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A CHARACTERIZATION OF REPRESENTABLE INTERVALS

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Abstract. In this note we provide a characterization, in terms of additional algebraic structure, of those strict intervals (certain cocategory objects) in a symmetric monoidal closed category $\mathcal{E}$ that are representable in the sense of inducing on $\mathcal{E}$ the structure of a finitely bicomplete 2-category. Several examples and connections with the homotopy theory of 2-categories are also discussed.

Introduction

Approached from an abstract perspective, a fundamental feature of the category of spaces which enables the development of homotopy theory is the presence of an object I with which the notions of *path* and *deformation* thereof are defined. When dealing with topological spaces, I is most naturally taken to be the closed unit interval $[0,1]$, but there are other instances where the homotopy theory of a category is determined in an appropriate way by an interval object I. For example, the simplicial interval $I = \Delta[1]$ determines — in a sense clarified by the recent work of Cisinski [3] — the classical model structure on the category of simplicial sets and the infinite dimensional sphere J is correspondingly related to the quasi-category model structure studied by Joyal [7]. Similarly, the category **2** gives rise to the natural model structure — in which the weak equivalences are categorical equivalences, the fibrations are isofibrations and the cofibrations are functors injective on objects — on the category **Cat** of small categories [9]. This model structure is, moreover, well-behaved with respect to the usual 2-category structure on **Cat** (it is a

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model **Cat**-*category* in the sense of [12]). One special property of the category **2**, which is in part responsible for these facts, is that it is a cocategory in **Cat**.

In this paper we study, with a view towards homotopy theory, one (abstract) notion of *strict interval object* — namely, a cocategory with object of coobjects the tensor unit in a symmetric monoidal closed category — of which **2** is a leading example. Every such interval I gives rise to a (higher-dimensional) sesquicategory structure on its ambient category and in some cases (such as when the monoidal structure is cartesian) this turns out to be a 2-category (indeed, a strict ω-category). It is our principal goal to investigate certain properties of such an induced 2-category structure in terms of the interval itself. In particular, our main theorem (Theorem 2.14) gives a characterization of those strict intervals I for which the induced 2-category structure is *finitely bicomplete* in the 2-categorical sense. A strict interval I with this property is said to be *representable* and the content of Theorem 2.14 is that a strict interval I is representable whenever it is a distributive lattice with top and bottom elements which are, in a suitable sense, its generators.

We note here that neither the closed unit interval in the category of spaces nor the simplicial interval in the category of simplicial sets are examples of strict interval objects in the sense of the present paper. For example, although the closed unit interval can be equipped with suitable structure maps, it fails to satisfy the defining equations for cocategories, which are only satisfied up to homotopy. Instead it is expected that these are examples of "weak ω-intervals" in the sense that they are weak co-ω-categories. As such, the present paper may be regarded as, in part, laying the groundwork for later investigation of these intervals and the corresponding weak higher-dimensional completeness properties of the model structures to which they give rise.

The plan of this paper is as follows. Section 1 is concerned with introducing the basic definitions and examples. In particular, we give the leading examples of strict intervals and explain the the resulting (higher-dimensional) sesquicategory structure and when it results in a strict 2-category. In Section 2 we recall the 2-categorical notion of finite bicompleteness and prove our main results including Theorem 2.14. Lack [12] has shown that every finitely bicomplete 2-category can be equipped with a model structure in which the weak equivalences are categorical equivalences and the fibrations are isofibrations and in Section 3 we briefly explain when, given the presence of a strict interval I which is representable, this model structure can be lifted, using a theorem due to Berger and Moerdijk [1], to the category of reduced operads.

NOTATION AND CONVENTIONS Throughout we assume, unless otherwise stated, that the ambient category $\mathcal{E}$ is a (finitely) bicomplete symmetric monoidal closed category (for further details regarding which we refer the reader to [14]). We employ common notation $(A \otimes B)$ and $[B, A]$ for the tensor product and internal hom of objects A and B, respectively. We denote the tensor unit by U (instead of the more common I) and the natural isomorphisms associated to the symmetric monoidal closed structure of $\mathcal{E}$ are denoted by $\lambda\colon U \otimes A \longrightarrow A$, $\rho\colon A \otimes U \longrightarrow A$, $\alpha\colon A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$, and $\tau\colon A \otimes B \longrightarrow B \otimes A$. Associated to the closed structure we denote the isomorphism

$[U, A] \longrightarrow A$ by ∂ and write $\varepsilon\colon [U, A] \otimes U \longrightarrow A$ for the evaluation map. We write iterated tensor products as associating to the left so that $A \otimes B \otimes C$ should be read as $(A \otimes B) \otimes C$.

We will frequently deal with pushouts and, if the following is a pushout diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ A \arrow[r, "f'"'] & P \end{tikzcd}$$

then, when $h\colon A \longrightarrow X$ and $k\colon B \longrightarrow X$ are maps for which $h \circ g = k \circ f$, we denote the induced map $P \longrightarrow X$ by $[h, k]$. Likewise, we employ the notation $\langle h, k\rangle$ for canonical maps into pullbacks.

Finally, we refer the reader to [11] for further details regarding 2-categories.

1. Intervals

The definition of *cocategory object* in $\mathcal{E}$ is exactly dual to that of category object in $\mathcal{E}$. In order to fix notation and provide motivation we will rehearse the definition in full. For us, the principal impetus for the definition of cocategories is that a cocategory in $\mathcal{E}$ provides (more than) sufficient data to define a reasonable notion of homotopy in $\mathcal{E}$ and this induced notion of homotopy is directly related to a 2-category structure on $\mathcal{E}$. In thinking about cocategory objects it is often instructive to view them as analogous to the unit interval in the category of topological spaces. However, the unit interval is *not* a cocategory object in the category of topological spaces and continuous functions.

1.1. The definition A *cocategory* $\mathbb{C}$ in a category $\mathcal{E}$ consists objects C_0 (*object of coobjects*), C_1 (*object of coarrows*) and C_2 (*object of cocomposable coarrows*) together with arrows

$$\begin{tikzcd}[column sep=huge] C_0 \arrow[r, bend left, "\perp"] \arrow[r, bend right, "\top"'] & C_1 \arrow[l, "i" description] \arrow[r, bend left, "\downarrow"] \arrow[r, bend right, "\uparrow"'] \arrow[r, "\star" description] & C_2 \end{tikzcd}$$

satisfying the following list of requirements.

- The following square is a pushout:

$$\begin{tikzcd} C_0 \arrow[r, "\perp"] \arrow[d, "\top"'] & C_1 \arrow[d, "\uparrow"] \\ C_1 \arrow[r, "\downarrow"'] & C_2. \end{tikzcd}$$

- The following diagram commutes:

$$\begin{tikzcd} C_0 \arrow[r, "\perp"] \arrow[dr, equal] & C_1 \arrow[d, "i"] & C_0 \arrow[l, "\top"'] \arrow[dl, equal] \\ & C_0. & \end{tikzcd}$$

- The following diagrams commute:

$$\begin{tikzcd} C_0 \arrow[r, "\perp"] \arrow[d, "\perp"'] & C_1 \arrow[d, "\star"] \\ C_1 \arrow[r, "\downarrow"'] & C_2, \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_0 \arrow[r, "\top"] \arrow[d, "\top"'] & C_1 \arrow[d, "\star"] \\ C_1 \arrow[r, "\uparrow"'] & C_2. \end{tikzcd}$$

- The following *co-unit laws* hold:

$$\begin{tikzcd} & C_1 \arrow[dl, equal] \arrow[d, "\star"] \arrow[dr, equal] & \\ C_1 & C_2 \arrow[l, "{[\perp\circ i, C_1]}"] \arrow[r, "{[C_1, \top\circ i]}"'] & C_1. \end{tikzcd}$$

- Finally, let the object C_3 (*the object of cocomposable triples*) be defined as the following pushout:

$$\begin{tikzcd} C_1 \arrow[r, "\uparrow"] \arrow[d, "\downarrow"'] & C_2 \arrow[d, "\downharpoonleft"] \\ C_2 \arrow[r, "\upharpoonright"'] & C_3, \end{tikzcd}$$

The *coassociative law* then states that the following diagram commutes:

$$\begin{tikzcd} C_1 \arrow[r, "\star"] \arrow[d, "\star"'] & C_2 \arrow[d, "{[\downharpoonleft\circ\downarrow, \upharpoonright\circ\star]}"] \\ C_2 \arrow[r, "{[\downharpoonleft\circ\star, \upharpoonright\circ\uparrow]}"'] & C_3. \end{tikzcd}$$

1.2. Remark. The map $\perp$ is the dual of the domain map, $\top$ is the dual of the codomain map, and $\downarrow$ and $\uparrow$ are dual to the first and second projections, respectively. This notation, and the other notation occurring in the definition, is justified by the interpretation of these arrows in the examples considered below. We refer to i and $\star$ as the *coidentity* and *cocomposition* maps, respectively.

If $\mathbb{C} = (C_0, C_1, C_2)$ is a cocategory object and A is any object of $\mathcal{E}$, then $A \otimes \mathbb{C} = (A \otimes C_0, A \otimes C_1, A \otimes C_2)$ is also a cocategory in $\mathcal{E}$. Moreover, if $\mathbb{C}$ is a cocategory in $\mathcal{E}$ and A is any object, then one obtains a category $[\mathbb{C}, A]$ in $\mathcal{E}$ by taking internal hom.

1.3. Remark. The composites

$$[I, A] \xrightarrow{t^*} [U, A] \xrightarrow{\partial} A,$$

with $t = \bot, \top$, are denoted by ∂_0 and ∂_1, respectively.

1.4. Cocategories with additional structure We will be concerned with cocategories which possess additional structure.

1.5. Definition. *A cocategory $\mathbb{C}$ in $\mathcal{E}$ is a* cogroupoid *if there exists a* symmetry *or* coinverse *map $\sigma\colon C_1 \longrightarrow C_1$ such that the following diagrams commute:*

$$\begin{tikzcd} C_0 \arrow[r, "\bot"] \arrow[dr, "\top"'] & C_1 \arrow[d, "\sigma"] & C_0 \arrow[l, "\top"'] \arrow[dl, "\bot"] \\ & C_1 & \end{tikzcd}$$

$$\begin{tikzcd} C_1 \arrow[r, "\star"] \arrow[d, "i"'] & C_2 \arrow[d, "{[\sigma, C_1]}"] \\ C_0 \arrow[r, "\top"'] & C_1 \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} C_1 \arrow[r, "\star"] \arrow[d, "i"'] & C_2 \arrow[d, "{[C_1, \sigma]}"] \\ C_0 \arrow[r, "\bot"'] & C_1 \end{tikzcd}$$

When $\mathbb{C}$ is a cogroupoid in $\mathcal{E}$ and A is an object of $\mathcal{E}$, $[\mathbb{C}, A]$ is a groupoid in $\mathcal{E}$.

