{j\in\mathcal{J}^{\mathrm{op}}}$ are acceptable commutative pro-algebras then

$$\hom\big(A, B\big) := \varprojlim_j \varinjlim_i \hom(A_i, B_j) \simeq \varprojlim_j \mathrm{Hom}_\otimes\Big(\varprojlim_i \mathcal{M}_{A_i}, \mathcal{M}_{B_j}\Big) \simeq \mathrm{Hom}_\otimes\Big(\varprojlim_i \mathcal{M}_{A_i}, \varprojlim_j \mathcal{M}_{B_j}\Big).$$

The first equivalence uses Theorem 1.8. ∎

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