1.6. Definition. *A cocategory object $\mathbb{C}$ in a category $\mathcal{E}$ is said to be a* strict interval *if the object C_0 of coobjects is the tensor unit U. When $\mathbb{C}$ is a strict interval we often write I instead of C_1 and I_2 instead of C_2. When a strict interval I is a cogroupoid it is said to be* invertible*.*

1.7. Remark. Because we will be dealing throughout exclusively with strict intervals the adjective "strict" will henceforth be omitted.

Cocategories in $\mathcal{E}$ together with their obvious morphisms form a category $\mathbf{Cocat}(\mathcal{E})$. There is also a category $\mathbf{Int}(\mathcal{E})$ of strict intervals in $\mathcal{E}$.

1.8. Examples of cocategories Before introducing some examples of cocategory objects it will be useful to first record the following lemma.

1.9. Lemma. *Assume $\mathcal{E}$ is an additive symmetric monoidal closed category. If we are given an object C together with arrows $i\colon C \longrightarrow U$, and $\bot, \top\colon U \longrightarrow C$ such that $i \circ \bot = 1_U = i \circ \top$, then there exists a map $\star\colon C \longrightarrow C_2$ such that the resulting structure is a strict interval.*

Proof. Set $\star\colon = \downarrow + \uparrow - (\uparrow \circ \bot \circ i)$. The axioms for a cocategory are then immediate. ∎

The following are examples of cocategories and intervals:

1. Every object A of a category $\mathcal{E}$ determines a cocategory object given by setting $A_i\colon = A$ for $i = 0, 1, 2$ and defining all of the structure maps to be the identity 1_A. This is said to be the *discrete cocategory on A*. The discrete cocategory on the tensor unit U is the terminal object in $\mathbf{Int}(\mathcal{E})$.

2. There is an (invertible) interval in $\mathcal{E}$ obtained by taking the object of coarrows to be $U + U$ with $\perp$ and $\top$ the coproduct injections. This is the initial object in $\mathbf{Int}(\mathcal{E})$. Indeed, a topos $\mathcal{E}$ is Boolean if and only if its subobject classifier Ω is an invertible interval with $\perp$ and $\top$ the usual "truth-values". (Observe that if a map $\sigma\colon \Omega \longrightarrow \Omega$ in a topos satisfies $\sigma^2 = 1_\Omega$, $\sigma(\top) = \perp$ and $\sigma(\perp) = \top$, then $\sigma = \neg$.)

3. In $\mathbf{Cat}$ the category $\mathbf{2}$ which is the free category on the graph consisting of two vertices and one edge between them is a cocategory object. Similarly, the free groupoid $\mathbf{I}$ on this graph is an invertible interval in $\mathbf{Cat}$ and in $\mathbf{Gpd}$ with the following structure:

$$\begin{tikzcd} \perp \arrow[r, bend right, "u"'] & \top \arrow[l, bend right, "d"'] \end{tikzcd}$$

such that u and d are inverse and where $\perp, \top\colon \mathbf{1} \rightrightarrows \mathbf{I}$ are the obvious functors. $\mathbf{I}_2$ is then the result of gluing $\mathbf{I}$ to itself along the top and bottom:

$$\begin{tikzcd} \perp \arrow[r, bend right] & \bullet \arrow[l, bend right] \arrow[r, bend right] & \top. \arrow[l, bend right] \end{tikzcd}$$

Cocomposition $\star\colon \mathbf{I} \longrightarrow \mathbf{I}_2$ is the functor given by $\star(\perp)\colon = \perp$ and $\star(\top)\colon = \top$, and the initial and final segment functors are defined in the evident way. Finally, $\sigma\colon \mathbf{I} \longrightarrow \mathbf{I}$ is defined by $\sigma(\perp)\colon = \top$ and $\sigma(\top)\colon = \perp$. We note that these examples also generalize to the case of internal categories in a suitably complete and cocomplete category $\mathcal{E}$.

4. Assume R is a commutative ring (with 1) and let $\mathbf{Ch}_{0\leq}(R)$ be the category of (nonnegatively graded) chain complexes of R-modules; then there exists an (invertible) interval $\mathbb{I}$ in $\mathbf{Ch}_{0\leq}(R)$ which we now describe. $\mathbb{I}^0$ is the chain complex which consists of R in degree 0 and is 0 in all other degrees. $\mathbb{I}^1$ is given by

$$\begin{array}{ccccc} \cdots \xrightarrow{\;d\;} 0 \xrightarrow{\;d\;} & R & \longrightarrow & R \oplus R \\ & x & \longmapsto & (x, -x), \end{array}$$

where $x \in R$. $\perp$ and $\top$ are the left and right inclusions, respectively. $i\colon \mathbb{I}^1 \longrightarrow \mathbb{I}^0$ is given by addition in degree 0 and the zero map in all other degrees. Cocomposition is given by Lemma 1.9. The symmetry $\sigma\colon \mathbb{I} \longrightarrow \mathbb{I}$ is given by taking additive inverse in degree 1 and by sending (x, y) to (y, x) in degree 0, for x, y in R.

5. Let R-**mod** denote the category of R-modules, for R a commutative ring. Assume given a set A together with two (not necessarily distinct) elements $\bot, \top \in A$. We obtain an interval, again appealing to Lemma 1.9, by applying the free R-module functor $\mathbf{Set} \longrightarrow R\text{-}\mathbf{mod}$ to these data and the canonical map $A \longrightarrow 1$.

6. Following the approach from Example (5) we obtain a further class of examples. Let B be a bialgebra over a commutative ring R with unit η and counit ϵ. Set $\bot \colon = \eta$, $\top \colon = \eta$ and $i \colon = \epsilon$. These data determine an interval in R-**mod** by Lemma 1.9.

7. Assuming $\mathcal{E}$ is a 2-category which is finitely cocomplete in the 2-categorical sense (as discussed, e.g., in Section 2 below) there exists for every object A of $\mathcal{E}$ a cocategory $(A \cdot \mathbf{2})$ obtained by taking the tensor of A with the category $\mathbf{2}$ (this fact can be found in its dual form in [16]). When $\mathcal{E}$ is simultaneously equipped with a **Cat**-enriched symmetric monoidal closed structure it follows that the 2-category structure on $\mathcal{E}$ is induced, in the sense of Theorem 1.20, by the interval $(U \cdot \mathbf{2})$. Note that the assumption of **Cat**-enrichedness is necessary.

8. Consider both the cartesian and the Gray monoidal closed structures on the category **2-Cat** of small 2-categories (cf. [4]). Because the tensor unit for both of these monoidal structures is the terminal object $\mathbf{1}$ it follows that intervals for one monoidal structure are the same as intervals for the other. E.g., the category $\mathbf{2}$, regarded as a 2-category with no non-identity 2-cells, is an interval in both of these monoidal structures.

As we have already noted, the topological unit interval $I = [0, 1]$ in **Top** fails to satisfy the co-associativity and co-unit laws on the nose and is therefore not an interval in the present sense.

1.10. Remark. The question of what kinds of cocategories can exist in a topos has been addressed by Lumsdaine [13] who shows that in a coherent category the only cocategories are "coequivalence relations". I.e., any such cocategory must have $\bot$ and $\top$ jointly epimorphic.

1.11. Induced sesquicategory and 2-category structures The first way in which we make use of the existence of an interval object in $\mathcal{E}$ is to define homotopy.

1.12. Definition. *Let I be an interval object in $\mathcal{E}$. A* homotopy (with respect to I) $\eta \colon f \Longrightarrow g$ from f to g, *for $f, g \in \mathcal{E}(A, B)$, is a map $\eta \colon A \otimes I \longrightarrow B$ such that the following diagram commutes:*

$$\begin{tikzcd}
A \otimes U \arrow[r, "A\otimes\bot"] \arrow[d, "\rho"'] & A \otimes I \arrow[d, "\eta"] & A \otimes U \arrow[l, "A\otimes\top"'] \arrow[d, "\rho"] \\
A \arrow[r, "f"'] & B & A \arrow[l, "g"]
\end{tikzcd}$$

1.13. Example. Notions of homotopy corresponding to several of the intervals from Section 1.8 are enumerated below.

1. The discrete interval on U generates the finest notion of homotopy (in terms of the number of homotopy classes of maps). I.e., there exists a homotopy between f and g with respect to this cocategory if and only if f and g are identical.

2. The initial object of $\mathbf{Int}(\mathcal{E})$ generates the coarsest homotopy relation: all maps are homotopic. Indeed, given maps f and g there exists, with respect to this cocategory, a unique homotopy $f \Longrightarrow g$.

3. In **Cat**, homotopies $f \Longrightarrow g$ with respect to $\mathbf{2}$ correspond to natural transformations $f \Longrightarrow g$ and, similarly, homotopies with respect to $\mathbf{I}$ correspond to natural isomorphisms.

4. In $\mathbf{Ch}_{0\leq}(R)$, $\mathbb{I}$ induces the usual notion of chain homotopy.

5. In the case of the interval $\mathbf{2}$ in **2-Cat**, homotopies with respect to the cartesian monoidal structure correspond to 2-natural transformations whereas homotopies with respect to the Gray monoidal structure correspond to pseudonatural transformations.

Recall that a *sesquicategory* [17] is a structure which satisfies all of the axioms of a 2-category with the exception of the interchange law. We will now see that, in the presence of a small amount of additional structure, the notion of homotopy from above induces on $\mathcal{E}$ the structure of a sesquicategory. We will also see that there are always at least two was to endow $\mathcal{E}$ with this additional structure (although there may, as we will see, be more than these two).

1.14. Definition. *Assume that I is an interval in $\mathcal{E}$. We say that a map $\Delta\colon I \longrightarrow I \otimes I$ is a* diagonal (for I) *if the following conditions are satisfied:*

1. *(I, Δ, i) is a comonoid (cf. Appendix A).*

2. *The diagram*

$$\begin{tikzcd} U \arrow[r, "\lambda^{-1}"] \arrow[d, "\diamond"'] & U \otimes U \arrow[d, "\diamond\otimes\diamond"] \\ I \arrow[r, "\Delta"'] & I \otimes I \end{tikzcd}$$

commutes, for $\diamond = \perp, \top$.

1.15. Example. Every interval I in $\mathcal{E}$ has an associated diagonal. For the comonoid comultiplication $\Delta\colon I \longrightarrow I\otimes I$ we first form, using the fact that I_2 is the pushout of $\perp$ along $\top$, the canonical map $[(\perp\otimes I)\circ\lambda^{-1},(I\otimes\top)\circ\rho^{-1}]\colon I_2\longrightarrow I\otimes I$. We then define

$$\Delta\colon = [(\perp\otimes I)\circ\lambda^{-1},(I\otimes\top)\circ\rho^{-1}]\circ\star$$

With these definitions the comonoid axioms follow from the counit and coassociativity laws for cocategories and the second condition from Definition 1.14 is immediate. In the case where the monoidal structure on $\mathcal{E}$ is cartesian, Δ is precisely the usual diagonal map and it is the only such map.

1.16. Example. If B is a bialgebra with coalgebra structure (B,Δ,ϵ) as in Example (6) from Section 1.8, then Δ is a diagonal.

1.17. Example. If Δ is a diagonal, then so is $\tau\circ\Delta$, where $\tau\colon I\otimes I\longrightarrow I\otimes I$ is the twist map.

Now, assume that I is an interval in $\mathcal{E}$ with diagonal Δ and equip $\mathcal{E}$ with the structure of a sesquicategory as follows. The 2-cells of $\mathcal{E}$ are homotopies with respect to I. I.e., we define

$$\mathcal{E}(A,B)_1\colon = \mathcal{E}(A\otimes I,B),$$

which endows $\mathcal{E}(A,B)$ with the structure of a category since $[I,B]$ is an internal category in $\mathcal{E}$. Explicitly, given φ in $\mathcal{E}(A,B)_1$, the domain of φ is defined to be the arrow $\varphi\circ(A\otimes\perp)\circ\rho^{-1}\colon A\longrightarrow B$ and the codomain is $\varphi\circ(A\otimes\top)\circ\rho^{-1}\colon A\longrightarrow B$. Given arrows $\eta\colon f\Longrightarrow g$ and $\gamma\colon g\Longrightarrow h$ in $\mathcal{E}(A,B)$, the vertical composite $f\Longrightarrow h$ is defined as follows. Since $\mathcal{E}$ is monoidal closed the following square is a pushout:

$$\begin{tikzcd} A\otimes U \arrow[r, "A\otimes\perp"] \arrow[d, "A\otimes\top"'] & A\otimes I \arrow[d, "A\otimes\uparrow"] \\ A\otimes I \arrow[r, "A\otimes\downarrow"'] & A\otimes I_2 \end{tikzcd}$$

and there exists an induced map $[\eta,\gamma]\colon A\otimes I_2\longrightarrow B$. Recalling the third clause from the definition of cocategory object, it is easily verified that $[\eta,\gamma]\circ(A\otimes\star)$ is the required vertical composite $(\gamma\cdot\eta)\colon f\Longrightarrow h$.

The horizontal composite $(\gamma * \eta)$ of a pair of 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\eta\Downarrow"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\gamma\Downarrow"] & C. \end{tikzcd}$$

is defined to be the composite

$$A \otimes I \xrightarrow{A \otimes \Delta} A \otimes (I \otimes I) \xrightarrow{\alpha} (A \otimes I) \otimes I \xrightarrow{\eta \otimes I} B \otimes I \xrightarrow{\gamma} C.$$

The proof that the structure defined above constitutes a sesquicategory is routine and is therefore left to the reader (the associativity and unit laws for horizontal composition following from the coassociative and counit laws for Δ). As such, we have the following:

1.18. PROPOSITION. *Suppose I is an interval object in $\mathcal{E}$ with diagonal Δ. Then $\mathcal{E}$ is a sesquicategory with the same objects and arrows, and with 2-cells the homotopies.*

By virtue of Example 1.15, it follows that any interval I in $\mathcal{E}$ induces a sesquicategory structure on $\mathcal{E}$. This will, however, *not* in general be a 2-category since the interchange law need not be satisfied.[1] Proposition 1.18 has the following evident corollary:

1.19. COROLLARY. *An interval I in $\mathcal{E}$ is invertible if and only if, for all objects A and B of $\mathcal{E}$, the category $\mathcal{E}(A, B)$ is a groupoid.*

We will now give necessary and sufficient conditions which characterize those cases in which the structure obtained in this way is a genuine 2-category.

1.20. THEOREM. *Assuming that I is an interval in $\mathcal{E}$ equipped with a diagonal Δ, then the following are equivalent:*

1. *$\mathcal{E}$ is a 2-category, when equipped with the sesquicategory structure from Proposition 1.18.*
2. *The diagram*

$$\begin{tikzcd} I \arrow[r, "\Delta"] \arrow[d, "\star"'] & I \otimes I \arrow[d, "\star\otimes\star"] \\ I_2 \arrow[r, "\Delta_2"'] & I_2 \otimes I_2 \end{tikzcd}$$

commutes, where $\Delta_2 \colon I_2 \longrightarrow I_2 \otimes I_2$ is the canonical map such that

$$\Delta_2 \circ \downarrow = (\downarrow \otimes \downarrow) \circ \Delta$$
$$\Delta_2 \circ \uparrow = (\uparrow \otimes \uparrow) \circ \Delta.$$

3. *Δ is equal to the induced diagonal from Example 1.15 and $\Delta = \tau \circ \Delta$ (i.e., Δ is cocommutative).*

[1] The author is grateful to the referee for pointing out this fact and thereby correcting an error in the original version of this paper, and for suggesting the connection with part (3) of Theorem 1.20.

Proof. First, to see that (1) implies (2), assume that the interchange law holds to show that the diagram commutes. Observe that we have the following diagram:

$$\begin{tikzcd}[column sep=huge]
U \arrow[r, bend left=60, "\downarrow\perp"] \arrow[r, "\downarrow\top" description] \arrow[r, bend right=60, "\uparrow\top"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \downarrow\circ\lambda"] \arrow[r, phantom, shift right=3ex, "\Downarrow \uparrow\circ\lambda"] & I_2 \arrow[r, bend left=60, "(I_2\otimes\downarrow\perp)\circ\rho^{-1}"] \arrow[r, "(I_2\otimes\downarrow\top)\circ\rho^{-1}" description] \arrow[r, bend right=60, "(I_2\otimes\uparrow\top)\circ\rho^{-1}"'] \arrow[r, phantom, shift left=3ex, "\Downarrow I_2\otimes\downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow I_2\otimes\uparrow"] & I_2 \otimes I_2
\end{tikzcd}$$

and therefore the interchange law implies that

$$\big((I_2\otimes\uparrow) * (\uparrow\circ\lambda)\big)\cdot\big((I_2\otimes\downarrow) * (\downarrow\circ\lambda)\big) = \big((I_2\otimes\uparrow)\cdot(I_2\otimes\downarrow)\big) * \big((\uparrow\circ\lambda)\cdot(\downarrow\circ\lambda)\big)$$

Since the left-hand side of this equation is equal to $\Delta_2\circ\star\circ\lambda$ and, using the fact that $(\uparrow\cdot\downarrow) = \star$, the right-hand side is equal to $(\star\otimes\star)\circ\Delta\circ\lambda$ it follows that the diagram in (2) commutes.

To see that (2) implies (3), observe that the following diagram commutes

$$\begin{tikzcd}
I \arrow[r, "\Delta"] \arrow[d, "\star"'] & I\otimes I \arrow[r, equal] \arrow[d, "\star\otimes\star"] & I\otimes I. \\
I_2 \arrow[r, "\Delta_2"'] & I_2\otimes I_2 \arrow[ur, bend right, "{[\perp\circ i, I]\otimes[I,\top\circ i]}"'] &
\end{tikzcd}$$

By the fact that Δ is a comonoid we therefore obtain

$$\Delta \;=\; \big[(\perp\circ i\otimes I)\circ\Delta, (I\otimes\top\circ i)\circ\Delta\big]\circ\star \;=\; \big[(\perp\otimes I)\circ\lambda^{-1}, (I\otimes\top)\circ\rho^{-1}\big]\circ\star$$

as required. Similarly, taking $[I,\top\circ i]\otimes[\perp\circ i, I]$ instead in the diagram above gives that $\Delta = \tau\circ\Delta$.

To see that (3) implies (1) note that it suffices to show that, given a diagram

$$\begin{tikzcd}
A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \arrow[r, bend left=50, "h"] \arrow[r, bend right=50, "k"'] \arrow[r, phantom, "\Downarrow\delta"] & C
\end{tikzcd}$$

the two ways of composing this diagram using vertical composition and whiskering agree (and, in particular, agree with the defined horizontal composite $\delta*\gamma$). First, one way of composing the diagram with whiskering and vertical composition is as $[\delta\circ(f\otimes I), k\circ\gamma]\circ(A\otimes\star)$. Using the fact that $f = \gamma\circ(A\otimes\perp)\circ\rho^{-1}$ and $k = \delta\circ(B\otimes\top)\circ\rho^{-1}$ a straightforward diagram chase shows that this is equal to $\delta*\gamma$. The other way of composing the diagram with whiskering and vertical composition gives $[h \circ \gamma, \delta \circ (g \otimes I)] \circ (A \otimes \star)$. Since $h = \delta \circ (B \otimes \perp) \circ \rho^{-1}$ and $g = \gamma \circ (A \otimes \top) \circ \rho^{-1}$ a diagram chase gives

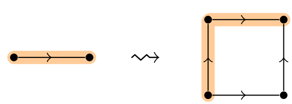
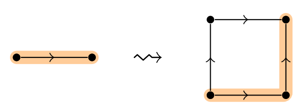
$$\begin{aligned}[h \circ \gamma, \delta \circ g \otimes I] \circ (A \otimes \star) &= \delta \circ (\gamma \otimes I) \circ \alpha \circ A \otimes [I \otimes \perp \circ \rho^{-1}, \top \otimes I \circ \lambda^{-1}] \circ A \otimes \star \\ &= \delta \circ (\gamma \otimes I) \circ \alpha \circ (A \otimes \tau) \circ (A \otimes \Delta).\end{aligned}$$

Therefore, since $\Delta = \tau \circ \Delta$ it follows that this way of composing the diagram is also equal to $\delta * \gamma$. ∎

1.21. Example. In the case where the monoidal structure is cartesian the equivalent conditions of Theorem 1.20 are easily seen to be satisfied. I.e., in the cartesian case, $\mathcal{E}$ is necessarily a 2-category.

1.22. Example. The discrete interval (1) and the Boolean interval (2) from Section 1.8 both necessarily give rise to a cocommutative Δ and therefore also give rise to (rather trivial) 2-categories.

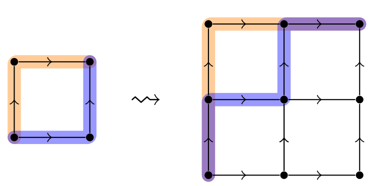
1.23. Example. It is easily seen that, for Δ the induced diagonal, we have $\tau \circ \Delta \neq \Delta$ in the category $\mathbf{Ch}_{0\leq}(R)$ of chain complexes. Diagrammatically, Δ and $\tau \circ \Delta$ are the maps

 and 

respectively. As such, it follows from Theorem 1.20 that $\mathbf{Ch}_{0\leq}(R)$ is neither a strict 2-category nor a strict ω-category. However, there does exist an invertible 2-cell $(\star \otimes \star) \circ \Delta \cong \Delta_2 \circ \star$ which is given by the map $\varphi\colon \mathbb{I} \otimes \mathbb{I} \longrightarrow \mathbb{I}_2 \otimes \mathbb{I}_2$ defined as follows:

$$\begin{aligned}\varphi_2(a) &:= (0, a, 0, 0)\\ \varphi_1(a,b,c,d) &:= (a+c, a, 0, b, 0, 0, 0, c, d, 0, 0, b+d), \text{ and}\\ \varphi_0(a,b,c,d) &:= (a, 0, b, 0, c, 0, 0, 0, d).\end{aligned}$$

Diagrammatically, this chain map is given by



where it is understood that the 2-cell of $\mathbb{I} \otimes \mathbb{I}$ is sent to the 2-cell in the upper left-hand corner of $\mathbb{I}_2 \otimes \mathbb{I}_2$.

1.24. Remark. As far as we know, it is an open question whether there exist examples, aside from the trivial ones mentioned in Example 1.22, of intervals in the non-cartesian monoidal setting for which the equivalent conditions of Theorem 1.20 hold.

1.25. Semistrict higher-dimensional structure Let arrows $f, g\colon A \longrightarrow B$ be given and let 2-cells $\gamma, \delta\colon f \Longrightarrow g$ also be given. A 3-cell $\varphi\colon \gamma \Longrightarrow \delta$ is given by an arrow $\varphi\colon A \otimes I \otimes I \longrightarrow B$ which is, regarded as a 2-cell, a homotopy $\gamma \Longrightarrow \delta$ such that

$$\begin{tikzcd}
A \otimes U \otimes I \arrow[r, "A\otimes\perp\otimes I"] \arrow[d, "\rho\circ(\rho\otimes i)"'] & A \otimes I \otimes I \arrow[d, "\varphi"'] & A \otimes U \otimes I \arrow[l, "A\otimes\top\otimes I"'] \arrow[d, "\rho\circ(\rho\otimes i)"] \\
A \arrow[r, "f"'] & B & A \arrow[l, "g"]
\end{tikzcd}$$

commutes. In general, given n-cells φ and ψ which are bounded by 0-cells A and B, an $(n+1)$-cell $\xi\colon \varphi \Longrightarrow \psi$ is given by an arrow $\xi\colon A \otimes I^{\otimes n} \longrightarrow B$ such that

$$\begin{tikzcd}
 & A \otimes I^{\otimes n} \arrow[dd, "\xi"'] & \\
A \otimes I^{\otimes k} \otimes U \otimes I^{\otimes n-1-k} \arrow[ur, bend left, "A\otimes I^{\otimes k}\otimes\perp\otimes I^{\otimes n-1-k}"] \arrow[d, "\rho^{n-1-k}\circ(\rho\otimes i^{\otimes n-1-k})"'] & & A \otimes I^{\otimes k} \otimes U \otimes I^{\otimes n-1-k} \arrow[ul, bend right, "A\otimes I^{\otimes k}\otimes\top\otimes I^{\otimes n-1-k}"'] \arrow[d, "\rho^{n-1-k}\circ(\rho\otimes i^{\otimes n-1-k})"] \\
A \otimes I^{\otimes k} \arrow[r, "\partial_0^{n-1-k}\varphi"'] & B & A \otimes I^{\otimes k} \arrow[l, "\partial_1^{n-1-k}\psi"]
\end{tikzcd}$$

commutes for $0 \leq k \leq n-1$. Composition of higher-dimensional cells must be specified depending on whether the cells in question meet at a 0-cell or at a higher-dimensional cell.

First, suppose given two $(n+1)$-cells φ and ψ such that $\partial_1^{n+1-k}\varphi = \partial_0^{n+1-k}\psi$ for $1 \leq k \leq n$. Then the following diagram commutes

$$\begin{tikzcd}
A \otimes I^{\otimes k-1} \otimes U \otimes I^{\otimes n-k} \arrow[r, "A\otimes I^{\otimes k-1}\otimes\perp\otimes I^{\otimes n-k}"] \arrow[d, "A\otimes I^{\otimes k-1}\otimes\top\otimes I^{\otimes n-k}"'] & A \otimes I^{\otimes n} \arrow[d, "\psi"] \\
A \otimes I^{\otimes n} \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

and therefore induces the map $[\varphi, \psi]\colon A \otimes I^{\otimes k-1} \otimes I_2 \otimes I^{\otimes n-k} \longrightarrow B$. We define the "vertical" composite of φ and ψ by

$$\psi *_k \varphi \colon = [\varphi, \psi] \circ A \otimes I^{\otimes k-1} \otimes \star \otimes I^{\otimes n-k}.$$

That $(\psi *_k \varphi) \circ (A \otimes I^{\otimes m} \otimes \diamond \otimes I^{n-1-m}) = \partial_\diamond^{n-m}(\psi *_k \varphi)$, for $\diamond = \perp, \top$, is straightforward in the cases where $m + 1 \geq k$ and is by the counit law when $m + 1 < k$.

Next, suppose given two $(n+1)$-cells φ and ψ such that $\partial_0^{n+1}\varphi = A$, $\partial_1^{n+1}\varphi = B = \partial_0^{n+1}\psi$, and $\partial_1^{n+1}\psi = C$. The "horizontal" composite of φ and ψ is given by the composite

$$A \otimes I^{\otimes n} \xrightarrow{A\otimes\Delta^{\otimes n}} A \otimes I^{\otimes 2n} \longrightarrow A \otimes I^{\otimes n} \otimes I^{\otimes n} \xrightarrow{\varphi\otimes I^{\otimes n}} B \otimes I^{\otimes n} \xrightarrow{\psi} C$$

where the second arrow is obtained by rearranging factors (using α and τ) in the obvious way so that the two $I^{\otimes n}$ in the codomain correspond to the original $I^{\otimes n}$ in the domain of $A \otimes \Delta^{\otimes n}$. That $(\psi *_0 \varphi) \circ (A \otimes I^{\otimes m} \otimes \diamond \otimes I^{n-1-m}) = \partial_\diamond^{n-m}(\psi *_0 \varphi)$, for $\diamond = \perp, \top$, is straightforward. That the associative and unit laws are satisfied for both the vertical and horizontal compositions just defined is essentially the same as the verification of these laws for the 2-dimensional sesquicategory structure. Furthermore, it is easily seen that the interchange law

$$(\psi' *_p \varphi') *_q (\psi *_p \varphi) = (\psi' *_q \psi) *_p (\varphi' *_q \varphi), \tag{1}$$

for $0 < q < p \leq n$ and $(n+1)$-cells φ, φ', ψ and ψ', holds. When $q = 0$ however (1) need not hold. Thus, it is only with respect to this case of the interchange law that $\mathcal{E}$ fails to be a strict ω-category. Thus, we have:

1.26. PROPOSITION. *Suppose I is an interval object in $\mathcal{E}$ with diagonal Δ. Then $\mathcal{E}$ is a weak higher-dimensional category in the sense that it satisfies all of the laws of a strict ω-category except for (1) in the case $q = 0$ which holds up to the existence of a higher-dimensional cell.*

In some cases Proposition 1.26 can be strengthened. First, we have the following two corollaries, the proofs of which are straightforward:

1.27. COROLLARY. *The equivalent conditions of Theorem 1.20 are satisfied if and only if $\mathcal{E}$ is a strict ω-category.*

1.28. COROLLARY. *There exists an invertible 2-cell $(\star \otimes \star) \circ \Delta \cong \Delta_2 \circ \star$ if and only if the interchange laws (1), for $q = 0$ and $(n+1)$-cells $\varphi, \varphi', \psi, \psi'$, hold up to the existence of an invertible $(n+2)$-cell.*

2. Representability

Henceforth we assume given an interval I which, together with its induced diagonal, gives rise to a 2-category (i.e., we assume that the equivalent conditions of Theorem 1.20 are satisfied). We now turn to the proof of our main Theorem 2.14 which gives necessary and sufficient conditions under which the 2-category structure on $\mathcal{E}$ is *finitely bicomplete* in the 2-categorical sense [16, 5]. We will also see that, when $\mathcal{E}$ is finitely bicomplete, I can be shown to possess additional useful structure. For example, we will see that such an interval is necessarily both a lattice and a *Hopf object* in the sense of Berger and Moerdijk [1].

First we recall the 2-categorical notion of finite (co)completeness due to Gray [5] and Street [16]. Namely, a 2-category $\mathcal{K}$ is *finitely complete* whenever it has all finite conical limits in the 2-categorical sense and, for each object A, the cotensor $(\mathbf{2} \pitchfork A)$ with the category $\mathbf{2}$ exists. Similarly, $\mathcal{K}$ is *finitely cocomplete* if and only if it possesses all finite conical colimits and tensors $(A \cdot \mathbf{2})$ with $\mathbf{2}$ exist. It is straightforward to verify that, when $\mathcal{E}$ possesses an interval I, the resulting 2-category possesses whatever conical limits and colimits $\mathcal{E}$ has in the ordinary 1-dimensional sense:

2.1. Lemma. *Assume that $\mathcal{E}$ possesses an interval I and regard $\mathcal{E}$ as a 2-category with respect to the 2-category structure induced by I. Then the conical (co)limit of a functor $F\colon \mathcal{C} \longrightarrow \mathcal{E}$ from a (small) category $\mathcal{C}$ exists if and only if the ordinary 1-dimensional (co)limit of F exists.*

In order to show that the 2-category structure on $\mathcal{E}$ induced by an interval I is finitely bicomplete it suffices, by Lemma 2.1, to prove that tensor and cotensor products with the category $\mathbf{2}$ exist. Indeed, if $(\mathbf{2} \pitchfork A)$ exists, it is necessarily isomorphic to the internal hom $[I, A]$ since the 2-natural isomorphism

$$\mathcal{E}(B, \mathbf{2} \pitchfork A) \cong \mathcal{E}(B, A)^{\mathbf{2}} \tag{2}$$

of categories restricts to a natural isomorphism of their respective collections of objects:

$$\mathcal{E}(B, \mathbf{2} \pitchfork A) \cong \mathcal{E}(B \otimes I, A).$$

Similar reasoning implies that when the tensor product $(A \cdot \mathbf{2})$ exists it is necessarily $(A \otimes I)$. Note though that it does not *a priori* follow that $[I, A]$ is $(\mathbf{2} \pitchfork A)$ in the sense of possessing the full 2-categorical universal property of $(\mathbf{2} \pitchfork A)$, and similarly for $(A \otimes I)$ and $(A \cdot \mathbf{2})$. This remark should be compared with the familiar fact that a 2-category with all 1-dimensional conical limits need not possesses all 2-dimensional conical limits (cf. [10]).

As the reader may easily verify, if I is an interval in $\mathcal{E}$, then there exist isomorphisms of categories

$$\mathcal{E}(B \otimes I, A) \cong \mathcal{E}\big(B, [I, A]\big)$$

natural in A and B. Thus, it follows that $\mathcal{E}$ possesses tensors with $\mathbf{2}$ if and only if it possesses cotensors with $\mathbf{2}$.

2.2. Definition. *An interval I in $\mathcal{E}$ is* representable *if cotensors with $\mathbf{2}$ exist with respect to the 2-category structure on $\mathcal{E}$ induced by I.*

Thus, an interval I is representable if and only if $\mathcal{E}$ is a finitely bicomplete 2-category with respect to the induced 2-category structure of Section 1.11. In particular, when I is representable the monoid structure is **Cat**-monoidal and I is necessarily obtained as in Example (7) from Section 1.8.

2.3. Injective boundaries An arrow $\varphi\colon B \otimes I \otimes I \longrightarrow A$ in $\mathcal{E}$ determines a square

$$\begin{tikzcd} \varphi_{00} \arrow[r, "\varphi_{\bullet 0}"] \arrow[d, "\varphi_{0\bullet}"'] & \varphi_{10} \arrow[d, "\varphi_{1\bullet}"] \\ \varphi_{01} \arrow[r, "\varphi_{\bullet 1}"'] & \varphi_{11} \end{tikzcd}$$

in $\mathcal{E}(B, A)$, where our notation should be clear (e.g., $\varphi_{0\bullet}$ is the result of precomposing φ with $(B \otimes \bot \otimes I) \circ (\rho^{-1} \otimes I))$. Because $\mathcal{E}$ is assumed to satisfy the equivalent conditions of Theorem 1.20 it follows that this diagram commutes. Consequently, there exists an induced homomorphism

$$\mathcal{E}(B,A)^\flat \xrightarrow{\partial} \mathcal{E}(B,A)^\natural$$

of double categories where $\mathcal{E}(B,A)^\natural$ is the double category with objects the objects of $\mathcal{E}(B,A)$, vertical and horizontal arrows both the arrows in $\mathcal{E}(B,A)$, and 2-cells commutative diagrams in $\mathcal{E}(B,A)$; and where $\mathcal{E}(B,A)^\flat$ has the same objects, horizontal and vertical arrows as $\mathcal{E}(B,A)^\natural$, but with 2-cells arrows $\varphi\colon B\otimes I\otimes I \longrightarrow A$. Note that horizontal composition of composable 2-cells φ,ψ in $\mathcal{E}(B,A)^\flat$ is given by $\psi\circ_h\varphi\colon = [\varphi,\psi]\circ(B\otimes\star\otimes I)$ and vertical composition is given by $\psi\circ_v\varphi\colon = [\varphi,\psi]\circ(B\otimes I\otimes\star)$. Let P be the pushout of $\star\colon I\longrightarrow I_2$ along itself. Then maps $B\otimes P\longrightarrow A$ are in bijective correspondence with the 2-cells of $\mathcal{E}(B,A)^\natural$. The maps $\Delta,\tau\circ\Delta\colon I\longrightarrow I\otimes I$ induce, by their definitions, a canonical map $P\longrightarrow I\otimes I$ and the action of ∂ on 2-cells is induced by precomposition with this map.

Moving from double categories to categories, ∂ restricts to a functor

$$\mathcal{E}(B,[I,A]) \xrightarrow{\Phi} \mathcal{E}(B,A)^{\mathbf{2}}$$

which acts by transpose under the tensor-hom adjunction. I.e., given an object f of $\mathcal{E}(B,[I,A])$, the arrow $\Phi(f)$ in $\mathcal{E}(B,A)$ is defined to be the transpose $\tilde{f}\colon B\otimes I\longrightarrow A$ of f. Similarly, for an arrow $\varphi\colon f\Longrightarrow g$ in $\mathcal{E}(B,[I,A])$, $\Phi(f)$ is obtained by projecting the transpose $\tilde{\varphi}$ to the commutative square $\partial(\tilde{\varphi})$:

$$\begin{tikzcd} \partial_0 f \arrow[r, "\partial_0\circ\varphi"] \arrow[d, "\tilde{f}"'] & \partial_0 g \arrow[d, "\tilde{g}"] \\ \partial_1 f \arrow[r, "\partial_1\circ\varphi"'] & \partial_1 g. \end{tikzcd} \tag{3}$$

The following lemma implies that if I is representable, then Φ is necessarily the natural isomorphism witnessing this fact.

2.4. Lemma. *If I is representable, then, for all objects A and B of $\mathcal{E}$, the functors Φ give isomorphisms of categories which are natural in A and B. Furthermore, the following diagram in* **Cat** *commutes:*

$$\begin{tikzcd} \mathcal{E}\big(B,[I,A]\big) \arrow[rr, "\Phi"] \arrow[dr, "\mathcal{E}(B,\partial_i)"'] & & \mathcal{E}(B,A)^{\mathbf{2}} \arrow[dl, "\partial_i"] \\ & \mathcal{E}(B,A) & \end{tikzcd}$$

when $i=0,1$.

In particular, representability of I is equivalent to Φ being an isomorphism of categories natural in A and B, which is equivalent to ∂ being an isomorphism of double categories which is similarly natural in A and B. Because naturality is immediate by definition all of these are equivalent to the canonical map $P\longrightarrow I\otimes I$ being an isomorphism.

2.5. Definition. *An interval I has* injective boundaries *if 2-cells φ in the double categories $\mathcal{E}(B,A)^\flat$ are completely determined by their boundaries. I.e., for any objects A and B of $\mathcal{E}$ and any 2-cells φ and ψ in $\mathcal{E}(B,A)^\flat$, $\partial(\varphi) = \partial(\psi)$ implies that $\varphi = \psi$.*

It is worth remarking that it suffices to test maps $I \otimes I \longrightarrow A$ in order to determine whether or not I has injective boundaries. In more homotopic terms, I has injective boundaries provided that for all paths f and g from a to b in A there is at most one homotopy rel endpoints $f \simeq g$. The following observation is a trivial consequence of the discussion above:

2.6. Lemma. *All representable intervals I have injective boundaries.*

2.7. Lattice structure of representable intervals We will now prove that if I is representable, then it is necessarily a unital distributive lattice in the sense of Appendix A.

2.8. Proposition. *If I is representable, then it possesses the structure of a unital distributive lattice such that $\bot$ is the unit for join $\vee\colon I \otimes I \longrightarrow I$ and $\top$ is the unit for meet $\wedge\colon I \otimes I \longrightarrow I$. Moreover, this structure is unique in the strong sense that meet and join are the canonical maps $I \otimes I \longrightarrow I$ such that both $\wedge$ and $\wedge \circ \tau$ are 2-cells $\bot \circ i \Longrightarrow 1_I$, and both $\vee$ and $\vee \circ \tau$ are 2-cells $1_I \Longrightarrow \top \circ i$.*

Proof. Because I is representable it follows that there exists a 2-natural isomorphism

$$\mathcal{E}(U,I)^{\mathbf{2}} \xrightarrow{\cong} \mathcal{E}\big(U,[I,I]\big) \tag{4}$$

of categories which is given at the level of objects by exponential transpose. In $\mathcal{E}(U,I)$ the following diagram commutes

$$\begin{tikzcd}
\bot \arrow[r, "\bot\circ i\circ\lambda"] \arrow[d, "\bot\circ i\circ\lambda"'] & \bot \arrow[d, "\lambda"] \\
\bot \arrow[r, "\lambda"'] & \top
\end{tikzcd}$$

Thus, applying (4) to this arrow of $\mathcal{E}\big(U,I\big)^{\mathbf{2}}$ yields a map $\boxdot\colon U \otimes I \longrightarrow [I,I]$. Denote by $\wedge\colon I \otimes I \longrightarrow I$ the transpose of the composite $\boxdot \circ \lambda^{-1}$ and observe, by definition and Lemma 2.4, that $\top$ is a unit for this operation and that the diagram

$$\begin{tikzcd}
U \otimes I \arrow[r, "\bot\otimes I"] \arrow[d, "\lambda"'] & I \otimes I \arrow[d, "\wedge"] & I \otimes U \arrow[l, "I\otimes\bot"'] \arrow[d, "\lambda"] \\
I \arrow[r, "\bot\circ i"'] & I & I \arrow[l, "\bot\circ i"]
\end{tikzcd} \tag{5}$$

also commutes. In the same way, applying the isomorphism (4) to the arrow

$$\begin{tikzcd} \bot \arrow[r, "\lambda"] \arrow[d, "\lambda"'] & \top \arrow[d, "\top\circ i\circ\lambda"] \\ \top \arrow[r, "\top\circ i\circ\lambda"'] & \top \end{tikzcd}$$

of $\mathcal{E}(U,I)^{\mathbf{2}}$ yields a map $\boxplus\colon U\otimes I \longrightarrow [I,I]$ for which the transpose $\vee\colon I\otimes I\longrightarrow I$ of $\boxplus\circ\lambda^{-1}$ is an operation which has as a unit $\bot$ and satisfies the dual of (5). Moreover, by Lemma 2.6, it follows that $\vee$ and $\wedge$ are the canonical maps $I\otimes I\longrightarrow I$ with these properties. For example, the idempotent law which states that $\vee\circ\Delta = 1_I$ holds since

$$\begin{aligned} \vee\circ\Delta &= [\vee\circ(\bot\otimes I)\circ\lambda^{-1}, \vee\circ(I\otimes\top)\circ\rho^{-1}]\circ\star \\ &= [1_I, \top\circ i]\circ\star \\ &= 1_I \end{aligned}$$

where the final equation is by the cocategory counit law. The other idempotent law is similar. Commutativity of the additional diagrams for distributive lattices also follow from Lemma 2.6 by a routine (but lengthy) series of diagram chases. ∎

Using join $\vee\colon I\otimes I\longrightarrow I$ we see that I is a commutative Hopf object in the sense of [1] (see Appendix A for the definition).

2.9. Corollary. *If I is representable, then it is a commutative Hopf object.*

Proof. As we have already seen (I,Δ,i) is a comonoid and both $(I,\vee,\bot)$ and $(I,\wedge,\top)$ are commutative monoids. In fact, I can be made into a commutative Hopf object using either of these monoid structures. To see this it remains to verify that $\vee$ and $\bot$, as well as $\wedge$ and $\top$, are comonoid homomorphisms. Since $\star\circ\bot = \downarrow\circ\bot$ it follows that $\bot$ is a homomorphism. $\vee$ is seen to be a homomorphism by testing on boundaries. A dual proof shows that $\wedge$ and $\top$ are also comonoid homomorphisms. ∎

2.10. Remark. We note that if $\varphi\colon I\longrightarrow H$ is an arrow in $\mathbf{Int}(\mathcal{E})$ between representable intervals, then it is necessarily also a morphism of Hopf objects provided that H and I are both equipped with "meet" (respectively, "join") Hopf object structures from the proof of Corollary 2.9.

2.11. The characterization of representable intervals We would now like to investigate the extent to which Proposition 2.8 characterizes representable intervals. For the remainder of this section, unless otherwise stated we do *not* assume that I is representable. We do however assume that there exist meet $\wedge\colon I\otimes I\longrightarrow I$ and join $\vee\colon I\otimes I\longrightarrow I$ operations which have $\bot$ and $\top$ as respective units and satisfy condition (8) from Appendix A (equivalently, both $\wedge$ and $\wedge\circ\tau$ are 2-cells $\bot\circ i\Longrightarrow 1_I$, and both $\vee$ and $\vee\circ\tau$ are 2-cells $1_I\Longrightarrow\top\circ i$).

Let us recall some double category machinery from [2]. An double category $\mathcal{D}$ is *edge symmetric* when it has the same horizontal and vertical edges.

2.12. Definition. *A* connection *on an edge symmetric double category $\mathcal{D}$ consists of maps*

$$\Gamma, \Gamma' \colon \mathcal{D}_1 \longrightarrow \mathcal{D}_2$$

such that $\Gamma(f)$ and $\Gamma'(f)$ have boundaries

$$\begin{tikzcd} \partial_0(f) \arrow[r, "f"] \arrow[d, "f"'] & \partial_1(f) \arrow[d, "1_{\partial_1(f)}"] \\ \partial_1(f) \arrow[r, "1_{\partial_1(f)}"'] & \partial_1(f) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \partial_0(f) \arrow[r, "1_{\partial_0(f)}"] \arrow[d, "1_{\partial_0(f)}"'] & \partial_0(f) \arrow[d, "f"] \\ \partial_0(f) \arrow[r, "f"'] & \partial_1(f) \end{tikzcd}$$

respectively, for a in $\mathcal{D}_1$, and such that Γ and Γ' satisfy several further conditions which we now describe. First, they are required to preserve identities in the sense that $\Gamma(1_a) = 1_{1_a} = \Gamma'(1_a)$ for a an object of $\mathcal{D}$. Next, it is required that, for arrows $f \colon a \longrightarrow b$ and $g \colon b \longrightarrow c$, $\Gamma(g \circ f)$ and $\Gamma'(g \circ f)$ are equal to the composites

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\Gamma(f)"] & b \arrow[r, "g"] \arrow[d, "1_b"] \arrow[dr, phantom, "1^v_g"] & c \arrow[d, "1_c"] \\ b \arrow[r, "1_b"'] \arrow[d, "g"'] \arrow[dr, phantom, "1^h_g"] & b \arrow[r, "g"'] \arrow[d, "g"] \arrow[dr, phantom, "\Gamma(g)"] & c \arrow[d, "1_c"] \\ c \arrow[r, "1_c"'] & c \arrow[r, "1_c"'] & c \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "1_a"'] \arrow[dr, phantom, "\Gamma'(f)"] & a \arrow[r, "1_a"] \arrow[d, "f"] \arrow[dr, phantom, "1^h_g"] & a \arrow[d, "f"] \\ a \arrow[r, "f"'] \arrow[d, "1_a"'] \arrow[dr, phantom, "1^v_g"] & b \arrow[r, "1_b"'] \arrow[d, "1_b"] \arrow[dr, phantom, "\Gamma'(g)"] & b \arrow[d, "g"] \\ a \arrow[r, "f"'] & b \arrow[r, "g"'] & c \end{tikzcd}$$

respectively. Finally, we require that Γ and Γ' are inverse to one another in the sense that $\Gamma(f) \circ_h \Gamma'(f) = 1^v_f$ and $\Gamma'(f) \circ_v \Gamma(f) = 1^h_f$.

We will make use of the following result in the proof of our main theorem.

2.13. Theorem. [Brown and Mosa (Corollary 4.4 in [2])] *On an edge symmetric double category $\mathcal{D}$, connections correspond to morphisms $\Theta \colon \Box\mathcal{D} \longrightarrow \mathcal{D}$ which are identity on objects and arrows, where $\Box\mathcal{D}$ is the double category which has the same objects and 1-cells as $\mathcal{D}$, but with 2-cells given by commutative squares.*

We now turn to our main theorem.

2.14. Theorem. *An interval I in $\mathcal{E}$ is representable if and only if it has injective boundaries and possesses binary meet and join operations such that both $\wedge$ and $\wedge \circ \tau$ are 2-cells $\bot \circ i \Longrightarrow 1_I$, and both $\vee$ and $\vee \circ \tau$ are 2-cells $1_I \Longrightarrow \top \circ i$.*

Proof. It follows from Proposition 2.8 and Lemma 2.6 that a representable interval possesses the required properties.

For the other direction of the equivalence it suffices, by the discussion in Section 2.3, to prove that $\partial\colon \mathcal{E}(B,A)^\flat \longrightarrow \mathcal{E}(B,A)^\natural$, which is immediately seen to be natural in A and B, is an isomorphism of double categories. For this, we first observe that using $\vee$ and $\wedge$ we obtain a connection on the double category $\mathcal{E}(B,A)^\flat$ by letting Γ and Γ' send $f\colon B\otimes I \longrightarrow A$ to the composites $f\circ(B\otimes\vee)\circ\alpha^{-1}$ and $f\circ(B\otimes\wedge)\circ\alpha^{-1}$, respectively. By definition, $\Gamma(f)$ and $\Gamma'(f)$ have the correct boundaries and, because I has injective boundaries, the remaining conditions on a connection are also satisfied. By Theorem 2.13, there exists a map Θ of double categories $\square\mathcal{E}(B,A)^\flat = \mathcal{E}(B,A)^\natural \longrightarrow \mathcal{E}(B,A)^\flat$ which is identity on objects and arrows. In particular, Θ is a section of ∂ and therefore, by injective boundaries, ∂ is an isomorphism of double categories. ∎

Although most of the examples of intervals studied earlier are already known to give rise to finitely bicomplete 2-category structures, it is nonetheless instructive to consider these cases in light of the theorem.

2.15. Example. Consider the following intervals:

1. The interval I obtained by taking the discrete cocategory on the tensor unit U is representable, with meet and join both the structure map $\lambda=\rho\colon U\otimes U\longrightarrow U$.

2. Using the isomorphism $(U+U)\otimes(U+U)\cong(U+U)+(U+U)$ it is easily seen that the interval $(U+U)$ satisfies the necessary and sufficient conditions from Theorem 2.14 for being representable and therefore gives rise to a finitely bicomplete 2-category.

3. In **Cat** both $\mathbf{2}$ and $\mathbf{I}$ are representable. Of course, this can be easily verified directly, but one can also check that the hypotheses of the theorem are satisfied. For instance, in both cases the meet map $\wedge$ is the functor which sends an object (s,t) of $\mathbf{2}\times\mathbf{2}$ to $\top$ if $s=t=\top$ and to $\bot$ otherwise.

4. We will now give an example of an interval giving rise to a 2-category structure, but which is *not* representable. Let us work in **Cat** with the cartesian monoidal structure. Let L be the free category on the graph with one vertex μ and one edge $\omega\colon\mu\longrightarrow\mu$. (I.e., it is the free monoid on a single generator.) Then L_2 is the free category with one vertex μ and two $l,r\colon\mu\longrightarrow\mu$ and $\downarrow(\omega)=l$, $\uparrow(\omega)=r$, $\star(\omega)=r\circ l$. This is an interval in **Cat** and it induces a trivial notion of homotopy. Namely, for functors $F,G\colon A\longrightarrow B$ if $F\simeq G$, then $F=G$ and there exists for each a in A a loop $\varphi_a\colon Fa\longrightarrow Fa$ such that

$$\varphi_b\circ Ff = Ff\circ\varphi_a$$

for $f\colon a\longrightarrow b$ in A. Roughly, this interval generates the same notion of homotopy as the discrete interval, *but* the data of a homotopy for L is not the same as the data of a homotopy for the discrete interval. As such, the resulting 2-category structures are not (*a priori*) the same.

The interval L can possess neither meet nor join operations since in this case $\bot = \top$ and so we would have

$$1_\mu = \bot = \omega \wedge \bot = \omega \wedge \top = \omega$$

which is false. Thus, by the characterization theorem it follows that L is not representable.

3. Homotopy theoretic consequences

The purpose of this section is to relate the considerations on intervals from the foregoing sections to several known results from homotopy theory. In particular, we show that, under suitable hypotheses on $\mathcal{E}$, if I is a representable interval in $\mathcal{E}$, then the "isofibration" model structure on $\mathcal{E}$ due to Lack [12] can be lifted to the category of (reduced) operads using a theorem of Berger and Moerdijk [1]. In order to apply the machinery of *ibid* it is first necessary to construct a *Hopf interval*, which is essentially a cylinder object equipped with the structure of a Hopf object. As such, the principal observation in this section is that, when $\mathcal{E}$ is cocomplete in the 1-dimensional sense, it is possible to construct the free Hopf object generated by the interval I. We refer the reader to [6] for more information regarding model categories.

Although we will not consider those intervals I which fail to be representable (or to give rise to 2-categories) in our discussion of homotopy theory below, we would like to mention that some effort has been made to investigate the homotopy theory of intervals arising in the setting of such categories as the category of chain complexes. In particular, Stanculescu [15] has employed intervals in his work on the homotopy theory of categories enriched in simplicial modules.

3.1. The isofibration model structure Now, assuming (as we will throughout the remainder of this section) that $\mathcal{E}$ is a finitely bicomplete symmetric monoidal closed category with a representable interval I, it follows from a theorem due to Lack [12] that $\mathcal{E}$ can be equipped with a model structure in which the weak equivalences are the categorical equivalences and the fibrations are isofibrations. Recall that an arrow $f\colon A \longrightarrow B$ in a 2-category is said to be a *categorical equivalence* if there exists a map $f'\colon B \longrightarrow A$ together with isomorphisms $f \circ f' \cong 1_B$ and $f' \circ f \cong 1_A$. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ in **Cat** is said to be an isofibration when isomorphisms in $\mathcal{D}$ whose codomains lie in the image of F can be lifted to isomorphisms in $\mathcal{C}$. This notion also makes sense in arbitrary 2-categories $\mathcal{E}$. We define a map $f\colon A \longrightarrow B$ in $\mathcal{E}$ to be an *isofibration* if, for any object E of $\mathcal{E}$, the induced map

$$\mathcal{E}(E, A) \xrightarrow{f_*} \mathcal{E}(E, B)$$

is an isofibration in **Cat**.

3.2. Definition. *Assume $\mathcal{E}$ is a finitely bicomplete 2-category with a model structure. Then $\mathcal{E}$ is a* model **Cat**-category *provided that if $p\colon E \longrightarrow B$ is a fibration and $i\colon X \longrightarrow Y$ is a cofibration, then the induced functor*

$$\mathcal{E}(Y,E) \xrightarrow{\langle p_*, i^* \rangle} \mathcal{E}(Y,B) \times_{\mathcal{E}(X,B)} \mathcal{E}(X,E)$$

is an isofibration which is simultaneously an equivalence if either p or i is a weak equivalence.

With these definitions, Lack [12] proved the following theorem:

3.3. Theorem. [Lack] *If $\mathcal{E}$ is a finitely bicomplete 2-category, then it bears the structure of a model* **Cat***-category in which the weak equivalences are the equivalences, the fibrations are the isofibrations and the cofibrations are those maps having the left-lifting property with respect to maps which are simultaneously fibrations and weak equivalences.*

We will refer to such a model structure on a 2-category $\mathcal{E}$ as the *isofibration model structure* on $\mathcal{E}$. Every object is both fibrant and cofibrant in this model structure. It is an immediate consequence of Theorem 3.3 that when $\mathcal{E}$ is a finitely bicomplete symmetric monoidal closed category with a representable interval I it is also a model **Cat**-category with the isofibration model structure.

3.4. The free Hopf interval generated by I In [1], a (commutative) *Hopf interval* in a symmetric monoidal model category is defined to be a cylinder object

$$\begin{tikzcd} U+U \arrow[rr] \arrow[dr, "\nabla"'] & & H \arrow[dl] \\ & U & \end{tikzcd}$$

on U such that H is a (commutative) Hopf object, and both maps $U + U \longrightarrow H$ and $H \longrightarrow U$ are homomorphisms of Hopf objects, where to be a cylinder object means that $U + U \longrightarrow H$ is a cofibration and $H \longrightarrow U$ is a weak equivalence. Here U has the trivial Hopf object structure given by by structure map $\lambda\colon U \otimes U \longrightarrow U$ and its inverse. On the other hand, $(U+U)$ is given the structure of a commutative Hopf object using the "meet" Hopf object structure described in Corollary 2.9 (cf. Example (2) from Section 1.8), which coincides with the Hopf object structure on described in *ibid.* We emphasize that a Hopf interval need not be an interval in the sense of Definition 1.6. The following lemma shows that we cannot in general expect I itself to be a Hopf interval in the isofibration model structure.

3.5. Lemma. *The following are equivalent:*

1. *I is invertible.*

2. *The structure map $\lambda\colon U \otimes I \longrightarrow I$, regarded as a 2-cell $\bot \Longrightarrow \top$, possesses an inverse $\neg\colon \top \Longrightarrow \bot$.*

3. *The meet operation, regarded as a 2-cell* $\perp \circ i \Longrightarrow 1_I$*, has an inverse. (Or, dually, the join operation has an inverse.)*

4. *The map* $i\colon I \longrightarrow U$ *is an equivalence.*

PROOF. (1) and (2) are equivalent since $\neg$ is defined using the existence of a coinverse map $\sigma\colon I \longrightarrow I$ to be $\sigma \circ \lambda$, and, going the other way, σ is defined in terms of $\neg$ as $\neg \circ \lambda^{-1}$. To see that (2) implies (3), define $\wedge'\colon I \otimes I \longrightarrow I$ to be the composite

$$I \otimes I \xrightarrow{I \otimes \lambda^{-1}} I \otimes (U \otimes I) \xrightarrow{I \otimes \neg} I \otimes I \xrightarrow{\quad \wedge \quad} I.$$

It is then easily seen that $\wedge'$ is the inverse of $\wedge$ as arrows in the category $\mathcal{E}(I, I)$.

That (3) implies (4) follows from the fact that $\wedge$ is a 2-cell $\perp \circ i \Longrightarrow 1_I$ and therefore the existence of an inverse for this homotopy implies that i is an equivalence. Going the other way, to see that (4) implies (2), assume given $k\colon U \longrightarrow I$ together with isomorphisms $\varphi\colon k \circ i \Longrightarrow 1_I$ and $\psi\colon 1_U \Longrightarrow i \circ k$. Then we define the inverse $\neg\colon \top \Longrightarrow \perp$ of λ in $\mathcal{E}(U, I)$ as follows. We first form the vertical composite $\xi\colon = (\varphi * \perp)$ which is, by definition, a 2-cell $k \Longrightarrow \perp$. Similarly, we define $\zeta\colon \top \Longrightarrow k$ to be the vertical composite $(\varphi^{-1} * \top)$. We then set $\neg\colon = (\xi \cdot \zeta)$. That $\neg$ is the inverse of λ is seen to hold by straightforward calculations. For example, that $\lambda \cdot \neg$ is $1_\top$ follows from the fact that, by the interchange law,

$$\lambda \cdot \neg = (\varphi * \lambda) \cdot \zeta.$$

Moreover, $(\varphi * \lambda) = (\varphi * \top)$ since $(i * \lambda) = 1_U$. Thus, $(\lambda \cdot \neg) = 1_\top$ by another application of the interchange law and the fact that $(\varphi \cdot \varphi^{-1}) = 1_I$. ■

3.6. REMARK. We mention a further equivalence, which we will not require and which is easily verified using representability of I. Namely, I is invertible if and only if it is a Boolean algebra.

By Lemma 3.5 it follows that, for example, $\mathbf{2}$ is not a Hopf interval in **Cat**. Nonetheless, when $\mathcal{E}$ it is possible to construct in the expected manner the free Hopf interval J generated by I. Specifically, where $\mathbf{I}$ is the category described in Example (3) from Section 1.8 and where $\mathbf{I} \cdot -$ denotes the tensor with $\mathbf{I}$ and exists since I is representable, we set $J\colon = \mathbf{I} \cdot U$. It is then immediate that J is an invertible interval with symmetry map $\sigma_J\colon J \longrightarrow J$ and there exists a morphism of intervals $\iota\colon I \longrightarrow J$ with which exhibits J as the free invertible interval generated by I in the sense that, for any morphism of intervals $\xi\colon I \longrightarrow H$ with H invertible, there exists a canonical map of intervals $\bar{\xi}\colon J \longrightarrow H$ extending ξ along the inclusion ι. Additionally, J classifies the invertible 2-cells in the 2-category structure induced by I as described in the following lemma (the proof of which is routine).

3.7. LEMMA. *Given morphisms* f *and* g *in* $\mathcal{E}(B, A)$ *together with a 2-cell* $\alpha\colon f \Longrightarrow g$ *in the 2-category structure induced by* I*,* α *is invertible if and only if there exists a canonical* *extension* $\bar{\alpha}\colon B \otimes J \longrightarrow A$ *such that the following diagram commutes:*

$$\begin{tikzcd} B \otimes J \arrow[rr, dotted, "\bar{\alpha}"] & & A \\ & B \otimes I \arrow[ul, "B\otimes\iota"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Using Lemma 3.7 it is possible to construct meet and join operations on J as such canonical extensions. For example, using representability of I, a straightforward calculation shows that, regarded as a 2-cell $\perp_J \circ i \Longrightarrow \iota$, the map $\iota \circ \wedge\colon I \otimes I \longrightarrow J$ has as its inverse the vertical composite $f \cdot (\iota \circ \vee)$ where f is defined to be

$$I \otimes I \xrightarrow{i \otimes I} U \otimes I \xrightarrow{\lambda} I \xrightarrow{\iota} J \xrightarrow{\sigma_J} J.$$

Thus, by Lemma 3.7 there exists a canonical extension $\bar{\wedge}\colon I \otimes J \longrightarrow J$. Applying the same trick one more time, using the symmetry map $\tau\colon I \otimes J \longrightarrow J \otimes I$, yields the required meet operation $\wedge_J\colon J \otimes J \longrightarrow J$. The construction of join is dual. This construction gives us the following lemma.

3.8. LEMMA. *J is a representable interval if I is.*

PROOF. We have seen that J possesses meet and join operations which satisfy the required equations by construction. Thus, by Theorem 2.14 it suffices to show that if $\varphi, \psi\colon B \otimes J \otimes J \rightrightarrows A$ are cells in $\mathcal{E}_J(B, A)^\flat$ which agree on their boundaries, then they are in fact equal. Since ι is a morphism of intervals and I is representable it follows that φ and ψ are equal upon precomposition with $(B \otimes \iota) \otimes \iota$. By construction of J it then follows that they are likewise equal upon precomposition with $(B \otimes J) \otimes \iota$. Finally, by Lemma 3.7, it follows that φ and ψ are equal. ∎

It follows from Lemma 3.8 that there also exists an isofibration model structure on $\mathcal{E}$ defined with respect to J. However, since only invertible 2-cells feature in the specification of the isofibration model structure, it follows, by Lemma 3.7, that this "J-model structure" coincides with the original "I-model structure". As such, we continue to simply refer to *the* isofibration model structure on $\mathcal{E}$ without reference to either interval I or J. Note though that the 2-category structures induced by I and J *do* in general differ. Namely, the 2-cells for J correspond exactly to the invertible 2-cells for I.

3.9. PROPOSITION. *$J = (J, \Delta_J, i_J, \wedge_J, \top_J)$ is the free commutative Hopf interval generated by I in the isofibration model structure.*

PROOF. Since J is an invertible interval it follows from Lemma 3.5 that $i_J\colon J \longrightarrow U$ is a weak equivalence. To see that $[\perp_J, \top_J]\colon U + U \longrightarrow J$ is a cofibration we first observe that, because U is cofibrant , rephrasing the usual argument in terms of the present setting, any weak equivalence $p\colon E \longrightarrow B$ is "full" in the sense of possessing the right-lifting property with respect to the map $[\perp, \top]\colon U + U \longrightarrow I$. Moreover, combined with the fact that any $h\colon I \longrightarrow E$ is invertible (as a 2-cell) whenever $p \circ h\colon I \longrightarrow B$ is, gives by Lemma 3.7 that any weak equivalence p has the right-lifting property with respect to $[\perp_J, \top_J]$ as well.

Observe that $(U+U)$ is the initial object in $\mathbf{Int}(\mathcal{E})$ and $[\bot_J, \top_J]$ is the coobject part of the induced canonical map $(U+U) \longrightarrow J$ of intervals. It therefore follows, by the remark following Corollary 2.9, that this is a morphism of Hopf objects. Similarly, i_J is a morphism of Hopf objects since it is the coobject part of the induced map into the terminal object U in $\mathbf{Int}(\mathcal{E})$.

Finally, for freeness, suppose given another Hopf interval H:

$$U+U \xrightarrow{[a,b]} H \xrightarrow{g} U$$

together with $\xi\colon I \longrightarrow H$ a morphism of commutative Hopf objects. Because g is a weak equivalence, there exists an arrow $g'\colon U \longrightarrow H$ together with an invertible 2-cell $\varphi\colon g' \circ g \Longrightarrow 1_H$. Then the vertical composite $(\varphi * a)\cdot(\varphi^{-1} * b)$ in $\mathcal{E}(U,H)$ is a 2-cell $b \Longrightarrow a$ which is, by the fact that $g\xi = i$, the inverse of ξ. Thus, by Lemma 3.7 there exists a canonical extension $\bar{\xi}\colon J \longrightarrow H$ of ξ which is a morphism of Hopf objects and commutes with $[a,b]$ and g. Finally, $\bar{\xi}$ is trivially the canonical map with these properties. ∎

Berger and Moerdijk [1] have shown that the existence of a commutative Hopf interval is one of several conditions which allow one to lift a model structure from a symmetric monoidal closed category $\mathcal{E}$ to the category of reduced operads over $\mathcal{E}$.

3.10. Theorem. [Berger and Moerdijk] *If a symmetric monoidal closed category $\mathcal{E}$ is a monoidal model category such that $\mathcal{E}$ is cofibrantly generated with cofibrant tensor unit U, $\mathcal{E}/U$ has a symmetric monoidal fibrant replacement functor and $\mathcal{E}$ possesses a commutative Hopf interval, then there exists a cofibrantly generated model structure on the category of reduced operads in which the weak equivalences and fibrations are "pointwise".*

Here that the model category is *monoidal* means that the appropriate internal form of the condition from Definition 3.2 is satisfied (cf. [6]). As such, in light of Theorem 3.10, we obtain the following corollary to Proposition 3.9:

3.11. Corollary. *Assume $\mathcal{E}$ possesses a representable interval I, then when the isofibration model structure is cofibrantly generated there is a model structure on the category of reduced operads over $\mathcal{E}$ in which the fibrations and weak equivalences are pointwise.*

Proof. By the fact that all objects in the isofibration model structure are fibrant and Proposition 3.9 it remains, in order to be able to apply Theorem 3.10, only to verify that the model structure is monoidal. For this, it suffices, by the definition of fibrations in the isofibration model structure, to note that if $f\colon E \longrightarrow B$ is a (trivial) fibration, then so is $f_*\colon [X,E] \longrightarrow [X,B]$ for any object X. First, that f_* is a fibration when f is follows from the tensor-hom adjunction. Next, that f_* is an equivalence when f is follows from the fact that the map

$$\mathcal{E}(E,B) \longrightarrow \mathcal{E}\big([X,E],[X,B]\big)$$

which sends an arrow $f\colon E \longrightarrow B$ to f_* is a functor and therefore preserves isomorphic 2-cells. ∎

A. Hopf objects and lattices in a symmetric monoidal category

In this appendix we provide the full definitions of comonoids, Hopf objects and distributive lattices in a symmetric monoidal category.

A.1. Monoids, comonoids and Hopf objects A *monoid* (M, η, m) in a symmetric monoidal category $\mathcal{E}$ is given by an object M of $\mathcal{E}$ together with arrows $\eta\colon U \longrightarrow M$ and $m\colon M \otimes M \longrightarrow M$ satisfying the following diagrams commute:

$$\begin{tikzcd}
M \otimes U \arrow[d, "M\otimes\eta"'] & M \arrow[l, "\rho^{-1}"'] \arrow[r, "\lambda^{-1}"] \arrow[d, "M"] & U \otimes M \arrow[d, "\eta\otimes M"] \\
M \otimes M \arrow[r, "m"'] & M & M \otimes M \arrow[l, "m"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \otimes (M \otimes M) \arrow[rr, "\alpha"] \arrow[d, "M\otimes m"'] & & (M \otimes M) \otimes M \arrow[d, "m\otimes M"] \\
M \otimes M \arrow[dr, "m"'] & & M \otimes M \arrow[dl, "m"] \\
& M &
\end{tikzcd}$$

A *comonoid* (G, ϵ, Δ) in a symmetric monoidal category $\mathcal{E}$ is given by an object G of $\mathcal{E}$ together with arrows $\epsilon\colon G \longrightarrow U$ and $\Delta\colon G \longrightarrow G \otimes G$ such that the following diagrams commute:

$$\begin{tikzcd}
M \otimes M \arrow[d, "\epsilon\otimes M"'] & M \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "M"] & M \otimes M \arrow[d, "M\otimes\epsilon"] \\
U \otimes M \arrow[r, "\lambda"'] & M & M \otimes U \arrow[l, "\rho"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
& M \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
M \otimes M \arrow[d, "\Delta\otimes M"'] & & M \otimes M \arrow[d, "M\otimes\Delta"] \\
(M \otimes M) \otimes M \arrow[rr, "\alpha^{-1}"'] & & M \otimes (M \otimes M)
\end{tikzcd}$$

A *(commutative) Hopf object* in $\mathcal{E}$ is a structure $(H, \eta, m, \epsilon, \Delta)$ such that (H, ϵ, Δ) is a comonoid, (H, η, m) is a (commutative) monoid, and the maps m and η are comonoid homomorphisms (cf. [1]). Here note that $H \otimes H$ is given the structure of a comonoid via the map, constructed using the symmetry τ, which (schematically) sends $x \otimes y$ to $\big((x \otimes y) \otimes (x \otimes y)\big)$.

A.2. Lattices Assume that (L, ϵ, Δ) is a comonoid in $\mathcal{E}$. Then L is a *lattice* if there are maps $\vee\colon L \otimes L \longrightarrow L$ and $\wedge\colon L \otimes L \longrightarrow L$ such that both $\vee$ and $\wedge$ are associative, commutative, and following diagrams commute:

$$\begin{tikzcd}
L \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "L"] & L \otimes L \arrow[d, "\vee"] \\
L \otimes L \arrow[r, "\wedge"'] & L
\end{tikzcd} \tag{6}$$

and

$$\begin{tikzcd}
L \otimes L \arrow[r, "\Delta \otimes L"] \arrow[d, "L \otimes \epsilon"'] & (L \otimes L) \otimes L \arrow[r, "\alpha^{-1}"] & L \otimes (L \otimes L) \arrow[d, "L \otimes \clubsuit"] \\
L \otimes U \arrow[dr, "\rho"'] & & L \otimes L \arrow[dl, "\diamondsuit"] \\
& L &
\end{tikzcd} \tag{7}$$

for $\clubsuit = \vee$ and $\diamondsuit = \wedge$, or $\clubsuit = \wedge$ and $\diamondsuit = \vee$.

A lattice L is *unital* if there exist maps $\top, \bot\colon U \rightrightarrows L$ such that $\top$ is a unit for $\wedge$, $\bot$ is a unit for $\vee$, and the following diagram commutes:

$$\begin{tikzcd}
U \otimes L \arrow[r, "t \otimes L"] \arrow[d, "\lambda"'] & L \otimes L \arrow[d, "\diamondsuit"] & L \otimes U \arrow[l, "L \otimes t"'] \arrow[d, "\rho"] \\
L \arrow[r, "t \circ \epsilon"'] & L & L \arrow[l, "t \circ \epsilon"]
\end{tikzcd} \tag{8}$$

for $\diamondsuit = \wedge$ and $t = \bot$, or $\diamondsuit = \vee$ and $t = \top$.

A lattice L is distributive if the further diagram commutes:

$$
\begin{tikzcd}
 & L \otimes (L \otimes L) \arrow[rr, "L\otimes\clubsuit"] \arrow[dl, "\alpha\circ(\Delta\otimes(L\otimes L))"'] & & L \otimes L \arrow[dr, "\diamondsuit"] & \\
\big((L \otimes L) \otimes L\big) \otimes L \arrow[d, "\alpha^{-1}\otimes L"'] & & & & L \\
\big(L \otimes (L \otimes L)\big) \otimes L \arrow[dr, "\tau\otimes L"'] & & & & L \otimes L \arrow[u, "\clubsuit"'] \\
 & \big((L \otimes L) \otimes L\big) \otimes L \arrow[rr, "\alpha^{-1}"'] & & (L \otimes L) \otimes (L \otimes L) \arrow[ur, "\diamondsuit\otimes\diamondsuit"'] &
\end{tikzcd}
\tag{9}
$$

for either $\clubsuit = \vee$ and $\diamondsuit = \wedge$, or $\clubsuit = \wedge$ and $\diamondsuit = \vee$.

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